



## Journal Article

### **Tower of subleading dual BMS charges**

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## Tower of subleading dual BMS charges

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**ABSTRACT:** We supplement the recently found dual gravitational charges with dual charges for the whole BMS symmetry algebra. Furthermore, we extend the dual charges away from null infinity, defining subleading dual charges. These subleading dual charges complement the subleading BMS charges in the literature and together account for all the Newman-Penrose charges.

**KEYWORDS:** Classical Theories of Gravity, Space-Time Symmetries

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**1 Introduction**

In recent work [1, 2], undertaken with the aim of providing a clear and explicit relation between the asymptotic BMS symmetry and gravitational charges of asymptotically flat spacetimes, we generalised the notion of BMS charges, as defined by Barnich and Troessaert [3], in two complementary ways. The Barnich-Troessaert BMS charges are derived from the general prescription of Barnich-Brandt [4] for defining asymptotic charges,<sup>1</sup> which in this case turns out to be given by the integral of the Hodge dual of a 2-form  $H$  over a 2-sphere at null infinity:<sup>2</sup>

$$\delta Q_0[\xi, g, \delta g] = \frac{1}{8\pi G} \lim_{r \rightarrow \infty} \int_S \star H[\xi, g, \delta g], \tag{1.1}$$

where  $\xi$  is the asymptotic symmetry generator,  $g$  is the background metric and  $\delta g$  is its variation. The variation symbol  $\delta$  denotes the fact that the charge is not necessarily integrable. This is generally due to gravitational flux at null infinity.

In ref. [1], we extended the notion of BMS charges by defining subleading BMS charges as a  $1/r$  expansion of the general prescription of Barnich-Brandt [4], so that<sup>3</sup>

$$\delta Q = \delta Q_0 + \frac{\delta Q_1}{r} + \frac{\delta Q_2}{r^2} + \frac{\delta Q_3}{r^3} + \dots, \tag{1.2}$$

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<sup>1</sup>There exists an equivalent formalism for defining asymptotic charges, developed by Wald and collaborators [5, 6]. Here, we shall continue to work in the framework of the Barnich-Brandt formalism.

<sup>2</sup>For an explicit expression for  $H$ , see equation (3.3).

<sup>3</sup>In principle, assuming analyticity, the tower of charges is infinite, with a charge at each order in the  $1/r$  expansion. However, for physical reasons the metric expansion may only be analytic up to a certain order in  $1/r$  (see e.g. refs. [7, 8]). The tower of BMS charges will then naturally truncate at some corresponding order.

and we showed that the  $O(1/r^3)$  term gives five of the ten non-linear Newman-Penrose (NP) charges [9]. Writing these charges in the Newman-Penrose formalism [10], we found that these five components, derived from the generalised BMS charge, correspond in some sense to the *real* part of the NP charges. An obvious question, then, is how do the other five *imaginary* parts of the NP charges fit into this understanding of generalised BMS charges?

In ref. [2], inspired by the situation for electromagnetism [11, 12], which allows for *electric* as well as *magnetic* charges, we defined new *dual* gravitational charges, associated with supertranslations, as the integral of the Barnich-Brandt 2-form  $H$  itself (as opposed to its dual):

$$\delta\tilde{\mathcal{Q}}_0[\xi, g, \delta g] = \frac{1}{8\pi G} \lim_{r \rightarrow \infty} \int_S H[\xi, g, \delta g]. \quad (1.3)$$

Together,  $\delta\mathcal{Q}_0$  and  $\delta\tilde{\mathcal{Q}}_0$  can be viewed as the real and imaginary parts of a leading-order supertranslation charge, which can succinctly be written in terms of the leading-order terms in a  $1/r$  expansion of the Newman-Penrose scalar  $\Psi_2$ , which is a certain null-frame component of the Weyl tensor, and  $\sigma$ , which parametrises the shear of the null congruence of  $\ell = \partial/\partial r$ . Thus, there is an attractive correspondence between, on the one hand, the real and imaginary parts of charges written in the complex Newman-Penrose formalism, and on the other hand “electric” and “magnetic” (or “dual”) BMS charges defined *à la* Barnich-Brandt.

In this paper, we shall address the problem of generalising the dual charge  $\delta\tilde{\mathcal{Q}}_0$  of ref. [2] to a tower of dual BMS charges (both  $SL(2, \mathbb{C})$  and supertranslation charges) as a series in powers of  $1/r$ , in the same manner as the Barnich-Troessaert charge  $\delta\mathcal{Q}_0$  was generalised in ref. [1]. The reason why in ref. [2] we were able to construct the dual charge at infinity (i.e. at the order  $1/r^0$ ) by integrating the Barnich-Brandt 2-form  $H$  as in equation (1.3) was that if one takes  $\delta g$  to be given by the action of the supertranslation generators, then  $\delta\tilde{\mathcal{Q}}_0[\xi, g, \delta_\xi g]$  defined in (1.3) vanishes on-shell. However, beyond the leading order, and including the  $SL(2, \mathbb{C})$  part of the BMS group, it was established in ref. [2] that the corresponding variations of the subleading terms in the  $1/r$  expansion of the integral of  $H$  do not vanish on-shell, and thus one does not get *bona fide* subleading charges by this means.

The question that we shall now address here is how does one generalise the dual charge (1.3) to the full BMS group and to subleading orders in a  $1/r$  expansion away from null infinity? Such a construction should provide an answer to the question raised by the results of ref. [1], i.e. it should presumably explain how the other five *imaginary* parts of the NP charges come about.

We shall construct a tower of *bona fide* dual gravitational BMS charges as a  $1/r$  expansion away from null infinity, and we shall show that this does, in particular, give rise at the order  $1/r^3$  to the five *imaginary* parts of the NP charges. The tower of dual charges is given in terms of a new 2-form  $\tilde{H}$ , such that

$$\delta\tilde{\mathcal{Q}}[\xi, g, \delta g] = \frac{1}{8\pi G} \int_S \tilde{H}[\xi, g, \delta g] = \delta\tilde{\mathcal{Q}}_0 + \frac{\delta\tilde{\mathcal{Q}}_1}{r} + \frac{\delta\tilde{\mathcal{Q}}_2}{r^2} + \frac{\delta\tilde{\mathcal{Q}}_3}{r^3} + \dots \quad (1.4)$$

Crucially, we construct  $\tilde{H}$  by requiring that the integral in (1.4) should vanish on-shell when  $\delta g$  is taken to be given by the action of the BMS asymptotic symmetry generators,

i.e.  $\delta\tilde{\mathcal{Q}}[\xi, g, \mathcal{L}_\xi g] = 0$ . It turns out that this condition uniquely defines  $\tilde{H}$ . Moreover, the more general condition that the central extension must be antisymmetric [4], i.e.  $\delta\tilde{\mathcal{Q}}[\xi, g, \mathcal{L}_\xi g] = -\delta\tilde{\mathcal{Q}}[\zeta, g, \mathcal{L}_\zeta g]$  is also satisfied. Furthermore, properties of the BMS group ensure the existence of a charge algebra [4]. The 2-form  $\tilde{H}$  that we find turns out to be equal to the Barnich-Brandt 2-form  $H$  only at leading order and only for supertranslations

$$\lim_{r \rightarrow \infty} (\tilde{H} - H) = 0. \tag{1.5}$$

Thus (1.4) gives the same leading-order result that we found in ref. [2], but now, we are able to extend the construction of dual gravitational charges to the full BMS group and to all subleading orders in a  $1/r$  expansion.

In section 2, we give some preliminary prerequisite information regarding asymptotically flat spacetimes. For a more detailed exposition of the notations and conventions we are using here, the reader is referred to section 2 of ref. [1]. In section 3, we define the dual gravitational charge corresponding to the full BMS group. We find the full dual BMS charge at leading order in section 4 and investigate the dual charges associated with supertranslations up to order  $1/r^3$  in a  $1/r$  expansion in section 5. The results of this section are analogous to those obtained for the subleading BMS charges defined in ref. [1]. Perhaps, most significantly, in section 5.3 we find that the dual charge at order  $1/r^3$  gives the imaginary parts of the NP charges, something that was missing in the analysis of ref. [1]. We finish with some comments in section 6.

## 2 Preliminaries

We define asymptotically flat spacetimes as a class of spacetimes for which Bondi coordinates  $(u, r, x^I = \{\theta, \phi\})$  may be defined, such that the metric takes the form [13, 14]

$$ds^2 = -Fe^{2\beta} du^2 - 2e^{2\beta} dudr + r^2 h_{IJ} (dx^I - C^I du)(dx^J - C^J du), \tag{2.1}$$

with the metric functions satisfying the following fall-off conditions<sup>4</sup> at large  $r$

$$\begin{aligned} F(u, r, x^I) &= 1 + \frac{F_0(u, x^I)}{r} + o(r^{-1}), \\ \beta(u, r, x^I) &= \frac{\beta_0(u, x^I)}{r^2} + o(r^{-2}), \\ C^I(u, r, x^I) &= \frac{C_0^I(u, x^I)}{r^2} + \frac{\log r}{r^3} D_J B^{IJ} + \frac{C_1^I(u, x^I)}{r^3} + o(r^{-3}), \\ h_{IJ}(u, r, x^I) &= \omega_{IJ} + \frac{C_{IJ}(u, x^I)}{r} + o(r^{-1}), \end{aligned} \tag{2.2}$$

where  $D_I$  is the standard covariant derivative associated with the unit round-sphere metric  $\omega_{IJ}$  with coordinates  $x^I = \{\theta, \phi\}$  on the 2-sphere.  $B^{IJ}$  and  $C_{IJ}$  are symmetric tensors

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<sup>4</sup>We require even weaker fall-off conditions for leading dual BMS charges, *viz.*  $F = 1 + o(r^0)$  and  $\beta = o(r^{-1})$ . However, we choose fall-off conditions such that we have both dual and Barnich-Troessaert BMS charges at leading order. In section 5, we impose stronger conditions in order to allow for NP charges.

with indices raised/lowered with the (inverse) metric on the 2-sphere. Moreover, a residual gauge freedom allows us to require that

$$h = \omega, \tag{2.3}$$

where  $h \equiv \det(h_{IJ})$  and  $\omega \equiv \det(\omega_{IJ}) = \sin \theta$ .

We assume, furthermore, that the components  $T_{00}$  and  $T_{0m}$  of the energy-momentum tensor in the null frame fall off as

$$T_{00} = o(r^{-4}), \quad T_{0m} = o(r^{-3}). \tag{2.4}$$

The Einstein equation then implies that

$$G_{00} = o(r^{-4}) \implies \beta_0 = -\frac{1}{32} C^2, \tag{2.5}$$

$$G_{0m} = o(r^{-3}) \implies C_0^I = -\frac{1}{2} D_J C^{IJ}, \tag{2.6}$$

where  $C^2 \equiv C_{IJ} C^{IJ}$ .

The asymptotic symmetry group corresponding to asymptotically flat spacetimes is the BMS group [13, 14], whose corresponding algebra is generated by

$$\xi = f \partial_u + \xi^I \partial_I - \frac{r}{2} (D_I \xi^I - C^I D_I f) \partial_r, \tag{2.7}$$

where

$$\xi^I = Y^I - \int_r^\infty dr' \frac{e^{2\beta}}{r'^2} h^{IJ} D_J f, \quad f = s + \frac{u}{2} D_I Y^I. \tag{2.8}$$

The  $Y^I$  are the set of conformal Killing vectors on the round unit 2-sphere, obeying

$$D_{(I} Y_{J)} = \frac{1}{2} D_K Y^K \omega_{IJ}, \tag{2.9}$$

and  $s(x^I)$  is an arbitrary function that depends only on the angular coordinates and generates angle-dependent translations in the  $u$ -direction, which are called supertranslations (ST). Thus,

$$\text{BMS} = \text{SL}(2, \mathbb{C}) \ltimes \text{ST}. \tag{2.10}$$

The Abelian part of the algebra, generated by the supertranslations, is

$$\xi = s \partial_u - \int_r^\infty dr' \frac{e^{2\beta}}{r'^2} h^{IJ} D_J s \partial_I - \frac{r}{2} (D_I \xi^I - C^I D_I s) \partial_r. \tag{2.11}$$

As will become clear in what follows, it will be convenient to define twisted/dualised objects, as follows: for some symmetric tensor  $X_{IJ}$ , we define its *trace-free* twist/dual by

$$\tilde{X}^{IJ} = X_K^{(I} \epsilon^{J)K}, \quad \epsilon_{IJ} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin \theta. \tag{2.12}$$

If  $X_{IJ}$  is, furthermore, trace-free, i.e.  $\omega^{IJ} X_{IJ} = 0$ , then  $X_K^{[I} \epsilon^{J]K} = 0$ , so  $\tilde{X}^{IJ}$  is symmetric without the need for explicit symmetrisation and we can simply write

$$\tilde{X}^{IJ} = X_K^I \epsilon^{JK}. \tag{2.13}$$

If  $X$  and  $Y$  are two symmetric trace-free tensors, then

$$X_{IK}\tilde{Y}^{JK} = -\tilde{X}_{IK}Y^{JK}. \quad (2.14)$$

Furthermore, if either one of the symmetric tensors  $X$  or  $Y$  is trace-free, then

$$X_{IJ}\tilde{Y}^{IJ} = -\tilde{X}_{IJ}Y^{IJ}. \quad (2.15)$$

### 3 Dual BMS charges

We define the dual BMS charge to be

$$\delta\tilde{Q}[\xi, g, \delta g] = \frac{1}{8\pi G} \int_S \tilde{H}[\xi, g, \delta g] = \frac{1}{8\pi G} \int_S d\Omega \frac{\tilde{H}_{\theta\phi}}{\sin\theta}, \quad (3.1)$$

where

$$\tilde{H} = \frac{1}{2} \left\{ \xi^c \nabla_b \delta g_{ac} - \frac{1}{2} \delta g_{bc} (\nabla_a \xi^c - \nabla^c \xi_a) \right\} dx^a \wedge dx^b. \quad (3.2)$$

This may be compared with the Barnich-Brandt 2-form  $H$  [4]:

$$H = \frac{1}{2} \left\{ \xi_b g^{cd} \nabla_a \delta g_{cd} - \xi_b \nabla^c \delta g_{ac} + \xi^c \nabla_b \delta g_{ac} + \frac{1}{2} g^{cd} \delta g_{cd} \nabla_b \xi_a + \frac{1}{2} \delta g_{bc} (\nabla_a \xi^c - \nabla^c \xi_a) \right\} dx^a \wedge dx^b. \quad (3.3)$$

We found a unique expression for  $\tilde{H}$  by parameterising the most general possible covariant 2-form, built from terms bilinear in  $\xi$  and  $\delta g$  and involving one covariant derivative, and determining the constant coefficients by requiring that its integral (3.1) should vanish on-shell when  $\delta g$  is given by the action of an asymptotic symmetry generator, i.e.  $\delta\tilde{Q}[\xi, g, \mathcal{L}_\xi g] = 0$ , where

$$\mathcal{L}_\xi g_{ab} = 2\nabla_{(a}\xi_{b)}. \quad (3.4)$$

(Essentially, this amounted to putting arbitrary coefficients for the terms in the expression (3.3) of the Barnich-Brandt 2-form, and solving for them by imposing the on-shell vanishing requirement.)

The general expression for the variation of a quantity  $\mathcal{Q}$  is

$$\delta\mathcal{Q} = \delta\mathcal{Q}^{(\text{int})} + \mathcal{N}, \quad (3.5)$$

where  $\delta\mathcal{Q}^{(\text{int})}$  is the integrable part, i.e. the “time derivative”, while the non-integrable term  $\mathcal{N}$  quantifies the flux out of the system. If  $\delta\mathcal{Q}[\xi, g, \mathcal{L}_\xi g] = 0$  on-shell, then we have a continuity equation and hence a charge corresponding to that asymptotic symmetry generator.

This reflects the fact that an asymptotic charge is not necessarily exactly conserved, viz. its time derivative is not necessarily zero, because of the existence of flux out of the system. Therefore, in this context we can define a charge if the quantity satisfies the analogue of a continuity equation, i.e. the charge changes by an amount given by the flux flowing out of the system. In appendix A, we show that  $\delta\tilde{Q}[\xi, g, \mathcal{L}_\xi g] = 0$ , giving rise to a charge  $\tilde{Q}^{(\text{int})}$ . Furthermore, in appendix B, we verify that  $\delta\tilde{Q}[\xi, g, \mathcal{L}_\xi g] = -\delta\tilde{Q}[\zeta, g, \mathcal{L}_\xi g]$ . This together with the fact that the asymptotic symmetry generators belong to the BMS group implies that the charges defined by the asymptotic symmetry generators belong to a charge algebra [4].<sup>5</sup>

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<sup>5</sup>See, in particular, Theorem 4 of ref. [4]. Note that this theorem is labelled Theorem 2 in the published version. For earlier results on this, in the context of the canonical formalism, we refer the reader to ref. [15]. We postpone a detailed study of the dual charge algebra to a future work.

In fact all the features of the new dual charges we have introduced in this paper are precisely analogous to those one encounters for the standard BMS charges, defined from the Barnich-Brandt 2-form. For example, at the leading  $1/r^0$  order the variation  $\delta\mathcal{Q}$  for the standard BMS charges also is non-integrable in general, owing to the presence of the Bondi news term  $\mathcal{N}$  [3]. One identifies the integrable part  $\delta\mathcal{Q}^{(\text{int})}$  of the variation as defining the BMS charge  $\mathcal{Q}^{(\text{int})}$  at infinity; it is conserved if the Bondi news vanishes.

Regarded as a  $1/r$  expansion away from null infinity, we have<sup>6</sup>

$$\delta\tilde{\mathcal{Q}}[\xi, g, \delta g] = \frac{1}{16\pi G} \int_S d\Omega \left\{ \delta\tilde{\mathcal{I}}_0 + \frac{\delta\tilde{\mathcal{I}}_1}{r} + \frac{\delta\tilde{\mathcal{I}}_2}{r^2} + \frac{\delta\tilde{\mathcal{I}}_3}{r^3} + o(r^{-3}) \right\}. \quad (3.6)$$

Hence, we find a tower of dual charges, which can be viewed as the charges dual to the BMS charges found in ref. [1]

$$\delta\mathcal{Q}[\xi, g, \delta g] = \frac{1}{8\pi G} \int_S \star H[\xi, g, \delta g] = \frac{1}{16\pi G} \int_S d\Omega \left\{ \delta\mathcal{I}_0 + \frac{\delta\mathcal{I}_1}{r} + \frac{\delta\mathcal{I}_2}{r^2} + \frac{\delta\mathcal{I}_3}{r^3} + o(r^{-3}) \right\}. \quad (3.7)$$

In particular, as we shall demonstrate in section 5.4,  $\tilde{\mathcal{I}}_3$  gives the five complementary NP charges [9] that were missing in the analysis of ref. [1].

We proceed to describe the leading-order dual charge for the full BMS group, before we investigate the subleading terms, corresponding to supertranslations only, in  $\delta\tilde{\mathcal{Q}}$  in the  $1/r$  expansion given in equation (3.6).

#### 4 Leading-order dual charges

In this section we derive the leading-order dual BMS charges. These charges ought to be viewed as duals of the Barnich-Troessaert charges found in ref. [3]. The dual charge as defined in ref. [2] by taking the integral of the Barnich-Brandt 2-form  $H$  (as opposed to the integral of its Hodge dual) cannot incorporate the  $\text{SL}(2, \mathbb{C})$  part of the BMS group. However, the dual charge defined in section 3 gives a charge for the full BMS group, as argued for in appendix A.

Using the definition of the asymptotic symmetry generator, given by equation (2.7) and the metric coefficients defined in equations (2.2), it is relatively simple to show that

$$\begin{aligned} \frac{\tilde{H}_{\theta\phi}}{\sin\theta} &= \frac{1}{2}\epsilon^{IJ} H_{IJ} \\ &= \frac{1}{2}\epsilon^{IJ} \left[ \xi^c \nabla_J \delta g_{Ic} - \frac{1}{2} \delta g_{Jc} (\nabla_I \xi^c - \nabla^c \xi_I) \right] \\ &= \frac{1}{2}\epsilon^{IJ} \left[ r \left( Y^K D_J \delta C_{IK} - \frac{1}{2} \delta C_{JK} (D_I Y^K - D^K Y_I) \right) + D_I (f \delta C_{0J}) - \frac{1}{4} D_I (Y_J \delta C^2) \right. \\ &\quad + D^K f D_I \delta C_{JK} + \frac{1}{2} f \partial_u C_{IK} \delta C_J^K + \frac{1}{2} Y^L \delta C_{JK} D_L C_I^K + \frac{1}{4} D_L Y^L C_{IK} \delta C_J^K \\ &\quad \left. + \frac{1}{2} C_{IL} \delta C_{JK} D^K Y^L - \frac{1}{2} C^{KL} \delta C_{JK} D_L Y_I \right] + O(1/r). \end{aligned} \quad (4.1)$$

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<sup>6</sup>See footnote 3.



Ignoring total derivatives, as these will integrate to zero, and freely integrating by parts, the expression above simplifies to

$$\begin{aligned}
 \frac{\tilde{H}_{\theta\phi}}{\sin\theta} &= \frac{1}{2}\epsilon^{IJ} \left[ \frac{1}{2}r\delta C_{JK}(D_I Y^K + D^K Y_I) - fD^K D_I \delta C_{JK} + \frac{1}{2}f\partial_u C_{IK}\delta C_J^K \right. \\
 &\quad + \frac{1}{4}Y^L \delta C_J^K D_L C_{IK} - \frac{1}{4}Y^L C_{IK} D_L \delta C_J^K \\
 &\quad \left. + \frac{1}{2}C_{IL}\delta C_{JK}D^K Y^L - \frac{1}{2}C^{KL}\delta C_{JK}D_L Y_I \right] + O(1/r) \\
 &= \frac{1}{2} \left[ r\delta\tilde{C}^{IJ}D_{(I}Y_{J)} - fD_I D_J \delta\tilde{C}^{IJ} + \frac{1}{2}f\partial_u C_{IJ}\delta\tilde{C}^{IJ} + \frac{1}{4}Y^K \delta(\tilde{C}^{IJ}D_K C_{IJ}) \right. \\
 &\quad \left. + \frac{1}{2}(C_{IK}\delta\tilde{C}^{JK} - C^{JK}\delta\tilde{C}_{IK})D_J Y^I \right] + O(1/r). \tag{4.2}
 \end{aligned}$$

Now, using equation (2.9) and the fact that  $\delta\tilde{C}^{IJ}$  is trace-free implies that the order  $r$  terms in the expression above vanish, as they should. Moreover, given that

$$C_{IK}\delta\tilde{C}_J^K = \frac{1}{2}C_{KL}\delta\tilde{C}^{KL}\omega_{IJ} - \frac{1}{4}\delta C^2\epsilon_{IJ}, \tag{4.3}$$

which can simply be derived from observing that the symmetric and antisymmetric parts of the expression on the left hand side of the above equation must be proportional to  $\omega_{IJ}$  and  $\epsilon_{IJ}$ , respectively, the expression for  $\tilde{H}_{\theta\phi}$  in equation (4.2) simplifies to

$$\frac{\tilde{H}_{\theta\phi}}{\sin\theta} = \frac{1}{2} \left[ -fD_I D_J \delta\tilde{C}^{IJ} + \frac{1}{2}f\partial_u C_{IJ}\delta\tilde{C}^{IJ} + \frac{1}{4}Y^K \delta(\tilde{C}^{IJ}D_K C_{IJ}) + \frac{1}{2}D_I \tilde{Y}^I \delta C^2 \right] + O(1/r), \tag{4.4}$$

where

$$\tilde{Y}^I = \epsilon^{IJ}Y_J. \tag{4.5}$$

In summary, we find that

$$\oint \tilde{\mathcal{Q}}_0 = \frac{1}{16\pi G} \int_S \left[ \delta \left( -fD_I D_J \tilde{C}^{IJ} + \frac{1}{4}Y^K \tilde{C}^{IJ} D_K C_{IJ} - \frac{1}{2}\tilde{Y}^I D_I C^2 \right) + \frac{1}{2}f\partial_u C_{IJ}\delta\tilde{C}^{IJ} \right]. \tag{4.6}$$

This dual charge may be compared with the charge in [3]<sup>7</sup>

$$\oint \mathcal{Q}_0 = \frac{1}{16\pi G} \int_S \left[ \delta \left( -2fF_0 + Y^K \left[ -3C_{1K} + \frac{1}{16}D_K C^2 \right] \right) + \frac{1}{2}f\partial_u C_{IJ}\delta C^{IJ} \right]. \tag{4.7}$$

<sup>7</sup>See equations (3.2) and (3.3) of ref. [3] with the following translations in notation:  $M = -1/2 F_0$  and  $N^I = -3/2 C_1^I$ .

## 5 Subleading dual charges

In the previous section, we computed the leading-order dual BMS charge for the full BMS group. In this section, for simplicity, we restrict ourselves to the most distinctive part of the BMS group, given by supertranslations, and compute the subleading charges. Thus, hereafter, the generators that will be of interest are those given by equation (2.11).

Furthermore, in this section we require stronger fall-off conditions:

$$\begin{aligned}
 F(u, r, x^I) &= 1 + \frac{F_0(u, x^I)}{r} + \frac{F_1(u, x^I)}{r^2} + \frac{F_2(u, x^I)}{r^3} + \frac{F_3(u, x^I)}{r^4} + o(r^{-4}), \\
 \beta(u, r, x^I) &= \frac{\beta_0(u, x^I)}{r^2} + \frac{\beta_1(u, x^I)}{r^3} + \frac{\beta_2(u, x^I)}{r^4} + o(r^{-4}), \\
 C^I(u, r, x^I) &= \frac{C_0^I(u, x^I)}{r^2} + \frac{C_1^I(u, x^I)}{r^3} + \frac{C_2^I(u, x^I)}{r^4} + \frac{C_3^I(u, x^I)}{r^5} + o(r^{-5}), \\
 h_{IJ}(u, r, x^I) &= \omega_{IJ} + \frac{C_{IJ}(u, x^I)}{r} + \frac{C^2 \omega_{IJ}}{4r^2} + \frac{D_{IJ}(u, x^I)}{r^3} + \frac{E_{IJ}(u, x^I)}{r^4} + o(r^{-4}). \tag{5.1}
 \end{aligned}$$

Further to the fall-off conditions (2.4), we require

$$T_{00} = o(r^{-5}), \quad T_{0m} = o(r^{-3}), \tag{5.2}$$

which implies

$$G_{00} = o(r^{-5}) \implies \beta_0 = -\frac{1}{32} C^2, \quad \beta_1 = 0, \tag{5.3}$$

$$G_{0m} = o(r^{-3}) \implies C_0^I = -\frac{1}{2} D_J C^{IJ}. \tag{5.4}$$

These stronger fall-off conditions are needed for the existence of NP charges, whose origin we explain in terms of subleading BMS and dual BMS charges in this section. These conditions allow us to define charges up to order  $1/r^3$ . In order to define yet further higher order charges we need to impose even stronger fall-off conditions.

### 5.1 Dual charge at $O(r^0)$

For the leading-order charge, the contribution of the supertranslations can be simply deduced from the general result (4.6) by turning off the  $SL(2, \mathbb{C})$  generators, i.e. the  $Y^I$ 's. Hence, from equation (2.8),  $f = s$  and charge (4.6) reduces to

$$\delta \tilde{\mathcal{I}}_0 = \delta \left( -s D_I D_J \tilde{C}^{IJ} \right) + \frac{s}{2} \partial_u C_{IJ} \delta \tilde{C}^{IJ}. \tag{5.5}$$

This agrees with the result of ref. [2] since at the leading order, the 2-forms  $H$  and  $\tilde{H}$  coincide

$$\lim_{r \rightarrow \infty} (\tilde{H} - H) = 0. \tag{5.6}$$

As emphasised in ref. [2], the leading-order dual charge is integrable if and only if

$$\partial_u C_{IJ} = 0, \tag{5.7}$$

i.e. the Bondi news vanishes. Recall from section 3.1 of ref. [1] that an equivalent statement holds for the Barnich-Troessaert charge  $\delta\mathcal{I}_0$  [3]:

$$\delta\mathcal{I}_0 = \delta(-2sF_0) + \frac{s}{2}\partial_u C_{IJ}\delta C^{IJ}. \quad (5.8)$$

Moreover, as discussed in ref. [2], the integrable parts of the two sets of charges may be written as the real and imaginary parts of a single expression<sup>8</sup>

$$\mathcal{Q}_0 = -\frac{1}{4\pi G} \int d\Omega s (\psi_2^0 + \sigma^0 \partial_u \bar{\sigma}^0). \quad (5.9)$$

More precisely (recalling that  $s$  is a real quantity),

$$\mathcal{Q}_0 = \mathcal{Q}_0^{(\text{int})} - i\tilde{\mathcal{Q}}_0^{(\text{int})}, \quad (5.10)$$

where, from equations (5.8) and (5.5), respectively,

$$\mathcal{Q}_0^{(\text{int})} = \frac{1}{16\pi G} \int d\Omega (-2sF_0), \quad \tilde{\mathcal{Q}}_0^{(\text{int})} = \frac{1}{16\pi G} \int d\Omega (-sD_I D_J \tilde{C}^{IJ}). \quad (5.11)$$

### 5.2 Dual charge at $\mathcal{O}(r^{-1})$

At the next order a simple, if rather tedious, calculation shows that up to total derivatives, which will vanish under integration,

$$\delta\tilde{\mathcal{I}}_1 = 0. \quad (5.12)$$

Recall (see section 3.2 of ref. [1]) that assuming

$$T_{01} = o(r^{-4}), \quad (5.13)$$

the Einstein equation implies that

$$\delta\mathcal{I}_1 = 0. \quad (5.14)$$

More generally, without assuming equation (5.13), we may equivalently define

$$\mathcal{Q}_1 = -\frac{1}{8\pi G} \int d\Omega s (\psi_2^1 - \bar{\delta}\psi_1^0), \quad (5.15)$$

since

$$\Re(\psi_2^1 - \bar{\delta}\psi_1^0) = -\frac{1}{2}\mathcal{I}_1, \quad \Im(\psi_2^1 - \bar{\delta}\psi_1^0) = 0. \quad (5.16)$$

### 5.3 Dual charge at $\mathcal{O}(r^{-2})$

A similar long, but simple, calculation finds that

$$\begin{aligned} \delta\tilde{\mathcal{I}}_2 = & s D_I D_J \delta \left( -\tilde{D}^{IJ} + \frac{1}{16} C^2 \tilde{C}^{IJ} \right) \\ & + s \left( \frac{1}{2} \left[ \partial_u D_{IJ} \delta \tilde{C}^{IJ} - \delta D_{IJ} \partial_u \tilde{C}^{IJ} \right] - \frac{1}{16} C_{IJ} \left[ \partial_u C^2 \delta \tilde{C}^{IJ} - \delta C^2 \partial_u \tilde{C}^{IJ} \right] + D_I (C_{1J} \delta \tilde{C}^{IJ}) \right. \\ & \left. - \frac{1}{16} D_I (D_J C^2 \delta \tilde{C}^{IJ}) - \frac{1}{2} D_I (C_{JK} D_L C^{KL} \delta \tilde{C}^{IJ}) \right). \end{aligned} \quad (5.17)$$

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<sup>8</sup>See section 4 of ref. [1] for a brief introduction to the Newman-Penrose scalars.

Note that this is very similar to  $\delta\mathcal{L}_2$  (see equation (3.12) of ref. [1]). In particular, the integrable part of one is obtained by taking the twist of the tensor fields in the other. The non-integrable part provides an obstruction to the conservation of the integrable charge, and in analogy with the nomenclature adopted for the non-integrable parts of the charges at subleading order in ref. [1], we may describe such terms here as “twisted fake news.”

In what follows, we consider whether  $s$  can be appropriately chosen such that the twisted fake news vanishes, leaving a conserved/integrable charge. In order to proceed, we need to know how  $C_{IJ}$  and  $D_{IJ}$  transform under the action of the asymptotic symmetry group; these are given by equations (2.30) and (2.32) of ref. [1], which we reproduce here for convenience

$$\delta C_{IJ} = s\partial_u C_{IJ} + \square s \omega_{IJ} - 2D_{(I}D_{J)}s, \quad (5.18)$$

$$\begin{aligned} \delta D_{IJ} = s\partial_u D_{IJ} + & \left[ \frac{1}{16}C^2\square s - \frac{1}{16}D^K C^2 D_{KS} - \frac{1}{2}C^{LM}D^K C_{KL}D_M s + C_1^K D_{KS} \right] \omega_{IJ} \\ & - 2C_{1(I}D_{J)}s - \frac{1}{4}C_{IJ}C^{KL}D_K D_L s - \frac{1}{8}C^2 D_I D_J s + \frac{1}{8}D_{(I}C^2 D_{J)}s + D_K C^{KL}C_{L(I}D_{J)}s. \end{aligned} \quad (5.19)$$

Moreover, assuming that  $T_{mm} = o(r^{-4})$ , we have [1]

$$\begin{aligned} \partial_u D_{IJ} = & \frac{1}{8}C_{IJ}\partial_u C^2 - \frac{1}{4}F_0 C_{IJ} - \frac{1}{2}D_{(I}C_{1J)} - \frac{1}{8}C_{IJ}D_K D_L C^{KL} \\ & + \frac{1}{32}D_I D_J C^2 + \frac{1}{2}D_{(I}(C_{J)K}D_L C^{KL}) - \frac{1}{8}D_I C^{KL}D_J C_{KL} \\ & + \frac{1}{4}\omega_{IJ} \left[ D_K C_1^K - \frac{5}{16}\square C^2 + D^M C^{KL} \left( D_K C_{LM} - \frac{1}{4}D_M C_{KL} \right) + C^2 \right]. \end{aligned} \quad (5.20)$$

In order to simplify the analysis, we begin by noting that since there is no Einstein equation for  $F_0$ , terms involving  $F_0$  in the non-integrable part of  $\delta\tilde{\mathcal{L}}_2$ , given in the second and third lines of equation (5.17), would have to vanish independently. Using the two equations (5.19) and (5.20), it is easy to see that

$$\begin{aligned} \delta\tilde{\mathcal{L}}_2^{(\text{non-int})}|_{F_0 \text{ terms}} &= -\frac{1}{8}sF_0 C_{IJ} \left[ \delta\tilde{C}^{IJ} - s\partial_u \tilde{C}^{IJ} \right] \\ &= \frac{1}{8}sF_0 \tilde{C}_{IJ} \left[ \delta C^{IJ} - s\partial_u C^{IJ} \right] \\ &= -\frac{1}{4}sF_0 \tilde{C}^{IJ} D_I D_J s, \end{aligned} \quad (5.21)$$

where, in the second equality, we have used equation (2.15) and, in the third equality, we have used (5.18). For arbitrary  $F_0$  and  $\tilde{C}^{IJ}$  (which is trace-free), the above expression vanishes if and only if

$$D_I D_J s = \frac{1}{2}\omega_{IJ}\square s. \quad (5.22)$$

This is precisely the condition that  $s$  is an  $\ell = 0$  or  $\ell = 1$  spherical harmonic (see appendix C of ref. [1]), which implies in particular that

$$\delta C_{IJ} = s\partial_u C_{IJ}, \quad \delta\tilde{C}_{IJ} = s\partial_u \tilde{C}_{IJ}. \quad (5.23)$$

Proceeding, while taking  $s$  to satisfy equation (5.22), the non-integrable part of  $\oint \tilde{\mathcal{I}}_2$  simplifies to

$$\begin{aligned} \oint \tilde{\mathcal{I}}_2^{(\text{non-int})} = & \left( C_{1I} D_J s - \frac{1}{16} D_I C^2 D_J s - \frac{1}{2} D_K C^{KL} C_{LI} D_J s \right) \delta \tilde{C}^{IJ} \\ & + s D_I (C_{1J} \delta \tilde{C}^{IJ}) - \frac{1}{16} s D_I (D_J C^2 \delta \tilde{C}^{IJ}) - \frac{1}{2} s D_I (C_{JK} D_L C^{KL} \delta \tilde{C}^{IJ}), \end{aligned} \quad (5.24)$$

which clearly forms a total derivative, and thus vanishes under integration. We therefore conclude that if  $s$  is an  $\ell = 0$  or  $\ell = 1$  spherical harmonic,

$$\oint \tilde{\mathcal{I}}_2^{(\text{non-int})} = 0 \quad (5.25)$$

and so

$$\tilde{\mathcal{I}}_2 = s D_I D_J \left( -\tilde{D}^{IJ} + \frac{1}{16} C^2 \tilde{C}^{IJ} \right). \quad (5.26)$$

However, up to total derivatives

$$\tilde{\mathcal{I}}_2 = D_I D_J s \left( -\tilde{D}^{IJ} + \frac{1}{16} C^2 \tilde{C}^{IJ} \right), \quad (5.27)$$

and this in fact vanishes upon use of equation (5.22). This analysis is, rather remarkably, completely analogous to that of  $\oint \mathcal{I}_2$  in ref. [1]: the non-integrable part can be made to vanish if and only if  $s$  is an  $\ell = 0$  or  $\ell = 1$  spherical harmonic, in which case the integrable charge itself turns out to be trivial.

Finally, we express the integrable parts of  $\oint \mathcal{I}_2$  and  $\oint \tilde{\mathcal{I}}_2$  as the real and imaginary parts, respectively, of a single charge written in terms of Newman-Penrose scalars. Defining

$$\mathcal{Q}_2 = \frac{1}{24\pi G} \int d\Omega s \bar{\delta}^2 \psi_0^0, \quad (5.28)$$

we find that this complex quantity may be written in terms of the integrable charges as

$$\mathcal{Q}_2 = \mathcal{Q}_2^{(\text{int})} - i \tilde{\mathcal{Q}}_2^{(\text{int})}, \quad (5.29)$$

where

$$\begin{aligned} \mathcal{Q}_2^{(\text{int})} &= \frac{1}{16\pi G} \int d\Omega s D_I D_J \left( -D^{IJ} + \frac{1}{16} C^2 C^{IJ} \right), \\ \tilde{\mathcal{Q}}_2^{(\text{int})} &= \frac{1}{16\pi G} \int d\Omega s D_I D_J \left( -\tilde{D}^{IJ} + \frac{1}{16} C^2 \tilde{C}^{IJ} \right). \end{aligned} \quad (5.30)$$

#### 5.4 Dual charge at $\mathcal{O}(r^{-3})$

Lastly, we consider the dual charge at order  $1/r^3$ . We find after some algebra that

$$\begin{aligned} \oint \tilde{\mathcal{I}}_3 = & -s D_I D_J \delta \tilde{E}^{IJ} \\ & + s \left( \frac{1}{2} \left[ \partial_u E_{IJ} \delta \tilde{C}^{IJ} - \delta E_{IJ} \partial_u \tilde{C}^{IJ} \right] - \frac{1}{4} D^K (C_{1I} C_{JK} \delta \tilde{C}^{IJ}) \right. \\ & + D^I \left( \left[ C_2^J - \frac{3}{4} (D_K D^{JK} - C^{JK} C_{1K}) - \frac{1}{64} C^2 D_K C^{JK} + \frac{1}{16} C^{JK} D_K C^2 \right] \delta \tilde{C}_{IJ} \right) \\ & + \frac{1}{4} D^K (\delta \tilde{C}^{IJ} D_I D_{JK}) - \frac{5}{4} D_I (D_{JK} D^K \delta \tilde{C}^{IJ}) + \frac{1}{16} D^K (\delta \tilde{C}^{IJ} C_{JK} D_I C^2) \\ & \left. - \frac{5}{64} \left[ \delta \tilde{C}^{IJ} D^K (C^2 D_I C_{JK}) - C_{JK} D_I (C^2 D^K \delta \tilde{C}^{IJ}) \right] \right). \end{aligned} \quad (5.31)$$

Assuming that  $T_{0m} = o(r^{-5})$ , the Einstein equation gives an equation for  $C_2^I$ , equation (2.18) of ref. [1]:

$$C_2^I = \frac{3}{4} (D_J D^{IJ} - C^{IJ} C_{1J}) + \frac{1}{64} C^2 D_J C^{IJ} - \frac{1}{16} C^{IJ} D_J C^2. \quad (5.32)$$

Substituting this equation into (5.31) gives

$$\begin{aligned} \delta \tilde{\mathcal{I}}_3 = & -s D_I D_J \delta \tilde{E}^{IJ} \\ & + s \left( \frac{1}{2} \left[ \partial_u E_{IJ} \delta \tilde{C}^{IJ} - \delta E_{IJ} \partial_u \tilde{C}^{IJ} \right] - \frac{1}{4} D^K (C_{1I} C_{JK} \delta \tilde{C}^{IJ}) \right. \\ & + \frac{1}{4} D^K (\delta \tilde{C}^{IJ} D_I D_{JK}) - \frac{5}{4} D_I (D_{JK} D^K \delta \tilde{C}^{IJ}) + \frac{1}{16} D^K (\delta \tilde{C}^{IJ} C_{JK} D_I C^2) \\ & \left. - \frac{5}{64} \left[ \delta \tilde{C}^{IJ} D^K (C^2 D_I C_{JK}) - C_{JK} D_I (C^2 D^K \delta \tilde{C}^{IJ}) \right] \right). \end{aligned} \quad (5.33)$$

Comparing this dual variation with the analogous term  $\delta \mathcal{I}_3$  given by equation (3.28) of ref. [1], we find that they are very similar. For the integrable parts, noting that we found (see equation (3.42) of ref. [1])

$$\mathcal{I}_3^{(\text{int})} = s D_I D_J \left( -E^{IJ} + \frac{1}{2} \text{tr} E \omega^{IJ} \right), \quad (5.34)$$

we see that the integrable parts are related by replacing the tensor fields in one by the twists of the fields in the other. (Note that  $\text{tr} \tilde{E} = 0$ .)

As with  $\delta \mathcal{I}_3$ , there exist non-integrable terms also, and one may consider, as we did previously at order  $1/r^2$ , whether there exists some choice of the function  $s$  such that the non-integrable part of  $\delta \tilde{\mathcal{I}}_3$ , given by the last three lines of equation (5.33), vanishes. We turn to this consideration in what follows. The variation of  $E_{IJ}$  under the action of a supertranslation is given by [1]

$$\begin{aligned} \delta E_{IJ} = & s \partial_u E_{IJ} + \left[ \frac{1}{4} D^{KL} D_K D_L s + \frac{3}{2} D_K D^{KL} D_L s - \frac{5}{4} C^{KL} C_{1K} D_L s - \frac{1}{64} C^2 C^{KL} D_K D_L s \right. \\ & \left. + \frac{3}{64} (C^{KL} D_K C^2 + 2C^2 D_K C^{KL}) D_L s \right] \omega_{IJ} + \frac{1}{2} C_{1(I} C_{J)K} D^K s - \frac{5}{2} D^K (D_{K(I} D_{J)} s) \\ & - \frac{1}{2} D^K s D_{(I} D_{J)K} + \frac{5}{32} D^K (C^2 C_{K(I} D_{J)} s) + \frac{5}{32} C^2 D^K s D_{(I} C_{J)K} - \frac{1}{8} C_{K(I} D_{J)} C^2 D^K s. \end{aligned} \quad (5.35)$$

Also, assuming that  $T_{mm} = o(r^{-5})$ , the Einstein equation implies that [1]

$$\begin{aligned} \partial_u E_{IJ} = & \frac{1}{2} D^K (C_{1(I} C_{J)K}) - \frac{1}{2} D^K D_{(I} D_{J)K} + \frac{5}{32} D^K (C^2 D_{(I} C_{J)K}) \\ & - \frac{1}{8} D^K (C_{K(I} D_{J)} C^2) + \frac{1}{2} \omega_{IJ} \left[ D^{KL} \partial_u C_{KL} - \frac{1}{4} C^2 F_0 - \frac{1}{2} C_1^K D^L C_{KL} \right. \\ & - C^{KL} D_K C_{1L} + \frac{1}{2} D^K D^L D_{KL} - \frac{1}{32} C^2 D^K D^L C_{KL} + \frac{5}{32} C^{KL} D_K D_L C^2 \\ & \left. - \frac{1}{16} C_{KL} D_M C^{MK} D_N C^{NL} + \frac{3}{32} C^{KL} D_K C^{MN} D_L C_{MN} \right]. \end{aligned} \quad (5.36)$$

Rewriting

$$s \left[ \partial_u E_{IJ} \delta \tilde{C}^{IJ} - \delta E_{IJ} \partial_u \tilde{C}^{IJ} \right] = - \left( \delta E_{IJ} - s \partial_u E_{IJ} \right) \delta \tilde{C}^{IJ} + \delta E_{IJ} \left( \delta \tilde{C}^{IJ} - s \partial_u \tilde{C}^{IJ} \right), \quad (5.37)$$

$\delta \tilde{\mathcal{L}}_3^{(\text{non-int})}$  simplifies to

$$\begin{aligned} \delta \tilde{\mathcal{L}}_3^{(\text{non-int})} &= \frac{1}{4} D^K \left[ \left( -C_{1I} C_{JK} + D_I D_{JK} + \frac{1}{4} C_{JK} D_I C^2 \right) s \delta \tilde{C}^{IJ} \right] \\ &\quad + \frac{5}{4} \left[ D^K (D_{JK} D_I s) \delta \tilde{C}^{IJ} - s D_I (D_{JK} D^K \delta \tilde{C}^{IJ}) \right] \\ &\quad + \frac{5}{64} \left[ s C_{JK} D_I (C^2 D^K \delta \tilde{C}^{IJ}) - D^K (C^2 D_I [s C_{JK}]) \delta \tilde{C}^{IJ} \right] \\ &\quad - \frac{1}{4} D^K \tilde{X}_{IJK} \left( \delta C^{IJ} - s \partial_u C^{IJ} \right), \end{aligned} \quad (5.38)$$

where

$$\tilde{X}_{IJK} = s C_{1I} \tilde{C}_{JK} - s D_I \tilde{D}_{JK} - 5 \tilde{D}_{JK} D_I s - \frac{1}{4} s \tilde{C}_{JK} D_I C^2 + \frac{5}{16} C^2 D_I (s \tilde{C}_{JK}) \quad (5.39)$$

and we have used equation (2.15). Note that the expression in the first line of equation (5.38) is a total derivative, which will integrate to zero. Moreover, integrating by parts and dropping total derivatives, the expressions on the second and third lines cancel. This leaves the expression on the fourth line, which, using equation (5.18), and integrating by parts, reduces to

$$\delta \tilde{\mathcal{L}}_3^{(\text{non-int})} = -\frac{1}{2} \tilde{X}_{IJK} D^K \left( D^I D^J s - \frac{1}{2} \omega^{IJ} \square s \right). \quad (5.40)$$

Note that  $\tilde{X}_{IJK}$  as defined in equation (5.39) is symmetric and trace-free in its indices ( $JK$ ). Thus, for arbitrary metric functions  $C_1^I$ ,  $C_{IJ}$  and  $D_{IJ}$ , the expression above vanishes if and only if

$$D^K \left( D^I D^J s - \frac{1}{2} \omega^{IJ} \square s \right) = \omega^{JK} U^I + \epsilon^{JK} W^I, \quad (5.41)$$

where the vectors  $U^I$  and  $W^I$  can be found by multiplying the above equation by  $\omega_{JK}$  and  $\epsilon_{JK}$ , respectively. Only the symmetric part in ( $JK$ ) is relevant here since  $\tilde{X}_{IJK}$  is symmetric in ( $JK$ ). Using the fact that

$$[D_I, D_J] V_K = R_{IJK}{}^L V_L, \quad R_{IJKL} = \omega_{IK} \omega_{JL} - \omega_{IL} \omega_{JK}, \quad (5.42)$$

we find that the trace-free, symmetric ( $JK$ ) projection of the left hand side of equation (5.41) becomes

$$W_{IJK} \equiv D_I D_{(J} D_{K)} s - \frac{1}{2} \omega_{I(J} D_{K)} \square s - \frac{1}{4} \omega_{JK} D_I \square s - \omega_{I(J} D_{K)} s + \frac{1}{2} \omega_{JK} D_I s = 0. \quad (5.43)$$

It is straightforward to see, by integrating the manifestly non-negative  $|W_{IJK}|^2$  over the 2-sphere and integrating by parts, that

$$\int d\Omega |W_{IJK}|^2 = -\frac{1}{4} \int d\Omega s \square (\square + 2)(\square + 6) s \tag{5.44}$$

and that therefore equation (5.43) is satisfied if and only if  $s$  is an  $\ell = 0$ ,  $\ell = 1$  or  $\ell = 2$  spherical harmonic.

In summary, the non-integrable part of  $\oint \tilde{\mathcal{L}}_3$  vanishes, in general, if and only if  $s$  is an  $\ell = 0$ ,  $\ell = 1$  or  $\ell = 2$  spherical harmonic. In this case,  $\oint \tilde{\mathcal{L}}_3$  is integrable, with the charge corresponding to

$$\tilde{\mathcal{L}}_3 = -s D_I D_J \tilde{E}^{IJ}. \tag{5.45}$$

This gives a charge that is itself trivially zero for  $\ell = 0$  or  $\ell = 1$  spherical harmonics, since they obey  $D_I D_J s = \frac{1}{2} \omega_{IJ} \square s$  and  $\tilde{E}^{IJ}$  is trace-free. Hence we obtain a non-trivial integrable charge if and only if  $s$  is an  $\ell = 2$  spherical harmonic. This complements the result we obtained in ref. [1], where we found that  $\oint \mathcal{L}_3$  is integrable and gives a non-trivial charge if and only if  $s$  is an  $\ell = 2$  spherical harmonic. As with previously considered subleading charges, we may define a single charge that encapsulates the integrable parts of  $\oint \mathcal{L}_3$  and  $\oint \tilde{\mathcal{L}}_3$  as its real and imaginary parts, respectively<sup>9</sup>

$$\mathcal{Q}_3 = \frac{1}{48\pi G} \int d\Omega s \bar{\partial}^2 \psi_0^1, \tag{5.46}$$

$$\tilde{\mathcal{Q}}_3 = \mathcal{Q}_3^{(\text{int})} - i \tilde{\mathcal{Q}}_3^{(\text{int})}, \tag{5.47}$$

where

$$\begin{aligned} \mathcal{Q}_3^{(\text{int})} &= \frac{1}{16\pi G} \int d\Omega s D_I D_J \left( -E^{IJ} + \frac{1}{2} \text{tr} E \omega^{IJ} \right), \\ \tilde{\mathcal{Q}}_3^{(\text{int})} &= \frac{1}{16\pi G} \int d\Omega s \left( -D_I D_J \tilde{E}^{IJ} \right). \end{aligned} \tag{5.48}$$

Now, choosing  $s$  to be an  $\ell = 2$  spherical harmonic so that the non-integrable parts of  $\oint \mathcal{L}_3$  and  $\oint \tilde{\mathcal{L}}_3$  vanish, from equation (5.46), we obtain a conserved charge<sup>10</sup>

$$\mathcal{Q}_3|_{s(\ell=2)} = \frac{1}{48\pi G} \int d\Omega \bar{Y}_{2,m} \bar{\partial}^2 \psi_0^1. \tag{5.49}$$

Integrating by parts we find that this is indeed equal to the NP charges [9]

$$\mathcal{Q}_3|_{s(\ell=2)} = \mathcal{Q}^{(NP)} = \frac{1}{4\sqrt{6}\pi G} \int d\Omega {}_2\bar{Y}_{2,m} \psi_0^1. \tag{5.50}$$

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<sup>9</sup>Note that there is an unimportant factor of 1/2 typographical error in equation (3.4) of ref. [1], which has lead to factor of 1/2 discrepancies in some other equations, such as equation (4.29) of ref. [1].

<sup>10</sup>Setting  $s = \bar{Y}_{2,m}$  is not strictly correct, because we should really be using a real basis of spherical harmonics, since  $s$  is real. However, it is more convenient to work with a complex basis of spherical harmonics. Clearly, this choice makes no substantive difference as we could equally think of choosing  $s$  to be the real and imaginary parts of  $\bar{Y}_{2,m}$ . See [1, 2] for a more extensive discussion of this point.



## 6 Discussion

In this paper, we have resolved two puzzles arising from earlier work [1, 2]; we have extended the notion of dual gravitational charges to subleading orders in a  $1/r$  expansion away from null infinity and found that at the order  $1/r^3$  this, together with the subleading charges proposed in ref. [1], accounts for all ten of the non-linear Newman-Penrose charges.

The tower of dual gravitational charges is constructed from a new 2-form  $\tilde{H}$  (see equation (3.2)) and can be viewed as being dual to the BMS charges constructed from the Hodge dual of the Barnich-Brandt 2-form  $H$  (see equation (3.3)). At the leading order, restricting to supertranslations, the two 2-forms coincide (see equation (5.6)). However, they are different at lower orders in a  $1/r$  expansion away from null infinity.

The Barnich-Brandt 2-form is derived by considering the linearised Einstein equation and defining a quantity that vanishes on-shell. The electric charge is the surface integral of a current that is conserved upon use of Maxwell's equation. Analogously the Barnich-Brandt 2-form defines a quantity that is the surface integral of a current that vanishes upon use of the linearised Einstein equation and may be viewed as the analogue of the electric charge.

On the other hand, we construct the dual charges using a 2-form that is a total derivative (see appendix A). In this sense it is analogous to a magnetic Komar charge and defines a charge without use of the Einstein equation. However, nevertheless we obtain charges that are non-trivial and account for the recently proposed dual gravitational charges [2], extending them to charges associated with the full BMS group, and the imaginary parts of the non-linear Newman-Penrose charges.

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## A Boundary terms

In this section, we prove that the variation of the dual charge (3.1) vanishes, using the fact that

$$\delta g_{ab} = 2\nabla_{(a}\xi_{b)}. \tag{A.1}$$

From the definition of the dual 2-form  $\tilde{H}$ , given in equation (3.2), and using equation (A.1)

$$\begin{aligned} 2\tilde{H}_{IJ} &= \xi^c\nabla_J\nabla_I\xi_c + \xi^c\nabla_J\nabla_c\xi_I + \nabla_J\xi^c\nabla_c\xi_I \\ &= R_{JIcd}\xi^c\xi^d + \nabla_J(\xi^c\nabla_c\xi_I) \\ &= \partial_J(\xi^c\nabla_c\xi_I), \end{aligned} \tag{A.2}$$

where we assume an antisymmetrisation in  $[IJ]$  on the right hand side for each equality. In the second equality we use the fact that

$$[\nabla_a, \nabla_b]V_c = R_{abc}{}^dV_d. \tag{A.3}$$

Thus, we conclude that  $\delta\tilde{Q}$  as defined in (3.1) vanishes.

## B Further properties of $\widetilde{H}$

In this section, we verify that

$$\oint \widetilde{Q}[\xi, g, \delta g = \mathcal{L}_\zeta g] = -\oint \widetilde{Q}[\zeta, g, \delta g = \mathcal{L}_\xi g] \quad (\text{B.1})$$

as one would expect.

Starting from equation (3.2),

$$\widetilde{H}[\xi, g, \delta g] = \frac{1}{2} \left\{ \xi^c \nabla_b \delta g_{ac} - \frac{1}{2} \delta g_{bc} (\nabla_a \xi^c - \nabla^c \xi_a) \right\} dx^a \wedge dx^b, \quad (\text{B.2})$$

we can write the first term as  $\nabla_b(\xi^c \delta g_{ac}) - \delta g_{ac} \nabla_b \xi^c$ , and hence we get

$$\widetilde{H}[\xi, g, \delta g] = \frac{1}{2} \left\{ \nabla_b(\xi^c \delta g_{ac}) + \frac{1}{2} \delta g_{bc} (\nabla_a \xi^c + \nabla^c \xi_a) \right\} dx^a \wedge dx^b. \quad (\text{B.3})$$

Thus we have

$$\widetilde{H}[\xi, g, \delta g] = -\frac{1}{2} d(\xi^c \delta g_{ac} dx^a) + \frac{1}{4} \delta g_{bc} g^{cd} (\nabla_a \xi_d + \nabla_d \xi_a) dx^a \wedge dx^b. \quad (\text{B.4})$$

If we take  $\delta g_{bc}$  to be a variation  $\delta_\zeta g_{bc}$  coming from a BMS generator  $\zeta$  with  $\delta_\zeta g_{bc} = \nabla_b \zeta_c + \nabla_c \zeta_b$ , and view  $(\nabla_a \xi_d + \nabla_d \xi_a)$  as defining a metric variation  $\delta_\xi g_{ad}$ , then we have

$$\widetilde{H}[\xi, g, \delta g] = d\omega + \frac{1}{4} (\delta_\xi g_{ac}) (\delta_\zeta g_{bd}) g^{cd} dx^a \wedge dx^b, \quad (\text{B.5})$$

where  $\omega = -\frac{1}{2} d(\xi^c \delta_1 g_{ac} dx^a)$ .

Equation (3.1) defines  $\oint \widetilde{Q}[\xi, g, \delta g]$  as the integral of  $\widetilde{H}[\xi, g, \delta g]$ . Thus, we conclude that

$$\oint \widetilde{Q}[\xi, g, \delta g = \mathcal{L}_\zeta g] = -\oint \widetilde{Q}[\zeta, g, \delta g = \mathcal{L}_\xi g]. \quad (\text{B.6})$$

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