ACTIVE VIBRATION CONTROL OF LIGHTWEIGHT FLOOR SYSTEMS

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(Dr. sc. ETH Zurich)

presented by

JAKOB FRIEDRICH BAADER

ing. méc. dipl. EPF (MSc ETH)
born on 15.07.1986
citizen of Gelterkinden, BL

accepted on the recommendation of:

Prof. Dr. Mario Fontana (ETH Zürich)
Prof. Dr. Thomas Gmütt (EPF Lausanne)
Dr. F. Weber (Maurer Switzerland GmbH)

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Abstract

Lightweight and wide-span structures such as bridges, deckings or floors are often susceptible to vertical vibrations. Depending on the amplitude and frequency, these vibrations can affect the serviceability or even lead to severe fatigue problems. The deadweight and stiffness play an important role in the dynamic behaviour of a structure. Stiffer and heavier structures usually have lower vibration amplitudes after an occurring excitation. However, stiffness and deadweight are closely linked to the amount of building material used for the structure. Therefore, a heavy and stiff structure has advantageous dynamic properties, but is not resource-efficient and has higher overall construction costs compared to a light structure. For every potentially vibrating structure, a compromise must be found between the amount of building material and the dynamic properties.

Many civil engineering structures such as bridges and high-rise buildings are equipped today with passive or semi-active vibration control devices, which considerably improve the serviceability of the structure. However, floors and substructures within buildings are usually not equipped with such devices. This study has shown that the application of vibration control can be advantageous, especially for wide-span and lightweight floor structures.

Floors and other flat structures can only be mitigated by inertia-based damping devices. The famous passive tuned mass damper (TMD) is usually applied for this type of vibration problem. However, this study shows that the use of active vibration control devices can be advantageous, especially when load scenarios are uncertain or structural components can change over time. Obviously, there are some implementation restrictions which need to be considered when designing the structure and the according vibration control method. These restrictions are shown in the first and second part of this study.

The control system is an important component for the damping efficiency of an active vibration control device. In this study, different control architectures were analysed and compared in simulations and experiments to find a powerful and robust controller for the vertical vibration problem. The most promising results were achieved by using controllers with an acceleration or a combined velocity feedback. The acceleration feedback showed a higher performance, while the combined velocity feedback provided a higher robustness and at the same time an reasonable performance. A reduction of the peak dynamic amplification factor by
more than 96% compared to the undamped system was achieved by a velocity-based control in simulations and experiments.

The problem of the overall height of a vibration mitigation device is analysed in the fourth part of this study. Initial experiments have shown that the stroke of today’s standard devices is limited by the elastic suspension of the damper mass. A novel construction with a wheel-suspended mass was successfully tested in the third and fourth experiments. This design allows exploiting the whole available space for the stroke of the damper mass. In combination with the high-force actuator and the novel dynamic stroke limitation, the system is very well adapted for application in buildings.

The final part of this study analyses the economic benefit of the application of active vibration control. Depending on the usage of the building, the floor layout and the applied materials, an economical benefit is possible by reducing the stiffness and mass of a floor. However, to guarantee the serviceability in this case, vibration control has to be implemented. The costs for the vibration control need to be lower than the projected benefit. The better serviceability thanks to the lower vibration amplitudes cannot be quantified in an objective manner and is therefore not considered in the economic analysis. However, the application of active vibration control devices leads to installation costs and running maintenance costs. Thus, the overall building cost strongly depends on the building layout and its considered lifetime.
Zusammenfassung


Die Steuerung des aktiven Schwingungsdämpfers ist ein entscheidender Punkt für die Wirksamkeit der Bedämpfung. Verschiedene Regler und Regelkreise wurden in dieser Studie analysiert und verglichen, um eine robuste und wirksame Bedämpfung für vertikale Strukturschwingungen zu finden. Die besten Resultate wurden mit Beschleunigungs- oder Geschwindigkeitsrückführungen der Hauptstruktur erreicht. Dabei zeigte die Beschleunigungsrückführung eine höhere Wirksamkeit, die Geschwindigkeitsrückführung jedoch eine bessere Robustheit bei gleichzeitig guter Wirksamkeit. Eine Reduktion von 96% des maximalen dynamischen Verstärkungs-
faktors im Vergleich zum unbedämpften System wurde mit der Geschwindigkeitsrückführung in Simulationen und Experimenten erreicht.


Acknowledgment

This PhD thesis was developed at the Institute of Structural Engineering (IBK) at the ETH Zürich under the supervision of Prof. Dr. M. Fontana. At this point, I would like to thank all of the people who supported me during this project.

First of all, I want to express my sincere thanks to my supervisor Prof. Dr. Mario Fontana for giving me the opportunity to work on this project. Although the topic of active vibration control is not a main research field of Prof. Dr. Fontana, his inputs and suggestions were always inspiring and very helpful. On many occasions, his different view on the topic helped me to take a step back and find another way.

Another key person during this thesis was Dr. Kurt Bitterli. Throughout this project, I was engaged part-time at his company K.Bitterli+Partner Ingenieure AG. Thus, I want to thank Dr. Bitterli and the whole team for the opportunity to work on this PhD thesis and for their support.

A KTI project (No 14871.1 PFIW-IW) about vibration control on conveyor bridges which I conducted together with Dr. Felix Weber at the EMPA created the basis of this thesis. Therefore, I would like to express my sincere thanks to Felix Weber. I strongly appreciated his visits and our technical discussions during this thesis. His profound knowledge about inertia-based vibration control were a great source of inspiration for me.

The experimental part of this thesis would not have been possible without the great support of the laboratory staff (Bauhalle-team). I want to thank in particular Patrick Morf for his help with the equipment and the interesting discussions.

A special thanks is also due to the co-examiners Prof. Dr. T. Gmür and Dr. F. Weber for their detailed lecture of this thesis and their helpful feedback.

Finally, I want to thank my parents for their support and good will. A special thanks is due my wife Annemarie for her everlasting encouragement. It would not have been possible without your support.
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Abbreviations and definitions

The following table provides definitions for some standard expressions used in this study. Some terminologies are explained in detail in this study while others are standard expressions in the field of structural dynamics or control theory. For more information, the interesting reader is referred to the standard literature of system identification and control [44, 45, 43, 36, 29] and dynamics of structures [59, 28, 18].

<table>
<thead>
<tr>
<th>Expression</th>
<th>Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>AMD</td>
<td>Active mass driver</td>
</tr>
<tr>
<td>ATMD</td>
<td>Active or adaptive tuned mass damper</td>
</tr>
<tr>
<td>Auxiliary structure</td>
<td>Spring-damper-mass system of the damping device, connected to the main structure. The auxiliary structure can be extended by an actuator or semi-active component.</td>
</tr>
<tr>
<td>Auxiliary system</td>
<td>Mathematical model of the auxiliary structure.</td>
</tr>
<tr>
<td>AVCD</td>
<td>Active vibration control device</td>
</tr>
<tr>
<td>Closed-loop system</td>
<td>Control structure with closed feedback loop.</td>
</tr>
<tr>
<td>CLTF</td>
<td>Closed-loop transfer function</td>
</tr>
<tr>
<td>CVF</td>
<td>Combined velocity feedback</td>
</tr>
<tr>
<td>Damping device</td>
<td>Auxiliary vibrating structure which is used to mitigate the main structure.</td>
</tr>
<tr>
<td>Dynamic amplification factor (DAF)</td>
<td>Frequency dependent amplification of the systems amplitude (normalised for static disturbances)</td>
</tr>
<tr>
<td>Fourier transform</td>
<td>Integral transformation from the time (or any other real coordinate) domain into the frequency domain.</td>
</tr>
<tr>
<td>Laplace transform</td>
<td>Integral transformation from the time domain into the complex frequency domain. The variable $s = j\omega$ is the complex frequency parameter.</td>
</tr>
<tr>
<td>LTI-system</td>
<td>Linear-time-invariant model of a dynamic process or structure.</td>
</tr>
<tr>
<td>OLTF</td>
<td>Open-loop transfer function</td>
</tr>
<tr>
<td>Main or primary structure</td>
<td>Vibrating structure which is desired to be mitigated by the vibration control device.</td>
</tr>
</tbody>
</table>
Expression Definition

<table>
<thead>
<tr>
<th>Main system</th>
<th>Mathematical model of the vibrating main structure.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Open-loop system</td>
<td>Control structure without feedback.</td>
</tr>
<tr>
<td>TMD</td>
<td>Tuned mass damper</td>
</tr>
<tr>
<td>Transfer function</td>
<td>Function to define the output of a dynamic system due to an arbitrary input.</td>
</tr>
</tbody>
</table>

Time-domain derivations are defined in the time and Laplace-domain as [43]:

\[
\dot{x}(t) = \frac{dx(t)}{dt} \iff sX(s) - x(0) \\
\ddot{x}(t) = \frac{d^2x(t)}{dt^2} \iff s^2 X(s) - sx(0) - \dot{x}(0)
\]

### Nomenclature

<table>
<thead>
<tr>
<th>variable</th>
<th>formula</th>
<th>unit</th>
<th>description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y(t))</td>
<td>(g(t) * u(t))</td>
<td>var.</td>
<td>Output of a dynamic system after input (u(t))</td>
</tr>
<tr>
<td>(g(t))</td>
<td></td>
<td>var.</td>
<td>Impulse response of a dynamic system</td>
</tr>
<tr>
<td>(u(t))</td>
<td></td>
<td>var.</td>
<td>Input to the dynamic system</td>
</tr>
<tr>
<td>(d(t))</td>
<td></td>
<td>var.</td>
<td>Disturbance on the dynamic system</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(s)</th>
<th>(j\omega)</th>
<th>(\sqrt{-1}) [rad/s]</th>
<th>Complex Laplace parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Y(s))</td>
<td>(G(s)U(s))</td>
<td>[-]</td>
<td>Laplace-transformed system output</td>
</tr>
<tr>
<td>(G(s))</td>
<td></td>
<td>[-]</td>
<td>Laplace-transformed impulse response, OLTf</td>
</tr>
<tr>
<td>(U(s))</td>
<td></td>
<td>[-]</td>
<td>Laplace-transformed system input</td>
</tr>
<tr>
<td>(D(s))</td>
<td></td>
<td>[-]</td>
<td>Laplace-transformed disturbance</td>
</tr>
<tr>
<td>(K(s))</td>
<td></td>
<td>[-]</td>
<td>Laplace-transformed controller matrix</td>
</tr>
<tr>
<td>(G_{CL}(s))</td>
<td></td>
<td>[-]</td>
<td>Laplace-transformed CLTF</td>
</tr>
<tr>
<td>variable</td>
<td>formula</td>
<td>unit</td>
<td>description</td>
</tr>
<tr>
<td>----------</td>
<td>---------</td>
<td>------</td>
<td>-------------</td>
</tr>
<tr>
<td>$c$</td>
<td>$2\xi_1\omega_1 m_1$</td>
<td>[Ns/m]</td>
<td>General viscous damping parameter</td>
</tr>
<tr>
<td>$c_1$</td>
<td>$2\xi_1\omega_1 m_1$</td>
<td>[Ns/m]</td>
<td>Viscous damping of the vibrating structure</td>
</tr>
<tr>
<td>$c_{1d}$</td>
<td>$2\xi_1\omega_1 m_1$</td>
<td>[Ns/m]</td>
<td>Desired viscous main damping</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$2\xi_2\omega_2 m_2$</td>
<td>[Ns/m]</td>
<td>Viscous damping of the auxiliary system</td>
</tr>
<tr>
<td>$c_{2d}$</td>
<td>$2\xi_2\omega_2 m_2$</td>
<td>[Ns/m]</td>
<td>Desired viscous auxiliary damping</td>
</tr>
<tr>
<td>$f$</td>
<td></td>
<td>[N]</td>
<td>Actuator force</td>
</tr>
<tr>
<td>$k_1$</td>
<td>$\omega_1^2 m_1$</td>
<td>[N/m]</td>
<td>Linear stiffness of the main system</td>
</tr>
<tr>
<td>$k_{1d}$</td>
<td>$\omega_1^2 m_1$</td>
<td>[N/m]</td>
<td>Desired main stiffness</td>
</tr>
<tr>
<td>$k_2$</td>
<td>$\omega_2^2 m_2$</td>
<td>[N/m]</td>
<td>Linear stiffness of the auxiliary system</td>
</tr>
<tr>
<td>$k_{2d}$</td>
<td>$\omega_2^2 m_2$</td>
<td>[N/m]</td>
<td>Desired auxiliary stiffness</td>
</tr>
<tr>
<td>$m$</td>
<td></td>
<td>[kg]</td>
<td>General mass property</td>
</tr>
<tr>
<td>$m_1$</td>
<td></td>
<td>[kg]</td>
<td>Modal mass of the vibrating structure</td>
</tr>
<tr>
<td>$m_2$</td>
<td></td>
<td>[kg]</td>
<td>Damper mass</td>
</tr>
<tr>
<td>$x_1$</td>
<td></td>
<td>[m]</td>
<td>Absolute position of the main mass</td>
</tr>
<tr>
<td>$x_2$</td>
<td></td>
<td>[m]</td>
<td>Absolute position of the auxiliary mass</td>
</tr>
<tr>
<td>$y_1$</td>
<td>$x_1$</td>
<td>[m]</td>
<td>Absolute position of the main mass</td>
</tr>
<tr>
<td>$y_2$</td>
<td>$x_2 - x_1$</td>
<td>[m]</td>
<td>Relative position between auxiliary and main mass</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>$\omega_1^2 / \omega_2$</td>
<td>[-]</td>
<td>Tuning of the damper resonance frequency</td>
</tr>
<tr>
<td>$\beta$</td>
<td>$s / \omega_1$</td>
<td>[-]</td>
<td>Normalised Laplace parameter</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$m_2 / m_1$</td>
<td>[-]</td>
<td>Normalised damper mass</td>
</tr>
<tr>
<td>$\xi_1$</td>
<td>$c_1 / 2 m_1 \omega_1$</td>
<td>[-]</td>
<td>Main modal viscous damping ratio</td>
</tr>
<tr>
<td>$\xi_{1d}$</td>
<td>$c_1 / 2 m_1 \omega_1$</td>
<td>[-]</td>
<td>Desired main modal viscous damping ratio</td>
</tr>
<tr>
<td>$\xi_2$</td>
<td>$c_2 / 2 m_2 \omega_2$</td>
<td>[-]</td>
<td>Auxiliary viscous damping ratio</td>
</tr>
<tr>
<td>$\xi_{2d}$</td>
<td>$c_2 / 2 m_2 \omega_2$</td>
<td>[-]</td>
<td>Desired auxiliary viscous damping ratio</td>
</tr>
<tr>
<td>$\omega$</td>
<td>$\sqrt{k_1 / m_1}$</td>
<td>[rad/s]</td>
<td>General frequency (pulsation)</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\sqrt{k_1 / m_1}$</td>
<td>[rad/s]</td>
<td>Main resonance frequency (standard modal frequency)</td>
</tr>
<tr>
<td>$\omega_2$</td>
<td>$\sqrt{k_2 / m_2}$</td>
<td>[rad/s]</td>
<td>Auxiliary resonance frequency</td>
</tr>
<tr>
<td>$\omega_d$</td>
<td>$\sqrt{k_2 / m_2}$</td>
<td>[rad/s]</td>
<td>Disturbance frequency</td>
</tr>
<tr>
<td>$S$</td>
<td>$(I + GK)^{-1}$</td>
<td>[-]</td>
<td>Sensitivity function</td>
</tr>
<tr>
<td>$T$</td>
<td>$(I + GK)^{-1}GK$</td>
<td>[-]</td>
<td>Complementary sensitivity function</td>
</tr>
<tr>
<td>$L$</td>
<td>$GK$</td>
<td>[-]</td>
<td>Loop transfer function</td>
</tr>
</tbody>
</table>
Introduction

The current document is divided into six parts. The first part provides a general introduction to the subject, in which the motivation and scope of this study are described. The modelling and simulation procedures as well as performance measures are shown in the second part. In the third part, different active vibration control methods are explained and compared. The fourth part is dedicated to the experimental work of this study, whereby the implementation and setup are shown in detail and experimental results are discussed. The fifth part presents a brief economic analysis using two different example building layouts. The final part provides a conclusion to this study and highlights areas for possible future work.

1.1 Motivation

Vibration problems of buildings and civil structures are today usually closely linked to the field of earthquake engineering. Seismic events can indeed easily lead to the collapse of a building or parts of it. Vibration problems within the building are usually less severe for the structural safety but very important for the serviceability of a structure. Many examples of floors in gymnastic and concert halls, operation theatres, bridges and other structures can be found in literature, where structural vibrations required a retrofit shortly after their construction. In the 1980s, people started to realise that the ongoing trend towards slender structures leads to dynamic problems. Prof. H. Bachmann from the ETH Zürich analysed several structures with dynamic problems. He realised that vibrations occur often on wide span and slender structures [9]. Common solutions at this time were to increase either the stiffness or the deadweight of the structure. First attempts with external damping devices (tuned mass dampers) were made. This analysis also created the basis of the specifications for floor dynamics in the Swiss building code [8, 3].

Bachmann [9] showed that mostly vertical vibrations of floors and catwalks are dominant. These structures are usually excited by human activities like walking or running. Today’s solution to these dynamic problems of civil structures are the
same as 30 years ago. One solution is to increase the bending stiffness of the structure, which will raise the resonance frequency but also increase the material consumption. The second solution is to add more weight to the structure to decrease its excitability. For this solution, an increase of the bending stiffness is indispensable to compensate the higher loads. Design limitations are reached for wide-span and slender structures.

An example calculation of a double deck gymnastic hall illustrates the problem of vibrations in buildings. The static calculation of the middle deck is carried out according to the Swiss building standards SIA 260 et sqq [3]. A distributed load of 5 kN is assumed on the simply-supported joist structure with a span of 16 m. The concrete decking of 12 cm is only taken into account as a load distributing plate between the main steel-joist structure with a spacing of 2 m. Figure 1.1 shows the utilisation factor for different profile heights for the HEA and HEB standard profiles. It is obvious that the requirements for the ultimate limit state (ULS) and the serviceability limit state (SLS) are fulfilled (utilisation < 1) for every profile stronger than a HEA 500. However, the fundamental resonance frequency condition of the Swiss building code [3] cannot be met even with a HEB 1000 profile. A possible elimination of the dynamic problems of such a structure can therefore have a strong benefit for the constructor of the building.

Fig. 1.1: Utilisation factors of a steel joists for different profile heights of the HEA and HEB families. Ultimate limit state (ULS), serviceability limit state (SLS) and frequency condition (FREQ) according to SIA 260 [3]. Values beyond 100% indicate a violation of the utilisation factor.
Novel materials like fine grain steel, timber and concrete composites with a high yield strength but a rather low stiffness to strength ratio will further reduce the deadweight of structures in the future. This is advantageous from an environmental protection perspective, because the overall material consumption is reduced. From the perspective of structural dynamics, the loss of weight will lead to more lightweight and therefore "lively" structures. Static and quasi-static loads can be easily encountered by pre-cambered or post-tensioned constructions. However, dynamic loads that lead to vibrations are only mitigated by the inherent damping of the building material. Due to the increased wear, excessive vibrations can also lead to fatigue and problems for the structural safety. The ongoing trend towards wide and open spaces in buildings with as few columns as possible leads to wide-span floor structures. This shows that dynamic problems of floors will become increasingly important in the future.

The application of external vibration mitigation devices like the famous tuned mass damper is still an exception in civil engineering. Today, only tall buildings and bridges are equipped with semi-active or passive vibration mitigation technologies. In the case of tall buildings, these devices are used to mitigate horizontal vibrations induced by wind or earthquakes. Wide-span bridges are equipped with damping devices to mitigate horizontal and vertical vibrations. Passive and semi-active tuned vibration control devices have a strong effect on the damping characteristic of a structure. However, a slight detuning will lead to remarkable lower damping efficiency. Bridges and other structures with a high deadload-to-payload ratio are less subjected to changes of the resonance frequency or vibrating mass. In the case of lightweight structures like floors and footbridges, the deadweight is usually equal or even lower than the payload. From the fully-loaded to the unloaded case, the dynamic properties of such structures can vary considerably. Thus, the application of passive tuned vibration mitigation devices is problematic.

Therefore, this study investigates the application of active vibration mitigation devices for vertical vibrations of lightweight civil engineering structures. Accordingly, the following questions are analysed:

- Is the dynamic serviceability criteria determining for wide-span and lightweight structures?
- Can the dynamic behaviour of lightweight floor systems be improved by applying active vibration control in the design stage?
• Is it possible to build a low-cost active vibration control device, that fulfils the stability and robustness criteria as well as (international) building regulations?

• Will the application of active vibration mitigation systems help to reduce the material consumption and therefore the overall building cost of a floor system?

1.2 Fundamentals on vibration in buildings

Mechanical vibrations are the result of a constant exchange of mechanical energy from kinetic energy to potential energy and back. This implies that at least an inertia (rotary or linear) and a source of potential energy (elastic spring, gravitational or electro-magnetic field) are involved. In real physical vibrating structures, every cycle transforms a part of the stored mechanical energy into thermal energy (heat) by friction and dissipation. This class of vibrating system will therefore reach a defined state where the stored mechanical energy is minimal. Stable physical systems have at least one state where the stored energy is minimal (cf. figure 1.2). As long as no external disturbance occurs, the system remains at its minimum energy state. After a disturbance, the energy content of the system is temporarily increased. Therefore, the system leaves the minimum energy state until the additional energy is dissipated again into thermal energy. Some vibrating systems also have local minimum energy states. It is, in the case of these complex systems, usually not predictable which energy state will be reached after an external disturbance. For the class of simple mechanical vibrating systems without dry friction, only one minimum energy state exists. It is reached if the position and speed are zero. If dry friction is taken into account, several local minimum energy points can exist where the velocity is zero and the system is in a non-zero resting position.
All dynamic systems have at least one resonance frequency which depends solely on the ratio between the back-driving force (stiffness, elasticity, ...) and the involved mass. Vibrating systems can be characterised by this exchange rate of kinetic and potential energy and the according energy dissipation ratio. The reciprocal of the resonance frequency is called pulsation time and defines the time between two consecutive maxima or minima of the kinetic or potential energy. The corresponding deflection form of the structure in space is called the mode shape. All physical vibration systems are subjected to a reduction of the amplitude in time. This phenomenon is called attenuation or damping and is usually due to internal friction or plasticity phenomena or external damping forces like hydro- and aerodynamic drag.

The excitation of a vibrating structure by means of an external force generates a travelling wave which is reflected at the structures border. Some waves are reflected in such a manner that their amplitude are annihilated, others are amplified. The amplified waves generate a stable waveform as long as the attenuation is low. The travelling speed of every type of wave is defined by the material properties. Longitudinal waves or material waves travel through a continuum (for example sound waves or pressure waves in a rod). Their amplitude is in line with their main travelling direction. Transversal waves travel along the interface of two continua (waves on the water surface) or within slender structures (rope or beam vibrations). Their amplitude is perpendicular to the main travelling direction of the wave. Mechanical transversal waves have usually higher amplitudes and lower resonance frequencies than longitudinal waves.

Transversal vibrations can be further divided into rope- or membrane-type and Euler-Bernoulli-type structures (cf. figure 1.3). This classification is based on the supports and the load-bearing behaviour of the structure.

Examples for a membrane-type structure are a rope which is held at both ends or a thin plate which is fixed around its border. These structures are characterised by a small (or null) bending and shear resistance, but a high tensile resistance. Another characteristic is their low thickness to span ratio. This leads to high amplitudes in the case of transversal loads, as the retaining force is achieved only by the second-order longitudinal tensile force. It is therefore necessary that the supports of the membrane structure are fixed parallel to the main span direction (cf. figure 1.3). In civil engineering, only few applications of this load-bearing structure are used, for example in suspension bridges and ropeways. This is mainly due to the large
deflection which is required for the mobilisation of the second-order tensile force. Pre- or post-stressed structures are not classed in this category here, as the resulting compressive force is absorbed by the structure itself.

Beam-type structures characterised by an important bending- and shear resistance. Transversal loads are propagated in the beam through shear and bending tensions (cf. figure 1.3). In the field of a simply-supported beam, the lower part is subjected to tension forces while the upper part is in compression. This leads usually to stiffer constructions than in the membrane-type case. As a result, only small transversal displacements and no resulting longitudinal forces of second order appear. Most horizontal structures in civil engineering are dimensioned for this type of load-bearing behaviour. Floors are built either isotropic (about equal spans) or an-isotropic (one main span) depending on the span and stiffness ratio. It should be noted that a pure isotropic structure is rarely used. Examples can be found in square cross-laminated timber or reinforced concrete constructions. For this reason, if not otherwise mentioned, a bending-beam-type load-bearing behaviour will be assumed for all example structures in this work.

Simple oscillating systems like a pendulum have only one resonance frequency. They can be modelled by a simple one-degree-of-freedom model. However, all continuous physical structures have an infinite number of resonance modes. It is an inherent property that all resonance modes of a structure are orthogonal [28]. This means
that no mode can be exactly build by a linear combination of the other modes. Thanks to this property, a multi-mode system can be modelled approximately by a linear combination of simple one-degree of freedom oscillators. Faster modes with high resonance frequencies require a higher amount of energy to be excited and present usually (case of a civil engineering structure) a higher modal damping. As a result of these facts a good approximation of the dynamic behaviour of the structure can be achieved by only taking into account some low-frequency modes.

Tall and slender buildings and bridges have their lowest pulsation times in the range of several seconds (transversal vibrations). Sub-structures within buildings, like floors for example, have higher resonance frequencies and therefore pulsation times between 0.1 and 1.0 second. This is also the range where humans are most sensitive to vibrations (cf. section 1.2.5). Longitudinal waves are rarely problematic in civil structures since their frequencies and amplitudes are usually above the detection threshold of human somatosensory and vestibular perception. An exception are sound waves which travel through structural elements and can be perceived easily. If necessary, these audible waves are mitigated by applying an intermediate layer of porous materials.

1.2.1 Influence of physical properties to the dynamic behaviour of a structure

The dynamic behaviour of a structure is mainly governed by its geometry, the support situation, its material properties and the applied external loading. It is assumed here that floors have uniform construction parameters and material properties over the whole area and that only bending-type vibrations are present. In this case the stiffness of the construction is defined by the second moment of inertia and the mobilised cross-section. Obviously, a stiff construction leads generally to smaller amplitudes. Another decisive geometrical key factor is the length between the supports or span. In the case of a simply-supported bending-beam-type structure for example, the influence of the span is in the power of 4 (cf. appendix D). Further, the support situation describes whether additional torsional or linear elasticity or damping is applied at the support location. This influences the total stiffness of the structure significantly. Obviously, for civil engineering structures, free body motion is prohibited.

The third significant factor which influences the excitability of a structure is the material property. Most important properties are the density (mass), linear elasticity
(stiffness) and inherent damping of the material. The stiffness (elasticity or Young’s modulus) and mass dominate the resonance modes, whereas the damping influences the decay rate of the amplitude over time of the vibrating structure. Building structures with a high deadweight like massive concrete structures for example, have usually low resonance frequencies. Thanks to the high mass, a moderate Young’s modulus and a relatively high inherent damping, concrete structures within buildings (floors, walls, ...) are rarely problematic. Steel structures are usually more slender and have a low inherent damping. This leads to resonance frequencies with little damping and therefore a problematic dynamic behaviour. Timber structures are usually lightweight and flexible and thus, prone to structural vibrations. A widely used solution is the increase of the deadweight of such structures. Especially timber-floor constructions are often filled with gravel or sand to increase the damping and decrease the excitability. Combinations of different materials, like steel-concrete or timber-concrete constructions, use the benefits of both materials. However, these structures are usually very optimised and utilised. If no additional damping is provided, such structures are often prone to vibrations. The choice of materials for floor structures is often predefined by the architect or the overall building construction type. Steel constructions require usually steel-concrete floors, whereas timber or concrete buildings will most likely have timber or concrete floors, respectively. The required static height of the floor structure depends therefore mainly on the span and the applied loading scenarios.

Another key factor for the dynamic behaviour of a floor are the loading scenarios. If unloaded, a floor structure has higher resonance frequencies than for a standard loading scenario. However, the loading scenario can vary considerably within the lifetime of a building and are usually not be predicable with sufficient precision during the planning phase of the building. This aspect is analysed in detail in the section 1.2.3.

In conclusion, two concepts lead to an advantageous dynamic behaviour of the structure:

**Concrete structures - heavy and high damping:** Heavy structures are difficult to excite. Thanks to the relatively high inherent damping, the low resonance frequencies are well dampened.

**Steel or timber structures - stiff and lightweight:** Stiff and lightweight structures have high resonance frequencies which are far from the range of human perception.
Floors are usually categorised in either low-frequency or high-frequency floors [25, 67]. A fundamental frequency below 8 to 10 Hz, leads in the case of human activities usually to a steady-state damped vibration of the structure (cf. figure 1.4, left). The same excitation forces and frequencies lead on a structure with a higher fundamental frequency to single vibration events for each impulse (cf. figure 1.4, right).

### 1.2.2 Sources and receivers in buildings

In this section, the two terms "source" and "receiver" are defined. It is obvious that, as long as no excitation acts on a dynamic system in zero-state, no motion occurs. On the other hand, a dynamic system in motion which fulfils the demanded serviceability is not problematic. It is therefore crucial to define the impact and the effect of a dynamic system. The term source is used in this work as an external disturbing force on the dynamic system with generates a movement. Most common vertical vibration sources on floors are rhythmic human movements (for example walking, running or jumping), stationary or moving machines within the building or external sources (earthquakes, traffic, etc.).

Table 1.1 lists common sources and their frequency and amplitude range. Obviously, stationary sources can be easily decoupled of the structure by means of an elastic mount. Thus, this category is of minor importance. Earthquakes act to the vibrating structure and payload as an increase of the deadweight. It should be noted that only the vertical component of an earthquake is taken into account for floor vibrations. Thanks to the relatively low occurrence rate of earthquake events, the importance as a source of vibration for the serviceability is negligible. Structure-borne noise and
<table>
<thead>
<tr>
<th>Source</th>
<th>Frequency range</th>
<th>Amplitude</th>
<th>Special</th>
</tr>
</thead>
<tbody>
<tr>
<td>Human walking (adult)</td>
<td>1.5 - 2.5 Hz</td>
<td>0.7 kN</td>
<td></td>
</tr>
<tr>
<td>Human running (adult)</td>
<td>2.0 - 4.0 Hz</td>
<td>1.0 kN</td>
<td></td>
</tr>
<tr>
<td>Stationary machines</td>
<td>var.</td>
<td>var.</td>
<td>fixed location</td>
</tr>
<tr>
<td>Moving machines</td>
<td>var.</td>
<td>var.</td>
<td>location unpredictable</td>
</tr>
<tr>
<td>Earthquakes (vertical)</td>
<td>var.</td>
<td>base accel.</td>
<td>Rare occasion</td>
</tr>
<tr>
<td>External source (traffic)</td>
<td>var.</td>
<td>var.</td>
<td>structure-borne</td>
</tr>
<tr>
<td>Falling objects</td>
<td>var.</td>
<td>var.</td>
<td>accidental load</td>
</tr>
</tbody>
</table>

**Tab. 1.1: Common sources of vibration on floors**

sound from outside sources (train, aircraft or road vehicles) can lead to vibrations of structures within a building. In most cases the traffic infrastructure can be decoupled from the building using simple decoupling methods. Human activities are a major source of vibration in buildings. Slow activities like walking generates disturbance forces with first harmonics at around 2 Hz at an amplitude of around 0.7 kN [41]. Running, jumping and dancing induces higher frequencies and amplitudes. Moving machines like fork lifts or overhead cranes generate disturbance forces by lifting and positioning heavy objects or by moving on the floor.

The term receiver is used in this study as a person or device located on the structure which is sensitive to vertical vibrations in the defined frequency range. The sensitivity range of humans is described in section 1.2.5. Examples for highly-sensitive devices are high-precision manufacturing machines, optical measurement devices or surgery-robots.

### 1.2.3 Loading and moveable weights on floors

In section 1.2.1, the influence of the deadweight to the dynamic behaviour of a structure is described. More distributed weight decreases generally the resonance frequencies of a structure. For non-uniformly distributed loads or loads which are not rigidly linked to the structure, a more complex influence to the structure is present. Non-uniformly distributed loads affect only some resonance modes for example or they can shift existing resonances modes. However, predictions of the resonance frequencies and modes require in this case a detailed calculation of the structure including the loading scenarios.

The structure-load interaction only needs to be taken into account if the resonance of the additional load is in the same magnitude as the considered resonance frequency.
of the structure. This can be the case for a group of people standing on a floor for example [41]. Stiffer connections, or connections with high damping between additional mass and structure can be modelled as rigid links without committing significant errors. This is the case for most furniture and other equipment usually found in buildings. Thus, these loads add directly to the deadweight of the floor. In this study all additional loads are modelled with a rigid link to the main structure.

Depending on the structure, the additional live load can achieve more than 100% of the deadweight of the structure. This is especially the case for lightweight structures. For rather heavy concrete structures, the variable part of the load is small compared to the deadweight.

### 1.2.4 Standards and building codes

**Switzerland**  Vibrations and resonance phenomena are hardly included in the Swiss building standards. The Swiss standard SIA 260 [3] lists minimum resonance frequencies for certain types of buildings. It is assumed that if the fundamental resonance frequency is above this threshold, no dynamic problem occurs. These threshold values are based on the work of Bachmann [9]. Table 1.2 lists the lower limitations of the fundamental frequency of different structures as provided in the standard [3]. It should be noted that no regulation is provided for buildings other than dancing and concert halls or gymasia and that damping effects are not included in the standard.

<table>
<thead>
<tr>
<th>Building</th>
<th>fundamental resonance frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Footbridges (vertical)</td>
<td>$f &lt; 1.6$ or $f &gt; 4.5$ Hz</td>
</tr>
<tr>
<td>Footbridges (horizontal, cross)</td>
<td>$f &gt; 1.3$ Hz</td>
</tr>
<tr>
<td>Footbridges (horizontal, longitudinal)</td>
<td>$f &gt; 2.5$ Hz</td>
</tr>
<tr>
<td>Structures in dancing or concert halls</td>
<td>$f &gt; 7.0$ Hz</td>
</tr>
<tr>
<td>Structures in gymasia and sports halls</td>
<td>$f &gt; 8.0$ Hz</td>
</tr>
</tbody>
</table>

**Tab. 1.2:** Allowed frequency range according to the Swiss building standard SIA 260 [3]

**Europe (Eurocode)**  The Eurocode standard EN 1990 [71] states that the comfort of the user and the functioning of the structure should be considered to achieve satisfactory vibration behaviour and that the natural frequency should be higher than an "appropriate" value. A reference to ISO 10137 is added for further guidance. The steel construction standard EN 1993-1-1 [72] specifies that "vibrations of
structures on which the public can walk should be limited to avoid discomfort to users". These limits should be defined for each project with the client.

The German national annexes to EN 1991-1-1 [70] states that structures which can be excited by human activities should be designed for this loading scenario. No further information is provided for the design of these structures. The British national annex to EN 1991-1-1 [69] provides a lower limit for the vertical natural frequency of a structure. Resonance phenomena should be avoided by designing structures with a vertical fundamental frequency greater than 8.4 Hz. Especially for wide-span structures this limitation is difficult to attain as shown in figure 1.1.

International  International standard ISO 10137 [73] includes vibration criteria for the serviceability limit states. The criteria are given for different receiver: human occupants, building contents and building structures. Human receivers are further subdivided in "active", "regular" and "sensitive" groups. However, calculations need to be run for every structure, a set of dynamic loads and different receiver locations. The received acceleration data are then used to obtain an equivalent vibration dose value. Model sensitivities must be analysed carefully to verify the results. These calculations require a high effort and are therefore often not suited for simple structures.

Steel Construction Institute (UK)  The steel construction institute (SCI) provides design guidelines for floor vibrations in the documentation SCI354 [67] for steel-joist floors. These guidelines are in accordance with ISO 10137 [73]. An estimation rule for the vertical acceleration is provided for a series of the lowest resonance modes of a beam-type floor. The maximum RMS-acceleration from steady-state and transient parts is then used to determine the response factor (R-factor). Generally, no element of a floor structure should have a fundamental resonance frequency below 3 Hz to avoid resonance effects from walking excitation.

Concrete Centre (UK)  The publication CCIP-016 of the Concrete Centre [87] provides calculation rules for determining the response factor (R-factor) of floors based on the vertical velocity or acceleration for transient or steady-state vibrations, respectively. The obtained R-factor is based on the maximum RMS-acceleration from transient and steady-state parts, similar to the guideline SCI 354.

United States  The design guide DG11 for "Vibrations of steel-framed structural systems due to human activity" [48] provides similar criteria as the SCI354 [67]. The
fundamental resonance frequency of the fully equipped floor and its sub-structures should be above 3 Hz. For structures with a fundamental resonance frequency between 3 and 8 Hz, an empirical calculation rule is provided to estimate the peak response acceleration for a generic walking excitation. Floors with fundamental resonance frequencies between 9 and 10 Hz should feature a minimum stiffness of 1 kN/mm for concentrated loads on unfavourable locations.

1.2.5 Human sensitivity to vibrations

The human sensitivity to vibrations depends strongly on the frequency and amplitude, but also on the duration of exposure and the direction. Different methods to compare the influence of a vibration to the human body have been invented. H. Reiher and F. J. Meister have published in 1931 a fundamental analysis on the human sensitivity to vibrations [64]. They analysed the effect of harmonic vibrations with different frequencies and amplitudes on standing or lying persons. It should be noted that the exposure limits are defined on the amplitude (peak) of the vibration. Categories for perceptible, well perceptible, strongly perceptible, probably harmful (long exposure) and harmful (extremely uncomfortable) were established based on their experimental work (cf. figure 1.5).
A slightly modified scale was published by Lenzen in 1966 [42]. Lenzen analysed the effects of transient vibration on humans. By shifting the amplitude scales of Reiher-Meister downwards, the Lenzen scale also takes into account transient vibration phenomena. The Lenzen method still uses only frequency and amplitude information without taking into account the vibration dose. It should be noted that the Reiher-Meister and Lenzen models apply to harmonic disturbances only.

Modern vibration exposure models include the international standard ISO 2631 [74] and the German standard DIN 4150 [53]. ISO 2631 [74] defines frequency weighing curves for horizontal and vertical whole-body vibrations (cf. figure 1.7). Other weighing curves apply for vibrations of body parts and rotational vibrations. A weighted running root-mean-square (RMS) is calculated in time domain from the acceleration measurement data. This method applies therefore not only to harmonic vibrations, but also higher order modes and non-harmonic accelerations are taken into account. The calculated RMS value is compared to health guidance curves (cf. figure 1.8). Alternatively, the fourth power vibration dose can be calculated and compared to the exposure limits from ISO 10137 [73].
Fig. 1.7: Frequency weighting function according to ISO 2631 [74]. The solid line ($W_k$) is applied to whole-body vibrations in vertical direction (head-foot), $W_d$ (dashed) is applied to horizontal vibrations (transverse). The dash-dotted line ($W_f$) is applied for motion sickness analysis only.

Fig. 1.8: Health guidance according to ISO 2631 [74]. The maximum exposure limit to the weighted acceleration depends strongly on the duration.
The German standard DIN 4150-2 [53] defines maximum exposure to vibration based on the weighted vibration velocity. The velocity is high-pass filtered to obtain the so-called $K_B$ value. The root-mean square of the maximum $K_B$ value over a time-step of 125ms ($K_{B_{FTm}}$) is compared to maximum exposure limits depending on day/night, place and duration.

1.2.6 Vibration of structures and transmissibility

Vibrations in buildings depend mainly on three key-factors: the source, the receiver and the transmission path between the two [73]. The source and receiver, as well as their location, are usually given by the utilisation of the building. The transmission path depends therefore solely on the dynamic behaviour of the vibrating structure.

Continuous real systems have an infinite number of resonance frequencies [28]. For every resonance frequency of such a structure, there exists a unique related mode shape. These properties are called the resonance mode of the structure. Figure 1.9 shows the first mode shapes of a simply-supported beam for example. Each mode of a physical system has at least one zero in the mode-shape function at the support point (rigid body motion is excluded here). For the simply-supported beam there are zeros at either end of the beam (cf. figure 1.9). Higher order modes may have several locations where the displacement of the structure is dynamically constraint to zero. These locations are called nodal points or node. In contrast, the points where the highest displacement of the vibrating structure is observed are called anti-nodes. If a harmonic disturbance force acts at a resonance frequency at exactly a node of the corresponding mode shape, the structure is not excited.
Let a floor be a beam-type structure which is simply supported at the border (cf. figure 1.10). The receiver is a person which is located in middle of the plate. Another person is walking from one edge towards the receiver and acts as an excitation source. For every location of the walking person, a transfer function can be defined from the source to the receiver. Some modes are not (fully) excited by the source at the current point and some of the excited modes are not felt by the receiver. The same is also applicable if a vibration mitigation device is applied to structure. The damping force can only be applied to vibration modes which have a significant displacement at the location of the vibration mitigation device. The transmissibility depends therefore not only on the dynamic behaviour of the structure, but also on the location and frequency content of the source and receiver.

**Fig. 1.10:** Transmissibility on a beam-type structure with vibration control: a) The excitation (walking person) is at the support. The structure is not excited and therefore the receiver (sitting person) does not feel any vibration. The vibration control is not activated. b) The excitation at the first resonance frequency is felt by the receiver and the vibration control is activated. c) The excitation at the second resonance frequency is not felt by the receiver, but the vibration control is activated.
The transmissibility depends strongly on the position of the disturbance source, the receiver and the damping device. The mode-shape function $q$ is used to display the percentage of the maximum amplitude for the given location of source, receiver and damping device.

**1.3 Inertia-based vibration mitigation technologies - State of the art**

Vibrations of wide-span structures such as floors can only be efficiently mitigated by applying external damping elements. Due to the large span and the lack of intermediate fixation points, the application of transversal or active tendon methods (cf. figure 1.14) cannot be applied for these structures. Vibration mitigation devices must therefore rely on the concept of the inertia-based damper, sometimes also called skyhook damping. These devices are characterised by an additional mass (auxiliary system) which is connected on the vibrating structure (main system). The damping characteristics are mainly determined by this connection between the two masses. If the connection is too weak, the auxiliary mass remains still while the main system vibrates. In contrast, if the connection is too rigid, the main and auxiliary mass vibrate together in parallel. The connection between the two system is therefore crucial for the functioning of the damping device. First devices had a purely passive connection consisting of elastic springs and damping elements. This system works only efficiently if the resonance of the auxiliary system is tuned to the resonance of the main system. It can be applied therefore only time-invariant systems. Later active and semi-active connection elements were added to increase the effectiveness of inertia-based damping devices. These three categories are
explained in detail in the following sections. Table 1.3 provides an overview of the possible applications for the three damping device categories.

<table>
<thead>
<tr>
<th>damping device</th>
<th>time-variant systems</th>
<th>harmonic excitations</th>
<th>transient excitations</th>
</tr>
</thead>
<tbody>
<tr>
<td>passive</td>
<td>no</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>semi-active</td>
<td>yes</td>
<td>yes</td>
<td>no</td>
</tr>
<tr>
<td>active</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>

Tab. 1.3: Comparison of the area of application of passive, semi-active and active damping devices

1.3.1 Passive dampers

Vibration problems became increasingly important in the beginning of the 20th century when faster, stronger and heavier machines were invented. It was during this time in 1911 when Frahm claimed a patent for a "device for damping vibrations of bodies" [26]. It is based on the idea of reducing the roll movement of ships (Frahm-tank [37]). Only few years later Stockbridge [76] patented in 1925 a damping device for overhead power lines which were affected by wind-induced vibrations.
Frahm’s invention was mathematically described, generalised and optimised by J. Ormondroyd and J.P. Den Hartog [57, 33] in the 1930s. By applying viscous damping to the vibration absorber, the damping effectiveness was significantly increased. The general concept of the passive methods is to tune the auxiliary system in such a way that kinetic energy is transferred from the main system to an auxiliary one. For multiple degree of freedom systems, the vibration modes are strictly orthogonal and thus, no energy transfer is possible between the modes. However, if the auxiliary system is correctly tuned to the main system, an energy exchange between the vibrating systems is possible thanks to the mutual excitation. The system is therefore called a "tuned mass damper" (TMD). For a long time, the TMD was the only inertia-based vibration control method. It is still the most commonly-applied concept for simple vibration control problems.

First applications of a TMD for civil structures were proposed by Lenzen in 1966 [42]. His work on different steel-joist concrete slabs in Kansas City with small dampers lead to the following simplified design criteria: the resonance frequency of the damper should be 1 Hz lower than the fundamental frequency of the structure and a viscous damping of 7.5% of the critical damping should be applied.

Allen and Swallow [1] presented in 1975 a "dynamic absorber" which is mounted under a steel-concrete-composite floor. It comprises a steel box, loaded with concrete blocks. Four commercial compression springs were used to support the box. The dynamic absorber was tuned at about 10% below the dominant resonance frequency of the floor.
Several applications of passive TMDs for floor vibrations were reported in the following period for example by Bachmann and Ammann [9] in 1987 on the upper floor of a two-story exhibition pavilion. Thornton, Cuoco and Velivasakis [15] have successfully applied TMDs on a gymnasium floor in 1990. Later Webster and Vaicajtis [86] used TMDs on a restaurant floor in 1992. Since then, numerous applications of TMDs to mitigate floor vibrations can be found in literature. However, the technology remained more or less unchanged since the first applications.

**Including inerters** The concept of the inerter has been published only recently by Smith in 2002 [68]. In system theory this device is a double integrator, closely linked to the inertia, but it has two connection ports similar to the capacity in electric systems. The ideal inerter has no mass, but a certain inertia $b$ in [kg]. This sort of connector device cannot transfer constant loads due to the stroke limitation of the device, similar to a dash-pot. It needs to be set therefore in parallel with an elastic spring for the case of the constant vertical load of the inertia-based vibration control system.

Including inerters to the design of TMD’s has been investigated recently in many publications (for example [61, 13, 46]). Mostly, the inerter-device is set either in parallel between the main and auxiliary mass or the link is closed from the auxiliary mass to the ground. If set in parallel, the inerter restrains the movement of the auxiliary system and thus, reduces its damping efficiency. If the inerter is connected directly to the ground, the resulting system has an increased auxiliary mass. Nevertheless, if the possibility of a rigid external link to the ground exits, it is preferable to create a direct rigid or damped connection to the main system in order to mitigation vibration modes. Under these aspects it is obvious that inerter do not contribute to the damping effectiveness of inertia-based vibration mitigation.

**1.3.2 Semi-active methods**

Semi-active inertia-based vibration control is characterised by an adaptive connection element in parallel between main and auxiliary mass. This component is usually either a device filled with a magneto-rheological fluid or elastomer, or a hydraulic cylinder with tuneable shunt valve. The semi-active actuator is used to provide a variable damping or friction and an additional quasi-stiffness. Obviously, the semi-active actuator is unable to provide real stiffness because no energy is stored in the device in contrast to a conventional spring. Therefore, no energy can be pushed into the device and the excitation of the auxiliary system depends solely
on the mutual excitation from the main system, similar to the passive vibration control device.

The concept of the semi-active vibration control has been described theoretically by Karnopp et al. in 1974 [39]. In the case of inertia-based vertical vibration mitigation, the damping device must be tuned to the vibrating main system to achieve a sufficient excitation of the auxiliary system. The most effective tuning was developed by Ormondroyd and Den Hartog in 1928 for the passive device [57]. Most semi-active tuned mass dampers are therefore tuned according to these design rules. By virtue of the semi-active element, the device is able to adapt to changes of the main system’s resonance frequency. Several examples of this method have been developed in recent past (for example [83, 85, 38, 80]).

Several other semi-active methods have been analysed and implemented. A very simple but promising solution is the tuned liquid column damper (TLCD) first described by Sakai et al. [65]. It comprises an U-shaped tube structure partly filled with a liquid. Due to a horizontal motion of the tube, the liquid is excited and mitigates the movement. A variable orifice [78] or propeller [17] allows controlling the amount of liquid passing from one end of the tube to the other. Applications of the passive TLCD can be found in high-rise buildings in Vancouver, CA (2001) or Philadelphia, US (2017). Other described methods include the use of shape-memory alloys, piezo-electric and magneto-strictive materials. However, these devices are less adapted to the application in buildings due to their small force and stroke capacities.

### 1.3.3 Active methods

In an inertia-based active vibration control device, the connection between main and auxiliary mass is built not only by passive elements (spring, dash-pot), but also by an additional actuator. The connection between the masses can therefore be adapted according to a predefined control law and different state measurements. On the other hand, the feedback control loop has to be designed carefully to avoid instabilities. First active vibration control devices were applied in the 1960s in aircrafts. During the same time period first active controlled electro-magnetic and hydraulic vibration control devices were designed [58, 56].

Like every other inertia-based vibration mitigation device, active devices require an auxiliary excitation to absorb mechanical energy from the vibrating main struc-
ture. A major benefit of the active control is that this auxiliary excitation can be
genenerated by the actuator and does not rely on a tuning. This allows building
tuning-independent devices with high damping capacities. This tuning-independent
vibration control method is usually called an "active mass driver" (AMD). In con-
trast, active tuned mass dampers (ATMDs) require a constant tuning to a varying
resonance of the main system. This concept requires much less energy but has,
on the other hand, an equal damping performance like a passive TMD. The ATMD
concept is therefore not analysed in detail within this study.

Different control laws have been applied for AMDs in structural control in the
past. Most promising control algorithms include the optimal linear-quadratic state
feedback (LQR/LQG), velocity feedback and acceleration feedback. The following
overview is grouped according to the applied control laws.

**Velocity feedback** In the civil engineering domain, the first attempts were made
by Hanagan and Murray in 1995 [31]. Their idea was to use an electro-magnetic
shaker (APS 400) to generate the required damping force. A velocity-dependent
feedback law by Balas [10] was chosen to generate additional damping of the
structure. The simple feedback control loop has been investigated for stability issues
in 2003 by Hanagan in a numerical model [32]. Nyawako et al. conducted a similar
study in 2009 [54]. Diaz and Reynolds [24] have included the actuator dynamics in
the Hanagan model to take into account possible instability issues. Nevertheless,
global stability limits were not provided for the velocity-based feedback control.

**Full state feedback** Linear-quadratic control (linear-quadratic regulator, LQR) of all
system states has first been implemented in a real active vibration mitigation in the
Kyobashi building in Tokio in 1991 [40]. This concept is well known and provides
stable controllers. However, robustness must be analysed carefully. LQR/LQG have
become the standard method (benchmark) to compare MIMO and MISO (multiple-
input-multiple-output, multiple-input-single-output) concepts in active vibration
control (for example in [62, 88])

**Acceleration feedback** Nishimura [51] published in 1992 a novel acceleration
feedback law for inertia-based vibration mitigation. His idea is based on a direct
negative acceleration feedback of the vibrating main structure. In order to achieve
a stable control loop, the viscous damping of the auxiliary system needs to be
relatively high (27% of the critical damping).
Several state-of-the-art reviews covering the current knowledge in active vibration control have been published ([34, 21, 55, 16]). For further information, the interested reader is referred to these publications.

1.4 Implementation considerations for (active) vibration control

The implementation of passive, semi-active or active vibration control methods requires careful considerations to find a suitable solution. First, the method for vibration control needs to be defined, which is usually given by the type of vibration problem. Four possible control methods are described in the following section. Furthermore, the implementation of semi-active or active methods requires additional considerations for the power consumption and for the case of power failures.

1.4.1 Vibration control methods

Vibration control methods can be grouped into four groups depending on their working principle. Figure 1.14 shows an example for each type.

a) Vibration isolation  The simplest method to control the vibrations of a structure is to isolate either the excitation source or the receiver from the structure. For this purpose, specialised rubber elements or soft springs are usually applied. Isolation of the excitation source is usually applied when vibrating machines are fixed in buildings. An example of a receiver isolation is the base isolation of buildings for earthquake safety reasons.

b) Active Tendon Type  The second group of vibration control methods is the "active tendon"-type structure. These methods use the vibration control device as a load-bearing part of the structure. Most famous examples are the active tendon in lattice type or actively controlled cable-stayed structures. The vibration control part of the structure must therefore have a high stability and stiffness. Usually active piezo-electric components or passive rubber elements are applied for such structures.

c) Transversal methods  The third group are the transversal methods. They are usually applied to mitigate wind-induced cable vibrations on cable-stayed bridges.
The vibration control device is not part of the load-bearing structure but stabilises its transversal motion.

d) Inertia-based vibration control (skyhook damping)  The fourth group is the inertia-based vibration control. The principle of action of these devices is that a damping force is generated between the vibrating structure and an auxiliary mass. The damping force acts equally to the main and auxiliary mass. It is therefore necessary to stabilise the auxiliary mass which reduces the efficiency of these vibration control devices. The most famous inertia-based vibration control device is the TMD. Another well-known application is the Stockbridge damper [76] on overhead power lines. A major advantage of the inertia-based vibration control is the independence from any structural linkage or bracing. Especially for bridges and wide-span structures, the inertia-based damping is the only applicable method. The drawback of the moving inertia and the stroke saturation need to be encountered either by a precise tuning or a suitable control of the device.

1.4.2 Location of the vibration control device

Chapter 1.2.6 shows that the transmissibility of vibrations from the source to the receiver depends strongly on their location on the structure. The implementation of vibration control devices requires therefore not only the verification of the resonance frequencies, but also a careful analysis of the vibration source, the receiver and their location on structure. Simple solutions of this problem can be found for the case when the location of the vibration source is fixed on the structure. In this case a vibration isolation reduces the excitation force and thus the movement of the structure. On the other hand, if the location of the receiver is fixed, for example a workplace in an office, the vibration control should be applied at the receiver location to reduce most of the disturbing structural vibrations. If neither the location of the vibration source nor the receiver is fixed, more complex solutions must be applied. The problem of the observability and governability of vibration modes is a well-known concept [12, 2, 30]. It can be extended by the control loop to find the optimal location for vibration control. A rather unpretentious, but nevertheless, for simple systems very effective method is either to place the vibration control device at the anti-node of the structures fundamental (or most disturbing) mode. However, for closely separated modes, a profound dynamic analysis of the structure with applied vibration control device is indispensable.
Fig. 1.14: From top-left in clockwise-direction: Vibration isolation of an electric motor by coil springs [52], Active vibration control of a braced tower structure [63], Pendulum damper (inertia-based) in the Taipei 101 building [14], Damping device for transversal cable vibrations (under construction) [75]
1.4.3 Device saturation

Every vibration mitigation device is subjected to several saturation effects. One limiting factor is the stroke saturation. The passive pendulum damper for example (cf. figure 1.14) hits the support cage if the excitation force exceeds the design limit. Especially low-frequency vibration control is subjected to this problem because high vibration amplitudes are required to generate moderate damping forces. Rubber damping elements are usually applied to protect the device from hitting stroke limitations. In the experimental part of this study, a dynamic saturation limitation was implemented in the control loop (cf. 4.3.4).

Another limitation is the power or force saturation of the actuator and amplifier. High damping forces are usually required to mitigate vibrations efficiently. If the applicable damping force is too low, the damping is not efficient and the saturation effect can lead to instabilities in the feedback loop. An over-estimation of the damping force is economically and ecologically not advantageous due to the increased costs and masses. Thus, for every vibration control device a trade-off must be considered.

If damping devices are set in series with the load bearing structure (cf. figure 1.14), the maximum occurring loading (ultimate limit state, ULS) must be lower than the load carrying capacity of the damping device. Dynamic factors need to be taken into account for the calculation of the dynamic forces.

1.4.4 Actuator type

One major design element of an active vibration mitigation device is the actuator. It should be chosen depending on the required stroke and force requirements. For the application in civil engineering, high forces and strokes are required. However, the frequencies and therefore also the velocities are relatively small. Usually one of the following actuator types is chosen:

**Hydraulic motor or piston** Hydraulic equipment has the highest energy and power densities among all actuator types. Control valves are standard parts and easily available. However, there is a lower size limit for the economic efficiency. A lot of the applied active and semi-active vibration mitigation devices in civil engineering work today with hydraulic components. Hydraulic pistons are among the best linear
actuators for this application. However, leakage problems, flammable liquids and high maintenance costs limit the application in environmentally sensitive or harsh places.

**Pneumatic motor or piston**  The major benefit of the pneumatic actuator is the simple implementation of the equipment. Control valves and compressors are standard parts and easily available. The stroke and especially the force are limited by the maximum compressibility of the air and the adiabatic temperature changes. Dew formation due to adiabatic expansion can block critical valves and thus, shut down the device. Pneumatic equipment is sensitive to dust and thus not well suited for harsh environments.

**Stacked piezo actuator**  Piezo crystals are subjected to a uni-directional volume change when an electric tension is applied at its border. This effect is fully reversible and can be used for micro-actuators. By stacking several piezo crystals together, the stroke of the actuator is increased. However, the stroke remains below 0.2% of the actuator length. Additionally, complex DC-generators are required for the application of the control voltages. Therefore, this actuator is rather inapt for vibration control devices in civil engineering.

**Linear electric motor**  Linear motors comprises a rail with permanent magnets and an electro-magnetic glider or inverse. Using an appropriate control of the electromagnets, an electro-magnetic force is created and the glider moves on the rail. Major problems of this actuator type are the exposed magnets, which can be contaminated with iron dust. Another weak point is the strong temperature increase due to the electric dissipation if no external cooling is applied. Additionally, the costs of linear electric motors are high compared to other solutions. For these reasons, linear motors are often used in laboratory equipment but not harsh environments.

**Rotary electric motor**  Different types of rotary electric motors are available on the market. Available machines include brushed or brushless motors for small applications with DC current and synchrone or asynchrone motors for larger applications with AC current. For all electric motors, there is an approximately proportional link between the size and torque. Thus, for a slow and high-torque application, reduction gears are inevitable, albeit which leads to an increased friction. Rotary electric motors are standard parts and easily available. The conversion between rotary movement of the motor and the linear motion of the auxiliary system should be established without any major loss in the device.
**Other actuator types** Other possible actuators are shape-memory alloys (SMA), magneto-strictive or electro-strictive materials. However, these actuators have very low force and stroke limits and are usually only applied in laboratory environments.

### 1.4.5 Power consumption

The power consumption of an active vibration control device is crucial for the lifetime cost and energy balance. Two parts govern the energy consumption of a device: the running permanent consumption for the measurement devices, decision loop and amplification and the actual power requirement for the generation of the damping force. For semi-active devices, the second part of the energy consumption is replaced by an adaptive damping device (magneto-rheological fluid or solid for example).

The power consumption of an active device for the generation of the damping force depends strongly on the applied control law. Generally, the more damping force is demanded, the higher is the power and consumption. Applying this damping force to a vibrating structure it is clear that a higher modal mass is damped slower than a lightweight one. It is the task of the designer to implement a vibration mitigation concept which fulfils the required specifications with an appropriate energy and power consumption.

### 1.4.6 Behaviour in power failure situations

The behaviour of a structure in unexpected and extraordinary situations must be considered carefully. If the structure is equipped with a vibration mitigation technique, the failure of these devices will lead to vibration problems. Depending on whether the vibration mitigation is applied for serviceability reason or fatigue problems, a malfunction will reduce the serviceability or can even lead to structural problems. In order to increase the performance of such systems, the electrical power supply can be supported by uninterruptible power supplies, batteries or similar. Vibration control devices for sensitive environments should therefore be added to an auxiliary power supply. It should be noted that in regular buildings and civil structures, the absence of electric power will lead to an interruption of the serviceability in any case.
Fire and earthquakes can be possible risks for the vibration mitigation devices. Fire-protective measures should be considered when implementing any vibration control device. However, if the device is applied only for serviceability reasons, no special precautions are required for the case of a fire. Earthquakes create strong horizontal body forces on the auxiliary mass due to the base acceleration. These forces need to be taken into account in the design of the vibration mitigation device.

1.5 Conclusion

This section showed the motivation and possible applications for active vibration control in civil engineering sub-structures within buildings. Some necessary fundamental definitions and concepts were presented to understand the physical basis of the mathematical definitions of the next chapter.

An overview of current civil engineering standards from different regions showed that vibrations of floors are usually not taken into account during the design phase of a building. Only few standards provide rules for the estimation of the fundamental resonance frequencies. However, a detailed dynamic analysis is not required for most cases today. Nevertheless, floor vibrations can have a significant impact on the serviceability of a building. Different standards show that human whole-body vibration perception is limited to a frequency band between approximately 4 and 10 Hz. Perception and exposure limits were defined in different standards in past. Today, the ISO 2631 is widely accepted as a reference based on the acceleration. Other methods are based on a response factor or velocity measurements.

The concept of the inertia-based vibration control device has been introduced in this section. It is obvious that this is the only possible method for the mitigation of floor type structures. Several applications of this method are shown in a state-of-the-art overview. It is shown that most existing control types (passive, semi-active and active methods) lack either of robustness or efficiency.

This section showed the benefits of the application of inertia-based active vibration mitigation on floors. Although some precautions must be taken into account, the active vibration control can generate a great benefit for the application of lightweight and wide-span floors by guaranteeing the serviceability even on dynamically problematic structures.
In order to analyse the dynamic behaviour of a structure, it is common to use linear time-invariant (LTI) systems. A purely reversible, elastic deformation of the structure is assumed for this class of systems. Thanks to the linearity, inverse dynamic models are obtained in most cases. For complex non-linear behaviour of a structure, quasi-linear models can be applied with a point-wise linearisation for any working condition. However, global stability cannot be determined for linearised systems. In this study, it is assumed that all structures have a strictly linear behaviour in the sense of the LTI-conditions [43]. All possible non-linearities and model uncertainties are treated as input disturbances.

2.1 LTI systems

An LTI system is a model of a dynamic process. All process parameters are assumed to be constant in time (within the time scale of the process) and depend linearly on the states of the process. For the interested reader, a mathematical definition of the linearity is given by Longchamp [43]. Furthermore, all systems in this study are assumed to be causal. This means that the output of the LTI system depends only on current or past inputs and the current state of the system. All systems are initially (for $t = 0$) in a stable resting position, called zero-state.

The output of an LTI system, initially in zero-state, is described by the convolution product of the impulse response of the system $g(t)$ and the input $u(t)$ [45]. Using the Laplace transform, the process is described in complex frequency domain. By virtue of this transformation, the rather complex convolution product is transformed in a simple multiplication. The output of a system in frequency domain is therefore the product of the Laplace-transformed system model and the Laplace-transformed input to the system. It should be noted that the LTI system $g$ is constant in time and $g(t)$ is the impulse response of the system $g$. 
\[ y(t) = g(t) * u(t) \quad \leftrightarrow \quad Y(s) = G(s) \cdot U(s) \] (2.1)

A system is BIBO-stable (Bounded-Input-Bounded-Output) if an input signal \( u \) with non-infinite amplitude creates a non-infinite output \( y \) [43]. The stability of a system is generally determined in the Laplace domain by the zeros of the denominator polynomial of \( G(s) \). A denominator zero (pole) is stable if its real part lies (strictly) on the left-hand side of the complex plane. This signifies a positive damping of the pole. Interested readers are referred to the standard literature of control theory and stability for further information (for example: [43, 44, 45, 79]).

2.2 Dynamic behaviour of floor systems

Floors are dynamic systems with inherent time-invariant physical properties within the time scale of the fundamental resonance oscillation period. Their dynamic behaviour can be described in a physical LTI system. It is a key-property of physical systems that mass, damping and elasticity coefficients are real and positive values. Like all real dynamic systems, floors have an infinite number of resonance modes and corresponding resonance frequencies. Depending on the location, the direction and the frequency content of an occurring disturbance force, certain resonance modes are excited, while others are not (cf. sec. 1.2.6). Floors are flat (2 dimensional) structures which are usually span between two or more supports. Depending on their setup, they behave like either a membrane- (isotropic plate with about equal span and fixed horizontal boundary conditions) or beam-type structure (an-isotropic structure or unequal span with free horizontal boundary conditions) according to section 1.2. For small deflections, a purely vertical motion of every point of the structure can be assumed (first-order approximation). In absence of any locally distributed loads, the first three modes of a simply-supported beam- and membrane-type structure can be easily calculated. These modes are shown in figure 2.1.

Figure 2.1 shows that higher order modes involve usually stronger curvature of the structure. Thus, a higher amount of energy required to obtain an equivalent deflection compared to lower order modes. Assuming a uniform excitation over the whole frequency band, the higher order modes play therefore only a minor role for the overall deflection. Additionally, in real physical structures, the higher
order resonance modes have usually an important modal damping. The dynamic behaviour of a structure can therefore be described with sufficient precision by taking into account only the first few resonance modes. The number of necessary modes depend on the structure and the desired modelling precision.

All real physical dynamic structures have a certain damping or attenuation which reduces vibration amplitudes over time. The kinetic energy of the modal movement is partially converted to thermal energy (heat). There are different types of damping in structures. Coulomb friction (dry friction) for example is characterised by a friction force, which is independent of the velocity and direction of the movement. Another type of damping is the viscous damping, which is present in most liquids for small Reynolds numbers. In this case the damping force is proportional to the relative velocity. Other possible damping forces are for example aerodynamic drag or eddy current forces. Real systems rarely present a single damping characteristic, but a combination of viscous, dry and other friction components depending on the vibration amplitude and mode.

Most processes and structures can be described with sufficient precision in a linear or (piecewise) linearised model. The linear velocity dependent viscous damping
is easy to implement in LTI systems which explains the vast use of this damping model. For damping forces far below the critical value most sort of damping can be linearised to fit a quasi-linear viscous damping.

The viscous damping force is defined as [59]:

\[
f_v(t) = c \frac{dx(t)}{dt} = 2\xi m\omega_0 \frac{dx(t)}{dt}
\]

In which \(\omega_0\) is the modal resonance frequency of the system and \(m\) is the modal mass. The value \(\xi\) is the damping ratio. It provides information about how strong the modal damping is relative to the critical damping where no oscillations are possible. For \(\xi = 1\), the damping is equal to the critical damping \(c_{crit} = 2m\omega_0\). Vibrating systems have an under-critical damping with \(\xi \ll 1\). Real oscillating systems with problematic resonance modes have usually damping ratios between 0.5% and 10%, which is also the case for most steel and timber structures. [41]

### 2.3 One-degree-of-freedom dynamic systems

This section provides a brief overview of the standard methods to define resonance frequencies and damping characteristics. The content of this section is general knowledge and can be found in most books on dynamics (for example [59, 18]). The general one-degree-of-freedom (1-DOF) model of a linear forced damped oscillator is shown in figure 2.2. The equation of motion is established using Newton’s method. Throughout this study, external forces and coordinates are defined positive in upward-direction. Thus, the force due to the elastic spring \(k\) and the viscous damping element \(c\) are negative for a positive position change of the mass \(m\).

\[
d(t) - c\ddot{x}(t) - kx(t) = m\ddot{x}(t)
\]

(2.2) can be normalised by the mass (inertia) of the system:

\[
\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{d(t)}{m}
\]

(2.3)

Simple solutions to this equation of motion can be found for a homogeneous \((d = 0)\) and undamped \((c = 0)\) system by using the general ansatz \(x(t) = A e^{\lambda t}\).

\[
\left(\lambda^2 + \frac{k}{m}\right)x(t) = 0
\]

(2.4)
This leads to $\lambda = \pm \sqrt{\frac{k}{m}} i$. The factor $\omega_0 = \sqrt{\frac{k}{m}} \in \mathbb{R}_{>0}$ is called the resonance pulsation. A division by $2\pi$ leads to the resonance frequency $f_0 = \frac{\omega_0}{2\pi}$ of the system.

$$x(t) = A_1 e^{\omega_0 t} + A_2 e^{-\omega_0 t} = A_1 (\cos(\omega_0 t) + i \sin(\omega_0 t)) + A_2 (\cos(-\omega_0 t) + i \sin(-\omega_0 t))$$

Given that the position $x(t)$ is a real value, $A_1 - A_2$ must be zero and therefore:

$$x(t) = \tilde{A} \cos(\omega_0 t)$$

A non-negligible modal damping has a strong influence on the dynamic behaviour of a vibrating system (strong attenuation) and also leads to lower resonance frequencies. The phenomena can be shown in the homogeneous 1-DOF model with viscous damping (from equ. 2.3):

$$\ddot{x}(t) + \frac{c}{m} \dot{x}(t) + \frac{k}{m} x(t) = 0$$

Solutions to this harmonic differential equation with constant real coefficients are found using a general ansatz $x(t) = A e^{\lambda t}$.

$$\lambda_{1,2} = -\frac{c}{2m} \pm \sqrt{\frac{c^2}{m^2} - 4 \frac{k}{m}}$$
Let $\xi = \frac{c}{2m\omega_0} \in \mathbb{R}_{>0}$ be the viscous damping ratio of the vibrating system.

$$\lambda_{1,2} = \omega_0 \left(-\xi \pm \sqrt{\xi^2 - 1}\right) \quad (2.9)$$

The negative first part of (2.9) defines the negative exponential attenuation of the amplitude of the system. The second part of equation (2.9) is complex for $\xi < 1$. The contribution of this part can be distinguished in three cases:

**Critical damping:** If $c = 2\omega_0m$ or $\xi = 1$, the second part of equation (2.9) is null and the equation of motion of the system is $x(t) = Ae^{-\xi\omega_0t}$. A vibration of the system is in this case not possible because an occurring initial displacement decays asymptotically to zero. The value $\xi = 1$ denotes the case for which the system is stabilised at the zero-position in minimum time.

**Over-critical damping:** If $c > 2\omega_0m$ or $\xi > 1$, the system is called over-damped. In this case equation (2.9) leads to two real negative exponents. The non-oscillating motion of the mass is governed by $x(t) = Ae^{\omega_0t}(\xi \pm \sqrt{\xi^2 - 1})$. As for the critical damping, no oscillation of the system is possible. An initial displacement decays asymptotically to zero, although, the decay rate is lower than for the critical damped system.

**Under-critical damping:** If $c < 2\omega_0m$ or $\xi < 1$, the system is called under-damped. Equation (2.9) leads to a negative real and an imaginary part which generates a vibrating motion. The motion is defined by $x(t) = Ae^{\omega_0t}(\xi \pm i\sqrt{1-\xi^2})$ and the appearing resonance frequency is therefore $\omega = \omega_0\sqrt{1-\xi^2}$.

The harmonic under-critical damped oscillation is governed by the following equation (cf. equation (2.5) and (2.6)):

$$x(t) = \tilde{A}e^{-\omega_0\xi t}\cos\left(\sqrt{1-\xi^2}\omega_0t\right) \quad (2.10)$$

A closed-form solution of the general equation of motion of the forced oscillator (2.3) can be established only in simple cases. If a harmonic external force $d(t) = D\cos(\omega t)$ is applied to the system this leads to:

$$\ddot{x}(t) + 2\xi\omega_0\dot{x}(t) + \omega_0^2x(t) = \frac{D}{m}\cos(\omega t) \quad (2.11)$$
In steady-state, the solution of the equation of motion will have the form \( x(t) = A \cos(\omega t - \phi) \). It should be noted that the amplitude \( A \) is strictly positive. Inserting the ansatz into (2.11) leads to:

\[
A \cos(\omega t - \phi)(\omega^2 - \omega_0^2) + 2\xi\omega_0\omega A \sin(\omega t - \phi) = \frac{D}{m} \cos(\omega t)
\] (2.12)

After some trigonometric transformations, following solutions are obtained:

\[
A = \frac{D}{m \sqrt{(\omega^2 - \omega_0^2)^2 + 4\xi^2\omega^2\omega_0^2}}
\]

\[
\tan(\phi) = \frac{2\xi\omega_0\omega}{\omega_0^2 - \omega^2}
\] (2.13)

The amplitude \( A \) is usually normalized by the deflection of the system for a given static disturbance. The absolute value of this ratio \(|A/A_{\text{stat}}|\) is called dynamic amplification factor (DAF). By definition, the DAF is equal to 1 for a static disturbance force. In order to compare the dynamic behaviour of systems the fundamental resonance frequency \( \omega_0 \) is used to standardize the representation. Let the normalized frequency be \( \beta := \frac{\omega}{\omega_0} \) and it follows from (2.13) that

\[
DAF = \frac{A k}{D} = \frac{1}{\sqrt{(\beta^2 - 1)^2 + 4\xi^2\beta^2}}
\] (2.14)

Obviously, the highest dynamic amplification for an undamped system (\( \xi = 0 \)) is given for \( \beta = 1 \). In this case the DAF tends towards infinity. For physical systems with \( \xi \neq 0 \), the maximum DAF is obtained for \( \beta = \sqrt{1 - \xi^2} \) as long as \( \xi < \frac{\sqrt{2}}{2} \). For small damping ratios (\( \xi \ll 1 \)) the maximum dynamic amplification is:

\[
DAF_{\text{max}} = DAF \bigg|_{\beta = \sqrt{1 - \xi^2}} = \frac{1}{\sqrt{4\xi^2 - 3\xi^4}} \approx \frac{1}{2\xi}
\] (2.15)

The lower the damping of the system is at resonance frequency, the higher is the dynamic amplification. The \( DAF_{\text{max}} \) value of the system is equal to the \( \infty \) norm for SISO systems (cf. section 2.6). Figure 2.3 shows the dynamic amplification and phase delay according to (2.13).

The 2 norm of a system is defined in section 2.6. A closed-form solution of equation (2.44) has been published by Crandall [20]. Some simple cases are given in appendix.
Fig. 2.3: Normalised dynamic amplification in the Bode diagram for different damping ratios $\xi$ according to equation 2.13. The maximum peak amplification is given by equation (2.15).

B. The output energy of a one-degree-of-freedom oscillator after an impulse-form excitation is therefore given by

$$E = \frac{1}{2} \frac{k}{c} = \frac{1}{4\xi m^2 \omega_0^3}$$  

(2.16)

Obviously, a stiff system (high $k$) with high resonance frequency and high viscous damping (high $c$) has a low output energy.

2.4 Multiple degrees-of-freedom dynamic systems

Real physical systems have an infinite number of resonance modes. Thus, the modelling of real structures requires usually multiple degrees-of-freedom models. The equation of motion is therefore established for systems with multiple degrees of freedom $x_k$ in matrix notation as:

$$[M]\ddot{x}(t) + [C]\dot{x}(t) + [K]x(t) = P(t)$$  

(2.17)
Each possible direction of motion or rotation is treated as a coordinate in a separate line of (2.17). At least one inertia property and one connection to another coordinate per line is required for a physical system.

Equation 2.17 is also the common equation of motion of continuous structures described in a lumped parameter model, for example in a finite element representation [28]. The above shown equation can therefore be taken as the general equation of motion of a forced damped oscillator. By normalising (2.17) by the mass matrix, it becomes:

$$\ddot{x}(t) + [M]^{-1}[C]\dot{x}(t) + [M]^{-1}[K]x(t) = [M]^{-1}P(t)$$ (2.18)

It should be noted that the mass matrix is generally square and positive definite. A non-square or (semi-)negative definite matrix would involve negative masses or directions which do not have an inertia. Therefore, the inverse matrix $[M]^{-1}$ exists in physical systems. High damping characteristics (linear or non-linear) will lead to a strong attenuation of a given vibration mode and therefore to an unproblematic dynamic behaviour in any case. It is therefore assumed that only little inherent damping is present in problematic vibrating systems (cf. section 2.3). The influence of the damping to the resonance frequency can therefore be neglected [59]. Thus, the resonance frequencies are governed mainly by:

$$\ddot{x}(t) + [M]^{-1}[K]x(t) = [M]^{-1}P(t)$$ (2.19)

A common ansatz to determine the homogeneous solution of (2.19) is $x(t) = A e^{\lambda t}$ [28]. In matrix notation, $A$ is the vector of the vibration amplitudes.

$$(-\lambda^2 + [M]^{-1}[K])x(t) = 0$$ (2.20)

A trivial solution of this expression is $x(t) = 0$. This solution has no physical meaning because all movements are restraint. Therefore, $x(t)$ must be not equal to zero at least once for $t \neq 0$ and the determinant of $(-\lambda^2 + [M]^{-1}[K])$ must be zero to fulfil (2.20). The free resonance modes (for $P(t) = 0$) are therefore given by the square roots of the eigenvalues of $[M]^{-1}[K]$. The according eigenvectors $[v]$ present the mode shape of the structure for the resonance frequency.

$$\lambda_k^2[v_k] = [M]^{-1}[K][v_k]$$ (2.21)

The expression $[M]^{-1}[K]$ is symmetric, of full rank and positive-definite for a mechanical system without rigid body movement. Thus, the eigenvalues are strictly positive and the according eigenvectors $v$ are orthogonal. The eigenvectors define
a set of "all possible movements" of the structure. Each normalised eigenvector \(|v_k| = 1\) represents the mode-shape of the corresponding resonance frequency. The movement of the structure can therefore be approximated by a linear combination of the eigenvectors. The desired precision of the simulation defines the number of required resonance modes. The eigenvectors are orthogonal and normalised (orthonormal vector space) and define the transformation between absolute and modal coordinates. Let \([V]\) be a matrix with the normalised eigenvectors \([v]\) as columns and \(y_k(t)\) the according amplitude. The vector \(y_k(t)\) contains the contribution of every mode shape \(v_k\) at the time \(t\).

\[
x(t) = \sum_k v_k y_k(t) = [V]y(t) \tag{2.22}
\]

Inserting (2.22) into (2.17) and pre-multiplying by \([V]^T\) yields:

\[
[V]^T[M][V] \ddot{y}(t) + [V]^T[K][V] y(t) = [V]^T P(t) \tag{2.23}
\]

Equation 2.23 shows the equation of motion in modal coordinates. The modal mass matrix and the modal stiffness matrix are diagonal [28]. Thus, the modes are independent from each other and the multi-mode system can be described a set of independent 1DOF systems, each with a modal mass, stiffness and excitation. By virtue of the orthonormality of the matrix \([V]\), (2.23) can be rearranged as:

\[
[I] \ddot{y}(t) + [V]^T[M]^{-1}[K][V] y(t) = [V]^T[M]^{-1} P(t) \tag{2.24}
\]

It is obvious that the expression \([V]^T[M]^{-1}[K][V]\) is equal to the diagonal matrix with the squared resonance frequencies \([\lambda_{kk}]\) on the diagonal axis.

Equation 2.23 shows that the excitation of the structure depends strongly on the mode shape. Depending on the location of the excitation force, certain modes are excited while others are not. An exciting force close to the nodal point of a mode \((V \approx 0)\) does not contribute to the chosen mode. As long as no damping is present and the resonance frequencies are distinct, the transformation into modal space is simple. If significant damping is present or the modes are closely separated, there is a mutual interaction between modes which results in an ill-conditioned eigenvalue problem.
2.5 Inertia-based vibration control devices - the two degree-of-freedom model

The two-degree-of-freedom (2DOF) model of a structure with inertia-based vibration control device comprises the main structure, which is in this case similar to the 1-DOF model from the previous section, and the auxiliary dynamic system of the damping device (cf. figure 2.4).

2.5.1 General model

The general two degree-of-freedom model of the vibrating structure with inertia-based vibration control device is shown in figure 2.4. It is assumed that only viscous damping is present and the damping of the vibrating structure is low \( (c_1 << c_{crit}) \). The vibrating structure is characterised by its modal mass \( (m_1) \), which is suspended on a spring-type elasticity \( (k_1) \) and a viscous damping \( (c_1) \). On the top of the vibrating structure, the auxiliary system is placed. It comprises the damping mass \( (m_2) \), which is connected by the linear elastic-spring \( (k_2) \) and the viscous damper \( (c_2) \). An additional element \( (f) \) is placed between the damper mass and the vibration structure. This can be any passive, semi-active or active element which is able to create a force between the two inertias. The dynamic behaviour of the two degree-
of-freedom system is described by the following equation of motion (cf. figure 2.4):

\[
m_1 \ddot{x}_1(t) + c_1 \dot{x}_1(t) + k_1 x_1(t) - c_2 \dot{x}_2(t) - \dot{x}_1(t) - k_2 (x_2(t) - x_1(t)) = d_1(t) - f(t)
\]
\[
m_2 \ddot{x}_2(t) + c_2 \dot{x}_2(t) - \dot{x}_1(t) + k_2 (x_2(t) - x_1(t)) = d_2(t) + f(t)
\]

(2.25)

Assuming that the initial states are zero, the Laplace transform [43] of (2.25) leads to:

\[
\left( m_1 s^2 + (c_1 + c_2) s + (k_1 + k_2) \right) X_1(s) - (c_2 s + k_2) X_2(s) = D_1(s) - F(s)
\]
\[- (c_2 s + k_2) X_1(s) + (m_2 s^2 + c_2 s + k_2) X_2(s) = D_2(s) + F(s)
\]

or in matrix notation:

\[
\begin{bmatrix}
  m_1 s^2 + c_1 s + k_1 + (c_2 s + k_2) \\
  -(c_2 s + k_2) \\
\end{bmatrix}
- \begin{bmatrix}
  -(c_2 s + k_2) \\
  m_2 s^2 + (c_2 s + k_2) \\
\end{bmatrix}
= \begin{bmatrix}
  D_1(s) + [-F(s)] \\
  D_2(s) + F(s) \\
\end{bmatrix}
\]

(2.26)

The matrix \([A]\) can be split in a sum of two sub-matrices. A diagonal matrix which incorporates the equation of motion of the two completely decoupled systems \((m_1 s^2 + c_1 s + k_1\) and \(m_2 s^2\)). The second part of the matrix \([A]\) comprises a symmetric matrix which creates the link \((c_2 s + k_2)\) between the two diagonal main systems. The actuator force \(F\) acts equally on the main and auxiliary mass, but with an inverse sign. This force can be used to generate coupling (for example a higher stiffness in parallel to \(k_2\)), but also decoupling forces (for example a reduction of \(k_2\)) between the two masses. Equation (2.26) can be normalised by the modal stiffness using the common definitions.

\[
\begin{bmatrix}
  A_{1,1} & A_{1,2} \\
  A_{2,1} & A_{2,2} \\
\end{bmatrix}
\begin{bmatrix}
  X_1(\beta) \\
  X_2(\beta) \\
\end{bmatrix}
= \begin{bmatrix}
  D_1(s) \\
  D_2(s) \\
\end{bmatrix}
+ \begin{bmatrix}
  -F(s) \\
  F(s) \\
\end{bmatrix}
\]

(2.27)

with

\[
A_{1,1} = k_1 \left( \beta^2 + 2\xi_1 \beta + 1 + 2\xi_2 \mu \alpha \beta + \alpha^2 \mu \right)
\]
\[
A_{1,2} = k_1 \left( -2\xi_2 \mu \alpha \beta - \alpha^2 \mu \right)
\]
\[
A_{2,1} = k_1 \left( -2\xi_2 \mu \alpha \beta - \alpha^2 \mu \right)
\]
\[
A_{2,2} = k_1 \left( \mu B^2 + 2\xi_2 \mu \alpha \beta + \alpha^2 \mu \right)
\]
The matrix \( A \) is square and of full rank if \( \mu \neq 0 \), which is always the case if an auxiliary system is present. The matrix \( A \) is therefore invertible and equation (2.27) yields:

\[
\begin{bmatrix}
X_1(\beta) \\
X_2(\beta)
\end{bmatrix} = [A]^{-1}
\begin{bmatrix}
D_1(\beta) - F(\beta) \\
D_2(\beta) + F(\beta)
\end{bmatrix}
\]  
(2.28)

The inverse matrix \( A(\beta) \) is called transfer function matrix \( G(\beta) \). It links the position of the inertias (output) to the sum of the input forces in the Laplace domain. The transfer function matrix is defined as:

\[
G = [A]^{-1} = \frac{1}{\det(A)}
\begin{bmatrix}
A_{2,2} & -A_{1,2} \\
-A_{2,1} & A_{1,1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\beta^4 + 2(\xi_1 + \alpha \xi_2 + (1 + \mu))(1 + \mu) + 4\alpha \xi_1 \xi_2 + 2(\xi_1^2 + \xi_2^2) + \alpha^2 \\
2\xi_2 \mu \alpha \beta + \mu \alpha^2 + 2\xi_2 \mu \alpha \beta + \alpha^2 \\
2\xi_2 \mu \alpha \beta + \alpha^2 + \beta^2 + 2(\xi_2 \mu \alpha + \xi_1) \beta + \alpha^2 \mu + 1
\end{bmatrix}
\]  
(2.29)

It should be noted that the transfer function matrix \( G(\beta) \) is symmetric and BIBO-stable for all physical vibration systems. All components of the transfer function matrix have the same poles (zeros of the characteristic polynomial \( \det(A) \)).

A coordinate change can be useful for certain applications (stroke limitations, etc.). The relative coordinates are therefore defined as:

\[
\begin{bmatrix}
y_1 \\
y_2
\end{bmatrix} := \begin{bmatrix}
x_1 \\
x_2 - x_1
\end{bmatrix} = \begin{bmatrix}
1 & 0 \\
-1 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
\]

2.5.2 Passive damping devices

Chapter 1.3 describes the passive vibration mitigation concepts using inertia type damping systems. Their common property is that no semi-active or active component is present. Therefore the actuator force \( F(s) = 0 \). The simplest passive vibration mitigation system is the vibration absorber, patented by Frahm in 1911 [26].

**Frahm’s Vibration Absorber**  
Frahm’s vibration absorber does not contain any damping element. The following model is therefore established based on (2.26). Any disturbance on the auxiliary mass is neglected in this model \( D_2 = 0 \).
Using the proposed normalisation and a tuning of $\alpha = 1$, equation (2.30) leads to:

$$
\begin{bmatrix}
\frac{1}{m_1 s^2 + k_1 + k_2 - k_2} & \frac{-k_2}{m_2 s^2 + k_2}
\end{bmatrix}
\begin{bmatrix}
X_1(s)
\end{bmatrix}
= 
\begin{bmatrix}
D(s)
\end{bmatrix}
\quad (2.30)
$$

$$
\begin{bmatrix}
X_1(\beta)
\end{bmatrix}
= 
\begin{bmatrix}
\frac{1}{k_1 (\beta^4 - \beta^2 (2 + \mu) + 1)}
\end{bmatrix}
\begin{bmatrix}
1 - \beta^2
\end{bmatrix}
\begin{bmatrix}
D(s)
\end{bmatrix}
\quad (2.31)
$$

Instead of one amplitude peak in the Bode diagram, the additional damping device leads to two peaks (zeros of the denominator polynomials at $\beta = \sqrt{\frac{\mu + 2 \pm \sqrt{\mu(\mu + 4)}}{2}}$) and a strong decrease of the existing peak at the resonance frequency $\beta = 1$ due to the zero of the numerator polynomial.

It is clear that the established mathematical model (2.31) does in reality contain a certain damping $c_1, c_2 \neq 0$. This damping of the main structure, as well as of the auxiliary system leads to a reduction of the peak amplitudes for the two poles, but also an amplification of the zero at resonance frequency.

**Tuned mass damper according to Den Hartog**  
J. P. Den Hartog developed together with J. Ormondroyd the optimal tuning and damping parameter for Frahm’s vibration absorber in 1928 [57]. Den Hartog published these results in the first edition of his famous book "Mechanical Vibrations" in 1934 [33]. He discovered that there are two fixed points in the dynamic amplification curve, which are independent of the auxiliary damping parameter for a given mass ratio and a negligible inherent damping of the structure. By setting these fix-points on an equal dynamic amplification, he was able to determine an optimal configuration in the sense of the $H_\infty$ norm of the system (cf. section 2.6). The vibration absorber, which is correctly tuned to the vibrating structure according to Den Hartog is called TMD. For small main system damping coefficients, these rules are valid with a negligible error. The optimal tuning parameter according to Den Hartog [33] is:

$$
\alpha = \frac{1}{1 + \mu}
$$

$$
\xi_2 = \sqrt{\frac{3\mu}{8(1 + \mu)^3}}
$$

(2.32)

J.P. Den Hartog did not find a closed-form solution for the damping ratio in 1934, although he provided numerical values (cf. figure 2.5). It was only in the 4th edition...
Fig. 2.5: Optimal damping ratio according to Den Hartog [33]


Several publications can be found about the optimal TMDs (cf. table 2.1). In 2003 exact $H_\infty$ solutions for the TMD were published by Asami and Nishihara [49]. These tuning parameters are exact in the sense of $H_\infty$, but are only slightly better (< 0.3%) than the fixed-point approach by Den Hartog. In real application this improvement of the DAF is usually not significant due to non-linearities and modelling uncertainties of the damping system. Asami and Nishihara [49] have also determined exact $H_\infty$ solutions for the case of a damped primary structure.
Optimum Criterion Tuning parameter $\alpha$ Auxiliary damping $\xi_2$

<table>
<thead>
<tr>
<th>Position $H_2$</th>
<th>(b)</th>
<th>$\sqrt{\frac{1+\mu/2}{1+\mu}}$</th>
<th>$\sqrt{\frac{\mu(1+3/4\mu)}{4(1+\mu)(1+\mu/2)}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(c)</td>
<td>$\frac{1}{1+\mu} \sqrt{\frac{2+\mu}{\mu}}$</td>
<td>$\frac{\mu(4+3\mu)}{8(1+\mu)(2+\mu)}$</td>
<td></td>
</tr>
<tr>
<td>$H_\infty$</td>
<td>(a)</td>
<td>$\frac{1}{1+\mu}$</td>
<td>$\frac{3\mu}{8(1+\mu)}$</td>
</tr>
<tr>
<td>(c)</td>
<td>$\frac{2}{1+\mu} \sqrt{\frac{2[16+23\mu+9\mu^2+2(2+\mu)\sqrt{1+3\mu}]}{3(64+80\mu+27\mu^2)}}$</td>
<td>$\frac{1}{4} \sqrt{\frac{8+9\mu-4\sqrt{1+3\mu}}{1+\mu}}$</td>
<td></td>
</tr>
<tr>
<td>Velocity $H_2$</td>
<td>(b)</td>
<td>$\sqrt{\frac{1}{1+\mu}}$</td>
<td>$\sqrt{\mu/4}$</td>
</tr>
<tr>
<td>$H_\infty$</td>
<td>(b)</td>
<td>$\sqrt{\frac{1+\mu/2}{1+\mu}}$</td>
<td>$\frac{3\mu(1+\mu+5/24\mu^2)}{8(1+\mu)(1+\mu/2)^2}$</td>
</tr>
<tr>
<td>(c)</td>
<td>$\sqrt{\frac{1}{1+\mu}}$</td>
<td>$\frac{3\mu}{8(1+\mu/2)}$</td>
<td>$\frac{1}{2} \sqrt{\frac{\mu}{2(1+\mu)}}$</td>
</tr>
</tbody>
</table>

Tab. 2.1: Optimal tuning parameter for different optimisation criteria according to Den Har-tog (a) [33], Warburton (b) [82] and Nishihara and Asami (c) [49, 4] for harmonic main mass excitation and undamped main systems.

### 2.5.3 Semi-active damping devices

Semi-active damping devices cannot be modelled with linear models due to their inherent non-linearities. The semi-active device must be either modelled as a linear active device or the design process must be made based on time-domain simulations. Nevertheless, a modelling as an active device overestimates the damping efficiency. Robustness limits are usually also too tight by taking into account only these models. On the other hand, a time-domain simulation provides only information for the applied loading scenario. Thus, a global stability statement is not possible.

The main drawback of the semi-active damping device is that no energy can be added to the system. Thus, the semi-active damping device is often designed as an adaptable tuned mass damper (ATMD). Several publications can be found about this topic [84, 16, 38]. However, it is possible to build a semi-active damping device with similar control laws as an active damping device, with the limitation that energy can only be extracted from the device [47]. Especially for transient excitations, the damping efficiency is therefore significantly lower compared to an active device. For long duration harmonic excitations the differences are smaller, because a sufficient mutual excitation of the auxiliary system is possible.
The commonly-applied concept of the ATMD requires a precise measurement or estimation of the current resonance frequency of the time-variant main vibrating system. The "quasi-stiffness" from the semi-active device is applied in such a manner that the auxiliary device is tuned according to the design rules of a passive TMD (cf. 2.1).

2.5.4 Active damping devices

General (semi-) active damping device The general model of the active damping device is established in chapter 2.5.1. It includes an additional actuator $F$. In order to mitigate the main system vibrations, a force is generated between the auxiliary and main modal mass. This force is acting equally on both masses and generates accelerations depending on the ratio of mass.

Obviously, this force needs to depend on the systems states (velocity and position of the mass). This type of control using a measurement of the actual system states is called closed-loop feedback. In the case of a two-degree-of-freedom system, a measurement of the position, velocity and acceleration is possible for each mass. Additionally, past values of these measurements are also available for the control. The desired damping force is built by a linear of the system states.

$$f(t) = \sum_i [K_{1i}x_i(t) + K_{2i}\dot{x}_i(t) + ...]$$ \hspace{1cm} (2.33)

Applying this force law, the general closed-loop 2-DOF model (2.28) becomes

$$\begin{bmatrix} X_1(\beta) \\ X_2(\beta) \end{bmatrix} = [A(\beta) + K(\beta)]^{-1} \begin{bmatrix} D_1(\beta) \\ D_2(\beta) \end{bmatrix}$$ \hspace{1cm} (2.34)

Depending on the components of K, a large variety of control laws $K$ can be created. Equation (2.34) shows that the controller K is directly added to the initial passive system $A$. A linear feedback of the relative position of the auxiliary mass to the main mass creates therefore an additional auxiliary stiffness for example. The mathematical basis of the control loop and various applicable control laws are provided in chapter three.

Two main concepts can be distinguished in active vibration control - the ATMD and the active mass driver (AMD).
Active tuned mass damper  The general concept of the ATMD is to adapt the vibration control device in real time to the dominant resonance frequency of the time-varying (non-LTI) system. For this purpose, a precise measurement or estimation of the current resonance frequency of the main system is required. The vibration mitigation depends, similar to the passive TMD, only on the mutual energy transfer from the main to the auxiliary system. This concept can be applied only for systems with dominant harmonic excitations. Transient vibrations can lead to errors in the frequency estimation and thus, to a reduced damping efficiency.

Active mass driver  The concept of the active mass driver (AMD) is to drive (accelerate) the auxiliary mass to create the desired current damping force. The energy transfer depends therefore solely on the feedback loop and works best if virtually no mutual energy transfer between the main and auxiliary system is present. This means that the elastic mount of the auxiliary system should be as soft as possible and present no auxiliary damping. As long as no stroke or actuator saturations are met, the damping force is proportional to the acceleration of the auxiliary mass. Constant or low-frequency disturbances require therefore high stroke limits or high auxiliary masses.

Different control strategies and control laws are shown in detail in chapter 3.

2.6 System norms and energy

LTI systems are often characterised by their norm. The most common system norms are the $\infty$ norm and the 2 norm. They can be derived from the general signal norm. The content of this section belongs to the general knowledge and is based on Doyle et al. [36] and Toscano [79].

2.6.1 Signal norms

The norm of a signal describes its content in a reduced order space. It is used to compare specific properties of a signal. The p norm of a piece-wise continuous signal $u(t)$ is defined in the time-domain as:

$$|u|_p := \left( \int_{-\infty}^{\infty} |u(t)|^p \, dt \right)^{1/p} \quad (2.35)$$
The 2 norm (equation 2.35 with $p = 2$) denotes the square-root of the area of the squared absolute signal value in the time plot.

$$|u|_2 = \sqrt{\int_{-\infty}^{\infty} |u(t)|^2 dt} \quad (2.36)$$

Equivalently the $\infty$ norm is the sum of all signal values powered by $\infty$ under the "$\infty$-root". This is equal to the highest absolute value of the signal in any other norm $k$. For the case of a simple movement in space, the $\infty$ norm is equal to the maximum value of the 2 norm (Euclidean norm).

$$|u|_{\infty} = \sup(|u|_k) \quad (2.37)$$

### 2.6.2 System norms

Equivalently to the signal norm, the system norm is defined in frequency domain. As in the previous section, the 2 norm and the $\infty$ norm are defined here for LTI systems. Let $g(t)$ be the impulse response signal of the BIBO-stable system $G(j\omega)$ in time domain. According to Parseval’s theorem, the 2 norm in frequency and time-domain are equal [79]:

$$|g|_2 = \sqrt{\int_{-\infty}^{\infty} |g(t)|^2 dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega} = |G(j\omega)|_2 \quad (2.38)$$

The 2 norm is therefore proportional to the square root of the area under the squared amplitude Bode plot for a single-input-single-output (SISO) system. Similar to the signal norm, the $\infty$ norm of a system is defined as the maximum possible amplification of a signal [79]:

$$|G(j\omega)|_{\infty} = \sup_{|U|_p \neq 0, p > 0} \left( \frac{|G(j\omega)U(j\omega)|_p}{|U(j\omega)|_p} \right) = \max_\omega |G(j\omega)| \quad (2.39)$$

In the case of a SISO system, the $\infty$ norm of the system $G(j\omega)$ can be found by applying a white noise input with constant spectrum ($|U(j\omega)|_p = 1$). The $\infty$-norm of a system is the supremum of the dynamic amplification function in any other norm ($0 < p < \infty$).

In the case of a systems with multiple inputs and outputs (MIMO-system), the 2 and $\infty$ system norm are defined as [79]
\[ |G(j\omega)|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}(G(j\omega)G^*(j\omega)) \, d\omega} \quad (2.40) \]

\[ |G(j\omega)|_{\infty} = \sup_{|U|_p \neq 0, p > 0} \left( \frac{|G(j\omega)U(j\omega)|_p}{|U(j\omega)|_p} \right) = \sup_{\omega} \sigma(G(j\omega)) \quad (2.41) \]

The function \( \sigma \) defines the maximum singular value of the transfer matrix function \( G(j\omega) \). Toscano [79] states therefore that the infinity norm is the largest possible frequency gain, which corresponds to the maximum of the largest singular value of \( G(j\omega) \).

### 2.6.3 Energy

The energy of a time-domain signal is proportional to the square of its amplitude. This close relation between 2 norm and signal energy allows defining the energy \( E \) of a signal \( f(t) \) in time and frequency domain using Plancherel’s (Parseval) theorem and the Fourier transformation \( F(\omega) \) of the signal \( f(t) \) [19]. It should be noted that the signal energy is the squared value of the 2 norm in time and frequency domain.

\[ E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = |F(\omega)|_2^2 \quad (2.42) \]

The expression \( |F(\omega)|^2 \) is called Energy Spectral Density (ESD) [19]. It shows the distribution of the signal energy in the spectrum.

Let \( y(t) \) be the output of an LTI system \( g(t) \) due to the system input \( u(t) \). The Laplace (or Fourier) transform of this yields \( Y(s) = G(s)U(s) \) with the Laplace parameter \( s = j\omega \). The output energy (2.42) of the system is therefore:

\[ E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(j\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)U(j\omega)|^2 \, d\omega \quad (2.43) \]

Two special cases are analysed usually. In the case one, the input is a harmonic signal, the second case treats a white noise (or impulse) input. Let \( U(\omega) = \hat{U}_k \delta(\omega - \omega_k) \) be the Laplace transfer of a harmonic input signal \( u(t) = \hat{U} \sin(\omega_k t) \). The output energy of the system \( G(j\omega) \) is therefore:

\[ E = \frac{\hat{U}_k^2}{2\pi} |G(j\omega_k)|^2 \]
The output energy is therefore the squared absolute value of the dynamic amplification at the excitation frequency multiplied by the squared excitation amplitude. For the second case, let the excitation (input) be a white noise signal (or impulse) with constant spectral amplitude. The system input is therefore $|U(j\omega)| = 1$ over the whole frequency band. The total output energy becomes therefore

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega = |G(j\omega)|^2$$  \hspace{0.5cm} (2.44)

It is obvious that the output energy due to an impulse-disturbance (2.38) is the squared 2 norm (2.44) of the system.

### 2.6.4 Mechanical energy balance and performance measurement

The vibration amplitude of a dynamic system is the result of a difference between input and output of power. An occurring power or energy input to a dynamic system conduct to a motion of the systems inertia. The applied power or energy is therefore stored in the form of kinetic or potential energy. However, the motion involves a certain dissipation until the final resting position of the inertia. In order to reach a steady state vibration with constant amplitude and frequency, a steady state input force with constant amplitude, frequency and phase shift is required. The resulting movement leads to an energy dissipation which equalises the input energy. If the amplitude of the input signal is increased, the amplitude of the system is increased. Thus the energy dissipation, which is approximately proportional to the stroke or velocity, is increased until an equilibrium is reached.

The power dissipation of a system is described for under-damped harmonic oscillators by the quality- or Q-factor. It is defined by the ratio between the energy stored in the system and the energy dissipated per cycle.

$$Q := 2\pi \frac{E_{\text{stored}}}{E_{\text{dissipation}}} = \frac{\omega}{P_{\text{dissipation}}}$$  \hspace{0.5cm} (2.45)

It is easy to show that the property Q is constant for a one-degree-of-freedom oscillator (cf. sect. 2.3) with linear viscous damping ($Q = 1/2\xi$). Thus, the higher the damping ratio, the lower is the Q-factor. The critical damping is reached for $Q = 1/2$. However, the application of inertia-based external damping leads to a non-constant Q-factor due to the mutual excitation between the two systems. It is
therefore preferable to apply a different performance measurement criterion. These performance measures can be based on the system norm or the power spectral density.

The amount of energy which can be dissipated over a certain time (average dissipation power) depends on the maximum stroke and auxiliary mass of the vibration control device. The power dissipation of the auxiliary system (one-degree-of-freedom oscillator) is given for a forced harmonic movement by:

\[ P_{diss}(t) = \dot{x}(t) \cdot F_{diss}(t) = \dot{x}^2(t) \cdot c \]  

(2.46)

It is therefore advantageous to have high strokes of the auxiliary system. However, the stroke amplitude \( Y_2 \) depends strongly on the auxiliary damping. A constant low value of \( \xi_2 \) leads to strong oscillations, yet a low power dissipation, while an exceeding value dissipates more energy but the oscillation amplitude is low. For non-controllable systems (passive systems), a trade-off must be designed. Semi-active and active systems are able to vary the damping values according to the current stroke amplitude to achieve a maximum dissipation. The dissipation capacity of a passive or semi-active device (with constant viscous damping parameter) is given for a harmonic movement \( (x(t) = X \sin(\omega t)) \) by:

\[ \bar{P}_{diss} = \frac{c}{T} \int_{t}^{t+T} \dot{x}^2(t) dt = \frac{cX^2\omega^2}{2} \]

(2.47)

The active device is usually limited by the maximum power saturation \( P_{max} \) of the actuator (or amplifier) which is given in the case of an electric motor by the heat radiation limit. If the actuator force is a limiting factor, the dissipation capacity is given by:

\[ \bar{P}_{diss} = \frac{F_{max}}{T} \int_{t}^{t+T} |\dot{x}(t)| dt = \frac{F_{max}^2X\omega}{\pi} \leq P_{max} \]

(2.48)

2.6.5 Energy transfer

Let \( S_{xx} \) be the power spectral density (PSD) of the signal \( x \). In the case of an LTI-system the output signal is linked to the input signal by the squared 2 norm of the system [79].

\[ S_{xx,i}(\omega) = |G_{ik}(\omega)|^2 S_{dd,k}(\omega) \]

(2.49)

The squared 2 norm of the system governs therefore the energy flux between the main and auxiliary system for every frequency. It is obvious that in the case of
vibration mitigation the PSD of the main system $S_{xx,1}$ due to a disturbance $S_{dd,1}$ should be minimal and thus, $|G_{11}|$ should be small for every frequency (cf. section 2.5.1). However, the component of $G$ are interconnected and cannot be chosen arbitrarily. $G$ must be optimized therefore in such a way that the energy flux from $d_1$ is lead to the auxiliary system instead of the main system.

### 2.7 Modelling errors and Sensitivity

Obviously, LTI-models incorporate only linear behaviour of the system. Any non-linearities are treated as disturbing effects. Friction, non-linear and second-order effects can lead therefore to modelling errors. Additionally, sensor noise and non-linearities due to the measurement system are not treated in these models. Mechanical systems have usually a very linear deformation behaviour. It is therefore state of the art to control treat systems as linear models. If an important dry friction is present, this modelling technique reaches its limits.

The sensitivity to these effects needs to be studied carefully. Especially, the acceleration feedback is very sensitive to errors in the measurement system. High-frequency noise and low-frequency drifts, as well as static offsets must be removed by an adequate filtering of the measurement signal.

### 2.8 Conclusion

This section presented the mathematical modelling of an LTI dynamic mechanical system. Basic concepts for LTI-modelling are shown and their limitations were discussed. Dynamic single- and multi-degree-of-freedom systems were introduced. Properties of the systems like stiffness, damping and modal movement were analysed. A general normalisation for dynamic systems by the fundamental resonance frequency and modal mass was introduced.

It was shown that multi-degree-of-freedom systems can be approximated by a series of single degree-of-freedom oscillators. The orthogonality of the modes and the therefore prohibited energy exchange between modes was discussed.

The concept of the inertia-based vibration control was mathematically described in this section. It was shown that the damping efficiency depends on the correct
excitation of the auxiliary system. For passive and semi-active devices, this excitation is achieved by a mutual energy exchange of both systems. Active devices can be excited by the additional actuator force and are therefore independent of the tuning.

The two main concepts of the active vibration mitigation (AMD and ATMD) were described in detail in this section. It was shown that the ATMD concept has generally a lower damping efficiency, but also a much lower power consumption compared to the AMD. However, by virtue of the actuator, an active vibration mitigation device is able to simulate both possible concepts.
In active vibration control, numerous state-of-the-art control strategies can be applied. This section provides a non-concluding overview of the usually applied algorithms in the first part. The second part of this section is dedicated to the novel control algorithms which were established during this study. In the framework of this study, only the system’s positions, velocities and acceleration and linear combinations of these measurements were used as feedback parameters. Other methods like feed-forward, pure delays and other control strategies were not applied for the obvious reason that no advantage can be draws from these techniques in the case of inertia-based vibration mitigation.

The control strategy is a crucial part of the active control of vibrations. It directly affects the damping efficiency of a device and can lead, if not correctly adapted to the structure, to instabilities and therefore to severe problems of the device and the structure itself.

The best controller provides a damping force to the main structure which is always exactly opposite to the actual occurring disturbance force. In the Bode diagram of the closed loop system, this phenomenon can be seen in the transfer function from the disturbance $D_1(t)$ to the acceleration of the auxiliary mass $\ddot{x}_2(t)$. At all frequencies where damping is required, the acceleration of the auxiliary mass should have a phase delay of $180^\circ$ relative to the disturbance. The closer the phase shift is to $180^\circ$ the higher is the damping efficiency. However, not only phase shift, but also the amplitude of the feedback force is important and needs to be taken into account.

All linear control strategies can be reduced to a simple block diagram shown in figure 3.1. The input disturbances $D_1$ and $D_2$ are not know and cannot be measured directly. The only way to gather information about the disturbance forces is via a measurement or estimation of the states (position and velocities) and acceleration of the main and auxiliary mass. By applying a linear state feedback controller $F = KX$, the following general closed-loop transfer function is established (3.1). It should be
Fig. 3.1: Control block diagram. The disturbances $D_1$ and $D_2$ act in parallel to the actuator force $F$ on the main and auxiliary mass. The resulting acceleration leads to the position of the main and auxiliary system $X_1$ and $X_2$ respectively.

noted that for the case of the ATMD a purely linear state feedback is not possible because this control method relies on a resonance frequency estimation. This estimation requires knowledge about the time between two peaks or zero-crossings, which is not a linear function.

\[
X = G^{-1}(D - F) \quad \Leftrightarrow \quad X = (A + K)^{-1} G_{CL} D \quad (3.1)
\]

The stability of the closed-loop transfer function (3.1) is there determined by the denominator polynomials of the matrix $(A + K)^{-1}$. For simple cases, the denominator polynomial is equal to the determinant of $(A + K)$. Thus, the zeros of the denominator polynomial, or the eigenvalues of $(A + K)$ need to be analysed for global stability.
3.1 State-of-the-Art control algorithms

This section provides an overview over some control architectures which have been successfully applied in recent past.

3.1.1 Adaptive tuned mass damper (ATMD)

The standard passive TMD has a low damping efficiency when the tuning of the components is not optimal [84]. For structures which have varying resonance frequencies due to loading or temperature sensitive building materials for example, TMDs are in most cases not optimally tuned. The tuning of passive TMDs can be adapted by means of a semi-active or active component in parallel to the spring. Weber et al. [83] have presented semi-active tuned mass damper based on a device with a magneto-rheological fluid. By adapting the damping of this device correctly, a "quasi-stiffness" can be provided. Obviously, the semi-active device can only produce a negative force (opposite to the travelling direction) and no pushing force like a real spring. Additionally, the adaptive damping of the semi-active device is used to apply an amplitude-dependent viscous damping force. This allows reducing the damping force for small amplitudes and thus, to increase the damping for small disturbance forces. The application of semi-active components is advantageous from the perspective of power and energy consumption because electric power is only required for the measurement and control equipment. The actuator itself is (apart from the control current) a purely passive device.

Weber et al. [84] have published a novel idea for ATMDs with an electric DC motor as an active device. Using active instead of semi-active parts also allows creating real stiffness characteristics and negative damping. This increased flexibility in the control also leads to an increased power and energy consumption.

The adaptive tuning of the ATMD requires an exact online estimation of the current resonance frequency of the structure. This is only possible by a constant measurement of the time between two maxima or zero-crossings. However, a harmonic vibration over several periods is often not desired on floor structures. In applications where mostly transient disturbances occur, the ATMD is therefore not well suited. Another drawback of the ATMD is that even if it is correctly tuned, the damping efficiency is similar to the passive TMD and it strongly depends on the mass ratio.
Let $\omega_1$ and $\omega_2$ be the resonance frequencies of the main and auxiliary system. Further, let $m_2$ and $\xi_2$ be the auxiliary mass and the auxiliary viscous damping ratio, respectively. Thus, the actuator force which creates the required additional stiffness $k_d$ and damping $c_d$ to generate a virtual TMD is given by:

$$f(t) = k_d(x_2(t) - x_1(t)) + c_d(\dot{x}_2(t) - \dot{x}_1(t))$$

$$= m_2 \left( \left( \frac{\omega_1}{1 + \mu} \right)^2 - \omega_2^2 \right) (x_2(t) - x_1(t))$$

$$+ \frac{2m_2\omega_1}{1 + \mu} \left( \sqrt{\frac{3\mu}{8(1 + \mu)^3}} - \xi_2 \right) (\dot{x}_2(t) - \dot{x}_1(t))$$

In the case of an ATMD, the value of $\omega_1$ needs to be measured constantly. A change of the modal mass due to an additional load will directly affect the resonance frequency, as well as the mass ratio. Assuming that the stiffness of the structure $k_1$ is constant, the mass ratio becomes:

$$\mu = \frac{m_2}{m_1} = \frac{m_2 \omega_1^{2, \text{meas}}}{k_1}$$

The measured resonance frequency of the main system should be filtered using a strong low-pass filter to remove undesired high-frequency changes.

### 3.1.2 State feedback - Optimal Control

State feedback with optimal control is one of the standard methods in control theory for MIMO systems. The content of this section is based on standard literature of control theory [44, 45, 36]. Optimal control aims to find a controller which minimises a time-domain cost function $J$. The most common case is to apply a linear state feedback to a model in state space representation. If the cost function $J$ is in the form of a quadratic function, the controller is called a linear-quadratic regulator (LQR). The LQR does not take into account the external disturbances during the design procedure. Disturbance rejection is simply achieved by the minimised cost function. If the modelling and measurement uncertainties are not negligible, the states of the system need to be estimated by a recursive Kalman filter. The combination of estimator and LQR is called linear-quadratic Gaussian control problem (LQG) [44]. The LQG regulator is, in contrast to the LQR, able to take into account disturbances on the system and measurement equipment. However,
disturbances must be modelled as coloured Gaussian noise, which requires precise knowledge of the occurring disturbances. These are standard control algorithms which are often applied to MIMO systems. For this section, it is assumed that all states of the systems are measurable with sufficient accuracy and precision.

The state space representation of a continuous LTI system is according to [44] usually defined as:

$$\Sigma_S : \begin{cases} \dot{x}(t) = A_{ss} x(t) + B_{ss} u(t) + E_{ss} d(t) \\ y(t) = C_{ss} x(t) + D_{ss} u(t) + F_{ss} d(t) \end{cases}$$

(3.4)

The vector $x(t)$ contains all states (positions and speeds of all masses in a mechanical system) of the system. The matrix $A_{ss}$ governs the system in open loop, $B_{ss}$ and $E_{ss}$ describe the influence of the control signals $u(t)$ and the disturbances $d(t)$ to the system. $y(t)$ is the vector of the output states, described by $C_{ss}$. The matrix $D_{ss}$ and $F_{ss}$ are the influence of the control signal (feed-through) and the disturbance to the measurements. In this study, the following common simplifications are applied: $C_{ss} = [I]$, $D_{ss} = [0]$ and $F_{ss} = [0]$ and thus $y(t) = x(t)$.

The goal of the LQR control algorithm is to minimise the state deviation from zero for certain states, as well as the minimisation of the control energy. For this purpose a quadratic cost function is established:

$$J = \int_{t_1}^{t_2} \left[ x^T(t) Q x(t) + u^T(t) R u(t) \right] dt$$

(3.5)

The diagonal matrix $Q$ defines a weighting between the states. The more important it is to control a state, the higher the value $q_{ii}$ is chosen. The diagonal matrix $R$ defines the weighting of the control energy of the states relative to the cost of the state deviation. The control value $u(t)$, and therefore also $R$, are scalar values for the case of a system with only one actuator. By applying a linear negative state feedback law $u(t) = -K x(t)$, the quadratic cost function yields:

$$J = \int_{t_1}^{t_2} \left[ x^T(t) \left( Q + K^T R K \right) x(t) \right] dt$$

(3.6)

This integral is solved easily by substituting the second state vector by the closed-loop transfer function $x(t) = (A_{ss} - B_{ss} K)^{-1} \dot{x}(t)$. Assuming a stable system, the states are defined for $t_1 = 0$ as $x(t_1) = x_0$ and for $t_2 \to \infty$ as $x(t_2) = 0$. 

3.1 State-of-the-Art control algorithms
\[
J = \int_{t_1}^{t_2} \left[ x^T(t) \left( Q + K^T R K \right) x(t) \right] dt \\
= \int_{t_1}^{t_2} \left[ x^T(t) \left( Q + K^T R K \right) (A_{ss} - B_{ss} K)^{-1} \dot{x}(t) \right] dt \\
= \frac{1}{2} x_0^T \begin{pmatrix} (Q + K^T R K)(A_{ss} - B_{ss} K)^{-1} \end{pmatrix} x_0
\]

The cost function \( J \) is minimal for every starting point \( x_0 \in \mathbb{R} \) if the gradient of \( N \) with respect to \( K \) is equal to the zero vector. This development can be found in appendix C.

\[
\nabla_K N = A^T B^{-T} R^T K + K^T R B^{-1} A - K^T R K + Q \\
= A^T P + PA - PBR^{-1} B^T P + Q = [0]
\] (3.8)

By taking advantage of the simplification \( K = R^{-1} B^T P \), the general case of the infinite-horizon, continuous time algebraic Riccati equation is obtained. In general, the Riccati equation has a numeric solution for stable initial systems which are observable and controllable. Finally, the matrix solution \( P \) of the Riccati equation is used to calculate the controller gain \( K \). If a solution \( K \) exists, this controller minimises the predefined cost function \( J \). The weighting matrices \( Q \) and \( R \) must be chosen carefully. If \( R \) is too high compared to \( Q \), the actuator output is governing the cost function and is therefore reduced, which results in a weak controller. It should be noted that this method provides an inherently stable controller without giving any information about the robustness of the closed-loop system.

In order to apply the state feedback control for the inertia-based vibration mitigation, the controllability of the system must be fulfilled [36]. In the case where only one actuator is present (\( u(t) \) is scalar), the input vector \( B \) and the controller \( K \) are vectors and \( R \) is scalar. The matrix \( A_{ss} \) in equation 3.9 shows that the states are strongly coupled. It is clear that not all states of the system are important for the user. Mainly the velocity of the main mass (floor) needs to be controlled.

It is easy to translate the general dynamic system from the transfer function matrix notation (2.26) into a state-space representation (3.4) using the following states:
\[
q_1(t) = \dot{x}_1(t)
\]
\[
q_2(t) = x_1(t)
\]
\[
q_3(t) = \dot{x}_2(t)
\]
\[
q_4(t) = x_2(t)
\]

The complete representation of the generalised 2-DOF system becomes:

\[
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t) \\
\dot{q}_3(t) \\
\dot{q}_4(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{-c_1+c_2}{m_1} & \frac{-k_1+k_2}{m_1} & \frac{c_2}{m_2} & \frac{k_2}{m_2} \\
1 & 0 & 0 & 0 \\
\frac{c_2}{m_2} & \frac{k_2}{m_2} & -\frac{c_2}{m_2} & -\frac{k_2}{m_2} \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
q_1(t) \\
q_2(t) \\
q_3(t) \\
q_4(t)
\end{bmatrix} +
\begin{bmatrix}
-\frac{1}{m_1} \\
0 \\
0 \\
-\frac{1}{m_2}
\end{bmatrix} f(t) +
\begin{bmatrix}
d_1(t) \\
d_2(t)
\end{bmatrix}
\]

Assuming that all states are directly accessible and measurable \((C_{ss} = [I], D_{ss} = [0]\) and \(F_{ss} = [0]\)), the controllability matrix according to Kalman [44] yields:

\[
S_S = [B \ AB \ A^2B \ A^3B] \quad (3.10)
\]

A short calculation shows that \(S_S\) is of rank 4 and the system \(\langle A_{ss}, B_{ss} \rangle\) is therefore fully controllable, although there is only one actuator. This means that every desired state vector \(q \in \mathbb{R}^4\) can be reached for a short period using an appropriate input force \(f(t)\). It is clear that this is only possible under the assumption that the disturbances \(d_1\) and \(d_2\) are zero.

For constant inputs \(f(t) = f\), the system reaches the stable state vector \(q = [0 \ 0 \ 0 \ \frac{f}{k_2}]^T\) for \(t \to \infty\). In the physical system this position for \(q_4 = x_2\) marks the distance between main and auxiliary mass (for a constant position of the main system \(x_1(t) = cte\)) which is altered by the actuator force against the auxiliary spring force. The position of the main mass cannot be constantly altered, except for \(f(t) \to \infty\), which case hurt stroke and force limitations in any case.

If linear state feedback is applied, the control force \(f(t)\) depends linearly on the state vector \(q(t)\) and the actuator force becomes:

\[
f(t) = [K_1 \ K_2 \ K_3 \ K_4][q_1 \ q_2 \ q_3 \ q_4]^T
\]
\[
= K_1q_1 + K_2q_2 + K_3q_3 + K_4q_4
\]

3.1 State-of-the-Art control algorithms
The components of $K$ are determined numerically by solving the Riccati equation (3.8). By inserting (3.11) into (3.9) closed-loop transfer function is therefore:

$$
\begin{bmatrix}
\dot{q}_1(t) \\
\dot{q}_2(t) \\
\dot{q}_3(t) \\
\dot{q}_4(t)
\end{bmatrix} =
\begin{bmatrix}
\frac{c_1+c_2+K_1}{m_1} & -\frac{k_1+k_2+K_2}{m_1} & \frac{c_2-K_4}{m_1} & \frac{k_2-K_4}{m_1} \\
1 & 0 & 0 & 0 \\
\frac{c_2+K_2}{m_2} & \frac{k_2-K_1}{m_2} & \frac{-c_2+K_1}{m_2} & \frac{-k_2+K_1}{m_2} \\
0 & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
q_1(t) \\
q_2(t) \\
q_3(t) \\
q_4(t)
\end{bmatrix}
+ \begin{bmatrix}
d_1(t) \\
0 \\
d_2(t) \\
0
\end{bmatrix}
$$

(3.12)

3.1.3 Direct Velocity feedback

The direct velocity feedback is an often used and very efficient control strategy. The controller needs simply to feed back the velocity of the main structure, multiplied by the feedback gain. The simplified one-degree-of-freedom model (for the case of a completely decoupled auxiliary system) of this feedback law becomes in closed loop (using $f(t) = c_d \dot{x}(t)$):

$$
X_1(s) = \frac{1}{m_1 s^2 + (c_1 + c_d)s + k_1} D_1(s)
$$

(3.13)

Obviously, the damping of the main structure can be altered when increasing $c_d$. However, if $c_d$ exceeds a certain value, the closed-loop system becomes unstable. Stability issues were investigated in numerical models by Hanagan [32], Nyawako [54] and Diaz [24]. This control strategy works best when the physical interaction between primary and auxiliary system is low and the damping device works as an AMD. This is achieved by applying a low stiffness and damping for the auxiliary system. If the connection between main and auxiliary mass is negligible, the second-order system becomes in closed loop:

$$
X(s) = \left( \begin{bmatrix}
A_{1,1}(s) & 0 & 0 & 0 \\
0 & m_2 s^2 & c_d \cdot s & 0 \\
0 & -c_d \cdot s & 0 & 0
\end{bmatrix} \right)^{-1} D(s) = \left( \begin{bmatrix}
\frac{1}{A_{1,1}(s)+c_d \cdot s} & 0 & 0 \\
\frac{c_d \cdot s}{A_{1,1}(s)m_2 s^2} & 0 & 0
\end{bmatrix} \right)^{-1} D(s)
$$

(3.14)

Equation (3.14) shows that a theoretical second-order integrator is established in the transfer function $X_2/D_1$. Low frequency disturbances are therefore strongly amplified. This requires an appropriate high-pass filtering of the measurement data to prevent the auxiliary system from hitting stroke and force limitations for low-frequency or static loads. However, high-pass filtering is only applicable if the
cut-off frequency is well-separated from the fundamental main system resonance frequency.

In order to encounter possible stability issues the feedback signal can be filtered a priori with inverse actuator dynamics [35]. Limitations of this method are modelling errors and changes of the auxiliary system by possible long-term effects which lead to a significant difference between model and actual auxiliary system.

It should be noted that the direct velocity feedback is able to mitigate several resonance modes of a system. This phenomenon is due to the total absence of a tuning to one resonance frequency.

3.1.4 Direct acceleration feedback

The direct main system acceleration feedback uses a feedback of the acceleration of the main system. If a disturbance acts on the main mass, an immediate acceleration occurs. Thus, it is natural to apply a damping force which is proportional to this acceleration to counteract the disturbance.

Obviously, by applying an acceleration-dependent feedback law, the main mass (inertia) can be virtually changed. However, this linear behaviour is only true if the dynamics of the systems are far from the current disturbance frequency.

By applying the control law \( f(t) = -gm_1 \ddot{x}_1(t) \), the closed-loop transfer function becomes:

\[
X(s) = \left( \begin{bmatrix}
  m_1 s^2 + (c_1 + c_2)s + k_1 + k_2 & -c_2s - k_2 \\
  -c_2s - k_2 & m_2 s^2 + c_2s + k_2
\end{bmatrix} \right) + \left( \begin{bmatrix}
  g \cdot m_1 s^2 & 0 \\
  0 & g \cdot m_1 s^2
\end{bmatrix} \right)^{-1} D(s)
\]

Equation (3.15) shows that a positive feedback gain \( g \) will virtually reduce the modal mass and thus the increase excitability. On the other hand, it increases the resonance frequency of the main system. A negative feedback gain \( g \) virtually reduces the resonance frequency of the main system, although it reduces the excitability due
to the increased modal mass. The system vibrates more strongly and faster, which is usually not desired. Appropriate low-pass filtering of the measurement data is required to prevent stability issues due to low phase margins in the high-frequency domain (cf. figure 3.2).

The stability of the closed-loop system is determined by the roots of the characteristic polynomial \( \det(A + K) \). Assuming a low inherent damping \((\xi_1 \approx 0)\), the stability analysis using the Routh-Hurwitz criterion (cf. appendix A) leads to the following stability constraints:

\[
\begin{align*}
&\frac{c_2(m_1 + m_2)}{m_1 m_2 (1-g)} > 0 \\
&\frac{c_2(k_1 m_2^2 + k_2 m_1^2 + k_2 m_2^2 + 2k_2 m_1 m_2 + gk_1 m_1 m_2)}{m_1^2 m_2^2 (1-g)^2} > 0 \\
&\frac{c_2^2 k_1^2 (m_2 + gm_1)}{m_1^2 m_2^2 (1-g)^3} > 0 \\
&\frac{c_2^2 k_2^2 k_1 (m_2 + gm_1)}{m_1^4 m_2^3 (1-g)^4} > 0
\end{align*}
\]

For any values of \( m_1, m_2, k_1, k_2, c_2 \in \mathbb{R}_{>0} \) and \( g \in \mathbb{R} \) these constraints can be simplified to (with \( \mu = m_2/m_1 \)):

\[
\begin{align*}
&\frac{1}{(1-g)} > 0 \\
&-\frac{k_1 \mu^2 + k_2 (1+\mu)^2}{k_1 \mu} < g \\
&\frac{\mu + g}{1-g} > 0 \\
&\mu + g > 0
\end{align*}
\]

and it is easily determined that the closed-loop system is stable for values of

\[-\mu < g < 1 \quad (3.16)\]

or \(-m_2 < g \cdot m_1 < m_1\). It should be noted that usually \( \mu \ll 1 \) and therefore the positive range of \( g \) is higher than the negative part.

Obviously, an arbitrary chosen auxiliary system with a moderate negative acceleration feedback only shifts the peak resonance frequency upwards as shown on figure 3.2. In order to attain an efficient damping of the structure, the acceleration feed-
back must therefore be carefully tuned to the fundamental main system resonance (cf. sect. 3.1.5).

3.1.5 Negative acceleration feedback according to Nishimura

This control strategy was described in detail by Nishimura et al. in 1992. In this paper [51] the authors analyse the influence of the negative feedback of the main structure's acceleration. By taking into account only the upper-left component of the closed-loop transfer matrix ($X_1/D_1$), an optimal configuration in the sense of the $H_\infty$ norm is established, based on the fixed point method proposed by Ormondroyd and Den Hartog [57]. Nishimura states in his thesis [50] that there exist two fixed-points in the amplification curve, which are independent of the auxiliary system's viscous damping. The amplification at these fixed points is the minimum possible $H_\infty$ norm of the system.

This approach is similar to Den Hartog's optimality criteria for the passive TMD. It is therefore unsurprising that similar results are obtained. A comparison of the corresponding main transfer functions of the TMD (3.17) and the acceleration feedback system (3.18) reveals that the only difference lies in the fourth-order term.
in $\beta$. Table 3.1 compares the tuning of a passive TMD and according to Nishimura. Obviously, the tuning is equal when taking into account the virtually-reduced modal mass.

Tuned Mass Damper

\[
\frac{X_1}{D_1} = \frac{\beta^2 + 2\xi_2\alpha\beta + \alpha^2}{k_1(\beta^4 + 2\alpha\xi_2(1 + \mu)\beta^3 + (\alpha^2\mu + \alpha^2 + 1)\beta^2 + 2\xi_2\alpha\beta + \alpha^2)}
\]  
(3.17)

Negative acceleration feedback

\[
\frac{X_1}{D_1} = \frac{\beta^2 + 2\xi_2\alpha\beta + \alpha^2}{k_1(\beta^4(1 - g) + 2\alpha\xi_2(1 + \mu)\beta^3 + (\alpha^2\mu + \alpha^2 + 1)\beta^2 + 2\xi_2\alpha\beta + \alpha^2)}
\]  
(3.18)

Due to the negative acceleration feedback, the main mass is virtually reduced. This results in a virtually higher mass ratio. Due to this reduction of the modal mass, the resonance frequency of the main system is increased by the factor $\sqrt{1 + g}$. In order to compensate this effect, the auxiliary system needs to be tuned to the new modal frequency, which is achieved by an increase of $\omega_2$ by the factor $\sqrt{1 + g}$ relative to the initial main system resonance frequency.

If the auxiliary resonance frequency and viscous damping are adjusted by passive spring and dash-pot elements, the required feedback force is therefore:

\[
f(t) = -gm_1\ddot{x}_1(t)
\]  
(3.19)
If, on the other hand, the adjustment of the stiffness and damping is made by the actuator, the overall feedback force yields (from (3.2)):

\[
f(t) = -gm_1\ddot{x}_1(t) - (c_2d - c_2)[\dot{x}_2(t) - \dot{x}_1(t)] - (k_2d - k_2)[x_2(t) - x_1(t)]
\] (3.20)

Obviously, the adjustment of the damping by the actuator reduces the overall power consumption because a part of the energy can be regenerated.

Nishimura et al. [51] also established a closed-form solution for the optimal feedback gain and the maximum DAF of the closed-loop system. Equation (3.21) shows that for \( g = \mu \) the DAF (\( \infty \) norm) tends to \( \infty \). For \( g = -1 \), equation 3.21 reaches the lower limit of \( 1/k_1 \) which is equal to the static deflection of the structure. For \( g = 0 \), the acceleration feedback is obviously zero and the \( \infty \) norm of the closed-loop system (3.21) is equal to the TMD. A comparison of (3.21) and (3.22) shows that only positive values of \( g \) lead to a reduction of the \( \infty \) norm.

Closed-loop acceleration feedback (Nishimura):

\[
|G_{CL,11}|_\infty = \frac{1}{k_1} \cdot \sqrt{\frac{2 + \mu - g}{\mu + g}}
\] (3.21)

Passive tuned mass damper:

\[
|G_{TMD,11}|_\infty = \frac{1}{k_1} \cdot \sqrt{\frac{2 + \mu}{\mu}}
\] (3.22)

The DAF of the auxiliary system \( \frac{x_2 - x_1}{D_1} \) at the main resonance frequency is equal for the TMD and the acceleration feedback according to Nishimura (cf. figure 3.3). However, the Nishimura method shows high amplifications in a broader frequency band [51].

\[
\sup (G_{CL,21}) = \frac{1}{k_1} \cdot \sqrt{\frac{\mu + 1}{\mu}}
\] (3.23)

Using equation 3.21, the required maximum feedback gain \( g \) for the desired peak-amplification between \( \frac{1}{k_1} \) and \( \frac{1}{k_1} \sqrt{\frac{2 + \mu}{\mu}} \) is deduced. The higher the gain \( g \) is chosen, the wider becomes the frequency range of the amplification of the auxiliary system and the stronger the high frequency band is amplified. Figure 3.3 shows that, by applying the acceleration feedback, a strong reduction of the peak amplitude of the main system is achieved. The main system phase graph reveals that a slow phase shift is obtained around the resonance frequency. This means that this feedback law is close to the optimal working point at resonance frequency. For higher disturbances
frequencies the Nishimura control law leads to a slight amplification of the main structure due to the strong auxiliary amplification.

The lower graph of figure 3.3 shows that the auxiliary system is strongly amplified over a large frequency band. An adequate filtering is therefore important to remove high-frequency noise. The phase shift of the filtering must be analysed in the significant band to prevent instability issues. Nishimura’s method has a high acceleration-dependent component and is therefore very sensitive to peaks or outliers in the acceleration measurement.

It is important to note that this control law works only efficient for systems with well-separated and time-invariant resonance frequencies. Due to the tuning, this vibration control law can only mitigate one resonance mode, similar to the TMD. Stability problems may arise on systems with multiple resonance modes. For additional resonance frequencies above the fundamental, the Nishimura method works similar to the direct acceleration feedback.

3.2 Novel control algorithms

The following section describes the advances and findings of this study. Based on novel concepts, different control laws have been deduced. Some promising
concepts have been tested in simulations and experiments. Other concepts have shown disadvantages either in the theoretical or practical implementation. These concepts are presented mainly for the sake of completeness.

3.2.1 Disturbance feedback control

If a precise knowledge of the disturbance force is accessible, an inertia-based ideal damping device is able to completely remove the dynamic behaviour of the main system. The control strategy is to simply counteract any occurring disturbance force. The general 2DOF system (cf. sect. 2.4) in frequency domain is given by (without the dependency on $s$ for simplicity reasons):

$$X_1 = G_{11}(D_1 - F) + G_{12}(D_2 + F)$$
$$X_2 = G_{21}(D_1 - F) + G_{22}(D_2 + F)$$

(3.24)

Applying the following actuator force $F = \frac{G_{11}D_1 + G_{12}D_2}{G_{11} - G_{12}}$, the closed loop system yields:

$$X_1 = 0 \cdot (D_1 + D_2)$$
$$X_2 = \frac{G_{11}G_{22} - G_{12}G_{21}}{G_{11} - G_{12}} \cdot (D_1 + D_2)$$

(3.25)

By applying this control law, the ideal main structure does not move at all in theory (remains at null-position), for all occurring external disturbances $D_1$ and $D_2$. However, the control force requires prior knowledge of the disturbance force and location, which is in the case of a pedestrian excitation, for example, not directly accessible in reality. Nevertheless, it is possible to estimate the disturbance forces by measuring the movement of the main and auxiliary system. This method is described in section 3.2.2.

It is clear that (3.25) cannot be implemented in reality. A constant disturbance load $D_1$ on the main system, for example, would require a constant negative force $F$ to keep $X_1$ at the position 0. This means that the auxiliary mass must be accelerated constantly in the opposite direction which will hurt the actuator saturations and stroke limitations. Instead of the desired "null-position", a certain movement of the main system can be allowed for constant or low-frequency disturbances. The desired dynamic behaviour in closed loop is named $G_d$. For a negligible auxiliary disturbance $D_2$, equation (3.24) leads to:
\[ X_1 = G_{11}D_1 - (G_{11} - G_{12})F = G_d D_1 \]  
\[ X_2 = G_{21}D_1 - (G_{21} - G_{22})F \]  

(3.26)

The actuator force \( F \) must therefore be:

\[ F = \frac{G_{11} - G_d}{G_{11} - G_{12}} D_1 \]  

(3.27)

In closed loop, this leads to:

\[ X_1 = G_d D_1 \]  
\[ X_2 = \left( G_{21} + \frac{(G_d - G_{11})(G_{21} - G_{22})}{G_{11} - G_{12}} \right) D_1 \]  

(3.28)

This shows that every dynamic behaviour \( G_d \) for the main system can be established in theory. However, in order to generate the required control force the auxiliary system will need important actuator forces and stroke limits, depending on the difference between the desired behaviour \( G_d \) and the open-loop behaviour \( G_{11} \). As mentioned above, this control law is applicable only in theory in the case of moving loads for obvious reasons. It is clear that this control law is also highly sensitive to modelling errors.

### 3.2.2 Estimated Disturbance feedback control

The real disturbance force and its location on the main system are usually unknown for the case of vertical floor vibrations with pedestrian excitation. However, the disturbance can be estimated using measurement data and the current and past system states and control force if a precise model of the system is available. It is easy to show that by inverting the closed loop transfer function \( X(s) = G(s)(D(s) + F(s)) \) an estimation of the disturbance is obtained. Obviously, the measured states and forces are delayed by at least one time step \( \tau \) in reality compared to the actual disturbance force. Let \( \hat{G}(s) \) be the model of the actual dynamic system \( G(s) \).

\[ \hat{D}(s) = \left( \hat{G}(s)^{-1}X_{meas}(s) + F_{meas}(s) \right) e^{-s\tau} \]  

(3.29)

Let \( G_d \) be the desired closed-loop behaviour of the main system. The dynamic behaviour of the auxiliary system is of minor interest. For the sake of simplicity the
disturbance of the auxiliary system $D_2$ is neglected. The closed-loop behaviour of the main system yields:

$$X_1(s) = G_{11}(s) \cdot (D_1(s) - F(s)) + G_{12}(s) \cdot F(s) = G_d(s)D_1(s)$$  \hspace{1cm} (3.30)

The actuator force must therefore be (from (3.30)):

$$F(s) = \frac{G_{11}(s) - G_d(s)}{G_{11}(s) - G_{12}(s)} \cdot D_1(s) \hspace{1cm} (3.31)$$

The required actuator force $F(s)$ to achieve the desired closed loop behaviour $G_d$ can be rewritten as a linear function of the states of the system by replacing the disturbance by its estimated value $\hat{D}_1(s)$ from (3.29).

$$F(s) = \frac{\hat{G}_{11}(s) - G_d(s)}{\hat{G}_{11}(s) - \hat{G}_{12}(s)} \cdot \left( X_{1,\text{meas}}(s) \cdot \frac{\hat{G}_{11}(s)}{\hat{G}_{11}(s)} + \left( 1 - \frac{\hat{G}_{12}(s)}{\hat{G}_{11}(s)} \right) F_{\text{meas}}(s) \right) e^{-st} \hspace{1cm} (3.32)$$

Assuming no measurement error, the measured values are: $X_{\text{meas}} = G_{11}D_1 + F(G_{12} - G_{11})$ and $F_{\text{meas}} = F$. Thus, the actuator force yields:

$$F(s) = - \frac{\hat{G}_{11}(G_d - G_{11})e^{-st}}{G_{11}(G_{11} - G_{12}) \left( \frac{e^{-st}(G_d - G_{11})(G_{11} - G_{11} - G_{12} + G_{12})}{G_{11}(G_{11} - G_{12})} - 1 \right)} \cdot D_1 \hspace{1cm} (3.33)$$
Obviously, (3.33) leads to a double integrator in the denominator of the \( X_2 \) transfer function. For this reason, an appropriate low-pass filter must be applied to prevent the auxiliary closed-loop system from static force offsets.

Inserting (3.33) into (3.24), the closed-loop transfer function yields:

\[
X_1 = \left( G_{11} + \frac{(G_{12} - G_{11})\hat{G}_{11}(G_d - G_{11})e^{-s\tau}}{G_{11}(G_{11} - G_{12}) (e^{-s\tau}(G_d - G_{11})(\hat{G}_{11} - G_{11}) + G_{12} - G_{12})} - 1 \right) D_1
\]

\[
X_2 = \left( G_{21} + \frac{(G_{22} - G_{21})\hat{G}_{11}(G_d - G_{11})e^{-s\tau}}{G_{11}(G_{11} - G_{12}) (e^{-s\tau}(G_d - G_{11})(\hat{G}_{11} - G_{11}) + G_{12} - G_{12})} - 1 \right) D_1
\]

(3.34)

Obviously, if the model \( \hat{G} \) is close (equal) to \( G \), (3.34) can be simplified to:

\[
X_1 = \left( G_{11} + (G_d - G_{11})e^{-s\tau} \right) D_1
\]

\[
X_2 = \left( \frac{(G_{11} - G_{12})G_{21} - (G_{11} - G_d)(G_{21} - G_{22})e^{-s\tau}}{G_{11} - G_{12}} \right) D_1
\]

(3.35)

The simplified closed-loop model (3.35) has been successfully applied in a simulation. The results are shown in chapter 3.3.1. However, the full closed-loop structure from equation 3.34 did not result in a stable closed-loop system due to an insufficient pole-zero cancellation. This phenomenon is due to numerical problems in the control loop. For this reason, the disturbance estimation feedback was not tested for robustness in chapter 3.3.3.

This control method depends strongly on a fast and accurate feedback of the measured position and force values, as well as on a precise model of the main and auxiliary structure. The influence of the feedback delay is shown in figure 3.5. Simulation results have shown that even for small differences between the actual and modelled system closed-loop stability is not achieved. Thus, the disturbance estimation feedback is not a suitable feedback law for the vibration mitigation on systems with varying properties.

3.2.3 Combined velocity feedback

The direct velocity feedback method has been described in section 3.1.3. Its benefits are an efficient damping with moderate actuator forces and its ability to mitigate
multiple resonance frequencies. It is stated that stability and optimality criteria have been analysed only in numerical models. This section provides both stability and optimality in the sense of $H_2$ and $H_\infty$ for simple structures with one degree of freedom in a closed for expression.

In real active damping devices, the auxiliary system has always a physical link to the primary system. Especially in the case of a vertical vibration mitigation device where the auxiliary mass must be hold in an equilibrium position. Due to this fact, the auxiliary system presents a resonance frequency which is usually strongly amplified in active closed-loop control. Let the primary system be a damped harmonic oscillator with one degree of freedom (cf. sect. 2.4) and the actuator force, a combination of the absolute velocity of the main and the relative velocity of the auxiliary system ($f(t) = K_1\dot{x}_1(t) - K_2(\dot{x}_2(t) - \dot{x}_1(t)))$. Thus, the actuator force is given in frequency domain by:

$$F(s) = \begin{bmatrix} K_1 + K_2 \\ -K_2 \end{bmatrix}^T \begin{bmatrix} X_1(s)s \\ X_2(s)s \end{bmatrix}$$

The actuator creates therefore a velocity-dependent force. It is natural to replace the controller variables $K_1$ and $K_2$ by an equivalent viscous damping of $K_1 = 2\xi_1 d m_1\omega_1$ and $K_2 = 2(\xi_2 d - \xi_2) m_2\omega_2$. The second part of the controller is reduced by the inherent viscous damping of the auxiliary system to reduce variables in the closed-

---

**Fig. 3.5:** Bode plot of a closed-loop example system $G_{CL,11}$ for different feedback delay times. The increasing delay time leads to a insufficient zero-pole cancellation at the main resonance frequency (1 rad/s in this example) and thus, to a stability issues.
loop transfer function. This leads in normalised notation to following actuator force:

$$ F(\beta) = 2k_1\beta \left[ \begin{array}{c} \xi_{1d} + (\xi_{2d} - \xi_2)\alpha\mu \\ -2(\xi_{2d} + \xi_2)\beta - \alpha^2\mu \\ -2(\xi_{2d} + \xi_2)\beta - \alpha^2\mu - \alpha\mu \end{array} \right]^T \begin{bmatrix} X_1(\beta) \\ X_2(\beta) \end{bmatrix} $$

(3.37)

The general closed-loop transfer function (3.1) yields for this case:

$$ X(\beta) = k_1 \left( \gamma + 2(\xi_1 + \xi_{1d} + \xi_{2d}\alpha\mu)\beta + \alpha^2\mu - 2\xi_{2d}\alpha\mu\beta - \alpha^2\mu \right)^{-1} D(s) $$

$$ \begin{bmatrix} \alpha^2\mu + \alpha^2 + 1 \\ -2(\xi_{2d} + \xi_2)\beta - \alpha^2\mu \end{bmatrix} $$

(3.38)

The Routh-Hurwitz criterion (cf. appendix A) is applied to find the stability boundaries of the closed-loop system. The characteristic denominator polynomial is given by the determinant of the matrix [A+K]. A set of inequalities is deduced from the Routh-Hurwitz criterion, which define the BIBO-stability boundaries. These inequalities can be simplified considerably by setting the inherent damping of the main system to zero. This simplification is strictly conservative if the inherent damping is low and does therefore not affect the resonance frequency significantly.

$$ \xi_{1d} + \xi_{2d}\alpha(1 + \mu) > 0 $$

$$ (\alpha^2\mu + \alpha^2 + 1)\xi_{1d} + \alpha(\alpha^2\mu^2 + 2\alpha^2\mu + \alpha^2 + \mu)\xi_{2d} > 0 $$

$$ \alpha\xi_{1d}^2 + (\alpha^2\mu^2 + \alpha^2 - 1)\xi_{1d}\xi_{2d} - \alpha\mu\xi_{2d}^2 > 0 $$

(3.39)

The first two inequalities are linear functions in $\xi_{1d}$ and $\xi_{2d}$, whereas the third one is a quadratic equation. (3.39) is solved for $\xi_{1d}$ to obtain upper and lower stability constraints.

$$ \xi_{1d} > -\xi_{2d}\alpha(1 + \mu) $$

(3.40)

$$ \xi_{1d} > -\xi_{2d}\alpha \left( \mu + \frac{\alpha^2(1 + \mu)}{(\alpha^2\mu + \alpha^2 + 1)} \right) $$

(3.41)

$$ \xi_{1d} > \xi_{2d} \left( \frac{1 - \alpha^2(1 + \mu)}{2\alpha} - \frac{\sqrt{\alpha^4(1 + \mu)^2 - 2\alpha^2(1 - \mu) + 1}}{2\alpha} \right) $$

(3.42)

$$ \xi_{1d} < \xi_{2d} \left( \frac{1 - \alpha^2(1 + \mu)}{2\alpha} + \frac{\sqrt{\alpha^4(1 + \mu)^2 - 2\alpha^2(1 - \mu) + 1}}{2\alpha} \right) $$

(3.43)

The first three conditions (3.40) to (3.42) are lower boundaries for $\xi_{1d}$. It is assumed that an increased damping of the main system and therefore a positive feedback value $\xi_{1d}$ is desired. Thus, the first three conditions are fulfilled for any
values $\xi_{1d} \geq 0$ and $\alpha, \mu, \xi_{2d} \in \mathbb{R}_{>0}$. Additionally, all three conditions have negative gradients for $\mu$ and $\alpha$ in $\mathbb{R}_{>0}$.

The fourth condition (3.43) is the upper stability criterion. Assuming that $\mu \ll 1$ and therefore $(1 + \mu) \approx (1 - \mu) \approx 1$, the square-root expression can be simplified to a simple binomial. For $\alpha < 1$ and $\mu = 0$, the following simplification is deduced from (3.43).

$$\xi_{1d} < \xi_{2d} \left( \frac{1}{\alpha} - \alpha \right)$$

(3.44)

Let $\eta$ be the ratio between the viscous feedback $\xi_{1d}$ and the total viscous damping of the auxiliary system $\xi_{2d}$. (3.44) is simplified to:

$$\eta_{stab} < \frac{1 - \alpha^2}{\alpha}$$

(3.45)

The upper stability plane (3.43) is shown in figure 3.7. Obviously, for low values of $\alpha$, higher feedback gains can be accepted and the dependency on $\mu$ is low. Figure 3.6 shows the stability condition plane for an example $\alpha = 0.5$. The gradient of the upper stability condition is positive in the direction of $\mu$. A conservative upper stability condition is therefore given for $\mu = 0$. 

---

**Fig. 3.6**: Stability region for $\alpha = 0.5$. Feedback gains within the grey area lead to a stable closed-loop system. Condition 1 to 4 denote the boundaries given by (3.40) to (3.43), the simplified stability condition is given in (3.45).
Equation (3.44) shows that two adjustments can be made at the auxiliary system to increase the upper stability border:

- $\alpha$: The tuning of the auxiliary system $\alpha$ can be reduced. However, there are physical limitations to this value. Decreasing the tuning ratio means reducing the stiffness of the auxiliary system's suspension. This suspension has mainly two functions: to hold the auxiliary mass in an equilibrium resting position and allow the auxiliary mass to move along its vertical path. A softer suspension has therefore a longer stroke in the initial position which limits the remaining stroke (cf. chapter 4.3).

- $\xi_{2d}$: The auxiliary viscous damping can be increased. This can be either achieved by adding a dash-pot between auxiliary mass and structure or by adding a viscous damping force to the controller output (actuator force).

Obviously, feedback gains close to the stability border lead to high peaks in the dynamic amplification function. A simple numeric example clarifies the trade-off between the feedback gains $\xi_{1d}$ and $\xi_{2d}$. The Bode plot of an example system is shown in figure 3.8 for different fraction of the stability feedback ratio $\eta_{\text{stab}}$. The closer the feedback gain is to the stability border, the lower is the dynamic
amplification at the main system resonance frequency. However, strong feedback gains lead to an increase of the dynamic amplification of the auxiliary system, which excites the main system on the other hand. Thus, a peak at the auxiliary system frequency appears. It should be noted that the phase delay at this auxiliary frequency is higher than the expected angle of \(-90^\circ = -\frac{\pi}{2}\) rad/s. This can be explained by the fact that this is not a natural resonance of the system but is induced by the external force of the auxiliary system. Obviously, the stroke of the auxiliary system is extremely high in this case.

Figure 3.8 suggests that a trade-off between stability and performance will lead to an optimal configuration.

**H\textsubscript{2}-optimal control** H\textsubscript{2}-optimal control aims to find a controller for which the 2 norm of a system is minimal. The well-known LQR controller is an application of the 2 norm optimisation. Let \(J\) be the squared 2 norm of a simple LTI system. \(J\) is also equal to the squared 2 norm of the system output to an impulse-form disturbance (\(|Y(j\omega)| = 1\)) in time and frequency domain (cf. sect. 2.6.3). This value is, for a
stable and damped system with \( \lim_{t \to \infty} y(t) = 0 \), equal to the energy of the system output \( y \). It is therefore also called output energy.

\[
J = \int_{-\infty}^{\infty} |y(t)|^2 dt = \int_{-\infty}^{\infty} |g * u(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(j\omega)|^2 d\omega = \|G(j\omega)\|^2_2
\]

Therefore, the 2 norm of the closed loop system can be calculated using the transfer function \( G_{CL,11} \). In certain cases a minimisation is possible in closed form. Let \( J \) be the 2 norm of the upper-left component of \( G_{CL} \), which is calculated in frequency domain as:

\[
J = \|G_{CL,11}(\beta)\|^2_2 = G_{CL,11}(\beta) \cdot \text{conj}(G_{CL,11}(\beta))
\]

where \( \text{conj} \) is the complex-conjugate expression. Simple closed-form solutions for this expression in frequency domain have been published [29, 20]. An extract is provided in the appendix B. To find optimal values of \( J \), the gradient according to \( \xi_{1d} \) and \( \xi_{2d} \) is set equal to zero. Equation (3.47) leads to a positive quadratic expression of \( J \). Therefore, there exists a global minimum. A simplification is made for small values of \( \mu \approx 0 \).

\[
\nabla J |_{\mu=0} = 0
\]

Partial derivatives are found for \( \partial J / \partial \xi_{1d} \) and \( \partial J / \partial \xi_{2d} \), which leads to a set of two equations. Under the assumption that \( \mu = 0 \) and \( \{\alpha \in \mathbb{R} \mid 0 < \alpha < 1\} \), the following closed-form solutions for the optimal feedback gains are found:

\[
\xi_{1d} = \frac{(p - 3)^2}{16} \sqrt{\frac{p + 3}{p - 1}}
\]

\[
\xi_{2d} = \frac{3 - p}{4} \sqrt{\frac{p + 3}{2(p + 1)}}
\]

\[
\eta = \frac{\xi_{1d}}{\xi_{2d}} = \frac{3 - p}{4} \sqrt{\frac{2(p + 1)}{p - 1}}
\]

with

\[
p = \sqrt{8\alpha^2 + 1} \quad p \in \mathbb{R} \mid 1 < p < 3
\]

It is obvious that these working points are optimal only for small values of \( \mu \). For higher values of \( \mu \) numerical values can be found. However, figure 3.10 reveals that the variation of \( J \) with respect to \( \mu \) is small for small values of \( \alpha \).
Fig. 3.9: Optimal $\xi_{1d}$ (top) and $\xi_{2d}$ (bottom) for a given pair of $\alpha$ and $\mu$ which minimise the 2 norm $J$ (output energy). $\xi_{1d}$ is quasi-independent of $\mu$, while $\xi_{2d}$ shows a dependency on $\mu$ for higher values of $\alpha$. 

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Figure 3.9 shows the optimal feedback gains for the different values of $\alpha$ and $\mu$. It should be noted that $\xi_{1d}$ is quasi-independent of $\mu$. This means that the optimal driving feedback gain is nearly constant for every mass ratio. The upper graph of figure 3.9 shows that for small values of $\alpha$, the optimal feedback gain $\xi_{1d}$ is strongly increased. For the theoretical case of $\alpha = 0$, the optimal driving feedback $\xi_{1d}$ tends towards $\infty$, whereas the retaining part $\xi_{2d}$ has a maximum value of $1/2$. Thus, the lower the tuning ratio $\alpha$ is, the higher is the damping efficiency. On the other hand, low auxiliary stiffness leads to strong amplifications of the auxiliary system.

**Fig. 3.10:** Minimal output energy $J$ for a given pair of $\alpha$ and $\mu$ with $\xi_{1d}$ and $\xi_{2d}$ according to figure 3.9. The encircled values denote the output energy of the passive TMD, according to Den Hartog and Ormondroyd [57]. These values are slightly above those of the CVF.

The evolution of the output energy $J$ is presented in figure 3.10 for different values of $\alpha$ and $\mu$. As long as $\alpha$ is small ($\alpha < 0.6$), the dependency of $J$ to $\mu$ can be neglected. The encircled values show the output energy of the passive TMD, tuned according to Den Hartog [33]. For the case of the TMD, the values of $\alpha$ and $\mu$ are linked by the optimal tuning ratio, whereas for the active feedback control, the values are independent and should be chosen according to physical limitations and design criteria. A strong decrease of the output energy $J$ is shown for low values of $\alpha$, which is achieved for a soft connection between main and auxiliary mass. However, in the case of vertical vibration mitigation the initial stroke due
to the dead weight of the auxiliary mass sets a lower limit to the softness of this connection.

![Optimal H₂ controller](image)

**Fig. 3.11:** Optimal feedback gains for combined velocity feedback which minimise the 2 norm for different values of $\alpha$ and $\mu$. Values for $\mu = 0$ are according to (3.49), values for $\mu \neq 0$ are derived numerically.

Figure 3.11 shows the optimal feedback gains $\xi_{1d}$ and $\xi_{2d}$. This is a combination of the two graphs on figure 3.9. This graph was used as a design basis for the verification in simulation and experiment (cf. sect. 3.3 and 4.3). The values on graph 3.11 for $\mu = 0$ were derived according to (3.49). Values with $\mu \neq 0$ were found using numerical methods. Obviously, for values of $\alpha > 0.6$ the influence of $\mu$ on $\xi_{2d}$ must be taken into account.

**H∞ optimal control**  The $\infty$ norm of a SISO system denotes the maximum peak of the dynamic amplification curve $DAF_{max}$ in frequency domain (cf. sect. 2.6). Figure 3.8 suggests that there exists an optimal adjustment of $\xi_{1d}$ and $\xi_{2d}$, which minimises both peaks to an equal height. The infinity norm of a system is found in most cases only using numerical methods. Ormondroyd and Den Hartog [57] solved the $\infty$ norm problem for the passive TMD with their discovery of the fixed-points in the dynamic amplification (cf. sect. 2.5.2). Figure 3.8 clearly shows that for the
case of the combined velocity feedback this phenomenon is not present. However, a first look reveals that for the presented example of $\alpha = 0.6$, the optimality is close to $\eta = 0.65 \eta_{\text{stab}}$. In this study, numerical values were deduced, which define an optimal working condition in the sense of the $\infty$ norm in closed-loop assuming that the auxiliary disturbance is small ($d_2(t) \approx 0$). The values were deduced by iterating over a set of well guessed controller parameters and retaining those with the lowest peak amplitude. Advanced numerical methods to find optimal $H_\infty$ controller were published for example by Stoorvogel [77]. However, the application of these methods usually requires higher computational costs and were therefore not applied within this study.

Figure 3.12 shows the optimal feedback gains for $H_\infty$ control for a given set of $\alpha$ and $\mu$. Obviously, the optimal values for $\xi_{1d}$ show a slight dependency on $\mu$, whereas the evolution of $\xi_{2d}$ is much stronger with $\mu$, especially for high values of $\alpha$.

The $H_\infty$ norm (or $DAF_{\max}$) values of the closed-loop system for a given set of $\mu$ and $\alpha$ are shown in figure 3.13. The graph shows that similar to the $H_2$ optimal controller, for low values of $\alpha$ a more efficient damping is achieved. For $\alpha \to 1$, the $DAF_{\max}$ is equal to the values of the passive TMD, because the driving feedback gain tends towards zero (and below). The optimal configurations for $H_\infty$-control are shown in figure 3.14. Both values, $\xi_{1d}$ and $\xi_{2d}$ show a significant dependency on $\mu$ for higher values of $\alpha$. It should be noted that for values above $\alpha > 0.6$, the driving feedback force $\xi_{1d}$ turns negative depending on the mass ratio. This still results in a stabilising controller when taking into account the extended stability criteria from figure 3.6. Figure 3.14 shows that the dependency of $\xi_{2d}$ on $\mu$ is increased for high values of $\alpha$ and the auxiliary damping ratio is approximately equal to the passive TMD according to Den Hartog [33].

**Stroke and actuator force saturation** The saturation of stroke and force should be avoided to guarantee stability and efficient damping of the closed loop system. A detailed analysis is therefore required for the design of a vibration control device. The actuator force $F(s)$ is deduced from the closed loop schema (cf. figure 3.1):

$$F(s) = K(s)X(s) = K(s)G_{CL}(s)D(s) \quad (3.50)$$
Fig. 3.12: Optimal $\xi_{1d}$ (top) and $\xi_{2d}$ (bottom) for a given pair of $\alpha$ and $\mu$ which minimize the $\infty$ norm or $DAF_{\text{max}}$. 

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Fig. 3.13: Maximum peak dynamic amplification for a given pair of $\alpha$ and $\mu$. For small values of $\alpha$, the dependency of $DAF_{max}$ on $\mu$ is low. The encircled values denote the $DAF_{max}$ of the passive TMD.

Fig. 3.14: Optimal feedback gains for combined velocity feedback which minimise the closed-loop $\infty$ norm. For higher values of $\mu$ and $\alpha$, $\zeta_{1d}$ turns negative and the viscous damping of the auxiliary system is strongly increased. The stars mark the optimal tuning according to Den Hartog [33].
For the case of the combined velocity feedback with a negligible auxiliary disturbance and inherent main system damping ($\xi_1, d_2(t) \approx 0$), the normalised actuator force is given by

$$F(\beta) = \frac{(2\xi_{1d} - 2\alpha\mu\xi_2 + 2\alpha\mu\xi_{2d})\beta^3 + 4\alpha\xi_2\xi_{1d}\beta^2 + 2\alpha^2\xi_{1d}\beta}{\beta^4 + (2\xi_{1d} + 2\alpha\xi_{2d} + 2\alpha\mu\xi_{2d})\beta^3 + (\alpha^2\mu + \alpha^2 + 1)\beta^2 + 2\alpha\xi_{2d}\beta + \alpha^2}D_1(\beta)$$

(3.51)

Obviously, the passive viscous damping parameter $\xi_2$ appears in the numerator of the actuator force expression. The power consumption depends therefore also on the passive viscous damping and thus, a low passive viscous damping of the auxiliary system reduces the actuator force and the overall consumed power.

The stroke of the auxiliary system is given by:

$$Y_2(s) = X_2(s) - X_1(s) = (G_{CL,21}(s) - G_{CL,11}(s))D_1 + (G_{CL,22}(s) - G_{CL,12}(s))D_2(s)$$

(3.52)

For the case of the combined velocity feedback with a negligible auxiliary disturbance and inherent main system damping ($\xi_1, d_2(t) \approx 0$), the normalised auxiliary stroke is given by

$$Y_2(\beta) = \frac{-\mu\beta^2 + 2\xi_{1d}\beta}{k_1\mu (\beta^4 + (2\xi_{1d} + 2\alpha\mu\xi_{2d} + 2\alpha\xi_{2d})\beta^3 + (\alpha^2\mu + \alpha^2 + \alpha)\beta^2 + 2\alpha\xi_{2d}\beta + \alpha^2)}D_1(\beta)$$

(3.53)

Obviously, the mass ratio parameter $\mu$ affects directly the stroke of the auxiliary mass. A higher mass ratio leads to lower amplitudes of the auxiliary system.

Using an estimation of the amplitudes and frequency content of the disturbance $d_1(t)$, the designer can estimate the maximum actuator force (3.51) and auxiliary stroke (3.53).

### 3.2.4 Selective feedback controller

The selective feedback controller model is based on the harmonic oscillator and the previously-described disturbance feedback method. It is obvious that the best controller provides a feedback force which is exactly the opposite of the disturbance. Figure 3.15 shows the Bode diagram of the one-degree-of-freedom oscillator. Obviously, the movement of the main mass is shift by a phase angle of $-\pi/2$ rad at resonance frequency. Thus, it is natural to use a velocity-dependent control in this frequency range. For disturbance frequencies above the fundamental resonance
the movement of the main mass is shift by an angle of \( \pi \) rad. Therefore, the disturbance is in phase with the acceleration of the main mass. In the lower part of the frequency spectrum, the movement of the main mass is approximately in phase with the disturbance. A negative acceleration or position-based feedback law is therefore best suited for these frequency ranges. The disturbance can be estimated by a summation of the three different frequency ranges with an appropriate filtering.

It is therefore natural to apply the different feedback laws only on the desired frequency range. This is in theory easily implemented by filtering the desired parts from the system output. Let \( H_1 \) be a low-pass filter, \( H_2 \) a band-pass filter and \( H_3 \) a high-pass filter.

\[
F(s) = H_1(s)X(s) + H_2(s)sX(s) + H_3(s)s^2X(s)
\]  
(3.54)

For simplicity reasons only first- and second-order Butterworth filters were applied. Let \( f_1 \) and \( f_2 \) be the lower and higher cut-off frequencies. Thus, (3.54) yields:

\[
F(s) = \left( \frac{2\pi f_1}{s + 2\pi f_1} \right)^2 g_1X_1(s) + \frac{s}{(s + 2\pi f_1)(s + 2\pi f_2)} g_2sX_1(s) + \frac{s}{s + 2\pi f_2} g_3s^2X_1(s)
\]  
(3.55)
This control law was implemented in simulations only. The relatively low performance and robustness led to the abandonment of this concept. These problems are inherent to this concept and due to the necessary close cut-off filtering.

### 3.2.5 Fixed-order robust control by convex optimisation

A promising control law was found in the fixed-order linear parametrised discrete controller according to Galdos [27]. The following part is based on the work of Galdos and has been presented in a paper at the SPIE conference 2016 [6]. A comparison of the simulation and experimental results is also shown in section 4.1. This rather complex controller synthesis method is well-adapted for SISO-control problems where only spectral models are available (measurement-based controller design). For this case, precise robustness and performance control is possible. Nevertheless, it was shown in this study that for the vertical vibration control problem this method is unfavourable due to the enormous calculation effort and the limitation to FIR-structured controllers. IIR-structured controllers have been tested during this study. However, this idea has been abandoned due to numerical issues during the optimisation procedure. The following chapter describes the applied method for the design of the controller used in the first experimental work (cf. chapter 4.1). The fixed-order discrete controller in Z-domain (Z-transform [43]) is build using a multiplication of the fixed order controller structure \( \phi(z) \) with parameter vector \( \rho \). This results in a causal finite-impulse transfer function in \( z^{-q} = e^{-jq\omega T} \).

\[
K(z) = \begin{bmatrix} \phi_1(z) \end{bmatrix}^T \cdot [\rho] = z^{-1}\rho_1 + z^{-2}\rho_2 + \ldots \quad (3.56)
\]

The actuator force \( f(t) \) is a combination of the two measurable states \( x_1(t) \) and \( x_2(t) \).

\[
F(z) = \begin{bmatrix} K_1(z) \\ K_2(z) \end{bmatrix}^T \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} = \rho^T \begin{bmatrix} \phi_1(z) \\ 0 \end{bmatrix} \begin{bmatrix} X_1(z) \\ X_2(z) \end{bmatrix} \quad (3.57)
\]

In order to find a suitable controller, the squared 2 norm of the difference between the upper-left closed-loop transfer function and a desired transfer function \( G_{CL,d} \) is minimised. Let \( J \) be the following objective function:

\[
J = \sum_k \|G_{CL,11}(z_k) - G_{CL,d}(z_k)\|_2^2 \quad \forall \ z_k = e^{j\omega_k T}, \omega_k \in [0, \omega_N] \quad (3.58)
\]
A first order Taylor series of the closed-loop transfer function $G_{CL,11}$ leads to a linear function in $\rho$. The objective function (3.58) is rewritten without the dependency on $z_k$ as:

$$J = \sum_k \|G_{CL,11}\|_\rho + \nabla G_{CL,11}\|_\rho (\rho - \bar{\rho}) - G_{CL,d}\|_2^2$$

$$= \sum_k \|\nabla G_{CL,11}\|_\rho + G_{CL,11}\|_\rho - \nabla G_{CL,11}\|_\rho - G_{CL,d}\|_2^2$$

$$= \sum_k \|Q_k\rho + P_k\|_2^2$$

$$= \sum_k [(Q_k\rho + P_k)^* (Q_k\rho + P_k)]$$

$$= \sum_k [\rho^* Q_k^* Q_k\rho + 2\Re [P_k^* Q_k\rho + P_k^* P_k]]$$

$$= \rho^* \sum_k [Q_k^* Q_k]\rho + 2\Re \left[ \sum_k [P_k^* Q_k^*] \rho + \sum_k [P_k^* P_k] \right]$$

Starting from an initial controller parameter vector $\rho = [0]$, the convex objective function $J$ is minimised iteratively under constraints to guarantee stability and robustness. The causality of the controller is determined by the structure vector $\phi(z)$. By virtue of its linear design, the controller is causal if all elements of $\phi(z)$ are causal.

**Stability** The closed-loop transfer function matrix of a generic MIMO system is given by:

$$\frac{Y(j\omega)}{U(j\omega)} = G_{CL}(j\omega) = G(I + K(j\omega)G(j\omega))^{-1} = G(j\omega)S(j\omega)$$

(3.60)

Let $L(j\omega) = K(j\omega)G(j\omega)$ be the loop transfer function. For an initially stable system $G(j\omega)$, the sensitivity function $S(j\omega) = (I + L(j\omega))$ determines the stability in closed loop. According to the generalised Nyquist criterion [23], closed loop stability is provided for a stable initial system if $\det(I + L(j\omega)) \neq 0$ and the image of $\det(I + L(j\omega))$ for $s = j\omega$ and $\omega \in [-\infty, \infty]$ does not encircle the origin of the complex plane. The iterative optimisation described above leads for initially stable systems to a stable closed loop system in a first step ($\rho = 0$). Thus, the image of $\det(I + L(j\omega))$ must not pass through the origin. This is controlled by the distance...
of the image for every frequency point to the origin of the complex plane. The distance of the image of the determinant of $I + L$ is given by:

$$\| \text{det} (I + L(j\omega)) \| = \| 1 + K_1(j\omega) (G_{11}(j\omega) - G_{12}(j\omega)) + K_2(j\omega) (G_{21}(j\omega) - G_{22}(j\omega)) \|$$

$$= \| 1 + \text{det}(L) \|$$

It should be noted that the determinant of $L$ is a linear function in $\rho$ for this special case. A graphical representation (cf. figure 3.16) shows that this expression is equal to the distance between the image of $\text{det} (L(j\omega))$ and the point (-1,0) in the complex plane. Therefore, closed-loop stability is provided if the image of $\text{det} L(j\omega)$ does not pass the critical point (-1,0). Let $w_1(j\omega)$ be the security margin which defines a circle with radius $\| w_1(j\omega) \|$ around the critical point (-1,0). Stability is guaranteed if the image of $\text{det}(L)$ remains on the initial side of the critical point and the following condition is fulfilled:

$$\| 1 + \text{det} (L(j\omega)) \| > \| w_1(j\omega) \|$$

The iterative approach allows working with linearised constraints. By approxim-
ing the left-hand side with a first-order Taylor series, (3.62) yields:

$$\Re e [(1 + L_{k-1})G\phi] \rho < -\|w_1(j\omega)\| - \Re e [(1 + L_{k-1})]$$  \hspace{1cm} (3.63)

A complete derivation of these linearised constraints is given in [5]. This leads for every iteration increment $k$ to a set of linear constraints in $\rho$.

**Robust Performance**  Robust performance, and therefore also stability, of a SISO closed loop system [36] is given if:

$$\| |S(j\omega)W_1(j\omega)| + |T(j\omega)W_2(j\omega)| | \|_{\infty} < 1$$ \hspace{1cm} (3.64)

where $S$ and $T$ are the sensitivity and complementary sensitivity function and $W_1$ and $W_2$ are the desired performance and robustness filters. If condition (3.64) is fulfilled, it is true for every frequency point $\omega_p \in \] - \infty, \infty[$. Thus, the following inequality must be true in the SISO case ($L$ is the loop transfer function matrix $L(j\omega) = K(j\omega)G(j\omega)$):

$$|W_1(j\omega_p)| + |L(j\omega_p)W_2(j\omega_p)| < |I + L(j\omega_p)| \hspace{1cm} \forall \hspace{0.2cm} \omega_p \in \] - \infty, \infty[$ \hspace{1cm} (3.65)

For the case of a MIMO system where $S$, $T$, $W_1$ and $W_2$ are matrices, the robust performance must be analysed using the maximum singular values $\sup \sigma$ of (3.64).

$$\sup \sigma (|S(j\omega_p)W_1(j\omega_p)| + |T(j\omega_p)W_2(j\omega_p)|) < \frac{\sup \sigma (W_1(j\omega_p)) + \sup \sigma (L(j\omega_p)W_2(j\omega_p))}{\inf \sigma (I + L(j\omega_p))} < 1$$ \hspace{1cm} (3.66)

Let $W_1 = w_1I$ and $W_2 = w_2I$ be two diagonal matrices with the performance and robustness filter parameters $w_1$ and $w_2$ on the diagonal axis. (3.66) is simplified to:

$$|w_1(j\omega)| + |w_2(j\omega)| \sup \sigma (L(j\omega)) < \inf \sigma (I + L(j\omega))$$ \hspace{1cm} (3.67)

Figure 3.17 shows the constraints in graphical representation. The minimum distance between $L(j\omega)$ and the critical point (-1,0) must be larger than the sum of the performance and robustness filters. The performance filter describes the distance to the critical point in the Nyquist diagram. Thus, it is equal to the modulus margin or inverse gain margin [43]. A performance filter of $w_1 = 0.5$ is approximately equal to a gain margin of 2, which is a common value for industrial applications. The robustness filter describes a multiplicative uncertainty to the measurement or due to noise. Obviously, the calculation of the singular values cannot be used as linear
constraints for the convex optimization. By multiplying both sides of equation (3.67), the calculation of the singular values can be avoided during the optimization. It is well-known that $\Pi_m [\sigma_m(A)] = \sigma_1 \cdot \sigma_2 = |\det(A)|$. A matrix with size 2x2 and full rank has two distinct singular values. Therefore, the robust stability condition leads to (without the dependency on $\omega$):

$$
\sup \sigma (I + L) [w_1 + w_2 \sup \sigma (L)] < |\det (I + L)| = |1 + \det(L)|
$$

(3.68)

For every iteration step $k$, it must be fulfilled:

$$
-1 - |\det(L_k)| \leq -|1 + \det(L_k)|
$$

$$
< - \sup \sigma (|I + L_{k-1}|) [w_1 + w_2 \sup \sigma (|L_{k-1}|)]
$$

(3.69)

Thus, by applying (3.61), the following set of linear constraints needs to be applied to the iterative optimization to guarantee the desired robust performance:

$$
-|\phi_1 (G_{11} - G_{12}) + \phi_2 (G_{21} - G_{22})| \rho_k
$$

$$
< 1 - \sup \sigma (|I + L_{k-1}|) [w_1 + w_2 \sup \sigma (|L_{k-1}|)]
$$

$$
|\phi_1 (G_{11} - G_{12}) + \phi_2 (G_{21} - G_{22})| \rho_k
$$

$$
< 1 - \sup \sigma (|I + L_{k-1}|) [w_1 + w_2 \sup \sigma (|L_{k-1}|)]
$$

(3.70)
3.2.6 Fixed-order control by genetic algorithm

The optimal controller gains of a fixed order controller can be found in an iterative convex optimisation as described in the previous chapter 3.2.5. A less complex optimisation technique is a genetic algorithm. The controller gain vector $\rho$ defines the "genome" of each individual. A set of different individuals generates the population. From the initial random population, the fittest individuals are selected. Together with a number of slightly different children of these individuals and some random mutants they form the next population, which is again tested for stability, robustness and performance. The complete algorithm is shown in figure 3.18. This iterative process is run until no evolution of the parental genome and fitness is observed over a given period. For further information, interested readers are referred to the standard literature about genetic algorithms and genetic programming.

Obviously, the controller $K_1$ and $K_2$ are formed as polynomial fractions in the Laplace domain. The structure of the polynomials is defined in the vector $\phi(s)$, whereas the vector $\rho = [\rho_1^T, \rho_2^T, \rho_3^T, \rho_4^T]^T$ contained the multiplication factors. Thus, a vector with the numerator and denominator coefficients of both controllers served as a genome. It is clear that the structure vector $\phi$ of the denominator has a higher order than the numerator vector to guarantee causality. During the iteration process,
the first component of the denominator vector must be fixed ≠ 0 to guarantee the causality.

\[ K_1 = \frac{\sum \phi_1(s)^T \rho_1}{\sum \phi_2(s)^T \rho_2} \]
\[ K_2 = \frac{\sum \phi_3(s)^T \rho_3}{\sum \phi_4(s)^T \rho_4} \]

(3.71)

The minimum \(\infty\) norm or 2 norm (cf. sect. 2.6) were used as a fitness criterion for performance. The genomes were sorted according their fitness and only the best ones were retained. Genomes which did not fulfil stability and robustness criteria were removed from the selection. Figure 3.19 shows an example evolution of the controller values (genome) and the according fitness performance during the iteration. Obviously, the progress of the fitness function is a purely heuristic process. The number of iterations needed or the final fitness value cannot be predicted.

The genetic algorithm produces stable and robust controllers with high performance rates. Nevertheless, a global minimum of the performance function cannot be guaranteed. Better performance may be reached after an infinite number of iterations. Another drawback is the necessity of a precise model of the main and auxiliary structure.
Fig. 3.19: First 100 generations and the corresponding controller parameter vector during the iteration (top). Each line presents an element of the genome $\rho$; Evolution of the maximum fitness (bottom), calculation time approximately 2 minutes for 100 generations.
3.3 Controller comparison

In the following section, the previously-described active vibration control methods are compared for performance, stroke and robustness to a change of the initial system. The following methods are taken into account (cf. section 3.1 and 3.2):

- ATMD - Active tuned mass damper, tuned according to Den Hartog [57]
- LQR - Optimal control using state feedback
- DVF - Direct velocity feedback
- DAF - Direct acceleration feedback
- Nishi - Acceleration feedback according to Nishimura
- EDF - Estimated disturbance feedback
- CVF - Combined velocity feedback
- SFC - Selective feedback control
- FCCO - Fixed-order robust control by convex optimisation
- FCGP - Fixed-order robust control by genetic programming

In order to create a comparable condition for all active control methods, the performance, stroke and robustness are tested for harmonic and transient (impulse-form) disturbances. The evaluation is made on basis of the $\infty$ norm, the 2 norm or the amplitude. Table 3.3 lists the applied parameters for the simulation results. All methods are tested on a two-degree-of-freedom system according to (2.26) with the system properties listed in table 3.2 (deviant parameters are listed in table 3.3). The applied controller transfer functions $K(s)$ are listed in the appendix E. A smaller comparison of control laws has been published in [7].
### Variable Value Unit Description

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_1$</td>
<td>1</td>
<td>[kg]</td>
<td>Main system modal mass</td>
</tr>
<tr>
<td>$k_1$</td>
<td>1</td>
<td>[N/m]</td>
<td>Main system modal stiffness</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>1</td>
<td>[rad/s]</td>
<td>Main system modal pulsation</td>
</tr>
<tr>
<td>$\xi_1$</td>
<td>0.5%</td>
<td>[-]</td>
<td>Main system modal viscous damping ratio</td>
</tr>
<tr>
<td>$\mu$</td>
<td>2%</td>
<td>[-]</td>
<td>Mass ratio</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>0.6</td>
<td>[-]</td>
<td>Auxiliary tuning ratio</td>
</tr>
<tr>
<td>$\xi_2$</td>
<td>5%</td>
<td>[-]</td>
<td>Auxiliary system viscous damping ratio</td>
</tr>
</tbody>
</table>

**Tab. 3.2:** System properties used for simulation purpose

#### 3.3.1 Damping performance and auxiliary stroke

The damping performance of an active vibration control method is quantified in this study using the $\infty$ norm and the 2 norm of the closed loop system. The $\infty$ norm or maximum dynamic amplification factor $DAF_{max}$ provides information about the dynamic behaviour of the system to harmonic disturbances, whereas the 2 norm is a measure of the energy content of the main system after a transient impulse-form disturbance (cf. section 2.6). For every active vibration control method, the closed-loop Bode diagram of the main and auxiliary system due to a disturbance $d_1(t)$ is provided in figures 3.20 to 3.22. Actual performance of MIMO systems should be evaluated based on the maximum singular values of the dynamic amplification due to a possible overlaying of different disturbance forces. However, given that the auxiliary disturbance $d_2(t)$ is usually negligible for vertical vibration control, an evaluation of the upper-left component of $G_{CL}$ is sufficient for this purpose.

The auxiliary stroke is approximately proportional to the actuator force for a purely harmonic disturbance. High auxiliary strokes require usually a larger height of the active damping device. It is therefore favourable to have lower strokes. The stroke is measured by the difference between the absolute position of the auxiliary mass and the absolute position of the main mass. The following definitions are applied:

$$y_1(t) = x_1(t)$$
$$y_2(t) = x_2(t) - x_1(t)$$

For the different active vibration control methods, the stroke is shown in an amplitude Bode diagram 3.20 to 3.22 for harmonic disturbances. The response to an impulse of energy 1 is given in time domain on figure 3.23 for all systems. The peak
auxiliary absolute position are shown as the 2 norm and $\infty$ norm of $G_{2,1}$ in table 3.4.

Table 3.4 provides the extracted performance information from figures 3.20 to 3.22. For each active vibration control method, the $\infty$ norm and 2 norm are listed for the main and auxiliary system. The maximum performance, which is equal to the minimum $\infty$ norm or 2 norm values of the main system, is attained using the fixed-order controller by genetic programming. This result can be explained by the fact that for this optimisation, the $\infty$ norm of the main system is the only selection criterion (fitness function). Robustness constraints are not applied in the genetic algorithm. However, this high performance is paid by an extensive stroke of the auxiliary mass. The lowest stroke of the auxiliary system is obtained for the ATMD, which results in a poor damping performance of the main system. The Nishimura acceleration feedback presents the highest equivalent damping ratio. This is due to the fact that the damping ratio was approximated form the logarithmic decrement of the first two maxima of the impulse response and that the Nishimura method has a strong damping in the first cycle. However, the main system is re-excited by the auxiliary system after the second cycle. Figure 3.23 shows that cycles two to four have about the same amplitude.

Figure 3.20 shows the comparison of the undamped (initial) system to the first three active controlled closed-loop systems. The ATMD shows the well-known two-lobes characteristic in the main amplitude for transient disturbances (passive TMD). The state-space LQR controller has a better damping performance in the sense of the $\infty$ norm for the main system. It shows a slight mutual amplification of the main mass at the auxiliary resonance frequency. This amplification is even higher for the direct velocity feedback, which presents a good damping performance at the main resonance frequency. The phase shift is for these four systems approximately equal.

Figure 3.21 compares the Bode plots of the direct acceleration feedback, combined velocity feedback, acceleration feedback according to Nishimura and the estimated disturbance feedback. The direct acceleration feedback leads to two new peaks, one just below the auxiliary resonance and a higher one at about 1.4 Hz. The damping performance is lower than for the most of the analysed methods. The combined velocity feedback shows the desired equal lobes between the main and auxiliary resonance frequency and a continuous, nearly linear phase shift. Acceleration feedback according to Nishimura also creates two-lobes amplitude behaviour where both lobes
have an approximately equal height. The phase shift is less smooth compared to the combined velocity feedback. A feedback of the estimated disturbance shows an improper pole-zero-cancellation at the main resonance frequency. This phenomenon is due to the time delay of the measurement-based estimation (cf. figure 3.5) and leads to stability issues when exceeding a certain limit. Another issue of the estimated disturbance feedback control is that the auxiliary stroke has a static offset which must be removed by a high-pass filter. The amplitude characteristic of the stroke $Y_2$ shows that for low frequencies the dynamic amplification does not tend towards zero.

Figure 3.22 presents the Bode diagram of the selective feedback control (SFC), the fixed-order control by convex optimisation (FCCO) and the genetic algorithm (FCGP). Obviously, SFC leads to a distinct maximum which is higher than the maxima of the passive TMD. The FCCO leads to a result which is approximately equal to the CVF. The genetic algorithm (FCGP) leads to an extremely small $\infty$ norm of the closed-loop system. However, this advantage in the main system damping leads to a strong amplification of the auxiliary system.

The impulse response of the closed-loop systems is shown in figure 3.23. For each active vibration control method, the response to a positive unity impulse $d_1$ on the main system is plotted. It should be noted that the units in these graphs are set that an impulse with energy 1 results in an amplitude of 1. The upper graph shows the absolute position $y_1 = x_1$, whereas the lower one shows the relative stroke $y_2 = x_2 - x_1$ respectively. Obviously, all systems show a stable behaviour with different damping characteristics. The best method (FCGP) is able to silence the main system within five seconds but requires a stroke of 40. Other methods, like the CVF or Nishi remove the movement within 10 to 15 s from the main system, requiring less than half of the auxiliary stroke compared to the FCGP.

### 3.3.2 Actuator force

High actuator force demands can lead to saturation effects of the actuator and thus, to stability issues in closed-loop. Figure 3.24 shows that the DAF and FCGP require the highest actuator forces. The other methods require actuator forces with a maximum amplitude of approximately 0.2 N for an impulse-form disturbance. Obviously, the actuator force amplitude decays slower for low damped systems. EDF has an initial actuator force which tends towards infinity. The reason for this behaviour is the delay in the loop which strongly amplifies the difference between...
**Fig. 3.20:** Bode plots with amplitude and phase of the main (top) and auxiliary system (bottom) for the undamped system (initial) and systems with additional damping using an active tuned mass damper (ATMD), full state-space linear-quadratic control (LQR) and direct velocity feedback (DVF).
Fig. 3.21: Bode plots with amplitude and phase of the main (top) and auxiliary system (bottom) with external damping using direct acceleration feedback (DAF), combined velocity feedback (CVF), acceleration feedback according to Nishimura (Nishi) and estimated disturbance feedback (EDF).
Fig. 3.22: Bode plots with amplitude and phase of the main (top) and auxiliary system (bottom) with external damping using selective feedback control (SFC), fixed-order control by convex optimisation (FCCO) and fixed-order control by genetic programming (FCGP).

3.3 Controller comparison
Fig. 3.23: Impulse response of the analysed controllers. For each controller, the $X_1$ and $X_2$ component after an impulse disturbance $D_1$ is shown. The abbreviations in the legend are explained in section 3.3.
estimated and actual disturbance. Additionally, EDF requires a positive force over the whole 50 s in the plot. This leads to a high auxiliary stroke, shown in figure 3.23.

### 3.3.3 Robustness

Robustness to changes of the main system (cf. chapter 1.2.3) is a key criterion of an active vibration control device. Possible changes of the modal mass and resonance frequency must not lead to instabilities or strong reduction of the damping performance under any circumstances. Chapter 1.2.3 defines possible changes of the modal mass for different types of floor structures. For the robustness criterion a change of the modal mass between $-20\%$ to $+100\%$ is assumed. This variation of the modal mass affects the damping as well as the resonance frequencies. Figure 3.25 shows the effect of a change of the modal mass $m_1$ in the range of $0.5m_1$ to $2.0m_1$ to the $\infty$ norm ($DAF_{\text{max}}$) and 2 norm (output energy).

Figure 3.25 shows that the fixed order controller by genetic programming (FCGP) presents by far the best performance and is stable for an increase of the main modal mass. However, slight reductions of the main modal mass lead to a loss of stability. The stability range of the SFC is even smaller. The other analysed methods lead to stable closed-loop behaviour even when doubling the main modal mass.

The $\infty$ norm of the combined velocity feedback (CVF), fixed-order control by convex optimization (FCCO) and acceleration feedback according to Nishimura have an approximately equal performance and robustness characteristics. For the nominal modal mass, these methods present good performance. With a rising modal main mass, the performance is strongly reduced.

The LQR method presents a quasi-linear characteristic. The increasing main mass reduces the damping performance. The LQR method presents overall a slightly higher $\infty$ and 2 norm and thus, a lower damping performance.

The ATMD is analysed in two different situations. The first setup shows the performance for a changing mass without adaptation which is equal to a standard passive TMD. This case is present if the disturbance is transient and faster than the adaptation algorithm. The second case treats an ATMD with adaptation and thus, the case where a long-duration disturbance excites the structure and the ATMD is correctly tuned. It should be noted that the 2 norm of the ATMD with adaptation is
Fig. 3.24: Actuator force due to an impulse disturbance $d_1$ for different active vibration control methods.
Fig. 3.25: Closed-loop robustness for a changing main mass between 50% and 200% of the modal mass \( m_1 \). Evolution of the \( \infty \) norm (top) and 2-norm (bottom) in relative and absolute values. The relative values are normed at the nominal mass \( m_1 = 1 \). Interrupted lines denote the end of stability for a method.
nearly constant and has a quasi-linear $\infty$ norm characteristic. The ATMD without adaptation presents the well-known robustness characteristics of the passive TMD.

3.3.4 Performance and robustness for multi-resonances

It is obvious that floors present usually multiple resonance frequencies. In the case of a beam-type structure, the resonance frequencies are at $\omega_{0,k} = k^2 \omega_0$ for $k \in [1, N]$ (cf. appendix D). Assuming that at the specific location of the active vibration control device the first three modes apply equally, the system presents three resonance frequencies as shown in figure 3.26.

The analysed main system consists therefore of a summation of three one-degree-of-freedom oscillators. The dynamic properties of the initial systems are given in table 3.5. The same controller architecture and values are applied like for the single-resonance system in section 3.3.1 and 3.3.3.

The control methods DAF, EDF, SFC, FCCO and FCGP are not closed-loop stable for the analysed multi-resonance system and are therefore not included in the performance analysis. The performance measures using the 2 norm and $\infty$ norm are given in table 3.6. For each initial peak, the corresponding maximum is presented also in this table. It should be noted that CVF and Nishimura present the best performance for the multi-resonance system. Additionally, the CVF has the lowest peak amplitude for the higher order modes. However, as already shown in chapter 3.3.1, the performance is directly connected to a strong movement of the auxiliary system.

Figure 3.27 shows the impulse response of the closed-loop stable systems. It is obvious that the CVF presents the strongest damping with a settling time of approximately 13 seconds. Nishimura’s method is with approximately 20 seconds in a good range. The other methods are unable to silence the closed-loop system within the plotted range of 50 seconds. The auxiliary stroke ($y_2$) of CVF and Nishimura have approximately equal peak amplitudes of $\pm 18m$.

3.4 Conclusion

In this section, different active vibration mitigation control strategies have been presented and compared. Existing solution from chapter 3.1 and novel ideas
Fig. 3.26: Closed-loop Bode plots of a multi-resonance system for different control methods. Not-listed control methods did not result in stable closed-loop systems.
Fig. 3.27: Impulse response of the closed-loop stable vibration mitigation methods for a multi-resonance system.
from chapter 3.2 have been tested in a simulation environment on MATLAB. The comparison is made based on an example vibrating system. In order to analyse the robustness in closed-loop mode, a varying modal mass is used in section 3.3.3. Real vibrating systems usually have several resonance frequencies. Thus, the performance and robustness of the described controllers was tested in a multi-resonance simulation.

The direct comparison of the damping performance and auxiliary stroke in section 3.3 showed that some methods present a high damping performance but on the other hand, a low robustness for a changing main system like for example the EDF, FCCO, FCGP. Other active damping methods like the DAF and SFC have low robustness and only mediocre performance. These methods are therefore not well suited for the application in inertia-based active vibration mitigation of floors. These five methods have also shown instability issues when applying to a multi-resonance model.

The remaining methods ATMD, LQR, DVF, Nishimura, CVF did not present any instabilities in the analysed cases. Among these methods, the CVF and Nishimura present the best performance results in the single resonance model, as well as for the multi-resonance simulation.
<table>
<thead>
<tr>
<th>Method</th>
<th>Parameter</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
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</thead>
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<td>ATMD</td>
<td>$\alpha$</td>
<td>variable</td>
<td>-</td>
<td>According to Den Hartog</td>
</tr>
<tr>
<td></td>
<td>$\xi_2$</td>
<td>variable</td>
<td>-</td>
<td>According to Den Harog</td>
</tr>
<tr>
<td></td>
<td>$\tau$</td>
<td>60 s</td>
<td>s</td>
<td>Settling time estimation</td>
</tr>
<tr>
<td></td>
<td>$Q_1$</td>
<td>1000</td>
<td>-</td>
<td>Weighting factor for X1</td>
</tr>
<tr>
<td></td>
<td>$Q_2$</td>
<td>1</td>
<td>-</td>
<td>Weighting factor for X2</td>
</tr>
<tr>
<td></td>
<td>$R$</td>
<td>0.01</td>
<td>-</td>
<td>Weighting factor for U</td>
</tr>
<tr>
<td>DVF</td>
<td>$c_d$</td>
<td>0.05 kg/s</td>
<td></td>
<td>Additional viscous damping ratio</td>
</tr>
<tr>
<td>DAF</td>
<td>$g$</td>
<td>-0.5 kg</td>
<td></td>
<td>Acceleration feedback gain</td>
</tr>
<tr>
<td>Nishi</td>
<td>$g$</td>
<td>-0.192 kg</td>
<td></td>
<td>Acceleration feedback gain</td>
</tr>
<tr>
<td></td>
<td>$\alpha$</td>
<td>0.88</td>
<td>-</td>
<td>Auxiliary tuning ratio</td>
</tr>
<tr>
<td></td>
<td>$\xi_2$</td>
<td>29.9%</td>
<td>%</td>
<td>Auxiliary</td>
</tr>
<tr>
<td>EDF</td>
<td>$G_d$</td>
<td>$\frac{1}{s^2+0.4s+1}$</td>
<td>-</td>
<td>Desired closed-loop behaviour</td>
</tr>
<tr>
<td></td>
<td>$\tau$</td>
<td>1 ms</td>
<td></td>
<td>Feedback loop delay</td>
</tr>
<tr>
<td></td>
<td>$f_{low}$</td>
<td>0.008 Hz</td>
<td></td>
<td>Feedback high-pass cut-off</td>
</tr>
<tr>
<td>CVF</td>
<td>$\xi_{1d}$</td>
<td>0.1130</td>
<td>-</td>
<td>Main feedback damping ratio</td>
</tr>
<tr>
<td></td>
<td>$\xi_{2d}$</td>
<td>0.2356</td>
<td>-</td>
<td>Auxiliary feedback damping ratio</td>
</tr>
<tr>
<td>SFC</td>
<td>$f_{low}$</td>
<td>0.3 Hz</td>
<td></td>
<td>Lower cut-off frequency</td>
</tr>
<tr>
<td></td>
<td>$f_{up}$</td>
<td>1.7 Hz</td>
<td></td>
<td>Upper cut-off frequency</td>
</tr>
<tr>
<td></td>
<td>$k_d$</td>
<td>-0.03 N/m</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$c_d$</td>
<td>0.4 kg/s</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$m_d$</td>
<td>-0.99 kg</td>
<td></td>
<td></td>
</tr>
<tr>
<td>FCCO</td>
<td>$w_1$</td>
<td>0.5</td>
<td></td>
<td>Performance filter (gain margin)</td>
</tr>
<tr>
<td></td>
<td>$w_2$</td>
<td>0.1</td>
<td></td>
<td>Robustness filter</td>
</tr>
<tr>
<td></td>
<td></td>
<td>100</td>
<td></td>
<td>Population size</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td></td>
<td>Parent survivors</td>
</tr>
<tr>
<td></td>
<td></td>
<td>8</td>
<td></td>
<td>Children per parent</td>
</tr>
<tr>
<td></td>
<td></td>
<td>10</td>
<td></td>
<td>Random mutants per generation</td>
</tr>
</tbody>
</table>

**Tab. 3.3:** Controller parameters used for simulation and comparison
<table>
<thead>
<tr>
<th>Method</th>
<th>$|D_1|_{\infty}$</th>
<th>$|D_1|_2$</th>
<th>Damping ratio</th>
<th>$|D_1|_{\infty}$</th>
<th>$|D_1|_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>100.0</td>
<td>50.0</td>
<td>0.5%</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ATMD</td>
<td>9.3</td>
<td>6.5</td>
<td>2.5%</td>
<td>48.9</td>
<td>132.0</td>
</tr>
<tr>
<td>LQR</td>
<td>6.5</td>
<td>3.5</td>
<td>7.8%</td>
<td>50.2</td>
<td>336.9</td>
</tr>
<tr>
<td>DVF</td>
<td>11.1</td>
<td>5.6</td>
<td>5.2%</td>
<td>126.2</td>
<td>339.2</td>
</tr>
<tr>
<td>DAF</td>
<td>12.1</td>
<td>7.7</td>
<td>3.8%</td>
<td>369.6</td>
<td>9 161.8</td>
</tr>
<tr>
<td>Nishi</td>
<td>3.4</td>
<td>2.2</td>
<td>66.1%</td>
<td>67.3</td>
<td>756.4</td>
</tr>
<tr>
<td>EDF</td>
<td>5.0</td>
<td>1.4</td>
<td>19.6%</td>
<td>69 290.3</td>
<td>24 671.9</td>
</tr>
<tr>
<td>CVF</td>
<td>3.3</td>
<td>2.2</td>
<td>13.8%</td>
<td>128.0</td>
<td>1 359.7</td>
</tr>
<tr>
<td>SFC</td>
<td>13.3</td>
<td>8.0</td>
<td>0.4%</td>
<td>295.3</td>
<td>3 132.3</td>
</tr>
<tr>
<td>FCCO</td>
<td>3.7</td>
<td>2.3</td>
<td>14.0%</td>
<td>114.5</td>
<td>1 048.4</td>
</tr>
<tr>
<td>FCGP</td>
<td>1.3</td>
<td>0.8</td>
<td>16.3%</td>
<td>1 308.1</td>
<td>15 460.1</td>
</tr>
</tbody>
</table>

Tab. 3.4: Performance measures of the applied active vibration control methods. The damping ratio is approximated from the logarithmic decrement between first and second amplitude maximum.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_{1,1}$</td>
<td>1 kg</td>
<td></td>
<td>Main system first modal mass</td>
</tr>
<tr>
<td>$\xi_{1,1}$</td>
<td>0.5</td>
<td>%</td>
<td>Main system first modal damping</td>
</tr>
<tr>
<td>$\omega_{1,1}$</td>
<td>1</td>
<td>rad/s</td>
<td>Main system first resonance frequency</td>
</tr>
<tr>
<td>$m_{1,2}$</td>
<td>1 kg</td>
<td></td>
<td>Main system second modal mass</td>
</tr>
<tr>
<td>$\xi_{1,2}$</td>
<td>0.1</td>
<td>%</td>
<td>Main system second modal damping</td>
</tr>
<tr>
<td>$\omega_{1,2}$</td>
<td>4</td>
<td>rad/s</td>
<td>Main system second resonance frequency</td>
</tr>
<tr>
<td>$m_{1,3}$</td>
<td>1 kg</td>
<td></td>
<td>Main system third modal mass</td>
</tr>
<tr>
<td>$\xi_{1,3}$</td>
<td>0.1</td>
<td>%</td>
<td>Main system third modal damping</td>
</tr>
<tr>
<td>$\omega_{1,3}$</td>
<td>9</td>
<td>rad/s</td>
<td>Main system third resonance frequency</td>
</tr>
</tbody>
</table>

Tab. 3.5: System properties for the mult-resonance system.
<table>
<thead>
<tr>
<th>Method</th>
<th>∞ norm</th>
<th>2 norm</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>100.0</td>
<td>54.3</td>
<td>100.0</td>
<td>31.3</td>
<td>6.2</td>
</tr>
<tr>
<td>ATMD</td>
<td>21.3</td>
<td>9.4</td>
<td>9.4</td>
<td>21.3</td>
<td>5.2</td>
</tr>
<tr>
<td>LQR</td>
<td>6.5</td>
<td>3.9</td>
<td>6.5</td>
<td>2.2</td>
<td>0.9</td>
</tr>
<tr>
<td>DVF</td>
<td>11.1</td>
<td>6.2</td>
<td>11.1</td>
<td>4.1</td>
<td>1.6</td>
</tr>
<tr>
<td>CVF</td>
<td>3.6</td>
<td>2.4</td>
<td>3.6</td>
<td>1.0</td>
<td>0.4</td>
</tr>
<tr>
<td>Nishi</td>
<td>3.3</td>
<td>2.8</td>
<td>3.3</td>
<td>2.7</td>
<td>1.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>∞ norm</th>
<th>2 norm</th>
<th>$P_1$</th>
<th>$P_2$</th>
<th>$P_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ATMD</td>
<td>49.1</td>
<td>135.4</td>
<td>49.1</td>
<td>22.7</td>
<td>5.3</td>
</tr>
<tr>
<td>LQR</td>
<td>52.1</td>
<td>357.8</td>
<td>52.1</td>
<td>3.6</td>
<td>1.1</td>
</tr>
<tr>
<td>DVF</td>
<td>138.2</td>
<td>376.5</td>
<td>138.3</td>
<td>5.0</td>
<td>1.7</td>
</tr>
<tr>
<td>CVF</td>
<td>139.4</td>
<td>1500.9</td>
<td>139.4</td>
<td>3.1</td>
<td>0.7</td>
</tr>
<tr>
<td>Nishi</td>
<td>67.2</td>
<td>808.8</td>
<td>67.2</td>
<td>29.1</td>
<td>12.3</td>
</tr>
</tbody>
</table>

**Tab. 3.6**: Performance measures for a multi-resonance system in closed-loop. $P_1$ to $P_3$ denote the peak amplification for the peaks 1 to 3 respectively.
Experimental Validation

Up to this point, only theoretical aspects of the active vibration mitigation of floors have been considered. The simulation results from chapter 3.3 showed that significant improvements of the dynamic behaviour of vibrating structures can be achieved. This chapter presents the experimental verifications of the theoretical results.

4.1 Active TMD - improving the TMD

The first experimental study aimed to improve the behaviour of an existing TMD and establish a proof of concept for the measurement equipment. For this purpose, an existing test rig at the laboratory of the ETH Zürich was used.

4.1.1 Implementation

The test rig for the proof of concept comprised an existing simply-supported concrete beam. It had a span of 17.4 metres and a total length of 19.0 metres. The details of the beam geometry are given in figure 4.1 and 4.2. The beam was supported on one side by pin-type bearing with one degree of freedom, whereas the opposite bearing is a roller-type support which allows two degrees of freedom (cf. figure 4.3). This results in a static well-defined simple support. The total dead weight of the beam is approximately 10.8 to. The modal mass of the fundamental mode was calculated by Bachmann and Dazio at 5300 kg [22], the first resonance frequency was measured at 1.97 Hz. The concrete beam is used annually for the laboratory work of the structural dynamics course at the ETH Zürich. It shows the students the efficiency of external damping devices. For this purpose, a passive TMD is applied at mid-span of the beam (cf. figure 4.1). The TMD is tuned according to Den Hartog [33]. It has an auxiliary mass of 310 kg and therefore a mass ratio of 5.85%. The auxiliary mass is supported by four coil springs, one at each corner of the mass. Between the springs four viscous damping elements are mounted which generate...
Fig. 4.1: Cross-section and view of the concrete beam used as test rig [22]. In mid span a passive TMD ("Schwingungstilger") is mounted.

The aim of this project was to improve the performance of the existing TMD by means of an active element. Thus, an actuator was placed between the main and auxiliary system. For simplicity reasons a standard electric DC motor (brushless maxon EC motor, 400 W) was chosen. The motor was fixed on the auxiliary system and connected to the main system by a steel rope. This connection was also used

<table>
<thead>
<tr>
<th>System</th>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Main system</td>
<td>Modal mass</td>
<td>$m_1$</td>
<td>5300</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td>First modal stiffness ratio</td>
<td>$k_1$</td>
<td>815</td>
<td>kN/m</td>
</tr>
<tr>
<td></td>
<td>First resonance frequency</td>
<td>$\omega_1$</td>
<td>12.4</td>
<td>rad/s</td>
</tr>
<tr>
<td></td>
<td>Approximate modal damping</td>
<td>$\xi_1$</td>
<td>1.3</td>
<td>%</td>
</tr>
<tr>
<td>Auxiliary system</td>
<td>Modal mass</td>
<td>$m_2$</td>
<td>310</td>
<td>kg</td>
</tr>
<tr>
<td></td>
<td>Mass ratio</td>
<td>$\mu$</td>
<td>5.85</td>
<td>%</td>
</tr>
<tr>
<td></td>
<td>Stiffness</td>
<td>$k_2$</td>
<td>42.4</td>
<td>kN/m</td>
</tr>
<tr>
<td></td>
<td>Stiffness ratio</td>
<td>$\alpha$</td>
<td>94.5</td>
<td>%</td>
</tr>
<tr>
<td></td>
<td>Approximate damping ratio</td>
<td>$\xi_2$</td>
<td>13.5</td>
<td>%</td>
</tr>
</tbody>
</table>

Tab. 4.1: Physical properties of the concrete beam according to Dazio [22].
**Fig. 4.2:** Photo of the beam during an experiment at the ETH Zürich. The background is blurred.

**Fig. 4.3:** Pin and roller-type supports of the beam.
as a reduction to increase the force between the systems and increase the rotation speed of the DC motor. Figure 4.5 shows the implementation of the actuator fixation with steel cable connection to the main system. The steel cable was fixed on the auxiliary system and wound around small rolls to create a reduction. The motor was used in capstan mode. Therefore, the cable was strapped around a cylindric part with a diameter of 20 mm. Using an electric motor with a nominal torque of 0.8 Nm this led to a nominal system force of 320 N. An initial acceleration of the auxiliary mass (310 kg) of approximately 0.1g was therefore possible. To generate an adequate control forces, the actuator was used in combination with an ESCON 70/10 motor controller by maxon motor in current regulation mode. The output of the controller (desired control force) needed therefore to be divided by torque constant of the motor and the transmission rate of the reduction and capstan to obtain the according current value.

4.1.2 Measurement equipment

In the first experimental study, the absolute positions of the main and auxiliary system \( x_1(t) \) and \( x_2(t) \) were measured. Two laser distance sensors Baumer OADM 20|6572/S14F were applied for this purpose. The sensor resolution was proportional to the maximum measurement stroke of these devices. In order to achieve the highest possible resolution, a total measuring stroke of 50 mm was applied. Thus, a
resolution of $< 0.05$ mm was achieved in theory. The sensor outputs were sampled with a frequency of 10 kHz in a first step using a LabView cDAQ-9138 measurement equipment with a NI 9329 BNC data acquisition card. In a second step the mean values over one control loop period ($T=0.005$ s) were used to generate the low-pass filtered current position measurement for the control loop. No further filtering of the measurements was required. For information purposes, the motor current and motor stroke were also measured.

The harmonic excitation force of the bridge was measured using an acceleration measurement of the APS-shaker mass. This measurement was multiplied by the moving shaker mass (30.9 kg) to obtain the excitation force. A MEMS inertial acceleration sensor (LIS344ALH by STMicroelectronics) was used for the measurement.

### 4.1.3 Control architecture

The control architecture for this experiment was derived using the optimal control by convex optimisation technique (cf. section 3.2.5). Thus, two high order discrete filter served as a controller to generate the control force. The optimisation technique described in section 3.2.5 was applied with the constant performance filter $w_1 = 0.5$ and the robustness filter $w_2 = 0.1$. Linearised robust performance constraint were used together with the quadratic optimisation tool by MATLAB.
The optimisation led to following two controllers:

\[
K_1(z) = \frac{-156.7651z^8 + 49.2317z^7 + 193.9438z^6 + 187.6031z^5}{z^9 - 0.0067z^8 - 0.0070z^7 - 0.0073z^6 - 0.0076z^5 - 0.0079z^4 - 0.0081z^3 - 0.0083z^2 - 0.0085z - 0.0087}
+ 161.6585z^4 + 42.7372z^3 - 148.4676z^2 - 168.6588z - 161.2828
\]

\[
K_2(z) = \frac{384.9279z^8 + 224.2021z^7 - 268.8191z^6 - 297.4495z^5}{z^9 - 0.0000z^8 + 0.0000z^7 + 0.0000z^6 + 0.0000z^5 + 0.0004z^4 + 0.0000z^3 + 0.0000z^2 + 0.0000z^1 + 0.0000z^0}
- 297.5040z^4 - 197.0236z^3 + 67.1239z^2 + 189.0474 + 195.4948
\]

(4.1)

The controllers were combined as follows:

\[
F(z) = K_1(z)X_1(z) + K_2(z)X_2(z)
\]

(4.2)

The control loop was run under LabView in soft real-time at a frequency of 200 Hz, which yields a loop time of 5 ms.

4.1.4 Results and discussion

The open-loop (passive) and closed-loop (active) system were tested using two different excitations. First, a harmonic excitation was applied using an APS 400 proof mass shaker by APS Dynamics and a laboratory frequency generator. Several measurements were conducted to generate the frequency response diagrams. In a second step, the concrete beam was analysed under an impact load. For this purpose, a 25 kg sand bag was dropped from a height of 1.0 m on the beam.

**Harmonic excitation** The frequency response functions of the upper-left (\(G_{11}\)) and lower-left (\(G_{21}\)) component in open and closed loop were measured using a harmonic excitation signal. The observed steady-state vibration amplitude was translated to the frequency response function (FRF), shown in figure 4.6. Both, the measured FRF of \(G_{11}\) and \(G_{21}\) are in good accordance with the simulation results. Due to the non-simulated dry friction of the auxiliary system, the damping efficiency is slightly lower than in the simulations. Therefore, the system \(G_{11}\) has a higher amplitude and the movement of the auxiliary mass \(G_{21}\) is lower than in simulations.

**Impulse excitation** The impulse response of the main system \(G_{11}\) was measured in open-loop and closed-loop mode using a 25 kg sandbag which was dropped
Fig. 4.6: Frequency response function of $G_{11}, G_{cl,11}$ (top) and $G_{21}, G_{cl,21}$ (bottom)
from a height of 1.0 m on the concrete beam. The time domain results are shown in figure 4.7. Obviously, a slight increase of the stroke is achieved in the active (closed-loop) mode. This results in a faster decay of the main system amplitude. It can be observed that the main system is perfectly silenced after approximately two seconds, whereas the passive TMD is unable to silence the system due to dry friction for these small amplitudes.

Discussion This first experiment showed that slight improvements of passive TMD can be achieved using additional actuators. However, it also showed that the standard TMD is already close to the optimal working condition of a closely tuned inertia-based device and that a significant performance improvement is difficult to achieve.

Nevertheless, it was shown in this experiment that the chosen setup and controller design method was not well adapted to the problem and the performance improvement was clearly unsatisfactory for an active vibration control device. This is mainly due to the high tuning ratio between main and auxiliary system. For this reason, the improvement of an existing TMD by a fixed-order controller design using convex optimisation techniques was abandoned. Significant performance improvements can be achieved using lower tuning ratios, as the next experiments will show.
4.2 Active mass driver AMD - proof of concept

After the first unsatisfactory attempt to increase the performance of a conventional TMD, a new concept was tested in January 2017 on the test rig at the ETH Zürich. For this purpose, an existing prototype of an ATMD (by Weber et al. [84]) was used on the same test rig as in the previous experiments (cf. section 4.1). The main purpose of this experimental verification was to compare the performance of the AMD with velocity feedback to a standard TMD. The second objective of these experiments were the verification of the measurement and control algorithms based on the acceleration measurements. Additionally, the performance of the acceleration feedback according to Nishimura was tested.

4.2.1 Implementation

The concrete beam test rig (cf. section 4.1) was installed in the laboratory at the ETH Zürich identical to the first experiment. The vibration control device was adopted from an existing prototype by Weber et al. [84] (cf. figure 4.8a). This prototype has a suspended mass of \( m_2 = 51.1 \) kg and a resonance frequency of approximately \( f_2 = 1.18 \) Hz. In contrast to the first experiment, this active vibration control device has a resonance frequency which is significantly different from the main systems fundamental frequency. Thus, the working principle of this device is the untuned AMD. The auxiliary mass comprised four steel plates with a size of 40 cm by 40 cm and a thickness of 10 mm. The elastic suspension was created by four tension coil springs. In order to increase the maximum stroke of the system, the upper fixation was extended to allow a stroke of \( \pm 55 \) mm. The actuator (brushless maxon EC motor, 400 W) was mounted on the suspended mass and connected by a rack pinion system to the fixed rack gear (cf. figure 4.8b).

The first experiments have shown that the existing power supply was not adapted to the new prototype working in AMD mode. Thus, a power supply with \( U_N = 48 \) V and \( P_N = 480 \) W was applied. The electric motor (Maxon EC60) was connected using a ESCON EC 70/10 motor driver with an additional shunt resistance DSR 70/30 by Maxon. The motor controller worked in current or torque control mode with the predefined adjustments.

The excitation of the main system (concrete beam) was made using an APS 400 proof mass shaker with a suspended mass of 30.9 kg. Sinusoidal disturbance forces
were generated using a laboratory waveform generator. Although the measurement of the disturbance force was recorded on the same device as the control loop was run, the calculated actuator force is based purely on the measured accelerations.

### 4.2.2 Measurement equipment

The measurement in the previous experiment was made using laser distance sensors. These sensors have provided good measurement results. However, a rigid basis for the measurements is required (ground). In the case of a real floor system in a building with multiple levels, this measurement is not possible in this way. In order to have a ground-independent measurement of the absolute position of the masses, accelerations were measured using MEMS inertial accelerometers. For this purpose, two LIS344ALH inertial sensors were placed on both sides of the concrete beam at mid-span. The measured acceleration was sampled at 25 kHz using the NI 9239 data acquisition system in an NI cDAQ-9138 laboratory computer. The mean value of the acceleration sensors over one control-loop period was used as the current acceleration measurement. A third accelerometer was placed on the top of the auxiliary mass (cf. figure 4.8b). On the top of the disturbance force generator (APS shaker), a fourth accelerometer was applied to record the disturbance force.

An integration of the acceleration measurements was used to obtain the absolute velocity of the main and auxiliary mass. An additional low-pass filtering was not required. However, in order to prevent a static gain and drift of the velocity
signal, the acceleration measurements were high-pass filtered using a second-order Butterworth filter with a cut-off frequency of 0.2 Hz. In a next step, a second integration with high-pass filter was used to obtain the absolute position measurements.

### 4.2.3 Control architecture

In the second experiment, different control algorithms were tested and compared. First of all, the virtual TMD with stiffness and damping provided by the actuator was tested. In a further step the direct velocity feedback (DVF, cf. section 3.1.3) and acceleration feedback according to Nishimura (Nishi, cf. section 3.1.5) were analysed. For this purpose, acceleration, velocity and position of both the main and auxiliary mass needed to be known.

#### Adaptive or virtual TMD

The prototype AMD had a resonance frequency of approximately $f_2 = 1.18$ Hz, whereas the main system's fundamental frequency was at $f_1 = 1.97$ Hz. In tuned-mass-damper-mode, the auxiliary resonance frequency must be close to the main system's frequency $f_{2,TMD} = \frac{f_1}{1+\mu} = 1.95$ Hz. Thus, an additional stiffness of $k_d = 4'870$ N/m was required. The passive viscous damping of the auxiliary system was measured using a settlement analysis. By measuring the decay rate, the viscous damping parameter was determined as $\xi_2 = 2.8\%$. According to Den Hartog [33], the damping ratio should be $\xi_{2d,TMD} = 5.9\%$. Thus,
an additional viscous damping of \( c_d = 53.0 \text{ kg/s} \) was required (cf. section 3.1.1). The applied control force was therefore:

\[
f(t) = 4870 (x_1(t) - x_2(t)) + 53 (\dot{x}_1(t) - \dot{x}_2(t))
\] (4.3)

**Direct velocity feedback**  The DVF (cf. section 3.1.3) uses only the measurement of the main system as a feedback parameter. The velocity was determined by integrating the main system acceleration measurement. Several feedback gains have been tested. Adequate performance and stability were found for feedback gains of \( c_d = 5000 \text{ kg/s} \).

\[
f(t) = 5000 \dot{x}_1(t)
\] (4.4)

**Acceleration feedback according to Nishimura**  The acceleration feedback method according to Nishimura (cf. section 3.1.5) required mainly an acceleration measurement of the main mass. Additionally, the auxiliary stiffness and damping were shifted using the actuator to obtain the optimal working condition. The feedback force yields:

\[
f(t) = -0.192 \cdot 5300 \ddot{x}_1(t) - 308.2 [\dot{x}_2(t) - \dot{x}_1(t)] - 3396.7 [x_2(t) - x_1(t)]
\] (4.5)

These three different control laws have been implemented as discrete IIR-filters, using a bilinear discretisation according to Tustin [43]. The control loop was programmed in LabView with a loop frequency of 200 Hz and therefore a loop sampling time of 5 ms.

### 4.2.4 Results and discussion

The applied control laws have been compared under harmonic and transient excitations. First, a harmonic excitation was applied using an APS 400 proof mass shaker by APS Dynamics and a standard frequency generator. Several measurements were conducted to generate the frequency response diagrams shown in figure 4.10. In a second step, the dynamic behaviour of the concrete beam was analysed in closed-loop under an impact load. For this purpose, a 25 kg sand bag was dropped onto the beam from a height of 1.0 m.
Harmonic excitation  Figure 4.10 shows the amplitude frequency response plot of the absolute position \( x_1 \) and \( x_2 \) for a harmonic disturbance \( d_1 \). Obviously, the undamped main system had a resonance peak at 1.97 Hz. The ATMD showed the characteristic two peaks with a lower amplitude. Due to the low mass ratio of \( \mu \approx 1\% \), the reduction using the TMD was small. Peak amplification dropped from 4.8e-5 to 1.6e-5 by 66%. The DVF method showed only one strongly damped peak, as expected from the simulation results. The chosen feedback value leaded to no visible main system amplification at the auxiliary frequency. Thus, a higher feedback value would have been possible and lead to an even lower main system acceleration. DVF leaded to a reduction by 81%. The best results in the sense of the maximal peak amplification (\( \infty \) norm) were obtained by applying the Nishimura feedback law. However, this feedback law showed a strong amplification of the auxiliary system over the whole analysed frequency band.

Transient excitation  The transient behaviour of the open- and closed-loop systems was tested using a sandbag drop test. Figure 4.11 shows the absolute positions of the main and auxiliary systems. The positions were integrated from the acceleration measurements. The undamped open-loop system showed a harmonic response with very little damping. The decay rate of the ATMD was much stronger and
Fig. 4.11: Impulse response after a sandbag of 25kg was dropped from 1m height on the main structure.

approximately equal to the DVF. However, the DVF showed a stronger decay in the first few oscillations. The auxiliary system had a stronger excitation for the DVF. The feedback law according to Nishimura had the fastest decay rate. Already after two oscillations, the main system was approximately silenced. However, this effectiveness is only possible thanks to a strong excitation of the auxiliary system by the oscillating actuator force.

Discussion  The results of these experiments have shown that ATMD and DVF are suitable control laws for the mitigation of the beam vibration. Both showed approximately equal dynamic behaviour in closed-loop. The Nishimura method presented a very strong mitigation of the main system vibration but required on the other hand a much higher auxiliary stroke and actuator force. The actuator was subjected to strong and fast oscillations in the first few oscillations, due to the unfiltered acceleration feedback of the higher order resonances of the beam. The experiment showed that the maximum stroke of 55mm of this device was not suitable for the mitigation of this system. For the next experiments, a novel prototype damping device was therefore designed.
4.3 AMD - advanced design

4.3.1 Implementation of the AMD device

The experiments with the AMD have shown the significant benefit of the active vibration control. The limiting factors of the device were the stroke and force limitations due to the simple hanging design. For the implementation as a vibration control device in a floor, the stroke and the total height of the device are limited to the space between the floor and ceiling of the level below. Thus, a novel design needed to be found. The following design criteria were defined:

- Total height of the device 30 cm
- Total stroke of ± 125 mm
- Usage of a standard DC motor as an actuator
- Suspended mass of approximately 50 kg
- No blocking or overloading of the elastic mount
- Auxiliary resonance frequency below 1.2 Hz
- Low friction and damping

Several designs have been analysed and evaluated. Obviously, a thin and wide steel plate as auxiliary mass is advantageous for the design. However, a simple hanging design with tension coil springs requires too much space to be implemented. The same argument holds for vertical compression coil springs. The static compression or stroke $x_{stat}$ of a spring with stiffness $k$ is directly linked to the resonance frequency $f_0$ via the deadweight of the suspended mass $m$.

$$ F = mg = kx_{stat} = (2\pi f_0)^2 m x_{stat} $$  \hspace{1cm} (4.6)

The static stroke is therefore independent of the suspended mass:

$$ x_{stat} = \frac{g}{(2\pi f_0)^2} \approx \frac{0.25}{f_0^2} \quad (4.7) $$
A minimum resonance frequency of 1.2 Hz requires therefore a static stroke of 17 cm. If a tension spring solution is chosen, the limiting factor is the total spring length which must not exceed the height of the device. If, in contrast, a compression spring is chosen, the limiting factor is the blocking length of the spring. Additionally, stability problems need to be taken into account when soft springs are used in compression mode. A solution was found in suspending the auxiliary mass by steel ropes (cf. figure 4.12). The ropes were guided by pulleys, fixed at the highest possible point of the device. The other end of the rope was fixed on a wheel with a diameter of $r_1 = 125$ mm. A smaller wheel with a diameter of $r_2 = 50$ mm was fixed on the same axle forming a reduction gear of $i = 25/10$. The small wheel was connected by steel ropes to coil tension springs. Thus, the energy conservation on the axle ($\theta$ is the rotation angle) yields $mr_1^2\dot{\theta}(t) - kr_2^2\theta(t) = 0$ and the stiffness must be $k = m\omega^2i^2$. Let $F_0$ be the prestressing of the tension coil spring. Therefore, the static stroke of the springs is:

$$x_{stat} = \frac{mg i - F_0}{i^2(2\pi f)^2}$$

(4.8)

The prestressing of the springs caused by the deadweight of the auxiliary mass must not be lifted by the maximum stroke of the auxiliary system. Thus, the spring stroke due to the maximum auxiliary stroke must be smaller than the static displacement. The maximum displacement of the auxiliary mass of 125 mm created an additional spring stroke of 50 mm with the designed reduction ratio of $i = 2.5$. Thus, the spring stroke is well below the initial stroke of 65 mm (cf. table 4.2).

The evaluation of the spring was generally based on two criteria: the stiffness and the maximum stroke. An iterative design procedure was applied. The general layout
<table>
<thead>
<tr>
<th>Variable</th>
<th>Value</th>
<th>Unit</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>3'638</td>
<td>N/m</td>
<td>Spring stiffness</td>
</tr>
<tr>
<td>$s_n$</td>
<td>159</td>
<td>mm</td>
<td>Maximum stroke</td>
</tr>
<tr>
<td>$F_0$</td>
<td>70</td>
<td>N</td>
<td>Factory prestressing of the spring</td>
</tr>
<tr>
<td>$x_{stat}$</td>
<td>65.0</td>
<td>mm</td>
<td>Initial static stroke (4.8)</td>
</tr>
<tr>
<td>$f_2$</td>
<td>1.086</td>
<td>Hz</td>
<td>Calculated auxiliary resonance frequency</td>
</tr>
</tbody>
</table>

Tab. 4.2: Physical properties of the chosen tension coil springs and physical properties of the AMD device.

The final design is shown in figure 4.14. A sheet-metal structure was designed which holds all installations and protects the moving parts. The ADM prototype has a total weight of 84 kg, which comprises the auxiliary mass (50 kg) and the case plus installations (approximately 34 kg). Therefore, the maximum force must be
Fig. 4.13: Drawing of the moving parts in the novel AMD-concept. Side-view (top) and isometric view (bottom). The auxiliary mass is suspended by four steel cables which are wound around the two side wheels. Four tension coil springs held the auxiliary mass in an initial position. The motor is connected via a secondary axis.
limited to approximately 335 N if the device is not fixed to the structure. Otherwise, the lifting force would detach the casing from the structure.

4.3.2 Implementation

For this experiment the same concrete beam was used as in experiment 1 and 2. The objective of the third experiment was to demonstrate the advantages of the CVF method and test the novel AMD-prototype. For this purpose, the AMD was placed in the mid-span location of the concrete beam. The electric motor of the AMD was connected to the ESCON 70/10 motor controller with an additional shunt resistance DSR 70/30 by Maxon Motors. Figure 4.15 shows the AMD device during the experiment.

4.3.3 Measurement equipment

A similar measurement and control equipment as in the previous experiment was used. The vertical acceleration of the main and auxiliary system was measured by LIS 344 ALH inertial MEMS acceleration sensors with a sensor range of ±2 g. Two sensors were placed in mid-span on the concrete beam. The mean value of both sensors was used for the control to reduce the sensor noise. A third acceleration sensor was placed on the auxiliary mass. A sampling period of 25 kHz was used to
Fig. 4.15: Setup with AMD device on the concrete beam during the experiment (top) and detailed view of the side wheel (bottom)
Controller parameter | $H_2$ | $H_\infty$
--- | --- | ---
$c_{1d} = 2\xi_1 d m_1 \omega_1$ | 25'917.2 | 18'697.4 |
$c_{2d} = 2(\xi_2 d - \xi_2) m_1 \omega_2$ | 51.57 | 43.90 |

Tab. 4.3: Combined velocity parameters for the experimental study for a $H_2$- and $H_\infty$-optimal controller respectively. For the implementation, a factor of 1.2 is added according to section 4.3.5.

generate a set of 125 samples per control-loop period of 2 ms. The average value of the 125 samples was used to remove high-frequency noise from the acceleration measurement. A first integration including a second-order high-pass Butterworth filter with a cut-off frequency of 0.3 Hz was used to generate absolute velocity signals of the main and auxiliary system. The absolute position values were obtained in a second integration step, again with an appropriate high-pass filtering to remove static offsets and drifts in the signal. Static and low-frequency velocities and position changes were removed from the signal due to the filtering. The cut-off frequencies need to be chosen carefully, especially when systems with low resonance frequencies are implied.

### 4.3.4 Control architecture

The main control architecture of the CVF is built in a very simple manner. It comprises a feedback force proportional to the velocity of the main and auxiliary mass.

$$f(t) = c_{1d} \dot{x}_1(t) + c_{2d} (\dot{x}_2(t) - \dot{x}_1(t))$$

The absolute velocities $\dot{x}_1$ and $\dot{x}_2$ were integrated from the corresponding acceleration measurement. The values of $c_{1d}$ and $c_{2d}$ were derived according to section 3.2.3.

**Force and stroke limitations** Force and stroke limitations are crucial for the safe operation of an active vibration control device. Force limitations are set directly at the controller output to protect the amplifier, current controller and motor from saturation effects. The stroke limitation has to be set more delicately, because a too restrict limitation will strongly reduce the damping performance of the device.

The maximum performance of an active vibration control device can be described for transient disturbances by its ability to store energy. The more energy is absorbed
by the AVCD, the less is transmit to the dynamic system and the lower is its vibration amplitude. The energy absorption capability of the AVCD is governed mainly by the auxiliary mass of the damper and the maximum stroke $y_{2,\text{max}}$.

$$E_{\text{max}} = \frac{1}{2} k_2 y_{2,\text{max}}^2 = \frac{1}{2} m_2 \omega_2^2 y_{2,\text{max}}^2 \quad (4.11)$$

If the stroke (or the velocity) is too high, the auxiliary mass must be slowed down to prevent the auxiliary system from hitting the stroke limitations. This deceleration force transfers energy back to main system, which is not desired. An ideal AVCD has a low inherent (passive) damping or friction force. By neglecting the inherent damping of the auxiliary system $\xi_2$, the total instantaneous energy in a passive auxiliary system is:

$$E(\dot{x}_2, y_2) = \frac{1}{2} m_2 \omega_2^2 y_2^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad (4.12)$$

By adding an actuator force $f(t)$, the total energy capacity is increased by the energy which can be dissipated on the path from the current position to $y_{2,\text{max}}$. It is assumed that the maximum relative velocity $\dot{y}_2$ is low compared to the maximum actuator speed. Therefore, the actuator force is limited only by the maximum electric current and the maximum actuator force $F$ is constant for all considered speeds. For higher speeds the power limitation of the actuator and shunt resistor would be limiting and the actuator force would also depend on the actuator speed ($F = P_{\text{max}}/\dot{y}_2$). For the case of a constant maximum actuator force, the limiting energy is therefore:

$$E_{\text{max}}(\dot{y}_2, y_2) = \frac{1}{2} m_2 \omega_2^2 y_{2,\text{max}}^2 + F \left| \frac{\dot{y}_2}{y_2} y_{2,\text{max}} - y_2 \right| \quad (4.13)$$

The limiting contour of the stored energy is

$$E(\dot{x}_2, y_2) \leq E_{\text{max}}(\dot{y}_2, y_2)$$

$$m_2 \omega_2^2 y_2^2 + m_2 \dot{x}_2^2 \leq m_2 \omega_2^2 y_{2,\text{max}}^2 + 2F \left| \frac{\dot{y}_2}{y_2} y_{2,\text{max}} - y_2 \right| \quad (4.14)$$

For the case where the relative speed is positive (the damper mass moves upwards, upper part of figure 4.16), this leads to:

$$m_2 \omega_2^2 y_2^2 + m_2 \dot{x}_2^2 \leq m_2 \omega_2^2 y_{2,\text{max}}^2 + 2F(y_{2,\text{max}} - y_2)$$

$$y_2^2 + \frac{2F y_2}{m_2 \omega_2^2} + \frac{x_2^2}{\omega_2^2} \leq y_{2,\text{max}}^2 + \frac{2F y_{2,\text{max}}}{m_2 \omega_2^2}$$

$$\left( y_2 + \frac{F}{m_2 \omega_2^2} \right)^2 + \left( \frac{\dot{x}_2}{\omega_2} \right)^2 \leq \left( y_{\text{max}} + \frac{F}{m_2 \omega_2^2} \right)^2 \quad (4.15)$$
Fig. 4.16: Energy stability border. Within the absolute stability border the auxiliary system will not reach the stroke limitation $y_{2,\text{max}}$ with high probability even in absence of any damping.

This results in a half-ellipse, centred at $-F/m_2\omega^2_2$. The semi major axis is therefore $y_{\text{max}} + F/m_2\omega^2_2$, whereas the semi-minor axis is $\omega y_{\text{max}} + F/m_2\omega^2$. For the case where the maximum actuator force is equal to the elastic force of the auxiliary mount at stroke limit $F = ky_{\text{max}} = m_2\omega^2_2y_{\text{max}}$, the centre of the half-ellipse is at $y_{\text{max}}$. If the relative speed $\dot{y}_2$ is negative (lower part of figure 4.16), the half-ellipse is shifted towards the right side equivalently.

From equation (4.15) it is visible that the velocity of the main structure $\dot{x}_1$ is not taken into account. In general, the velocity of the auxiliary mass is higher than the velocity of the main mass. Therefore, $\dot{y}_2 = \dot{x}_2 - \dot{x}_1 \approx \dot{x}_2$. If the auxiliary system reaches a state where (4.14) is not fulfilled, it will hit with high probability the stroke limitation. If the main system velocity $\dot{x}_1$ is changed by an appropriated...
disturbance in such a way that it reduces the stroke $y_2$ it is possible that stroke limitation can be avoided even if (4.15) is not fulfilled. For example if the auxiliary system reaches the lower stroke limitation with still considerable speed and in the same time the main system is accelerated downwards, the stroke is reduced and the auxiliary mass will not hit the stroke limitation. The same phenomenon is also possible in the opposite way that if the auxiliary mass is close to one of the limitations and an external disturbance accelerates the main system in the opposite direction, the auxiliary system will be saturated.

It is therefore necessary to introduce a security margin which takes into account the velocity of the main system. This can be easily done by adding a security factor $\psi \leq 1$ to the semi-minor axis. This leads to the following stability criterion:

$$\left(y_2 + \frac{F}{m_2\omega_2^2}\right)^2 + \left(\frac{\dot{x}_2}{\psi\omega_2}\right)^2 \leq \left(y_{max} + \frac{F}{m_2\omega_2^2}\right)^2$$  \hspace{1cm} (4.16)

If equation 4.16 is fulfilled, the active mass damper can work without any further measures. If the auxiliary velocity exceeds the given limit, as a first measure the driving output force should be reduced. If the damper mass is nevertheless approaching the ultimate limit, an deceleration force must be applied. In this study, a linear increasing force was used.
4.3.5 Results and discussion

**AMD-verification** The dynamic behaviour of the novel AMD was tested to verify the properties. For this purpose, the prototype was placed on a rigid structure. The movement of the auxiliary mass was measured for a set of sinusoidal actuator forces. The simulation of the following first order system shows a good accordance in the significant band (cf. figure 4.18).

\[
\frac{Y_2(s)}{F(s)} = \frac{1}{60(s^2 + 2 \cdot 0.18(2\pi 1.08)s + (2\pi 1.08)^2)}
\]  

(4.17)

In the frequency band below the resonance frequency, the measured amplitude was slightly below the values of the model. This is due to the dry friction in the system. Time-domain data showed a strong distortion of the sinusoidal signal in this frequency band, as well as a significant stick-slip behaviour. It should be noted that the simulated mass is at 60 kg and thus, 10 kg higher than the physical auxiliary mass. This is due to the additional inertia of the motor and the side wheels which must also be accelerated. However, the force acting on the main system remains at 50 kg. Thus, the demanded actuator force was increased by a factor of \(\frac{60}{50} = 1.2\).
Experimental results In this experiment, the AMD prototype as well as the developed CVF were tested. First, a harmonic excitation was applied using an APS 400 proof mass shaker by APS Dynamics and a laboratory frequency generator. Several measurements were conducted to generate the frequency response diagrams. In a second step, the concrete beam was analysed under an impact load. For this purpose a 25 kg sand bag was dropped onto the beam from a height of 1.0 m.

Harmonic excitation Figure 4.19 shows the amplitude frequency response diagram of the absolute main and auxiliary system. Both controller values for $H_2$- and $H_\infty$-optimal showed excellent damping performances. The FRF of the Nishimura feedback method from the second experiment is added to figure 4.19 to illustrate that the amplification factors of the closed-loop main system are on an equivalent level. The CVF $H_\infty$-optimal feedback method showed two approximately equal maximums. The CVF $H_2$-optimal method showed only one maximum approximately at the auxiliary resonance frequency. This behaviour is in accordance with the simulation results from section 3.3. It should be noted that below 1.2 Hz, the APS-shaker was unable to provide a sufficient disturbance force to outrun the dry friction of the auxiliary system (cf. figure 4.18). Thus, the frequency response is in this frequency band approximately equal to the undamped system and does not show the same characteristics as the simulation results.
Transient excitation  The impulse response after an impact of a 25 kg sandbag from a height of 1m is shown in figure 4.20. The absolute position of the main system showed a very strong damping in closed-loop. Both, the $H_\infty$- and $H_2$-optimal controller were able to silence the main system within less than three oscillations. Comparing this result to the Nishimura results from the second experiment, an improvement of the damping performance is obvious. However, the increased damping performances are gained by an increased auxiliary stroke. The last graph in figure 4.20 shows the actuator force in the time domain. Obviously, the force limitation was hit during this experiment. However, no instabilities were observed. It should be noted that the actuator forces required by Nishimura’s method are much more fluctuating due to the acceleration feedback. A low-pass filtering of the acceleration measurement was tried using a second-order Butterworth filter. However, the phase delay due to the filtering leaded to instabilities of the closed-loop system. Due to the close separation of the higher order modes, an adequate filtering was not possible without a strong phase shift in the interesting frequency band.
**Discussion**  The results of this experiment have proven that the novel designed AMD prototype and the developed CVF method are suitable to mitigate vertical vibrations of a beam-type structure. An excellent damping performance and robustness even in the case of actuator saturation leaded to optimal results for transient and harmonic disturbances.

A comparison to the results of the Nishimura method from the second experiment showed approximately equal damping performance for all tested active control laws. However, strong fluctuations of the actuator forces were detected using Nishimura’s method. This behaviour is due to the direct acceleration feedback component of the controller and the higher order modes of the beam.
4.4 AMD - Implementation on a floor

After the successful tests of the novel active vibration control device with CVF on the laboratory concrete beam, a last experiment on a more realistic structure was performed. For this purpose, a quadratic floor-like timber structure with an area of 6m x 6m was used as main vibrating structure.

4.4.1 Implementation

The main structure for this study comprises a 6mx6m cross-laminated timber (CLT) slab with a thickness of 260 mm. This slab was used for a long-duration outdoor study of timber and glue by Prof. A. Frangi and Marcel Muster in 2017 and 2018 at the ETH Zürich. The CLT-stack was built by five layers of 20mm thickness, two layers of 30mm and five layers of 20mm. The single layers were glued together crosswise in the factory to form quadratic elements of 3 m x 3 m. On site four of these elements were glued together face-on-face to form a slab of 6 m x 6 m. The objective of this experiment was to study the behaviour of the glued face-to-face joint. For this reason the slab was supported on the four corner points to create a maximum bending moment in the glued interface. In order to further increase the loading, twelve big-bags were deposed on the top of the slab, each containing approximately 800 kg of gravel. Figure 4.21 shows the timber slab in front of the laboratory building at the ETH Zürich.

Resonance modes of the structure  Prior to the experiments, a modal analysis was performed to estimate the first resonance frequencies and the according mode shapes. A system identification helped to verify the resonance frequencies with measurement data. Figure 4.22 shows the four first mode shapes obtained from the FEM-analysis (AxisVM X13 R4g). Table 4.4 lists the according frequencies. The difference between mode 2 and 3 (both plate bending) is due to the modelling of the CLT structure which implies a slightly higher stiffness in one direction. This is due to an inherent limitation of the applied FEM programme.

To verify the obtained FEM-results, a system identification was performed based on vibration data using a step-form excitation. The data were collected by Marcel Muster in spring 2017 using a weight of 198 kg which was fixed in mid-span at a small string. The vertical acceleration of the mid-span point was recorded after cutting the string. This resulted in a step-form excitation. A representative
Fig. 4.21: Photo of the timber slab used for the experiments. The slab is supported at the four corners. Additional load is applied by big-bags filled with gravel. A weather-protecting roof is under construction. (Photo by Marcel Muster, IBK ETHZ 2017)

Fig. 4.22: First four resonance modes of the loaded timber slab fixed at the four corners. The calculation was performed under Axis VM13 using shell-elements.
<table>
<thead>
<tr>
<th>Mode No</th>
<th>Frequency [Hz]</th>
<th>Modal mass [kg]</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.74</td>
<td>12'625</td>
<td>Movement of the middle point</td>
</tr>
<tr>
<td>2</td>
<td>7.03</td>
<td>3'255</td>
<td>Movement of two sides</td>
</tr>
<tr>
<td>3</td>
<td>7.19</td>
<td>3'344</td>
<td>Movement of two sides</td>
</tr>
<tr>
<td>4</td>
<td>13.41</td>
<td>3'498</td>
<td>Movement of four sides</td>
</tr>
</tbody>
</table>

**Tab. 4.4:** First four resonance modes obtained from the FEM analysis. The mode shapes are shown in figure 4.22. The differences between mode 2 and 3 are due to modelling limitations of the CLT in the FEM-programme.

**Fig. 4.23:** System identification for the loaded deck in the middle position (left), acceleration in mid-span after a step-form disturbance of 1'980 N (right)

measurement series was chosen and the system’s input and output were analysed using spectral FFT-data. To reduce truncation error a Hamming window with length of the time series was overlaid.

The identification of the physical system using a vertical acceleration measurement in mid-span shows a first resonance frequency at approximately 4 Hz. The second, third and fourth resonances are not detected, which is obvious from figure 4.22. The mid-span excitation cannot excite these modes. Figure 4.23 shows clearly that the acceleration measurement data was low-pass filtered with a cut-off frequency of 100 Hz.

The harmonic excitation of the main structure was achieved using an APS proof mass shaker. The shaker was placed in middle of the plate for the first experiments. The second series of the experiments was run with an excitation at the location of the acceleration sensor 1 according to figure 4.25. Test with a transient step-form excitation were performed only in the middle of the plate. For this purpose, a
additional weight (198 kg of steel plates) was connected by a string in the middle of the plate. By cutting the string, a step-form disturbance was achieved. Due to the acceleration measurement and the high-pass filtering, the resulting change in position is not detectable in the results.

The AMD prototype from section 4.3 was used for this experiment. It was placed either in the middle of the plate or in mid-span between the two front supports (location of the sensor 1 in figure 4.25).

### 4.4.2 Measurement equipment

The vertical acceleration of the timber slab was measured at five points using inertial MEMS acceleration sensors LIS344ALH by STMicroelectronics with a sensor range of ± 2 g. Four sensors were placed in mid-span between the supports, one was located in the middle of the plate. The measurement equipment was transferred directly from the third experiment (cf. section 4.3).
4.4.3 Control architecture

Exactly the same control architecture was applied as in the third experiment on the concrete beam (cf. section 4.3). A controller gain value of $c_{1d} = 135'000$ kg/s and $c_{2d} = 250$ kg/s were used. These values correspond approximately the $H_{\infty}$-optimal controller layout of the CVF according to section 3.2.3. Given that the first resonance frequency is present in both setups (location 1 and 5 according to figure 4.25), the same controller properties were applied in both experiments.

4.4.4 Results and discussion

The CVF method was applied to a mainly single-resonance system (position 5 according to figure 4.25) and a multi-resonance system (position 1) subjected to harmonic disturbances. The transient response was measured in the middle of the plate (position 5) only.
Harmonic excitation  The measured frequency responses are shown in figure 4.26. Obviously, the damping device is able to reduce the peak dynamic amplification of the first resonance (upper graph) from 6.8e-6 to 9.0e-7 by 87%. However, the nodal point at approximately 14 Hz is slightly amplified. The low-frequency band below 1 Hz shows a strong amplification. This phenomenon is mainly due to the undesired low-frequency content of the acceleration measurement which is strongly amplified in the double integration. As already described in section 4.3, the AMD is unable to overcome the static friction in this frequency band. Therefore, no amplification from the auxiliary system was detected.

The measured frequency response at the location 1 (lower graph on figure 4.26) shows that the first peak dynamic amplification at approximately 4 Hz is reduced from 6.0e-6 to 7.2e-7 by 88%. The second peak at approximately 8 Hz is reduced from 1.2e-6 to 2.5e-7 by 79%. This peak is shifted slightly upwards which is probably due to the low-pass filter with a cut-off frequency of 30 Hz. The third peak is slightly amplified, which is again due to the filtering of the high-frequency content and the resulting phase delay. A slight amplification at approximately 1 Hz is visible in frequency response of the auxiliary system.

Transient excitation  Figure 4.27 shows the time-domain acceleration response of the timber plate in mid-span (position 5 according to figure 4.25) and the damping device. The measurements are overlaid to produce the step disturbance at exactly 1 s. Obviously, the CVF feedback system is able to silence the closed-loop system within approximately two seconds, while the vibration of the undamped system is still detectable after 10 seconds. The acceleration of the auxiliary system reaches a value of 5 m/s² or approximately 0.5 g. Higher accelerations are restricted by the actuator force limitation at 240 N. The bottom graph shows that this limit is slightly touched for the applied excitation.

Discussion  This experiment has shown the benefits of the active vibration control using the CVF method. A strong damping performance with good robustness was achieved for the single- and multi-resonance systems. As predicted in the simulations, several resonance frequencies were mitigated by the AMD prototype. However, the filtering of the measurement noise reduced the effect of the higher order modes.
Fig. 4.26: Measured frequency response diagram (amplitude Bode plot) of the plate middle point (top graph, position 5 acc. figure 4.25) and side point (bottom graph, position 1 acc. figure 4.25). The higher order modes at 8 and 15 Hz are clearly visible on the lower graph as expected from the mode shapes on figure 4.22.
Fig. 4.27: Measured time-domain acceleration step response of the damped and undamped main system (top) and auxiliary system (bottom) in the middle of the plate.
This part analyses the economic impact of an active vibration mitigation on the building costs. It is obvious that this chapter uses only approximative values and estimations. A more detailed cost-benefit analysis is only possible based on a real project.

Only a small part of the overall building costs is generally due to the structural components. Depending on the purpose of the building or structure and the location, a significant part of the costs is due to planning, demolition, provision of local public infrastructure, secondary contract work, non-structural components of the building, installations and other works. However, for structures like multi-story parking garages, warehouses and industrial buildings, the structural costs can account for 50 to 75% of the overall building costs.

In the first part, the construction and installation costs of an AVCD are estimated. The second part analyses the possible application and cost benefits on two example building layouts.

### 5.1 Active vibration control device - A cost analysis

The cost structure of an AVCD can be split approximately in three parts (cf. equation 5.1). The first part \( A_1 \) are the fixed costs of the device which are approximately independent of the device-size. This part contains the project and design costs, the costs for the control and mechanics, assembly and transportations costs and the project overhead. The second part \( A_2 \) is the cost of the auxiliary mass which is directly proportional to the required amount of material. The third part \( A_3 \) contains all time-dependent running costs such as maintenance, spare parts and the energy consumption. A simplified formula has therefore the following form.

\[
C(t, m_2) = A_1 + A_2 \cdot m_2 + A_3 \cdot t
\]  \hspace{1cm} (5.1)
5.1.1 AVCD construction costs

The costs of the active vibration control device depend strongly on the number of units fabricated. The most expensive components are the DC motor and the measurement and control electronics, which costs about CHF 300.- per unit (for a large number of units). The cost of the auxiliary mass (steel plates) is in the range of approximately CHF 1.20 per kg. Including the labor, a total cost of approximately CHF 600.-/unit is estimated excluding the auxiliary mass. Obviously, if higher number of units are fabricated, the cost per unit can drop to a lower limit of approximately CHF 400.-/unit.

5.1.2 Maintenance and energy costs

The estimated maintenance time of a five-story building with two AVCD per story (10 units in total) is estimated one man-day (approx. CHF 1000.-) every two years. The simple and robust construction of the AVCD requires usually no expensive spare parts. Thus, the maintenance costs are estimated to CHF 50.-/year and unit.

The equivalent permanent power consumption of one device is estimated to 20 W. Obviously, peak power demand can reach up to several 100 Watts, nevertheless, most of the time the power consumption is low. This is equal to 175 kWh per year. One device will generate therefore energy costs of approximately CHF 15.- per year. Total operating costs of an AVCD is therefore approximately CHF 65.-/year including maintenance and energy costs. The lifetime of the building is defined in this study to be 20 years. Thus, each unit generates costs of CHF 1300.- within this time span.

5.2 Industry building 7.5x20.0m

Let the example building be a five-story industrial and warehouse building with a layout of 7.5 m x 20 m. The floors are built by a steel joist structure and prefabricated concrete elements (t=15cm) which are simply supported on the steel joists. An example payload of 8 kN/m² is applied together with the deadweight of the concrete plates. This structure requires according to the Swiss SIA standards [3] a four-meter spaced HEB 360 (S235) steel structure. By changing the steel quality to S355, the size of the steel beams can be reduced to HEB 300. The production costs of a structure in S355 are slightly higher due to the elevated prime costs and a
higher manufacturing time (reduced cutting speeds, increased tool wear, etc). Let the cost increase be 10% of the initial cost in S235. However, the table 5.1 shows that the overall costs for the structure in steel S355 is approximately 10% lower than in S235 due to the material savings.

Table 5.1 shows that the increased steel grade leads to lower overall costs of the structure. However, by reducing the profile size, the second moment of inertia, and thus the stiffness, is significantly reduced. A simple calculation of a simply-supported beam model (according to Euler-Bernoulli in appendix D) shows that this has a direct influence on the resonance frequency of the structure (figure 5.2). Using the smaller profile HEB 300, the resonance frequencies are approximately 1.5 Hz lower than for a HEB 360. Increased static deflections can be reduced by pre-cambering the steel beams. However, the dynamic effects can be mitigated only by applying an external damping element. Let the modal mass of the first resonance frequency be approximately 50% of the total mass. The deadweight of the construction is approximately 50 to, whereas the payload can reach up to 120 to. The AVCD is designed for 50% of the payload plus the deadweight of the structure. Thus, the first modal mass is approximately 55 to. By applying two active vibration control
Tab. 5.1: Comparison of an example steel structure in S235 and S355

<table>
<thead>
<tr>
<th></th>
<th>HEB 360</th>
<th>HEB 300</th>
</tr>
</thead>
<tbody>
<tr>
<td>Steel Grade</td>
<td>S 235</td>
<td>S 355</td>
</tr>
<tr>
<td>Section modulus $M_{y,Rd}$ [kNm]</td>
<td>600.5</td>
<td>631.8</td>
</tr>
<tr>
<td>Second moment of area $I_y$ $[10^6 , mm^4]$</td>
<td>431.9</td>
<td>251.7</td>
</tr>
<tr>
<td>Weight/m'</td>
<td>[kg/m']</td>
<td>142</td>
</tr>
<tr>
<td>Costs</td>
<td>[CHF/kg]</td>
<td>3.50</td>
</tr>
<tr>
<td>Costs</td>
<td>[CHF/m']</td>
<td>497.00</td>
</tr>
<tr>
<td>Total steel weight per story [kg]</td>
<td>6 390</td>
<td>5 265</td>
</tr>
<tr>
<td>Total steel costs per story [CHF]</td>
<td>22 365.00</td>
<td>20 270.25</td>
</tr>
</tbody>
</table>

Fig. 5.2: Beam-type resonance frequencies for different payloads and an influence width of 4.0m.

devices with a desired mass ratio of 1%, each AVCD must have an auxiliary mass of 275 kg.

The estimation of the construction costs has shown that a reduction of approximately CHF 2'000.- per story can be achieved using a higher-grade steel. On the other hand, the reduction leads to a significantly reduced bending stiffness of the construction and therefore to lower fundamental resonance frequencies. The application of active vibration control devices leads to installation costs of approximately CHF 1’860.- per story as shown in the previous sections. Including the operating costs over the lifetime of 20 years, the lower steel grade is still slightly more cost efficient. However, different additional costs for the handling, transportation of the heavier
profiles and more corrosion protection due to the larger profile surface are not taken into account. On the other hand, the AVCD requires electric power. The additional installation costs of the electric connection will approximately balance out these cost savings.

The solution in HEB 360, S235, leads according to figure 5.2 to significantly higher resonance frequencies. However, slight vibrations are possible if corresponding excitations are present. Thus, the application of AVCDs is still advantageous even thought, the overall costs are slightly higher.

### 5.3 Medium-rise office building

The Swiss fire-safety standards [81] define structures below a height of 30m as medium-rise buildings. High-rise buildings above 30m are subjected to tighten standards. Furthermore, in urban areas without a land-use plan, the construction regulations often limit the building height to 30 m. It is therefore often advantageous or necessary to establish buildings below this limit. However, the usable space within the building should be maximized.

This example works with an open-layout medium-rise office building with a square layout of 30x30m. The building is separated in two parts by the stair-case and elevator (cf. figure 5.3). Thus, two separate floors with an area of 360 $m^2$ can be used. Let the required free room height in the building be 2.9m.

A concrete-steel component floor is designed for this building with an in-situ cast concrete decking. Design guidelines [66] provide approximate ratio length/profile height of 30 to 40 and a concrete thickness ratio (profile distance over thickness) of 20 to 30. A higher concrete thickness is advantageous for the dynamic behaviour of the structure. The designed layout requires therefore a steel profile with a height of 350 mm and a concrete plate of approximately 200 mm for a span of 12 m and a spacing of 4.2m. However, a short calculation including the deadweight and a payload of 2 kN/m2 leads to a required profile IPE 450, S235. The additional decking of 200mm leads to a total thickness of 650 mm. By choosing a higher steel grade (S355) and a different profile type, an equal bending resistance can be achieved using a concrete decking of only 120 mm and a HEB 220 steel profile. Thus, the total height is reduced from 650 mm to 340 mm by 310 mm per story. Table 5.2 shows the results of the static calculations for both structures.
The first resonance frequency is estimated using a beam-type model from the design guideline [66] and a plate-type FEM model. The resonance frequencies of the initial structure with an IPE450, S235 present a fundamental frequency of approximately 3.8 Hz, whereas the fundamental frequency of the smaller profile is approximately 2.3 Hz. Obviously, the dynamic behaviour of both structures is not optimal. A resonance frequency of 2.3 Hz, as well as some higher modes, are certainly excited by human activities.

Figure 5.3 shows the location of the six active vibration control devices. Each device is equipped with an auxiliary mass of 230 kg ($\mu = 1\%$). Using the estimated lifetime costs plus the costs of the auxiliary mass, the total cost per unit is CHF 2'180.- per unit. A total of six units lead to a cost surplus of CHF 13'080.- per story. Comparing this to table 5.2, a benefit of approximately CHF 3'800.- per story is achieved.

Obviously, both proposed structures fulfil the ULS and SLS-criterion according to the Swiss standards [3]. However, the lower profile height leads to a much lighter structure, which requires an additional external vibration mitigation. The stiffer profile is, due to its higher deadweight and resonance frequency much less subjected
Steel profile | IPE450 | HEB220
--- | --- | ---
Concrete thickness [mm] | 200 | 120
Steel Grade | S235 | S355
Effective slab width [m] | 4.2 | 4.2
Design bending moment $M_{y,Ed}$ [kNm] | 756.0 | 550.3
Section modulus $M_{ypl,Rd}$ [kNm] | 892.0 | 614.8
Second moment of area $I_y$ [$10^6 \, mm^4$] per slab | 1528.4 | 366.4
Steel deadweight [kg/m$^2$] | 77.6 | 71.5
Concrete volume [m$^3$/m$^2$] | 0.84 | 0.50
Costs steel [CHF/kg] | 3.50 | 3.85
Costs concrete [CHF/m$^3$] | 300 | 300
Total costs [CHF/m$^2$] | 523.6 | 425.3
Total steel costs/story (two floors) [CHF] | 89 760 | 72 908

| First resonance freq. (estimation) [Hz] | 3.83 | 2.29
| First resonance freq. (FEM) [Hz] | 4.01 | 2.37

Tab. 5.2: Comparison of an example steel structure in S235 and S355

to vibrations. However, the cost balance shows that it is beneficial to implement the lighter decking with active vibration control.

Additionally, the lower profile height allows building one story more within the boundary of a 30m height building. A profile IPE450 plus the concrete layer of 200 mm leads to a total story height of 3.6 m, whereas the lower profile HEB 220 add up with a concrete layer of 120 mm to a total story height of 3.26 m. Thus, eight stories of 3.6 m are approximately equal to nine stories of 3.26 m each. The yearly rental income of the additionally 720 m2 office space exceeds the yearly maintenance costs of the vibration control devices by far.

5.4 Conclusion

This chapter has analysed the costs and benefits of an application of active vibration control based on two example buildings. The analysis has shown that the costs of an active vibration control system can be equalized in some cases by the reduced material consumption. In the second example, a strong reduction of the building cost and an increase of the floor area in the building can be achieved by applying active vibration control of floors.
It should be noted that the applied costs are only rough estimations. The final building costs depend also strongly on auxiliary aspects like the building layout and utilization, the customers’ requests and wishes, the materialisation, the location and accessibility of the site, limitations in the building permit, etc.
Conclusion

This part presents an overall conclusion of this study and lists the most important findings. The main motivation for this study was to analyse whether the dynamic serviceability of lightweight floors can be improved by inertia-based vibration control. A comparison between passive and active systems served to show the huge advantages in performance and robustness of the active vibration control. Questions for the implementation consideration were analysed in detail.

The first part presented a broad overview of the dynamics of structures. It was shown that especially lightweight and wide-span structures are subjected to vertical vibrations. On the other hand, these structures are usually very cost- and resource-efficient. The dynamic properties of these structures depend strongly on the geometric layout, the support situation, the bending stiffness and the internal damping of the structure. Nevertheless, the dynamics of floor structure also strongly depend on the deadload and payload distribution and excitation forces. Part one introduced therefore the source-transmission-receiver-process as a transfer function from the excitation source to the observer or receiver. This process depends strongly on the dynamics of the structure and the location of the source and receiver. The best damping results are achieved if the vibration control device is in the same location as the observer (if possible). For the case of a moving observer, a trade-off between the dominant mode shapes should be considered as the location of the damping device. Modes with nodal points close to location of the damping device are nearly not mitigated.

A brief overview of the most important national and international standards for vibrations in buildings showed that the topic is usually not considered in the design of structures. However, the ongoing trend towards more lightweight structures and open-space-layouts will increase the necessity of profound dynamic studies in civil engineering. The dynamics of structures can be significantly improved by applying external damping devices. For the class of wide-span flat structures, only inertia-based damping devices can be applied for obvious reasons. Other external vibration control methods require a fixed-point or column for a proper working.
The second part of this study is dedicated to the mathematical modelling of dynamic systems and some basic formulations of the control theory. The brief introduction to LTI systems with one and multiple degrees of freedom has shown that the dynamics of floor structures can be modelled as a linear combination of single-degree-of-freedom systems. Thus, only the decisive (lowest) resonance modes need to be taken into account. The extension of the dynamic model with an inertia-based damping device has been introduced and the mathematical models were established for passive and active vibration control systems.

In the third part of this study, several-state-of-the-art control laws for active vibration control have been described. Together with some novel ideas which were developed in this study, the performance, robustness and limits of application have been compared. A comparison has shown that a genetic algorithm provides best damping performance (in absence of any robustness filters). However, other control laws have only slightly lower performance values. For example, the acceleration feedback according to Nishimura and the novel CVF show similar damping performance but much better robustness values.

The fourth part describes the experimental work of this study. Several experiments have been run to verify the results from theoretical findings and simulations. First experiments have been conducted on a concrete beam (bending-beam) with one dominant resonance frequency. The last experiment was conducted on a plate-type structure with a bi-axial load bearing. The novel concept of the CVF has been successfully tested on a system with multiple resonance frequencies. For the experimental part, a novel AMD prototype has been invented. Its design allows considerably reducing the required height and thus maximising the stroke of the auxiliary mass. Additionally, the stroke of the auxiliary mass and the actuator force are controlled to prevent saturation effects of the device. These measurements are used to limit dynamically the actuator force.

In the fifth part, a short economic analysis is provided. It is shown that under certain circumstances, the active vibration control can reduce the building costs by virtue of a reduced amount of building material. It strongly depends on different aspects if a cost benefit can be achieved. Thus, the costs and benefits must be analysed for every project.
6.1 Summary of important findings

This study allowed creating some important novel or unconventional finding in the field of the active vibration control using inertia-based damping devices. The most important findings are listed in the following paragraphs.

**Design limits of passive and semi-active vibration control** Passive vibration control devices are well adapted to time-invariant and harmonically excited systems. Semi-active devices show a good performance and robustness even in time-variant systems but have low performance in transient excitations. However, mitigating efficiently vibrations on a light-weight floor with different loading scenarios and multiple resonance modes requires an active vibration control device.

**Active vibration control - Combined velocity control** A novel control law, named combined velocity control, was established in this study. The goal of this control law is to establish a trade-off between a retaining viscous damping force of the auxiliary system and the driving viscous damping force of the main system. $H_{\infty}$- and $H_2$-optimal configurations have been derived from theoretical models. Simulations have confirmed the performance and efficiency of the control law. Finally, the combined velocity control has been verified on a beam-type and a plate-type structure. Results have shown a good accordance between simulations and experiments.

**Implementation and its limits** In contrast to a conventional passive TMD, the damping performance of an active vibration control device with CVF is quasi-independent of the mass ratio. However, smaller mass ratios lead to higher strokes of the auxiliary system. A trade-off between actuator stroke (or speed) and auxiliary mass must be defined during the design process. The maximum damping performance is directly linked to the auxiliary stroke and mass, as well as the actuator force. If the auxiliary system is saturated, the damping efficiency is reduced. However, stability issues due saturation effects were not detected in simulations or experiments using the CVF.

6.2 Limits of the application of active vibration control

It was shown in this study that every inertia-based vibration control device has a power dissipation saturation limit which depends on the travel stroke, the auxiliary mass, the auxiliary damping characteristics and the actuator force limitation. If a
harmonic excitation source with constant amplitude excites a vibrating structure and the amplitude exceeds the device limit, a constant part of the excitation force is dissipated in the vibration control device (saturation power). The remaining part of the excitation force leads to an excitation of the structure. It is therefore the duty of the designer to apply a suitable vibration control solution. For this purpose, the maximum occurring excitation force amplitudes and the according frequency bands need to be known.

6.3 Outlook and future work

This study has analysed the theoretical aspects of the active vibration control of lightweight floor structures. The experimental work has shown the efficiency of the novel combined velocity method. However, the prototype active vibration control device still has some weak points which needs improvement before the implementation in a real building.

- Complex mechanical part (requires maintenance)
- High inherent auxiliary damping. An equivalent damping of 18% of the natural damping has been identified. To reduce the actuator force, the damping needs to be reduced.
- Dry friction of the auxiliary system. The dry friction introduces non-linearities which are not modelled and are a possible source of instabilities.
- Filtering introduces phase delay. The phase delay is a possible source of instabilities. More sophisticated filtering techniques may reduce the phase delay.
Appendices

A Routh-Hurwitz Stability Criterion

A dynamic system is stable if the zeros of the denominator polynomial with real coefficients lie in the left-hand side of the complex plane \([43]\). This means that the real part of the zeros must be negative. Let \(A(s) = \sum a_n s^n\) with \(a_n > 0\) be the denominator polynomial of a system. The according square k-by-k Hurwitz matrices \(H_k\) with \(k \in [1, n]\) are therefore:

\[
H_k = \begin{bmatrix}
    a_{k-1} & a_{k-3} & a_{k-5} & a_{k-7} & \cdots \\
    a_k & a_{k-2} & a_{k-4} & a_{k-6} & \cdots \\
    0 & a_{k-1} & a_{k-3} & a_{k-5} & \cdots \\
    0 & a_k & a_{k-2} & a_{k-4} & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix} \quad (A.1)
\]

Obviously, elements with negative indices are replaced with zeros. Closed-loop stability is given if and only if all determinants of the k-by-k sub-matrices \(H_k\) are strictly positive.

\[
\det(H_k) > 0 \quad \forall \quad k \in [1, n] \quad (A.2)
\]
This section provides some closed form solutions for the 2 norm of a SISO system which are used in this work. The integrals are part of the book "Random vibration in mechanical systems" [20]. A more extensive table of integrals is found in the book "Theory of Servomechanisms" [29]. Let 

\[ H(s) = \frac{B_0}{A_1 s + A_0} \]

(B.3)

\[ H(s) = \frac{B_1 s + B_0}{A_2 s^2 + A_1 s + A_0} \]

(B.4)

\[ H(s) = \frac{B_2 s^2 + B_1 s + B_0}{A_3 s^3 + A_2 s^2 + A_1 s + A_0} \]

(B.5)

\[ H(s) = \frac{B_3 s^3 + B_2 s^2 + B_1 s + B_0}{A_4 s^4 + A_3 s^3 + A_2 s^2 + A_1 s + A_0} \]

(B.6)
This section provides the development of the integral-derivative of the cost function $J$ for the LQR control.

$$J = \int_{t_1}^{t_2} [x^T(t) (Q + K^T R K) x(t)] \, dt$$

$$= -\frac{1}{2} x_0^T (Q + K^T R K)(A_{ss} - B_{ss} K)^{-1}) x_0$$

Assuming that the matrix $\bar{Q} = Q + K^T R K$ is quadratic, diagonal and at least positive semi-definite and the matrix $\bar{A} = A - BK$ is quadratic and positive definite. In this case of an LTI system the matrices $\bar{Q}$ and $\bar{A}$ are constant in time and we know that:

$$J = \int_{t_1}^{t_2} x^T(t) \bar{Q} x(t) \, dt$$

$$= \int_{t_1}^{t_2} x^T(t) \bar{Q}^{-1} x(t) \, dt$$

$$= \frac{1}{2} \int_{t_1}^{t_2} \dot{x}^T(t) \bar{A}^{-T} \bar{Q}^{-1} \bar{A} \dot{x}(t) + x^T(t) \bar{Q} \bar{A}^{-1} \dot{x}(t) \, dt$$

$$= \frac{1}{2} \left[ x(t)^T \bar{Q} \bar{A}^{-1} x(t) \right]_{t_1}^{t_2}$$

For an infinite time horizon and a stable system, we know that $\lim_{t_2 \to \infty} x(t) = 0$. In this case we can write:

$$J = x_0^T \frac{\bar{Q} \bar{A}^{-1}}{2} x_0$$

The differential of $dJ$ is:

$$dJ(K) = x_0^T d(\bar{Q} \bar{A}^{-1}) x_0$$

$$= x_0^T \left( d\bar{Q} \bar{A}^{-1} + \bar{Q} \, d(\bar{A}^{-1}) \right) x_0$$

$$= x_0^T \left( d\bar{Q} \bar{A}^{-1} \bar{A} - \bar{Q} \, d\bar{A} \bar{A}^{-1} \right) x_0$$

$$= x_0^T \left( d\bar{Q} \, (d\bar{A})^{-1} \bar{A} - \bar{Q} \right) \bar{A}^{-1}(-B)dK \bar{A}^{-1} x_0$$

(C.9)

If either $S = 0$, $\bar{A}^{-1} = 0$ or $B = 0$, the energy function is minimal. Obviously, only the first expression leads to a reasonable result. For the case where $\bar{Q}$ and $\bar{A}$ are positive definite.
$
\bar{A}$ are positive definite the expression in brackets must be zero. Using the already mentioned definition $\bar{Q} = Q + KT RK$ and $\bar{A} = A - BK$ this leads to:

$$
S = d\bar{Q} (d\bar{A})^{-1} \bar{A} - \bar{Q}
= (K^T RdK + dK^T RK)(-BdK)^{-1}(A - BK) - Q - K^T RK
= - K^T RB^{-1} A + K^T RK - ((dK)^{-1}B^{-1}(A - BK))^T (dK^T RK) - Q - K^T RK
= - K^T RB^{-1} A - A^T B^{-T} RK + K^T RK - Q
$$

(C.10)

Applying the definition $K = R^{-1} B^T P$ and the fact that $R$ is diagonal (C.10) yields:

$$
S = - K^T RB^{-1} A - A^T B^{-T} RK + K^T RK - Q
= - P^T A - A^T P + P^T BR^{-1} B^T P - Q
$$

(C.11)

If this equation is fulfilled, the cost function is minimal. The solution of this algebraic Riccati equation in $P$ leads to the desired controller which minimises the cost function.
D Vibrations of a Euler-Bernoulli beam

The Bernoulli-Euler bending beam is a special case of the Timoshenko beam theory. It is characterised by a negligible shear force and an important bending moment. The length and position of the neutral axis of the beam remains unchanged under load and the cross-section is perpendicular to the neutral axis [60].

The dynamic behaviour of the Bernoulli-Euler beam is governed by the following Euler-Lagrange equation [28], which describes the difference between kinetic and potential energy. The third term describes the external energy due to the point load $q$.

$$L = \frac{\rho S (\ddot{y}(x,t))^2}{2} - \frac{EI}{2} \left( \frac{\partial^2 y(x,t)}{\partial x^2} \right)^2 + p(x,t)y(x,t)$$ (D.12)

Lagrange’s equation of the second kind for a function with second-order derivatives is given by [60]:

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = \frac{\partial L}{\partial y} + \frac{\partial^2 L}{\partial x^2} \left( \frac{\partial}{\partial \left( \frac{\partial^2 y}{\partial x^2} \right)} \right)$$

$$\rho S \ddot{y}(x,t) = p(x,t) - \frac{\partial^2}{\partial x^2} \left( EI \frac{\partial^2 y}{\partial x^2} \right)$$ (D.13)

Equation (D.13) can be solved using a separation of variables. This is the standard solution method for this kind of problem. For any case where the coordinates $x$ are constant in time, the following approach is valid:

$$y(x,t) = q(x) \cdot r(t)$$ (D.14)

A time-invariant mode-shape function $q(x)$ is defined which depends only on the position $x$ along the beam. The function $r(t)$ is a time-dependant unknown function at a fixed position. By inserting (D.14) in (D.13) and assuming that $E$ and $I$ are constant over the whole length of the beam, we find:

$$EIr(t) \frac{d^4 q(x)}{dx^4} + \rho S \ddot{r}(t)q(x) = p(x,t)$$ (D.15)

A simple closed-form solution to this differential equation can be found for the free harmonic vibration ($p(x,t) = 0$). For this case a sinusoidal ansatz $r(t) = \sin(\omega t + \phi)$ is inserted in (D.15). The sinusoidal ansatz defines an arbitrary vibration with a
pulsation $\omega$ and a constant phase delay $\phi$. The equation must be valid for any time-invariant mode-shape function $q(x) \neq 0$.

$$\frac{d^4 q(x)}{dx^4} - \frac{\rho S \omega^2}{EI} q(x) = 0 \quad (D.16)$$

The case of the simply supported beam leads to the following four well known boundary conditions for the mode shape function $q(x)$:

1) No deflection at $x = 0$ $q(0) = 0$
2) No deflection at $x = L$ $q(L) = 0$
3) No curvature at $x = 0$ $q''(0) = 0$
4) No curvature at $x = L$ $q''(L) = 0$

The standard solution which fulfils the boundary conditions is:

$$q(x) = \sin \left( \frac{\pi k}{L} x \right) \quad \forall k \in \mathbb{R}_+ \quad (D.17)$$

It should be noted that the maximum amplitude (Euclidean norm) of the mode-shape function is normalised to 1. By substituting (D.17) and its fourth differential into (D.16), the equation of motion becomes:

$$\sin \left( \frac{\pi k}{L} x \right) \left( \left[ \frac{\pi k}{L} \right]^4 - \frac{\rho S \omega^2}{EI} \right) = 0 \quad (D.18)$$

The equation (D.18) must be true for any value of $x \in [0, L]$. Therefore, it is obvious that:

$$\omega_k = \sqrt{\frac{EI}{\rho S} k^2 \pi^2 \frac{\pi^2}{L^2}} \quad (D.19)$$

The trivial solution $k = 0$ has no physical meaning. The values $\omega_k$ are called natural or resonance frequencies and the corresponding function $q_k(x)$ is called the mode shape. Two fundamental properties of the mode shape functions are their uniqueness and orthogonality [28]. No linear combination of any mode shape can produce another mode shape function. Therefore, the motion of a structure can be modelled as a linear combination of one-degree-of-freedom (1DOF) oscillators, each with a specific resonance frequency and amplitude.

$$y(x, t) = \sum_k [q_k(x) \cdot \sin(\omega_k t + \phi_k)] \quad (D.20)$$
The modal amplitude depends strongly on the mass which is accelerated by the external excitation source for a given frequency. The fraction of the accelerated mass is called modal participation factor or - when multiplied by the total mass of the structure - the modal mass. It can be calculated by applying a kinetic energy balance for every mode. The kinetic energy of the vibrating beam must be equal to the energy in a 1DOF oscillator with the same modal properties and an equal amplitude. It is assumed that the material properties and cross-section are constant over the total length of the beam $L$.

$$E_{\text{kin}} = \frac{1}{2} m_{\text{mod}} \dot{y}_{\text{max}}^2 = \frac{1}{2} \int_0^L \rho(x) \cdot S(x) \cdot \dot{y}_{\text{max}}^2(x, t) \, dx$$

$$= \frac{\rho S}{2} \int_0^L \omega_k^2 q^2(x) \, dx$$

$$= \frac{\rho S \omega^2}{2} \int_0^L \sin^2 \left( \frac{\pi k}{L} x \right) \, dx$$

$$= \frac{\rho S L \omega^2}{4} = \frac{m \dot{y}_{\text{max}}^2}{4} \quad \forall k \in \mathbb{R}_+ \tag{D.21}$$

For the case of a simply-supported Euler-Bernoulli beam, every resonance mode has therefore the same modal mass of $m_{\text{mod}} = m/2$. More complex structures like floors have usually different modal participant factors for every mode. The generalised modal mass is established by ($q$ is the normalised mode shape function):

$$m_k = \int_\Omega \rho q_k^2 \, d\Omega \quad \tag{D.22}$$

The calculation of the Euler-Bernoulli beam shows the difficulties which arise already in simple examples. More complex systems are only accessible by finite element (FE) calculation and other numerical techniques. The interested reader is referred to specific literature [28].
### Controller

The controllers are implemented as a 2x2-matrix using the following definition: \( K = \begin{bmatrix} K_1 & K_2 \\ -K_1 & -K_2 \end{bmatrix} \)

<table>
<thead>
<tr>
<th>Method</th>
<th>( K_1 )</th>
<th>( K_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>ATMD</td>
<td>0.002097s + 0.01202</td>
<td>-0.002097s - 0.01202</td>
</tr>
<tr>
<td>LQR</td>
<td>0.09999s + 1.6e-05</td>
<td>-0.003162s + 2.242e-08</td>
</tr>
<tr>
<td>DVF</td>
<td>0.05s</td>
<td>0</td>
</tr>
<tr>
<td>DAF</td>
<td>-0.5s^2</td>
<td>0</td>
</tr>
<tr>
<td>CVF</td>
<td>0.2305s</td>
<td>-0.004454s</td>
</tr>
<tr>
<td>Nishimura</td>
<td>-0.192s^2 + 0.0055s + 0.008332</td>
<td>-0.0055s - 0.008332</td>
</tr>
<tr>
<td>EDF</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>SFC</td>
<td>-0.99s^6 - 71.42s^5 - 313s^4 + 1459s^3 + 3388s^2 - 278.6s -800.2s^4 + 139.3s^3 + 5185s^2 + 2.344e04s + 2.667e04</td>
<td>0</td>
</tr>
<tr>
<td>FCCO</td>
<td>(-0.1036s^8 - 414.5s^7 - 4.974e08s^6 + 9.89e11s^5 - 3.988e15s^4 + 7.959e18s^3 - 7.892e21s^2 + 1.598e25s - 7.098e12) (s^8 + 1.467e04s^7 + 1.093e08s^6 + 4.108e11s^5 + 1.046e15s^4 + 1.644e18s^3 + 1.495e21s^2 + 9.398e23s + 8.638e25)</td>
<td>(0.07244s^8 - 7.23e04s^7 + 3.092e08s^6 - 2.109e12s^5 + 4.264e15s^4 - 7.35e18s^3 + 1.203e22s^2 - 3.084e23s - 2.054e11) (s^8 + 1.467e04s^7 + 1.093e08s^6 + 4.108e11s^5 + 1.046e15s^4 + 1.644e18s^3 + 1.495e21s^2 + 9.398e23s + 8.638e25)</td>
</tr>
<tr>
<td>FCGP</td>
<td>(s^2(-6.367s^2 + 9.248s + 6.923)) (s^2 + 0.4883s + 0.05185)</td>
<td>(s^2(-0.0006803s^2 + 0.8458s + 0.9714)) (s^2 - 3.387s - 1.506)</td>
</tr>
</tbody>
</table>

**Tab. D.1**: Applied controllers for simulation and comparison in section 3.3. The estimated disturbance feedback is not implemented using a separate controller \( K \) due to the feedback delay. Instead, the closed-loop transfer function is simulated in a discretised form.
Bibliography


Curriculum Vitae

Jakob Friedrich Baader

Date of birth 15th July 1986
Place of birth Liestal, Switzerland
Nationality Swiss
Marital status married

since 2015 PhD student at ETH Zurich (part-time)
Institute of Structural Engineering (IBK)
in the group of Prof. Dr. M. Fontana

since 2011 Project manager and structural engineer (part-time) at
K.Bitterli + Partner Ingenieure AG, Gelterkinden

2011 Master degree in mechanical engineering EPF Lausanne
Diploma thesis at the automatic control laboratory
(laboratoire d’automatique, LA) entitled "Simulation
of a car equipped with an active suspension system"

2006 - 2011 Studies in mechanical engineering at the EPF Lausanne
Specialisation in control systems, design and production

2006 Baccalaureate