A new moment-independent measure for reliability-sensitivity analysis

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A new moment-independent measure for reliability-sensitivity analysis

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Chair of Risk, Safety and Uncertainty Quantification

ETH Zürich

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Outline

Reliability-sensitivity analysis

The question:

What are the input variables that influence the most the reliability of the system?

The challenges:

- The concept of importance needs to be clearly defined
- Dependence is difficult to treat
- Many available measures are computationally expensive

Reliability-sensitivity analysis

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A few examples

FORM importance factors $$\boldsymbol{\alpha} = -\frac{\nabla G_{\boldsymbol{U}^*}}{\| \nabla G_{\boldsymbol{U}^*} \|}, \quad IF_i = \alpha_i^2$$

Partial derivatives $$S_{j,i} = \frac{\partial P_f}{\partial \theta_{j,i}}, \quad \text{where} X_i \sim F_{X_i}(x_i, \boldsymbol{\theta})$$

Goal-oriented measure $$S_{\Psi}^i = \frac{\mathrm{Var}\left[\mathbf{1}_{\boldsymbol{\mathcal{D}_f}}(\boldsymbol{X})\right] - \mathbb{E}_{X_i}\left[\mathrm{Var}\left[\mathbf{1}_{\boldsymbol{\mathcal{D}_f}}(\boldsymbol{X}|X_i)\right]\right]}{\mathrm{Var}\left[\mathbf{1}_{\boldsymbol{\mathcal{D}_f}}(\boldsymbol{X})\right]}$$

Moment-independent sensitivity (Borgonovo)

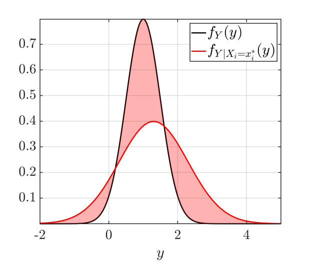
Borgonovo global sensitivity measure

Borgonovo, 2007

Rationale:

Measure the difference between unconditional and conditional $f_Y(y)$

$$\delta_i = \frac{1}{2}\mathbb{E}_{X_i}\left[\int_{\mathcal{D}_Y} \left| f_Y(y) - f_{Y|X_i=x_i}(y) \right| dy\right]$$



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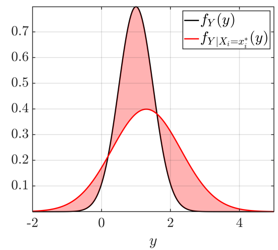


Properties

- Global: the entire input space is considered
- Bounded: $\delta_i \in [0,\ 1]$
- Useful for model selection ($\delta_i = 0 \iff X_i$ is unimportant)
- Expensive to estimate (need $f_Y(y)$ and $f_{Y|X_i}(y)$)

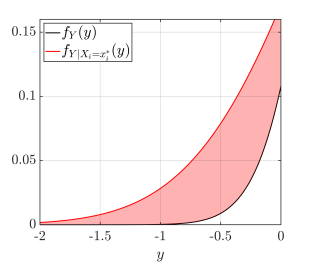
Extension to sensitivity-reliability

Same idea, but focus on the failure domain

Rationale:

Measure the difference between unconditional and conditional $f_Y(y)$ in the failure domain

$$d_i(x_i) = \int_{Y \leq 0} \left| f_Y(y) - f_{Y|X_i=x_i}(y) \right| dy$$



Two measures

$$\delta_i^{\mathsf{RSA}_1} = \frac{1}{2} \left(\frac{\mathbb{E}\left[d_i(x_i)\right]}{\int_{Y \leq 0} f_Y(y) dy} \right) = \frac{\mathbb{E}\left[d_i(x_i)\right]}{2P_f} \qquad \delta_i^{\mathsf{RSA}_2} = \frac{\sqrt{\mathrm{Var}\left[d_i(x_i)\right]}}{2P_f}$$

Properties of the proposed measure

Two possible measures

$$\delta_i^{\mathsf{RSA}_1} = \frac{\mathbb{E}\left[d_i(x_i)\right]}{2P_f}$$

- Bounded: $\delta_i^{\mathsf{RSA}_1} \in [0,\ 1]$
- Well defined also with dependence
- Describes the overall effect of a variable

$$\delta_i^{\mathsf{RSA}_2} = \frac{\sqrt{\mathrm{Var}\left[d_i(x_i)\right]}}{2P_f}$$

- Unbounded
- Comes for free...
- Provides information on the local effect of the variable

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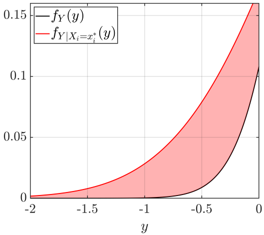
Estimation through Monte Carlo simulation

Monte-Carlo based estimation

Ingredients

- Full response PDF $Y \sim f_Y(y)$
- Conditional response PDF $f_{Y|X_i=x_i}(y)$
- How to estimate their difference?



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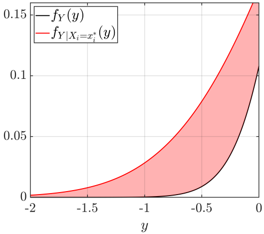

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Solution

Approximate conditional sampling from a large MCS sample
(Same approach as *histogram-based* Borgonovo estimators)

Approximate conditional sampling

Idea: discretize a large sample through conditional 2D histograms

Ingredients: a sample from the failure domain $\mathcal{D}_f$ $\mathcal{X} = \left\{\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(N_{\mathsf{MC}})}\right\}$ and $\mathcal{Y} = \left\{y^{(1)} = g\left(\boldsymbol{x}^{(1)}\right), \ldots, y^{(N_{\mathsf{MC}})} = g\left(\boldsymbol{x}^{(N_{\mathsf{MC}})}\right)\right\}$ such that $y^{(i)} \leq 0, \quad \forall i$

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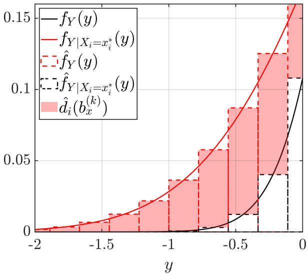
Algorithm

- Select n_y bins and store their edges
- For each input variable:
  - Bin the sample along X_i: $\{b_x^{(1)}, \ldots, b_x^{(n_x)}\}$
  - Take the difference between the empirical PDF of $\mathcal{Y}$ and that of the subsets:
    $$\hat{d}_i(b_x^{(i)}) = \sum_{k=1}^{n_y} \left| \hat{f}_Y^{(k)} - \hat{f}^{(k)}_{Y|X_i = b_x^{(i)}} \right|$$



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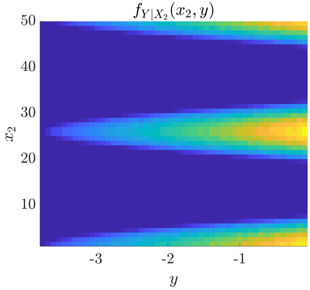

$$\delta_i^{\mathsf{RSA}_1} = \frac{\mathbb{E}\left[d_i\left(x_i\right)\right]}{2P_f} \qquad \delta_i^{\mathsf{RSA}_2} = \frac{\sqrt{\mathrm{Var}\left[d_i(x_i)\right]}}{2P_f}$$

Computational burden

The challenge: although simple, for low probability of failure, in multiple dimensions this may require $\mathcal{O}(10^{6-8})$ model runs

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The solution: AK-MCS – Kriging + active learning

1. Start from a small initial set $\mathcal{X}_{\text{ED}} = \{\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(N_{\text{ini}})}\}$
2. Train a Kriging surrogate $\hat{g}(\boldsymbol{x}) \approx g(\boldsymbol{x})$
3. Estimate P_f with MCS
4. Adaptively enrich the $\mathcal{X}_{\text{ED}}$:

$$\boldsymbol{x}^* = \arg\min_{\boldsymbol{x}} \mathbb{P}(\text{sign}(\hat{g}(\boldsymbol{x})) \neq \text{sign}(g(\boldsymbol{x})))$$

5. Repeat 2-4 until convergence

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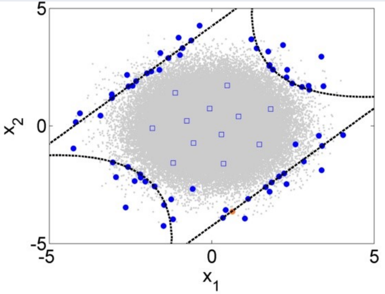
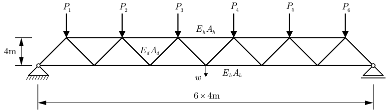


Calculate the $\delta_i^{\text{RSA}_{1,2}}$ with the final $\hat{g}(\boldsymbol{x})$ on a very large MCS sample

Elastic truss structure

Blatman *et al.* , 2007

Problem statement



10 independent variables

- 4 describing the bars properties
- 6 describing the loads

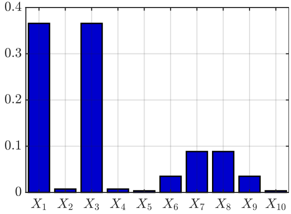
- Response quantity: max. deflection
- Black-box FEM model

Probabilistic model

Parameters	Name	Distribution	Mean	Std. Deviation
Young's modulus	E_1, E_2 (Pa)	Lognormal	2.10×10^{11}	2.10×10^{10}
Hor. bars section	A_1 (m^2)	Lognormal	2.0×10^{-3}	2.0×10^{-4}
Vert. bars section	A_2 (m^2)	Lognormal	1.0×10^{-3}	1.0×10^{-4}
Loads	P_1-P_6 (N)	Gumbel	5.0×10^{4}	7.5×10^{3}

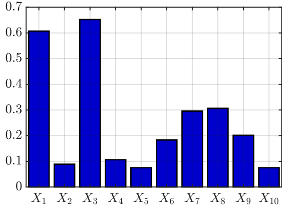
Truss: results

Importance factors



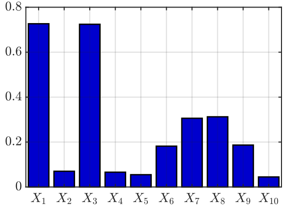
Cost: 150 model runs

$\delta_i^{\mathsf{RSA}_1}$



Cost: ~ 40 model runs

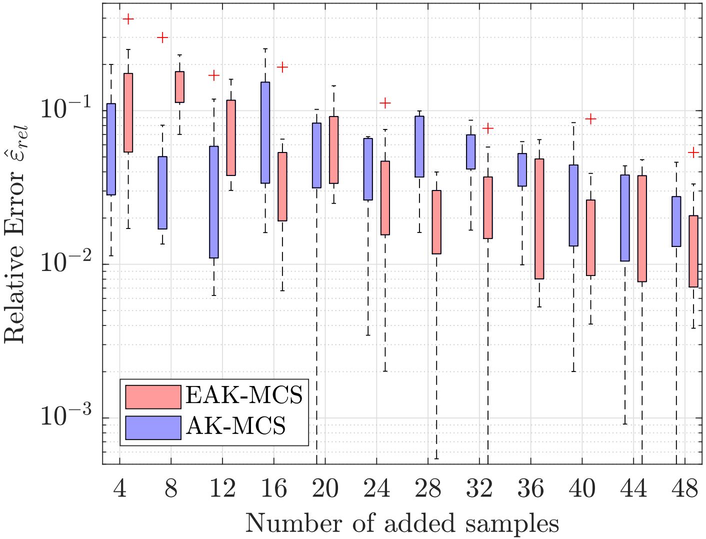
$\delta_i^{\mathsf{RSA}_2}$



Cost: ~ 40 model runs

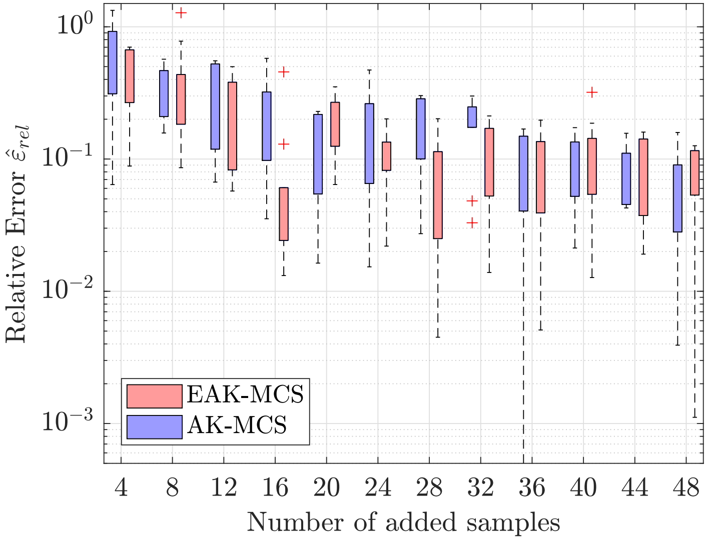
Note: Probability of failure: $P_f \approx 10^{-3}$

Truss: convergence



$\delta_i^{\mathsf{RSA}_1}$

Truss: convergence



$\delta_i^{\mathrm{RSA}_2}$

Summary and Outlook

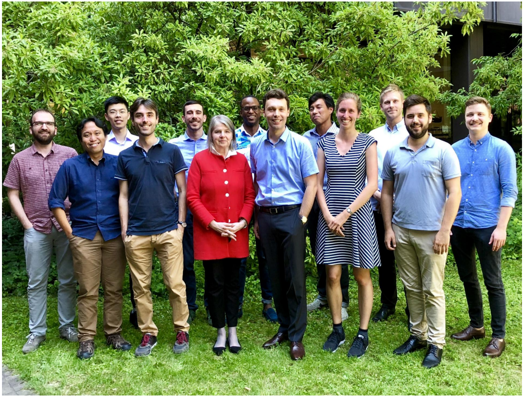
Summary

- A novel set of reliability-sensitivity indices has been proposed
- The two different indices provide complementary information about the problem
- The proposed measure does not require independence of the input distribution
- Thanks to a novel adaptive algorithm, the indices can be calculated with a manageable number of model runs

Outlook

- Extensions of the active-learning algorithm to parallel computing
- Development of dedicated surrogate strategies to improve the accuracy of the tails of the conditional distributions

Questions ?



Chair of Risk, Safety & Uncertainty Quantification

www.rsuq.ethz.ch

The Uncertainty Quantification Software

www.uqlab.com

UQLab

Thank you very much for your attention!

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