YANGIAN SYMMETRY FOR THE ACTION OF THE $\mathcal{N} = 4$ SUPERSYMMETRIC YANG–MILLS THEORY

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Abstract

A Yangian algebra based on the superalgebra $\text{psu}(2,2|4)$, $\text{Y}(\text{psu}(2,2|4))$, has been identified as the symmetry algebra governing the behavior of numerous observables of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory in the planar limit. It appeared in the study of the scattering amplitudes as the dual superconformal symmetry, it was shown to be a symmetry of the smooth Wilson loops, and made appearance in the investigation of the spectrum of the theory.

In this thesis we show that the $\text{Y}(\text{psu}(2,2|4))$ is also a symmetry of the classical action of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. In this way we relate the discoveries made on the level of various observables to the usual notion of the symmetry of the theory as the invariance of its action. This is achieved first by understanding the behavior of the equations of motion of the theory under the action of the Yangian generators. This leads us to propose the notion of a weak and a strong invariance of the equations of motion. The strong invariance under the Yangian symmetry is subsequently argued to be equivalent to the invariance of the action. We then use it to derive how the Yangian generators are to be applied to the action of the theory in order to demonstrate that it is indeed left invariant.

We show that the symmetry can be preserved also in the presence of gauge-fixing terms. To this end we introduce new types of Yangian-like generators, which combine local and global transformations. We then derive the identity satisfied by the action of the $\mathcal{N} = 4$ supersymmetric Yang–Mills theory with the addition of the BRST gauge-fixing terms.

Finally we propose Slavnov–Taylor identities for the Yangian symmetry. These are the identities satisfied by the correlation functions of the fundamental fields of the theory. We demonstrate that our proposed identities hold in numerous examples due to invariance of the action, also taking into account the effects of gauge-fixing. We also argue why the symmetry should hold in the quantum theory.
Zusammenfassung

Die Yang’sche Algebra $Y(\text{psu}(2,2\mid 4))$ basierend auf der Superalgebra $\text{psu}(2,2\mid 4)$ wurde als die Symmetriealgebra identifiziert, die das Verhalten zahlreicher Observablen der planaren $\mathcal{N} = 4$ supersymmetrischen Yang–Mills-Theorie bestimmt. Sie erschien in der Untersuchung der Streuamplituden als die duale superkonforme Symmetrie. Es wurde gezeigt, dass sie eine Symmetrie der glatten Wilson’schen Schleifen ist. Sie erschien bei der Untersuchung des Spektrums der Theorie.

In dieser Arbeit zeigen wir, dass $Y(\text{psu}(2,2\mid 4))$ auch eine Symmetrie der klassischen Wirkung der $\mathcal{N} = 4$ supersymmetrischen Yang–Mills-Theorie ist. Auf diese Weise verbinden wir die gemachten Entdeckungen auf der Ebene verschiedener Observablen mit der üblichen Vorstellung einer Symmetrie als Invarianz der Wirkung einer Theorie. Dies wird zuerst erreicht, indem man die Wirkung der Yang’schen Generatoren auf die Bewegungsgleichungen versteht. Dies führt uns dazu, die Vorstellung einer schwachen und einer starken Invarianz der Bewegungsgleichungen vorzuschlagen. Die starke Invarianz unter der Yang’schen Symmetrie wird später als gleichwertig zur Invarianz der Wirkung betrachtet. Wir verwenden sie dann, um abzuleiten, wie die Yang’schen Generatoren auf die Wirkung der Theorie angewendet werden müssen, um zu zeigen, dass die Wirkung invariant bleibt.

Wir zeigen, dass die Symmetrie auch in Gegenwart von Eichfixierenden Termen erhalten werden kann. Zu diesem Zweck führen wir neue Arten von Yangian-ähnlichen Generatoren ein, die lokale und globale Transformationen kombinieren. Wir leiten dann die Identität her, die durch die Wirkung der $\mathcal{N} = 4$ supersymmetrischen Yang–Mills-Theorie inklusive der BRST-Eichfixierungsterme befriedigt wird.

Schliesslich schlagen wir Slavnov–Taylor-Identitäten für die Yang’sche Symmetrie vor. Dies sind die Identitäten, denen die Korrelationsfunktionen der fundamentalen Felder der Theorie genügen. Wir zeigen, dass unsere vorgeschlagenen Identitäten aufgrund der Invarianz der Wirkung in zahlreichen Beispielen erfüllt sind, wobei auch die Effekte der Eichfixierung berücksichtigt werden. Wir argumentieren auch, warum die Symmetrie in der Quantentheorie gelten sollte.
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Chapter 1

Introduction

History and motivation

The three decades of 1950s to 1970s have been enormously successful for the area of particle physics. Both theoretical as well as experimental research allowed the construction and verification of the Standard Model (see e.g. [1] for a short review and further references), a quantum field theory perfectly describing the interactions of fundamental particles. Up to this date the theory firmly withstands all its experimental tests (see e.g. [2] for a recent discussion). The most recent one was the 2012 discovery of the Higgs boson [3], a particle necessary to give masses to the other ones, almost 50 years after its theoretical prediction [4]. However, there still exist some unresolved issues within the fundamental physics: gravity cannot be described on par with the strong and electroweak interactions which are captured by the Standard Model; we do not have a candidate for a dark matter particle [5], even though it seems necessary to explain the current state of the Universe; we do not know the precise mechanism giving masses to neutrinos [6], but we know they are not massless. On top of that, Standard Model has a large number of parameters which we need to tune by hand. All those aspects suggest that a more fundamental theory should exist, one which would resolve those puzzles. However, without any experimental signatures, the theoretical physics community does not have a clear direction as to what such theory should look like.

Although an inconvenience, the lack of experimental markers does not stop the theoretical progress. For even if the work does not directly relate to the world around us, it still helps us understand the rules governing it better, the assumption being that some properties of the models under study are universal. Additionally, theoretical works can help in pointing the experimenters in the right directions, thereby fueling the theory-experiment cooperation. On the opposite end of the spectrum, some developments of theoretical physics catch the interest of mathematicians, leading to progress in that discipline. One could trace back such an interplay as far back as to Newton’s invention of calculus [7], but more recent examples include Fields medals being awarded for physics-inspired work to Edward Witten (1990), Wendelin Werner (2006) and Stanislav Smirnov (2010) [8].

One huge such theoretical development started also in the 1970s, in parallel with the development of the constituent theories of the Standard Model. String Theory (see e.g. [9] for an introduction) indeed originated as a theory of strong interactions aimed at describing partons. Although Quantum Chromodynamics (QCD) has later been universally accepted as the correct description thereof (see e.g. Wilczek’s 2004 Nobel Lecture [10]), and partons have been identified as quarks and gluons, String Theory has very much remained in focus of theoretical physicists. To this day it still remains a strong candidate for a theory of everything [11], especially as the only
framework capable of describing both gauge theories as well as gravity. As the Standard Model constituent theories are indeed Yang–Mills gauge theories (For example, QCD is a Yang–Mills theory with a gauge group $SU(3)$. This gauge group describes local symmetries of the theory which relate equivalent physical configurations.), the initial hope was that one could reproduce it from string theory. Although no direct link between string theory and phenomenology has been established to date, with many of the obstacles stemming from the fact that for consistency (supersymmetric) String Theory requires a ten-dimensional spacetime, it certainly led to many interesting discoveries of a mathematical nature (for example, the explanation of Monstrous Moonshine [12] or the description of Mirror Symmetry [13]).

AdS/CFT correspondence

A very important result within string theory was obtained in the mid-1990s. Following the discovery that the entropy of black hole scales with its (horizon’s) area, and not volume as expected [14], physicists including e.g. ‘t Hooft [15] advocated the holographic principle (see [16] for a review). The holographic principles states that the description of a volume of a spacetime in a gravitational theory should be encoded in the degrees of freedom on its boundary. It was first realized in [17]. Its most concrete realization however is the AdS/CFT correspondence proposed in [18] and further developed in [19]. As the name indicates, the correspondence relates a gravitational theory formulated on an Anti-de Sitter (AdS) background, a solution of Einstein’s equations with a negative cosmological constant, to a Conformal Field Theory (CFT) in a flat Minkowski spacetime of one dimension less than AdS. In the form advocated in [18], a supersymmetric string theory defined on the product manifold \( \text{AdS}_d \times M^{10-d} \) (where \( M \) is a compact manifold)- which in a certain limit becomes a (supersymmetric) gravity theory - maps to a non-gravitational supersymmetric CFT on flat spacetime of dimension \( d-1 \). One usually pictures it as the CFT degrees of freedom living on the conformal boundary of the AdS spacetime (which is indeed flat Minkowski spacetime of one dimension less; the translations in the remaining dimension of AdS are then identified with the renormalization group (RG) flow [20]). The AdS/CFT correspondence is a weak/strong duality, meaning that a strongly interacting, complicated regime of one of the theories is mapped to a weakly interacting limit of its counterpart. That renders proving the conjectured duality very difficult. On the other hand, assuming duality between various pairs of theories allows one to make predictions in the regimes normally inaccessible with the usual computational tool of perturbation theory.

That computational approach has evolved into some independent branches of high energy theoretical physics, which apply AdS/CFT to other research areas. A common use is the study of strongly-coupled field theories of direct experimental interest (condensed matter systems [21] or QCD [22]) by mapping them to some engineered superstring theories which in this limit become tractable supergravities. Indeed, holographic superconductors [23] have been actively studied, and one of the huge initial successes of the field was the computation of \( \frac{\eta}{S} \) (viscosity per entropy unit) of the quark-gluon plasma of QCD [24]. These approaches also motivate a more general name of gauge/gravity duality, as the gauge theories considered are not always conformal, and the gravity backgrounds deviate from pure AdS.

The arguably best-studied example of the AdS/CFT correspondence however, already introduced in [18] (see also [25] or [26] for a pedagogical introduction), postulates the duality between the type IIB String Theory on the \( \text{AdS}_5 \times S^5 \) space and our titular \( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory in 4 dimensions (\( \mathcal{N} = 4 \text{sYM} \)). Due to the relationships between the couplings of the
theories, the non-perturbative regime of the gauge theory is captured by the free classical string theory, whereas non-perturbative free quantum strings are described by the perturbative regime of the gauge theory in its planar limit (when rank (size) of the gauge group becomes large; see Chapter 2 for the discussion thereof). It is hence indeed a weak/strong duality, as stated before. The duality is conjectured to hold for all values of the couplings, but the proof is still missing.

The $\mathcal{N} = 4$ supersymmetric Yang–Mills theory

Having introduced the background, we can now motivate the research presented in this thesis, as well as understand how it fits in the bigger picture. The study in this work will be exclusively dedicated to the gauge theory side of the AdS/CFT correspondence in its original form. Let us thus concentrate on the $\mathcal{N} = 4$ sYM theory, first described in [27]. Even considered separately from the gauge/gravity duality, the $\mathcal{N} = 4$ sYM is a theory worth studying for a number of its unusual properties which make it stand out among the four-dimensional gauge theories.

To start with, it possesses the maximal amount of supersymmetry ($\mathcal{N} = 4$) which a four-dimensional theory can possess without incorporating gravity. Knowing that supersymmetry transformations change the spin of the particle by $\pm \frac{1}{2}$, this result is very intuitive: there are exactly 4 steps of size $\frac{1}{2}$ when going down in allowed spin (projection) values from $+1$ to $-1$, which are the possible values for a massless vector field. The spin-$\pm \frac{1}{2}$ states are then fermions, whereas spin-0 states are scalars. It also is a unique such theory (up to the choice of gauge group), with its field content and the action completely fixed by the symmetries (This is unlike in the general $\mathcal{N} = 2$ supersymmetric gauge theories where the amount of supersymmetry still allows a lot of freedom in specifying the matter content, see e.g. [28]).

Furthermore, its conformal symmetry persists also at the quantum level. Even though classically in four dimensions Yang–Mills theories like Quantum Electrodynamics (QED) or QCD are also conformal, upon taking into account the quantum corrections an energy scale emerges, breaking this symmetry. The initially dimensionless coupling becomes dimensionful due to dependence on this energy scale (often denoted $\mu$) and changes (runs) as the energy changes (see e.g. [29], [30] or [31]). In $\mathcal{N} = 4$ sYM however this is not the case: the Yang–Mills coupling $g_{YM}$ is independent of the energy scale $\mu$, and hence conformal symmetry is preserved also on the quantum level. It can be seen easily at 1-loop level using the formula [31]:

$$\beta_{1\text{-loop}} = -\mu \frac{\partial g_{YM}}{\partial \mu} = - \frac{g_{YM}^3}{16\pi^2} \left( \frac{11}{3} N - \frac{1}{6} \sum_{i\in\text{scalars}} C_i - \frac{2}{3} \sum_{j\in\text{fermions}} \tilde{C}_j \right).$$

Since all the fields are in the adjoint representation with the Casimirs given by $N$, and we have 6 scalars and 4 fermions, we see that the $\beta_{1\text{-loop}} = 0$. This result has been argued to persist to all loops [32], thus resulting in the four-dimensional conformal field theory on the quantum level.

Combining the supersymmetry and the conformal symmetry one discovers that the $\mathcal{N} = 4$ sYM theory is invariant under a Lie superalgebra commonly denoted as $\text{psu}(2,2|4)$ (see e.g. [33] for a detailed review), consisting of 30 bosonic and 32 fermionic generators. This persisting large symmetry algebra allows one to constrain observables in the quantum theory by demanding that they remain invariant under the transformations that $\text{psu}(2,2|4)$ describes. The pivotal example was the determination of the AdS/CFT S-Matrix\footnote{This is the S-Matrix behind the scattering of magnons in the spin-chain description of two-point function (see Chapter 2) and is not to be confused with the S-Matrix describing scattering processes in $\mathcal{N} = 4$ sYM.} in [34] together with the Bethe equations al-
lowing one to compute the spectrum of the theory (dimensions of various operators). Another example is the cusp anomalous dimension (the UV-divergence of Wilson loops with an Euclidean cusp) (see [35] for a review), which notably was successfully compared to the string theory predictions, conjectured to be valid for the strongly-coupled gauge theory. Both these results relied on integrability (see [36] for a pedagogical introduction and [37] for an extensive review of the results in the context of $\mathcal{N} = 4$ sYM techniques, and this phenomenon was usually associated with field theories only in two dimensions, and even there it is highly unusual. Here on the other hand it appeared in the planar limit of a four-dimensional gauge theory. The uniqueness of this property makes $\mathcal{N} = 4$ sYM worth studying. Finally, let us briefly discuss another aspect of $\mathcal{N} = 4$ sYM which we will not consider in this thesis, but which deserves mentioning due to its importance: the study of the scattering amplitudes ([38] for a textbook introduction and [39] for a review). The theory’s remarkable simplicity resulted in the development of new techniques and computations being performed to previously unaccessible levels of accuracy, with a hope that some universal features of $\mathcal{N} = 4$ sYM could help us improve the study of Standard Model theories, furnishing an interesting collaboration between pure theory and phenomenology (see e.g. [40]).

At that point let us return to the context of gauge/gravity duality. An important observation rendering the duality plausible is the comparison of the symmetries at both sides. For both the sigma model on $\text{AdS}_5 \times S^5$ as well as $\mathcal{N} = 4$ sYM, the algebra of symmetries turns out to be the superalgebra $\mathfrak{psu}(2,2|4)$ (see [41] for a very detailed overview of the string theory side) which we already mentioned before.

Stunningly, the study of the string theory side in [42] has revealed a larger – infinite – number of symmetries of the string theory side of the duality. Viewed as a two-dimensional field theory, it could have been subjected to a number of tools existing for such models (see [43] for an accessible introduction), rendering the discovery of increased symmetry easier than on gauge theory side. If we however believe that the AdS/CFT correspondence holds, then we should find a similar infinitely-dimensional algebra also in the $\mathcal{N} = 4$ sYM. Alternatively, we could consider its existence as another validation of the conjecture. The $\mathcal{N} = 4$ sYM would then be a unique example of a four-dimensional field theory with an infinite number of global symmetries. The stunning results we have mentioned before, and their reliance on integrability, heavily pointed to the existence of such an algebra, as it is believed that integrable field theories ought to possess infinite number of symmetries ([44]; field theories have infinitely many degrees of freedom, but it is hard to compare infinities hence this statement is usually not made as precise as Liouville’s theorem in classical mechanics). This infinitely dimensional symmetry algebra has soon been discovered and identified as the Yangian algebra based on the $\mathfrak{psu}(2,2|4)$ symmetry, $Y(\mathfrak{psu}(2,2|4))$ [45], thus matching – in terms of dimensionality – the result obtained on the string theory side and providing a stronger reason for why the duality should hold. We will give more details on that in Chapters 2 and 3.

Objective and results of the thesis

Until now the Yangian symmetry has only been found on the level of the observables - correlators, scattering amplitudes or Wilson loops ([46], [47], [48]; see also a more detailed discussion in Chapter 2). One would however desire it to be a symmetry of the action, from which symmetries of observables would follow. This way of studying symmetries of theories is natural in view of the Noether’s theorem [49], as well as the Ward identities (see [50], [51] for original works or [29] for a pedagogical introduction) in quantum theories.

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2This statement being made up to the possible deformations, see Chapter 5.
The main goal of this thesis is to show that indeed $Y(\text{psu}(2,2|4))$ can and should be viewed as an honest symmetry of the action of the planar $\mathcal{N} = 4$ sYM theory, that is that for a $\widehat{J} \in Y(\text{psu}(2,2|4))$ we have:

$$\widehat{J} \cdot S_{\mathcal{N}=4\text{sYM}} = 0.$$  

Once the action $S$ of the theory is shown to be invariant, the existence of the symmetry in various observables is classically expected. In order to argue that this result is not only important for the full understanding of the research discipline, but can be used in practice, we will subsequently demonstrate that the correlation functions of fundamental fields of the $\mathcal{N} = 4$ sYM are invariant under this symmetry by means of the Slavnov–Taylor identities (named after the authors of [52] and [53]). In order to give meaning to this computation, we will also verify that the Yangian symmetry can be preserved even after gauge fixing - a procedure of removing the redundancies necessary in quantization of gauge theories. Due to the interplay of global and gauge symmetries it required a verification - and is not obvious even for the usual Lie-type symmetries. Considerations of gauge fixing will prompt us to introduce novel classes of Yangian-like symmetries, which are possessed by all the gauge theories, but nevertheless necessary to establish the invariance of the gauge-fixed $\mathcal{N} = 4$ sYM under the real Yangian symmetry. We will also make some known results from the BRST (Becchi-Rouet-Stora-Tyutin, [54]) gauge fixing more explicit for further reference. To show the flexibility of our formalism, we also demonstrate that it can be applied to a close cousin of the $\mathcal{N} = 4$ sYM, the $\beta$-deformed supersymmetric Yang–Mills [55]. The deformed theory provides another example of an integrable four-dimensional field theory, as well as a further verification of the AdS/CFT correspondence [56], [57]. We will explicitly demonstrate that the deformation can be reabsorbed into the symmetry generators, thus restoring the full $\text{psu}(2,2|4)$ symmetry of the undeformed $\mathcal{N} = 4$ sYM, subsequently promoting it to the Yangian. In the final chapters we will make the initial steps of investigating the Yangian symmetry at the quantum level. In practical computations, it is obscured by the regularization. We will however provide some heuristic arguments as to why the symmetry is not anomalous, as well as show that the 1-loop integrand of the 3-point correlation function preserves the Yangian symmetry.

This thesis is based on the articles [58], [59], [60] and [61].

**Structure of the thesis**

The content of the thesis is as follows:

- Chapter 2 introduces the $\mathcal{N} = 4$ sYM theory, its global symmetry algebra $\text{psu}(2,2|4)$ and the gauge symmetries, as well as discusses in more detail the occurrences of integrability in its study.

- Chapter 3 serves as the mathematical introduction to the Yangian algebras, providing the minimal mathematical formalism necessary in their study.

- Chapter 4 describes in detail why establishing Yangian invariance of the action is problematic and has not been achieved so far. This Chapter also introduces an equation-of-motion based formalism to circumvent some of the issues. There we prove by explicit calculation that the equations of motions of $\mathcal{N} = 4$ sYM satisfy various identities which we argue to finally be equivalent to the Yangian invariance of the action.
• Chapter 5 introduces a close cousin of $\mathcal{N} = 4$ sYM, the $\beta$-deformed sYM, where integrable structures have been discovered, and uses the formalism introduced in 4 to argue that this theory possesses classically the same symmetry algebra as $\mathcal{N} = 4$ sYM.

• Chapter 6 starts with the results obtained for equations of motion in Chapter 4 and uses them to show that indeed the action of $\mathcal{N} = 4$ sYM is invariant under the generators of Yangian of $\mathfrak{psu}(2, 2|4)$.

• Chapter 7 investigates the interplay between global and local symmetries and introduces new types of (non-local) gauge symmetries.

• Chapter 8 discusses Slavnov–Taylor identities which follow from invariance of the action under the Yangian symmetries.

• Chapter 9 verifies that the correlation functions of fundamental fields indeed satisfy the Slavnov–Taylor identities. First we show it in a simplified setting where gauge-fixing is ignored, later on also including its effects. We also introduce a graphical notation which simplifies the computations.

• Chapter 10 provides some insights towards the loop-level of $\mathcal{N} = 4$ sYM. We show that the integrand of the one-loop correlation function satisfies the Slavnov–Taylor identity proposed in Chapter 8 and argue that the Yangian symmetry may not be anomalous, as we cannot find a suitable anomaly operator.

• Finally in Chapter 11 we list possible further research projects stemming from the material discussed in this thesis.

• The Appendices A, C, B, D and E provide various useful identities and demonstrate computations in more detail.
Chapter 2

\( \mathcal{N} = 4 \) supersymmetric Yang–Mills theory

In this chapter we will discuss the field content, the action as well as the Lie-type symmetries of the \( \mathcal{N} = 4 \) supersymmetric Yang–Mills (\( \mathcal{N} = 4 \) sYM) theory \[27\]. This chapter will serve as a reminder of important facts, as well as allow us to set the notation for the subsequent work.

2.1 Field content and the action

The \( \mathcal{N} = 4 \) sYM theory consists of the following fields: a gauge field \( A_{\dot{\alpha}\alpha} \), fermionic fields \( \Psi^a_\alpha \) and their Hermitian conjugates \( \bar{\Psi}^{\dot{\alpha}a} \) as well as scalars \( \Phi^{ab} \). All the fields are in the adjoint representation of the gauge group which we will take to be \( SU(N) \) unless otherwise specified:

\[
Z_I = \sum_{X=1}^{N^2-1} Z^X_I t^X, \quad (2.1.1)
\]

where \( t^X \) are the matrices that generate the gauge symmetry algebra \( \mathfrak{su}(N) \) and \( Z_I \) stands for any of the fields of the theory:

\[
Z_I = \{ A_{\dot{\alpha}\alpha}, \Psi^a_\alpha, \bar{\Psi}^{\dot{\alpha}a}, \Phi^{ab} \}, \quad (2.1.2)
\]

and the multi-index \( I \) includes all the possible indices discussed above as well as position dependence.

The above fields feature three types of indices; Greek dotted and undotted indices \( \alpha, \dot{\alpha} = 1, 2 \) are the vector indices of either of \( \mathfrak{su}(2) \) factors making up the four-dimensional Lorentz symmetry \( \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \), whereas the Latin indices \( a, b, \ldots = 1, \ldots, 4 \) indicate the internal (R) \( \mathfrak{su}(4) \) symmetry and in case of scalar fields are antisymmetric:

\[
\Phi^{ab} = -\Phi^{ba}. \quad (2.1.3)
\]

For the uniform notation, since we are not using the fourdimensional vector indices on the gauge fields, we will also not use them for the spacetime coordinates. The positions \( x \) and the derivatives \( \partial \) will therefore feature bi-spinor indices and satisfy \( \partial_{a\dot{\alpha}} x^{\dot{\theta}\theta} = \delta^\theta_\theta \delta^\dot{\theta}_\dot{\alpha} \). In general, one

\[\text{Due to the isomorphism } \mathfrak{su}(4) \cong \mathfrak{so}(6) \text{ one frequently encounters scalars with a single index: } \Phi^m = \sigma^{m_{ab}} \Phi^{ab}. \quad \text{We will not use this notation in this thesis.}\]
can recover the usual vector indices with the help of Pauli matrices (where $\sigma^{\alpha\dot{\alpha}}$ is a $2 \times 2$ identity matrix):

$$x^\mu = \frac{1}{\sqrt{2}} \sigma^{\alpha\dot{\alpha}} x^\alpha \dot{\alpha},$$

(2.1.4)

and similarly for gauge fields, derivatives and other objects.

Specifically for the $\mathcal{N} = 4$ supersymmetry, the scalar fields satisfy additionally the following reality condition:

$$\bar{\Phi}_{ab} = \frac{1}{2} \epsilon_{abcd} \Phi^{cd},$$

(2.1.5)

where the fourdimensional totally antisymmetric $\epsilon$ symbol satisfies $\epsilon^{1234} = \epsilon_{1234} = 1$. We will use an analogous convention also for the twodimensional $\epsilon$ symbol: $\epsilon^{12} = \epsilon_{12} = 1$. It will be used to contract the spinor indices, instead of raising and lowering the indices of the respective fields.

The coupling between the gauge field $A_{\alpha\dot{\alpha}}$ and fermions and scalars will come through the covariant derivative, as usually in the gauge theories. The covariant derivative is given by:

$$D_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}} + ig_{YM} A_{\alpha\dot{\alpha}},$$

(2.1.6)

where $g_{YM}$ is the Yang–Mills coupling constant. Classically $g_{YM}$ can be completely reabsorbed by the field redefinitions, so that it appears as the overall rescaling of the action. Therefore we will not write it from now on.

Having introduced the covariant derivative, we are in a good position to discuss the gauge transformations of the fields. Let us thus introduce the collective notation $\mathcal{Y}_I$:

$$\mathcal{Y}_I = \{D_{\alpha\dot{\alpha}}, \Psi^a_\alpha, \bar{\Psi}^\dot{b}_\beta, \Phi_{ab}\}.$$  

(2.1.7)

By itself, the covariant derivative is of course not a field of the theory. However, it transforms uniformly under gauge transformations, just like fermions and scalars do - and unlike the bare gauge connection:

$$\mathcal{Y}_I \rightarrow U \mathcal{Y}_I U^{-1}, \quad A_{\alpha\dot{\alpha}} \rightarrow U A_{\alpha\dot{\alpha}} U^{-1} - i U \partial_{\alpha\dot{\alpha}} U^{-1},$$

(2.1.8)

where $U = U(x)$ is an element of the gauge group. On the level of algebra thus the gauge transformations take the form:

$$\mathcal{Y}_I \rightarrow \mathcal{Y}_I + [\mathcal{X}, \mathcal{Y}_I], \quad A_{\alpha\dot{\alpha}} \rightarrow A_{\alpha\dot{\alpha}} + i[D_{\alpha\dot{\alpha}}, \mathcal{X}]$$

(2.1.9)

where $\mathcal{X}$ now is the gauge transformation parameter (an adjoint–valued field). We will pay more attention to gauge transformations in Chapter 7 before introducing BRST gauge fixing of the $\mathcal{N} = 4$ sYM action in the same chapter.

Since all the fields are in the adjoint representation of the gauge group, the covariant derivative acts as:

$$D_{\alpha\dot{\alpha}} \mathcal{Y}_I = [D_{\alpha\dot{\alpha}}, \mathcal{Y}_I] = \partial_{\alpha\dot{\alpha}} + i[A_{\alpha\dot{\alpha}}, \mathcal{Y}_I].$$

(2.1.10)

If $\mathcal{Y}_I$ is taken to be the covariant derivative\footnote{As a notational assignment we will also write $D_{\alpha\dot{\alpha}} A_{\beta\dot{\beta}} = F_{\alpha\dot{\alpha}\beta\dot{\beta}}$.} the resulting expression becomes the field strength $F$, which naturally separates into chiral and antichiral parts:
2.2 Lie-type symmetries of $\mathcal{N} = 4$ SYM: $\mathfrak{psu}(2,2|4)$ Lie algebra

With the Lagrangian (2.1.12) at hand, we can now discuss the symmetries of the $\mathcal{N} = 4$ SYM. The symmetry algebra turns out to be $\mathfrak{psu}(2,2|4)$, a Lie superalgebra with 32 fermionic and 30 bosonic generators.

The bosonic generators of $\mathfrak{psu}(2,2|4)$ can be conveniently divided into the fourdimensional conformal part consisting of Lorentz generators ($\mathfrak{su}(2)$ rotations) $L^\alpha_\beta$, $L^\dot{\alpha}_\dot{\beta}$, translations $P_{a\dot{a}}$, dilatations $D$ and special conformal transformations $K^{a\dot{a}}$, as well as the $\mathfrak{su}(4)$ $R$-symmetry rotations generators $R^a_b$. The feature that makes $\mathcal{N} = 4$ SYM stand out among other gauge theories is that the conformal symmetry survives also at the quantum level, which is equivalent to the all-loop vanishing of the $\beta$ function.

The fermionic part of $\mathfrak{psu}(2,2|4)$ consists of 16 supersymmetry generators $Q_{a\dot{a}}$, $\bar{Q}^\dot{b}_\beta$ as well as 16 superboosts $S^{a\dot{a}}$, $S^\dot{b}_\beta$ necessary to close the algebra. The commutation relations of $\mathfrak{psu}(2,2|4)$ are listed in Appendix A.1. Unfortunately, this maximal supersymmetry does not allow for the off-shell formulation\footnote{Introducing $\mathcal{N} = 4$ superspace is possible, but imposing constraints is equivalent to imposing the equations of motion.} \cite{62, 63}. It is however possible to write down the $\mathcal{N} = 4$ SYM action in terms of $\mathcal{N} = 1$ superfields, thus making part of the symmetry manifest. We will encounter this formulation in Chapter 5 on $\beta$-deformed sYM.

We already mentioned in section 2.1 the gauge symmetries of $\mathcal{N} = 4$ SYM. Let us now describe the interplay of the two types of invariances: gauge and $\mathfrak{psu}(2,2|4)$ in the representation of the symmetry algebra we will be working with.

The simplest of the generators is the momentum $P_{a\dot{a}}$. We will implement its action on the fields with the covariant derivative:

$$P_{a\dot{a}} \cdot Z_I = i[D_{a\dot{a}}, Z_I] = i\partial_{a\dot{a}}Z_I - [A_{a\dot{a}}, Z_I].$$

(2.2.1)
We see that in addition to the usual partial derivative normally used to generate translations it also features the gauge field. The advantage of this choice is that while acting on gauge-invariant (resp. covariant) objects we will again map them to manifestly gauge-invariant (covariant) ones. For other spacetime symmetries in order to maintain gauge-invariance it is then sufficient to also use covariant derivatives. The Lorentz generators act as:

\[
\begin{align*}
L^\delta_{\beta} \cdot Z_I &= -ix^{\beta\dot{\alpha}} D_{\delta\dot{\alpha}} Z_I + \frac{i}{2} \delta^\delta_\delta x^{\alpha\dot{\alpha}} D_{\alpha\dot{\alpha}} Z_I + (L_{\text{spin}})^\delta_{\delta} \cdot Z_I, \\
\bar{L}^{\dot{\alpha}_{\dot{\gamma}}} \cdot Z_I &= -ix^{\dot{\beta}\dot{\gamma}} D_{\dot{\alpha}_{\dot{\gamma}}} Z_I + \frac{i}{2} \delta^\dot{\alpha}_{\dot{\gamma}} x^{\beta\dot{\beta}} D_{\beta\dot{\beta}} Z_I + (\bar{L}_{\text{spin}})^{\dot{\alpha}_{\dot{\gamma}}} \cdot Z_I, 
\end{align*}
\]

where \(L_{\text{spin}}\) and \(\bar{L}_{\text{spin}}\) denote the spin contribution of the operator which acts non-trivially only on the spacetime indices of spinor fields

\[
\begin{align*}
(L_{\text{spin}})^\delta_{\delta} \cdot \Psi^c &= -i \delta^\delta_\delta \Psi^c, \\
(\bar{L}_{\text{spin}})^{\dot{\alpha}_{\dot{\gamma}}} \cdot \bar{\Psi}^{id} &= -i \delta^{\dot{\alpha}_{\dot{\gamma}}} \bar{\Psi}^{id}.
\end{align*}
\]

Similarly, the dilatation generator \(D\) acts as:

\[
D \cdot Z_I = -i (x^{\alpha\dot{\alpha}} [D_{\alpha\dot{\alpha}}, Z_I] + \Delta_Z Z_I),
\]

with \(\Delta_\phi = \Delta_{\bar{\phi}} = 1, \Delta_\psi = \Delta_{\bar{\psi}} = \frac{3}{2}, \Delta_A = 0\).

Observe that for the gauge field (covariant derivative), there is no spin contribution to the Lorentz transformation, and its dilatation coefficient \(\Delta_A\) does not match its conformal dimension. These assignments are tied of course to the choice of our gauge-preserving representation. Indeed, the covariant derivative by itself is not a field, it always has to act on something, whereas the gauge connection \(A_{\alpha\dot{\alpha}}\) is not gauge-covariant, hence it cannot appear by itself. That prohibits the appearance of any terms with either single \(D_{\alpha\dot{\alpha}}\) or \(A_{\alpha\dot{\alpha}}\). It is easy however to verify, that the commutation relations are preserved on all of the fields. Furthermore, the field strength \(F\) receives a correct conformal dimension \(\Delta_F = 2\) as well as a usual transformation of its spinor indices, which is due to the action of a derivative on the explicit position dependence \(x^{\alpha\dot{\alpha}}\) in the generators. This is because even though in our construction the generators act only on the fields, once we want to commute parts of the expressions, we take into account the position and derivatives present in the formulae for Poincaré generators. Let us show that indeed the conformal dimension of the field strength comes as expected:

\[
D \cdot F_{\alpha\dot{\alpha},\beta\dot{\beta}} = D \cdot (-i [D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}]) \\
= (-i)^2 \left( [x^{\gamma\dot{\gamma}} [D_{\gamma\dot{\gamma}}, D_{\alpha\dot{\alpha}}], D_{\beta\dot{\beta}}] + [D_{\alpha\dot{\alpha}}, x^{\gamma\dot{\gamma}} [D_{\gamma\dot{\gamma}}, D_{\beta\dot{\beta}}]] \right) \\
= (-i)^2 \left( x^{\gamma\dot{\gamma}} [D_{\gamma\dot{\gamma}}, [D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}]] - \delta^\gamma_\gamma \delta^\dot{\gamma}_{\dot{\gamma}} [D_{\gamma\dot{\gamma}}, D_{\alpha\dot{\alpha}}] + \delta^\gamma_\gamma \delta^\dot{\gamma}_{\dot{\gamma}} [D_{\gamma\dot{\gamma}}, D_{\beta\dot{\beta}}] \right) \\
= (-i) \left( x^{\gamma\dot{\gamma}} [D_{\gamma\dot{\gamma}}, F_{\alpha\dot{\alpha},\beta\dot{\beta}}] + 2F_{\alpha\dot{\alpha},\beta\dot{\beta}} \right).
\]

In the same manner, the correct behavior of the field strength under Lorentz transformations can be recovered.

We postpone the explicit formulae for the remaining bosonic generators to the Appendix \(A\) and proceed with the supersymmetry transformations \(Q\) and \(\bar{Q}\). Their action on the fields is given by:
\[ Q_{a\beta} \cdot \Phi_{cd} = \delta_a^c \psi_{\beta}^d - \delta_a^d \psi_{\beta}^c, \]
\[ Q_{a\beta} \cdot \bar{\Phi}_{cd} = \varepsilon_{acde} \psi_{\beta}^e, \]
\[ Q_{a\beta} \cdot \psi^c_{\delta} = -2 \delta_a^c F_{\delta \beta} + i \varepsilon_{\beta \delta}[\Phi^{ce}, \bar{\Phi}_{ac}], \]
\[ Q_{a\beta} \cdot \bar{\psi}_{\delta d} = 2i[D_{\beta \gamma}, \bar{\Phi}_{ad}], \]
\[ Q_{a\beta} \cdot A_{\delta \gamma} = -i \varepsilon_{\beta \delta} \bar{\psi}_{\gamma a}. \]  \hspace{1cm} (2.2.6)

and analogous formulae for \( \bar{Q}_a^a \). Observe that in this formulation the supersymmetry generators act nonlinearly in some cases, that is they map one field to a product of fields. This is both expected – due to inexistence of \( N = 4 \) superspace explained above – and necessary in order to preserve the commutation relations of \( \mathfrak{psu}(2,2|4) \), for example \( \{ Q_{aa}, \bar{Q}_b^{b \gamma} \} = 2 \delta_a^a \mathcal{P}_{ab} \). One can easily verify that the \( \mathcal{P} \) appearing on the right-hand side is the one generated by the covariant derivative introduced in (2.2.1). An artifact however is that the commutation relations are satisfied only up to field-dependent gauge transformations, as in:

\[ [P_{\beta \delta}, P_{\gamma \gamma}] \cdot Z_I = -i[F_{\beta \delta \alpha \gamma}, Z_I], \]
\[ [Q_{a \beta}, P_{\gamma \delta}] \cdot Z_I = i \varepsilon_{\beta \delta}[\psi_{\gamma a}, Z_I], \]
\[ \{ Q_{a \beta}, Q_{c \delta} \} \cdot Z_I = 2i \varepsilon_{\beta \delta}[\bar{\psi}_{a c}, Z_I]. \] \hspace{1cm} (2.2.7)

At that point this mixing of global and gauge symmetries is a mild inconvenience. None of these terms will matter while acting on gauge-invariant observables and it will be easy to spot them while working with gauge-covariant ones.

Indeed, the gauge transformations form an ideal which we can quotient out. Let \( G[Z] \) denote the gauge transformation sourced by a field \( Z \), so that for example:

\[ [P_{\beta \delta}, P_{\gamma \gamma}] \cdot Z_I = -i[F_{\beta \delta \alpha \gamma}, Z_I]. \] \hspace{1cm} (2.2.8)

Then one can verify that they indeed satisfy the relation:

\[ [G[Z_I], J] = -G[J \cdot Z_I], \] \hspace{1cm} (2.2.9)

and in the usual Lie algebra we can indeed quotient out by it. Later on we will however encounter more complications due to those gauge transformations. As already alluded to earlier, we will treat them in more detail in Chapters 7 and 9. The generators of \( \mathfrak{psu}(2,2|4) \) not explicitly mentioned here can be found in the Appendix A.

2.3 Planar limit

As already alluded to in the Introduction 1, a majority of the unexpected results in the context of \( \mathcal{N} = 4 \) sYM has been obtained in the planar limit. That is also the case for the work done in this thesis. The planar limit in gauge theories was introduced as a means of reorganizing perturbation theory in 1974 by ’t Hooft (64), see also (65) for a pedagogical introduction by the same author). It amounts to taking the rank of gauge group, \( N \), to be very large, simultaneously rescaling the Yang–Mills coupling constant \( g_{YM} \).

\[ ^4 \text{Recall that we reabsorbed } g_{YM} \text{ in the field definition and hence we do not see it explicitly in the classical computations.} \]
\[ N \to \infty, \quad g_{YM}^2 N = \lambda \text{ fixed.} \]

\( \lambda \) is commonly referred to as 't Hooft coupling. The name planar limit stems from the fact, that in this limit, only Feynman diagrams of planar topology (or more specifically, ones which can be drawn on the surface of lowest possible genus) survive, with the remaining ones being suppressed by powers of \( \frac{1}{N} \).

Unless otherwise stated, the following discussions of \( \mathcal{N} = 4 \) sYM will always assume the planar limit to be taken. Most of the time however it will be implicit, and no \( \frac{1}{N} \) expansion will be performed. The understanding of how the planar limit should be taken into account while considering the invariance of the \( \mathcal{N} = 4 \) sYM under Yangian symmetry was one of the challenges in this work, since planar limit cannot be taken on the level of the action \( (2.1.13) \). We will solve this issue in Chapter 4 while explicitly working in the planar limit in Chapters 8, 9 and 10.

2.4 Integrability results in \( \mathcal{N} = 4 \) sYM

We now have all the necessary prerequisites to discuss first discoveries that lead the community to believe that \( \mathcal{N} = 4 \) sYM in the planar limit is an integrable field theory (see \([66]\) for an extensive review). We have already mentioned some of the results in the Introduction \(^1\), but now we can present them in a more formal manner.

The notion of integrability is clear for mechanical systems (via Liouville’s theorem; \([67]\), as well as two-dimensional field theories (see for example \([68]\) and \([69]\)). In case of higher-dimensional field theories like \( \mathcal{N} = 4 \) sYM, there exists no unequivocally accepted definition. Let us thus not rely on a strict definition, but rather follow two (not unrelated) concepts - exact solvability or increased symmetry. Exact solvability describes the possibility of obtaining solutions which are nonperturbative in the coupling constant (in case of \( \mathcal{N} = 4 \) sYM it is the ’t Hooft coupling \( \lambda \)). The increased symmetry refers to the existence of hidden symmetries of the theory, giving rise to infinitely-dimensional (a necessity for field theories) symmetry algebras. We will encounter both hints towards integrability in this section. The discussion here will serve a purpose of an overview and motivation for further chapters, rather than rigorous presentation of the results. Readers interested in a more detailed presentation will find references in the respective parts to both original articles and reviews.

2.4.1 Dilatation operator

Since \( \mathcal{N} = 4 \) sYM possesses conformal symmetry, determination of its spectrum means finding conformal dimensions of the operators \( O_i \) in the theory. This corresponds to finding eigenvalues \( \Delta \) of dilatation operator \( D \) and in turn results in knowledge of two-point correlators, since in a CFT:

\[
\langle O_i(x) \bar{O}_j(y) \rangle = \frac{\delta_{ij}}{|x - y|^{2\Delta_{O_i}}}. \tag{2.4.1}
\]

The question can of course be reversed, and one can equally well compute two-point functions and read off the scaling dimensions \( \Delta \). This task can be then mapped to a spin-chain picture, where the constituents of \( O \) are viewed as spins on a chain and dilatation operator becomes a

\( ^5\)We take into account only the leading term in \( \frac{1}{N} \) expansion. At first one could be surprised that increasing the number of degrees of freedom can simplify the results, but this phenomenon is very common i.e. in the thermodynamic limit of statistical physics.
2.4. INTEGRABILITY RESULTS IN $\mathcal{N} = 4$ SYM

Hamiltonian (see [70] for the initial study). In the simplest case one considers the smallest closed (at a given loop order) subsector of $\mathcal{N} = 4$ SYM, the $\mathfrak{su}(2)$ sector consisting of two complex scalars $Z$ and $X$ (complex linear combinations of $\Phi^{ab}$). The classical dimension of an operator consisting of scalars is of course given by its length $L$ and the task is to compute the quantum corrections. In the spin-chain picture, the vacuum is taken to be $O = Z^L$ (we can view them as spin-downs), and the $X$ fields are considered as excitations (spin-ups):

$$O = \text{Tr}(ZZZXZZ \cdots ZXXZ) \leftrightarrow \downarrow \downarrow \uparrow \uparrow \downarrow \cdots \downarrow \uparrow \downarrow.$$

The dilatation operator then becomes a Hamiltonian of the chain, which at 1 loop in the planar limit turns out to be an integrable Heisenberg XXX chain (see e.g. [71] for an introduction):

$$H_{XXX} = \sum_{i=1}^{L} (I_{i,i+1} - P_{i,i+1}),$$

where $P_{ab}$ is a permutation operator on sites $a$ and $b$. The study continued later on to larger subsectors and higher loops ([46], [72], [73], [74]) - reaching the all-loop Bethe Ansatz [34]. That allows to find exact values of quantum-corrected operators’ conformal dimensions. The correction to the classical value (which for operators made up of scalars only is equal to their length $L$) is what is referred to as the anomalous dimension.

2.4.2 Scattering amplitudes and the dual superconformal symmetry

Another important discovery in the studies of $\mathcal{N} = 4$ SYM was done in the field of scattering amplitudes (see [38] for a pedagogical introduction). Of direct phenomenological interest in Standard Model computations, due to the theory’s simplified structure they are also being investigated in the context of the $\mathcal{N} = 4$ SYM, hope being they could shed some light on the more realistic models [40].

Tree level amplitudes in $\mathcal{N} = 4$ SYM are invariant under the whole $\mathfrak{psu}(2, 2|4)$ inherited from the invariance of the action. If we denote the momentum of the $i$th scattering particle by $p_i$, and subsequently introduce dual coordinates $x_i = p_{i+1} - p_i$, then it turns out that the amplitude will also be invariant under another, independent copy of $\mathfrak{psu}(2, 2|4)$, expressed in coordinates $x$ [75]. This symmetry has been dubbed dual superconformal symmetry. Making dual superconformal symmetry manifest has been a part of the amplituhedron [76] research program. Additionally, for the $\mathcal{N} = 4$ SYM amplitudes it was possible to write down all the tree-level superamplitudes ($N^k$MHV with arbitrary $k$) using the BCFW construction [38].

2.4.3 Identification of Yangian symmetry

Due to the existence of a well-developed framework, the tower of increased symmetries has first been found in the string theory dual of the $\mathcal{N} = 4$ SYM, the type IIB string on $\text{AdS}_5 \times S^5$ [42]. Shortly thereafter, it has been suggested that similarly large symmetries also hold in $\mathcal{N} = 4$ SYM ([77], [45]), following the conjectured duality between the two theories. They have been identified as elements of the Yangian algebra $Y(\mathfrak{psu}(2, 2|4))$, which we discuss in Chapter 3. These works also aimed at constructing conserved multi-local charges corresponding to Yangian generators, but only succeeded in doing so at $\lambda = 0$. It would be interesting to understand whether their approach can

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The invariance at loop level is obscured due to regularization.
be generalized beyond that limit. On the other hand, the dual superconformal symmetry (or to be precise, the closure of usual and dual superconformal symmetries) discovered in the context of amplitudes has also been subsequently understood as the Yangian of $\mathfrak{psu}(2, 2|4)$, and the larger symmetry algebra has been found in the context of other observables in $\mathcal{N} = 4$ sYM: spin-chains or Maldacena–Wilson loops.
Chapter 3

Yangian algebras

This chapter will introduce the main mathematical concept of this thesis, namely Yangian algebras. As already mentioned in the previous chapter, integrability of field theories is linked to the existence of infinitely dimensional symmetry algebra of the theory. One of the types of such algebras which very often features in the integrable theories is the Yangian of a Lie algebra [80], [81]. As opposed to the Lie-type symmetry algebras, Yangians are not universally known, hence a separate chapter dedicated to them. For a more detailed pedagogical introduction we refer the reader to [82].

3.1 Structures of Hopf algebras

Let us start with a usual Lie (super-)algebra $\mathfrak{g}$, and consider its universal enveloping algebra $A = U(\mathfrak{g})$, whose elements are polynomials in the generators $J$ of $\mathfrak{g}$, modulo the commutation relations. The enveloping algebra comes with the multiplication map $\mu : A \otimes A \to A$ whose identity element we denote by 1, as well as the unit map $\eta : K \to A$ such that $\eta(c) = c1$ for any $c \in K$. The $K$ is a field we work over, usually taken to be $\mathbb{C}$. The multiplication satisfies the associativity condition:

$$\mu(\mu \otimes 1) = \mu(1 \otimes \mu). \quad (3.1.1)$$

This property can conveniently be illustrated with a following Diagram 3.1.

![Diagram 3.1: Associativity of the multiplication map $\mu$.]](image)

Introduction of further structures turns the algebra into a bialgebra. These are the coproduct $\Delta : A \to A \otimes A$ and the counit $\epsilon : A \to K$.

The coproduct is assumed to be coassociative (see also Diagram 3.2):
(3.1.2)
which guarantees that inside the n-fold coproduct:

\[ \Delta^n = \ldots (\Delta \otimes 1 \otimes 1)(\Delta \otimes 1) \Delta \]  

(3.1.3)
we can change the position of \( \Delta \)'s. This will become clear while doing explicit computations in the following chapter.

In the physics terminology, the coproduct yields the action of the element of \( A \) on the two-particle state, its coassociativity having the interpretation of the consistency condition while acting on three-particle state.

The counit, which we can think of as the dual of the identity element \( e \) of a group \( G \), satisfies

\[ (1 \otimes \epsilon)\Delta(a) = (\epsilon \otimes 1)\Delta(a) \forall a \in A. \]

The last map necessary in order to obtain a Hopf algebra is the antipode \( \Sigma : A \rightarrow A \). The remaining compatibility conditions the above introduced maps need to satisfy are best illustrated with the Diagram 3.3.

3.2 Yangian algebras

We are now in possession of all the necessary ingredients to introduce the Yangian \( Y \) of a given Lie algebra \( \mathfrak{g} \): \( Y(\mathfrak{g}) \). Indeed, let \( \mathfrak{g} \) be a finite-dimensional simple Lie (super-)algebra. Let us denote its generators by \( J^A \). They satisfy the commutation relations:
3.2. YANGIAN ALGEBRAS

\[ [J^A, J^B] = i f^{AB} C J^C, \]  
(3.2.1)

where with the square bracket we denote the graded commutator (which becomes an anticommutator if both generators J are fermionic):

(3.2.2)

The grading \(|A|\) of a generator \(J^A\) is:

\[ |A| = \begin{cases} 0 & \text{for bosonic generators} \\ 1 & \text{for fermionic generators.} \end{cases} \]  
(3.2.3)

In the context of Yangian algebras, we will call the generators of the underlying Lie algebra \(\mathfrak{g}\) level-0 generators, a terminology made clear by introducing level-1 generators \(\hat{J}^A\) satisfying:

\[ [J^A, \hat{J}^B] = i f^{AB} C \hat{J}^C. \]  
(3.2.4)

With level-1 generators at hand we can produce an infinite tower of higher-level generators by repetitive commutations, schematically:

\[ [J^{(n)}, J^{(m)}] \simeq J^{(n+m)}, \]  
(3.2.5)

where \(n, m = 0, 1, \ldots\) denote levels of the generators. Due to this property we can view Yangian as (half of) a loop algebra. For consistency, the generators need to satisfy additional constraints, namely Serre identities:

\[ [\hat{J}^A, [\hat{J}^B, J^C]] + \text{cyclic} = f^{AD} E f^{BF} H f^{CG} L f^{DF} G J^E J^H J^L. \]  
(3.2.6)

The most important feature of Yangian algebra for our work is the coproduct. For level-0 and level-1 generators it takes the following form:\(^2\)

\[ \Delta J^A = 1 \otimes J^A + J^A \otimes 1, \quad \Delta \hat{J}^A = 1 \otimes \hat{J}^A + \hat{J}^A \otimes 1 + f^{AB} C J^B \otimes J^C. \]  
(3.2.7)

We see the difference between the action of level-0 and level-1 generators: the latter obtain what we will call a bi-local term consisting of a simultaneous action of two level-0 generators, antisymmetrised by the structure constants \(f_{BC}^A\). For our work it will be useful to state the \((n-1)\)-folded coproduct, that is how a generator acts on an \(n\)-fold tensor product of representations (\(n\)-particle state in physics terminology):

\[ \Delta^{(n-1)} J^A = \sum_{i=1}^{n} J_i^A, \quad \Delta^{(n-1)} \hat{J}^A = \sum_{i=1}^{n} \hat{J}_i^A + f^{AB} C \sum_{i<j}^{n} J_i^B J_j^C, \]  
(3.2.8)

where the index \(i,j\) indicates site in the tensor product on which the generator acts. For completeness, the antipode acts on the level-0 generators as:

\(^1\)In a whole loop algebra, the generators can also have negative levels, whereas Yangian starts at level-0.

\(^2\)Raising and lowering of the indices of the structure constants \(f_{BC}^A\) requires a nondegenerate bilinear form on \(\mathfrak{g}\). As the dual Coxeter number for \(\mathfrak{psu}(2,2|4)\) vanishes, so does its Killing form, which would be the usual choice. Nevertheless, a nondegenerate bilinear form on \(\mathfrak{psu}(2,2|4)\) exists and can be used, i.e. \(\kappa_{FG}^{AB} = s \text{Tr}_F(J^A J^B)\). See Appendix C for details.
\begin{equation}
\Sigma(J^A) = -J^A, \tag{3.2.9}
\end{equation}

while on the level-1 generators as:

\begin{equation}
\Sigma(\hat{J}^A) = -\hat{J}^A + f^A_{\ BC} J^B J^C. \tag{3.2.10}
\end{equation}

To finish this part, let us introduce Sweedler’s notation, which allows to compactify the notation while considering bilocal actions. Following Sweedler, we can write the bilocal part of the coproduct as:

\begin{equation}
f^A_{\ BC} J^B \otimes J^C := J^{(1)} \otimes J^{(2)} = -J^{(2)} \otimes J^{(1)}, \tag{3.2.11}
\end{equation}

therefore:

\begin{equation}
\Delta \hat{J} = \hat{J} \otimes \mathbb{I} + \mathbb{I} \otimes \hat{J} + J^{(1)} \otimes J^{(2)}. \tag{3.2.12}
\end{equation}

That furnishes the physics-independent discussion of the Yangian algebras.

We will encounter later on particular realizations thereof, which will pose some complications specific to a given theory. We will treat them in their respective sections.

To finalize, let us mention that the above construction is known as (Drinfeld’s) first realization. There exist two another ones, based respectively on Chevalley-Serre basis, as well as what is known as RTT realization, based on the RTT relation known from integrable systems. We will however only use the first realization, referring the interested reader to the cited sources, e.g \cite{82}.
Chapter 4

Symmetries of equations of motion

In the two previous chapters we have introduced the $\mathcal{N} = 4$ sYM theory as well as the concept of Yangian algebras, pointing out that various observables of the theory possess Yangian symmetry. This symmetry has however never been established on the level of the action, which is the usual first step while treating the invariances of physical models. Indeed, one would normally start with invariance of the action, and from it derive identities satisfied by the observables.

As already alluded to in the Introduction however, there are several obstructions to establishing the Yangian symmetry of the action of $\mathcal{N} = 4$ sYM. The first difficulty is that the integrability and the Yangian symmetry of $\mathcal{N} = 4$ sYM have only appeared in the planar limit (see Chapter 2). The action however generates both planar and non-planar contributions to various observables and it is not clear how to take the planar limit on the level of either the action or the generators.

The second difficulty is the cyclicity of the action (2.1.13) imposed by the trace. While acting with the usual Lie-type generators (level-0 Yangian, as introduced in Chapter 3), the cyclicity is maintained in the sense that it does not matter which site we pick as the first one - we can close the trace back afterwards and obtain the same results:

$$\text{tr}(J \cdot (Z_1 Z_2 Z_3 \ldots Z_n)) = \text{tr}(J \cdot (Z_n Z_1 Z_2 \ldots Z_{n-1})). \quad (4.0.1)$$

Using cyclicity of the trace and antisymmetry of Swedler’s notation we see that these expressions indeed differ: the difference amounts to $\text{tr}(2(J^{(1)} Z_1)(J^{(2)} Z_2)Z_3 + 2(J^{(1)} Z_1 Z_2)(J^{(2)} Z_3))$. This issue has been beautifully resolved in [47] while considering the Yangian symmetry of the color-ordered scattering amplitudes, but the argument relied on linearity of the representation, which is not the case in our example. We will however come back to this argument in Chapter 6 and find that it can be generalized to the case of non-linear representations.

This chapter is based on [58] and [60].
CHAPTER 4. SYMMETRIES OF EQUATIONS OF MOTION

4.1 Equations of Motions in $\mathcal{N} = 4$ sYM

In order to circumvent the problem of cyclicity, we will first look at the equations of motion of the theory. To be precise, while saying “equations of motion” we will most often mean variations of the action with respect to a field. The equations of motion follow by demanding that such variation is equal to 0 (setting on-shell). Let us introduce a notation:

$$\tilde{Z}_A = \frac{\delta S}{\delta Z^A}, \quad (4.1.1)$$

where $S$ is the action (unless otherwise stated the action (2.1.13) of $\mathcal{N} = 4$ sYM) and $A$ is a multi-index, containing spinor and $\mathfrak{su}(4)$ indices as well as the position dependence. With this notation, the equations of motion are $\tilde{Z}_A = 0$. Let us now list the equations of motion which result from varying the action (2.1.13):

$$\tilde{\Phi}_{lk} = \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\beta\delta} D_{\beta\dot{\alpha}} D_{\delta\dot{\gamma}} \bar{\Phi}_{kl} - \frac{i}{2} [\bar{\Phi}_{ab}, [\Phi^{ab}, \bar{\Phi}_{kl}]] + \frac{i}{2} \varepsilon_{klgh} \varepsilon^{\alpha\beta} \{\bar{\Psi}^g_{\alpha}, \bar{\Psi}^h_{\beta}\} + i \varepsilon^{\dot{\alpha}\dot{\gamma}} \{\bar{\Psi}_{\dot{\alpha}k}, \bar{\Psi}_{\dot{\gamma}l}\};$$

$$\tilde{\bar{\Psi}}_{l}^{\dot{k}} = i \varepsilon^{\dot{k}\dot{\alpha}} \varepsilon^{\beta\gamma} D_{\beta\dot{\alpha}} \Psi_{l}^{\dot{\gamma}} - i \varepsilon^{\dot{k}\dot{\alpha}} [\Phi^{\dot{e}e}, \bar{\Psi}_{\dot{e}l}];$$

$$\tilde{\bar{\Psi}}_k^{\dot{\lambda}} = i \varepsilon^{\dot{k}\dot{\alpha}} \varepsilon^{\beta\gamma} D_{\beta\dot{\alpha}} \bar{\Psi}_{\dot{\lambda}k} - i \varepsilon^{\dot{k}\dot{\alpha}} [\bar{\Phi}_{\dot{e}e}, \Psi_{\dot{e}k}];$$

$$\tilde{A}_\lambda^\alpha = \varepsilon^{\lambda\beta} \varepsilon^{\dot{\epsilon}\dot{\delta}} \left(-\varepsilon^{\alpha\gamma} D_{\alpha\dot{e}} F_{\gamma\beta} - \varepsilon^{\dot{\epsilon}\dot{\gamma}} D_{\beta\dot{e}} F_{\dot{\epsilon}\dot{\gamma}} - \frac{i}{2} [\bar{\Phi}_{\dot{e}e}, D_{\beta\dot{e}} \Phi^{\alpha\beta}] - \{\Psi^\alpha_{\beta}, \bar{\Psi}_{\dot{e}a}\} \right). \quad (4.1.2)$$

4.2 Lie-type symmetries of the equations of motion

For the superconformal generators introduced in section 2.2 it is easy to confirm that they are symmetries of the theory by applying them to the action $S$ directly. The representations of generators act on the fields non-linearly, but the change of length poses no difficulty here. Also, cyclicity will be preserved. In this section we want to see, what consequences the invariance of the action has however for the equations of motions. Let us denote with $J$ an arbitrary Lie-type generator such that:

$$J \cdot S = 0. \quad (4.2.1)$$

This is the statement one usually has in mind while saying that a theory possesses a certain symmetry. From (4.2.1) we can then derive conservation laws via Noether’s theorem, which allow us to simplify the study of the theory. If a symmetry survives quantization (more on it in chapter 10), the invariance of the action will lead to identities between correlation functions in the QFT (see Chapter 8). For the case of $\mathcal{N} = 4$ sYM, invariance of the action (2.1.13) can be checked directly using the representations of $J$ such as (2.2.1), (2.2.2), (2.2.3), (2.2.4), (2.2.6).

We can rewrite the above statement (4.2.1) making the appearance of the variations of the action $\tilde{Z}$ explicit.

$$J \cdot S = (J \cdot Z^I) \frac{\delta S}{\delta Z^I} = (J \cdot Z^I) \tilde{Z}_I = 0. \quad (4.2.2)$$

1Notable exceptions being of course non-Lagrangian theories, i.e. certain CFTs, where we know the symmetries even though we cannot currently write down the action.
As already discussed in Chapter 2, the field index \( I \) refers to all dependencies of all fields including the gauge degrees of freedom as well as the full coordinate dependence\(^{2}\). As such it is evident that the statement is necessarily off-shell for it becomes trivial when the equations of motion \( \hat{Z} = 0 \) are imposed. In other words, a symmetry implies that \((4.2.2)\) holds without making use of the equations of motion, even though conservation laws following from it usually require the use of the equations of motion.

**Strong invariance of the equations of motion.** We now wish to proceed and take a variation of \((4.2.2)\) with respect to a field \( Z_K \).

\[
J \cdot Z^I \frac{\delta^2 S}{\delta Z^K \delta Z^I} + \frac{\delta (J \cdot Z^I)}{\delta Z^K} \frac{\delta S}{\delta Z^I} = 0.
\]  

\((4.2.3)\)

This statement can readily be expressed in a more concise form as

\[
J \cdot \hat{Z}_K = -\frac{\delta (J \cdot Z^I)}{\delta Z^K} \hat{Z}_I.
\]  

\((4.2.4)\)

It now has an open field index \( K \) which means that it is localised at some point \( x \) of spacetime and that there is one statement for each component of each fundamental field. In the case of a gauge theory, the statement is no longer gauge-invariant but rather gauge-covariant. Here the open gauge indices imply an \( N \times N \) matrix of relationships. The equality \((4.2.4)\) tells us how an equation of motion transforms, when acted upon with a symmetry generator \( J \). Importantly, the statement is still off-shell. What is the qualitative difference between \((4.2.2)\) and \((4.2.4)\)? Variation of \((4.2.2)\) with respect to \( Z_K \) can easily be integrated back (which we will indeed do later on), the only degree of freedom being a field-independent term. Now, since all the generators map fields to fields, there is no possibility of a term completely independent of all the fields to appear. Hence those statements are equivalent. We will demonstrate it explicitly later on. We shall already however denote \((4.2.4)\) as strong invariance of the equations of motion. Again, verification of this statement for \( N = 4 \) SYM is straight-forward using the concrete equations of motion \((4.1.2)\) and the representations of \( J \).

**Weak invariance of the equations of motion.** We can now apply the equations of motion \( \hat{Z} = 0 \) to the statement \((4.2.4)\) in order to remove the r.h.s.

\[
J \cdot \hat{Z}_K \approx 0.
\]  

\((4.2.5)\)

Note that even though the l.h.s. contains \( \hat{Z} \), it does not vanish automatically because of the variation within \( J \) which hits \( \hat{Z} \) before the e.o.m. are imposed. By construction, this statement is on-shell, which is expressed by the symbol ‘\( \approx \)’ here and in the following. Both relationships \((4.2.4)\) and \((4.2.5)\) state that the variation of the equations of motion is proportional to the equations of motion. In other words, they imply that symmetries map solutions to solutions. Nevertheless, the second version of the statement is clearly weaker because it does not predict the specific linear combinations on the r.h.s.. We therefore call \((4.2.5)\) weak invariance of the equations of motion.

Let us now reason in the opposite direction and ask ourselves the following question: Suppose we have a transformation \( J \) such that \((4.2.5)\) holds. To what extent can we consider \( J \) a symmetry

\(^{2}\)Due to the implicit integration over all space, one can expect that partial integration is necessary to confirm the statement. Analogously, one should take cyclicity of the trace over the gauge degrees of freedom into account.
of the theory? When can we promote this transformation to a symmetry of the action of the theory? The following example shows that (4.2.5) can hardly be considered a sufficient condition for a symmetry. Weak invariance of the equations of motion is only a necessary condition. We propose that the strong version (4.2.4) is a sufficient condition for J to be a symmetry of the action.

Example. Let us now demonstrate that the strong invariance condition (4.2.4) indeed allows to differentiate which invariance of the equations of motion stems from a true symmetry of the action. To this end consider a simple example, the free complex scalar field $\phi$ defined by the following action

$$S = \int d^d x \bar{\phi} \partial^2 \phi.$$  \hspace{1cm} (4.2.6)

The equations of motion following from (4.2.6) are the wave equations

$$\ddot{\phi} = \partial^2 \phi = 0, \quad \ddot{\bar{\phi}} = \partial^2 \bar{\phi} = 0.$$  \hspace{1cm} (4.2.7)

The above equations of motion are weakly invariant as in (4.2.5) under a global complex rescaling of the fields

$$\phi \mapsto e^{\rho + i\theta} \phi, \quad \bar{\phi} \mapsto e^{-\rho - i\theta} \bar{\phi},$$  \hspace{1cm} (4.2.8)

with $\rho$, $\theta$ real parameters. However, the above transformation leaves the action invariant (4.2.1) only for pure rotations $\rho = 0$.

Can we observe the distinction of $\rho$ and $\theta$ on the level of equations of motion, using only the strong invariance formula (4.2.4)? Let us introduce the generators of the infinitesimal form of the above transformations, separating the complex rotation $R$ and scaling $S$

$$R \cdot \phi = i\theta \phi, \quad R \cdot \bar{\phi} = -i\theta \bar{\phi},$$

$$S \cdot \phi = \rho \phi, \quad S \cdot \bar{\phi} = \rho \bar{\phi}.$$  \hspace{1cm} (4.2.9)

It is then a simple exercise to see that $R$ indeed satisfies (4.2.4), whereas for $S$ one finds (4.2.4) with the opposite sign on the r.h.s.. Hence as claimed, (4.2.4) allows us to verify whether a given symmetry of the equations of motion is also a symmetry of the action.

### 4.3 Yangian symmetry of the equations of motion

We now turn to investigation of the Yangian invariance of the equations of motion, in analogy to their invariance under Lie-type symmetries. Indeed, due to an open index $K$ in $\bar{Z}_K$ working with equations of motions solves the issue of apparent incompatibility of the action of $\mathcal{N} = 4$ sYM with the Yangian symmetry. Since we cannot start with the invariance of the action under Yangian generators, our starting point will be the weak invariance. Having established that one, we will propose a level-1 equivalent of the formula (4.2.4) and verify that it is satisfied by the equations of motion of $\mathcal{N} = 4$ sYM.

Our analysis of Yangian symmetry can be simplified by the fact that only a single level-one generator along with the level-zero generators is sufficient to generate the whole Yangian algebra; all other level-one generators follow from the adjoint property (3.2.4), and the higher-level ones from the Serre-relation (3.2.6). We are thus free to choose a particular generator for which the resulting expressions simplify as much as possible. Arguably, this is the level-one momentum $\hat{P}$
which is also known as the dual conformal generator. Its coproduct reads (including all appropriate signs due to fermionic terms)\(^3\)

\[
\Delta \hat{P}_{\beta\dot{\alpha}} = \hat{P}_{\beta\dot{\alpha}} \otimes 1 + 1 \otimes \hat{P}_{\beta\dot{\alpha}} - L^\gamma_{\beta} \wedge P_{\gamma\dot{\alpha}} - \bar{L}^\gamma_{\dot{\alpha}} \wedge P_{\beta\gamma} - D \wedge P_{\beta\dot{\alpha}} - \frac{i}{2} Q_{c\beta} \wedge \bar{Q}^{c\dot{\alpha}}, \tag{4.3.1}
\]

where the anti-symmetric tensor product \(\wedge\) of any two objects \(X\) and \(Y\) is defined in analogy with the graded commutator from Chapter 3 as

\[
X \wedge Y := X \otimes Y - (-1)^{|X||Y|} Y \otimes X. \tag{4.3.2}
\]

Note that the action of \(\hat{P}\) conveniently only needs the dilatation \(D\) next to the super Poincaré generators \(L, \bar{L}, Q, \bar{Q}\) and \(P\). All of their representations are reasonably simple compared to the representations of the superconformal boosts \(S, \bar{S}\) and \(K\), as can be seen in Appendix A. In fact we will encounter further simplifications due to the choice of \(\hat{P}\) later on. Expressions for the coproducts of the level-one generators \(\hat{J}\) with \(J \in \{Q, \bar{Q}, R\}\) – which in turn contain the more complicated level-0 generators – as well as the level-one bonus symmetry \(\hat{B}\) introduced in [66] can be also found in the Appendix A.

Having circumvented the issues related to cyclicity of the action and its incompatibility with level-1 generators by working with the equations of motion, another major complication in considering Yangian symmetry within an (interacting) field theory is that symmetry representations are often non-linear. By definition, the representation is still a linear map between observables; here, non-linearity refers to the fact that a single field can be mapped to a product of fields. In a gauge theory, the covariant derivative is a major source of non-linearity, see e.g. (2.2.1), (2.2.2). Furthermore, the representation of supersymmetry (2.2.6) contains non-linear terms which are not due to covariant derivatives.

The issue is that the concept of non-linear representations is in competition with the definition of tensor product representations via the coproduct (recall (3.2.7) and (3.2.8)): In a linear representation, each field corresponds to a single tensor factor. The representation acts on fields one-to-one thus preserving the structure of the tensor product. For non-linear symmetries, the representation changes the number of fields and consequently the structure of the tensor product, see [74] for a discussion and construction of some aspects of non-linear representations. In fact, the representation does not split into sub-representations with a definite number of fields, but there is only the indecomposable representation on polynomials of the fields. For the local action of level-zero generators this complication is minor and one can still view the full representation as the sum of representations on component fields. Conversely, the construction of the bi-local action for the level-one generators is less evident, cf. some comments in [83]. Here, the action of the first constituent generator \(J^{(1)}\) changes the tensor product on which the second constituent generator \(J^{(2)}\) is supposed to act. This mainly refers to the precise definition of the bounds in the double sum in (3.2.7) given that \(n\) is not well-defined anymore. For example, it is conceivable that \(J^{(2)}\) acts on the output of \(J^{(1)}\) corresponding to an overlapping action. This leaves some ambiguity for the precise definition of a non-linear level-one representation. Unfortunately, this becomes a rather serious issue when taking gauge symmetry into account: for instance, the covariant derivative consists of a partial derivative and a gauge potential, and as such it relates terms of a different number of constituent fields. In that sense, the non-linear terms of the Yangian representations must be chosen delicately such that they will not violate gauge symmetry.

\(^3\)The overall factor of the bi-local terms depends on the coefficient of the invariant form \(\kappa_{AB}\) (see Chapter 3), and it thus varies with conventions.
We will thus need representations which are analogous to the actions defined in (3.2.7). In particular, they should reduce to the ordinary coproduct rule (3.2.7) when restricting to linear terms. To that end, it will make sense to distinguish the local and bi-local contribution in the level-one representation, and write

\[ \hat{J} = \hat{J}_\text{loc} + \hat{J}_\text{biloc} \quad \text{where} \quad \hat{J}_\text{biloc} := J^{(1)} \otimes J^{(2)}. \]  
\[ (4.3.3) \]

Here, the tensor product symbol \( \otimes \) should be interpreted such that the first factor acts on a field which is to the left of the field on which the second factor acts. Formally, we define the tensor product acting on a sequence of fields by the rule

\[ (J^{(1)} \otimes J^{(2)}) \cdot (\cdots Z^I \cdots Z^J \cdots) := \cdots + \cdots (J^{(1)} \cdot Z^I) \cdots (J^{(2)} \cdot Z^J) \cdots + \cdots, \]  
\[ (4.3.4) \]

where the dots on the r.h.s. represent further terms due to pairwise contractions with the omitted fields in the sequence. The local part \( \hat{J}_\text{loc} \) will just act on all fields homogeneously. Note that any overlapping contributions that could be part of some alternative definition of a bi-local action can and should be interpreted as local contributions.

### 4.3.1 Weak invariance of the equations of motion

We start by considering the weak notion of a symmetry of the equations of motion (4.2.5): given a generator \( J \) and its action on the fields \( Z \), it must leave the equation of motion \( \ddot{Z} = 0 \) invariant

\[ J \cdot \ddot{Z} \approx 0. \]  
\[ (4.3.5) \]

For the Yangian level-one generators \( \hat{J} \), the non-trivial coproduct poses two problems: First, if we want to act with \( \hat{J}_\text{biloc} \) on the equations of motion, we have to fix an ordering prescription for each term that appears; second, we need to specify how \( \hat{J}_\text{loc} \) acts on a single field.

Gladly, we can impose a “natural” ordering for the fields within the equations of motion for Yang–Mills theories with (S)U(\( N \)) gauge group. All the fields in \( N = 4 \) sYM can be treated as \( N \times N \) matrices and the (non-commutative) matrix product provides the ordering within a monomial of the fields. The equations of motion inherit this matrix structure supposing that all structure constants of the gauge group are written in terms of commutators of the fields. This is also where the large-\( N \) (planar) limit comes into play: the possibility to write arbitrary (adjoint) combinations of the fields in terms of matrix polynomials requires a (S)U(\( N \)) gauge group with correspondingly large \( N \) \(^4\). Note that 't Hooft’s (double) line notation for adjoint fields, which is prominently used in the discussion of the planar limit, directly depicts our ordering prescription.

For what concerns the single-field action \( \hat{J}_\text{loc} \), we will fix it by first computing the bi-local action of \( \hat{J}_\text{biloc} = J^{(1)} \otimes J^{(2)} \) on some equations of motion, which is completely determined by the conformal representation. We shall then require that the local terms eliminate all remaining terms

\[ [\hat{J}_\text{biloc} + \hat{J}_\text{loc}] \cdot \ddot{Z} \approx 0. \]  
\[ (4.3.6) \]

\(^4\)For matrix products of \( N \) fields or more, there are certain identities of polynomials related to antisymmetrisation which eventually introduce some ambiguity for the ordering of fields. Arguably this ambiguity does not apply to the equations of motion when \( N_c > 3 \) (at least). Nevertheless, we will later want to establish Yangian symmetry for arbitrary correlators of the fields, in which case an arbitrarily large \( N \) will be necessary. Of course there is the option to reverse-engineer the ordering rule such that the level-one generators are directly represented on fields contracted by structure constants (as long as \( N \) is sufficiently large), however any such rule will be rather messy and cumbersome.
4.3. YANGIAN SYMMETRY OF THE EQUATIONS OF MOTION

The fact that suitable local terms can be found will be a first test for Yangian symmetry. Subsequently, we will consider the other equations of motion. It will of course only be sensible for the local part $\tilde{J}_{\text{loc}}$ to be the same for all of them.

In the following we will focus on the level-one momentum generator $\tilde{J} = \hat{P}$ whose bi-local action is determined by the coproduct in (4.3.1). We will first compute the action of this bi-local term on the Dirac equation (4.1.2):

$$\tilde{\Psi}^{i\dot{\kappa}} = i\bar{\varepsilon}^{i\dot{\alpha}\gamma} D_{\dot{\beta}\dot{\gamma}} \Psi^i - i\varepsilon^{i\dot{\alpha}\epsilon} [\Phi^{i\epsilon}, \bar{\Psi}_{\dot{\alpha}e}] \approx 0. \quad (4.3.7)$$

The choice is motivated by the equation’s simplicity. It consists of only 2 terms (up to sums over indices), both of length 2. Hence finding a suitable local part for $\hat{P}$ should be easiest in this case. We act on the Dirac equation with the coproduct (4.3.1) by means of the tensor product action (4.3.4) using the superconformal representation (2.2.1), (2.2.2), (2.2.3), (2.2.4) and (2.2.6) to obtain

$$\hat{P}_{\beta\dot{\alpha}, \text{biloc}} \cdot \tilde{\Psi}^{i\dot{\gamma}} = i\varepsilon^{i\dot{\gamma}\dot{\epsilon}} \{D_{\dot{\beta}\dot{\tau}} \Phi^{\epsilon\epsilon}, \bar{\Psi}_{\dot{\alpha}\tau}\} + i\varepsilon^{i\dot{\gamma}\dot{\epsilon}} \{\Phi^{i\epsilon\epsilon}, D_{\dot{\beta}\dot{\alpha}} \bar{\Psi}_{\dot{\alpha}\tau}\} + i\varepsilon^{i\dot{\gamma}\dot{\epsilon}} \{[\Phi^{i\epsilon\epsilon}, \bar{\Phi}_{\alpha\epsilon\tau}], \Psi^i\}. \quad (4.3.8)$$

This expression does not vanish on-shell on its own. However, we can make an ansatz for a single-field action $\hat{P} \cdot Z$ based on all terms with appropriate quantum numbers and symmetries, and add their contribution to the overall action of $\hat{P}$. By choosing the coefficients of the ansatz as

$$\hat{P}_{\beta\dot{\alpha}} \cdot \Phi^{cd} := 0,$$
$$\hat{P}_{\beta\dot{\alpha}} \cdot \Psi^{\epsilon\dot{\delta}} := -\varepsilon_{\beta\delta} \{\Phi^{\epsilon\dot{\gamma}}, \bar{\Psi}_{\dot{\alpha}\tau}\},$$
$$\hat{P}_{\beta\dot{\alpha}} \cdot \bar{\Psi}_{\dot{\gamma}\dot{d}} := -\varepsilon_{\dot{\alpha}\dot{\gamma}} \{\Phi_{cd}, \bar{\Psi}^{\epsilon\dot{\gamma}}\},$$
$$\hat{P}_{\beta\dot{\alpha}} \cdot A_{\beta\dot{\gamma}} := \frac{i}{5} \varepsilon_{\dot{\alpha}\dot{\gamma}\dot{\beta}} \{\Phi^{\epsilon\epsilon}, \bar{\Phi}_{\alpha\epsilon\tau}\},$$

we get that

$$\hat{P}_{\beta\dot{\alpha}} \cdot \tilde{\Psi}^{i\dot{\gamma}} = -\delta_{\dot{\alpha}}^{\dot{\gamma}} \varepsilon_{\beta\epsilon} \{\Phi^{i\epsilon\epsilon}, \bar{\Psi}_{\dot{\alpha}e}\} \approx 0. \quad (4.3.9)$$

This shows that $\hat{P}_{\beta\dot{\alpha}}$ is a weak symmetry of the Dirac equation of $\mathcal{N} = 4$ sYM.

One comment on the structure (4.3.9) of the single-field action is in order: It may appear unconventional as it is formulated in terms of anti-commutators in places where commutators are usually expected. However, this configuration is actually natural considering the action of a level-one Yangian generator $\tilde{J}_{\text{biloc}}$ on a commutator

$$f^A_{BC}(J^B \otimes J^C) \cdot [Z^I, Z^J] = f^A_{BC}(J^B \cdot Z^I \cdot J^C \cdot Z^J - J^B \cdot Z^J \cdot J^C \cdot Z^I) = f^A_{BC}[J^B \cdot Z^I, J^C \cdot Z^J]. \quad (4.3.11)$$

This means that level-one generators typically map commutators to anti-commutators.

With the above single-field action of (4.3.9) it is now possible to show that all the equations of motion of $\mathcal{N} = 4$ sYM are weakly invariant under $\hat{P}$

$$\hat{P}_{\beta\dot{\alpha}} \cdot \tilde{\Psi}^{i\dot{\gamma}} = -\delta_{\beta}^{\dot{\gamma}} \varepsilon_{\epsilon\dot{\alpha}} \{\Phi^{i\epsilon\epsilon}, \bar{\Psi}_{\alpha\epsilon}\},$$
$$\hat{P}_{\beta\dot{\alpha}} \cdot \bar{\Phi}^{i\dot{\epsilon}} = -\delta_{\beta}^{\dot{\epsilon}} \varepsilon_{\dot{\alpha}\epsilon} \{\Phi_{cd}, \bar{\Psi}^{i\dot{\epsilon}}\},$$
$$\hat{P}_{\beta\dot{\alpha}} \cdot \bar{\Phi}_{d\epsilon} = -i\varepsilon_{\dot{\alpha}\dot{\epsilon}} \varepsilon_{\beta\lambda} \{\Phi_{d\epsilon}, \bar{\lambda}^{\lambda\kappa}\} + \frac{5}{2} \varepsilon_{d\epsilon\tau\alpha\kappa} \{\Psi^i_{\beta\epsilon}, \bar{\Psi}^{i\dot{\delta}}\}. \quad (4.3.10)$$

Note that all explicit $x$-dependence originating from the generators $L$, $\hat{L}$ and $D$ cancels out exactly. This convenient feature is related to the fact that $\hat{P}$ commutes with $P$ in the Yangian algebra.
the answer is positive, and this novel relationship reads

\[ \hat{P}_{\dot{a}\dot{b}} \cdot \hat{A}^{\dot{b}\dot{c}} = -\frac{i}{2} \delta_\alpha^\gamma \delta_\beta^\delta \{ \Phi^{e\delta}, \Phi_{e\delta} \} - \frac{5}{2} \delta_\alpha^\gamma \delta_\beta^\delta \{ \Psi^{e\gamma}, \Psi_{e\gamma} \} - \frac{5}{2} \delta_\alpha^\gamma \delta_\beta^\delta \{ \Psi^{e\delta}, \Psi_{e\delta} \} + \frac{1}{2} \delta_\alpha^\gamma \delta_\beta^\delta \{ \Psi^{e\lambda}, \Psi_{e\lambda} \} + \frac{1}{2} \delta_\alpha^\gamma \delta_\beta^\delta \{ \Psi^{e\delta}, \Psi_{e\delta} \}. \] (4.3.12)

Moreover, we have verified explicitly that they are weakly invariant under the level-one generators \( \hat{J} \) with \( J \in \{Q, Q, R\} \) as well as the level-one bonus symmetry \( \hat{B} \) which extends the Yangian of \( \mathfrak{psu}(2,2|4) \). This also fixes the single-field actions of these generators, and we present our results in Appendix A.

Up to some issues w.r.t. the closure of the Yangian algebra onto gauge transformations, to be discussed in Chapter 7, we conclude that the Yangian of \( \mathfrak{psu}(2,2|4) \) is a weak symmetry of the classical planar equations of motion of \( \mathcal{N} = 4 \) sYM. As previously discussed, this represents a necessary condition for Yangian symmetry in planar \( \mathcal{N} = 4 \) sYM. It is reassuring to see that it is met, and we can continue with the construction of a sufficient condition.

### 4.3.2 Strong invariance of the equations of motion

Next we would like to promote the above results to a strong invariance of the equations of motion. As we argued in section 4.2, this would amount to an honest statement of Yangian symmetry in classical planar \( \mathcal{N} = 4 \) sYM. To that end, we need an expression which predicts the exact form of the l.h.s. of \( J \cdot \hat{Z} \approx 0 \) as a linear combination of the \( \hat{Z} \) (which are zero on shell).

Let us start from the ordinary superconformal symmetry. As we have already shown, the invariance of the action under the generators \( J \) of \( \mathfrak{psu}(2,2|4) \) implies an off-shell relationship for the equations of motion of (4.2.4)

\[ J \cdot \hat{Z}_K = -\hat{Z}_I \frac{\delta(J \cdot Z^I)}{\delta Z_K}. \] (4.3.13)

On the r.h.s. of this equality we have a sum of terms \( \hat{Z} \) whose the coefficients are completely determined by the superconformal representation on the fields. Note that the sequence of fields for the term on the r.h.s. takes a particular form which is not completely evident from the above expression. It can be inferred from the colour structure which provides the correct adjacency information due to contraction of the field indices \( I \) and \( K \). In particular, the contraction of indices \( I \) implies to take a colour trace. Let us explain it by means of an example: suppose \( J \cdot Z^I = Z^L \hat{Z}^M Z^N \) yields some cubic combination of fields, the term on the r.h.s. would take the form

\[ \hat{Z}_I \frac{\delta(J \cdot Z^I)}{\delta Z_K} = \delta^L_K Z^M Z^N \hat{Z}_I + \delta^M_K Z^N \hat{Z}_I Z^L + \delta^N_K \hat{Z}_I Z^L Z^M. \] (4.3.14)

It is straightforward to generalise this expression to any number of fields.

We can now ask ourselves if we can find an analogous expression for level-one Yangian generators: an off-shell relationship for the action of \( \hat{J} \) on \( \hat{Z}_K \), whose right-hand side is a sum of \( \hat{Z}_I \) with coefficients completely determined by the action of the generators on the fields of the theory. With some inspiration from the expected structures related to level-one symmetry, we find that the answer is positive, and this novel relationship reads

\[ \hat{J} \cdot \hat{Z}_K = -\hat{Z}_I \frac{\delta(\hat{J} \cdot Z^I)}{\delta Z_K} + \hat{Z}_I \left[ J^{(1)} \land \frac{\delta}{\delta Z^I} \right] \cdot (J^{(2)} \cdot Z^I). \] (4.3.15)
The anti-symmetric tensor product $\wedge$ was defined in (4.3.2) and it acts in analogy to (4.3.4). The insertion of the new field $\hat{Z}_I$, however, follows different rules based on the colour structure associated to the indices $I, K$ in analogy to (4.3.14). For example, assume that $J^{(2)} \cdot Z^I = Z^L Z^M Z^N$ again is a cubic combination of fields. Then the bi-local term in (4.3.15) expands to

$$
\hat{Z}_I \left[ (J^{(1)} \wedge \frac{\delta}{\delta Z^K}) \cdot (J^{(2)} \cdot Z^I) \right] = -\delta_K^I \left[ (J^{(1)} \cdot Z^M) Z^N \hat{Z}_I + Z^M (J^{(1)} \cdot Z^N) \hat{Z}_I \right] \\
+ \delta_K^M \left[ Z^N \hat{Z}_I (J^{(1)} \cdot Z^L) - (J^{(1)} \cdot Z^N) \hat{Z}_I Z^L \right] \\
+ \delta_N^M \left[ \hat{Z}_I (J^{(1)} \cdot Z^L) Z^M + \hat{Z}_I Z^L (J^{(1)} \cdot Z^M) \right].
$$

Note that the colour trace implied by the contraction of $I$ is cut open by the operator $\delta/\delta Z^K$, so that $\hat{Z}_I$ can appear in the middle of the resulting polynomial. This expression also generalises to any number of fields, and it can be non-zero only if $J^{(2)} \cdot Z^I$ consists of (at least) two fields. In other words, the bi-local term in (4.3.15) only sees the non-linear contributions of $J^{(2)}$.

First, we shall show that the above relationship holds for the Dirac equation in $\mathcal{N} = 4$ sYM. The l.h.s. of (4.3.15) has been computed in (4.3.10). We need to show that it matches with the combination of terms on the r.h.s. of (4.3.15). The first term involves a variation of $\hat{P} \cdot Z^I$ by $\bar{\Psi}$. The only single-field action (4.3.9) containing a field $\bar{\Psi}$ is the one for $Z^I = \Psi$. Thus we can set $Z^I$ to $P \cdot Z^I$ in the first term. The second term turns out to yield no contribution for a combination of reasons: First of all, $J^{(2)} \cdot Z^I$ must yield an expression non-linear in the fields. One option is $J^{(2)} = Q, \bar{Q}$ acting on $Z^I = \Psi, \bar{\Psi}$, which however never produces any terms containing $Z^K = \bar{\Psi}$. Another option is $J^{(2)} = L, \bar{L}, D$, but all non-linear terms have an explicit $x$-dependence which will eventually cancel against other terms. It remains to check $J^{(2)} = P$. It must act on $Z^I = \bar{\Psi}$ if the result is to contain $Z^K = \bar{\Psi}$. The only other field is $Z^J = A$ (within $D\bar{\Psi}$) which can be acted upon by $J^{(1)} = L, \bar{L}, D$. However, all these terms are explicitly $x$-dependent and as such they are needed to cancel against other terms. Hence the second term does not contribute, and we are left with

$$
\hat{P}_{\beta \hat{a}} \cdot \bar{\Psi} d^\gamma \gamma = -\text{tr} \left[ \bar{\Psi}_{\gamma d} \frac{\delta (\hat{P}_{\beta \hat{a}} \cdot \Psi^f)}{\delta \Psi_{\gamma d}} \right] = \varepsilon_{\beta \epsilon} \text{tr} \left[ \bar{\Psi}_{\gamma d} \frac{\delta [\Phi_{g f \epsilon}, \bar{\Psi}_{\gamma d}]}{\delta \Psi_{\gamma d}} \right] \\
= \varepsilon_{\beta \epsilon} \text{tr} \left[ \{ \bar{\Psi}_{\gamma d}, \Phi_{g f \epsilon} \} \right] = \delta_{\alpha \beta} \varepsilon_{\gamma d} \{ \Phi_{d f}, \bar{\Psi}_{\gamma d} \}. \tag{4.3.17}
$$

This agrees precisely with (4.3.10). We have also verified that the relation (4.3.15) holds exactly for all equations of motion of $\mathcal{N} = 4$ sYM and for the level-one generators $\hat{J}$ with $J \in \{ P, Q, \bar{Q}, R, B \}$, i.e. it correctly reproduces the r.h.s. of all terms in (4.3.12) and corresponding relations for the other generators.

Next, let us now explain the meaning of the two terms appearing on the r.h.s. in (4.3.15). The first of them is a direct counterpart of the one from the level-zero formula (4.3.13). The other term should be viewed as a result of the non-trivial coproduct of the level-one generators (3.2.7). Interestingly, the action of the two constituent operators $J^{(1)}$ and $J^{(2)}$ is overlapping, as $J^{(1)}$ acts purely on the output of $J^{(2)}$. We will thus call this contribution the overlapping term. Note that the overlapping term is a purely non-linear effect.

To conclude this discussion, we would like to emphasise that unlike its level-zero counterpart (4.3.13), the level-one formula (4.3.15) for invariance of the equations of motion has not been valid.

\[\text{This is in agreement with the fact the overlapping contribution from two linear operators } J^{(1)} \text{ and } J^{(2)} \text{ essentially boils down to their commutator. The latter is equivalent to a local term and could therefore be absorbed into the definition of the local part of the bi-local operator.}\]
derived from first principles. We have merely verified that it holds exactly for all equations of motion of $\mathcal{N} = 4$ sYM and for several level-one generators $\J$. In other words, we have derived a strong form of invariance of the equations of motion for level-one generators. This invariance amounts to a set of non-trivial off-shell identities which hold in classical planar $\mathcal{N} = 4$ sYM. Such identities are hard to get hold of, and independently of how we obtained them and of their precise form, they clearly indicate a (hidden) property of classical planar $\mathcal{N} = 4$ sYM which is equivalent to a global symmetry.

4.3.3 $\mathcal{N} < 4$ sYM verification of the strong invariance

In this section we would like to address a possible concern that our results discussed above, especially the formula (4.3.15), are a mere result of level-zero (superconformal) and gauge symmetries of the theories. If that was the case, our work would not provide any criterion for establishing Yangian symmetry of physical models. As we will now demonstrate, this is fortunately not the case and for formula (4.3.15) to hold increased symmetry is indeed required.

To this end let us consider a pure sYM theory, but keep $\mathcal{N}$ arbitrary. The Lagrangian is again given by:

$$
\mathcal{L} = -\frac{1}{2} \varepsilon^\alpha\varepsilon^\gamma\tr(F_{\alpha\gamma}F_{\epsilon\kappa}) - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\dot{\kappa}\dot{\lambda}}\tr(\bar{F}_{\dot{\alpha}\dot{\kappa}\dot{\kappa}\dot{\lambda}} - \bar{F}_{\dot{\alpha}\dot{\kappa}\dot{\lambda}\dot{\kappa}})
$$

$$
+ i\varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^\beta\tr(\bar{D}_{\dot{\alpha}a}D_{\beta\dot{\alpha}}\Psi^d_\gamma) - \frac{1}{4} \varepsilon^{\dot{\alpha}\dot{\gamma}}\varepsilon^{\dot{\beta}\dot{\gamma}}\tr(D_{\dot{\beta}\dot{\alpha}}\Phi_{ef} D_{\dot{\beta}\dot{\gamma}}\Phi^e_f)
$$

$$
+ \frac{i}{2} \varepsilon^{\alpha\gamma}\tr(\Phi_{ef}\{\Psi^e_a, \Psi^f_\gamma\}) + i\varepsilon^{\dot{\alpha}\dot{\gamma}}\tr(\Phi^e_f\{\bar{\Psi}^e_{\dot{a}e}, \bar{\Psi}^f_\gamma\})
$$

$$
+ \frac{1}{16} \tr((\bar{\Phi}_{ab}, \Phi^e_f)[\Phi^{ab}, \Phi_{ef}]).
$$

The action (4.3.18) looks exactly like (2.1.13) of $\mathcal{N} = 4$ sYM. The only difference is the range of indices $a, b, \ldots = 1, \ldots, \mathcal{N}$, where $\mathcal{N} \leq 4$. As the reality condition (2.1.5) exists only for $\mathcal{N} = 4$, the scalar potential is written here in a different form, valid also for $\mathcal{N} \neq 4$. Moreover, for $\mathcal{N} = 1$ the scalar fields do not exist at all, which here is implicitly taken care of by the anti-symmetry of their indices, $\Phi^{ab} = -\Phi^{ba} = \Phi^{11} = 0$. The symmetry algebra of (4.3.18) is $\mathfrak{psu}(2, 2|\mathcal{N})$ and for any $\mathcal{N} \geq 0$ its Yangian exists. What we want to demonstrate is that $\mathcal{Y}[\mathfrak{psu}(2, 2|\mathcal{N})]$ is a symmetry of the theory only for the special value $\mathcal{N} = 4$. To that end it is already enough to show that for other values of $\mathcal{N}$ the weak invariance of the equations of motion does not hold, as its failure will assure that all the stronger criteria do not stand either.

We thus can repeat the computation from 4.3.1 keeping $\mathcal{N}$ arbitrary. The action of all the generators carries over from Chapter 2 and 4.3 (see equations (2.2.6) and (4.3.9) for supersymmetry and local Yangian action respectively), of course up to the restricted range of indices mentioned above (e.g. the local action of $\bar{\P}$ vanishes completely for $\mathcal{N} = 1$). For general $\mathcal{N} \geq 1$ we act with $\bar{\P}$ on the Dirac equation of motion

$$
\bar{\Psi}^{\dot{d}\dot{\iota}} = i\varepsilon^{\dot{\alpha}\dot{\iota}}\varepsilon^\beta\bar{D}_{\dot{\alpha}\dot{\iota}}\Psi^d_\beta - i\varepsilon^{\dot{\alpha}\dot{\iota}}\{\Phi^{\dot{d}\dot{e}}, \bar{\Psi}_{\dot{d}\dot{e}}\}.
$$

On shell we are left with a term of the form

$$
\bar{\P}_{\alpha\beta}\bar{\Psi}^{\dot{d}\dot{\iota}} \approx (4 - \mathcal{N})\varepsilon^{\dot{\iota}\dot{\kappa}}\{\bar{F}_{\dot{\beta}\dot{\kappa}}, \Psi^d_\alpha\}.
$$

Even though we have the principal freedom to adjust the local action of $\bar{\P}$ to the case $\mathcal{N} < 4$, it is easy to see that this cannot remove the residual term: In order to cancel the term, one would
need extra contributions of the form $\hat{P}A \sim \hat{F}$ or $\hat{P}\psi \sim D\psi$. However, these would merely produce commutators $[\hat{F},\psi]$ rather than the desired anti-commutator $\{\hat{F},\psi\}$.

For $\mathcal{N} = 0$ this argument of course fails, since there are no fermions in this theory. We have to therefore work with a slightly more difficult (because it features terms of length 3) Yang–Mills equation of motion. The generator $\hat{P}$ in this case will also have a trivial local action, just like for $\mathcal{N} = 1$, whereas the bilocal part of the coproduct does not have the supersymmetry generators, with the other terms staying the same:

$$\Delta \hat{P}^{\mathcal{N}=0}_{\alpha\dot{\beta}} = \hat{P}_{\alpha\dot{\beta}} \otimes 1 + 1 \otimes \hat{P}_{\alpha\dot{\beta}} - L^\gamma_\alpha \wedge P_{\gamma\dot{\beta}} - \bar{L}^\gamma_\dot{\beta} \wedge P_{\alpha\gamma} - D \wedge P_{\alpha\dot{\beta}} \quad (4.3.21)$$

Applying the generator $\hat{P}$ to the Yang–Mills equation of motion we encounter some cancellations, but in the end we are left with a term of the form:

$$\hat{P}^{\mathcal{N}=0}_{\alpha\dot{\beta}} \tilde{A}^{\gamma\dot{\kappa}} \approx \delta^\gamma_\alpha \delta^\dot{\kappa}_\dot{\beta} \left( \varepsilon^{\rho\epsilon} \varepsilon^{\phi\delta} \{F_{\rho\phi}, F_{\epsilon\delta}\} + \varepsilon^{\hat{\rho}\hat{\epsilon}} \varepsilon^{\hat{\phi}\hat{\delta}} \{\bar{F}_{\hat{\rho}\hat{\phi}}, \bar{F}_{\hat{\epsilon}\hat{\delta}}\} \right)$$

$$+ \frac{1}{4} \delta^\gamma_\alpha \delta^\dot{\kappa}_\dot{\beta} \varepsilon^{\rho\phi} \{F_{\rho\phi}, F_{\epsilon\delta}\} + \frac{1}{4} \delta^\rho_\alpha \delta^\hat{\rho}_\beta \varepsilon^{\gamma\dot{\kappa}} \varepsilon^{\phi\delta} \{F_{\rho\phi}, F_{\epsilon\delta}\} \quad (4.3.22)$$

Also in this case this term cannot be removed by any adjustments to the single field action. Thus we see that the value $\mathcal{N} = 4$ is special as it warrants the cancellation of the residual term and the on-shell invariance of the equations of motion of $\mathcal{N} = 4$ sYM. More importantly, we observe that superconformal and gauge symmetries by themselves are not sufficient to warrant our results presented in the preceding sections, and indeed Yangian invariance of a theory is a non-trivial statement.
Chapter 5

Yangian symmetry in $\beta$-deformed sYM

Having finished the previous chapter with a negative example, where our formalism correctly shows the lack of Yangian symmetry in the pure $\mathcal{N} < 4$ sYM, we will now apply it to a theory where integrable structures have been found. As the integrability has been proven an indispensable tool in verifying the AdS/CFT correspondence, the search has begun to find examples beyond the $\mathcal{N} = 4$ sYM and ABJM gauge theories and their respective duals. One way of obtaining different integrable theories starting with some initial ones is by the means of deformations.

For $\mathcal{N} = 4$ SYM such deformations – modifications of the superpotential – were first discussed by Leigh and Strassler in [55]. It was early observed that one of the deformations, the real-$\beta$-deformation, preserves integrability of $\mathcal{N} = 4$ SYM (see [56] and references therein) in the planar limit. Its gravity dual was found by Lunin and Maldacena [57], and later demonstrated to stem from TsT transformations of the original $AdS_5 \times S^5$ background [84], [85]. In this way, the AdS/CFT duality is demonstrated to apply to a larger set of theories.

The real-$\beta$-deformed theory is manifestly an $\mathcal{N} = 1$ SYM. In [86], [87] it was demonstrated however, that the manifest $SU(3) \times U(1)$ R-symmetry of $\mathcal{N} = 4$ SYM expressed in the $\mathcal{N} = 1$ language (see Section 5.1) does survive the deformation, albeit in a twisted way. Thus the $\mathcal{N} = 4$ supersymmetry is not necessarily broken, but rather hidden. This result is backed up by the study of amplitudes in [88]. The conclusion there was that the amplitudes in the twisted theory can be easily obtained from the ones of $\mathcal{N} = 4$ SYM by a procedure that depends only on the external legs, irrespectively of the internal structure, even though the vertices get deformed too.

As we already alluded to before, the superconformal symmetry by itself is not sufficient to account for integrability of a field theory. For the parent $\mathcal{N} = 4$ SYM the correct infinite-dimensional algebra has been identified as the Yangian $Y(\mathfrak{psu}(2,2|4))$ [77], as recalled and demonstrated in previous Chapters. It is natural to expect that the integrability of the real-$\beta$-deformed theory will be explained by a suitable deformation of this Yangian algebra. It was indeed shown in [89] that some closed subsectors of it do enjoy a twisted Yangian symmetry, resonating well with the results of [90] on the twisted R-matrix. In this chapter we will slightly modify and use the equations-of-motion based formalism to show that the Yangian symmetry holds in the classical $\beta$-deformed sYM as well as in the undeformed $\mathcal{N} = 4$ sYM. This will demonstrate that the formalism based on equations of motion can be applied to variety of theories. This chapter is based on [59].

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1What we consider here is however not a q-deformed algebra, since the commutation relations will not be deformed. The deformation will influence only the Hopf structure, i.e. will be visible on the level of the coproduct.
5.1 \( \beta \)-deformed \( \mathcal{N} = 4 \) SYM

The action of the real-\( \beta \)-deformed \( \mathcal{N} = 4 \) SYM [55] is most conveniently expressed in the \( \mathcal{N} = 1 \) language, the field content being three chiral and one vector superfield. Working with component fields, \( \phi^i \) are the three complex scalar fields, \( \psi^i_{\alpha} \) their superpartners (\( i = 1, 2, 3 \)). The gauge field \( A_{a\dot{a}} \) has the gluino \( \psi^4_{\alpha} \) as its superpartner and acts as a connection for the covariant derivative \( D_{a\dot{a}} = \partial_{a\dot{a}} + iA_{a\dot{a}} \). All the fields are in the adjoint of the gauge group \( U(N) \).

Written out explicitly, the Langrangian of the theory takes the form:

\[
\mathcal{L} = \text{tr} \left( -\frac{1}{2} \varepsilon^{\alpha\dot{\gamma}\kappa} \varepsilon^{\gamma\dot{\kappa}} \text{tr}(F_{a\gamma} F_{\dot{a}\kappa}) - \frac{1}{2} \varepsilon^{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\kappa}} \varepsilon^{\dot{\gamma}\dot{\kappa}} \text{tr}(\bar{F}_{\dot{a}\dot{\gamma}} \bar{F}_{\dot{a}\dot{\kappa}}) - \frac{1}{2} \varepsilon^{\alpha\beta\dot{\gamma}\dot{\kappa}} \varepsilon^{\dot{\gamma}\dot{\kappa}} [D_{a\dot{a}}, \bar{\psi}_{\dot{\alpha}}][D_{\beta\dot{\beta}}, \phi^i] + \\
- \frac{1}{2} [\phi^i, \phi^j]_{\beta_{ij}} [\bar{\phi}_i, \bar{\phi}_j]_{\beta_{ij}} + \frac{1}{4} [\phi^i, \bar{\phi}_i][\phi^j, \bar{\phi}_j] + \varepsilon^{\alpha\beta\dot{\gamma}\dot{\kappa}} [D_{a\dot{a}}, \psi^4_{\alpha}] + \\
+ \varepsilon^{\alpha\beta\dot{i}\dot{j}} \bar{\psi}_{\dot{i}\dot{j}} [D_{\beta\dot{\beta}}, \psi^i_{\alpha} + i\left( \varepsilon^{\alpha\beta}[\psi^4_{\alpha}, \psi^i_{\beta}]_{\bar{\beta}_{ij}} + \varepsilon^{\dot{\alpha}\dot{\beta}}[\bar{\psi}_{\dot{4}\dot{\alpha}}, \bar{\psi}_{\dot{i}\dot{j}}\dot{\beta}]\right) + \\
+ \frac{i}{2} \left( \varepsilon_{ijk} \varepsilon^{\alpha\beta}[\psi^i_{\alpha}, \psi^j_{\beta}]_{\bar{\beta}_{ij}} \phi^k + \varepsilon_{ijk} \varepsilon^{\dot{\alpha}\dot{\beta}}[\bar{\psi}_{\dot{i}\dot{j}}, \bar{\psi}_{\dot{k}\dot{\beta}}]\phi^k \right) \right). \tag{5.1.1}
\]

In the above we have introduced a \( \beta \)-deformed graded commutator:

\[
[f_i, g_j]_{\beta_{ij}} = \varepsilon^{i\pi\beta_{ij}} f_i g_j - (-1)^{|f_i||g_j|}\varepsilon^{i\pi\beta_{ji}} g_j f_i, \tag{5.1.2}
\]

where \( \beta_{ij} = -\beta_{ji}, \beta_{12} = \beta_{23} = \beta_{31} = \beta \in \mathbb{R} \).

In order to make the connection with the action of the \( \mathcal{N} = 4 \) SYM introduced in Chapter 2 we need to relate the complex \( \mathcal{N} = 1 \) formulation scalars \( \phi^i \) with the real scalar fields \( \bar{\phi}^{ab} \) introduced earlier on. This is given by:

\[
\phi^{ab} = \frac{i}{2} \begin{bmatrix}
0 & -\bar{\phi}_3 & \bar{\phi}_2 & \phi^1 \\
\bar{\phi}_3 & 0 & -\bar{\phi}_1 & \phi^2 \\
-\bar{\phi}_2 & \bar{\phi}_1 & 0 & \phi^3 \\
-\phi^1 & -\phi^2 & -\phi^3 & 0
\end{bmatrix}. \tag{5.1.3}
\]

Taking \( \beta = 0 \) in the Lagrangian [5.1.1] we recover the maximally symmetric \( \mathcal{N} = 4 \) SYM theory. In the present form however, only \( SU(3) \times U(1) \) subgroup of the \( SU(4) \) R-symmetry is manifestly present, with the \( U(1) \) factor corresponding to the vector superfield and the \( SU(3) \) to 3 chirals.

For an arbitrary real \( \beta \) the action [5.1.1] is invariant under \( \mathcal{N} = 1 \) supersymmetry with the charges \( Q_\alpha, \bar{Q}_{\dot{\alpha}} \), which in the undeformed theory correspond to \( Q_{4\alpha}, \bar{Q}_{4\dot{\alpha}} \) (see Appendix D).

As was shown in [55], the theory remains conformal even on a quantum level – a property shared with the full \( \mathcal{N} = 4 \) SYM – and thus yields another example of a superconformal quantum field theory, as already alluded to in the Introduction.

5.2 Untwisting the twist

As already discussed in the Introduction, the \( \beta \)-deformed theory shares a lot of similarities with its parent, the \( \mathcal{N} = 4 \) SYM. In this section we want to show that the deformed theory in fact possesses the full (albeit deformed and nonlocal) \( \mathcal{N} = 4 \) supersymmetry. To this end it is enough
to demonstrate that the theory is invariant under the full $SU(4)$ R-symmetry group. All the remaining supersymmetry generators can be then recovered from the $Q_\alpha = Q_{4\alpha}$ and $\bar{Q}_\alpha = \bar{Q}_{\dot{\alpha}}$ via:

$$Q_{4\alpha} = [R_{4i}, Q_\alpha]$$  \hspace{1cm} (5.2.1)

and similarly for the conjugate ones.

Corresponding to the $\mathcal{N} = 1$ supersymmetry, the action is invariant under the $R_{44}$ component of the R-symmetry (in this case the $U(1)$ generator). In the case of the real-$\beta$ deformation however, the symmetry is actually larger, as all the diagonal elements $R_{cc}$ survive. The additional $U(1)$ symmetries enabled the construction of the background of the gravity dual [57] (see also [86] and [87] for a discussion of extended symmetry of the theory).

We will now show that all the remaining generators $R_{ab}$ can be promoted to the symmetries of the action (5.1.1). To this end, we will first give their action for $\beta = 0$ case, which is just the $\mathcal{N} = 4$ SYM theory expressed in the $\mathcal{N} = 1$ language. In the second step we will see how to modify their action in the case of nonvanishing $\beta$.

The gauge field is annihilated by all the generators $R_{ab}$. Their action on the fermionic fields is given by:

$$R^a_b \psi^c_\alpha = \delta^c_b \psi^a_\alpha - \frac{1}{4} \delta^a_b \psi^c_\alpha.$$  \hspace{1cm} (5.2.2)

All the four fields $\psi^a_\alpha$ are treated here on an equal footing and an analogous formula holds for the conjugate fermions $\bar{\psi}^a_{\dot{\alpha}}$. The action on the scalars can then be obtained by relating the complex fields $\phi^i$ to the hermitian scalars with antisymmetric $su(4)$ indices most frequently used in the studies of $\mathcal{N} = 4$ SYM: $\phi^i \propto \Phi^4$ (see Appendix D for details). Each of these indices transforms then as given by (5.2.2). Interestingly, that results in mixing of the fields $\phi^i$ and $\bar{\phi}^j$ under the action of $R_{ab}$, for example $R^4_3 \phi^1 = \bar{\phi}^2$. With those expressions one can then verify that for $\beta = 0$ the action (5.1.1) is annihilated by all $R_{ab}$.

Next we want to generalize this result to the case of the nonvanishing deformation parameter. This can be achieved by a suitable modification of the coproduct of the generators, i.e. the action on products of fields. In the real-$\beta$-deformed SYM this is achieved by the Drinfeld-Reshetikhin twist of the comultiplication [91]:

$$\Delta_{\mathcal{F}} = \mathcal{F} \Delta \mathcal{F}^{-1},$$  \hspace{1cm} (5.2.3)

with $\mathcal{F}$ given in terms of $SU(4)$ charges of the fields:

$$\mathcal{F} = e^{i\pi \beta (h_1 \wedge h_2 + h_2 \wedge h_3 - h_1 \wedge h_3)},$$  \hspace{1cm} (5.2.4)

where $h_i = R^i_1 + R^4_4$ (no sum over $i$).

We will omit the subscript $\mathcal{F}$ from now on. As alluded to above, for the diagonal R-symmetry generators the modification acts as identity:

$$\Delta R^c_c = \mathbb{1} \otimes R^c_c + R^c_c \otimes \mathbb{1}.$$  \hspace{1cm} (5.2.5)

The coproduct of nondiagonal elements however changes to

\footnote{For the ordinary Lie algebra generators $J^a$ the coproduct is usually trivial, that is $\Delta J = J \otimes \mathbb{1} + \mathbb{1} \otimes J$. From there follows the most common action of the generator on a product of fields: $J(Z_1 Z_2 \ldots Z_n) = \sum_{i=1}^{\alpha} Z_1 \ldots (J Z_i) \ldots Z_n$.}
\[ \Delta R^a{}_b = K_{ab} \otimes R^a{}_b + R^a{}_b \otimes K_{ba} \]  
(5.2.6)
where \( K_{ba} = K_{ab}^{-1} \). The element \( K_{ab} \) is group-like:

\[ \Delta K_{ab} = K_{ab} \otimes K_{ab}. \]  
(5.2.7)

This property warrants a different notation for generator(s) \( K \).

Its action on the fields of the theory amounts to a multiplication by a \( \beta \)-dependent phase:

\[ K_{ab} \phi^i = e^{i \pi r(a,b,i) \beta} \phi^i \]
\[ K_{ab} \bar{\phi}_i = e^{-i \pi r(a,b,i) \beta} \bar{\phi}_i \]
\[ K_{ab} \psi^i{}_{\alpha} = e^{i \pi r(a,b,i) \beta} \psi^i{}_{\alpha} \]
\[ K_{ab} \bar{\psi}_{i\alpha} = e^{-i \pi r(a,b,i) \beta} \bar{\psi}_{i\alpha} \]
\[ K_{ab} Z = Z, \]  
(5.2.8)
where \( Z \) stands for all the remaining fields. The function \( r(a,b,i) \) is given explicitly in the Appendix.

Observe that the coproduct (5.2.7) does not cope well with cyclicity. It is a usual problem with nontrivial coproducts, since due to appearance of trace in (5.1.1) and any other Lagrangian the action of a theory is cyclic. To circumvent this problem, in the previous chapter 4 we introduced an equation-of-motion-based formalism. We will sketch it here. For a usual Lie-type symmetry with a trivial coproduct which leaves the action invariant:

\[ J^S := (J Z_I) \frac{\delta S}{\delta Z_I} = 0, \]  
(5.2.9)
where \( J \) is an algebra generator, we can differentiate (5.2.9) with respect to an arbitrary field \( Z_C \) to obtain:

\[ J \frac{\delta S}{\delta Z_K} = - \frac{\delta (J Z_I)}{\delta Z_K} \frac{\delta S}{\delta Z_I}. \]  
(5.2.10)

Notice that the equality (5.2.10), contrary to (5.2.9), contains no cyclic objects. What (5.2.10) states is that the result of \( J \) acting on an equation of motion \( \frac{\delta S}{\delta Z_C} \) is a particular combination of other equations of motion with coefficients determined purely by a representation of \( J \) on fields \( Z_A \) of the theory. We argued in chapter 4 that (5.2.10) is equivalent to the invariance of the action. The question now is what is the equivalent of (5.2.10) in case of a twisted coproduct (5.2.7). Surprisingly, the answer is that (5.2.10) holds in this case in an unchanged form, with no explicit appearance of generators \( K_{ab} \) (even though they do contribute while acting on an equation of motion) - a fact we confirmed by direct computation. Indeed, it can actually be shown that their action amounts to an overall factor while acting on equations of motion and hence becomes unobservable. Heuristically this can be explained by noting that the kinetic term is always undeformed. All the transformations are thus fully determined by the single-field action of \( R^a{}_b \). Having shown that all the R-symmetry generators are symmetries of the action, we now may use them to construct the missing supercharges according to (5.2.1) and thus argue that indeed the real-\( \beta \)-deformed SYM possesses a \( \mathcal{N} = 4 \) supersymmetry. The formula (5.2.10) holds then also for the generators \( Q_{aa} \) and \( \bar{Q}_{\dot{\alpha}a} \), as we verified again by explicit calculation.

Concluding this paragraph, let us mention that it may actually be directly shown that the action obtained by integrating the Lagrangian (5.1.1) is invariant under all R-symmetry generators \( R^a{}_b \).
5.3. TWISTED YANGIAN

without resorting to the equations of motion. To this end, we cannot rely on picking an arbitrary cyclic representative, but rather need to consider the action as the averaged sum of all the possible ones. This of course is a trivial operation under the trace, but in order to act with a generator of transformation, we need to cut the trace open at some point. A trivial coproduct respects cyclicity of the trace and hence the choice of the opening point is irrelevant, but that ceases to be true for coproducts like (5.2.6). We hence rewrite the terms forming the action as:

$$\text{tr} (Z_1 Z_2 \ldots Z_n) \rightarrow \frac{1}{n} \sum_{\sigma \in Z_n} Z_{\sigma(1)} Z_{\sigma(2)} \ldots Z_{\sigma(n)},$$  \hspace{1cm} (5.2.11)

and then act on them using iterated coproducts. A heuristic picture to have in mind is cutting open the closed (due to trace) chain of fields at every possible point. Under so defined application of the generators, the action is invariant.

5.3 Twisted Yangian

Having established a twisted $\mathcal{N} = 4$ superconformal symmetry of the real-\(\beta\)-deformed theory, we are now in position to observe what happens to the Yangian symmetry. Indeed, for the $\mathcal{N} = 4$ SYM the Yangian of $\text{psu}(2,2|4)$ underlies its integrability. Twisted Yangian algebra has been identified in [89] as the symmetry of some subsectors of the real-\(\beta\)-deformed SYM. We would now like to promote this discussion to a full theory, taking into account the nonlinearities of the symmetries.

In the case of $\mathcal{N} = 4$ SYM, the simplest level-1 Yangian generator is the level-1 momentum $\hat{P}_{\alpha \dot{\alpha}}$, whose coproduct is given by:

$$\Delta \hat{P}_{\alpha \dot{\alpha}}, \mathcal{N}=4 = I \otimes \hat{P}_{\alpha \dot{\alpha}} + \hat{P}_{\alpha \dot{\alpha}} \otimes I + D \wedge P_{\alpha \dot{\alpha}} + P_{\beta \dot{\beta}} \wedge L^\beta _\alpha + P_{\alpha \dot{\beta}} \wedge \bar{L}^\dot{\beta}_{\dot{\alpha}}$$

$$- \frac{1}{2} Q_{\alpha} \wedge \bar{Q}_{\dot{\alpha}} - \frac{1}{2} Q_{\alpha i} \wedge \bar{Q}_{\dot{\alpha}}^i$$

$$= I \otimes \hat{P}_{\alpha \dot{\alpha}} + \hat{P}_{\alpha \dot{\alpha}} \otimes I + h^{\alpha \beta}_{M N} J^M \otimes J^N,$$  \hspace{1cm} (5.3.1)

where $h^{M L}_{\mathcal{N} N}$ are the $\text{psu}(2,2|4)$ structure constants. Since the generators of conformal algebra and $Q_{\alpha}$ and $\bar{Q}_{\dot{\alpha}}$ are insensitive to the $\beta$-deformation, the only possible twists in the coproduct for $\hat{P}_{\alpha \dot{\alpha}}$ can appear in the terms containing $Q_{\alpha i}$ and $\bar{Q}_{\dot{\alpha}}^i$. Indeed, following the methods developed in [58] we checked via explicit computations that the modified coproduct:

$$\Delta \hat{P}_{\alpha \dot{\alpha}}, \beta \neq 0 = I \otimes \hat{P}_{\alpha \dot{\alpha}} + \hat{P}_{\alpha \dot{\alpha}} \otimes I + D \wedge P_{\alpha \dot{\alpha}} + P_{\beta \dot{\beta}} \wedge L^\beta _\alpha + P_{\alpha \dot{\beta}} \wedge \bar{L}^\dot{\beta}_{\dot{\alpha}}$$

$$- \frac{1}{2} Q_{\alpha} \wedge \bar{Q}_{\dot{\alpha}} - \frac{1}{2} Q_{\alpha i} \wedge \bar{Q}_{\dot{\alpha}}^i K_{i \alpha}^{-1} \otimes \bar{Q}_{\dot{\alpha}}^i K_{i \dot{\alpha}} - \frac{1}{2} Q_{\alpha i} \bar{Q}_{\dot{\alpha}}^i K_{i \alpha}^{-1} \otimes Q_{\alpha i} K_{i \dot{\alpha}}$$  \hspace{1cm} (5.3.2)

maps equations of motion of the theory to each other, provided the single-field action of $\hat{P}_{\alpha \dot{\alpha}}$ is given by eq. (5.3.3)

$$\hat{P}_{\alpha \dot{\alpha}} \phi^i = 0$$

$$\hat{P}_{\alpha \dot{\alpha}} \phi_{\dot{i}} = 0$$

$$\hat{P}_{\alpha \dot{\alpha}} \psi^{i \dot{\beta}} = i \epsilon_{\alpha \beta} \epsilon^{\dot{i} \dot{jk}} \{ \bar{\psi}_{\dot{j} \dot{\alpha}}, \phi_{\beta} \}_j^k - i \epsilon_{\alpha \beta} \{ \phi^{i}, \bar{\psi}_{\dot{i} \dot{\alpha}} \}$$
\[
\begin{align*}
\hat{P}_{\alpha\dot{\alpha}} \bar{\psi}_{i\dot{\beta}} &= i \epsilon_{\dot{\alpha}\dot{\beta}} \epsilon_{ijk} \{ \psi^j_\alpha, \phi^k_\beta \} \bar{\phi}_i, \psi^4_\alpha \\
\hat{P}_{\alpha\dot{\alpha}} \bar{\psi}^4_\beta &= i \epsilon_{\alpha\beta} \bar{\psi}_{i\dot{\alpha}}, \phi^i_\beta \\
\hat{P}_{\alpha\dot{\alpha}} \bar{\psi}_{4\dot{\beta}} &= i \epsilon_{\dot{\alpha}\dot{\beta}} \{ \psi^i_\alpha, \bar{\phi}_i \} \\
\hat{P}_{\alpha\dot{\alpha}} D_{\beta\dot{\beta}} &= - \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} \{ \phi^i, \bar{\phi}_i \}.
\end{align*}
\] (5.3.3)

Observe that the modified part is no longer antisymmetric. Furthermore, due to the group-like nature of twist generators \( \mathbb{K} \) the coproduct is no-longer bilocal, but actually may encompass all the fields from the product. This is in contrast with the undeformed theory, but already made its appearance at level-0.

In the case of undeformed Yangian algebras we showed that the following formula, an level-1 analogue of (5.2.10), is equivalent to the invariance of the action and utilises only equations of motion, thus circumventing the issue of the cyclicity:

\[
\frac{\delta S}{\delta Z_A} = - \frac{\delta S}{\delta Z_B} + h^{MN}_{LM} \frac{\delta S}{\delta Z_B} \left( J^L \wedge \frac{\delta}{\delta Z_A} \right) (J^N Z_B).
\] (5.3.4)

In case the of real-\( \beta \)-deformed theory, this equality gets a predictable deformation for the \( \hat{P}_{\alpha\dot{\alpha}} \) generator:

\[
\hat{P}_{\alpha\dot{\alpha}} \frac{\delta S}{\delta Z_A} = - \frac{\delta S}{\delta Z_B} + f^{\alpha\dot{\alpha}}_{ABC} \frac{\delta S}{\delta Z_B} \left( J^C \wedge \frac{\delta}{\delta Z_A} \right) (J^N Z_B) +
\]

\[
\frac{\delta S}{\delta Z_B} \left( Q_{\alpha\dot{\alpha}} K_{i\dot{\alpha}}^{-1} \wedge \frac{\delta}{\delta Z_A} \right) (\bar{Q}_{\dot{\alpha}} i K_{i\dot{\alpha}} Z_B) + \frac{\delta S}{\delta Z_B} \left( \bar{Q}_{\dot{\alpha}} i K_{i\dot{\alpha}} \wedge \frac{\delta}{\delta Z_A} \right) (Q_{\alpha\dot{\alpha}} K_{i\dot{\alpha}} Z_B),
\] (5.3.5)

where \( f^{ABC} \) are \( \text{psu}(2,2|1) \) structure constants. Indeed, we checked that the equations of motion of the real-\( \beta \)-deformed SYM satisfy (5.3.5).

Eventually, we see that the properly defined level-1 Yangian generator \( \hat{P}_{\alpha\dot{\alpha}} \) is a symmetry of real-\( \beta \)-deformed \( \mathcal{N} = 4 \) SYM. All the other level-1 generators can be obtained by commuting level-0 ones with \( \hat{P}_{\alpha\dot{\alpha}} \), as follows from (3.2.4) and (3.2.6). We thus obtain an infinitely-dimensional symmetry algebra also for the case of the twisted theory, which explains its observed integrability. On the level of algebra, the symmetry is just the Yangian \( Y(\text{psu}(2,2|4)) \), the difference with the \( \mathcal{N} = 4 \) SYM being only the Hopf structure.
Chapter 6
Yangian invariance of the action

We would now like to address Yangian invariance of the action. Our starting point is the strong invariance of the equations of motion (4.3.15) which we claimed to be a valid statement of symmetry. In the following we will rearrange the terms in the relationship such that they take the form of a Yangian invariance of the action

\[ \hat{J} \cdot S_{N=4sYM} = 0. \]

The goal is to find a precise formulation of this statement that holds for planar \( N = 4 \) sYM. This chapter is based on [60] and [61].

6.1 Notation

In order to perform the rearrangements, we will need a concise notation for the various terms that arise. First of all, we decompose all objects w.r.t. the number of fields that they contain. The action of \( N = 4 \) sYM has quadratic, cubic and quartic terms, and the superconformal representation has terms which preserve the number of fields as well as terms that increase the number of fields by one unit

\[ S = \frac{1}{2} S_2 + \frac{1}{3} S_3 + \frac{1}{4} S_4, \quad J = J_0 + J_1. \]  

We choose a particular level-one generator which minimises the non-linear terms, such as the level-one momentum \( \hat{J} = \hat{P} \), see also Appendix A for further calculational simplifications. The relevant local and bi-local terms in \( \hat{J} = \hat{J}_{\text{loc}} + \hat{J}_{\text{biloc}} \) then expand as follows

\[ \hat{J}_{\text{loc}} = \hat{J}_{\text{loc},[1]}, \quad \hat{J}_{\text{biloc}} = J_{[0]}^{(1)} \otimes J_{[0]}^{(2)} + J_{[1]}^{(1)} \otimes J_{[0]}^{(2)} + J_{[0]}^{(1)} \otimes J_{[1]}^{(2)}. \]  

Next we need to write out somewhat more explicitly how the representations act on the individual fields in the action. The action is a polynomial in the fields whose ordering matters. The contribution \( J_{[0]} \) to the representation maps one field to one, and it is natural to denote the action on the field at site \( k \) by \( J_{[0],[k]} \). For the higher contributions \( J_{[m]} \), \( m > 0 \), which map one field to

---

1. We define the expansion coefficients \( O_{[n]} \) of traced polynomials \( O \) of the fields with an explicit symmetry factor of \( 1/n \). Conversely, there is no symmetry factor for the expansion coefficients \( X_{[n]} \) and \( J_{[n]} \) of open polynomials \( X \) and operators \( J \), respectively. Even though these different assignments may be confusing at times, they will avoid many combinatorial factors.

2. Even though there is in principle a bi-local term which adds a field at both insertion points, this term will not contribute in practice.

3. To avoid excessive clutter we may suppress the symbol ‘\( \cdot \)’ when an operator acts on a specific field.
$m + 1$ fields, the situation is not as evident, see [74] for discussions of this matter in a similar context. Here we make the choice that $J_{[m],k}$ acts on the field at site $k$ and replaces it by an appropriate sequence of $m + 1$ fields. Consequently, the fields at sites $1, \ldots, k - 1$ are mapped to themselves, whereas the fields at sites $k + 1, k + 2, \ldots$ are shifted by $m$ steps to sites $k + m + 1, k + m + 2, \ldots$, e.g.

$$J_{[1],3}(Z^I Z^J Z^K Z^L Z^M) := Z^I Z^J (J_{[1]} Z^K) Z^L Z^M.$$  \hspace{1cm} (6.1.3)

In general, for an open polynomial (example thereof being the variation of the action $\tilde{Z}$) we can write the expansion in the number of fields:

$$X = \sum_n X_{[n]}.$$  \hspace{1cm} (6.1.4)

Then the action of a non-linear generator $J = \sum_m J_{[m]}$ reads:

$$J \cdot X = \sum_{n,m} J_{[m]} \cdot X_{[n]} = \sum_n \sum_{m=0}^n J_{[m]} \cdot X_{[n-m]} = \sum_n (J \cdot X)_{[n]}$$  \hspace{1cm} (6.1.5)

with the expansion coefficients

$$(J \cdot X)_{[n]} = \sum_{m=0}^n \sum_{k=1}^{n-m} J_{[m],k} X_{[n-m]}.$$  \hspace{1cm} (6.1.6)

When considering the Yangian symmetry of the equations of motion in Chapter 4 we also learned that the correct way of acting on an open polynomial is:

$$\hat{J} \cdot X := \sum_{n,m} \sum_{k=1}^n \hat{J}_{[m],k} X_{[n]} + \sum_{n,m,p} \sum_{k=2}^{n-1} \sum_{j=1}^{k-1} J_{[p],j} J_{[m],k}^2 X_{[n]},$$  \hspace{1cm} (6.1.7)

which also takes the nonlinearities into account.

The change of length has a relevant implication on the commutation of two operators $J^1$ and $J^2$, since we have to correctly take the insertion points into account. The commutator of two non-linear actions $J^1$ and $J^2$ has the expansion

$$[J^1, J^2] = \sum_n [J^1, J^2]_{[n]}.$$  \hspace{1cm} (6.1.8)

where the coefficients read

$$[J^1, J^2]_{[n],j} = \sum_{m=0}^n \sum_{k=1}^{m+1} J_{[n-m],k+j-1} J_{[m],j}^2 - \sum_{m=0}^n \sum_{k=1}^{m+1} J_{[n-m],k+j-1} J_{[1],j}. $$  \hspace{1cm} (6.1.9)

Note that the non-overlapping terms drop out from the commutator as usual due to the non-linear commutation relation

$$J_{[l],j}^1 J_{[m],k}^2 = J_{[m],k+l}^2 J_{[l],j}^1 \quad \text{if } j < k.$$  \hspace{1cm} (6.1.10)
Cyclicity. All the terms in the action are gauge-invariant by means of a trace. The trace establishes a neighbouring relationship between the last and the first site, i.e. periodic boundary conditions. Moreover, the trace is cyclic, i.e. it is invariant under cyclic shifts. In order to deal with periodic boundary conditions, we introduce $U$ as the operator that performs a cyclic shift by one site to the left; e.g., it shifts the second site to the first and the first site to the last. Cyclicity of a traced operator $\mathcal{O}$ is then expressed as

$$U\mathcal{O} = \mathcal{O} \quad \text{or expanded} \quad U\mathcal{O}_{[n]} = \mathcal{O}_{[n]}.$$  

(6.1.11)

Note that we will also work with polynomials $\mathcal{O}$ in the fields which are periodic but not cyclic. The cyclic shift operator shifts the insertion of an operator $J$ according to the rule

$$J_{[m],k} U\mathcal{O}_{[n]} = \begin{cases} UJ_{[m],k+1}\mathcal{O}_{[n]} & \text{for } k \neq n, \\ Um+1J_{[m],1}\mathcal{O}_{[n]} & \text{for } k = n. \end{cases}$$  

(6.1.12)

It is worth emphasising that a term of the form $U^k J_{[m],1}$ with $k \leq m$ never appears on the r.h.s. of (6.1.12). Such a term cannot be written with the shift $U$ residing to the right of $J$, the shift must remain to the left of $J$. This is because the term has a form which is not covered by $J_{[m],k}$ alone and which is different from (6.1.3). In this term, the output of $J_{[m]}$ extends past the last site of the polynomial and continues periodically to the first site or beyond. An example analogous to (6.1.3) could be written as

$$UJ_{[1]}(Z^I Z^J Z^K Z^L Z^M) := (J_{[1]} Z^I) Z^J Z^K Z^L Z^M = (J_{[1]} Z^I)_2 Z^J Z^K Z^L Z^M (J_{[1]} Z^I)_1,$$  

(6.1.13)

where $(J_{[m]} Z^I)_j$ denotes the $j$-th site of the output of $J_{[m]}$ on $Z$ (with an implicit sum over all $m$-tuples of fields in the polynomial $J_{[m]} Z$ in similarity to Sweedler’s notation).

6.2 Level zero

Equipped with this notation, we will first address superconformal symmetry of the action. This will provide us with some identities that we shall need later when we consider Yangian symmetry.\footnote{We would like to remind the reader of a general issue w.r.t. homogeneous non-linear representations acting on periodic objects: On the one hand, the periodic shift operator $U$ classifies periodic objects according to the eigenvalue $e^{2\pi i n/L}$ where $n = 0, \ldots, L - 1$. On the other hand, the non-linear representation changes the length $L$ of the object. Now the spectra of $U$ for different lengths $L$ are largely distinct, the only eigenvalue which is present for all lengths is 1 corresponding to cyclic objects. Therefore non-linear representation can be homogeneous, i.e. commute with the shift operator $U$, only on the subspace of cyclic objects. In other words, the periodic object on which they act must be cyclic and the representation must be constructed such that the result is cyclic as well.}

First, we decompose the statement $J \cdot S = 0$ by the number of fields, and we obtain the following set of 4 statements

$$\frac{1}{2} J[0] \cdot S[2] + \frac{1}{3} J[0] \cdot S[3] + \frac{1}{2} J[1] \cdot S[2] = \frac{1}{4} J[0] \cdot S[4] + \frac{1}{3} J[1] \cdot S[3] = \frac{1}{4} J[1] \cdot S[4] = 0.$$  

(6.2.1)

When making the fields explicit, the first statement reads

$$\frac{1}{2} J[0],_1 S[2] + \frac{1}{2} J[0],_2 S[2] = 0.$$  

(6.2.2)
Using cyclicity of the trace, we may as well write this even more concisely as \( J_{[0],3}S_2 \simeq 0 \) where the symbol ‘\( \simeq \)’ denotes equality up to cyclic permutations. The cubic relationship following from superconformal symmetry reads

\[
\frac{1}{3}J_{[0],1}S_3 + \frac{1}{3}J_{[0],2}S_3 + \frac{1}{3}J_{[0],3}S_3 + \frac{1}{2}J_{[1],1}S_2 + \frac{1}{2}J_{[1],2}S_2 \simeq 0. \tag{6.2.3}
\]

This relationship can be rewritten in two alternative ways using cyclicity: Collecting terms we arrive at the simpler form \( J_{[0],1}S_3 + J_{[1],1}S_2 \simeq 0 \). However, we can also write the relationship in a manifestly cyclic fashion as

\[
\frac{1}{3}J_{[0],1}S_3 + \frac{1}{3}J_{[0],2}S_3 + \frac{1}{3}J_{[0],3}S_3 + \frac{1}{3}J_{[1],1}S_2 + \frac{1}{3}J_{[1],2}S_2 = 0. \tag{6.2.4}
\]

Here it is necessary to use the cyclic shift operator \( U \) for one term to distribute the sequence of fields resulting from \( J_{[1]} \) over the last and first site. Observe also that due to this explicit rewriting, the coefficients of all the terms are now identical – and equal to \( \frac{1}{3} \), which is the inverse of the length of the contributing polynomials.

It is also useful to consider the strong invariance of the equations of motion \( 4.3.13 \). For any \( Z^K \) specifying the equation of motion, it yields a relationship consisting of a polynomial of fields. In order to handle the identities for all fields \( Z^K \) at the same time, we prepend this field to the relationship polynomial and sum over all fields

\[
\mathcal{Y} := Z^K \left[ (J \cdot Z^I) \frac{\delta^2 S}{\delta Z^I \delta Z^K} + \frac{\delta(J \cdot Z^I)}{\delta Z^K} \frac{\delta S}{\delta Z^I} \right] = 0. \tag{6.2.5}
\]

In this definition we explicitly do not take the colour trace such that the above field \( Z^K \) will always reside at site 1 of the polynomial by construction. The expansion \( \sum_n \mathcal{Y}_{[n]} \) of the open polynomial \( \mathcal{Y} \) in the number \( n \) of fields takes the form

\[
\mathcal{Y} = \sum_n \mathcal{Y}_{[n]}, \quad \mathcal{Y}_{[n]} = \sum_{m=0}^{n-2} \sum_{j=2}^{n-m} J_{[m],j}S_{[n-m]} + \sum_{m=0}^{n-2} \sum_{j=1}^{m+1} U^{j-1}J_{[m],1}S_{[n-m]}. \tag{6.2.6}
\]

Here, the symmetry factors \( 1/(n-m) \) for \( S_{[n-m]} \) in \( 6.1.1 \) have been cancelled by varying the component of the (cyclic) action \( S_{[n-m]} \) consisting of \( n-m \) fields by \( Z^K \). Note that the statements \( \frac{1}{2}\mathcal{Y}_{[2]} = 0 \) and \( \frac{1}{3}\mathcal{Y}_{[3]} = 0 \) are precisely the above \( 6.2.2 \) and \( 6.2.4 \), respectively. In fact, the expression \( \mathcal{Y} \) is cyclic; by rewriting the first term using \( 6.1.12 \) such that \( J_{[m]} \) will always act on site 1, we can make cyclicity manifest

\[
\mathcal{Y}_{[n]} = \sum_{m=0}^{n-2} \sum_{j=1}^{n} U^{j-1}J_{[m],1}S_{[n-m]}. \tag{6.2.7}
\]

Indeed, this is because

\[
\sum_{m=0}^{n-2} \sum_{j=2}^{n-m} J_{[m],j}S_{[n-m]} = \sum_{m=0}^{n-2} \sum_{j=2}^{m+1} U^{j-1}J_{[m],1}S_{[n-m]} \equiv 0. \tag{6.2.8}
\]

\[\text{footnote}{\text{We assume that } \delta S/\delta Z^K \text{ pulls one of the } n-m \text{ fields and moves the empty spot within the polynomial to site 1. The subsequent variation } \delta/\delta Z^I \text{ pulls another field (but not from the empty site 1) and replaces it by } J \cdot Z^I.}\]
where we separated out the $U^m$ factor to stress that the shift operator never breaks up the effect of the action of $J$, just as it happens in the first term of (6.2.6). The remaining iterations of the shift operator distribute it everywhere within the $S$ but for the first place, which is taken care of by the second term of (6.2.6). Of course, we assume cyclic invariance of the action $S$. Therefore, we lose no information by identifying terms by cyclic permutations, and we find a more concise statement

$$\mathcal{Y}_{[n]} \simeq n \sum_{m=0}^{n-2} J_{[m],[n]} S_{[n-m]} \simeq 0.$$  \hspace{1cm} (6.2.9)

To wrap this discussion up, we can rewrite the above definition (6.2.5) as

$$\mathcal{Y} = Z^K \frac{\delta (J \cdot S)}{\delta Z^K}.$$  \hspace{1cm} (6.2.10)

As the operation $Z^K (\delta / \delta Z^K)$ just counts the number of fields, it is not surprising that the symmetry variation of the action expands precisely to the coefficients $\mathcal{Y}_{[n]}$

$$J \cdot S = \sum_n \frac{1}{n} \mathcal{Y}_{[n]}.$$  \hspace{1cm} (6.2.11)

Therefore, the above transformations are a bit of a detour in this case, but they will help us find a corresponding expression $\hat{J} \cdot S$ for the level-one Yangian generators $\hat{J}$ acting on the action $S$.

6.3 Level one

We would now like to construct a suitable level-one Yangian representation on the action, $\hat{J} \cdot S$, such that invariance amounts to $\hat{J} \cdot S = 0$. We will approach this construction by expanding in the number of fields,

$$\hat{J} \cdot S = \sum_n \frac{1}{n} (\hat{J} \cdot S)_{[n]},$$  \hspace{1cm} (6.3.1)

where the prefactors $1/n$ account for cyclic symmetry. We will start with the simplest cases at $n = 2, 3$ and later address arbitrary lengths. Before doing so, we derive some more identities which are needed to transform the expressions.

**Commuting constituents.** Consider the commutator

$$H := \frac{1}{2} [J^{(1)}, J^{(2)}],$$  \hspace{1cm} (6.3.2)

which we can expand in fields as usual as $H = H_{[0]} + H_{[1]}$ with

$$H_{[m],[k]} = \sum_{l=0}^{m} \sum_{j=0}^{l} J^{(1)}_{[m-l],[k+j]} J^{(2)}_{[n],[l],[k]}.$$  \hspace{1cm} (6.3.3)

The leading term $H_{[0]}$ contains the combination $f^{A}_{BC} f^{BCD}$ which is proportional to the dual Coxeter number of the level-zero algebra. The dual Coxeter number for our superconformal algebra $\text{psu}(2,2|4)$ is zero (see Appendix C), consequently the linear term vanishes, $H_{[0]} = 0$. The nonlinear term $H_{[1]}$ is also zero for $\mathcal{N} = 4$ sYM by explicit computation. One can relate this finding to the G-identity for $\mathcal{N} = 4$ sYM found in [79]. Altogether, the constituent operators of the bi-local level-one generator commute (when summed over all pairs as implied by Sweedler’s notation)

$$H = 0.$$  \hspace{1cm} (6.3.4)
**Two fields.** Next, we address level-one Yangian symmetry. The starting point is the strong invariance statement \((4.3.15)\) of the equations of motion which we have already shown to hold. We will treat it analogously to the strong invariance of the equation of motion at level-zero \((4.3.15)\). We prepend the field that selects the equation of motion to the relationship polynomial and denote the resulting expression by

\[
\hat{Y} := Z^K \left[ \tilde{J} \cdot \tilde{Z}_K + \tilde{Z}_I \frac{\delta (\tilde{J} \cdot Z^I)}{\delta Z^K} - \tilde{Z}_I \left( J^{(1)} \wedge \frac{\delta}{\delta Z^K} J^{(2)} \cdot (J^{(2)} \cdot Z^I) \right) \right] = 0. \tag{6.3.5}
\]

It can be expanded in fields leading to some additional identities of a similar kind as \((6.2.2), (6.2.4)\)

\[
\hat{Y} = \sum_n \hat{Y}^{(n)} \tag{6.3.6}
\]

However, the relationship \(\hat{Y}^{(2)} = 0\) at the level of two fields is empty, the first non-trivial relationship is at three fields.

What are the implications of Yangian symmetry at two fields? We use the fact that the combination \(Y^{(2)}\) in \((6.2.7)\) is exactly zero and apply another generator \(J\) to it to obtain a new relationship

\[
J^{(1)}_{[0,1]} Y^{(2)}_{[2]} = J^{(1)}_{[0,1]} J^{(2)}_{[0,1]} S_{[2]} + J^{(1)}_{[0,1]} J^{(2)}_{[0,2]} S_{[2]} = 0. \tag{6.3.7}
\]

The first term has the form \(H_{[0,1]} S_{[2]}\) and therefore it vanishes by \((6.3.4)\). The second term is the bi-local part of the level-one representation on the quadratic part of the action. Furthermore, there is no linear contribution to the local part of the level-one representation. Altogether, \((6.3.7)\) boils down to the statement

\[
\tilde{J}^{[0]} \cdot S_{[2]} = \left( J^{(1)}_{[0]} \otimes J^{(2)}_{[0]} \right) \cdot S_{[2]} = 0. \tag{6.3.8}
\]

In other words, the level-one symmetry of the action at quadratic order in the fields is a plain consequence of level-zero symmetry and the vanishing of the dual Coxeter number. This agrees with the fact that the equations of motions are trivially invariant at linear order.

**Three fields.** The strong invariance relationship \((4.3.15)\) at three fields reads \(\hat{Y}^{(3)} = 0\) with

\[
\hat{Y}^{(3)} := \tilde{J}^{[1,2]} S_{[2]} + U \tilde{J}^{[1,1]} S_{[2]} + \tilde{J}^{[1,1]} S_{[2]} + J^{(1)}_{[0,2]} J^{(2)}_{[0,2]} S_{[3]} - J^{(2)}_{[0,2]} J^{(1)}_{[0,2]} S_{[2]} + U J^{(2)}_{[0,1]} J^{(1)}_{[1,1]} S_{[2]} \tag{6.3.9}
\]

We would now like to reformulate this combination such that it looks more like the cubic term in the expansion of \(\tilde{J} \cdot S\). To that end, let us first address cyclicity of \(\hat{Y}^{(3)}\). The first three terms are manifestly cyclic; the latter three are not. We therefore look at the violation of cyclicity

\[
(U - 1) \hat{Y}^{(3)} = -J^{(1)}_{[0,2]} Y^{(2)}_{[3]} - U J^{(1)}_{[1,1]} Y^{(2)}_{[2]} + H_{[0,2]} S_{[3]} + U H_{[1,1]} S_{[2]} \tag{6.3.10}
\]

Here we have made use of the algebraic identities \((6.1.9)\) and \((6.1.12)\) and the implicit anti-symmetry in Sweedler's notation \((3.2.11)\). All remaining terms could then be collected in the combinations \(Y\) \((6.2.7)\) and \(H\) \((6.3.3)\) which are zero as discussed above. Therefore the relationship \(\hat{Y}^{(3)} = 0\) is effectively cyclic; all the non-cyclic contributions to the relationship are zero for

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*Importantly, the relationship \(Y_{[2]} = 0\) holds without assuming the equation of motion to hold or without applying cyclic permutations.*
lesser reasons than Yangian symmetry. It also allows us to compare modulo cyclic identifications (‘\(\simeq\)’) without losing relevant information

\[
\hat{\mathcal{Y}}_3 \simeq 3\hat{J}_{[1],1}S_{[2]} + J_{[0],2}^{(1)}J_{[0],3}^{(2)}S_{[3]} - J_{[0],2}^{(2)}J_{[1],1}^{(1)}S_{[2]} + J_{[0],1}^{(2)}J_{[1],1}^{(1)}S_{[2]}.
\]  

(6.3.11)

It is a good point to compare our result to the one obtained by Drummond el al. in [47]. While establishing the Yangian invariance of color-ordered scattering amplitudes in \(\mathcal{N} = 4\) sYM, they showed that the cyclicity violations can be written as:

\[
(U - 1)\hat{J}^A = -if^{A BC}f^{BCD}J_1^D + 2f^{A BC}J_1^BJ^C,
\]  

(6.3.12)

where \(J_1^A\) denotes a generator \(J^A\) acting only on the original site 1. We now see that the first term on the right-hand side of (6.3.12) corresponds to our \(H_{[0]}\) and hence vanishes. The other term is proportional to \(J^C\) acting on the whole amplitude, and thus is a direct counterpart of our \(\mathcal{Y}^C\), only in the context of the scattering amplitudes, where all the generators \(J\) act linearly. We can thus view our result – now for 3 fields, but later on for arbitrary number of them – as a non-linear completion of the earlier result of [47].

Let us now turn to the reformulation of our result to make it resemble more what we expect the invariance of the action too look like. Due to the effective cyclicity of \(\hat{Y}_3\), it is reasonable to define \(\hat{J} \cdot S)_{[3]}\) to be proportional to it. To figure out the factor of proportionality, we will compare the terms in (6.3.11) to some canonical terms in \((\hat{J} \cdot S)_{[3]}\). The first term in (6.3.11) is equivalent to the local part of the level-one representation on \(S_{[2]}\)

\[
\hat{J}_{\text{loc},[1]} \cdot S_{[2]} \simeq 2\hat{J}_{[1],1}S_{[2]},
\]  

(6.3.13)

while the second term in (6.3.11) is equivalent to the bi-local part on \(S_{[3]}\)

\[
\hat{J}_{\text{biloc},[0]} \cdot S_{[3]} := J_{[0],1}^{(1)}J_{[0],2}^{(2)}S_{[3]} + J_{[0],1}^{(1)}J_{[0],3}^{(2)}S_{[3]} + J_{[0],2}^{(1)}J_{[0],3}^{(2)}S_{[3]} \simeq J_{[0],1}^{(1)}J_{[0],2}^{(2)}S_{[3]}.
\]  

(6.3.14)

Here, the prefactor of the last term is equal to 1 (and not 3) due to cancellations stemming from the antisymmetry in Sweedler’s notation once we identify states cyclically.

The remaining two terms in (6.3.11) have a special form, one may interpret them as a non-linear bi-local contribution where both legs overlap. Such terms are not provided by the usual coproduct rule for tensor product representations, but our non-linear representation is not exactly a tensor product, and hence it is conceivable to have them

\[
\hat{J}_{\text{o’lap},[1]} \cdot S_{[2]} := \frac{2}{3}J_{[0],2}^{(1)}J_{[1],1}^{(2)}S_{[2]} - \frac{2}{3}J_{[0],1}^{(1)}J_{[1],1}^{(2)}S_{[2]}.
\]  

(6.3.15)

The (insignificant) prefactors in this definition were chosen to agree with a desirable form of expression further below. Finally, we notice that \(\hat{Y}_3\) does not contain the standard bi-local contribution to the representation on \(S_{[2]}\). However, this term is zero modulo cyclic identifications

\[
J_{[1],1}^{(1)}J_{[0],2}^{(2)}S_{[2]} + J_{[0],1}^{(1)}J_{[1],2}^{(2)}S_{[2]} \simeq 0.
\]  

(6.3.16)

Altogether we derive an invariance statement for the action at cubic order in the fields

\[
\frac{1}{3}(\hat{J} \cdot S)_{[3]} \simeq \frac{1}{2}\hat{J}_{\text{loc},[1]} \cdot S_{[2]} + \frac{1}{3}\hat{J}_{\text{biloc},[0]} \cdot S_{[3]} + \frac{1}{3}\hat{J}_{\text{o’lap},[1]} \cdot S_{[2]} \simeq \frac{1}{3}\hat{Y}_{[3]} \simeq 0.
\]  

(6.3.17)

Note that the prefactors of all terms agree with the symmetry factors \(1/m\) of the \(S_{[m]}\) which is acted upon.\(^7\) We have verified explicitly that (6.3.17) holds for the level-one momentum \(\hat{J} = \hat{P}\).

\(^7\)For the novel overlapping terms this is a choice which was used to fix the previously chosen prefactors in its definition (6.3.15).
Non-linear level-one invariance. Now let us turn to a general number of fields \( n \) in order to understand the precise structure of the unconventional terms. The combination \( \hat{J}_{[n]} \) of the strong invariance condition \([4.3.15] \) can be written in our present notation as

\[
\hat{J}_{[n]} = \sum_{m=0}^{n-2} \sum_{j=2}^{n-m} \hat{J}_{[m],j} \mathcal{S}_{[n-m]} + \sum_{m=0}^{n-2} \sum_{j=1}^{n-m+1} U^{j-1} \hat{J}_{[m],1} \mathcal{S}_{[n-m]}
\]

\[+ \sum_{m=0}^{n-2} \sum_{l=0}^{m} \sum_{j=2}^{n-m-1} \sum_{k=j+1}^{n-m} J^{(1)}_{[l],j} J^{(2)}_{[m-l],k} \mathcal{S}_{[n-m]}
\]

\[+ \sum_{m=0}^{n-2} \sum_{l=0}^{m} \sum_{j=1}^{n-m} \sum_{k=j+1}^{n-m} \left( U^{j-1} J^{(1)}_{[m-l],k} - \sum_{j+1}^{m-1} U^{j-1+m-l} J^{(1)}_{[m-l],j} \right) J^{(2)}_{[l],1} \mathcal{S}_{[n-m]}.
\]

(6.3.18)

Using the same identities as at the level of 3 fields, we find that violations of cyclicity are given by terms containing \( \mathcal{Y} \) and \( H \)

\[(U - 1)\hat{J}_{[n]} = \sum_{m=0}^{n-2} \left(1 + U^{m+1}\right) \left[ J^{(1)}_{[m],1} \mathcal{Y}^{(2)}_{[n-m]} - H_{[m],1} \mathcal{S}_{[n-m]} \right].
\]

(6.3.19)

This means that \( \hat{\mathcal{Y}} \) is in fact cyclic provided that the action is invariant under level-zero:

\[J^1 \cdot \mathcal{S} = 0 = J^2 \cdot \mathcal{S}
\]

(6.3.20)

and that the combined generator \( H \) defined in \([6.3.2]\) is zero, meaning that the constituent generators commute, possibly up to a sum implicit in the Sweedler’s notation:

\[[J^1, J^2] = 0.
\]

(6.3.21)

We may thus interpret the statement \( \hat{\mathcal{Y}} = 0 \) as level-one invariance of the action.

To bring this statement somewhat closer to the expected form of \( \hat{J} \cdot \mathcal{S} \) we introduce a modified combination \( \hat{\mathcal{Y}}' \) with some convenient extra terms which are zero due to \( \mathcal{Y} = H = 0 \)

\[
\hat{\mathcal{Y}}'_{[n]} := \hat{J}_{[n]} + \sum_{m=0}^{n-2} \left[ J^{(1)}_{[m],1} \mathcal{Y}^{(2)}_{[n-m]} + H_{[m],1} \mathcal{S}_{[n-m]} \right]
\]

\[= \sum_{m=0}^{n-2} \sum_{j=1}^{n} U^{j-1} \hat{J}_{[m],1} \mathcal{S}_{[n-m]}
\]

\[+ \sum_{m=0}^{n-2} \sum_{l=0}^{m} \sum_{k=l+2}^{m-l+2} \sum_{j=l+2}^{m-l} U^{j-1} J^{(1)}_{[m-l],k} \mathcal{S}_{[n-m]}
\]

\[+ \sum_{m=0}^{n-2} \sum_{l=0}^{m} \sum_{k=1}^{n-l+k} \sum_{j=2}^{m+1} \left[ 1 - \sum_{j=k+1}^{m+1} \right] U^{j-1} J^{(1)}_{[m-l],k} \mathcal{S}_{[n-m]}.
\]

(6.3.22)

The terms in the first line of the result represent the local terms of the level-one representation. The terms in the second line are clearly bi-local with two non-overlapping insertions. The terms in the third line take a similar form as the bi-local terms, but here the insertion of \( J^{(1)} \) is within the range of \( J^{(2)} \) and thus their actions overlap\[\footnote{Consequently, the combination of both insertions is effectively local, nevertheless we will still associate them to the bi-local part due to their structural similarity.}\]
Some comments regarding the bi-local and overlapping terms are in order: Even through the prefactors for all terms in (6.3.22) are 1, those of the non-linear bi-local terms appear somewhat unnatural: when compared to their linear counterparts, they are off by a factor of \((n - m)/n\). On the one hand, this may be worrisome in view of gauge symmetry because such relative factors could easily upset the composition of covariant derivatives and thus spoil this essential symmetry. On the other hand, such factors can be compensated by the relative length of the objects and thus by the multiplicity of operator insertions. Moreover, the overlapping terms have no analog in the conventional level-one representation based on the coproduct rule.

How to make sense of this behaviour? Clearly, the novel non-linear representation of level-one symmetry on cyclic polynomials can have new features, which do not need to follow the coproduct rule strictly. What matters is that the above form follows by elementary transformations from the strong invariance relationship (4.3.15). The latter holds in planar \(\mathcal{N} = 4\) sYM for the level-one momentum \(\hat{J} = \hat{P}\). It therefore makes sense to view \(\hat{J}^0 = 0\) as given by (6.3.22) as the definition of the non-linear level-one invariance of the action. Whether or not it has the expected from, it certainly does describe a non-trivial relationship of planar \(\mathcal{N} = 4\) sYM. In any case, the structures in (6.3.22) clearly deserve further theoretical scrutiny.

**Cyclic level-one representation.** As the expression (6.3.22) is effectively cyclic, we may furthermore identify terms related by cyclic permutation. This yields a simpler expression

\[
\hat{J}^0_{[n]} \simeq \sum_{m=0}^{n-2} n\hat{J}^0_{[m],1}S_{[n-m]} + \sum_{m=0}^{n-2} \sum_{l=0}^{m} \sum_{k=1}^{n-m+2l+1} (k - l - \frac{1}{2}n + \frac{1}{2}m - 1)J^{(1)}_{[m-l],k}J^{(2)}_{[l],1}S_{[n-m]}.
\]

(6.3.23)

Note that all bi-local terms could be combined in a uniform expression. Here the first and the last \(l + 1\) terms labelled by \(k = 1, \ldots, l + 1\) and \(k = n - m + l + 1, \ldots, n - m + 2l + 1\) represent overlapping bi-local terms whereas the insertions do not overlap in the remaining middle range \(k = l + 2, \ldots, n - m + l\).

Let us comment on this formula and explain the origin of the prefactor \((k - l - \frac{1}{2}n + \frac{1}{2}m - 1)\). The non-overlapping terms are more intuitive, hence we will start with them. Performing the sum over \(j\) in the (6.3.22) we obtain the factor of \((k - l - 1)\), which features within \((k - l - \frac{1}{2}n + \frac{1}{2}m - 1)\). The difference is \(\frac{1}{2}(n - m)\), which is half of the length of the part of the action \(S\) we act on. This is indeed to be expected, since we are cyclically identifying the terms. As the action \(S\) by itself is cyclic, and we can exchange \(J^{(1)}\) and \(J^{(2)}\) (up to a minus sign), in order to account for overcounting, we subtract \(\frac{1}{2}(n - m)\).

Regarding the overlapping terms, they correspond to \(k = 1, \ldots, l + 1\) and \(k = n - m + l + 1, \ldots, n - m + 2l + 1\), even though originally in (6.3.22) the sum ran only over the first part thereof. However, the object that \(J^{(1)}_{[m-l],k}\) acts on, \(J^{(2)}_{[l],1}S_{[n-m]}\), has only length \(n - m + l\), and hence for the insertion point of generators we take \(k\) modulo \(n - m + 1\), whereas in the prefactor \((k - l - \frac{1}{2}n + \frac{1}{2}m - 1)\) it appears unbounded. Combining the terms where \(J^{(1)}\) is inserted at the same point, the resulting prefactor is \(-2 + 2k - l\): precisely what we would obtain from (6.3.22).

Since invariance of the action can and should be expressed modulo cyclicity of the trace, we may write the symmetry statement as

\[
\hat{J} \cdot S \simeq 0 \quad \text{with} \quad \hat{J} \cdot S \simeq \sum_n \frac{1}{n} \hat{J}^0_{[n]}.
\]

(6.3.24)
For the future use we will express the invariance of the action, (6.3.24), in a way which explicitly separates overlapping terms from the non-overlapping ones we would normally expect:

\[ \hat{J} \cdot \mathcal{S} \simeq \sum_{n,m} \hat{J}_{[m],1} \mathcal{S}_{[n]} + \sum_{n,m,l} \sum_{k=2}^{n} \frac{2k - n - 2}{n + m + l} \hat{J}_{[m],k} \hat{J}_{[l],1} \mathcal{S}_{[n]} \]

\[ + \sum_{n,m,l} \sum_{k=1}^{l+1} \frac{2k - l - 2}{n + m + l} \hat{J}_{[m],k} \hat{J}_{[l],1} \mathcal{S}_{[n]} \]

\[ - \sum_{n,m,l} \sum_{k=1}^{m+1} \frac{2k - m - 2}{n + m + l} \hat{J}_{[l],k} \hat{J}_{[m],1} \mathcal{S}_{[n]} \]  

(6.3.25)

In this presentation, the first line is the single-field action, in analogy with (6.2.7). The second one consists of the non-overlapping bilocal combinations, which already appeared in the work with equations of motion. The last two lines contain the terms where the constituent generators \( \hat{J}_1 \) and \( \hat{J}_2 \) overlap.

This form is a complete expression for level-one invariance of the action including all standard and non-standard terms. We have verified explicitly by computer algebra that it holds for planar \( \mathcal{N} = 4 \) sYM for the level-one generators \( \hat{J} \) with \( J \in \{P, Q, \bar{Q}, R, B\} \). Unfortunately, the calculations produce hundreds of intermediate terms which are subject to cyclic identifications and integrations by parts. Only the invariance under the level-one bonus symmetry \( \hat{B} \) has a reasonably simple structure, and we shall show it explicitly below.

As such we have not yet fully established that the extended symmetries generate a Yangian algebra; we would need to show that the adjoint property (3.2.4) as well as the Serre-relation (3.2.6) hold. A complication, to be addressed in Chapter 7, is that the algebra is mixed with gauge transformations of a novel kind. Independently of which algebraic relations the symmetry generators obey, they give rise to novel relations for planar \( \mathcal{N} = 4 \) sYM.

Level-one bonus symmetry. In order to show invariance of the action under the level-one bonus symmetry \( \hat{B} \) introduced in [66], see also [92], we first introduce the additional algebraic structures, see Appendix A for further details. The coproduct depends only on the odd level-zero generators

\[ \Delta \hat{B} = \hat{B} \otimes 1 + 1 \otimes \hat{B} - \frac{1}{4} S^{ab} \wedge Q_{ba} - \frac{1}{4} \bar{S}_{\dot{b}}^{\dot{a}} \wedge \bar{Q}_{\dot{a}}^{\dot{b}}, \]  

(6.3.26)

and the single-field action is trivial

\[ \hat{B} \cdot \mathcal{Z} = 0. \]  

(6.3.27)

Heuristically, this is because dimension of \( \hat{B} \) is equal to 0, while the fact that it commutes with \( P \) (as an element of the bigger algebra \( u(2,2|4) \)) dictates that it should not have an explicit position dependence. It should thus map a single field to a single field, but this is inconsistent with the parity-inverting property discussed in Chapter 4, which requires at least two fields. This result can also be verified by demanding the weak invariance of the equations of motion.

\footnote{By construction, these symmetries form some algebra. If it is not of Yangian kind, it will inevitably be much larger than that. So the default assumption is that the algebra is as small as possible, and thus of Yangian kind. Nevertheless it intrinsically interesting to understand the actual symmetry algebra and its relations in detail.}
Finally, the representation of the superconformal boosts \( S \) and \( \bar{S} \) are almost completely given in terms of the supersymmetries \( \bar{Q} \) and \( Q \), respectively

\[
S^{ab} \sim i x^{\alpha i} \bar{Q}^i_a b + S^{ab}, \quad \bar{S}_{a}^\gamma \sim -i \bar{x}^{\beta \dot{i}} Q_{a \beta} + \bar{S}_{a}^\gamma .
\]  

(6.3.28)

Both of the additional operators \( S' \) and \( \bar{S}' \) act non-trivially only on a single type of field

\[
S^{ab} \cdot \Psi^{\delta}_{\epsilon} = -2 \delta^{a}_{\delta} \phi^{bc}, \quad S'_{a}^\gamma \cdot \bar{\Psi}^{\dot{\epsilon} d} = 2 \delta_{\dot{\epsilon}}^\gamma \bar{\phi}_{ad} .
\]  

(6.3.29)

Based on these relationships, we can in fact show that almost all contributions to the invariance condition are trivially zero. First of all, by combining (6.3.26) and (6.3.28) one can observe that the \( x \)-dependent terms due to \( \bar{S} \wedge \bar{Q} \) and \( S \wedge Q \) are purely bosonic and they do not involve partial derivatives. Further, the only non-zero contributions to the invariance condition can originate from the \( x \)-independent operators \( S' \) and \( \bar{S}' \) as well as when \( S \) and \( \bar{S} \) act on a partial derivative which subsequently removes the \( x \)-dependence

\[
S^{ab} \cdot (\partial_{\delta \gamma} Z) = i \partial_{\alpha \delta} (x^{\alpha i} \bar{Q}^i_a b \cdot Z) + \partial_{\delta \gamma} (S^{ab} \cdot Z)
\]

\[
= i x^{\alpha i} \partial_{\delta \gamma} (\bar{Q}^i_a b \cdot Z) + \partial_{\delta \gamma} (S^{ab} \cdot Z) + i \delta^{a}_{\delta} \bar{Q}^i_a b \cdot Z ,
\]

(6.3.30)

First we consider overlapping terms in the action of \( \hat{B} \): Overlapping terms require the first level-zero generator to act non-linearly. However all non-linear contributions from the odd generators act on fermions only and they produce two bosons without derivatives. When acting further on the result, all \( x \)-dependent contributions must cancel by the above arguments, and, as we have seen, bosons without derivatives cannot generate \( x \)-independent terms. Moreover, extra terms cannot be generated from the action of the first generator because \( S' \) and \( \bar{S}' \) are purely linear and any partial derivatives acting on the original field can be pulled out from the calculation. Hence there are no overlapping terms. Analogously, the single-field action of \( \hat{B} \) is zero.

Let us now consider the various terms in the action: The quadratic terms \( S_{[2]} \) can only contribute via overlapping terms and therefore they are all trivially invariant on their own. Furthermore, all the quartic terms \( S_{[4]} \) are purely bosonic and they do not involve partial derivatives. Since the operators \( S' \) and \( \bar{S}' \) act on fermions only, and the absence of partial derivatives prevent the generation of further \( x \)-independent terms, also all terms in \( S_{[4]} \) are invariant on their own. It remains to consider the cubic terms \( S_{[3]} \).

By elementary transformations we can summarise the (non-linear) action \( (6.3.23) \) on the cubic terms as

\[
\sum_n \hat{S}_{[n]} \simeq J_1^{(1)} J_2^{(2)} S_{[3]}
\]

\[
\simeq - \frac{1}{4} (S^{ab})_1 (Q_{a \alpha} b)_2 S_{[3]} - \frac{1}{4} (Q_{a \alpha})_1 (S^{ab})_2 S_{[3]}
\]

\[
- \frac{1}{4} (S_{b \dot{\alpha}}^\gamma) (Q_{a \beta} b)_2 S_{[3]} - \frac{1}{4} (Q_{a \beta} b) (S_{b \dot{\alpha}}^\gamma) S_{[3]}. 
\]  

(6.3.31)

Effectively, only \( x \)-independent terms can potentially contribute, therefore we need to consider only two types of terms from this expression: The generators \( S \) and \( \bar{S} \) can act on a derivative \( \partial Z \) in such a way that the derivative eliminates the \( x \)-dependence from the action of \( S \) and \( \bar{S} \). Alternatively, \( S \) and \( \bar{S} \) can act on fermions and yield terms via the extra operators \( S \) and \( \bar{S} \). Curiously, the cubic terms in the action \( (2.1.13) \) split into two corresponding classes, namely terms with fermions and terms with derivatives

\[
L_{[3]} = - i \epsilon^{\dot{\alpha} \dot{\gamma} \dot{\epsilon} \dot{\delta}} \epsilon^{\dot{\beta} \dot{\delta}} \epsilon^{\dot{\epsilon} \dot{\delta}} \epsilon^{\dot{\lambda}} \text{tr} ([A_{\lambda \dot{\alpha}}, D_{\dot{\beta} \dot{\delta}} A_{\dot{\epsilon} \dot{\delta}}] A_{\dot{\delta} \dot{\gamma}}) - i \frac{1}{2} \epsilon^{\dot{\alpha} \dot{\gamma} \dot{\epsilon} \dot{\delta}} \epsilon^{\dot{\beta} \dot{\delta}} \text{tr} ([\Phi^{(e)}, D_{\dot{\alpha} \dot{\beta}} \bar{\Phi}_{(e)}] A_{\dot{\delta} \dot{\gamma}})
\]
\[ + \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left\{ \Psi^d, \Psi_{\overline{\kappa} d}\right\} A_{\hat{\alpha} \hat{\alpha}}\right) + \frac{i}{2} \varepsilon^{\alpha \gamma} \text{tr}\left(\left\{ \Psi^e, \Psi^f_{\gamma}\right\} \Phi_{\epsilon f}\right) + \frac{i}{2} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left\{ \Psi_{\hat{a} e}, \Psi_{\hat{\gamma} f}\right\} \Phi_{\epsilon f}\right). \quad (6.3.32) \]

Let us first consider the fermionic terms on the second line on which \( S \) and \( \bar{S} \) can effectively act only via the extra operators \( S' \) and \( \bar{S}' \). By considering the types which can potentially be generated along with the parity-reversing nature of the level-one generators, we find terms of the types
\[ \varepsilon^{\alpha \gamma} \text{tr}\left(\left\{ \Psi^e, \Psi^f_{\gamma}\right\} \Phi_{\epsilon f}\right), \quad \varepsilon^{\alpha \gamma} \text{tr}\left(\left\{ \Phi_{\epsilon f}, \Phi_{\epsilon f}'\right\} F_{\alpha \gamma}\right), \quad \text{tr}\left(\left\{ \Phi_{\epsilon f}, \Phi_{\epsilon f}'\right\} \left[ \Phi_{\epsilon d}, \Phi_{\epsilon f}\right]\right), \quad \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left\{ A_{\hat{\alpha} \hat{\alpha}}, \Phi_{\epsilon f}\right\} [A_{\hat{\beta} \gamma}, \Phi_{\epsilon f}]\right), \quad (6.3.33) \]
as well as
\[ \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(D_{\hat{\gamma} \delta}\{ \Phi_{\epsilon f}, \Phi_{\epsilon f}\} A_{\hat{\beta} \hat{\alpha}}\right). \quad (6.3.34) \]

Now the former six terms are trivially zero because they all involve simultaneous symmetrisation and anti-symmetrisation of indices. Only the last term is non-zero. It can only be generated from the term \( \{ \Psi, \bar{\Psi}\} A \) in the action and the contributions read
\[ \frac{1}{2} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left\{ S^n f, \Psi_{\gamma}^{\overline{\kappa} d}\right\} A_{\hat{\alpha} \hat{\alpha}} + \left\{ S^i f, \bar{Q}_{f \gamma}, \Psi_{\hat{\gamma}}^{\hat{d} \kappa}\right\} A_{\hat{\beta} \hat{\alpha}}\right). \quad (6.3.35) \]
However, by explicit computation, the two terms cancel precisely.

By similar arguments as above, the purely bosonic cubic terms of the action can yield terms of the kinds
\[ \varepsilon^{\alpha \gamma} \text{tr}\left(\left[ \Psi^e, \Psi^f_{\gamma}\right] \Phi_{\epsilon f}\right), \quad \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left[ \Psi_{\hat{a} e}, \Psi_{\hat{\gamma} f}\right] \Phi_{\epsilon f}\right), \quad \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left[ \Psi^d, \bar{\Psi}_{\overline{\kappa} d}\right] A_{\hat{\alpha} \hat{\alpha}}\right). \quad (6.3.36) \]
As before the former two terms have incompatible symmetrisations and are zero. Only the last term is non-zero. By acting on the term \( \left[ \Phi, D\Phi\right] A \) we find
\[ - \frac{i}{24} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left[ D_{\hat{\alpha} \hat{\beta}}\left(S^{\kappa g} \Phi_{\epsilon f}\right), Q_{g \delta} \Phi_{\epsilon f}'\right] A_{\hat{\alpha} \hat{\alpha}} + \left[ D_{\hat{\alpha} \hat{\beta}} \bar{S}_g^{\delta} \Phi_{\epsilon f}, Q_{g \delta} \Phi_{\epsilon f}'\right] A_{\hat{\beta} \hat{\alpha}}\right), \quad (6.3.37) \]
which evaluates to
\[ - \frac{1}{2} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left[ \Phi^f_{\beta}, \bar{\Psi}_{\hat{\alpha} f}\right] A_{\hat{\alpha} \hat{\alpha}}\right). \quad (6.3.38) \]
Conversely, the action on the term \( \left[ A, DA\right] A \) yields
\[ \frac{i}{12} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \varepsilon^{\delta \epsilon} \varepsilon^{\zeta \lambda} \text{tr}\left(\left[ D_{\hat{\alpha} \hat{\beta}} S^{\kappa g} A_{\zeta \zeta}, Q_{g \delta} A_{\delta \kappa}\right] A_{\hat{\alpha} \hat{\alpha}} + \left[ D_{\hat{\alpha} \hat{\beta}} \bar{S}_g^{\delta} A_{\zeta \zeta}, Q_{g \delta} A_{\delta \kappa}\right] A_{\hat{\beta} \hat{\alpha}}\right), \quad (6.3.39) \]
which amounts to
\[ \frac{1}{2} \varepsilon^{\hat{\alpha}} \varepsilon^{\beta \gamma} \text{tr}\left(\left[ \Phi^f_{\beta}, \bar{\Psi}_{\hat{\alpha} f}\right] A_{\hat{\alpha} \hat{\alpha}}\right). \quad (6.3.40) \]
Therefore, both remaining contributions cancel, and altogether this proves that the bonus level-one Yangian generator \( \bar{B} \) is a symmetry of planar \( \mathcal{N} = 4 \) sYM.

**Heuristic argument** After showing by the explicit computation that the action of the \( \mathcal{N} = 4 \) sYM is invariant under the Yangian symmetry, let us give a heuristic argument why that should be the case. We will make it more formal and precise in Chapter 10 after we have understood the role of gauge symmetry, but we are already in position to sketch it.

Let us define
\[ \mathcal{A}[\bar{P}_{\hat{a} \hat{a}}] := \bar{P}_{\hat{a} \hat{a}} \cdot \mathcal{S}, \quad (6.3.41) \]
and try to constrain $\mathcal{A}[\hat{P}_{a\dot{a}}]$. Recall the commutation relations satisfied by level-one generators:

$$[J^A, \hat{J}^B] = f^{AB}_C \hat{J}^C,$$

(6.3.42)
as well as invariance of the action $\mathcal{S}$ under all $\mathfrak{psu}(2,2|4)$ generators $J$. To start with, let us take $J = D$, the dilatation generator, which will give us the mass dimension of $\mathcal{A}[\hat{P}_{a\dot{a}}]$. Since $[D, \hat{P}] = \hat{P}$, we can write in two different ways:

$$[D, \hat{P}_{a\dot{a}}] \cdot \mathcal{S} = \left\{ \begin{array}{l} D \cdot \hat{P}_{a\dot{a}} \cdot \mathcal{S} - \hat{P}_{a\dot{a}} \cdot D \cdot \mathcal{S} = D \cdot \hat{P}_{a\dot{a}} \cdot \mathcal{S} = D \cdot \mathcal{A}[\hat{P}_{a\dot{a}}] \\ \hat{P}_{a\dot{a}} \cdot \mathcal{S} = \mathcal{A}[\hat{P}_{a\dot{a}}], \end{array} \right.$$ (6.3.43)
or simply $D \cdot \mathcal{A}[\hat{P}_{a\dot{a}}] = \mathcal{A}[\hat{P}_{a\dot{a}}]$, so that $\mathcal{A}[\hat{P}_{a\dot{a}}]$ has mass dimension equal to 1.

Similarly considering other generators $J$, we see that $\mathcal{A}[\hat{P}_{a\dot{a}}]$ will be translationally invariant (commutation with $P$) supersymmetric (with $Q$ and $\bar{Q}$) vector (with Lorentz generators $L$ and $\bar{L}$) operator. Furthermore, it will be a scalar of the $\mathfrak{su}(4)$ R-symmetry (due to commutations with $R$).

As the translationally-invariant operators appear as integrals of local operators over spacetime, we are effectively looking for a local dimension-5 $\mathfrak{su}(4)$-scalar supersymmetric vector operator. Furthermore, it should have negative $SU(N)$ parity, a fact we already encountered in Chapter 4 (essentially, odd number of commutators should be replaced with anticommutators and the other way around).\footnote{That is instead of the usual $\{\bar{\Psi}, \Psi\}$ we would have $[\bar{\Psi}, \Psi]$ or $\{\Phi, \Phi\}$ instead of $[\Phi, \Phi]$ etc.}

Using the results from the study of representations of $\mathfrak{psu}(2,2|4)$ (see e.g. \cite{33}, \cite{46} or \cite{107}; we will also list all the operators satisfying the constraints from bosonic generators in Chapter 10 and one can explicitly verify that they are not supersymmetric) it can then be shown, as we will discuss in detail in Chapter 10, that no such operator exists and hence $\mathcal{A}[\hat{P}_{a\dot{a}}] = 0$ or:

$$\hat{P}_{a\dot{a}} \cdot \mathcal{S} = 0.$$ (6.3.44)

We have thus found another argument that the action of $\mathcal{N} = 4$ sYM is Yangian-invariant.

This is in contrast to the pure $\mathcal{N} = 1$ sYM which we already encountered in Chapter 4. In this theory, the operator $\epsilon^\alpha \epsilon^{\dot{\alpha}} tr \left( [\bar{\Psi}_1, \Psi_1] F_{\alpha\dot{\alpha}3\dot{3}} \right)$ does satisfy the requirements discussed above (operators of this type exist of course also in $\mathcal{N} = 4$ sYM, but they are not annihilated by all the supersymmetry generators).

The heuristic argument suggests therefore that $\mathcal{N} = 1$ sYM theory need not be invariant under level-one generators (which is indeed the case). Of course a full computation – or use of our weaker criteria of invariance of equations of motion – is needed to confirm that the coefficient in front of it is not $0$\footnote{Indeed, consider this argument applied to chiral symmetry in Yang–Mills theory with fermions. Classically, the action is invariant under this symmetry, even though the operator $F_{\mu\nu} F^{\mu\nu}$ satisfies all the conditions. Only at the quantum level however it appears as an anomaly, classically its coefficient being 0.}

Similar arguments can also be applied to the generator $\hat{B}$ in the context of $\mathcal{N} = 4$ sYM and we will mention them in Chapter 10.
Chapter 7

Variety of gauge symmetries and gauge fixing

In the previous chapter we have established the Yangian symmetry of the action of $\mathcal{N} = 4$ sYM. The non-linearities of our representation turned out to be a challenge, but the fact that they were a result of gauge-transformations mixing in with the $\mathfrak{psu}(2, 2|4)$ algebra generators was not of utmost importance.

In a quantum field theory context proving invariance of the action is just a first step. We would like to use it in order to constrain some observables, hence simplifying our computations. However, in a gauge theory context any such computation requires working in a specific gauge. Of course, its choice is entirely up to us, but once a gauge is chosen, it has to be kept fixed. Among different methods of gauge fixing there exist (see [30], [29] or [31] for an introduction) we will work with the Fadeev-Popov procedure [93], which will result in the appearance of BRST symmetry [54] as a remainder of the gauge symmetry.

It will then turn out that the existence of gauge/BRST symmetries allows us to discover new types of bi-local symmetries, located inbetween gauge and Yangian generators. Should we have expected them? A short exercise shows that yes. Recall from Chapter 4 that, as expected, $\hat{P}$ is not a symmetry of $\mathcal{N} = 1$ sYM. On the other hand, from the level-1 generators’ adjoint property (3.2.4) we expect that $[P, \hat{P}] = 0$. We already discussed in Chapter 2 that in our representation, the relations are only satisfied up to field-dependent gauge transformations. The right-hand side of the $[P, \hat{P}]$ need not be zero, but essentially has to be trivial. Since it however acts bi-locally due to the level-1 generator, it cannot be a usual gauge transformation. The short exercise now is to apply the generator $[P, \hat{P}]$ to the action of $\mathcal{N} = 1$ sYM. It turns out that it is indeed a symmetry, even though the constituent $\hat{P}$ was not. That result strongly suggests the existence of bi-local gauge transformations which can indeed be promoted to the symmetries of the action. Of course, in the end they do not constrain the action more than the corresponding level-0 and gauge transformations. The construction will however prove necessary in the following chapters. This chapter builds on [61].

7.1 Bi-local commutators

Before arriving at the bi-local gauge transformations, we first need to introduce some technical results related to the algebra. In the following we consider the commutator of a generic bi-local generator $J^1 \otimes J^2$ with another local generator $J^3$. In here, the expression $J^1 \otimes J^2$ also includes the
possible local terms, as will be made explicit in the equations (7.1.1) and (7.1.2). This notational choice makes the study of commutation relations with another generator easier and more concise. We will derive the expression by acting on an open polynomial $\mathcal{X}$. Subsequently, we will discuss the algebra for cyclic polynomials which bears some complications.

Open polynomials. We first act with the commutator of bi-local and local operators on a generic open polynomial $\mathcal{X}$ using the expressions (6.1.5), (6.1.6) and (6.1.7). The calculation is straightforward but it requires some patience due to the various non-linear terms. By considering the bi-local terms we find that all non-linear contributions combine nicely into commutators of local generators (6.1.8), (6.1.9). We find the unsurprising result

$$
[ J^3, J^1 \otimes J^2 ] \cdot \mathcal{X} = ( [ J^3, J^1 ] \otimes J^2 ) \cdot \mathcal{X} + ( J^1 \otimes [ J^3, J^2 ] ) \cdot \mathcal{X} + \text{local}. \quad (7.1.1)
$$

The remaining local contributions to the relation depend on the precise definition of the local terms in the three bi-local operators. The local terms from the commutator can be expressed as

$$
[ J^3, J^1 \otimes J^2 ]_{[n],j} = \sum_{m=0}^{n} \sum_{k=1}^{m+1} J^3_{[n-m],k+j-1} ( J^1 \otimes J^2 )_{[m],j} - \sum_{m=0}^{n} \sum_{k=1}^{m+1} ( J^1 \otimes J^2 )_{[n-m],k+j-1} J^3_{[m],j} - \sum_{m=1}^{n} \sum_{l=0}^{n-m} \sum_{k=2}^{l+1} \sum_{i=1}^{k-l} J^1_{[n-m-l],i+j-1} J^2_{[l],k+j-1} J^3_{[m],j}. \quad (7.1.2)
$$

The terms on the first line correspond to the ordinary commutator of the local parts, cf. (6.1.9), whereas the second term originates from both components of the bi-local generator acting on the non-linear result of the local generator.

Cyclic polynomials. The commutator algebra for the bi-local action on a cyclic polynomial $\mathcal{O}$ is hardly as straightforward as the open polynomial counterpart. According to (7.1.1) one would expect

$$
[ J^3, J^1 \wedge J^2 ] \cdot \mathcal{O} \simeq ( [ J^3, J^1 ] \wedge J^2 ) \cdot \mathcal{O} + ( J^1 \wedge [ J^3, J^2 ] ) \cdot \mathcal{O} + \text{local}. \quad (7.1.3)
$$

Unfortunately, this relationship is very hard to evaluate for a variety of reasons: The intermediate expressions involve three generators $J^1$, $J^2$ and $J^3$ which can act on different positions within the polynomial. The insertion of generators commutes with each other unless they overlap. Due to cyclic symmetry, only the relative insertion points matter and cyclic symmetry allows to cyclically permute (non-overlapping) generators. In addition, each of the three generators as well as the polynomial consists of terms of different length. Altogether this amounts a six-fold sum with non-trivial boundary conditions for each term. Moreover, the validity of the statement depends on certain algebraic constraints, which are equally hard to spot in the residual terms. In fact, almost all cancellations between terms are due to these constraints.

In order to streamline the calculation, we shall address a bi-local generator $Q \otimes Q$ composed from two equal fermionic generators $Q$. Anti-symmetry of the expression is then manifest. The action (6.3.25) on a cyclic polynomial $\mathcal{O}$ reduces somewhat to

$$
( Q \otimes Q ) \cdot \mathcal{O} \simeq \sum_{n,m} ( Q \otimes Q )_{[m],1} \mathcal{O}_{[n]} \quad \text{for equal fermionic generators}. \quad (7.1.3)
$$
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\[\begin{align*}
&+ \sum_{n,m,l} \sum_{k=1}^{n} \frac{2k - n - 2}{2(n + m + l)} Q_{[m],k+l} Q_{[l],1} O_{[n]} \\
&+ \sum_{n,m,l} \sum_{k=1}^{l+1} \frac{2k - l - 2}{n + m + l} Q_{[m],k} Q_{[l],1} O_{[n]}.
\end{align*}\]  

(7.1.4)

Note that the specialisation to \(Q \otimes Q\) is in fact not a restriction. Due to linearity of all expressions in each generator \(J^1\) and \(J^2\), we can recover a corresponding relationship for \(J^1 \wedge J^2\) by means of a replacement

\[\ldots Q \ldots Q \ldots \rightarrow \ldots J^1 \ldots J^2 \ldots - \ldots J^1 \ldots .\]  

(7.1.5)

This replacement is to be understood for every line and every term of the calculation when read from left to right. Any signs due to exchange statistics of the fermionic generators \(Q\) is reflected by the explicit anti-symmetry of \(J^1\) and \(J^2\). Moreover, we can now denote the third generator \(J^3\) by \(J\) and avoid some of the index structure while paying attention to the fermionic nature of the generators \(Q\).

The statement we need to show thus reduces \((7.1.3)\) to

\[\begin{align*}
[J, Q \otimes Q] \cdot O &\simeq ([J, Q] \wedge Q) \cdot O. \\
&= (7.1.6)
\end{align*}\]

Here we have assumed that the local terms in \((7.1.3)\) have been absorbed into the definition of the resulting bi-local generator \([J, Q] \wedge Q\) which is defined by the corresponding relation \((7.1.1)\) on open polynomials \(\mathcal{X}\) without local term

\[\begin{align*}
[J, Q \otimes Q] \cdot \mathcal{X} &\simeq ([J, Q] \wedge Q) \cdot \mathcal{X}. \\
&= (7.1.7)
\end{align*}\]

Hence, the local contributions to \([J, Q] \wedge Q\) are completely defined by \((7.1.2)\).

By means of a lengthy calculation, we find\(^1\)

\[\begin{align*}
\sum_{n,m,p} \sum_{k=1}^{n} \frac{2k - n - 2}{n + m + p} &\left( \frac{2k - n - 2}{n + m} \right) J_{[p],k+m} Q_{[m],1} (Q \cdot O)_{[n]} - \frac{1}{2} \{Q, Q\}_{[m],1} O_{[n]} \\
\sum_{n,m,p} \sum_{k=1}^{m+1} \frac{2k - m - 2}{n + m + p} &\left( \frac{2k - m - 2}{n + m} \right) J_{[p],k} Q_{[m],1} (Q \cdot O)_{[n]} - \frac{1}{2} \{Q, Q\}_{[m],1} O_{[n]} \\
- \sum_{n,m,p} \sum_{k=1}^{p+1} \frac{2k - p - 2}{n + m + p} &\left( \frac{2k - p - 2}{n + m} \right) Q_{[m],k} J_{[p],1} (Q \cdot O)_{[n]} - \frac{1}{2} \{Q, Q\}_{[m],1} J_{[p],1} O_{[n]}.
\end{align*}\]  

(7.1.8)

We have arranged all residual terms as combinations of generators acting on \(Q \cdot O\) and as combinations of generators involving \(\{Q, Q\}\) acting on \(O\). If we impose the restrictions \((6.3.20), (6.3.21)\) used in deriving the form of the non-linear bi-local action on cyclic polynomials \((7.1.3)\), namely \(Q \cdot O \simeq 0\) and \(\{Q, Q\} = 0\), we find that the commutator algebra comes out as expected for cyclic polynomials. In other words, \((6.3.25)\) indeed defines a proper algebraic representation of the bi-local operator \(J^1 \wedge J^2\) on cyclic polynomials \(O\) which is compatible with the corresponding relation \((7.1.1)\) on open polynomials provided that the constraints \((6.3.20), (6.3.21)\) hold.

\(^1\)Note that all remaining terms vanish if \(J\) has a purely linear action, i.e. for \(p = 0\), hence they are clearly effects of non-linear actions.

\(^2\)The remaining terms are somewhat reminiscent of the terms in \((J \wedge Q) \cdot Q \cdot O\).
CHAPTER 7. VARIETY OF GAUGE SYMMETRIES AND GAUGE FIXING

7.2 Bi-local gauge symmetries

As already discussed in Chapter 2, the level-zero algebra closes only modulo field-dependent gauge transformations. Consider for example the momentum generator $P_\mu$ for which we can express the resulting gauge transformations in generality as

$$[P_{\alpha\dot{\alpha}}, J] = [P_{\alpha\dot{\alpha}}, J_\mathfrak{g}] + G[J \cdot A_{\alpha\dot{\alpha}}].$$  (7.2.1)

Here, the first term represents the result of the straight level-zero algebra $\mathfrak{g}$ with the ideal of gauge transformations quotiented out. The second term in the algebra relations specifying a gauge transformation is largely unproblematic on its own because it clearly disappears when acting on gauge-invariant objects. For instance, if both $P_\mu$ and $J$ are symmetries of the action, so is their commutator. The additional gauge transformation term then does not make a difference as the action is gauge-invariant by construction.

The appearance of gauge transformations in the level-zero algebra naturally has an impact on the algebra involving level-one generators (3.2.4) and (3.2.6). At the very least, one would expect gauge transformations to appear there as well, however, the situation turns out to be more involved. Let us consider the bi-local part of the level-one momentum generator

$$\hat{P} = P^{(1)} \otimes P^{(2)}.$$  (7.2.2)

A commutator with the level-zero momentum generator $P_\mu$ yields (7.1.1)

$$[P_{\alpha\dot{\alpha}}, \hat{P}] = G[P^{(1)} \cdot A_{\alpha\dot{\alpha}}] \wedge P^{(2)} + \text{local}.$$  (7.2.3)

All the regular non-gauge terms cancel as they should within the Yangian algebra, see (3.2.4), but the contributions from gauge transformations remain. However, the resulting bi-local action is not a gauge transformation as such; it merely involves gauge transformations alongside level-zero transformations. Consequently, the action of the commutator cannot be expected to vanish simply on gauge-invariant objects, and we need to understand in what sense it can be considered trivial. Gladly, some substantial simplifications come about due to the partial gauge transformation, and we will show in the following Section 7.2.1 in more generality that the resulting bi-local term annihilates gauge-invariant objects which are invariant under level-zero transformations at the same time. As the latter properties apply to the action $S$ by construction (irrespectively of whether the complete Yangian algebra is a symmetry or not) the resulting bi-local term must be part of an algebraic ideal, which can be discarded to recover the plain Yangian algebra.

This property puts us in a good position to argue that the adjoint property (3.2.4) holds modulo (bi-local) gauge transformations, and that the non-gauge level-one generators transform in the adjoint representation of the non-gauge level-zero generators. The same should apply to the Serre relations (3.2.6): modulo (multi-local) gauge transformations, one can expect to find precisely the Yangian relations (3.2.6) because the non-gauge terms follow (3.2.6). Nevertheless it would be desirable to confirm explicitly all the Yangian commutation relations (3.2.4) and (3.2.6) and to derive the concrete decorations due to gauge transformations.

7.2.1 Bi-local gauge symmetries

We have seen above that commutators at level one yield terms of the kind $G[X] \otimes J$, where $G[X]$ is a gauge transformation with gauge parameter $X$ and $J$ is some level-zero transformation.
7.2. BI-LOCAL GAUGE SYMMETRIES

Definition. We will now discuss how such a bi-local generator based on a sequence of fields $\mathcal{X}$ acts on a generic sequence $Z_1 \cdots Z_n$ of $n$ fields without derivatives. Using the special form of a gauge transformation, we can immediately write the (purely bi-local) action as

$$(G[\mathcal{X}] \otimes J)_{\text{biloc}} \cdot (Z_1 \cdots Z_n) = \sum_{k=1}^{n} \mathcal{X} Z_1 \cdots Z_{k-1} (J \cdot Z_k) Z_{k+1} \cdots Z_n$$

$$- \sum_{k=1}^{n} Z_1 \cdots Z_{k-1} \mathcal{X} (J \cdot Z_k) Z_{k+1} \cdots Z_n.$$  \hspace{1cm} (7.2.4)

Here the first term is a bi-local bulk-boundary term while the second one is purely local. Interestingly, we can remove the latter completely by defining the local action of $G[\mathcal{X}] \otimes J$ as

$$(G[\mathcal{X}] \otimes J) \cdot Z := \mathcal{X} J \cdot Z.$$  \hspace{1cm} (7.2.5)

The combined bi-local and local action can then be written as the action of $J$ dressed by the sequence $\mathcal{X}$

$$(G[\mathcal{X}] \otimes J) \cdot (Z_1 \cdots Z_n) = \mathcal{X} J \cdot (Z_1 \cdots Z_n).$$  \hspace{1cm} (7.2.6)

So far, we have ignored derivative terms. Gladly, the local and bi-local terms conspire to yield a consistent expression on (non-linear) covariant derivatives

$$(G[\mathcal{X}] \otimes J) \cdot (D_{\alpha \dot{\alpha}} Z) = \mathcal{X} J \cdot (D_{\alpha \dot{\alpha}} Z).$$  \hspace{1cm} (7.2.7)

Therefore the form of the action on polynomials (7.2.6) also holds in the presence of covariant derivatives.

Evidently, the bi-local generator $J \otimes G[\mathcal{X}]$ with the opposite ordering of constituent generators behaves analogously:

$$(J \otimes G[\mathcal{X}]) \cdot Z := -J \cdot Z \mathcal{X} \quad \Rightarrow \quad (J \otimes G[\mathcal{X}]) \cdot (Z_1 \cdots Z_n) = -J \cdot (Z_1 \cdots Z_n) \mathcal{X}.\hspace{1cm} (7.2.8)$$

For the anti-symmetric combinations $G[\mathcal{X}] \wedge J$ appearing within level-one commutators, one thus obtains

$$(G[\mathcal{X}] \wedge J) \cdot (Z_1 \cdots Z_n) = \{ \mathcal{X}, J \cdot (Z_1 \cdots Z_n) \}.\hspace{1cm} (7.2.9)$$

With this form, the remaining local term in (7.2.3) can now be fixed by a direct computation as

$$[P_{a\dot{a}}, \hat{P}] = G[P^{(1)} \cdot A_{a\dot{a}}] \wedge P^{(2)} + G[\hat{P} \cdot A_{a\dot{a}}].\hspace{1cm} (7.2.10)$$

Symmetry. A relevant observation is that the above bi-local generators preserve the form of the polynomial on which they act. If we apply one of them to the equations of motion $\ddot{Z} \approx 0$ corresponding to some field $Z$, we find

$$(G[\mathcal{X}] \wedge J) \cdot \ddot{Z} = \{ \mathcal{X}, J \cdot \ddot{Z} \}.\hspace{1cm} (7.2.11)$$

Supposing that $J$ is a symmetry of the equations of motion, we know that $J \cdot \ddot{Z} \approx 0$ and consequently

$$(G[\mathcal{X}] \wedge J) \cdot \ddot{Z} \approx 0.\hspace{1cm}$$

This is a necessary requirement for $G[\mathcal{X}] \wedge J$ being a symmetry, cf. the discussion in [60].

Let us therefore check whether any gauge-invariant action $S$ invariant under a local generator $J$ is also invariant under the bi-local transformation $G[\mathcal{X}] \wedge J$ by means of the bi-local action [6.3.25].
on $\mathcal{S}$. Expanding the gauge transformations in terms of an operator $I[\mathcal{X}]$ to insert a sequence of fields $\mathcal{X}$ between any two fields of the polynomials and collapsing some telescoping sums, we find

$$
(G[\mathcal{X}] \land J) \cdot \mathcal{S} \simeq 2I[\mathcal{X}] \cdot (J \cdot \mathcal{S}) + 2I[J \cdot \mathcal{X}] \cdot \mathcal{S}.
$$

(7.2.12)

Here the first term is analogous to (7.2.9), and it vanishes if $J$ is a symmetry of the action. The second term inserts the expression $J \cdot \mathcal{X}$ at all places in the action, and one can hardly expect it to be a symmetry. To make it vanish, $J \cdot \mathcal{X}$ must be zero, and according to (2.2.9) this is the case if the two constituent operators $G[\mathcal{X}]$ and $J$ commute as they should according to the constraint (6.3.21).

Interestingly, this term vanishes for the residual bi-local transformations $G[\mathcal{P}(1) \cdot \mathcal{A}^{\alpha} \dot{\mathcal{A}}^{\dot{\alpha}}] \wedge \mathcal{P}(2)$ arising from the level-one Yangian algebra (7.2.3) because $\mathcal{P}(2) \cdot \mathcal{P}(1) \cdot A^{\alpha} \dot{A}^{\dot{\alpha}} = 0$. Therefore the bi-local gauge transformation is a symmetry of our model. It also explains, why the commutator $[\mathcal{P}, \hat{\mathcal{P}}]$ turned out to be a symmetry of a $\mathcal{N}=1$ sYM discussed in the beginning of this Chapter.

**Gauge covariance.** Another relevant question which we have not yet addressed is gauge covariance of the bi-local action (6.3.25) of $J^1 \otimes J^2$ on a cyclic polynomial $\mathcal{O}$. By specialising (7.1.3) to $J^3 = G[\mathcal{X}]$ we find

$$
[G[\mathcal{X}], J^1 \wedge J^2] \cdot \mathcal{O} \simeq ([G[\mathcal{X}], J^1] \wedge J^2) \cdot \mathcal{O} + (J^1 \wedge [G[\mathcal{X}], J^2]) \cdot \mathcal{O}
$$

(7.2.13)

provided that the constraints (6.3.20), (6.3.21) hold as they should, namely $J^1 \cdot \mathcal{O} = J^2 \cdot \mathcal{O} = [J^1, J^2] = 0$. We transform further using the commutators of gauge transformations (2.2.9), (7.2.12),

$$
[G[\mathcal{X}], J^1 \wedge J^2] \cdot \mathcal{O} \simeq -2I[J^1 \cdot \mathcal{X}] \cdot (J^2 \cdot \mathcal{O}) + 2I[J^2 \cdot \mathcal{X}] \cdot (J^1 \cdot \mathcal{O}) + 2I[[J^1, J^2] \cdot \mathcal{X}] \cdot \mathcal{O}.
$$

(7.2.14)

All resulting terms vanish due to the prerequisites for $J^1$, $J^2$ and $\mathcal{O}$. This shows that the bi-local action (6.3.25) is gauge-covariant in the sense that its result on a gauge-invariant cyclic polynomial is again gauge-invariant subject to the constraints (6.3.20), (6.3.21).

### 7.3 Faddeev – Popov gauge fixing and appearance of BRST symmetry

The content of this section is a standard material as taught in an advanced Quantum Field Theory course, hence we will keep it concise and refer the reader to standard references, e.g. [30], [29].

The following introduction is based on [31]. As no spinor fields explicitly appear from now on, we will restore the usual Lorentz vector indices.

Let us first sketch the Faddeev-Popov procedure. We want to evaluate the following path integral:

$$
\int DA e^{i f d^4 x (-\frac{1}{4} F_{\mu\nu} F^{\mu\nu})}
$$

(7.3.1)

Now, in order to integrate only over the physically different configurations in the above path integral, we insert a Dirac delta localizing it on a particular gauge orbit, i.e. restricting only to configurations satisfying $G(A_\mu) = 0$, where $G$ is a suitably chosen function. Having taken into account the Jacobian of a transformation, we can write:

---

3We omit the scalars and fermionic fields appearing in (2.1.13) as it is only the gauge field that is important in this discussion.
\[ 1 = \int DA \delta(G(A^A)) \det \frac{\delta(G(A^A))}{\delta A}, \quad (7.3.2) \]

where \( A^A \) is a gauge-transformed field: \( A^A = A + i[D, \Lambda] \), where we supressed the vector as well as gauge algebra indices.

We subsequently insert this identity into the path integral \([7.3.1]\):

\[
\int DA \int DA \delta(G(A^A)) \det \frac{\delta(G(A^A))}{\delta A} e^{i \int d^4 x (\frac{1}{4} F_{\mu \nu} F^{\mu \nu})}. \quad (7.3.3)
\]

Now, recall that we can write a determinant of a matrix \( M \) as an integral over Grassmanian variables \( \bar{C}, C \):

\[
\det M = \int d\bar{C} dC e^{\text{MC}}. \quad (7.3.4)
\]

Final step is the observation made by 't Hooft. Take the gauge condition \( G \) to be \( G(A) = \partial_\mu A^\mu - \omega(x) \) with an arbitrary (matrix-valued) function \( \omega(x) \). It does not change the determinant or influence the ghost fields; it only shifts the \( \delta \) function:

\[
\delta(\partial_\mu A^A_\mu) \rightarrow \delta(\partial_\mu A^{A,\mu} - \omega(x)). \quad (7.3.5)
\]

Let us keep that in mind and first spell out the matrix \( M \):

\[
M = \frac{\delta G(x)}{\delta A(y)} = -i \partial_\mu D^\mu \delta^4(x - y). \quad (7.3.6)
\]

We see that it produces a Lagrangian for the Grassmann fields \( \bar{C}, C \), which in case of a non-Abelian gauge-theory is interacting:

\[
S_{\text{gh}} = \int d^d x [D_\mu; C] \partial^\mu \bar{C}, \quad (7.3.7)
\]

where the subscript \( gh \) stands for (Faddeev - Popov) ghosts. The fields \( \bar{C} \) and \( C \) are anti-commuting scalars, violating spin-statistics theorem, going by that name. They should not be confused with the ghost states which violate unitarity.

Going now back to the 't Hooft’s trick \([7.3.5]\), we multiply the path integral \([7.3.3]\) by

\[
\int D\omega e^{-\frac{i}{2} \int d^4 x \omega^2}. \quad (7.3.8)
\]

Clearly, since the initial path integral does not depend on \( \omega(x) \), this merely changes the normalization. Due to the shifted Dirac delta however, we can now perform the integral over \( \omega \), adding to the action the gauge-fixing lagrangian:

\[
S_G = -\int d^d x \frac{1}{2 \xi} (\partial_\mu A^\mu). \quad (7.3.9)
\]

It imposes the gauge fixing, or more accurately actually a family of \( R_\xi \) gauges, leaving us the freedom to pick the most convenient value of \( \xi \). Observe that we can alternatively impose the gauge fixing using an auxillary (commuting scalar) field \( B \), known as Nakanishi - Lautrup field. It will enter the action purely algebraically and integrating it out will imediately give back the last result. However, it will make the BRST symmetry in the next section \( off-shell \).
The final action is then:

\[ S = S_{\text{YM}} + S_{\text{gh}} + \int d^d x \left( -B \partial_\mu A^\mu + \frac{1}{2} \xi B^2 \right) . \]  
\[ (7.3.10) \]

Of course, the procedure works in exactly the same manner for other gauge-theories. We already mentioned before that adding the scalars and fermions to restore the original \( \mathcal{N} = 4 \) sYM causes no difficulty. Furthermore, we can also fix the gauge in Chern–Simons-like theories (i.e. ABJM) using Faddeev–Popov procedure. It makes hence sense to write:

\[ S_{\text{gf}} = \int d^d x \left( [D_\mu, C] \partial_\mu \bar{C} - B \partial_\mu A^\mu + \frac{1}{2} \xi B^2 \right) , \]  
\[ (7.3.11) \]

and then:

\[ S = S_0(\mathcal{Z}) + S_{\text{gf}}(\mathcal{Z}, C, \bar{C}, B), \]  
\[ (7.3.12) \]

where now \( S_0 \) is an action of any gauge theory.

### 7.4 BRST symmetry

Due to the appearance of the bare gauge field \( A \) and non-covariant partial derivatives \( \partial \), the terms \( S_{\text{gf}} \) break the gauge invariance of the theory. A remnant thereof however survives; it is known as the BRST symmetry and it is generated by the action of a fermionic generator \( Q \) on the fields of the theory

\[ Q \cdot \mathcal{Z} = G[C] \cdot \mathcal{Z}, \quad Q \cdot C = CC, \quad Q \cdot \bar{C} = iB, \quad Q \cdot B = 0, \]  
\[ (7.4.1) \]

where \( \mathcal{Z} \) denotes any of the fields of the original model. For \( \xi \neq 0 \), one might integrate out the auxiliary field \( B \) whose equation of motion is algebraic, \( B \approx \xi^{-1} \partial^\mu A_\mu \), but this would obscure some statements about irrelevant contributions.

BRST symmetry has the special property that it squares to zero

\[ QQ = \frac{1}{2} \{ Q, Q \} = 0, \]  
\[ (7.4.2) \]

and therefore it defines a cohomology. The latter is useful in specifying physical objects which have to be closed and carry ghost number zero. In particular the action is BRST-closed. For the original action \( S_0 \) this follows from gauge symmetry,

\[ Q \cdot S_0 = G[C] \cdot S_0 = 0, \]  
\[ (7.4.3) \]

and for the gauge-fixing terms \( S_{\text{gf}} \) it follows from BRST-exactness

\[ S_{\text{gf}} = -Q \cdot K_{\text{gf}}, \quad K_{\text{gf}} = \int d^d x \text{ tr} [iA_\mu \partial^\mu \bar{C} + i\xi B\bar{C}] . \]  
\[ (7.4.4) \]

In fact, the latter feature is important because BRST-exactness indicates that the gauge-fixing terms effectively do not contribute to physical processes.

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\(^4\)We will make signs due to the fermionic statistics of \( Q \) explicit. However, we will keep assuming that the generators \( J \) are bosonic, so that any signs for fermionic generators \( J \) and \( \hat{J} \) (e.g. in commutators with \( Q \)) are implicit and need to be inserted manually.
7.4.1 Local symmetries

Our investigations are based on the assumption that the original action is invariant under some symmetries \( J \)
\[ J \cdot S_0 = 0. \tag{7.4.5} \]
It is clear that gauge fixing breaks some of these symmetries, \( J \cdot S_{gf} \neq 0 \), so that altogether \( J \cdot S \neq 0 \).
In particular, we have argued that the level-zero algebra closes onto gauge transformations which are no longer exact symmetries of the full system. In order to let gauge fixing preserve a symmetry \( J \) we need to show at least that the variation of the action is BRST-exact
\[ J \cdot S = -Q \cdot \mathcal{K}[J]. \tag{7.4.6} \]

In order to determine the precise form of \( \mathcal{K}[J] \) we need to fix the action of the level-zero generators \( J \) on the additional fields \( C, \bar{C} \) and \( B \). A seeming complication is that these fields typically do not form proper multiplets under the level-zero algebra.\(^7\) This complication could perhaps be resolved in particular situations by adding further unphysical fields to complete the multiplets, but it would inevitably be a rather complicated solution. The convenient alternative is to declare all these fields singlets under all level-zero generators. In other words, we declare the level-zero representation on the unphysical fields to be trivial\(^8\),
\[ J \cdot C = J \cdot \bar{C} = J \cdot B = 0. \tag{7.4.7} \]
Furthermore, we do not modify the level-zero representation of the original fields of the model so that the original part of the action \( S_0 \) remains invariant according to (7.4.5). It also nicely ensures that BRST symmetry commutes with the level-zero symmetry
\[ [Q, J] = 0. \tag{7.4.8} \]
In this case, it is straight-forward to show using (7.4.4) that (7.4.6) holds
\[ J \cdot S_{gf} = -Q \cdot \mathcal{K}[J] \tag{7.4.9} \]
with the compensator \( \mathcal{K}[J] \) given by acting with the level-zero generator \( J \) on \( K_{gf} \)
\[ \mathcal{K}[J] = J \cdot K_{gf} = i \int d^d x \text{ tr} [J \cdot A_{\mu} \partial^\mu \bar{C}]. \tag{7.4.10} \]

By construction, gauge fixing breaks gauge symmetry. However, gauge transformations arise from the level-zero algebra, and the full amount of gauge symmetry must be preserved in the same

\(^5\)In this definition, the natural sign assignment due to statistics reads \( J \cdot S_{gf} = -(-1)^{|J|} Q \cdot \mathcal{K}[J] \). It follows by substitution of \( S_{gf} = -Q \cdot K_{gf} \) and by the assumption that (qualitatively) \( K[J] \sim J K_{gf} \). Then this matches with the natural sign due to permutation of \( Q \) and \( J \): \( -JQ \cdot K_{gf} \sim -(-1)^{|J|} QJ \cdot K_{gf} \).

\(^6\)Note that the additional term on the r.h.s. spoils the on-shell invariance of the equations of motion \( \mathcal{Z} \approx 0 \) such that \( \mathcal{Z} \cdot \mathcal{Z} \neq 0 \). However, there will be some well-prescribed terms involving \( \mathcal{K}[J] \) and \( Q \) to compensate the remaining terms.

\(^7\)In particular, this is rather evident in a supersymmetric theory when the level-zero algebra includes supersymmetry.

\(^8\)This includes the curious statement that the unphysical fields carry no momentum, no energy and no angular momentum. However, the assignment is formal and counts only towards the notion and representation of level-zero symmetry. Of course, the fields still depend non-trivially on \( x \).

\(^9\)There may be other permissible representations as it should not matter much how the unphysical fields transform in the end.
sense as the level-zero symmetries. By extending the action of gauge transformations to the ghost fields as

\[ G[\mathcal{X}] \cdot C = G[\mathcal{X}] \cdot \bar{C} = G[\mathcal{X}] \cdot B = 0, \quad (7.4.11) \]

the gauge generator \( G[\mathcal{X}] \) commutes with the BRST generator \( Q \). The corresponding compensator for gauge invariance of the action

\[ K[G[\mathcal{X}]] = G[\mathcal{X}] \cdot K_{gf} = -Q \cdot K[G[\mathcal{X}]] \]

reads

\[ K[G[\mathcal{X}]] = G[\mathcal{X}] \cdot K_{gf} = i \int d^d x \, \text{tr} \left[ G[\mathcal{X}] \cdot A_\mu \partial^\mu \bar{C} \right] = -\int d^d x \, \text{tr} \left[ D_\mu \mathcal{X} \partial^\mu \bar{C} \right]. \quad (7.4.12) \]

### 7.4.2 Bi-local symmetries

Before discussing the level-one Yangian symmetries, let us introduce a new class of bi-local generators involving BRST generators \( Q \). This case will be instructive as these generators are considerably simpler than the full level-one Yangian generators. Furthermore, they introduce some additional terms in the gauge-fixed invariance statement. They will also be needed for the level-one Yangian generators and later they will be relevant in formulating identities for quantum correlators due to the level-one symmetries.

**Bi-local BRST symmetry.** We start by recalling that we have introduced bi-local transformations based on gauge transformations in Section 7.2.1. Noting that BRST transformations can be viewed as gauge transformations \( Q \sim G[C] \) using the ghost field \( C \) as gauge transformation parameter, it is conceivable that \( Q \otimes Q \) and \( Q \wedge J \) will be further symmetries of the planar gauge theory. We shall call them bi-local BRST symmetries.

We first derive their action on single fields by the condition that the bi-local generators commute with plain BRST symmetry \( Q \). This is compatible with the absence of bi-local terms due to the bi-local algebra relations

\[ [Q, Q \otimes Q] = \{Q, Q\} \wedge Q = 0, \]
\[ \{Q, Q \wedge J\} = \{Q, Q\} \wedge J - Q \wedge [Q, J] = 0, \quad (7.4.13) \]

which follow from nilpotency of \( Q \) and from commutation with \( J \). We find the following local contributions

\[
(Q \otimes Q) \cdot Z = \frac{1}{2} [C, Q \cdot Z], \\
(Q \otimes Q) \cdot C = CCC, \\
(Q \otimes Q) \cdot B = 0, \\
(Q \otimes Q) \cdot \bar{C} = 0, \\
(Q \wedge J) \cdot Z = \{C, J \cdot Z\}, \\
(Q \wedge J) \cdot C = 0, \\
(Q \wedge J) \cdot B = 0, \\
(Q \wedge J) \cdot \bar{C} = 0. \quad (7.4.14)
\]

It turns out that \( Q \otimes Q \) annihilates any ghost-free gauge-invariant cyclic polynomial via the representation (6.3.25). For invariance under \( Q \wedge J \), the cyclic polynomial must also be invariant under

\[ \{Q, Q\} \wedge Q = 0 \]

\[ \{Q, Q\} \wedge J = \{Q, Q\} \wedge J - Q \wedge [Q, J] = 0 \]

for all \( Q \) and \( J \). We fix these degrees of freedom to a convenient choice.

---

10We assume the gauge transformation parameter \( \mathcal{X} \) to be an internal field of the theory (or an open polynomial). For an external field \( A \) the action on the ghost would have to be replaced by \( G[A] \cdot C = [A, C] \).

11The generator \( Q \) we used as a trick in Section 7.1 was an unspecified fermionic generator, but the formulae of course also hold if \( Q \) is taken to be the BRST transformation as it satisfies all the required properties.

12Note that the BRST generator \( Q \) commutes with the level-zero generator \( J \) as well as with itself, \( \{Q, Q\} = 0 \), and thus makes the bi-local generators compatible with cyclicity. Furthermore, the BRST generator \( Q \) is fermionic, and thus the bi-local combination \( Q \otimes Q \) is anti-symmetric on its own.

13There is some freedom in assigning the action on the ghosts \( B \) and \( \bar{C} \). Such a freedom is related to adding a local generator and thus not interesting. We fix these degrees of freedom to a convenient choice.
These statements follow in analogy to (7.2.12)\textsuperscript{14}. In particular, the original action is invariant under the bi-local BRST symmetries
\[(Q \otimes Q) \cdot S_0 \simeq 0, \quad (Q \wedge J) \cdot S_0 \simeq 0. \quad (7.4.15)\]

Gauge-fixed invariance statements. To compensate for the broken invariance of the gauge-fixing term $S_{gf}$, the procedure from level zero turns out to be insufficient for bi-local generators for the following reason: We need to compensate $(J^1 \wedge J^2) \cdot S_{gf} = -(J^1 \wedge J^2) \cdot Q \cdot \mathcal{K}_{gf}$. Supposing that $Q$ commutes with $J^1 \wedge J^2$ we could rewrite this term as $-Q \cdot (J^1 \wedge J^2) \cdot \mathcal{K}_{gf} = -Q \cdot \mathcal{K}[J^1 \wedge J^2]$ and declare the compensator to read $\mathcal{K}[J^1 \wedge J^2] = (J^1 \wedge J^2) \cdot \mathcal{K}_{gf}$. Unfortunately, the commutator algebra (7.1.3) on cyclic polynomials cannot be trusted because $\mathcal{K}_{gf}$ is not invariant under $J^1$ and $J^2$. For a similar reason, we cannot even be sure that $(J^1 \wedge J^2) \cdot S_{gf}$ is BRST-exact.

We address the above issue by computing the action of $Q \otimes Q$ on $S_{gf} = -Q \cdot \mathcal{K}_{gf}$. To understand the result better, we shall make no assumptions on the fermionic generator $Q$ and on the precise form of cyclic polynomial $\mathcal{K}_{gf}$ at first. In a calculation analogous to (7.1.8) we find
\[(Q \otimes Q) \cdot Q \cdot \mathcal{K}_{gf} \simeq Q \cdot (Q \otimes Q)_{loc} \cdot \mathcal{K}_{gf} + \frac{1}{2} (Q \wedge \{Q, Q\}) \cdot \mathcal{K}_{gf} + \frac{1}{2} (Q \wedge \{Q, Q\})_{loc} \cdot \mathcal{K}_{gf} \]
\[-\frac{1}{3} \sum_{n,m,l,p} \sum_{k=3}^{n} \sum_{j=2}^{k-1} Q_{\mu m, k-p+l} Q_{\nu j, p} Q_{\mu j, p} Q_{\mu j, p} K_{gf, \mu n}. \quad (7.4.16)\]

Among the resulting terms we find the commutator of $Q$ with $Q \otimes Q$, see (7.4.13), albeit with a prefactor $1/2$. Two other terms represent the purely local contributions from bi-local generators, and there is a tri-local term without overlapping contributions in the second line. Now we have constructed $Q \otimes Q$ such that it commutes with $Q$; this eliminates the two terms involving $\{Q, Q\} \wedge Q$. Furthermore, the tri-local term requires polynomials of length 3 or more, but the compensator $\mathcal{K}_{gf}$ in (7.4.4) is purely quadratic. Therefore in our case, the above relation reduces to
\[(Q \otimes Q) \cdot Q \cdot \mathcal{K}_{gf} \simeq Q \cdot (Q \otimes Q)_{loc} \cdot \mathcal{K}_{gf}. \quad (7.4.17)\]
This shows that the usual form (7.4.6) of gauge-fixed symmetry relationship holds
\[(Q \otimes Q) \cdot S_{gf} \simeq -Q \cdot \mathcal{K}[Q \otimes Q], \quad (7.4.18)\]
where the compensator is given by merely the local part of the bi-local generator $Q \otimes Q$ acting on $\mathcal{K}$
\[\mathcal{K}[Q \otimes Q] \simeq (Q \otimes Q)_{loc} \cdot \mathcal{K} \simeq i \int d^4x \text{ tr}[(Q \otimes Q) \cdot A_\mu \partial^\mu C]. \quad (7.4.19)\]

The symmetry statement for the mixed bi-local BRST generator $Q \wedge J$ turns out to be somewhat more involved. We first need to generalise the identity (7.4.16) to $(Q \wedge J) \cdot Q \cdot \mathcal{K}_{gf}$. We do this by the same trick employed in 7.1 and replace all triplets of $Q$ in every term of (7.4.16) (in their order of appearance) by the following linear combinations
\[(Q, Q, Q) \rightarrow (J, Q, Q) - (Q, J, Q) + (Q, Q, J). \quad (7.4.20)\]
\textsuperscript{14}The expression (7.2.12) applied to $Q \otimes Q$ may seem to suggest a non-zero result proportional to $I[Q \cdot C] \cdot S_0$. However, here one has to pay attention to the action $Q \cdot C = CC$ in the overlapping terms, which is almost a gauge transformation, but only up to a factor of $1/2$. An explicit calculation then shows that all terms $I[CC] \cdot S_0$ vanish.
Again, we eliminate terms due to \(\{Q, Q\} = [Q, J] = 0\) as well as tri-local contributions on \(K_{\text{gf}}\) and find
\[
-(Q \wedge J) \cdot Q \cdot K_{\text{gf}} + (Q \otimes Q) \cdot J \cdot K_{\text{gf}} \simeq J \cdot (Q \otimes Q)_{\text{loc}} \cdot K_{\text{gf}} + Q \cdot (Q \wedge J)_{\text{loc}} \cdot K_{\text{gf}}. 
\]
(7.4.21)

By rearranging the terms, we can express the relationship as \(Q \wedge J\) acting on \(S_{\text{gf}}\)
\[
(Q \wedge J) \cdot S_{\text{gf}} \simeq Q \cdot K[Q \wedge J] - (Q \otimes Q) \cdot K[J] + J \cdot K[Q \otimes Q]
\]
(7.4.22)

with the additional compensator defined as before as the action of the local part of \(Q \wedge J\)
\[
K[Q \wedge J] \simeq (Q \wedge J)_{\text{loc}} \cdot K \simeq i \int d^d x \, \text{tr} \left[ (Q \wedge J) \cdot A_{\mu} \partial^{\mu} \bar{C} \right]. 
\]
(7.4.23)

The physical meaning of the two additional terms in (7.4.22) as compared to (7.4.6) remains somewhat obscure at this point. One may argue that it is sufficient that these terms, just like the regular term, are based on the ‘lesser’ generators \(J\) and \(Q \otimes Q\) in the hierarchy of all symmetries. Moreover these terms are fully determined in terms of other relations, so they do not introduce any additional degrees of freedom. Most importantly, there exists a compensator \(K[Q \wedge J]\) to formulate an exact symmetry relationship. Later in the investigation of identities for quantum correlation functions we shall find a different justification for the above particular combination of terms.

In conclusion, the bi-local BRST generators are curious additional symmetries of gauge-fixed planar gauge theories with invariance of the gauge-fixed action \(S\) expressed as (7.4.18), (7.4.22)
\[
0 \simeq (Q \otimes Q) \cdot S + Q \cdot K[Q \otimes Q], \\
0 \simeq (Q \wedge J) \cdot S - Q \cdot K[Q \wedge J] + (Q \otimes Q) \cdot K[J] - J \cdot K[Q \otimes Q]. 
\]
(7.4.24)

We have explicitly checked validity of these relations in gauge-fixed planar \(N = 4\) sYM. Moreover, they can be expected to be present in other non-integrable planar gauge symmetries after BRST gauge fixing.

**Yangian symmetry.** Let us now turn our attention to the level-one Yangian generators \(\hat{J}\). We will not change the definition of the local terms in \(\hat{J}\), therefore the assumption of Yangian symmetry of the original action remains valid
\[
\hat{J} \cdot S_0 \simeq 0. 
\]
(7.4.25)

As before, we define the single-field action on ghosts to be trivial
\[
\hat{J} \cdot C = \hat{J} \cdot \bar{C} = \hat{J} \cdot B = 0, 
\]
(7.4.26)

This assignment ensures that the generator commutes with BRST transformations,
\[
[Q, \hat{J}] = 0. 
\]
(7.4.27)

Towards obtaining a symmetry statement for the gauge-fixed action, we can again generalise the relationship (7.4.16) using the trick of 7.1 for the replacement
\[
(Q, Q, Q) \rightarrow (J^{(1)}, J^{(2)}, Q) - (J^{(1)}, Q, J^{(2)}) + (Q, J^{(1)}, J^{(2)}).
\]
(7.4.28)

The resulting gauge-fixed invariance relationship turns out to have the form
\[
\hat{J} \cdot S_{\text{gf}} \simeq -Q \cdot K[\hat{J}] - (Q \wedge J^{(2)}) \cdot K[J^{(1)}] + J^{(1)} \cdot K[Q \wedge J^{(2)}]
\]
(7.4.29)
with the compensator taking the established form

\[ K[\hat{J}] \simeq \hat{J}_{\text{loc}} \cdot K_{\text{gf}} \simeq i \int d^d x \, \text{tr} \left[ \hat{J} \cdot A_\mu \partial^\mu \bar{C} \right]. \] (7.4.30)

The invariance property of the gauge-fixed action \( S \) therefore reads\(^{15}\)

\[ 0 \simeq \hat{J} \cdot S + Q \cdot K[\hat{J}] + (Q \wedge J^{(2)}) \cdot K[J^{(1)}] - J^{(1)} \cdot K[Q \wedge J^{(2)}]. \] (7.4.31)

This proves that the gauge-fixing procedure does not spoil the Yangian invariance of planar gauge theories. The same should hold for the bi-local generators involving gauge transformations discussed in \( \text{7.2.1} \) whose gauge fixing proceeds completely analogously.

\(^{15}\)The necessity of additional terms in the invariance statement for gauge-fixed ABJM theory was pointed out by Matteo Rosso.
Chapter 8

Slavnov – Taylor identities

8.1 Slavnov–Taylor identities

After having fixed the gauge and formulated statements for Yangian symmetries in the gauge-fixed theory, our next goal is to formulate corresponding identities for quantum correlators, so called Slavnov–Taylor identities. These will eventually reduce to unambiguous identities on the physical contributions, but they may also constrain the unphysical degrees of freedom to some extent.

In the following we will derive and propose the Slavnov–Taylor identities for the various symmetries we have encountered so far. As the level-one generators heavily rely on level-zero and (extended) gauge symmetries, we should start with the simpler kinds of symmetries. Unfortunately, we have no firmly established tools to perform all the required transformations for bi-local generators.\footnote{These tools would have to make reference to the planar limit which is essential for the bi-local symmetries to work. We are not aware of a suitable set of tools which specifically address the planar path integral.} Therefore we shall test the proposed identities for correlation functions in the following Chapter.\footnote{This chapter is based on [61].}

8.1.1 Local symmetries

We start with the identities for local symmetries which can be derived as usual from the path integral. We act with a local generator $J$ onto the integrand of the path integral and obtain the identity

$$\langle J \cdot O + iJ \cdot S \cdot O \rangle = 0.$$ (8.1.1)

This identity is due to the fact that $J$ acts as a total variation within the path integral, and it can either hit the operator $O$ or the implicit phase factor $\exp(iS)$. If we specialise this identity to a BRST generator $J = Q$ we find that

$$\langle Q \cdot O \rangle = 0$$ (8.1.2)

because the action itself is BRST-closed, $Q \cdot S = 0$. For the other local generators, however, further terms are needed because the gauge-fixed action is not exactly invariant, but only up to a BRST-exact term (7.4.6). We can cancel the latter term by adding the identity (8.1.2) applied to the composite operator $i\mathcal{K}[J] \ O$.

$$\langle J \cdot O - i\mathcal{K}[J] Q \cdot O + i(J \cdot S + Q \cdot \mathcal{K}[J]) \ O \rangle = 0.$$ (8.1.3)
Here we have combined the two terms constituting the gauge-fixed invariance condition (7.4.6) for $J$. By dropping them we arrive at the Slavnov–Taylor identity corresponding to the generator $J$

$$\langle J \cdot \mathcal{O} - i \mathcal{K}[J] Q \cdot \mathcal{O} \rangle = 0.$$  \hspace{1cm} (8.1.4)

This identity also holds for gauge symmetries $J = G[\mathcal{X}]$ when supplemented by the corresponding BRST compensators $\mathcal{K}[G[\mathcal{X}]]$.

### 8.1.2 Bi-local total variations

The goal is to formulate a Slavnov–Taylor identity corresponding to the level-one Yangian generator $\hat{J}$. However, we cannot proceed as above because it is not evident how to derive the equivalent of (8.1.1) for a bi-local generator consisting of two field variations ordered in a particular way involving the planar limit. Hence we need to find the equivalent of (8.1.1) for bi-local generators.

The central idea underlying the identity (8.1.1) is that the generator $J$ is a variational operator which can act either on the operator $\mathcal{O}$ or on the action within the phase factor $\exp(iS)$. For some bi-local generator $J^1 \wedge J^2$ we have two variations which should individually act either on the operator or on the phase factor. When both of them act on the same object $\mathcal{O}$, the result should be a product of the actions of the individual generators $J^1$ and $J^2$ which we shall denote by $(J^1 \cdot \mathcal{O}_1) \wedge (J^2 \cdot \mathcal{O}_2)$. This product of operators will somehow incorporate the planar topology of $\mathcal{O}$ when acting on two independent objects $\mathcal{O}_1$ and $\mathcal{O}_2$. It is not evident how to define this combination, not even whether there is a proper definition in the first place, so the notation $(J^1 \cdot \mathcal{O}_1) \wedge (J^2 \cdot \mathcal{O}_2)$ shall serve as a placeholder and we shall use it for book-keeping purposes only.\(^2\)

More concretely, we propose the bi-local generalisation of (8.1.1) as

$$\langle (J^1 \wedge J^2) \cdot \mathcal{O} + i(J^1 \wedge J^2) \cdot S \mathcal{O} \rangle + \langle i(J^1 \cdot S) \wedge (J^2 \cdot \mathcal{O}) + i(J^1 \cdot \mathcal{O}) \wedge (J^2 \cdot S) - (J^1 \cdot S) \wedge (J^2 \cdot S) \mathcal{O} \rangle = 0.$$  \hspace{1cm} (8.1.5)

Here we suppose that $J^1 \wedge J^2$ induces a total variation of the path integral\(^3\) so that the sum of all these variations is zero. Note that the last term originates from the constituents of $J^1 \wedge J^2$ acting on different actions within the exponential factor $\exp(iS)$. The above conjecture is our best guess for a total variational identity involving bi-local generators. Conceivably, it is not entirely correct or correct only under certain conditions. In particular, we do not claim to understand how to define the terms on the second line. It would be very desirable to better understand and/or formally derive the above variational identity (8.1.5). For the time being, it will help us setting up Slavnov–Taylor identities for the bi-local generators. In the following Chapter 9 we will test the predicted equations for tree-level correlation functions involving up to four external fields in order to justify our proposal a posteriori.

\(^2\)The idea behind book-keeping by means of a bi-local combination of the kind $\mathcal{O}_1 \wedge (\mathcal{O}_2 - \mathcal{O}_3)$ is that a certain contribution to $\mathcal{O}_2$ would always appear in the same place within a planar correlator as does a structurally equivalent contribution to $\mathcal{O}_3$. Now, if a symmetry implies that $\mathcal{O}_2 = \mathcal{O}_3$, the above bi-local combination, independently of how it may be defined, is zero due to linearity. If we can make all bi-local terms $\mathcal{O}_1 \wedge \mathcal{O}_2$ cancel, there will be no need for a precise definition.

\(^3\)It would be helpful to understand under which conditions on $J^1$ and $J^2$ this assumption holds and how far violations could break symmetries.
8.1.3 Bi-local symmetries

Armed with the variational identities \((8.1.1), (8.1.5)\) for local and bi-local generators, we can now try construct identities for bi-local generators \(J^1 \wedge J^2\) to predict their action on some operator \(\mathcal{O}\) within a correlator

\[
\langle (J^1 \wedge J^2) \cdot \mathcal{O} \rangle = \ldots . \tag{8.1.6}
\]

Here \(\mathcal{O}\) must be some operator of circular topology in the planar sense, i.e. it must consist of a single trace of fields.

**Bi-local BRST symmetry.** It will be easiest to start with the additional bi-local symmetries based on BRST transformations introduced in \(7.4.2\). This is because the constituent BRST transformations are exact symmetries which do not require compensating terms. We propose that the above form of Slavnov–Taylor identity \((8.1.4)\) equivalently applies to the bi-local transformation \(Q \otimes Q\)

\[
\langle (Q \otimes Q) \cdot \mathcal{O} \rangle - i\langle \mathcal{K}[Q \otimes Q] Q \cdot \mathcal{O} \rangle = 0. \tag{8.1.7}
\]

This is consistent with the proposed variational identity \(8.1.5\) because all terms on the second line vanish due to the fact that the action is BRST-closed, \(Q \cdot \mathcal{S} = 0\). The remainder of the derivation works precisely as in the case of a generic local generator \(J\), and it uses the symmetry statement \(7.4.18\) for the gauge-fixed action.

We will later test the identity for tree-level correlators involving up to four external fields, and find that it successfully predicts all the arising terms. However, it is conceivable that further terms are needed to accommodate for a larger number of fields in \(\mathcal{O}\).

For the mixed bi-local generator \(Q \wedge J\), we have to bear in mind that the constituent \(J\) does not annihilate the action right away, but it requires a BRST compensator \(K[J]\). In order to cancel the arising terms, we will need a cascade of further compensating terms. We start with the formula \(8.1.7\) applied to a generator \(Q \wedge J\) - hence dropping the terms \(Q \cdot \mathcal{S}\), since the action is BRST invariant:

\[
0 = \langle (Q \wedge J) \cdot \mathcal{O} + i(Q \wedge J) \cdot \mathcal{S} \mathcal{O} - i (J \cdot \mathcal{S}) \wedge (Q \cdot \mathcal{O}) \rangle \tag{8.1.8}
\]

The first term is the action of our composite generators on the external legs of the correlator, and we wish to keep it. The second term could be cancelled due to the invariance of the gauge-fixed action under \(Q \wedge J\), which we thus now subtract from the above, sitting within a correlator with \(\mathcal{O}\):

\[
0 = -i\langle ((Q \wedge J) \cdot \mathcal{S} - Q \cdot \mathcal{K}[Q \wedge J] + (Q \otimes Q) \cdot \mathcal{K}[J] - J \cdot \mathcal{K}[Q \otimes Q])\mathcal{O} \rangle \tag{8.1.9}
\]

In the next step, we apply the identity \(8.1.7\) to a composite operator \(\mathcal{K}[J]\mathcal{O}\):

\[
0 = i\langle (Q \otimes Q) \cdot \mathcal{K}[J]\mathcal{O} + \mathcal{K}[J](Q \otimes Q) \cdot \mathcal{O} - (Q \cdot \mathcal{K}[J]) \wedge (Q \cdot \mathcal{O}) \rangle \\
+ \langle \mathcal{K}[Q \otimes Q]([Q \cdot \mathcal{K}[J]]\mathcal{O} - \mathcal{K}[Q \otimes Q]\mathcal{K}[J]Q \cdot \mathcal{O}) \rangle, \tag{8.1.10}
\]

where changes of sign follow from fermionic nature of \(\mathcal{K}[J]\).

\(^4\)Even though we do not understand the nature of such terms, we hope that we can apply symmetry statements to eliminate them.
Next we use the exact invariance of correlators under BRST symmetry $Q$ in order to move the generator from $\mathcal{K}[Q \wedge J]$ to $\mathcal{O}$ in (8.1.9):

$$0 = i\langle Q \cdot \mathcal{K}[Q \wedge J | \mathcal{O} + \mathcal{K}[Q \wedge J | Q \cdot \mathcal{O} \rangle. \quad (8.1.11)$$

Finally we apply the level-0 Slavnov–Taylor identity to the generator $\mathcal{K}[Q \otimes Q] \mathcal{O}$:

$$0 = -i\langle J \cdot \mathcal{K}[Q \otimes Q | \mathcal{O} + \mathcal{K}[Q \otimes Q J | \mathcal{O} + iJ \cdot S \mathcal{K}[Q \otimes Q | \mathcal{O} \rangle. \quad (8.1.12)$$

We see that we have now achieved our goal - cancelation of the undefined terms of the form $\mathcal{O}_1 \wedge \mathcal{O}_2$:

$$0 = -i\langle (J \cdot S + Q \cdot \mathcal{K}[J]) \wedge (Q \cdot \mathcal{O}) \rangle, \quad (8.1.13)$$

as well as cancelling all the terms where a generator (either $J$ or $Q$) acts on a compensator:

$$0 = \langle \mathcal{K}[Q \otimes Q] (J \cdot S + Q \cdot \mathcal{K}[J]) \mathcal{O} \rangle, \quad (8.1.14)$$

and the remaining terms cancel pairwise.

We are hence left with a Slavnov–Taylor identity for the bi-local generator $Q \wedge J$ in a form we can directly apply to correlation functions:

$$\langle (Q \wedge J) \cdot \mathcal{O} + i\mathcal{K}[J] (Q \otimes Q) \cdot \mathcal{O} \rangle + \langle (-i\mathcal{K}[Q \wedge J] - \mathcal{K}[Q \otimes Q \mathcal{K}[J]) Q \cdot \mathcal{O} - i\mathcal{K}[Q \otimes Q] J \cdot \mathcal{O} \rangle = 0. \quad (8.1.15)$$

So we see that we need a number of terms to cancel off the effects of the gauge-fixing terms. However, it can be argued that all these terms are based on simpler symmetries acting on the operator $\mathcal{O}$. In [9.3] we will demonstrate in an example that the above identity holds indeed.

**Yangian symmetry** Having investigated the Slavnov–Taylor identities for the bi-local BRST symmetry $Q \otimes Q$ as well as the mixed BRST–level-0 symmetry $J \wedge Q$, we are now in position to evaluate the identity for the full level-1 Yangian generator $J^1 \wedge J^2$. We start again with the proposition (8.1.5), which we will rewrite using the identities (8.1.1) and (8.1.5), as well as using the invariance of the gauge-fixed action $\mathcal{S}$ under various local and bi-local symmetries, see Chapter 7. The steps will be analogous to the ones we have performed for the $Q \wedge J$, but due to the non-invariance of the gauge-fixed action under both constituents, the number of compensators will increase substantially. Also, just like the invariance under $Q \otimes Q$ was necessary for the validity of Slavnov–Taylor identity under $Q \wedge J$, here we will see that all the lower-rank symmetries are required, that is also the mixed $Q \wedge J$ one. That of course does not come as a surprise given its appearance in the invariance statement (7.4.31). We again start with the formula (8.1.5) and add lower-type identities for various composite operators, which the reader should now recognize:

$$0 = \langle \hat{J} \mathcal{O} + i\hat{J} \mathcal{S} \mathcal{O} - (J^{(1)} S \otimes J^{(2)} S) \mathcal{O} + iJ^{(1)} \mathcal{O} \wedge J^{(2)} S \rangle$$

$$0 = \langle i(Q \wedge J^{(2)}) \mathcal{O} \mathcal{K}[J^{(1)}] + i(Q \wedge J^{(2)}) \mathcal{K}[J^{(1)}] \mathcal{O} \rangle + \langle i(Q \mathcal{O} \wedge J^{(2)} \mathcal{K}[J^{(1)}]) + i(Q \mathcal{K}[J^{(1)}] \wedge J^{(2)} \mathcal{O}) \rangle - \langle (Q \wedge J^{(2)}) \mathcal{S} \mathcal{O} \mathcal{K}[J^{(1)}] + (Q \mathcal{O} \wedge J^{(2)} S) \mathcal{K}[J^{(1)}] + (Q \mathcal{K}[J^{(1)}] \wedge J^{(2)} S) \mathcal{O} \rangle$$
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We subsequently combine the terms so as to arrange them into combinations that are 0 due to the invariance of the action under various symmetries. Also, remember that due to the antisymmetry already present in Sweedler’s notation we have that $J^{(1)} \wedge J^{(2)} = 2 J^{(1)} \otimes J^{(2)}$. With that in mind, we can write:

\[
0 = \langle \hat{J} \mathcal{O} + i (Q \wedge J^{(2)}) \mathcal{O} \mathcal{K}[J^{(1)}] - \frac{1}{2} (Q \otimes Q) \mathcal{O} \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] \\
- \frac{3}{2} Q \mathcal{O} \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] + Q \mathcal{O} \mathcal{K}[Q \wedge J^{(2)}] \mathcal{K}[J^{(1)}] + i Q \mathcal{O} \mathcal{K}[\hat{J}] \\
- i (J^{(1)} \mathcal{O} \mathcal{K}[Q \wedge J^{(2)}] - J^{(1)} \mathcal{O} \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(2)}]) \rangle
\]

\[
0 = \langle \hat{J} \mathcal{S} + Q \mathcal{K}[\hat{J}] + (Q \wedge J^{(2)}) \mathcal{K}[J^{(1)}] - J^{(1)} \mathcal{K}[Q \wedge J^{(2)}]) \mathcal{O} \\
- \langle (Q \wedge J^{(2)}) \mathcal{S} - Q \mathcal{K}[Q \wedge J^{(2)}] + (Q \otimes Q) \mathcal{K}[J^{(2)}] - J^{(2)} \mathcal{K}[Q \otimes Q]) \mathcal{O} \mathcal{K}[J^{(1)}] \\
- \frac{3}{2} \langle (Q \otimes Q) \mathcal{S} + Q \mathcal{K}[Q \otimes Q] \mathcal{O} \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] \\
- \langle ((J^{(1)} \mathcal{S} + Q \mathcal{K}[J^{(1)}]) \otimes (J^{(2)} \mathcal{S} + Q \mathcal{K}[J^{(2)}])) \mathcal{O} \\
+ i (J^{(1)} \mathcal{O} \wedge J^{(2)} \mathcal{S} + Q \mathcal{K}[J^{(2)]})\rangle \\
- \langle (Q \mathcal{O} \wedge (J^{(2)} \mathcal{S} + Q \mathcal{K}[J^{(2)]}) \mathcal{K}[J^{(1)}] \\
+ i (Q \mathcal{O} \wedge J^{(2)} \mathcal{K}[J^{(1)]}) \\
- \langle (J^{(1)} \mathcal{S} + Q \mathcal{K}[J^{(1)]}) \mathcal{O} \mathcal{K}[Q \wedge J^{(2)]}) \\
+ i (\mathcal{O} \mathcal{K}[Q \otimes Q] J^{(2)} \mathcal{S} + Q \mathcal{K}[J^{(2)]}) \mathcal{K}[J^{(1)]}) \\
+ \langle \mathcal{O} \mathcal{K}[Q \otimes Q] J^{(2)} \mathcal{K}[J^{(1)]}) \rangle
\]

The first part is the desired form of the Slavnov–Taylor identity:

\[
0 = \langle \hat{J} \mathcal{O} + i (Q \wedge J^{(2)}) \mathcal{O} \mathcal{K}[J^{(1)}] - \frac{1}{2} (Q \otimes Q) \mathcal{O} \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] \\
- \frac{3}{2} Q \mathcal{O} \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] + Q \mathcal{O} \mathcal{K}[Q \wedge J^{(2)}] \mathcal{K}[J^{(1)}] + i Q \mathcal{O} \mathcal{K}[\hat{J}] \\
- i (J^{(1)} \mathcal{O} \mathcal{K}[Q \wedge J^{(2)}] - J^{(1)} \mathcal{O} \mathcal{K}[Q \otimes Q] \mathcal{K}[J^{(2)]}) \rangle
\]
product of two compensators $\mathcal{K}[J^1]$ and $\mathcal{K}[J^2]$. Due to the implicit sum in Sweedler’s notation those factors actually compensate a factor of 2 which would appear whenever $Q$ acts on either of the compensators, since (recall the fermionic nature of the BRST compensators for bosonic generators $J$):

$$Q \cdot (\mathcal{K}[J^2]\mathcal{K}[J^1]) = (Q \cdot \mathcal{K}[J^2])\mathcal{K}[J^1] - \mathcal{K}[J^2](Q \cdot \mathcal{K}[J^1]) = 2(Q \cdot \mathcal{K}[J^2])\mathcal{K}[J^1]. \quad (8.1.19)$$

Another point to mention is the vanishing of the term $J^{(1)} \cdot \mathcal{K}[J^{(2)}]$. This is a consequence of the trivial action of the level-0 generators $J$ on the ghost fields (and $\bar{C}$ in particular) and the commutativity of $J^{(1)}$ and $J^{(2)}$, since:

$$J^{(1)} \cdot \mathcal{K}[J^{(2)}] = i \int d^d x \text{tr} \left[ (J^{(1)} J^{(2)} \cdot A_\mu) \partial^\mu \bar{C} \right] = \frac{i}{2} \int d^d x \text{tr} \left[ ([J^{(1)}, J^{(2)}] \cdot A_\mu) \partial^\mu \bar{C} \right] = 0. \quad (8.1.20)$$

Altogether we see that we have managed to obtain the Slavnov–Taylor identities for various bi-local symmetries of the action $S$. In the next Chapter we will verify them in concrete examples. A caveat is in order, however. The results we have derived hold for the operator $\mathcal{O}$ being a product of fundamental fields located at different spacetime points (with the trace taken). As such, it would be important to generalize them to correlation functions of multiple single-trace operators. In the context of $\mathcal{N} = 4$ sYM such functions of course are being studied, as we have already discussed. The concrete understanding of their transformation properties under the action of Yangian generators is however still missing.
Chapter 9

Yangian invariance of Correlation Functions

For correlation functions, symmetries manifest as Ward–Takahashi identities. In the gauge-fixed quantum theory, they follow by applying the Slavnov–Taylor identities to operators $\mathcal{O}$ consisting of a number of fields $Z_k(x_k)$ at different spacetime points. Our goal in this chapter, building on [61], is mainly to check whether the predictions due to the proposed Slavnov–Taylor identity are correct. With some gained confidence, one could later use the Ward–Takahashi identities towards constraining correlation functions in integrable planar gauge theories.

We would like to point out that analogous invariance results have been obtained in the context of scattering amplitudes starting with [75, 47]. Apart from the different kind of considered observables, the main difference between our analysis and previous studies lies in the underlying methods. Previous studies relied on existing expressions for scattering amplitudes or argued that constructions of scattering amplitudes preserve Yangian (dual conformal) symmetry. Further investigations used representations of scattering amplitudes in terms of advanced geometric concepts such as twistors, Graßmannians, etc. [94, 95, 96, 97]. In our analysis we merely rely on invariance of the action and the structure of planar Feynman diagrams.

Note that analogous conclusions on Yangian invariance of planar correlation functions have been drawn in [98] within an integrable bi-scalar theory [99]. The main difference w.r.t. this analysis is that the bi-scalar theory is not a gauge theory and the representation of level-zero (conformal) symmetry is purely linear. Therefore almost all of the complications related to planar gauge theories that we shall encounter do not apply in the simplified bi-scalar field theory model.

9.1 Propagators

The simplest non-trivial correlation functions involve two fields; at tree level they are the propagators. It is essential to understand the Ward–Takahashi identities corresponding to both local and bi-local generators because they will be needed for showing the identities for correlators with more than two legs. As the structure of propagators is rather simple, we will discuss them including the BRST terms right from the beginning. It is very easy to obtain the simplified results ignoring gauge fixing by putting all the $\mathcal{K}[J] = 0$.

Local symmetries. A first identity relating the ghost and auxiliary propagators comes from the Slavnov–Taylor identity (8.1.2) for BRST symmetry. We apply it to two fields $\mathcal{O} = Z_1Z_2$ at tree
and thus restrict to linear contributions to obtain
\[ \langle Q_{[0]} \cdot Z_1 Z_2 \rangle + \langle Z_1 Q_{[0]} \cdot Z_2 \rangle = 0. \]  
(9.1.1)
Now the only linear contributions to BRST variations are
\[ Q_{[0]} \cdot A_\mu = \partial_\mu C \quad \text{and} \quad Q_{[0]} \cdot \bar{C} = B, \]  
(9.1.2)
where the partial derivative is the 0-field part of the covariant derivative appearing in the full expression.

Considering that a correlator can be non-zero only if the overall ghost number is zero, a non-trivial statement follows only for \( \mathcal{O} = \bar{C}(x) A_\mu(y) \). It amounts to
\[ \langle B(x) A_\mu(y) \rangle = \langle \bar{C}(x) \partial_\mu C(y) \rangle = \frac{\partial}{\partial y^\mu} \langle \bar{C}(x) C(y) \rangle. \]  
(9.1.3)
Apart from relating the auxiliary to the ghost propagator, the relationship states that the auxiliary field contracts only to the gauge degrees of freedom within the gauge field \( A_\mu \). A simple corollary is that \( \langle B F_{\mu\nu} \rangle = 0 \) at tree level due to incompatible symmetrisations of the spacetime indices.

For a level-zero symmetry \( J \) we apply the Slavnov–Taylor identity (8.1.4) to two fields \( \mathcal{O} = Z_1 Z_2 \) and expand to tree level
\[ \langle J_{[0]} \cdot Z_1 Z_2 \rangle + \langle Z_1 J_{[0]} \cdot Z_2 \rangle = i \langle 1/2 K[J]_{[2]} Q_{[0]} \cdot Z_1 Z_2 \rangle + i \langle 1/2 K[J]_{[2]} Z_1 Q_{[0]} \cdot Z_2 \rangle. \]  
(9.1.4)
The l.h.s. of this equation is the symmetry variation of a propagator. The equation shows that the propagator respects the symmetry manifestly up to terms involving linear BRST variations of the fields. One can convince oneself that all terms in (9.1.4) are trivially zero when one of the fields is \( \bar{C} \). A non-vanishing r.h.s. for (9.1.4) thus requires one of the fields to be a gauge potential. Explicitly, if the other field is a gauge potential as well, we find with (7.4.10)
\[ \langle J_{[0]} \cdot A_\mu(x) A_\nu(y) \rangle + \langle A_\mu(x) J_{[0]} \cdot A_\nu(y) \rangle \]
\[ = i \frac{\partial}{\partial x^\mu} \int d^4 z \langle C(x) \partial^\rho \bar{C}(z) \rangle \langle J_{[0]} \cdot A_\mu(z) A_\nu(y) \rangle \]
\[ + i \frac{\partial}{\partial y^\nu} \int d^4 z \langle C(y) \partial^\rho \bar{C}(z) \rangle \langle A_\mu(x) J_{[0]} \cdot A_\rho(z) \rangle. \]  
(9.1.5)
and otherwise for a generic gauge-covariant field \( Z \)
\[ \langle J_{[0]} \cdot A_\mu(x) Z(y) \rangle + \langle A_\mu(x) J_{[0]} \cdot Z(y) \rangle = i \frac{\partial}{\partial x^\mu} \int d^4 z \langle C(x) \partial^\rho \bar{C}(z) \rangle \langle J_{[0]} \cdot A_\rho(z) Z(y) \rangle. \]  
(9.1.6)
We note that the extra terms are total derivatives and as such constitute gauge variations.

**Bi-local symmetries.** For the corresponding level-one Ward–Takahashi identity we recall from [60] that level-one invariance of the quadratic part of the action follows from the corresponding level-zero invariance together with the vanishing of the dual Coxeter number of the level-zero symmetries. Indeed, we can derive a relationship from the level-zero identity (8.1.4) for the generator

\( \bar{C} \) never produces the ghost \( C \), so \( \bar{C} \) cannot contract with any other field. Furthermore, \( J_{[0]} \) produces a gauge-covariant field, so it will not contract with \( B \sim Q_{[0]} \cdot \bar{C} \).
\[ J = J^{(1)} \] and the operator \( \mathcal{O} = A_\mu(x) J^{(2)} \cdot A_\nu(y) \). Since \( Z = J_{[0]} \cdot A_\nu \) is gauge covariant, we can read off the result from (9.1.6) as
\[
\langle \hat{J} \cdot (A_\mu(x) A_\nu(y)) \rangle = \langle J^{(1)}_{[0]} \cdot A_\mu(x) J^{(2)}_{[0]} \cdot A_\nu(y) \rangle = i \frac{\partial}{\partial x^\mu} \int d^4 z \langle C(x) \partial^\nu \bar{C}(z) \rangle \langle J^{(1)}_{[0]} \cdot A_\mu(z) J^{(2)}_{[0]} \cdot A_\nu(y) \rangle
\]
\[
= i \frac{\partial}{\partial y^\nu} \int d^4 z \langle C(y) \partial^\rho \bar{C}(z) \rangle \langle J^{(1)}_{[0]} \cdot A_\mu(x) J^{(2)}_{[0]} \cdot A_\nu(z) \rangle.
\] (9.1.7)

The second form follows in a similar fashion by exchanging the role of the two fields. Together the two identities imply that the correlator is a total derivative in both points
\[
\langle J^{(1)}_{[0]} \cdot A_\mu(x) J^{(2)}_{[0]} \cdot A_\nu(y) \rangle = - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} R[\hat{J}](x, y)
\] (9.1.8)
with the remainder function \( R[\hat{J}] \) taking the form
\[
R[\hat{J}](x, y) = \int d^4 z_1 d^4 z_2 \langle J^{(1)}_{[0]} \cdot A_\rho(z_1) J^{(2)}_{[0]} \cdot A_\sigma(z_2) \rangle \langle C(x) \partial^\rho \bar{C}(z_1) \rangle \langle C(y) \partial^\sigma \bar{C}(z_2) \rangle.
\] (9.1.9)

This function has been evaluated explicitly for \( \hat{J} = \hat{P} \) in \( N = 4 \) sYM in [79] with a very simple structure as \( R[\hat{P}]_\mu \sim (x - y)_\mu / (x - y)^2 \).

Let us point out one curiosity regarding the function \( R \): On the one hand, the expression (9.1.9) heavily depends on ghost field propagators and therefore manifestly depends on the gauge-fixing procedure. On the other hand, the function must be independent of the chosen gauge because the underlying correlator on the l.h.s. of (9.1.8) involves gauge-covariant fields only.

Considering more general gauge theories, we can speculate that the function \( R \) may be even more universal. This is based on the fact that gauge fields have mass dimension 1 and in (9.1.8) their vector index is used up for formulating the total derivatives. Consequently, the function \( R[\hat{J}] \) has the same mass dimension as the level-one generator \( \hat{J} \). If we furthermore expect it transform analogously to \( \hat{J} \), the only reasonable form appears to be
\[
R[\hat{J}] \sim (J_x - J_y) \log(x - y)^2,
\] (9.1.10)
where \( J_x \) is the level-zero generator corresponding to \( \hat{J} \) acting on a spacetime point \( x \) (rather than on fields), see also [100]. In the following we will not need the precise form for \( R[\hat{J}] \), but it would be desirable to understand better the above form (9.1.10) and whether there is a manifestly gauge-invariant way to derive it.

Let us finish by checking the bi-local Slavnov–Taylor identity (8.1.18) applied to two fields \( A_\mu(x) A_\nu(y) \) at tree level. Note that all compensators for bi-local operators are cubic at least, and consequently do not contribute at this level. For the remaining terms we find
\[
\langle \hat{J}_{[0]} \cdot (A_\mu(x) A_\nu(y)) \rangle = - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} R[\hat{J}](x, y),
\]
\[
-i \langle \frac{1}{2} \mathcal{K}[J^{(1)}]_{[2]} (Q \wedge J^{(2)})_{[0]} \cdot (A_\mu(x) A_\nu(y)) \rangle = 2 \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} R[\hat{J}](x, y),
\]
\[
\frac{1}{2} \langle \frac{1}{2} \mathcal{K}[J^{(2)}]_{[2]} \frac{1}{2} \mathcal{K}[J^{(1)}]_{[2]} (Q \otimes Q)_{[0]} \cdot (A_\mu(x) A_\nu(y)) \rangle = - \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu} R[\hat{J}](x, y).
\] (9.1.11)

Indeed, they sum to zero, which serves as a first confirmation for the identity (8.1.18).
9.1.1 Diagrammatic notation

Although the above computations were tractable, in view of the upcoming chapters, let us present them in a diagrammatic way. The defining property of propagators:

\[
\langle Z_1 S_2,1 \rangle \langle Z_2 S_2,2 \rangle = i \langle Z_1 Z_2 \rangle. \tag{9.1.12}
\]

reads

\[
\text{---} = i \hspace{1cm}, \tag{9.1.13}
\]

where the straight line is the propagator and the shaded circle is the insertion of the quadratic part of the action \( S_{[2]} \) — inverse of which is the propagator. Later on we will encounter also \( S_{[n]} \), and the notation as well as terms themselves should be familiar from previous chapters. Invariance of the propagator under BRST transformations takes the form (9.1.1)

\[
0 = \text{---} = i \hspace{1cm} + i \hspace{1cm}. \tag{9.1.14}
\]

In these diagrams the (red) triangle represents the BRST generator \( Q \). This will be used to tacitly shift BRST generators from one end of a propagator to the other

\[
\text{---} = - \hspace{1cm} \text{---}. \tag{9.1.15}
\]

Invariance of the propagator under level-zero transformations (9.1.4) consists of the following four diagrams (sometimes we will however ignore the effects of the gauge fixing to make the computations more tractable, see (9.2.9)):

\[
0 = \text{---} = \text{---} + \text{---} - \text{---} - \text{---} = i \hspace{1cm} + i \hspace{1cm} + \text{---} + \text{---}. \tag{9.1.16}
\]

Here, the (purple) rectangle represents the BRST compensator \( K[J] \) associated to the enclosed generator \( J \). The latter two terms require special attention due to exchange statistics of the BRST generator \( Q \) and the compensator \( K[J] \) which cannot be expressed well in the diagrams alone: Here we assume that the sign of a diagram is determined relative to the sequence \( \ldots K[J] \ldots Q \ldots \) in the corresponding mathematical expression. If an expression has the opposite ordering \( \ldots Q \ldots K[J] \ldots \), its diagram will receive an extra sign. This is in fact the reason for the negative sign for the terms in the first line which originate from the combination \( +Q \cdot K[J] \). Their sign in the second line is flipped due to (9.1.15).

9.2 Level-zero symmetries of three-point functions

Next, we will discuss Ward–Takahashi identities for correlators of three fields. As before, we start with a level-zero generator \( J \) to gain some experience in the occurring structures and required transformations. The Slavnov–Taylor identity (8.1.4) for three fields reads

\[
\langle J \cdot (Z_1 Z_2 Z_3) \rangle - i \langle K[J] Q \cdot (Z_1 Z_2 Z_3) \rangle = 0. \tag{9.2.1}
\]

When expanding out in the number of fields, we find the following contributions

\[
0 = \langle J_{[0]} \cdot (Z_1 Z_2 Z_3) \rangle_{[1]} + \langle J_{[1]} \cdot (Z_1 Z_2 Z_3) \rangle_{[0]}
\]
where \( \langle \ldots \rangle_{[n]} \) represents a correlator with \( n \) three-vertices inserted (we may suppress this notation for correlators without vertices). The former two terms represent the ordinary (non-linear) variation of the three-point function, while the latter three compensate for effects due to gauge fixing. As before, terms involving \( Q_{[0]} \) produce total derivatives when acting on external gauge fields and nothing otherwise. Terms involving \( Q_{[1]} \) yield a new kind of contribution which is not a total derivative, but which is implied by proper non-linear gauge transformations.

By formally writing this expression in terms of propagators and vertices and by using the symmetry of propagators derived in 9.1, one can rewrite the above expression as

\[
0 = -i\langle \frac{1}{3}J_{[0]} \cdot S_{[3]} Z_1 Z_2 Z_3 \rangle + i\langle \frac{1}{2}J_{[1]} \cdot S_{[2]} Z_1 Z_2 Z_3 \rangle + \langle \frac{1}{3}Q_{[0]} \cdot K_{[J]}[3] Z_1 Z_2 Z_3 \rangle + \langle \frac{1}{2}Q_{[1]} \cdot S_{[2]} \frac{1}{2}K_{[J]}[2] Z_1 Z_2 Z_3 \rangle. \tag{9.2.3}
\]

These terms are easily combined to

\[
- i\langle \frac{1}{3}(J \cdot S + Q \cdot K_{[J]}[3]) Z_1 Z_2 Z_3 \rangle + \langle \frac{1}{2}(Q \cdot S)[3] \frac{1}{2}K_{[J]}[2] Z_1 Z_2 Z_3 \rangle, \tag{9.2.4}
\]

both of which are zero due to level-zero and BRST symmetry of the action, respectively, see 7.4.1. This confirms that the above Ward–Takahashi identity \( 9.2.1 \) holds at tree level by means of elementary transformations and symmetries of the action.

Unfortunately, the above transformation requires a lot of patience and care. Let us therefore present some useful identities and walk through a simplified transformation where we ignore all terms due to gauge fixing. We will also expand and use the diagrammatical representation for the arising terms which helps to visually understand the confirmation of the identity.

Some useful identities in transforming the expressions are as follows: A correlator of three fields at tree level requires the insertion of a cubic vertex

\[
\langle Z_1 Z_2 Z_3 \rangle_{[1]} = i\langle Z_1 S_{[3],2} \rangle_{[0]} \langle Z_2 S_{[3],1} \rangle_{[0]} \langle Z_3 S_{[3],3} \rangle_{[0]} = i. \tag{9.2.5}
\]

The central vertex in the diagram corresponds to the cubic part \( S_{[3]} \) of the action. Note that its symmetry factor \( \frac{1}{3} \) is compensated by three sets of contractions which are equivalent due to the cyclic symmetry of the action. Here we have restricted to planar contributions which allows contractions between the external fields and the fields of the vertex in opposite ordering.

Second, the non-linear contribution of a symmetry generator on an external field at tree level requires no further vertex and it splits up into two propagators

\[
\langle J_{[1]} \cdot Z_1 Z_2 Z_3 \rangle = \langle (J_{[1]} \cdot Z_1)_2 Z_2 \rangle \langle (J_{[1]} \cdot Z_1)_1 Z_3 \rangle = \frac{1}{2}. \tag{9.2.6}
\]
The (blue) semi-circle in the diagram represents the level-zero symmetry generator $J$. It acts on the (single) field which is on the straight side, and yields (one or several) fields on the circular side.

Using the above rules, the symmetry variation of the three-field correlator can be transformed step-by-step as follows (for simplicity first ignoring the gauge fixing terms; we will reinstall them in the next Paragraph)\footnote{The reduction of prefactors from 3 to $1/3$ and $1/2$ in the second but last line is due to two effects: $J_{[3-n]} \cdot S_{[n]}$ produces $n$ equivalent terms and there are 3 equivalent contractions with the external fields up to cyclic permutations.}

\[
\langle J \cdot (Z_1 Z_2 Z_3) \rangle \simeq 3 \langle J_{[0]} \cdot Z_1 Z_2 Z_3 \rangle + 3 \langle J_{[1]} \cdot Z_1 Z_2 Z_3 \rangle \\
\simeq 3i \langle J_{[0]} \cdot Z_1 S_{[3,2]} \rangle \langle Z_2 S_{[3,1]} \rangle \langle Z_3 S_{[3,3]} \rangle + 3 \langle (J_{[1]} \cdot Z_1) Z_2 \rangle \langle (J_{[1]} \cdot Z_1) Z_3 \rangle \\
\simeq -3i \langle Z_1 J_{[0]} \cdot S_{[3,2]} \rangle \langle Z_2 S_{[3,1]} \rangle \langle Z_3 S_{[3,3]} \rangle \\
-3i \langle S_{[2,1]} Z_1 \rangle \langle (J_{[1]} \cdot S_{[2,2]}) Z_2 \rangle \langle (J_{[1]} \cdot S_{[2,2]}) Z_3 \rangle \\
\simeq -i \langle Z_1 Z_2 Z_3 \frac{1}{3} J_{[0]} \cdot S_{[3]} \rangle - i \langle Z_1 Z_2 Z_3 \frac{1}{3} J_{[1]} \cdot S_{[2]} \rangle \\
= -i \langle Z_1 Z_2 Z_3 \frac{1}{3} (J \cdot S)_{[3]} \rangle = 0. \tag{9.2.7}
\]

The sign ‘$\simeq$’ here and in the following implies to equivalence modulo cyclic permutations of the three external fields. Using diagrams we can write the first class of terms as

\[
\langle J_{[0]} \cdot (Z_1 Z_2 Z_3) \rangle = i \quad \begin{array}{c}
\text{3} \\
2
\end{array} + i \quad \begin{array}{c}
3 \\
2
\end{array} + i \quad \begin{array}{c}
3 \\
1
\end{array} = -i \quad \begin{array}{c}
3 \\
2
\end{array} - i \quad \begin{array}{c}
3 \\
2
\end{array} - i \quad \begin{array}{c}
3 \\
1
\end{array}. \tag{9.2.8}
\]

The transformation towards the second line makes use of the linearised symmetry of the propagators discussed in \ref{9.1}

\[
\begin{array}{c}
\text{J} \\
2
\end{array} = - \begin{array}{c}
\text{J} \\
2
\end{array}. \tag{9.2.9}
\]

The second class of terms corresponds to the diagrams

\[
\langle J_{[1]} \cdot (Z_1 Z_2 Z_3) \rangle = \begin{array}{c}
\text{3} \\
2
\end{array} + \begin{array}{c}
3 \\
2
\end{array} + \begin{array}{c}
3 \\
1
\end{array} = -i \quad \begin{array}{c}
3 \\
2
\end{array} - i \quad \begin{array}{c}
3 \\
2
\end{array} - i \quad \begin{array}{c}
3 \\
1
\end{array}. \tag{9.2.10}
\]
Combining all terms, we find

\[ \langle J_0 \cdot (Z_1 Z_2 Z_3) \rangle_{[1]} + \langle J_1 \cdot (Z_1 Z_2 Z_3) \rangle_{[0]} = -i \langle \frac{1}{3} (J \cdot S)[3] Z_1 Z_2 Z_3 \rangle_{[0]} = 0, \quad (9.2.11) \]

where the (green) vertex with embedded symmetry generator represents the variation of the action w.r.t. this symmetry. Such vertices represent a sum of terms which is zero by invariance of the action under this symmetry, \( J \cdot S = 0 \). This proves the level-zero Ward–Takahashi identity modulo gauge-fixing terms at the level of diagrams.

Let us now use the diagrammatic technique to demonstrate that also the Ward–Takahashi identity with the gauge-fixing terms included \((8.1.4)\) holds.

**BRST symmetry.** We first consider the identity for BRST symmetry. As BRST symmetry itself does not require compensators, the consideration is equivalent to \(9.2\) without gauge fixing where \( J \) is replaced by \( Q \). In short, invariance of the three-point function follows using invariance of the propagator \((9.1.1)\) as

\[ \langle Q \cdot O \rangle \approx 3i \begin{array}{c} 3 \\ \end{array} = -3i \begin{array}{c} 3 \\ \end{array} - 3i \begin{array}{c} 3 \\ \end{array} \approx -i \begin{array}{c} 3 \\ \end{array} = -i \langle \frac{1}{3} (Q \cdot S)[3] O \rangle = 0. \quad (9.2.12) \]

**Level-zero symmetry.** Next we discuss the level-zero Ward–Takahashi identity for three external fields \((9.2.2)\) using diagrams. It consists of the following terms (for convenience, we identify the diagrams with labels a–k)

\[ \langle J_0 \cdot O \rangle \approx 3i \begin{array}{c} a \\ \end{array}, \]

\[ \langle J_1 \cdot O \rangle \approx 3 \begin{array}{c} b \\ \end{array}, \]

\[ -i \langle \frac{1}{3} K[J][3] Q_0 \cdot O \rangle \approx 3i \begin{array}{c} c \\ \end{array}, \]

\[ -i \langle \frac{1}{2} K[J][2] Q_1 \cdot O \rangle \approx -3i \begin{array}{c} d \\ \end{array} - 3i \begin{array}{c} e \\ \end{array}, \]

\[ -i \langle \frac{1}{2} K[J][2] Q_0 \cdot O \rangle \approx 3 \begin{array}{c} f \\ \end{array} - 3 \begin{array}{c} g \\ \end{array} - 3 \begin{array}{c} h \\ \end{array}. \quad (9.2.13) \]
In diagrams c, g, h we have used the linearised BRST symmetry of propagators (9.1.15) to shift the BRST generator to the central vertex.

We can transform diagrams a and f by adding the linearised level-zero invariance of the gauge-fixed propagator (9.1.16)

\[
0 = -3 i \simeq -3 i
\]

(9.2.14)

This effectively replaces them by diagrams i and j with the opposite sign. Then we add the BRST invariance of the action in (9.2.12) together with insertion of a spectator vertex \( K[J] \) to cancel most of the terms related to gauge fixing

\[
0 = 3 i = \langle \frac{1}{2} K[J] \rangle_{2} \frac{1}{3} (Q \cdot S)_{[3]} \langle J \rangle
\]

(9.2.15)

For diagrams d, e and k we have removed a quadratic vertex by means of (9.1.13). The remaining 4 terms represent the level-zero invariance of the action at three fields with gauge fixing. We add the corresponding transformation of the action

\[
0 = i \langle J \cdot O \rangle = i \langle \frac{1}{2} (J \cdot S + Q \cdot K[J])_{[3]} \langle J \rangle \rangle
\]

(9.2.16)

For diagram b we have again removed a quadratic vertex. Note that the ordering of \( K[J] \) and \( Q \) in \( Q \cdot K[J] \) is opposite to the ordering in all the above expressions, hence the corresponding diagrams receive an extra sign flip. When adding up all the diagrams, we finally get zero

\[
\langle J \cdot O \rangle - i \langle K[J] Q \cdot O \rangle = 0.
\]

(9.2.17)

### 9.3 Bi-local symmetries of three-point functions

In the following we will discuss the Ward–Takahashi identities corresponding to bi-local symmetries for three external fields \( O := \text{tr}(Z_{1} Z_{2} Z_{3}) \). Again, we will work modulo cyclic permutations of the external fields by means of the equivalence relation ‘\( \simeq \)’. 
9.3. BI-LOCAL SYMMETRIES OF THREE-POINT FUNCTIONS

Yangian symmetry without gauge-fixing  As in case of the level-0 symmetries, we will also here start in a simplified setting, where we ignore gauge-fixing. This amounts to setting \( K[J] = 0 \).

In order to streamline the calculation, we first define a manifestly cyclic variant \((J^1 \wedge J^2)\)′ of the bi-local generator \(J^1 \wedge J^2\) on the external fields \(O_{[n]} := Z_1 \cdots Z_n\):

\[
(J^1 \wedge J^2)' \cdot O_{[n]} := (J^1 \wedge J^2) \cdot O_{[n]} + \sum_{k=1}^{n} \frac{n+1-2k}{n} (J^1 \cdot (J^2_k O_{[n]}) - J^2 \cdot (J^1_k O_{[n]}) - [J^1_k, J^2_k] O_{[n]})
\]

\[
= \sum_{k=1}^{n} (J^1 \wedge J^2)_k O_{[n]} + \sum_{k=1}^{n} \sum_{j=1}^{n-1} \frac{2j-n}{n} J^1_{k+j} J^2_k O_{[n]}.
\] (9.3.1)

It differs from the original definition by terms which contain the commutator of two local generators \([J^1, J^2] = 0\) as well as terms which are local transformations of some combinations of fields. It will be consistent to use the above definition of bi-local actions on fields within the Slavnov–Taylor identity (8.1.18). The additional terms cancel against each other upon use of the corresponding local identity (8.1.4). Due to manifest cyclicity it now makes sense to compare modulo cyclic permutations

\[
(J^{(1)} \otimes J^{(2)})' \cdot O_{[n]} \simeq n(J^{(1)} \otimes J^{(2)})_1 O_{[n]} + \sum_{j=1}^{n-1} (j - \frac{1}{2} n) J^{(1)}_{j+1} J^{(2)}_1 O_{[n]}.
\] (9.3.2)

For three external fields this expression reduces to

\[
\langle \hat{J}' \cdot (Z_1 Z_2 Z_3) \rangle \simeq 3 \langle Z_1 Z_2 \hat{J} \cdot Z_3 \rangle + \langle J^{(1)} \cdot Z_1 J^{(2)} \cdot Z_2 Z_3 \rangle.
\] (9.3.3)

Upon evaluation of the correlators at tree level, we find the following diagrams

\[
\langle \hat{J}' \cdot (Z_1 Z_2 Z_3) \rangle \simeq 3 \hspace{1cm} + i \hspace{1cm} + i \hspace{1cm} + i .
\] (9.3.4)

The double semi-circle represents the local contribution to the level-one generator \(\hat{J}\) (with no linear contribution), while the single semi-circles without and with decoration (dot) correspond to the level-zero generators \(J^{(1)}\) and \(J^{(2)}\), respectively.

\[
\hat{J} \langle Z_1 Z_2 Z_3 \rangle \simeq -3i \hspace{1cm} + i \hspace{1cm} + i \hspace{1cm} - i ,
\]

\[
\simeq -i \langle \frac{1}{3} (\hat{J} \cdot S)[3] Z_1 Z_2 Z_3 \rangle.
\] (9.3.5)

---

3Here, \(J_k\) is the generator \(J\) acting on the field(s) at the \(k\)-th site of the correlator (rather than the \(k\)-th field within the polynomial). It is the fully non-linear generator which may change the length of the polynomial.

4This perfect cancellation can be viewed as a mild verification of the form (8.1.18) of the bi-local Slavnov–Taylor identity.
The combination of insertions is proportional to the cubic combination of local terms which arises in Yangian invariance of the action (6.3.25)

\[
(J \cdot S)[3] \simeq 3(J_{[1,1]} S_{[2]} + J_{[0,2]}^2 J_{[0,1]}^1 S_{[3]} + (J_{[0,1]}^2 - J_{[0,2]}^1) J_{[1,1]}^1 S_{[2]} \simeq 0.
\]

(9.3.6)

Note that the natural ordering for the fields on the insertion is opposite to the ordering of external fields. In fact, this reversal of ordering is necessary to make the signs match up: Shifting a local term from the external legs to the internal vertices typically flips the sign. Shifting a bi-local term, however, induces one sign flip for each propagator. Hence, an extra sign flip is needed to match the sign of the transformation of local terms. It arises due to changing the order in combination with the anti-symmetry of the bi-local terms. The latter anti-symmetry also implies that the interchange of \(J^{(1)}\) and \(J^{(2)}\) is equivalent to a flip of sign.

In conclusion, the level-one Ward–Takahashi identity for correlators of three fields is equivalent to the cubic contribution to the level-one symmetry of the action (modulo gauge fixing).

Finally, we would like to point out the structural difference between the bi-local action on a collection of external fields (9.3.1) and the one on cyclic polynomials (6.3.25) such as the action \(S\). Both expressions have similar terms with similar coefficients, but their details differ. In particular, the former derives directly from the bi-local action on open polynomials (6.1.7), and consequently there are no overlapping terms. We have seen above that both types of expressions are indeed related by the Slavnov–Taylor identity (8.1.18), and it seems natural to apply these particular actions for each of the objects that are acted upon. Curiously, the conjectured bi-local variational identity (8.1.5) which was used to derive the identity (8.1.18) seems to implicitly translate between the two forms without making direct reference to either. It would be good to understand this issue better.

Bi-local BRST symmetry. We now move towards including the effects of gauge-fixing. For bilocal generators this is substantially more involved, hence on the way we will perform the intermediate steps of analyzing \(Q \otimes Q\) and \(Q \wedge J\) generators. We start with the bi-local BRST generator \(Q \otimes Q\) introduced in 7.4.2. Since the BRST component operators \(Q\) have no compensators \(K[Q]\), the derivation is identical to the one of the behavior of the 3-point function under local symmetry, together with a proper treatment of the gauge-fixed local contributions \(K[Q \otimes Q]\) (it is only the constituents which require no compensators, not the composite bilocal operator). The Ward–Takahashi identity for \(Q \otimes Q\) then reads

\[
\langle (Q \otimes Q) \cdot \mathcal{O} \rangle - i \langle K[Q \otimes Q] Q \cdot \mathcal{O} \rangle = 0.
\]

(9.3.7)

The relevant invariance of the action takes the form

\[
0 = \langle \frac{1}{3} ((Q \otimes Q) \cdot S + Q \cdot K[Q \otimes Q]) \rangle = \langle \frac{1}{3} ((Q \otimes Q) \cdot S + Q \cdot K[Q \otimes Q]) \rangle \mathcal{O} \rangle.
\]

(9.3.8)

5 The different forms of applicable bi-local actions may be related to the different quantum field theory functionals underlying the different objects: Correlation functions are based on the partition function (or its connected component in view of the planar limit) which is a functional of field sources. Conversely, the action (and likewise the effective action) is a functional of the fields themselves. It is conceivable that the Legendre transformation, which translates between these two types of objects, naturally maps one type of bi-local representation on cyclic objects to the other (while there is no qualitative change for local representations).
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\[
\simeq 3i \quad - \quad \quad -i \quad +i \quad -3 \quad .
\]

**Mixed bi-local symmetry.** Now we will verify the mixed symmetry \( Q \wedge J \) introduced in \[7.4.2\] including the gauge-fixing corrections caught by the Slavnov–Taylor identity \( (8.1.15) \)

\[
\langle (Q \wedge J) \cdot \mathcal{O} + i\mathcal{K}[J] (Q \otimes Q) \cdot \mathcal{O} \rangle \\
+ \langle (-i\mathcal{K}[Q \wedge J] + \mathcal{K}[J] \mathcal{K}[Q \otimes Q]) Q \cdot \mathcal{O} - i\mathcal{K}[Q \otimes Q] J \cdot \mathcal{O} \rangle = 0. \quad (9.3.9)
\]

This Ward–Takahashi identity consists of five different contributions totalling 22 diagrams. In order to streamline the presentation, we will tacitly remove quadratic vertices by means of \( (9.1.13) \) and shift BRST generators \( Q \) towards the centre of the diagram by means of \( (9.1.15) \). The diagrams turn out to be

\[
\langle (Q \wedge J) \cdot \mathcal{O} \rangle \simeq 3 \quad +i \quad -i \quad ,
\]

\[
i\langle \mathcal{K}[J] (Q \otimes Q) \cdot \mathcal{O} \rangle \simeq 3i \quad +3i \quad +i \quad ,
\]

\[
- \quad \quad + \quad - \quad - \quad ,
\]

\[
+ i \quad - i \quad - i \quad + i \quad ,
\]

\[
- i \langle \mathcal{K}[Q \wedge J] Q \cdot \mathcal{O} \rangle \simeq 3i \quad ,
\]

\[
\langle \mathcal{K}[J] \mathcal{K}[Q \otimes Q] Q \cdot \mathcal{O} \rangle \simeq -3 \quad -3 \quad +3 \quad ,
\]

\[
- i \langle \mathcal{K}[Q \otimes Q] J \cdot \mathcal{O} \rangle \simeq -3i \quad . \quad (9.3.11)
\]
Some remarks are in order: For the first set of diagrams with the regular application of the generator $Q \wedge J$ (as opposed to the earlier cases of $\hat{J} = J^{(1)} \otimes J^{(2)}$ and $Q \otimes Q$) we note that the two underlying generators, $Q$ and $J$, are of different types and we have to explicitly consider both orderings.

Another subtlety concerns the signs due to statistics of the operators. Diagrams of the second term involve several fermionic operators $Q$ and terms $K[J]$ whose ordering matters. We assume the reference ordering within mathematical expressions to be $(K[J], Q, \hat{Q})$ where $\hat{Q}$ corresponds to the decorated BRST operator (triangle with dot) in the diagrams but otherwise acts as an ordinary BRST operator $Q$. Likewise, the reference ordering for fermionic operators $Q$ and terms $K[J], K[Q \otimes Q]$ within diagrams in the fourth term is assumed to be $(K[J], K[Q \otimes Q], Q)$.

Next we would like to shift level-zero generators $J$ acting on external fields towards the centre of the diagrams $b_1, c_1, d_1, e_1$ and $q_1$. To this end we add a couple of terms each of which is zero using the extended invariance relation (9.1.16) of the gauge-fixed propagator

\[ 0 = -3 + 3i - 3 + 3i - 3 + 3i - 3 . \] (9.3.12)

Effectively this replaces diagrams $b_k, c_k, d_k, e_k$ and $q_k$ with $k = 1, 2$ by the corresponding diagrams with $k = 3, 4$. Note that the signs of the diagrams with $k = 2, 4$ are superficially different from the underlying relation (9.1.16). This is because the fermionic operators and terms are ordered as $(Q, K[J], \hat{Q})$ and $(K[Q \otimes Q], K[J], Q)$. Therefore they require one elementary permutation to be brought to the assumed ordering corresponding to a sign flip.

Furthermore, we add the following combination of terms to our set of diagrams which is zero by means of the invariance of the gauge-fixed action under $Q \otimes Q$ (9.3.8)

\[ 0 = -3 = -\langle K[J] \frac{1}{3} ((Q \otimes Q) \cdot S + Q \cdot K[Q \otimes Q])_{[3]} O \rangle \]
\[ \tag{9.3.13} \]

Note that we need to explicitly average over all cyclic permutation in (9.3.8) due to the extra vertex \( \mathcal{K}[J] \) which breaks this symmetry.

Finally, we add the invariance of the gauge-fixed action under (7.4.24)

\[ \tag{9.3.14} \]

with the four contributions

\[ \tag{9.3.15} \]

\[ -i \langle \frac{1}{3} (Q \cdot \mathcal{K}[Q \wedge J])_{[3]} \mathcal{O} \rangle \approx -3i \]
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\[ i \left\langle \frac{1}{3} ((Q \otimes Q) \cdot \mathcal{K}[J])[3] \mathcal{O} \right\rangle \simeq 3i \begin{array}{c}
 r \\
 -i \\
 -i \\
 +i \\
 t
\end{array} \] \quad (9.3.16)

Altogether we find that all terms cancel. More explicitly, every one of the 35 distinct diagrams is labelled by a letter, and it appears twice with equal but opposite coefficients. This shows that the Ward–Takahashi identity \((9.3.9)\) indeed holds in this case.

What is more, observe that the bi-local correlator

\[ -i \left\langle (J \cdot S + Q \cdot \mathcal{K}[J]) \wedge (Q \cdot \mathcal{O}) \right\rangle, \quad (9.3.17) \]

for which we have merely provided a superficial description in \(8.1.2\) is apparently represented truthfully by diagrams \(b, c, d, e\) in \((9.3.12)\). All of this gives us some confidence that the considerations in \(8.1\) apply indeed, and that we can trust the Slavnov–Taylor identities for bi-local symmetries.

Yangian symmetry. We have also performed the corresponding calculation for invariance of the gauge-fixed three-point correlator under a level-one Yangian generator. Unfortunately, it involves a substantial increase in combinatorics compared to the previous calculations due to the various types of elements that contribute. For example, the Ward–Takahashi identity \((8.1.18)\) expands to 54 diagrammatical terms. Let us nevertheless sketch how to show also its validity. It will turn out that the answer lies precisely in the result \((8.1.17)\) which we recall here:

\[
0 = \langle \hat{J} \mathcal{O} + i (Q \wedge J^{(2)}) \mathcal{O} \mathcal{K}[J^{(1)}] - \frac{1}{2} (Q \otimes Q) \mathcal{O} \mathcal{K}[J^{(2)}] \mathcal{K}[J^{(1)}] - \frac{i}{2} [Q \mathcal{K}(Q \otimes Q) \mathcal{K}[J^{(1)}] - J^{(1)} \mathcal{K}[Q \wedge J^{(2)}]) \mathcal{O} \rangle
\]

The aforementioned 54 diagrams are a result of drawing the first part, which is the Slavnov–Taylor identity one would be using in a real-life computation \((8.1.18)\).

Now, in order to cancel those diagrams we add terms which are equal to 0, as we already did before. Previously, we have been adding them in a way which might have resembled educated
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However, the results of the last section on \( Q \wedge J \) can serve as an inspiration, since we observed that some terms combine into what looks like:

\[
-i \langle (J \cdot S + Q \cdot K[J]) \wedge (Q \cdot O) \rangle,
\]

which is a vanishing term we added in \([8.1.13]\) (even though we did not know how to precisely define it). In the end what we realize is that the pictures we added correspond precisely to the terms used in establishing the Slavnov–Taylor identity. It turns out that instead of educated guesswork presented in the preceding computations, we can use the results of Chapter 8 in order to perform the computations in a more controlled, almost algorithmic way, thus demonstrating the validity of Slavnov–Taylor identities.

The number of pictures contributing to \([8.1.17]\) is humongous and not too much benefit comes from presenting all of them; let us nevertheless observe some beautiful examples of multi-term cancellations on a few kinds of diagrams.

The first one will be terms with a 3-vertex inserted and generators \( J^{(1,2)} \) acting linearly. We have already encountered this term in the computation of a non-gauge-fixed theory, \([9.3.4]\). There we just pulled the generators towards the central vertex, using the invariance of the propagators. We are no longer allowed to do it (due to gauge fixing), and hence we need to find terms which will produce exactly this diagram, albeit with an opposite sign. Looking at the formula \([8.1.17]\) we immediately see what we need to do:

\[
-i \langle \hat{J} \cdot O + \cdots \rangle \simeq -i \left( \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \right),
\]

\[
-i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) + i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) + \cdots
\]

\[
+i \langle \hat{J} \cdot S + \cdots O \rangle \simeq -i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) + \cdots
\]

\[
+i \langle J^{(1)} \cdot O \wedge (J^{(2)} \cdot S + \cdots) \rangle \simeq -i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) - 2i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) - i \left( \begin{array}{c}
\begin{array}{c}
\end{array} \right) + \cdots,
\]

where the factor of 2 comes from both possible orderings of \( J^{(1,2)} \). We see that those terms precisely cancel.

Another interesting cancellation comes about when considering terms featuring compensators of both superconformal generators \( J^{(1,2)} \), \( K[J^{(1)}]K[J^{(2)}] \), possibly with the action of \( Q \) on either of them. These terms are purely due to gauge fixing but also clearly of a bilocal origin and it will be interesting to see how they cancel within our Slavnov–Taylor identity.
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\[ -\frac{1}{2} \left\langle (Q \otimes Q) \cdot O \right\rangle_{K[J^{(2)}]K[J^{(1)}]} \simeq i + \cdots \]

\[ -\frac{i}{2} \left\langle (Q \otimes S) O K[J^{(2)}]K[J^{(1)}] \right\rangle \simeq -i + \cdots \]

\[ -\left\langle (Q O \wedge Q) K[J^{(2)}]K[J^{(1)}] \right\rangle \simeq + i + \cdots \]

(9.3.20)

Using the BRST invariance of the propagator to change which field Q acts on we again see that all the terms perfectly cancel. The factors of \( \frac{1}{2} \) disappear due to sum implicit in Sweedler’s notation.

We can do the similar computation for all the remaining classes of terms obtaining perfect cancellations and thus verifying the Slavnov–Taylor identities proposed.

9.4 Level-1 symmetry of the four-point function

We can use the very same Slavnov–Taylor identity to show the invariance of the four-point function under Yangian symmetry. The distinguishing feature in comparison to the three-point function will be the appearance of the inner propagator in some of the graphs, just like in the famous S-channel diagram:

\[ \left\langle Z_1 Z_2 Z_3 Z_4 \right\rangle \simeq + \cdots . \]

Unlike in the field of Scattering Amplitudes, where the external legs of the graph are on-shell (see [101] for an overview of on-shell methods in this context), the Correlation Functions are completely off-shell. Therefore the inner propagator satisfies the same identities as the outside ones. In the end then, the computation for the four-point function is of course more involved than for the three-point one, but this is only due to the number of the terms - or more precisely, more possible topologies - and not any novel features. Nevertheless, we will only perform it without the gauge fixing terms.

Again, we start with the level-one generator acting on four fields modulo cyclic permutations

(9.3.2)

\[ \left\langle \hat{J}' \cdot (Z_1 Z_2 Z_3 Z_4) \right\rangle \simeq 4 \left\langle \hat{J} \cdot Z_1 Z_2 Z_3 Z_4 \right\rangle + 2 \left\langle J^{(1)} \cdot Z_1 J^{(2)} \cdot Z_2 Z_3 Z_4 \right\rangle. \]

(9.4.1)
As mentioned many times, all our results are obtained for $N \to \infty$, but here this limit is most easily observed. Indeed, the non-planar $u$ channel diagram pictured in Figure 9.1 will not appear in the expansion of (9.4.1):

\[ \text{Figure 9.1: The } u\text{-channel contribution to the 4-point function which is suppressed in the planar limit.} \]

Expanding the Slavnov–Taylor identity (9.4.1) thus in terms of only planar diagrams, we find the following terms at tree level

\[
\langle \hat{J} \cdot (Z_1 Z_2 Z_3 Z_4) \rangle \simeq -2 \begin{array}{c}
\text{Diagram 1}
\end{array} - 2 \begin{array}{c}
\text{Diagram 2}
\end{array} + 2i \begin{array}{c}
\text{Diagram 3}
\end{array}
\]

\[
+ 2i \begin{array}{c}
\text{Diagram 4}
\end{array} - 2i \begin{array}{c}
\text{Diagram 5}
\end{array} - 2i \begin{array}{c}
\text{Diagram 6}
\end{array} + 2i \begin{array}{c}
\text{Diagram 7}
\end{array}
\]

\[
+ 2 \begin{array}{c}
\text{Diagram 8}
\end{array} + 4i \begin{array}{c}
\text{Diagram 9}
\end{array} + 4i \begin{array}{c}
\text{Diagram 10}
\end{array}.
\]

(9.4.2)

Here we have used the level-zero symmetry of the propagator to bring the individual terms to some standard form which makes them easier to compare.

To show the Yangian invariance of the four point function we now add a carefully balanced sum of terms to (9.4.2) which all involve a composite cubic interaction vertex which is zero by means of level-zero or level-one symmetry (see 9.2 and 9.3) or the composite operator representing the commutator $J^{(1)} J^{(2)} = 0$

\[
\begin{array}{c}
\text{Diagram 11}
\end{array} + \begin{array}{c}
\text{Diagram 12}
\end{array} + \begin{array}{c}
\text{Diagram 13}
\end{array} + \begin{array}{c}
\text{Diagram 14}
\end{array} + i \begin{array}{c}
\text{Diagram 15}
\end{array} + i \begin{array}{c}
\text{Diagram 16}
\end{array} + i \begin{array}{c}
\text{Diagram 17}
\end{array} - i \begin{array}{c}
\text{Diagram 18}
\end{array} - i \begin{array}{c}
\text{Diagram 19}
\end{array} - i \begin{array}{c}
\text{Diagram 20}
\end{array} + i \begin{array}{c}
\text{Diagram 21}
\end{array}.
\]

\[
\begin{array}{c}
\text{Diagram 22}
\end{array} - \begin{array}{c}
\text{Diagram 23}
\end{array} - \begin{array}{c}
\text{Diagram 24}
\end{array} - \begin{array}{c}
\text{Diagram 25}
\end{array} - \begin{array}{c}
\text{Diagram 26}
\end{array} - \begin{array}{c}
\text{Diagram 27}
\end{array} - \begin{array}{c}
\text{Diagram 28}
\end{array} + i \begin{array}{c}
\text{Diagram 29}
\end{array} + i \begin{array}{c}
\text{Diagram 30}
\end{array} + i \begin{array}{c}
\text{Diagram 31}
\end{array} + i \begin{array}{c}
\text{Diagram 32}
\end{array} + i \begin{array}{c}
\text{Diagram 33}
\end{array} + i \begin{array}{c}
\text{Diagram 34}
\end{array} + i \begin{array}{c}
\text{Diagram 35}
\end{array} - i \begin{array}{c}
\text{Diagram 36}
\end{array} - i \begin{array}{c}
\text{Diagram 37}
\end{array} - i \begin{array}{c}
\text{Diagram 38}
\end{array} - i \begin{array}{c}
\text{Diagram 39}
\end{array} - i \begin{array}{c}
\text{Diagram 40}
\end{array} - i \begin{array}{c}
\text{Diagram 41}
\end{array} + i \begin{array}{c}
\text{Diagram 42}
\end{array}.
\]

(9.4.3)
In writing these terms, we have again made use of simple identities introduced in 9.2 and 9.3, such as level-zero symmetry of propagators, removal of quadratic vertices, vanishing of the dual Coxeter number, implicit anti-symmetry of the level-zero generators and cyclic permutations. By summing up all terms, we find that almost all diagrams cancel.

We arrive at a collection of terms with a central quartic vertex which match precisely with the quartic term in the invariance of the action (6.3.25)

\[
(J \cdot S)_{[4]} \simeq 4J_{[1,1]} S_{[3]} + 2J^{(2)}_{[0,2]} J^{(1)}_{[0,1]} S_{[4]} + (J^{(2)}_{[1,1]} - J^{(2)}_{[1,2]}) J^{(1)}_{[1,1]} S_{[2]} + (J^{(2)}_{[0,1]} - J^{(2)}_{[0,2]} + J^{(2)}_{[0,3]} - J^{(2)}_{[0,4]}) J^{(1)}_{[1,1]} S_{[3]} \simeq 0.
\]

(9.4.4)

We can subtract these terms

\[
0 = i \langle \hat{J} \cdot S_{[4]} Z_1 Z_2 Z_3 Z_4 \rangle
\]

\[
\simeq 4i \langle \hat{J} \cdot S_{[2]} - 2i \hat{J} \cdot S_{[2]} + \hat{J} \cdot S_{[2]} \rangle - \langle \hat{J} \cdot S_{[2]} \rangle
\]

\[
+ i \langle \hat{J} \cdot S_{[2]} \rangle - i \langle \hat{J} \cdot S_{[2]} \rangle - i \langle \hat{J} \cdot S_{[2]} \rangle + i \langle \hat{J} \cdot S_{[2]} \rangle
\]

(9.4.5)

and find that all terms cancel

\[
\langle \hat{J}' \cdot (Z_1 Z_2 Z_3 Z_4) \rangle = 0.
\]

(9.4.6)

This proves the level-one Ward–Takahashi identity for four external fields modulo gauge-fixing terms.
Chapter 10

The fate of Yangian symmetry at quantum level

All the results obtained in the previous chapters have been classical: either we have been working with the classical action of the theory, or we have investigated tree-level contributions to the correlation functions. Though they are certainly novel, one would like to extend them to the quantum field theory. The appearance of loops in Feynman diagrams makes the discussion however much more obscure. This is due to the fact that loop computations involve integrals which are divergent and hence need to be regularized. All the regularization procedures break (at least superficially) some of the symmetries: cut-off regularization breaks relativistic invariance, dimensional regularization most often will break supersymmetry (However see [102] for discussion). This chapter is based on [61].

10.1 Loop effects

At loop level many new issues may come into play: Proper treatment of loops in correlation functions will require renormalisation. Likewise, the symmetry generators need to be regularised and potentially renormalised. At the end of the day, there are three conceivable outcomes for the Slavnov–Taylor identities (8.1.18) corresponding to Yangian symmetries at loop level:

- Yangian symmetry is manifest: the tree-level form of the Slavnov–Taylor identities remains valid without changes at loop level.

- Yangian symmetry is anomalous: the Slavnov–Taylor identities receive anomalous contributions, but they constitute exact statements for the quantum theory.

- Yangian symmetry is broken: conceivably, quantum effects completely spoil the Slavnov–Taylor identities, and no meaningful conclusions due to Yangian symmetry can be drawn for quantum observables.

We can speculate that our identities will at least continue to hold as they stand at the level of loop integrands, i.e. before performing loop integrals. Divergences in the loop integrals and the ensuing renormalisation procedure in conjunction with gauge fixing and unphysical degrees of freedom could well render Yangian symmetry anomalous in some sense. Nevertheless it is also conceivable that the unusual kinds of transformations which came to use will not be compatible with loop integrands and thus spoil the identities beyond repair. However, the many successes of
integrability in $\mathcal{N} = 4$ sYM, at the loop level and even at finite coupling strength, see [37], suggest that the identities will remain largely intact in the full quantum theory. Let us therefore inspect the simplest non-trivial variation of a correlation function at loop level.

### 10.2 Three-point function at one loop: superconformal symmetry

Before tackling the Yangian symmetry of the one-loop integrand, we will learn how to work with it by examining the level-0 (superconformal) symmetry.

\[
\frac{1}{3} \langle J \cdot (Z_1 Z_2 Z_3) \rangle_{(1)} \simeq
\]

\[
-i a + i b - i c - i a' - i b' - i c' + i a + i b - i c.
\]

(10.2.1)

After using invariance of the propagator and inserting the quadratic part of the action according to (9.1.13) the above expression vanishes. This can be seen by adding the following combination of zero-diagrams:

\[
0 \simeq -i a - i b - i c.
\]

(10.2.2)

Up to gauge-fixing we thus see that the integrand of the one-loop three-point function is invariant under a level-0 symmetry. That was of course to be expected, but shows that our diagrammatic formalism can be lifted to a loop level.

### 10.3 Three-point function at one loop: Yangian symmetry

We are now ready to once again investigate the Yangian symmetry, this time at quantum level. The one-loop correction to the two-point function is the simplest quantum correction to a correlation
function. However, we have argued in Chapter 9 that two-point functions are more or less trivially invariant under Yangian level-one symmetries (up to gauge fixing). We therefore consider the three-point function at one loop as the next-to-simplest correlator.

As could have been seen in the last Chapter 9, a proper treatment of gauge fixing bloats the combinatorics without altering the conclusions concerning invariance. We shall therefore ignore gauge-fixing effects in the following analysis. Furthermore, we will disregard regularisation and renormalisation which would be needed to properly eliminate divergences at loop level. Effectively, we will thus consider loop integrands rather than loop integrals. On the one hand, this restricts the significance of the result. On the other hand, we will demonstrate that the non-linear representation of Yangian symmetry is structurally compatible with loop-level planar diagrams. The many non-trivial cancellations in our example will make it appear likely that Yangian symmetry does not break in a bad way quantum mechanically. It would thus remain to argue against quantum anomalies which we shall do later in this section.

We start with the level-one symmetry variation of the three-point function at one loop in terms of the diagrams introduced in Chapter 9

\[
\langle \hat{J}' \cdot (Z_1 Z_2 Z_3) \rangle_{(1)} \approx -i - 3 - i + i + 3 - i - 3 + i + 3i + i - i - 3 - i + i - i.
\]  

(10.3.1)

We have already applied several identities to transform the diagrams to equivalent ones without pointing out these transformations explicitly: Quadratic vertices were eliminated by means of Green’s identity (9.1.13). Level-zero invariance is used to shift linearised generators across propagators, see (9.2.9). Furthermore we consider diagrams modulo cyclic permutations, and we can exchange the two constituent level-zero generators (with or without dot) at the expense of a sign flip. Finally, we drop tadpole diagrams (any diagram where a propagator connects a point to itself).\(^1\)

\(^1\)The loop integral of this propagator merely amounts to some (potentially divergent) number, which is equivalent to a suitably chosen local counterterm and can therefore be removed.
We need to show that all these diagrams sum up to zero. We achieve this by adding further diagrams which we already know to sum to zero due to invariance of the action under level-zero and level-one symmetries (green circles) as well as commutativity of the level-zero constituents of the level-one generators (green boxes):

\[ 0 \simeq -3i \]

\[ + \frac{i}{2} \]

\[ - \frac{i}{2} \]

\[ - i \]

\[ + \frac{i}{2} \]

\[ - \frac{i}{2} \]

\[ - \frac{1}{2} \]

\[ + \frac{1}{2} \]

\[ - 3i \]

\[ - 3i \]

\[ - 3i \]

\[ + i \]

\[ - i \]

\[ + i \]

\[ - i \]

\[ + i \]

\[ - i \]

\[ + i \]

\[ - i \]

\[ - 3 \]

\[ - 3 \]

\[ + \]

\[ - \]

\[ + \]

\[ - \]

\[ + \frac{3}{4} \]

\[ - \frac{3}{4} \]

\[ + \frac{1}{4} \]

\[ - \frac{1}{4} \]

\[ - \frac{1}{2} \]

\[ - \frac{1}{2} \]

\[ + \frac{1}{2} \]

\[ + \frac{1}{2} \]

\[ - \]

\[ - \]

\[ + \]

\[ + \]
10.4 Anomalies

Each of these terms can be expanded in terms of elementary diagrams as before. They yield between 2 and 28 individual diagrams which can be transformed and brought to one of 90 standard forms using the transformation rules mentioned above, see Appendix E for a complete expansion of all terms. By careful inspection we indeed confirm that all the diagrams cancel

$$\langle \hat{J}' \cdot (Z_1 Z_2 Z_3) \rangle_{(1)} = 0.$$  

(10.4.3)

In our example, we find no surprises for Yangian symmetry of loop integrands nor does the invariance require modification of the symmetry representation. However, it remains to be seen whether gauge fixing or regularisation will change this conclusion.

10.4 Anomalies

Finally, we would like to comment on anomalies, see \[103\], in the interpretation as a non-invariance of the quantum mechanical path integral measure \[104, 105\]. For ordinary symmetries, the anomaly could be understood a deformation of the total variational identity (8.1.1)

$$\langle J \cdot \mathcal{O} + iJ \cdot \mathcal{S} + i\mathcal{A}[J] \mathcal{O} \rangle = 0.$$  

(10.4.1)

Here the additional term \( \mathcal{A}[J] \) represents the potentially non-trivial variation of the path integral measure by the generator \( J \). A corresponding deformation of the conjectured variational identity (8.1.5) for a bi-local generator \( J_1 \wedge J_2 \) reads

$$0 = \langle (J_1 \wedge J_2) \cdot \mathcal{O} + i(J_1 \wedge J_2) \cdot \mathcal{S} \mathcal{O} + i\mathcal{A}[J_1 \wedge J_2] \mathcal{O} \rangle$$

(10.4.2)

$$+ \langle i(J_1 \cdot \mathcal{S}) \wedge (J_2 \cdot \mathcal{O}) + i(J_1 \cdot \mathcal{O}) \wedge (J_2 \cdot \mathcal{S}) + i\mathcal{A}[J_1] \wedge (J_2 \cdot \mathcal{O}) + i(J_1 \cdot \mathcal{O}) \wedge \mathcal{A}[J_2] \rangle$$

$$+ \langle -(J_1 \cdot \mathcal{S}) \wedge (J_2 \cdot \mathcal{S}) \mathcal{O} - \mathcal{A}[J_1] \wedge (J_2 \cdot \mathcal{S}) \mathcal{O} - (J_1 \cdot \mathcal{S}) \wedge \mathcal{A}[J_2] \mathcal{O} - \mathcal{A}[J_1] \wedge \mathcal{A}[J_2] \mathcal{O} \rangle.$$  

Now the level-zero symmetry is non-anomalous in the principal examples of Yangian symmetric planar gauge theories, \( \mathcal{A}[J] = 0 \). Dropping all terms in the bi-local variation containing the level-zero anomaly \( \mathcal{A}[J] \), we are left with the genuine anomaly \( \mathcal{A}[J_1 \wedge J_2] \) of a bi-local symmetry. So the question of a Yangian anomaly translates to the presence of this latter term.

A key aspect of anomalies is locality and cohomology of the symmetry algebra: In a renormalisable quantum field theory, the anomaly term \( \mathcal{A}[J] \) is local. Moreover, the anomaly typically joins the variation of the action in the combination \( J \cdot \mathcal{S} + \mathcal{A}[J] \)

$$\langle J \cdot \mathcal{O} + i(J \cdot \mathcal{S} + \mathcal{A}[J]) \mathcal{O} \rangle = 0.$$  

(10.4.3)

So the actual question is whether this combination vanishes or can be made to vanish by adjusting the action appropriately. Now the action is manifestly local and so is its variation. However, a local anomaly is not necessarily the variation of a local term, which makes this question a problem of cohomology. For the bi-local Yangian symmetry we can collect the terms of the above variational identity as

$$0 = \langle \hat{J} \cdot \mathcal{O} \rangle + i\langle (\hat{J} \cdot \mathcal{S} + \mathcal{A}[J]) \mathcal{O} \rangle$$

$$+ \langle i(\hat{J} \cdot \hat{S}) \wedge (\hat{J}_1 \wedge \hat{J}_2) \mathcal{O} + i(\hat{J} \cdot \hat{S}) \wedge (\hat{J}_1 \wedge \hat{J}_2) \mathcal{O} \rangle + i\mathcal{A}[\hat{J}_1] \wedge (\hat{J}_2 \cdot \mathcal{O}) + i(\hat{J}_1 \cdot \mathcal{O}) \wedge \mathcal{A}[\hat{J}_2] \rangle$$

$$+ \langle -(\hat{J} \cdot \hat{S}) \wedge (\hat{J}_1 \wedge \hat{J}_2) \mathcal{O} - \mathcal{A}[\hat{J}_1] \wedge (\hat{J}_2 \cdot \mathcal{O}) - (\hat{J} \cdot \hat{S}) \wedge \mathcal{A}[\hat{J}_2] \mathcal{O} - \mathcal{A}[\hat{J}_1] \wedge \mathcal{A}[\hat{J}_2] \mathcal{O} \rangle.$$
CHAPTER 10. THE FATE OF YANGIAN SYMMETRY AT QUANTUM LEVEL

\[ + i \langle (J^{(1)} \cdot S + A[J^{(1)}]) \wedge (J^{(2)} \cdot O) \rangle - \langle (J^{(1)} \cdot S + A[J^{(1)}]) \otimes (J^{(2)} \cdot S + A[J^{(2)}]) O \rangle. \] (10.4.4)

One should therefore ask whether \( \hat{J} \cdot S + A[\hat{J}] = 0 \) together with \( J \cdot S + A[J] = 0 \). Importantly, is \( A[\hat{J}] \) local? Is the appropriate cohomology for the Yangian algebra trivial? Can one construct a local anomaly term \( A[\hat{J}] \) consistent with the relations of Yangian symmetry? And even if so, is the resulting prefactor in \( A[\hat{J}] \) zero? Answering these questions should help in settling Yangian symmetry as a symmetry of a planar quantum gauge theory.

10.5 Anomalies in \( \mathcal{N} = 4 \) sYM

We are now in position to formalize the heuristic argument presented in Chapter 6, understanding it now as a lack of anomaly. We will rely on the results of Chapter 7 regarding the interplay of Yangian and gauge symmetries, and also use some results from representation theory, basics of which we present in the Appendix.

Ignoring potential subtleties regarding regularisation, gauge fixing, bi-local total variations and locality of \( A[\hat{J}] \), we can consider the anomaly of the level-one momentum \( \hat{P} \) in \( \mathcal{N} = 4 \) sYM theory. Such an anomaly should obey a consistency relation along the lines of

\[ J \cdot A[\hat{P}] = \hat{P} \cdot A[J] + A[[J, \hat{P}]] \] (10.5.1)

with the level-zero generators \( J \) originating from the algebra at level one. Following the arguments in Section 7.2.1 and generalising (7.2.10), the commutator \( [J, \hat{P}] \) should be a combination of level-one generators following from the Yangian algebra as well as bi-local and local gauge transformations

\[ [J, \hat{P}] = [J, \hat{P}]_Y + G[P^{(1)} \ldots \widehat{P}^{(2)}] \wedge P^{(2)} + G[\hat{P} \ldots]. \] (10.5.2)

According to (7.2.12) one can expect the anomalies of bi-local gauge transformations to reduce to anomalies of level-zero symmetries and a term with two level-zero generators

\[ A[G[P^{(1)} \ldots \wedge P^{(2)}] \simeq 2I[P^{(1)} \ldots] \cdot A[P^{(2)}] + 2A[I[P^{(2)} \cdot P^{(1)} \ldots]]. \] (10.5.3)

In \( \mathcal{N} = 4 \) sYM there are no anomalies for superconformal and gauge symmetries and moreover the commutator \( P^{(2)}P^{(1)} \) vanishes. Hence the above consistency requirement reduces to

\[ J \cdot A[\hat{P}] = A[[J, \hat{P}]_Y], \] (10.5.4)

where the commutator is evaluated in the plain Yangian algebra. Since the level-one generators transform in the adjoint representation of the level-zero algebra, \( \hat{P} \) has (almost) the same quantum numbers as the momentum generator \( P \). Effectively, this implies that \( A[\hat{P}] \) is a translation-invariant supersymmetric Lorentz-vector SU(4)-singlet gauge-invariant operator of dimension 1. Furthermore, we can argue that the anomaly term has negative CP-parity just as the level-one Yangian generators.\(^2\) Translation-invariant operators are integrals over local operators, and supersymmetric operators reside at the top of supermultiplets. Before integration over spacetime,

\(^2\) The CP-parity operation maps the adjoint fields \( Z \) to their negative transpose \(-Z^T\) and therefore effectively reverses the order of the trace in the planar limit. The anti-symmetric combination of the bi-local generators implies a negative parity.
we therefore consider Lorentz-vector SU(4)-singlet gauge-invariant local operators of dimension 5 and negative CP-parity at the top of some supermultiplet.

Now it is not difficult to argue that no such local operators exist based on the representation theory of the superconformal algebra $\mathfrak{psu}(2,2|4)$, see [107]: Long supermultiplets (with no shortening condition) span a range of conformal dimension 8 (4$N'$/2, as the generators $Q$ and $\bar{Q}$ have dimension $+\frac{1}{2}$), hence supersymmetric local operators at dimension less than 10 could only originate from short supermultiplets. The only short supermultiplets relevant in the planar limit are the $1/2$-BPS (i.e. annihilated by half of the supersymmetry generators) single-trace multiplets. Negative CP-parity restricts to the $1/2$-BPS multiplet at dimension 3 which also possesses no suitable supersymmetric state. For example, it is easy to see that the SU(4) charge of the corresponding superconformal primary operators cannot be eliminated by the application of merely 4 supersymmetry operators to go from the primary at dimension 3 to a descendant at dimension 5.

We can also argue less abstractly by enumerating local operators with the desired properties: Dropping the requirement of supersymmetry, there are 15 Lorentz-vector SU(4)-singlet gauge-invariant local operators of dimension 5 and negative CP-parity. Eight of these are ordinary local operators\footnote{It suffices to sketch their form using ordinary conventions for the fields on $\mathcal{N} = 4$ sYM as well as the composition of spacetime, internal and spinor indices.}

\[
\begin{align*}
\text{tr} &\left\{\Phi^{ab}, \bar{\Phi}^{cd}\right\}[D_\mu \bar{\Phi}_{ab}, \bar{\Phi}^{cd}], & \Gamma^{\alpha\beta}_\mu \text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]\Phi^{ab}\bar{\Phi}_{ab}, \\
\Gamma^{\alpha\beta}_{\mu[abcd]} &\text{tr} &\left\{\bar{\Psi}_\alpha, \Psi_\beta\right\}\Phi^{ab}\bar{\Phi}^{cd}, & \Gamma^{\alpha\beta}_\mu &\text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]\Phi^{ab}\bar{\Phi}_{ab}, \\
\Gamma^{\alpha}_\nu &\text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]F^{\nu}, & \Gamma^{\alpha\beta}_\nu &\text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]\bar{F}^{\mu\nu}, \\
\Gamma^{\alpha\beta}_{ab\mu\nu} &D^{\nu} &\text{tr} &\left\{\Phi^{ab}, \bar{\Phi}_{ab}\right\}F_{\mu\nu}, & \Gamma^{\alpha\beta}_{ab\mu\nu} &D^{\nu} &\text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]\bar{F}_{\mu\nu},
\end{align*}
\]

four are conformal descendants

\[
\begin{align*}
\text{tr} &\left\{\Phi^{ab}, \bar{\Phi}_{ab}\right\}F_{\mu\nu}, & \Gamma^{\alpha\beta}_{ab\mu\nu} &D^{\nu} &\text{tr} &\left[\bar{\Psi}_\alpha, \Psi_\beta\right]\bar{F}_{\mu\nu},
\end{align*}
\]

and three are trivial modulo the equations of motion\footnote{It suffices to state the leading term with the largest number of derivatives as the others are linear combinations of the previously mentioned terms.}

\[
\begin{align*}
\text{tr} &\left[\Phi^{ab}\bar{\Phi}_{ab}D^{\nu}F_{\mu\nu}\right] + \ldots, & \text{tr} &\left[\Phi^{ab}\bar{\Phi}_{ab}D^{\nu}\bar{F}_{\mu\nu}\right] + \ldots, \\
(\Gamma_\nu\Gamma_{ab\mu})^{\alpha\beta} &\text{tr} &\left[D^{\nu}\bar{\Psi}_\alpha, \Psi_\beta\right]\Phi^{ab} + \ldots, & (\Gamma_\nu\Gamma_{ab\mu})^{\alpha\beta} &\text{tr} &\left[D^{\nu}\bar{\Psi}_\alpha, \Psi_\beta\right]\Phi^{ab} + \ldots.
\end{align*}
\]

In the above, the spinor indices of (one of the) $\mathfrak{su}(2)$ and the $\mathfrak{su}(4)$ have been combined into a single index $\tilde{\alpha}, \tilde{\beta}$, to be acted upon by the $\Gamma$ matrices. All of these operators belong to long superconformal multiplets: There are 1, 4, 1 relevant long superconformal multiplets of primary dimension 3, 4, 5, respectively [106]. Furthermore, there are 2 and 1 extra superconformal multiplets of primary dimension 4.5 and 5, respectively, which are proportional to the equations of motion. In the following table, we list the quantum numbers of the superconformal primaries as well as the combination of supercharges and momentum generators to turn them into operators of the desired type (see [33] or [26] for the explanation of the Dynkin labels and the unitary representations of}
Altogether these account for 8 conformal primaries, 4 conformal descendants (P) and 3 operators proportional to the equations of motions. As members of long supermultiplets, they are clearly not the highest and thus not supersymmetric states.

Therefore, there are no suitable operators $\mathcal{A}[^\hat{\mathcal{P}}]$ in planar $\mathcal{N} = 4$ sYM theory, and consequently, Yangian symmetry is non-anomalous (modulo subtleties alluded to above and supposing the superconformal and gauge symmetries are non-anomalous). In fact, the absence of suitable operators $\mathcal{A}[^\hat{\mathcal{P}}]$ not only implies the absence of anomalies, but even better, it implies (without subtleties) that the classical action must be invariant, as was shown explicitly in [60]. This constitutes an alternative proof of Yangian symmetry of planar $\mathcal{N} = 4$ sYM which is merely based on the representation content of the theory together with the Yangian algebra relations.

By the same logic, the level-one bonus symmetry $\hat{\mathcal{B}}$ [68, 108, 109] can be argued to be non-anomalous: The anomaly $\mathcal{A}[\hat{\mathcal{B}}]$ must be a translation-invariant Lorentz-singlet SU(4)-singlet gauge-invariant operator of dimension 0 and negative parity. Such an operator does not exist as one can show by straight-forward enumeration of local operators at dimension 4, see also (10.5.8). Even though this argument is much easier than the one for $\hat{\mathcal{P}}$, it may also be more fragile at the quantum level: Namely, the generator $\hat{\mathcal{B}}$ is based on superconformal boosts as opposed to $\hat{\mathcal{P}}$ which merely uses rotations, translations, supersymmetries and scale transformations. The issue is that special conformal symmetries are typically superficially broken by the quantisation procedure whereas rotations, translations and supersymmetries are known to hold manifestly. Therefore, an anomaly of $\hat{\mathcal{P}}$ is most closely linked to the anomaly of scale transformations, whereas the anomaly of $\hat{\mathcal{B}}$ requires understanding of the anomaly of special superconformal transformations.

<table>
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<td>[0, 1, 0]</td>
<td>Q^2, Q^2</td>
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<td>1</td>
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<td>[1, 0]</td>
<td>[1, 0, 0]</td>
<td>Q</td>
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Chapter 11

Conclusions and outlook

In this thesis we have managed to resolve the uncertainties regarding the origin of Yangian symmetry in the various observables of $\mathcal{N} = 4$ sYM by demonstrating it can be tracked back to the invariance of the action of the theory. We have achieved that by first analysing the classical equations of motion of the model and showing that on-shell they are left invariant by the level-1 generators of $Y(\text{psu}(2, 2|4))$. It turned out that the correct representation which takes gauge invariance into account is non-linear in fields. We then strengthened the result by showing that under the action of the Yangian generators the equations of motion satisfy an off-shell identity, which we claimed to be equivalent to the invariance of the action, by analogy with a similar statement for the usual Lie-type symmetries. The major difference between the equations of motion and the action $S$ of $\mathcal{N} = 4$ sYM is that the latter is cyclic due to the trace over the gauge degrees of freedom. Usually, Yangian symmetries are considered incompatible with the cyclicity due to their nontrivial coproduct which requires ordering of the fields. Due to the exact commutativity of the constituent level-0 generators featuring in the coproduct of the level-1 ones, $[J^{(1)}, J^{(2)}] = 0$, as well as invariance of the action of $\mathcal{N} = 4$ sYM under all the level-0 generators we managed to reformulate the strong invariance of the equations of motion to the full invariance of the action:

$$\hat{J} \cdot S_{\mathcal{N}=4sYM} = 0.$$ 

A corollary of this result is a construction of a novel length-changing representation of Yangian generators on cyclic polynomials. An unusual property is that it features overlapping terms, where one of the constituent level-0 generators acts on the output of the other one. This (purely nonlinear) effect has not been encountered before, but it was both necessary to assure invariance of the actions, as well as predicted by the strong invariance of the equations of motion. Using our result we also confirmed that the bonus level-1 generators $\hat{B}$, which is not a part of $Y(\text{psu}(2, 2|4))$ but which was found to be a symmetry of the scattering amplitudes also can be viewed as the symmetry of the action.

Having established the classical result, we proceeded towards the quantum theory. The first necessary step in that direction was to make sure that we preserve the Yangian symmetry in presence of the gauge-fixing terms. That is an important question, especially given the fact that our representation mixes global and local (gauge) symmetries. Fortunately, the answer was affirmative. In the process, we discovered novel types of bi-local gauge transformations of the form $G[\mathcal{A}_1] \land G[\mathcal{A}_2]$ and $G[\mathcal{X}] \land J$ (respectively $Q \otimes Q$ and $Q \land J$ where $Q$ generates BRST transformations). They are satisfied provided gauge transformations and level-0 generators are symmetries, hence they do not constrain theory more than their constituents (unlike the full Yangian generators). However, they...
proved to be a necessary ingredient in the next step, namely in establishing the Slavnov–Taylor identities.

Having showed that the action is invariant under Yangian generators, we want to use it to constrain observables, and Slavnov–Taylor identities allow to translate between invariance of the action and the correlation functions. We proposed the Slavnov–Taylor identities stemming from the Yangian invariance, taking into account the complications arising from the gauge fixing. Afterwards, we verified that indeed tree-level correlation functions as well as (unintegrated) 1-loop correlation functions satisfy them. We also discussed why the Yangian symmetry should not be anomalous.

**Further directions**

The research may continue in numerous directions from here. Let us start with the projects purely in the context of $\mathcal{N} = 4$ sYM, and then move towards the broader picture of AdS/CFT correspondence.

**Scattering Amplitudes** The first interesting possibility is obtaining the invariance of the scattering amplitudes under dual superconformal/Yangian symmetry starting from our Slavnov–Taylor identities and using the LSZ reduction [110], [111]. Given the fact that the representations used to obtain those results are different (symmetry of scattering amplitudes being realized linearly) it would be most valuable to understand, how they are nevertheless related. A possible additional issue which one would need to resolve is the generalization of the LSZ formula to massless states.

**Wilson Loops** Another research direction worth pursuing is the application of the Slavnov–Taylor identities to the (Maldacena–)Wilson loops in the theory. They have been shown in [18] and [79] to indeed be Yangian invariant up to one loop (where gauge fixing effects are not yet visible), but with the new tools developed in the research leading to this thesis their investigation should now be simplified.

**Algebraic identities** On a more algebraic level, recall from the Chapter 3 that for consistency the level-1 generators are supposed to satisfy the Serre relations, as well as transform in the adjoint of $\text{psu}(2, 2|4)$. To some extent we have checked and understood the latter property, having explained in what sense i.e. the commutator $[P, \hat{P}]$ is trivial. We have not however touched upon the Serre relations. Their verification is rendered very difficult again due to the mixing of the Yangian and gauge transformations.

**Fishnet diagrams** Next interesting issue is connected to a recently discovered integrability and Yangian symmetry of the fishnet diagrams, stemming from the double scaling limit of $\gamma$-deformed sYM [99], [112], [98]. In a sense, it is a natural continuation of the study of the $\beta$-deformed theory described in Chapter 5. As the resulting model however is not supersymmetric and possibly not conformal on quantum level [113], the precise understanding of how the Yangian symmetry survives nevertheless would certainly be important.

**Correlation Functions** A very fruitful area of integrability-based research into $\mathcal{N} = 4$ sYM is the study of correlation functions. Unlike the objects we covered in Chapter 9 the ones we mean...
now are correlation functions of gauge invariant local operators, i.e.:

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\cdots\mathcal{O}_n(x_n) \rangle,$$  \hspace{1cm} (11.0.1)

where $\mathcal{O}(x) = \text{Tr}(Z_1(x)Z_2(x)\cdots Z_k(x))$. For $n = 2$ they are fixed by the conformal symmetry and have been well understood via mapping to the spin-chain picture. Three-point functions are also constrained, but their understanding is still not complete, even in spite of the recent progress [114], [115]. The construction there however to some extent still resembles the spin-chain picture of two-point function and hence is based on (two copies of) the $\mathfrak{su}(2|2)$ algebra and possibly its Yangian. It would however be interesting to see whether one could constrain them using the Yangian $Y(\mathfrak{psu}(2,2|4))$, without any introduction of a ground state.

**String theory dual**  As we alluded to in the Introduction, the $\mathfrak{psu}(2,2|4)$ symmetry algebra has been established and well understood on both sides of the AdS/CFT. With the Yangian symmetry now also defined for the action of $\mathcal{N} = 4$ sYM, the infinite number of symmetries for both theories has been established. It would however be valuable to understand, how those infinitely-dimensional symmetry algebras are precisely related. Yangian-type symmetries have been identified for certain observables on the string theory side [116], but full $Y(\mathfrak{psu}(2,2|4))$ has not been constructed.
Appendix A

\textbf{psu}(2, 2|4) superconformal symmetry and its Yangian

In this appendix we summarise the \( \mathcal{N} = 4 \) superconformal algebra \textbf{psu}(2, 2|4) and its representation on the fields. Some of the information and formulae which follow have already been introduced in the thesis, but here we collect them all in one place. This Appendix mostly builds on [60].

A.1 \textbf{psu}(2, 2|4) algebra

\textbf{Algebra.} The supersymmetry algebra is spanned by the supersymmetry generators \( Q_{a\dot{\gamma}} \), \( \bar{Q}^{\dot{a}c} \) and the momentum generator \( P_{\gamma\dot{\alpha}} \). The special conformal generators are given by two fermionic generators \( S^{\alpha c} \), \( \bar{S}_{\dot{\alpha}}^{c} \) and the bosonic generator \( K^{\alpha\dot{\gamma}} \). Furthermore, the superconformal algebra includes the Lorentz and internal rotation generators \( L^{\alpha\gamma} \), \( \bar{L}^{\dot{\alpha}\dot{\gamma}} \), \( R^{a}_{\ c} \) (whose trace over the indices vanishes) and the dilatation generator \( D \). Finally, we will also need gauge transformations to discuss the gauge-covariant representations. These are generated by \( G[X] \) where the field \( X \) serves as the gauge parameter matrix.

Although we do not explicitly refer to real algebras, a suitable set of reality conditions for \( \mathcal{N} = 4 \) sYM is given by

\begin{align}
(P_{\gamma\dot{\alpha}})^\dagger &= P_{\alpha\gamma}, \\
(K^{\alpha\dot{\gamma}})^\dagger &= K^{\gamma\dot{\alpha}}, \\
(Q_{a\dot{\gamma}})^\dagger &= \bar{Q}^{\dot{a}c}, \\
(S^{\alpha c})^\dagger &= \bar{S}_{\dot{c}}^{\dot{a}} \tag{A.1.1}
\end{align}

as well as

\begin{align}
(L^{\alpha\gamma})^\dagger &= \bar{L}^{\dot{\alpha}\dot{\gamma}}, \\
(R^{a}_{\ c})^\dagger &= R_{\ c}^{\ b}, \\
D^\dagger &= D, \\
G[X]^\dagger &= G[X^\dagger] \tag{A.1.2}
\end{align}

The below representations will be unitary w.r.t. these reality conditions.

In the following we will list the most relevant algebra relations. The Lorentz and internal algebra relations take the form

\begin{align*}
[L^{\alpha\beta}, L^{\gamma\delta}] &= i(\delta^{\gamma}_{\delta}L^{\alpha\beta} - \delta^{\beta}_{\delta}L^{\gamma\beta}) + G[\ldots], \\
[\bar{L}^{\dot{\alpha}\dot{\beta}}, \bar{L}^{\dot{\gamma}\dot{\delta}}] &= i(\delta^{\dot{\gamma}}_{\dot{\delta}}\bar{L}^{\dot{\alpha}\dot{\beta}} - \delta^{\dot{\beta}}_{\dot{\delta}}\bar{L}^{\dot{\alpha}\dot{\gamma}}) + G[\ldots], \\
[R^{a}_{\ b}, R^{c}_{\ d}] &= i(\delta^{c}_{\ d}R^{a}_{\ b} - \delta^{a}_{\ d}R^{c}_{\ b}), \tag{A.1.3}
\end{align*}

Here, the Lorentz algebra relations involve gauge transformations \( G[\ldots] \), where the omitted gauge parameter is a term of the form \( x x D A \) representing the field strength contracted with the Killing spinors of the two rotations. We do not present the long list of algebra relations with the remaining
generators, as these can easily be inferred as the transformations of spinor indices compatible with the above relations.

The algebra of the scaling generator $D$ measures the scaling dimension $\Delta_J$ of the other generators $J$

$$[D, J] = -i\Delta_J J + G[\ldots]$$

with the dimensions

$$\Delta_L = \Delta_K = \Delta_R = \Delta_D = 0, \quad \Delta_Q = \Delta_Q = -\Delta_S = -\Delta_S = \frac{1}{2}, \quad \Delta_P = -\Delta_K = 1.$$  \hspace{1cm} (A.1.5)

The remaining relations involving the fermionic generators are trivial modulo gauge transformations.

The non-trivial relations of the fermionic generators read

$$[P_{\beta\dot{\alpha}}, P_{\dot{\beta}\alpha}] = -iG[D_{\dot{\alpha}\beta}A_{\dot{\beta}\alpha}],$$

$$[P_{\beta\dot{\alpha}}, K^{\alpha\gamma}] = i\delta^\gamma_\beta L^\gamma_{\alpha\beta} + i\delta^\gamma_\beta i\delta^\beta_\alpha D + G[\ldots],$$

$$[K^{\beta\dot{\alpha}}, K^{\gamma\dot{\alpha}}] = G[\ldots].$$ \hspace{1cm} (A.1.6)

Again, these relations may involve some gauge transformation $G[\ldots]$ in addition to the pure conformal generators.

The non-trivial relations of the fermionic generators read

$$\{Q_{\alpha}, \Phi^d_x\} = 2\delta^d_\alpha P_{\alpha\gamma}, \quad \{Q_{\gamma}, S^d_{\alpha}\} = 2i\delta^\gamma_\alpha R^d_{\alpha\gamma} - 2i\delta^\gamma_\alpha L^\gamma_{\alpha\beta} - i\delta^\gamma_\delta \delta^\delta_\alpha D,$$

$$\{S^d_{\alpha}, S^e_{\alpha}\} = 2\delta^d_\alpha K^{\alpha\gamma}, \quad \{Q_{\alpha}, S^d_{\alpha}\} = 2i\delta^\gamma_\alpha R^d_{\alpha\gamma} + 2i\delta^\gamma_\alpha L^\gamma_{\alpha\beta} + i\delta^\gamma_\delta \delta^\delta_\alpha D.$$ \hspace{1cm} (A.1.7)

while the non-trivial mixed relations read

$$[P_{\beta\dot{\alpha}}, S^\gamma_{\dot{\alpha}}] = \delta^\gamma_\beta \bar{Q}^\gamma_{\dot{\alpha}} - \varepsilon_{\dot{\alpha}\dot{\beta}} G[x^{\gamma\dot{\alpha}}\Psi^\beta_{\dot{\beta}}],$$

$$[K^{\gamma\dot{\alpha}}, Q_{ad}] = -\delta^\gamma_\alpha \bar{S}^\gamma_{ad} + G[\ldots],$$

$$[K^{\gamma\dot{\alpha}}, Q_{ad}] = \delta^\gamma_\alpha \bar{S}^\gamma_{ad} + G[\ldots].$$ \hspace{1cm} (A.1.8)

The remaining relations involving the fermionic generators are trivial modulo gauge transformations

$$[P_{\beta\dot{\alpha}}, Q_{d\gamma}] = i\varepsilon_{\beta\gamma} G[\bar{\Psi}^{\dot{\alpha}d}],$$

$$\{Q_{\beta\dot{\alpha}}, Q_{d\gamma}\} = 2i\varepsilon_{\alpha\gamma} G[\bar{\Phi}_{bd}],$$

$$\{Q_{\beta\dot{\alpha}}, S_{d\gamma}\} = G[\ldots],$$

$$\{Q_{\beta\dot{\alpha}}, S_{d\gamma}\} = G[\ldots],$$

$$[\bar{S}^\alpha_{\dot{b}}, K^{d\dot{\gamma}}] = G[\ldots].$$ \hspace{1cm} (A.1.9)

**Representation.** The representation of the supersymmetries on the fields reads

$$Q_{\alpha\beta} \cdot \Phi^{cd} = \delta^c_\alpha \bar{Q}^d_{\beta\gamma} - \delta^d_\alpha \bar{Q}^c_{\beta\gamma},$$

$$Q_{\alpha\beta} \cdot \Phi^{d\gamma} = \varepsilon_{acde} \Psi^{\gamma}_{\beta\alpha},$$

$$Q_{\alpha\beta} \cdot A^{\dot{\alpha}\dot{\beta}} = -i\varepsilon_{\beta\gamma} \Psi^\gamma_{\dot{\alpha}\dot{\beta}},$$

$$Q_{\alpha\beta} \cdot \Psi_{\dot{\alpha}\dot{\beta}} = -2i\delta^e_\alpha F_{\beta\dot{\alpha}} + i\varepsilon_{\beta\dot{\alpha}\dot{\beta}} [\Phi^{ce}, \Phi_{\dot{e}}],$$

$$Q_{\alpha\beta} \cdot \Psi_{\dot{\beta}d} = 2iD_{\beta\gamma} \bar{\Phi}_{ad},$$

$$Q_{\alpha\beta} \cdot \Psi_{\dot{\alpha}\dot{\gamma}} = 2iD_{\beta\gamma} \bar{\Phi}_{ad},$$

$$Q_{\alpha\beta} \cdot \Psi_{\dot{\beta}d} = 2iD_{\beta\gamma} \bar{\Phi}_{ad},$$

$$Q_{\alpha\beta} \cdot \Psi_{\dot{\alpha}\dot{\gamma}} = 2iD_{\beta\gamma} \bar{\Phi}_{ad}. \hspace{1cm} (A.1.10)$$
The standard rules (2.2.1) and (2.2.4) for the representation of the momentum and dilatation generators read explicitly

\[ P_{\beta\bar{\alpha}} \cdot \Phi^{cd} = iD_{\beta\bar{\alpha}} \Phi^{cd}, \quad D \cdot \Phi^{cd} = -ix_{\beta\bar{\alpha}} D_{\beta\bar{\alpha}} \Phi^{cd} - i\Phi^{cd}, \]
\[ P_{\beta\bar{\alpha}} \cdot \Psi^{c} = iD_{\beta\bar{\alpha}} \Psi^{c}, \quad D \cdot \Psi^{c} = -ix_{\beta\bar{\alpha}} D_{\beta\bar{\alpha}} \Psi^{c} - \frac{3}{2} i\Psi^{c}, \]
\[ P_{\beta\bar{\alpha}} \cdot \bar{\Psi}_{\gamma d} = iD_{\beta\bar{\alpha}} \bar{\Psi}_{\gamma d}, \quad D \cdot \bar{\Psi}_{\gamma d} = -ix_{\beta\bar{\alpha}} D_{\beta\bar{\alpha}} \bar{\Psi}_{\gamma d} - \frac{3}{2} i\bar{\Psi}_{\gamma d}, \]
\[ P_{\beta\bar{\alpha}} \cdot A_{\delta\gamma} = iD_{\beta\bar{\alpha}} A_{\delta\gamma}, \quad D \cdot A_{\delta\gamma} = -ix_{\beta\bar{\alpha}} D_{\beta\bar{\alpha}} A_{\delta\gamma}. \]  

(A.1.11)

For the Lorentz generators one finds\footnote{1}{In analogy to the assignment \( \Delta_A = 0 \) for scaling transformations, the spacetime indices of the gauge field \( A \) are not transformed explicitly. See\footnote{2}{2} for explanation.}

\[ \mathcal{L}^{\beta}_{\delta} \cdot \Phi^{ce} = -ix_{\beta\delta} D_{\beta\delta} \Phi^{ce} + \frac{i}{2} \delta^{\beta}_{\delta} x^{\kappa\lambda} D_{\kappa\lambda} \Phi^{ce}, \]
\[ \mathcal{L}^{\beta}_{\delta} \cdot \Psi^{c} = -ix_{\beta\delta} D_{\beta\delta} \Psi^{c} + \frac{i}{2} \delta^{\beta}_{\delta} x^{\kappa\lambda} D_{\kappa\lambda} \Psi^{c} - i\delta^{\beta}_{\delta} \Psi^{c}, \]
\[ \mathcal{L}^{\beta}_{\delta} \cdot \bar{\Psi}_{\gamma e} = -ix_{\beta\delta} D_{\beta\delta} \bar{\Psi}_{\gamma e} + \frac{i}{2} \delta^{\beta}_{\delta} x^{\kappa\lambda} D_{\kappa\lambda} \bar{\Psi}_{\gamma e}, \]
\[ \mathcal{L}^{\beta}_{\delta} \cdot A_{\epsilon\gamma} = -ix_{\beta\delta} D_{\beta\delta} A_{\epsilon\gamma} + \frac{i}{2} \delta^{\beta}_{\delta} x^{\kappa\lambda} D_{\kappa\lambda} A_{\epsilon\gamma}. \]  

(A.1.12)

analogously for the conjugate Lorentz generators

\[ \mathcal{L}^{\alpha}_{\gamma} \cdot \Phi^{cd} = -ix_{\alpha\gamma} D_{\alpha\gamma} \Phi^{cd} + \frac{i}{2} \delta^{\alpha}_{\gamma} x^{\beta\delta} D_{\beta\delta} \Phi^{cd}, \]
\[ \mathcal{L}^{\alpha}_{\gamma} \cdot \Psi^{c} = -ix_{\alpha\gamma} D_{\alpha\gamma} \Psi^{c} + \frac{i}{2} \delta^{\alpha}_{\gamma} x^{\beta\delta} D_{\beta\delta} \Psi^{c} - i\delta^{\alpha}_{\gamma} \Psi^{c}, \]
\[ \mathcal{L}^{\alpha}_{\gamma} \cdot \bar{\Psi}_{\epsilon d} = -ix_{\alpha\gamma} D_{\alpha\gamma} \bar{\Psi}_{\epsilon d} + \frac{i}{2} \delta^{\alpha}_{\gamma} x^{\beta\delta} D_{\beta\delta} \bar{\Psi}_{\epsilon d} - i\delta^{\alpha}_{\gamma} \bar{\Psi}_{\epsilon d}, \]
\[ \mathcal{L}^{\alpha}_{\gamma} \cdot A_{\delta\epsilon} = -ix_{\alpha\gamma} D_{\alpha\gamma} A_{\delta\epsilon} + \frac{i}{2} \delta^{\alpha}_{\gamma} x^{\beta\delta} D_{\beta\delta} A_{\delta\epsilon}. \]  

(A.1.13)

as well as for the internal rotation generators

\[ R^{a}_{\ b} \cdot \Phi^{cd} = i\delta^{c}_{b} \Phi^{ad} + i\delta^{d}_{b} \Phi^{ac} - \frac{i}{2} \delta^{a}_{b} \Phi^{cd}, \]
\[ R^{a}_{\ b} \cdot \Psi^{c} = i\delta^{c}_{b} \Psi^{a} - \frac{i}{4} \delta^{a}_{b} \Psi^{c}, \]
\[ R^{a}_{\ b} \cdot \bar{\Psi}_{\gamma d} = -i\delta^{\gamma}_{d} \bar{\Psi}_{\gamma b} - \frac{i}{4} \delta^{\gamma}_{d} \bar{\Psi}_{\gamma d}, \]
\[ R^{a}_{\ b} \cdot A_{\delta\gamma} = 0. \]  

(A.1.14)

The representation of the special superconformal generators can be summarised as

\[ S^{ab} \cdot \Phi^{cd} = ix^{ak}_{\ a} \bar{Q}_{k}^{\ b} \cdot \Phi^{cd}, \quad S_{a} \cdot \Phi^{cd} = -ix^{ak}_{\ a} Q_{a\beta} \cdot \Phi^{cd}, \]
\[ S^{ab} \cdot \Psi^{c} = ix^{ak}_{\ a} \bar{Q}_{k}^{\ b} \Psi^{c} - 2\delta^{c}_{b} \Phi^{ac}, \quad S_{a} \cdot \Psi^{c} = -ix^{ak}_{\ a} Q_{a\beta} \cdot \Psi^{c}, \]
\[ S^{ab} \cdot \bar{\Psi}_{\gamma d} = ix^{ak}_{\ a} \bar{Q}_{k}^{\ b} \bar{\Psi}_{\gamma d}, \quad S_{a} \cdot \bar{\Psi}_{\gamma d} = -ix^{ak}_{\ a} Q_{a\beta} \cdot \bar{\Psi}_{\gamma d} + 2\delta^{\gamma}_{d} \phi_{ad}, \]
\[ S^{ab} \cdot A_{\delta\gamma} = ix^{ak}_{\ a} \bar{Q}_{k}^{\ b} A_{\delta\gamma}, \quad S_{a} \cdot A_{\delta\gamma} = -ix^{ak}_{\ a} Q_{a\beta} \cdot A_{\delta\gamma}. \]  

(A.1.15)

whereas the one of the special conformal generators takes the explicit form

\[ K^{\alpha}_{\ gamma} \cdot \Phi^{cd} = ix^{ak}_{\ a} D_{\beta\gamma} \Phi^{cd} + ix^{ak}_{\ a} \Phi^{cd}, \]
\[ K^{\alpha}_{\ gamma} \cdot \Psi^{c} = ix^{ak}_{\ a} D_{\beta\gamma} \Psi^{c} + ix^{ak}_{\ a} \Psi^{c} + i\delta^{k}_{\ c} \chi^{k}_{\ a} \Psi^{c}, \]
\[ K^{\alpha}_{\ gamma} \cdot \bar{\Psi}_{\epsilon d} = ix^{ak}_{\ a} D_{\beta\gamma} \bar{\Psi}_{\epsilon d} + ix^{ak}_{\ a} \bar{\Psi}_{\epsilon d} + i\delta^{k}_{\ \epsilon} \chi^{k}_{\ a} \bar{\Psi}_{\epsilon d}, \]
\[ K^{\alpha}_{\ gamma} \cdot A_{\delta\epsilon} = ix^{ak}_{\ a} D_{\beta\gamma} A_{\delta\epsilon}. \]  

(A.1.16)
Finally, the representation of a gauge transformation by the field $X$ is defined by
\[
G[X] \cdot \Phi^{cd} = [X, \Phi^{cd}],
G[X] \cdot \Psi^{c\delta} = [X, \Psi^{c\delta}],
G[X] \cdot \overline{\Psi}_{\gamma d} = [X, \overline{\Psi}_{\gamma d}],
G[X] \cdot A_{\gamma \delta} = iD_{\gamma \delta} X.
\] (A.1.17)

**A.2 Level-one generators**

In this appendix we will give explicit expressions for the coproducts as well as single-field actions of the level-one Yangian generators
\[
\hat{P}_{\beta \dot{\alpha}}, \quad \hat{Q}_{a \beta}, \quad \hat{\bar{Q}}^{b \dot{\alpha}} \quad \hat{R}_{a b} \quad \text{and} \quad \hat{B}.
\] (A.2.1)

These are the level-one generators which commute with the ordinary momentum $P$ (up to gauge artefacts) and hence their single-field action can be expected to have no explicit dependence on the position $x$.

**Level-one momentum.** The level-one momentum $\hat{P}$ has the following single field action:
\[
\hat{P}_{\beta \dot{\alpha}} \cdot \Phi^{cd} := 0,
\hat{P}_{\beta \dot{\alpha}} \cdot \Psi^{c\delta} := -\varepsilon_{\beta \delta} \{\Phi^{ce}, \overline{\Psi}_{\dot{\alpha} e}\},
\hat{P}_{\beta \dot{\alpha}} \cdot \overline{\Psi}_{\delta d} := -\varepsilon_{\dot{\alpha} \gamma} \{\overline{\Phi}_{de}, \Psi^{e \gamma}\},
\hat{P}_{\beta \dot{\alpha}} \cdot A_{\gamma \delta} := \frac{i}{2} \varepsilon_{\gamma \delta} \{\Phi^{ef}, \overline{\Phi}_{ef}\}.
\] (A.2.2)

The coproduct reads:
\[
\Delta \hat{P}_{\beta \dot{\alpha}} = \hat{P}_{\beta \dot{\alpha}} \otimes 1 + 1 \otimes \hat{P}_{\beta \dot{\alpha}}
+ P_{\gamma \dot{\alpha}} \wedge L^{\gamma \beta} + P_{\beta \dot{\gamma}} \wedge \overline{L}^{\gamma \dot{\alpha}} + \frac{1}{2} \tilde{Q}_{c \beta} \wedge \tilde{Q}^{c \dot{\alpha}}.
\] (A.2.3)

**Level-one supersymmetry.** The level-one supersymmetries $\hat{Q}$ and $\hat{\bar{Q}}$ have a non-trivial single-field action only on one of the fermionic fields:
\[
\hat{Q}_{a \beta} \cdot \Psi^{c\delta} = -\frac{1}{2} \delta_{a}^{c} \varepsilon_{\beta \delta} \{\Phi^{ef}, \overline{\Phi}_{ef}\},
\hat{\bar{Q}}^{b \dot{\alpha}} \cdot \overline{\Psi}_{\delta d} = -\frac{1}{2} \delta_{d}^{b} \varepsilon_{\dot{\alpha} \gamma} \{\Phi^{ef}, \overline{\Phi}_{ef}\}.
\] (A.2.4)

The single-field action on all the remaining fields $Z$ of the theory is:
\[
\hat{Q}_{a \beta} \cdot Z = 0, \quad \hat{\bar{Q}}^{b \dot{\alpha}} \cdot Z = 0.
\] (A.2.5)

The coproduct for $\hat{Q}$ is now:
\[
\Delta \hat{Q}_{a \beta} = \hat{Q}_{a \beta} \otimes 1 + 1 \otimes \hat{Q}_{a \beta}
+ Q_{a \gamma} \wedge L^{\gamma \beta} + \frac{1}{2} Q_{a \beta} \wedge D - \frac{1}{2} \tilde{Q}_{c \beta} \wedge \tilde{Q}^{c \dot{\alpha}} + i \gamma_{\beta} \wedge \tilde{S}_{\dot{\alpha} \gamma} - Q_{c \beta} \wedge R^{c \dot{\alpha}}.
\] (A.2.6)

and the corresponding expression for $\hat{\bar{Q}}$ reads:
\[
\Delta \hat{\bar{Q}}^{b \dot{\alpha}} = \hat{\bar{Q}}^{b \dot{\alpha}} \otimes 1 + 1 \otimes \hat{\bar{Q}}^{b \dot{\alpha}}
+ \bar{Q}^{b \dot{\gamma}} \wedge \bar{L}^{\gamma \dot{\alpha}} + \frac{1}{2} \hat{\bar{Q}}^{b \dot{\alpha}} \wedge D + i \gamma_{\dot{\alpha}} \wedge \bar{S}^{b \gamma} + \hat{\bar{Q}}^{c \dot{\alpha}} \wedge R^{b \dot{\alpha}}.
\] (A.2.7)
A.2. LEVEL-ONE GENERATORS

Level-one internal rotations. The generator $\hat{R}^a_b$ has a trivial single-field action

$$\hat{R}^a_b \cdot Z = 0. \quad (A.2.8)$$

Heuristically, this is easy to understand: The generator $\hat{R}$ carries zero mass dimension so its action on a field $Z$ should have the same mass dimension as $Z$, i.e. 1 for bosonic fields or $\frac{3}{2}$ for fermionic fields. Furthermore, the action should have no explicit position dependence as argued above. Together, this implies that the result can only be a single field. However, the parity-inverting property of the level-one generators requires to have at least two fields in the result.

The following coproduct thus defines this symmetry completely:

$$\Delta \hat{R}^a_b = \hat{R}^a_b \otimes 1 + 1 \otimes \hat{R}^a_b + R^a_c \wedge R^c_b - \frac{1}{2} \delta^a_b \delta^c_d \wedge Q_{c \gamma} - \frac{1}{2} \delta^a_c \delta^d_b \wedge Q_{a \gamma} + \frac{1}{8} \delta^a_b \delta^c_d \wedge Q_{c \gamma} - \frac{1}{8} \delta^a_c \delta^d_b \wedge Q_{a \gamma}. \quad (A.2.9)$$

Level-one bonus symmetry. The last generator to introduce here is a level-1 counterpart of a generator $B$ ($B$ can be viewed as a u(1) automorphism of $\mathfrak{psu}(2,2|4)$). The generator $B$ acts on the fields as follows:

$$B \cdot \Phi^{cd} = 0, \quad B \cdot \Psi^c_\delta = -\frac{i}{2} \Psi^c_\delta, \quad B \cdot \bar{\Psi}^d_{\dot{\gamma}} + \frac{i}{2} \bar{\Psi}^d_{\dot{\gamma}} + \frac{i}{2} \bar{\Psi}^d_{\dot{\gamma}}.$$

As can be easily verified, it is not a symmetry of $N = 4$ SYM. It has however been shown in [66] that $\hat{B}$ is a symmetry of the S-matrix of planar $N = 4$ SYM, leading to an interesting conclusion, that its full symmetry algebra is somewhere inbetween $\mathfrak{u}(2,2|4)$ and $\mathfrak{u}(2,2|4)$. In this work we have demonstrated that $\hat{B}$ is also a symmetry of the action of $N = 4$ SYM.

For the same reasons as for $\hat{R}$, the level-one bonus symmetry $\hat{B}$ has a trivial single-field action

$$\hat{B} \cdot Z = 0. \quad (A.2.11)$$

Its coproduct reads:

$$\Delta \hat{B} = \hat{B} \otimes 1 + 1 \otimes \hat{B} - \frac{1}{4} \delta^{\dot{\alpha}}_b \wedge Q_{\dot{\alpha} \beta} - \frac{1}{4} \delta^a \wedge Q_{a \beta}. \quad (A.2.12)$$
Appendix B

Representation theory of $\mathfrak{psu}(2, 2|4)$

In this Appendix we sketch the basics of the representation theory of $\mathfrak{psu}(2, 2|4)$. These are especially useful for the justification of the lack of anomalies in Chapter 10. The discussion will be based on [26] and [33] and we refer the reader to those sources for further details.

$\mathfrak{su}(2)$

The Lie algebra $\mathfrak{su}(2)$, two copies of which are spanned by the generators $L$ and $\bar{L}$ of $\mathfrak{psu}(2, 2|4)$, has rank 1 and as such its representations are labeled by one number, a half-integer spin $j$. The spin is an eigenvalue of the Casimir operator

$$C = -\frac{1}{4} ((L_1^1 - L_2^2)^2 + (L_1^2 - L_2^1)^2 - (-L_1^1 - L_2^2)^2)$$

(B.0.1)

such that: $C|j\rangle = j(j + 1)|j\rangle$. Alternatively, we can replace spin $j$ with an integer Dynkin label $s$ such that $s = 2j$.

Each field and state of $\mathcal{N} = 4sYM$ can then be assigned its $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ Dynkin labels $(s_1, s_2)$. For example, the spinors $\Psi^a_\alpha$ and $\bar{\Psi}^{\dot{a}}_\dot{\alpha}$ carry labels $(1, 0)$ and $(0, 1)$ respectively. The vector gauge field $A_{a\dot{a}}$ has labels $(1, 1)$ and a more familiar vector index for it can be restored by contraction with the extended Pauli matrices $\sigma^{\mu\dot{\alpha}} = (\mathbb{I}_2, \tilde{\sigma})^T$. This exhibits the isomorphism between $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{so}(4)$.

$\mathfrak{su}(4)$

The $\mathfrak{su}(4)$ algebra describes the R-symmetry of $\mathcal{N} = 4$ sYM. It is a rank-3 algebra and as such has three Dynkin labels $[p, q, r]$. They can be best read off from the Young tableaux of the representation: $r$ is the length of the third row, $q + r$ - of the second row, and $p + q + r$ - of the topmost one. Hence the following Young tableaux in Figure B.1 corresponds to Dynkin labels $[2, 3, 1]$.

In this notation, the fundamental representation is denoted as $[1, 0, 0]$ and the antifundamental $[0, 0, 1]$: fermionic fields $\Psi$ and $\bar{\Psi}$ take value in these representations respectively. The scalars are valued in the antisymmetric product of two fundamental representations, which is $[0, 1, 0]$ - its Young tableaux is depicted in Figure B.2.
APPENDIX B. REPRESENTATION THEORY OF $\mathfrak{psu}(2, 2|4)$

\[ \begin{array}{} & & & & \downarrow \end{array} \]

Figure B.1: Young tableaux corresponding to $[2, 3, 1]$ representation of $\mathfrak{su}(4)$.

\[ \begin{array}{} & & \end{array} \]

Figure B.2: Young tableaux corresponding to $[0, 1, 0]$ representation of $\mathfrak{su}(4)$ in which the scalar fields $\Phi^{ab}$ are valued.

$\mathfrak{psu}(2, 2|4)$

The bosonic subalgebra of $\mathfrak{psu}(2, 2|4)$ is not compact, hence the unitary representations are infinite-dimensional. The maximal compact subalgebra is the $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(4) \oplus \mathfrak{u}(1)$, with the $\mathfrak{u}(1)$ factor corresponding to the dilatations. Under this compact subalgebra, the irreducible representations (irreps) of $\mathfrak{psu}(2, 2|4)$ decompose into infinitely many finite dimensional representations. One of them will be the lowest one, annihilated by the generators $K$, $S$ and $\bar{S}$. The higher irreps will then be obtained by the action of $P$, $Q$ and $\bar{Q}$.

Let us now concentrate on only lowest weight irreps. As common in the theories with enhanced supersymmetry, some lowest weight states will be annihilated by a subset of the raising generators $Q, \bar{Q}$. Such states are called BPS states and they lead to what is known as short multiplets. An example of a $\frac{1}{2}$-BPS state (where $\frac{1}{2}$ indicates that it is annihilated by half of the supercharges $Q, \bar{Q}$) is:

\[ |L\rangle = \text{tr}(\Phi^{12})^L, \tag{B.0.2} \]

where $L$ is an integer. Such state will vanish under the action of $Q_3$, $Q_4$, $\bar{Q}_\alpha^1$ and $\bar{Q}_\alpha^2$, which indeed make up half of the supersymmetry generators. States which do not satisfy any shortening condition give rise to what is called long multiplets.

For a more detailed discussion of these issues see [107].
Appendix C

The fundamental representation of $u(2, 2|4)$

As mentioned in Chapter 3, the usual Killing form on $(ps)u(2, 2|4)$ vanishes, hence we cannot use it to raise and lower the indices on structure constants.

Indeed, usually a Killing form on the algebra would be introduced using the adjoint representation in which:

$$\rho_{Ad}(J^A) = (f^A)^B_C,$$  \hspace{1cm} (C.1)

where we use the bracket to make the matrix indices explicit. Then the Killing form is given by:

$$K^{AB} = sTr(\rho_{Ad}(J^A)\rho_{Ad}(J^B)) = (-1)^{|C|} f^{AC}_E f^{BE}_C.$$  \hspace{1cm} (C.2)

It can be checked by an explicit computation that for $psu(2, 2|4)$:

$$(-1)^{|C|} f^{AC}_E f^{BE}_C = 0$$  \hspace{1cm} (C.3)

and hence Killing form $K^{AB}$ vanishes.

Lowering of the indices however is necessary to define to coproduct of the level-1 Yangian generators, see (3.2.7). In this Appendix we will sketch how it nevertheless is possible to find a nondegenerate form on $psu(2, 2|4)$, thus circumventing this problem. We will follow the exposition given in [47].

Start by introducing the matrices $E^{A,B}$ whose elements are all 0 except for a single 1 in $A$th row and $B$th column.

For the generators $J^A \in \{P_\alpha\dot{\alpha}, L^\alpha_\beta, \tilde{L}^\dot{\alpha}_\beta, K^{\alpha\dot{\alpha}}, R^a_b, Q_{\alpha\dot{a}}, \bar{Q}_\dot{\alpha}^a, S^{a\alpha}, \bar{S}^{\dot{a}a}\}$ represent them as:

$$\rho_F(J^A) = \begin{pmatrix} E^\alpha_\beta - \frac{1}{2} \delta^\alpha_\beta 1 & E^{a\dot{\alpha}}_\beta & E_{\alpha\dot{\alpha}} \cr E^{\alpha\dot{\alpha}}_\beta & E^\dot{\alpha}_\beta & E^{\dot{\alpha}a}_\beta \cr E^a_\alpha & E^{a\alpha}_\beta & E^{a\dot{\alpha}}_\beta - \frac{1}{4} \delta^a_\beta \end{pmatrix},$$  \hspace{1cm} (C.4)

where the generator can be identified by the index structure, whereas for the remaining generators take:

$$\rho_F(D) = \begin{pmatrix} \frac{1}{2} 1 0 0 \\
0 - \frac{1}{2} 1 0 \\
0 0 0 0 \end{pmatrix}, \quad \rho_F(B) = \begin{pmatrix} 0 0 0 \\
0 0 0 \\
0 0 - \frac{1}{2} 1 \end{pmatrix}, \quad \rho_F(C) = \begin{pmatrix} \frac{1}{2} 1 0 0 \\
0 \frac{1}{2} 1 0 \\
0 0 \frac{1}{2} 1 \end{pmatrix}.$$  \hspace{1cm} (C.5)

Demanding $B = C = 0$ reduces the algebra to $psu(2, 2|4)$. 

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With these representation we can introduce the nondegenerate bilinear form $\kappa_F^{AB}$ as:

$$
\kappa_F^{AB} = sTr(\rho_F(J^A), \rho_F(J^B)).
$$

(C.6)

Its inverse $\kappa_{F,AB}$ defined by $\kappa_F^{AB} \kappa_{F,BC} = \delta_A^C$ can now be used to lower the index on the structure constants necessary to define the Yangian coproduct.
Appendix D

Yangian symmetry of $\beta$-deformed sYM: Appendix

In this Appendix we explicitly spell out the formulae necessary to reproduce the results of Chapter 5. This Appendix follows [59].

D.1 Twisted R-symmetry

After the Drinfeld-Reshetikhin twist (5.2.3) is applied to the coproduct, the latter becomes nontrivial, as given by (5.2.6). The function $r(a, b, i)$ appearing in the action (5.2.8) is given by:

$$r(a, b, i) = (\delta^a_4 + \delta^b_4) \sum_{c=1}^{4} (\varepsilon_{abc} + (1 - \delta^a_4 - \delta^b_4)(-2\varepsilon_{abc} + (1 - |\varepsilon_{abc}|)\text{sig}(a, b))). \quad (D.1)$$

The antisymmetric function $\text{sig}(a, b) = -\text{sig}(b, a)$ is then given by: $\text{sig}(1, 2) = \text{sig}(2, 3) = \text{sig}(3, 1) = 1$, $\text{sig}(a, 4) = 0$.

D.2 Action of supersymmetry generators

D.2.1 Nondeformed $\mathcal{N} = 1$ supersymmetry

The manifest $\mathcal{N} = 1$ SUSY generators $Q_\alpha$ and $\bar{Q}_{\dot{\alpha}}$ act on the fields in the following way:

$$Q_\alpha \phi^i = \sqrt{2} i \psi^i_{\alpha} \quad (D.1)$$
$$Q_\alpha \bar{\phi}_i = 0 \quad (D.2)$$
$$Q_\alpha \psi^i_{\dot{\beta}} = -\frac{1}{\sqrt{2}} \varepsilon_{\alpha\beta} \varepsilon^{ijk}[\bar{\phi}_j, \bar{\phi}_k]_{\beta, jk} \quad (D.3)$$
$$Q_\alpha \bar{\psi}_{i\dot{\beta}} = -\sqrt{2} i \sigma^\mu_{\alpha\dot{\beta}}[D_\mu, \bar{\phi}_i] \quad (D.4)$$
$$Q_\alpha \psi_{4\beta} = -\frac{1}{\sqrt{2}} \varepsilon^{i\kappa}[D_\alpha \bar{\gamma}, D_{\beta\dot{k}}] - \frac{1}{\sqrt{2}} \varepsilon_{\alpha\beta}[\phi^i, \bar{\phi}_i] \quad (D.5)$$
$$Q_\alpha \bar{\psi}_{4\dot{\beta}} = 0$$
$$Q_\alpha D_\beta \gamma = -\sqrt{2} \varepsilon_{\alpha\beta} \bar{\psi}_{4\dot{\gamma}}, \quad (D.6)$$
and analogous formulae for \( \bar{Q}_{\dot{a}} \).

**D.2.2 The hidden supersymmetry generators**

The action of \( Q_i^\alpha \) and \( \bar{Q}_{i\dot{\alpha}} \) is to be obtained by commuting the manifest supersymmetry generators with R-symmetry generators: \( Q_{i\alpha} = [R^i, Q_\alpha] \). For the sake of completeness, since they appear explicitly in the deformed coproduct for the level-1 momentum generator [5.3.2], we present the results explicitly.

\[
\begin{align*}
Q_{i\alpha} \phi_j &= \sqrt{2} \, i \delta_i^j \psi_4^\alpha, \\
Q_{i\alpha} \bar{\phi}_j &= \sqrt{2} \, i \epsilon_{ijk} \psi_k^\alpha, \\
Q_{i\alpha} \psi_j^\beta &= \frac{1}{\sqrt{2}} \delta_i^j \left( \epsilon^{\dot{\alpha}\dot{\beta}} [D_{\alpha\dot{\alpha}}, D_{\beta\dot{\beta}}] + \epsilon_{\alpha\beta} [\phi^k, \bar{\phi}_k] \right) - \sqrt{2} \epsilon_{\alpha\beta} [\phi^j, \bar{\phi}_j]_{\beta j}, \\
Q_{i\alpha} \bar{\psi}_j^\dot{\alpha} &= \sqrt{2} \, i \epsilon_{ijk} [D_{\alpha\dot{\alpha}}, \phi^k], \\
Q_{i\alpha} \bar{\psi}_j^4 &= -\epsilon_{\alpha\beta} \sqrt{2} \epsilon_{ijk} [\phi^j, \phi^k]_{\beta j}, \\
Q_{i\alpha} D_{\beta\dot{\alpha}} &= \sqrt{2} \epsilon_{\alpha\beta} \psi_{i\dot{\beta}}. 
\end{align*}
\]

Again similar formulae hold for \( \bar{Q}_{\dot{a}i} \).

The coproduct for the generators \( Q_{i\alpha} \) gets twisted:

\[
\Delta Q_{i\alpha} = \mathbb{K}_{i\alpha} \otimes Q_{i\alpha} + Q_{i\alpha} \otimes \mathbb{K}_{i\alpha}^{-1}. 
\]

The twist generator \( \mathbb{K}_{i\alpha} \) has the following action on fields:

\[
\begin{align*}
\mathbb{K}_{i\alpha} \phi^j &= e^{\pi \text{sig}(i,j)\beta} \phi^j, \\
\mathbb{K}_{i\alpha} \bar{\phi}_j &= e^{-\pi \text{sig}(i,j)\beta} \bar{\phi}_j, \\
\mathbb{K}_{i\alpha} \psi^j_\gamma &= e^{\pi \text{sig}(i,j)\beta} \psi^j_\gamma, \\
\mathbb{K}_{i\alpha} \bar{\psi}_j^\dot{\alpha} &= e^{-\pi \text{sig}(i,j)\beta} \bar{\psi}_j^\dot{\alpha}, \\
\mathbb{K}_{i\alpha} Z &= Z, 
\end{align*}
\]

where the \( \text{sig}(a, b) \) function has been defined in [D.1]

**D.3 Equations of motion**

Since we work with equations of motion, we list the variations of the action here:

\[
\frac{\delta S}{\delta D_{\gamma\dot{\kappa}}} = -\frac{1}{2} \epsilon^{\alpha\beta} \epsilon^{\gamma\kappa} \epsilon^{\dot{\alpha}\dot{\beta}} \epsilon^{\dot{\gamma}\dot{\kappa}} [D_{\alpha\beta}, [D_{\dot{\alpha}\dot{\beta}}, D_{\gamma\dot{\kappa}}]] - \frac{1}{2} \epsilon^{\alpha\gamma} \epsilon^{\dot{\alpha}\dot{\kappa}} ([\bar{\phi}_i, [D_{\alpha\dot{\alpha}}, \phi^i]] + [\phi^i, [D_{\alpha\dot{\alpha}}, \bar{\phi}_i]]) \\
+ \epsilon^{\alpha\gamma} \epsilon^{\dot{\alpha}\dot{\kappa}} \{\bar{\psi}_j^\dot{\beta}, \psi^4_\alpha\} + \epsilon^{\alpha\gamma} \epsilon^{\dot{\alpha}\dot{\kappa}} \{\bar{\psi}_i^\dot{\beta}, \psi^4_\alpha\} \\
\frac{\delta S}{\delta \phi^i} = \frac{1}{2} \epsilon^{\alpha\beta} \epsilon^{\dot{\alpha}\dot{\beta}} [D_{\alpha\dot{\alpha}}, [D_{\beta\dot{\beta}}, \phi^i]] - \frac{1}{2} [\bar{\phi}_j, [\phi^i, \phi^j]]_{\beta j} + \frac{1}{4} [\phi^i, [\phi^j, \bar{\phi}_j]]
\]

\[(D.1)\]
\[ + \epsilon^{\alpha\beta}\{\psi^A_{\alpha}, \psi^\dagger_{\beta}\} + \frac{i}{2} \epsilon^{ijk}\epsilon^{\dot{\alpha}\dot{\beta}}\{\bar{\psi}_{j\dot{\alpha}}, \bar{\psi}_{k\dot{\beta}}\}\beta_{jk} \] (D.2)

\[ \frac{\delta S}{\delta \psi^A_{\dot{\alpha}4}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\beta}\dot{\alpha}}[D_{\beta\dot{\beta}}, \psi^A_{\alpha}] + i\epsilon^{\dot{\alpha}\dot{\beta}}[\bar{\psi}_{i\dot{\beta}}, \phi^i] \] (D.3)

\[ \frac{\delta S}{\delta \psi^i_{\dot{\alpha}a}} = \epsilon^{\alpha\beta} \epsilon^{\dot{\beta}\dot{\alpha}}[D_{\beta\dot{\beta}}, \psi^i_{\alpha}] - i\epsilon^{\dot{\alpha}\dot{\beta}}[\bar{\psi}^4_{4\dot{\beta}}, \phi^i] + \frac{i}{2} \epsilon^{ijk}\epsilon^{\dot{\alpha}\dot{\beta}}[\bar{\psi}_{j\dot{\beta}}, \phi_k]\beta_{ij} \] (D.4)

together with their respective conjugates. Putting fields on-shell corresponds of course to setting \( \frac{\delta S}{\delta Z_I} = 0 \), where \( Z_I \) stands for any of the fields.
Appendix E

Level-one invariance of the three-point function at one loop

In this appendix following [61] we show the Yangian invariance of the three-point function at one loop explicitly (modulo gauge fixing and regularisation). Recall that symmetry variation of the correlator yields

\( \langle \hat{J} \cdot (Z_1 Z_2 Z_3) \rangle_{(1)} \)

\[ \approx -i \begin{array}{c}
\text{diagram 1}
\end{array} - 3 \begin{array}{c}
\text{diagram 2}
\end{array} + \begin{array}{c}
\text{diagram 3}
\end{array} + i \begin{array}{c}
\text{diagram 4}
\end{array} \]

\[ -i \begin{array}{c}
\text{diagram 5}
\end{array} - i \begin{array}{c}
\text{diagram 6}
\end{array} + i \begin{array}{c}
\text{diagram 7}
\end{array} - 3 \begin{array}{c}
\text{diagram 8}
\end{array} - \begin{array}{c}
\text{diagram 9}
\end{array} + \begin{array}{c}
\text{diagram 10}
\end{array} \]

\[ - i \begin{array}{c}
\text{diagram 11}
\end{array} - i \begin{array}{c}
\text{diagram 12}
\end{array} + i \begin{array}{c}
\text{diagram 13}
\end{array} + 3i \begin{array}{c}
\text{diagram 14}
\end{array} + i \begin{array}{c}
\text{diagram 15}
\end{array} - i \begin{array}{c}
\text{diagram 16}
\end{array} \]

\[ - 3 \begin{array}{c}
\text{diagram 17}
\end{array} - \begin{array}{c}
\text{diagram 18}
\end{array} - \begin{array}{c}
\text{diagram 19}
\end{array} + i \begin{array}{c}
\text{diagram 20}
\end{array} \]

\[ - 3 \begin{array}{c}
\text{diagram 21}
\end{array} + \begin{array}{c}
\text{diagram 22}
\end{array} + \begin{array}{c}
\text{diagram 23}
\end{array} - i \begin{array}{c}
\text{diagram 24}
\end{array} \]

We can cancel all diagrams by adding the following terms, all of which are zero by invariance of the action and commutativity of the level-zero generators

\[ -3i \begin{array}{c}
\text{diagram 25}
\end{array} \approx +i \begin{array}{c}
\text{diagram 26}
\end{array} - i \begin{array}{c}
\text{diagram 27}
\end{array} + i \begin{array}{c}
\text{diagram 28}
\end{array} + 3 \begin{array}{c}
\text{diagram 29}
\end{array} - \begin{array}{c}
\text{diagram 30}
\end{array} + \begin{array}{c}
\text{diagram 31}
\end{array} \]

\[ + 3 \begin{array}{c}
\text{diagram 32}
\end{array} - \begin{array}{c}
\text{diagram 33}
\end{array} + \begin{array}{c}
\text{diagram 34}
\end{array} + 3 \begin{array}{c}
\text{diagram 35}
\end{array} + \begin{array}{c}
\text{diagram 36}
\end{array} - \begin{array}{c}
\text{diagram 37}
\end{array} \]

\[ + i \begin{array}{c}
\text{diagram 38}
\end{array} \approx + i \begin{array}{c}
\text{diagram 39}
\end{array} - i \begin{array}{c}
\text{diagram 40}
\end{array} + i \begin{array}{c}
\text{diagram 41}
\end{array} + \frac{1}{2} \begin{array}{c}
\text{diagram 42}
\end{array} + \frac{1}{2} \begin{array}{c}
\text{diagram 43}
\end{array} + \frac{1}{2} \begin{array}{c}
\text{diagram 44}
\end{array} \]

\[ - i \begin{array}{c}
\text{diagram 45}
\end{array} \approx + i \begin{array}{c}
\text{diagram 46}
\end{array} + i \begin{array}{c}
\text{diagram 47}
\end{array} + i \begin{array}{c}
\text{diagram 48}
\end{array} - \frac{1}{2} \begin{array}{c}
\text{diagram 49}
\end{array} - \frac{1}{2} \begin{array}{c}
\text{diagram 50}
\end{array} - \frac{1}{2} \begin{array}{c}
\text{diagram 51}
\end{array} \]
\begin{align}
-i & \simeq -i - 2i - 3i + i , \quad \text{(E.5)} \\
+ \frac{i}{2} & \simeq + \frac{i}{2} + \frac{i}{2} + \frac{1}{2} - \frac{i}{2} + \frac{1}{2} , \quad \text{(E.6)} \\
- \frac{i}{2} & \simeq - \frac{i}{2} + \frac{i}{2} - \frac{1}{2} + \frac{1}{2} - \frac{i}{2} , \quad \text{(E.7)} \\
- \frac{1}{2} & \simeq + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - i - \frac{i}{2} - \frac{i}{2} , \quad \text{(E.8)} \\
+ \frac{1}{2} & \simeq - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} - i + \frac{i}{2} + \frac{i}{2} , \quad \text{(E.9)} \\
-3i & \simeq +i - i + i + 3 + 3 - 3 - 3 , \quad \text{(E.10)} \\
-3i & \simeq -i - i + i + 3 + 3 - 3 - 3 , \quad \text{(E.11)} \\
-3i & \simeq +i + i - i + 3 - 3 + 3 + 3 , \quad \text{(E.12)} \\
+3 & \simeq -i + i - i + 3 + 3 - 3 - 3 , \quad \text{(E.13)} \\
+i & \simeq +i + i + i , \quad \text{(E.14)} \\
-i & \simeq -i + i - i , \quad \text{(E.15)} \\
+i & \simeq +i - i - i , \quad \text{(E.16)} \\
-i & \simeq -i - i + i , \quad \text{(E.17)} \\
+i & \simeq +i - i - i , \quad \text{(E.18)} \\
-i & \simeq -i + i + i , \quad \text{(E.19)} \\
+i & \simeq -i + i + i , \quad \text{(E.20)}
\end{align}
\[-i + i - i - i - i,\]  
(E.20)

\[-3 + \frac{3}{2} - \frac{3}{2} + \frac{3}{2} - \frac{3}{2},\]  
(E.21)

\[-3 - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} + \frac{3}{4},\]  
(E.22)

\[+ \frac{3i}{4} - \frac{3i}{4} - \frac{3i}{4} + \frac{3i}{4},\]  
(E.23)

\[-3 - i - i - i - i,\]  
(E.24)

\[-3 - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} + \frac{3}{4},\]  
(E.25)

\[-3 + \frac{3}{4} - \frac{3}{4} + \frac{3}{4} - \frac{3}{4},\]  
(E.26)

\[-3 - \frac{3}{4} + \frac{3}{4} - \frac{3}{4} + \frac{3}{4},\]  
(E.27)

\[-\frac{3}{4} - \frac{3}{4} + \frac{3}{4} + \frac{3}{4} - \frac{3}{4},\]  
(E.28)
\begin{equation}
\frac{1}{4} + \frac{1}{4} \simeq -\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4},
\end{equation}
(E.29)

\begin{equation}
-\frac{1}{4} \simeq -\frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4},
\end{equation}
(E.30)

\begin{equation}
-\frac{1}{2} \simeq + \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + i \frac{1}{2},
\end{equation}
(E.31)

\begin{equation}
-\frac{1}{2} \simeq + \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - i \frac{1}{2},
\end{equation}
(E.32)

\begin{equation}
+ \frac{1}{2} \simeq - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} - \frac{1}{2} - i \frac{1}{2},
\end{equation}
(E.33)

\begin{equation}
+ \frac{1}{2} \simeq - \frac{1}{2} - \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + i \frac{1}{2},
\end{equation}
(E.34)

\begin{equation}
- \frac{1}{8} \simeq + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + i \frac{1}{8},
\end{equation}
(E.35)

\begin{equation}
- \frac{1}{8} \simeq + \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - i \frac{1}{8},
\end{equation}
(E.36)

\begin{equation}+ \frac{1}{8} \simeq - \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \frac{1}{4} + i \frac{1}{8},
\end{equation}
(E.37)

\begin{equation}+ \frac{1}{8} \simeq - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} + \frac{1}{4} + i \frac{1}{8},
\end{equation}
(E.38)

\begin{equation}
\frac{i}{8} \simeq + \frac{i}{4} + \frac{i}{4} + \frac{i}{4} + \frac{i}{4},
\end{equation}
(E.39)

\begin{equation}
- \frac{i}{8} \simeq - \frac{i}{4} - \frac{i}{4} - \frac{i}{4} - \frac{i}{4}.
\end{equation}
(E.40)
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