On the disconnection of a discrete cylinder by a random walk

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Abstract. We investigate the large N behavior of the time the simple random walk on the discrete cylinder $(\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}$ needs to disconnect the discrete cylinder. We show that when $d \geq 2$, this time is roughly of order N^{2d} and comparable to the cover time of the slice $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$, but substantially larger than the cover timer of the base by the projection of the walk. Further we show that by the time disconnection occurs, a massive "clogging" typically takes place in the truncated cylinders of height $N^{d-\epsilon}$. These mechanisms are in contrast with what happens when $d = 1$.

0. Introduction

Consider simple random walk on an infinite discrete cylinder having a base modeled on a d-dimensional discrete torus of side length N. In this note we investigate the following question of H.J. Hilhorst: what is the asymptotic behavior for large N of the time needed by the walk to disconnect the cylinder? When $d = 1$, it is straightforward to argue that this time is roughly of order N^2 and comparable to the time for the projection of the process to cover the base. We show here that things behave differently when $d \geq 2$, and that in a suitable sense a massive clogging occurs inside the cylinder by the time the disconnection happens.

Before discussing our results any further, we describe the model more precisely. For integer $N \geq 1$, we consider the state space

$$E = (\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}\,, \tag{0.1}$$

that we tacitly endow with its natural graph structure. We say that a finite subset $S \subseteq E$ *disconnects* E if, for large M, $(\mathbb{Z}/N\mathbb{Z})^d \times [M, \infty)$ and $(\mathbb{Z}/N\mathbb{Z})^d \times (-\infty, -M]$ are contained in two distinct connected components of $E \backslash S$.

We denote with P_x, $x \in E$, the canonical law on $E^{\mathbf{N}}$ of the simple random walk on E starting at x, and with $(X_n)_{n \geq 0}$ the canonical process. We are principally interested in the disconnection time of E:

$$T_N = \inf\{n \geq 0;\ X_{[0,n]} \text{ disconnects } E\}\,. \tag{0.2}$$

Under P_x, $x \in E$, the Markov chain $X_.$ is irreducible, recurrent, and it is plain that

$$T_N < \infty,\ P_x\text{-a.s., for all } x \in E\,. \tag{0.3}$$

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As a comparison consider $\widetilde{C}_N$, the cover time of $(\mathbb{Z}/N\mathbb{Z})^d$ by the projection of $X_.$ on the base, i.e. the first time the projection of $X_.$ has visited all points of $(\mathbb{Z}/N\mathbb{Z})^d$, as well as C_N the cover time of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$ by X. It is also plain that:

$$\widetilde{C}_N \le T_N \le C_N \,. \tag{0.4}$$

Cover times of finite graphs have been extensively investigated, cf. for instance [1], [2], [4], [6], [7], and the references therein, and one knows that for any $d \ge 1$,

$$\frac{\log \widetilde{C}_N}{\log N} \underset{N\to\infty}{\longrightarrow} d \vee 2, \text{ in } P_0\text{-probability}\,, \tag{0.5}$$

(much more is known, see the above references). Our first main result states that:

Theorem 1. $(d \ge 2)$

$$\textit{In } P_0\textit{-probability, } \lim_N \frac{\log T_N}{\log N} = \lim_N \frac{\log C_N}{\log N} = 2d\,. \tag{0.6}$$

In fact, cf. Remark 2.6, (0.6) also holds when $d = 1$. As a consequence of (0.5) and (0.6), we thus see that unlike what happens when $d = 1$, there is a substantial discrepancy between $\widetilde{C}_N$ and T_N when $d \ge 2$.

Our second main result shows a massive "clogging" in the cylinder by the time disconnection occurs when $d \ge 2$. Let us denote with $d(x, A)$, for $x \in E$, $A \subseteq E$, the minimal length of a nearest neighbor path from x to A. We have:

Theorem 2. $(d \ge 2)$

$$\textit{For all } \epsilon, \eta \in (0,1), \max_{x\in(\mathbb{Z}/N\mathbb{Z})^d\times[-N^{d-\epsilon},N^{d-\epsilon}]} d(x, X_{[0,T_N]})/N^\eta \underset{N\to\infty}{\longrightarrow} 0\,,$$
$$\textit{in } P_0\textit{-probability}\,. \tag{0.7}$$

So Theorem 2 (see also Theorem 3.1) shows that by time T_N the walk pretty much fills up the truncated cylinder $(\mathbb{Z}/N\mathbb{Z})^d \times [-N^{d-\epsilon}, N^{d-\epsilon}]$, when N is large. Once again this can be contrasted with the $d = 1$ situation, where with non-vanishing probability points in $(\mathbb{Z}/N\mathbb{Z})^d \times [-N^{1-\epsilon}, N^{1-\epsilon}]$ at distance of order N from $X_{[0,T_N]}$ do occur, cf. Remark 3.2.

We now give some indications of the proofs of Theorems 1 and 2. The proof of Theorem 1 consists of an upper bound, cf. Theorem 1.1, and a lower bound, cf. Theorem 2.1. The upper bound is simpler to prove. It is a direct consequence of the fact that T_N is smaller than C_N, cf. (0.4), and the estimates we derive on this cover time. It is instructive that this rather primitive strategy captures the correct rough order of magnitude of T_N. The lower bound is more delicate. The rough idea of the proof is that for $\gamma \in (0,1)$, and large N, one must find a box of size N^γ in E containing about $O(N^{d\gamma})$ points of the trajectory $X_{[0,T_N]}$, since $X_{[0,T_N]}$ disconnects E, cf. Lemma 2.4. We use here isoperimetric controls of Deuschel-Pisztora [8]. Now if γ is chosen small enough, with high probability $X_.$ puts at most about $(\log N)\, N^{2\gamma}$ points in any box of side length N^γ by times "slightly smaller" than

N^{2d}, cf. (2.26). For $d \geq 3$, these are much fewer points than the required $O(N^{d\gamma})$ points to produce disconnection. This yields a lower bound on T_N in case $d \geq 3$. The argument for $d = 2$ is of a similar flavor. However, a considerable refinement is required in this case. We now find a collection $\mathcal{E}_*$ of $O((\log N)^{2\alpha})$ disjoint sub-boxes of size $\ell = L(\log N)^{-\alpha}$, with centers on a common $O(\ell)$-sub-grid of some L-size box, such that the two-dimensional projection in a suitable direction of the intersection of $X_{[0,T_N]}$ with any of these sub-boxes contains at least $c\ell^2$ points, cf. Lemma 2.5. When γ is small and α is chosen smaller than $3/4$, we show that the probability that such an event happens within the first $N^{4-\delta}$ steps of the walk tends to zero as N goes to infinity, cf. (2.42), (2.43), thus yielding the lower bound on T_N, when $d = 2$.

As for the proof of the "clogging effect", cf. Theorem 2 or Theorem 3.1, the main idea is to rely on the lower bound on T_N of Theorem 2.1 and show that before time T_N in a uniform fashion for $x \in (\mathbb{Z}/N\mathbb{Z})^d \times [-N^{d-\epsilon}, N^{d-\epsilon}]$, the walk comes "often enough" within distance N of x, giving each time an opportunity to come closer to x.

Let us now explain how this article is organized.

In Section 1, we provide further notations and definitions. The main objective is Theorem 1.1, that provides an upper bound on C_N and hence also on T_N.

In Section 2, we prove a lower bound on T_N in Theorem 2.1. We derive controls on excursions of the process in Proposition 2.2, which we then combine with a geometric lemma, cf. Lemma 2.4 for $d \geq 3$, or its finer version Lemma 2.5 for $d = 2$.

In Section 3 we show in Theorem 3.1 that clogging takes place by time T_N, when $d \geq 2$.

Let us finally explain the convention we use concerning constants. We denote with c or c' positive constants depending on d, with value changing from place to place. The numbered constants $c_0, c_1, \ldots$ will be fixed and refer to the value at their first appearance in the text. Dependence of constants on additional parameters will appear in the notation; for instance $c(\delta)$ will denote a positive constant depending on d and δ.

1. The upper bound

The main objective of this section is to begin the proof of Theorem 1 of the introduction and more specifically to provide in Theorem 1.1 an upper bound on the disconnection time T_N. We begin with some additional notations.

We denote with π_N the canonical projection from $\mathbb{Z}^{d+1}$ on E, cf. (0.1). For $x \in \mathbb{Z}^{d+1}$, resp. $x \in E$, we let x^{d+1} stand for the last component, resp. the projection on $\mathbb{Z}$, of x. We denote with $|\cdot|$ and $|\cdot|_\infty$ the Euclidean and ℓ_∞-distances on $\mathbb{Z}^{d+1}$, or the corresponding induced distances on E. We write $B(x, r)$ or $B_\infty(x, r)$ for the corresponding open balls with radius $r > 0$ and center $x \in \mathbb{Z}^{d+1}$, or $x \in E$. For A and B subsets of E or of $\mathbb{Z}^{d+1}$ we denote with $A + B$ the set of points of the form $x + y$, with x in A and y in B. For a subset U of $\mathbb{Z}^{d+1}$ or E, we denote with $|U|$ the cardinality of U and with ∂U the boundary of U:

$$\partial U = \{x \in U^c;\ \exists y \in U,\ |x - y| = 1\}. \tag{1.1}$$

We let $(\theta_n)_{n\geq 0}$, and $(\mathcal{F}_n)_{n\geq 0}$, respectively stand for the canonical shift and filtration for the process $(X_n)_{n\geq 0}$ on $E^{\mathbb{N}}$. For $U \subseteq E$, H_U, T_U are the entrance time and exit time in or from U:

$$H_U = \inf\{n \geq 0, X_n \in U\}, \quad T_U = \inf\{n \geq 0, X_n \notin U\}. \tag{1.2}$$

For simplicity we write H_x in place of $H_{\{x\}}$. We denote with Q_x, $x \in \mathbb{Z}^{d+1}$, the canonical law on $(\mathbb{Z}^{d+1})^{\mathbb{N}}$ of the simple random walk on $\mathbb{Z}^{d+1}$. We will use, when this causes no confusion, the same notations as above for the canonical process, the canonical shift, the entrance or exit times for the simple random walk on $\mathbb{Z}^{d+1}$.

We now turn to the main objective of this section: the derivation of an upper bound on the disconnection time T_N. As explained in the Introduction, we simply use the fact that T_N is smaller than the cover time by X of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$, and estimate from above this cover time.

Theorem 1.1. $(d \geq 2)$

$$\forall \delta > 0, \lim_{N\to\infty} P_0\Big[\frac{\log T_N}{\log N} \leq \frac{\log C_N}{\log N} \leq 2d + \delta\Big] = 1, \tag{1.3}$$

(as a matter of fact (1.3) *also holds for* $d = 1$*, cf. Remark* 1.4 *below).*

Proof. We introduce two subsets of E, namely the truncated cylinders

$$B = (\mathbb{Z}/N\mathbb{Z})^d \times [-N, N], \text{ and } \widetilde{B} = (\mathbb{Z}/N\mathbb{Z})^d \times [-2N+1, 2N-1]. \tag{1.4}$$

We then consider the sequence of successive returns to B and departures from $\widetilde{B}$ of the walk:

$$\begin{aligned} &R_1 = H_B, \ D_1 = T_{\widetilde{B}} \circ \theta_{R_1} + R_1, \text{ and for } k \geq 1, \\ &R_{k+1} = R_1 \circ \theta_{D_k} + D_k, \ D_{k+1} = D_1 \circ \theta_{D_k} + D_k, \end{aligned} \tag{1.5}$$

so that

$$0 \leq R_1 \leq D_1 \leq \cdots \leq R_k \leq D_k \leq \cdots \leq \infty,$$

and these inequalities except maybe for the first one are strict, P_x-a.s., for any $x \in E$. The proof of Theorem 1.1 will use the next

Lemma 1.2. *For any* $N \geq 1$*,* $y \in B$*, and* $x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$*,*

$$P_y[H_x < T_{\widetilde{B}}] \geq cN^{-(d-1)}. \tag{1.6}$$

Proof. We define the subset of $\mathbb{Z}^{d+1}$:

$$U = (-2N, 2N)^{d+1} \cap \mathbb{Z}^{d+1}. \tag{1.7}$$

The probability in (1.6) is bigger than

$$Q_v[H_u < T_U], \tag{1.8}$$

where $v \in \{0, \dots, N-1\}^d \times \{-N, \dots, N\}$ and $u \in \{0, \dots, N-1\}^d \times \{0\}$ satisfy $\pi_N(v) = y$, $\pi_N(u) = x$.

For D a subset of $\mathbb{Z}^{d+1}$, we denote with $g_D(\cdot, \cdot)$ the Green function of the simple random walk killed when exiting D:

$$g_D(w, w') = E^{Q_w}\Big[\sum_{n=0}^{T_D - 1} 1\{X_n = w'\}\Big], \text{ for } w, w' \in \mathbb{Z}^{d+1}, \tag{1.9}$$

and for simplicity write $g(\cdot, \cdot)$ for $g_{\mathbb{Z}^{d+1}}(\cdot, \cdot)$. It follows from the strong Markov property at the stopping time $H_u \wedge T_U$, that:

$$Q_v(H_u < T_U) = \frac{g_U(v, u)}{g_U(u, u)} . \tag{1.10}$$

Now by standard estimates

$$c\, g(w, w') \le g_U(w, w') \le g(w, w'), \text{ for all } w, w' \in \{-N, \dots, N\}^{d+1} \subseteq U , \tag{1.11}$$

(the second inequality is immediate, for the first inequality we refer to (1.82) and (1.83) in Antal [3], when $|w - w'| \le cN$, with c small, the general case follows for instance with the invariance principle), and (1.6) then follows from classical bounds on $g(\cdot, \cdot)$ in dimension $d + 1$, cf. Lawler [12], p. 31. □

We now return to the proof of Theorem 1.1. Consider $x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$. From the strong Markov property, we see that for $k \ge 0$, with the convention $D_0 = 0$,

$$\begin{aligned} P_0[H_x > D_{k+1}] &= E_0\big[H_x > R_{k+1}, P_{X_{R_{k+1}}}[H_x > T_{\widetilde{B}}]\big] \\ &\overset{(1.6)}{\le} P_0[H_x > R_{k+1}](1 - cN^{-(d-1)}) \le P_0[H_x > D_k](1 - cN^{-(d-1)}) \\ &\overset{\text{induction}}{\le} (1 - cN^{-(d-1)})^{k+1} . \end{aligned} \tag{1.12}$$

We thus have the following estimate on the cover time of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$:

$$P_0\big[\max_{x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}} H_x > D_k\big] \le N^d (1 - cN^{-(d-1)})^k, \text{ for } k \ge 1 . \tag{1.13}$$

On the complement of the event inside the above probability, we have:

$$T_N \overset{(0.4)}{\le} C_N = \max_{x \in (\mathbb{Z}/N\mathbb{Z})^d \times \{0\}} H_x \le D_k , \tag{1.14}$$

and hence for $\epsilon > 0$ and large N,

$$P_0[T_N \le C_N \le D_{[N^{(d-1)+\epsilon}]}] \ge 1 - N^d e^{-cN^\epsilon} \underset{N \to \infty}{\longrightarrow} 1 . \tag{1.15}$$

We now control the tail of D_k for large k. To this end we note that:

$$\begin{aligned} &\text{under } P_0,\ X^{d+1}_{\cdot} \text{ has same distribution as } Y_{\cdot}, \text{ the random walk on } \mathbb{Z}, \\ &\text{starting in 0, with jump distribution } \tfrac{1}{2(d+1)} (\delta_{-1} + \delta_1) + \tfrac{d}{d+1}\, \delta_0 . \end{aligned} \tag{1.16}$$

Note that the above random walk $Y_{\bullet}$ is obtained by delaying a simple random walk on $\mathbb{Z}$ with a geometric clock of parameter $\frac{1}{d+1}$ at each site of $\mathbb{Z}$. Coming back to P_0, the strong Markov property yields that

$$\begin{aligned}&\text{under } P_0,\, D_1,\, D_2 - D_1, \dots, D_{k+1} - D_k, \dots \text{ are independent variables}\\&\text{and for } k \geq 1,\, D_{k+1} - D_k \text{ have the same distribution as the sum of}\\&\text{two independent variables respectively distributed like the entrance}\\&\text{time of } Y \text{ in the set } \{N\} \text{ and } T_{\widetilde{B}} \circ \theta_{R_2} \text{ under } P_0\,.\end{aligned} \tag{1.17}$$

As an immediate consequence we see that for $k \geq 1$,

$$\begin{aligned}&\text{under } P_0,\, D_{k+1} - D_1 \text{ has the same law as the sum of two independent}\\&\text{variables } U_k \text{ and } V_k \text{ respectively distributed as the entrance time of } Y\\&\text{in the set } \{kN\} \text{ and as the sum of } k \text{ independent variables } T_{\widetilde{B}} \circ \theta_{R_2}\\&\text{under } P_0\,.\end{aligned} \tag{1.18}$$

We then note that:

Lemma 1.3.

$$\sup_{x \in \widetilde{B}} E_x\Big[\exp\Big\{\frac{c}{N^2}\, T_{\widetilde{B}}\Big\}\Big] \leq c',\ \textit{for } N \geq 1\,. \tag{1.19}$$

Proof. This is a consequence of Khaśminskii's lemma, cf. [10], and the estimate

$$\sup_{z \in \widetilde{B}} E_z[T_{\widetilde{B}}] \leq cN^2\,, \tag{1.20}$$

see for instance Lemma 1.1 of [13], p. 292. □

With standard Cramer-type estimates, it now follows from (1.18), (1.19), that for some positive constant c and any $\epsilon > 0$:

$$P_0\big[V_{[N^{(d-1)+\epsilon}]} > c\, N^{(d+1)+\epsilon}\big] \underset{N\to\infty}{\longrightarrow} 0\,. \tag{1.21}$$

It also follows from the remark below (1.16) and Example 6.6 in Chapter 7 of Durrett [9], p. 369, that with hopefully obvious notations:

$$P^Y\big[H_{[N^{(d-1)+\epsilon}]N} > N^{2d+3\epsilon}\big] \underset{N\to\infty}{\longrightarrow} 0\,. \tag{1.22}$$

We thus find that for $\epsilon > 0$ and large N:

$$\begin{aligned}P_0[T_N \leq C_N \leq N^{2d+4\epsilon}] &\overset{(1.15)}{\geq} P_0\big[D_{[N^{(d-1)+\epsilon}]} \leq N^{2d+4\epsilon}\big] - N^d\, e^{-cN^{\epsilon}}\\ &\overset{(1.17),(1.18)}{\geq} P_0[D_1 \leq N^{2d+3\epsilon}]\, P_0\big[U_{[N^{(d-1)+\epsilon}]} \leq N^{2d+3\epsilon}\big] P_0\big[V_{[N^{(d-1)+\epsilon}]} \leq N^{2d+3\epsilon}\big]\\ &\quad - N^d\, e^{-cN^{\epsilon}} \underset{N\to\infty}{\longrightarrow} 1,\ \text{as } N \to \infty\,,\end{aligned} \tag{1.23}$$

using (1.21), (1.22) in the last step, together with the fact that $D_1 = T_{\widetilde{B}}$, P_0-a.s.. Since ϵ is an arbitrary positive number, the claim (1.3) now follows. □

Remark 1.4.

1) When $d = 1$, Theorem 1.1 remains true. One only needs to replace $N^{-(d-1)}$ with $(\log N)^{-1}$ in (1.6) of Lemma 1.2, see for instance Proposition 1.6.7 of [12]. Inserting this new lower bound in (1.13), (1.14), the proof of Theorem 1.1 goes otherwise unchanged.
2) We refer to Dembo-Peres-Rosen-Zeitouni [6], and also to Lawler [11], where the asymptotic analysis of the cover time of a ball of radius N by the two-dimensional simple random walk is analyzed. This problem has some common flavor with the investigation of the large N behavior of C_N, which in this note comes as a subsidiary issue to the asymptotic analysis of T_N.

2. The lower bound

The main object of this section is to derive a lower bound on T_N, cf. Theorem 2.1. In combination with Theorem 1.1 this completes the proof of Theorem 1 of the Introduction, in particular showing that when $d \geq 2$, the cover time of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$ is in principal order comparable to T_N.

Theorem 2.1. *($d \geq 2$)*

$$\forall \delta > 0, \lim_{N\to\infty} P_0\Big[\frac{\log T_N}{\log N} \geq 2d - \delta\Big] = 1\,, \tag{2.1}$$

(as a matter of fact (2.1) *holds also for* $d = 1$*, cf. Remark* 2.6 *below).*

Proof. We denote with P^N the law of the walk with initial distribution ν_N the uniform measure on B, see (1.4). Thanks to translation invariance,

$$T_N \text{ has same distribution under } P_0 \text{ and } P^N\,. \tag{2.2}$$

The claim (2.1) will hence follow if we replace P_0 with P^N. We introduce the positive numbers

$$\delta, \gamma \in (0,1) \text{ and } \delta' = \frac{\delta}{3}\,, \alpha \in (0, 3/4), \tag{2.3}$$

and for $N \geq 2$, $x \in E$, the numbers

$$L = [N^\gamma],\ \ell = 1000[L/(\log N)^\alpha], \tag{2.4}$$

and the subsets of E, see (1.4) and the beginning of Section 1 for the notations,

$$B(x) = x + B,\ \widetilde{B}(x) = x + \widetilde{B}\,, \tag{2.5}$$

$$C(x) = B_\infty(x, L),\ \widetilde{C}(x) = B_\infty(x, 2L)\,. \tag{2.6}$$

$$D(x) = B_\infty(x, \ell),\ \widetilde{D}(x) = B_\infty(x, 2\ell). \tag{2.7}$$

Let us briefly explain the strategy of the proof. When starting in $C(x)$ the walk spends a time of order $N^{2\gamma}$ in $C(x)$ until it exits $\widetilde{C}(x)$. We are first going to show that when γ is small, cf. (2.22), with probability tending to 1 as N goes to infinity, the time spent by the trajectory $X_{[0,N^{2d-\delta}]}$ in any $C(x)$ is at most $O(N^{2\gamma}\log N)$, cf. (2.26). In essence this will correspond to showing that it is unlikely any $C(x)$ gets visited too often or any such visit lasts too long.

Then we will see, cf. Lemma 2.4, that when $d \geq 3$, for large N, any set disconnecting E has at least $O(N^{d\gamma})$ points in some $C(x)$. This and (2.26) will show that when $d \geq 3$, with probability tending to 1 as N goes to infinity T_N is bigger or equal to $N^{2d-\delta}$.

In case $d = 2$, taking L and ℓ as in (2.4), we show, cf. Lemma 2.5, that for some $c = c(\gamma, \alpha) > 0$, any set S disconnecting E has a collection $\mathcal{E}_*$ of $c(L/\ell)^2$ points y on a common $O(\ell)$-sub-grid $\mathcal{L}_{x_*}$ of the same box $C(x_*)$, with the following property: within each sub-box $D(y)$ centered at $y \in \mathcal{E}_*$, at least $c\ell^2$ of the segments that are intersection of $D(y)$ with translates of the i_*-th coordinate axis, meet S. It suffices to consider the $O(\exp(c(\log N)^{2\alpha\vee 1}(\log\log N))$ possible collections $\mathcal{E}_*$ of this type in the slab corresponding to $|x_*^3| \leq N^4$. For any of these x_*, $\mathcal{E}_*$, and i_*, the trajectory $X_.$ hits more than $c|\mathcal{E}_*|\ell^2$ segments in $D(y)$, $y \in \mathcal{E}_*$, during its first $O(\log N)$ excursions from $C(x_*)$ to $\widetilde{C}(x_*)$, with probability of at most $\exp(-c|\mathcal{E}_*|^{1/3}\log\ell)$. Taking $\alpha < 3/4$, this is $O(\exp(-c(\log N)^{\beta}))$ for some $\beta > 2\alpha \vee 1$, cf. (2.43). If γ is small, then with probability tending to 1 as N goes to infinity, there are at most $c_0 \log N$ excursions of $X_.$ from $C(x)$ to $\widetilde{C}(x)$ by time $N^{2d-\delta}$, cf. (2.41), so (2.43) shows that T_N is at least $N^{2d-\delta}$.

Our first goal is to prove (2.26). We denote with $R_k^x \leq D_k^x$, $k \geq 1$, the successive times of return to $B(x)$ and departure from $\widetilde{B}(x)$, defined analogously as in (1.5) with B and $\widetilde{B}$ replaced by $B(x)$ and $\widetilde{B}(x)$. In what follows, when this causes no confusion, we will simply drop the superscript x and write R_k, D_k in place of R_k^x, D_k^x, for simplicity.

We want to investigate the number of returns to $C(x)$ and departures from $\widetilde{C}(x)$ performed by $X_.$ during each time interval $[R_k, D_k - 1]$, keeping in mind that for large N,

$$X_. \text{ lies in } B(x)^c \subset \widetilde{C}(x)^c \text{ during each time interval } [0, R_1 - 1], \text{ and } [D_k, R_{k+1} - 1],\ k \geq 1\,. \tag{2.8}$$

We thus define the sequence of stopping times:

$$\begin{aligned} &R_1' = D_1 \wedge (H_{C(x)} \circ \theta_{R_1} + R_1),\ \ D_1' = D_1 \wedge (T_{\widetilde{C}(x)} \circ \theta_{R_1'} + R_1')\,, \\ &\text{and for } m \geq 1\,, \\ &R_{m+1}' = D_1 \wedge (H_{C(x)} \circ \theta_{D_m'} + D_m'),\ \ D_{m+1}' = D_1 \wedge (T_{\widetilde{C}(x)} \circ \theta_{R_{m+1}'} + R_{m+1}')\,. \end{aligned} \tag{2.9}$$

The number of returns to $C(x)$ and departures from $\widetilde{C}(x)$ during $[R_1, D_1 - 1]$ is then

$$N_1^x = \sum_{m\geq 1} 1\{D_m' < D_1\}\,, \tag{2.10}$$

and the corresponding number during $[R_k, D_k - 1]$, $k \geq 2$, is

$$N_k^x = N_1^x \circ \theta_{R_k^x}\,,$$

where the above equality holds matter-of-factly for $k = 1$ as well. We will use the following, (see (2.3) for the notation):

Proposition 2.2. $(d \geq 2,\ \delta \in (0,1),\ 0 < \gamma \leq \frac{\delta'}{(d-1)})$
There is a constant $c_0 \geq 1$ *such that*

$$\lim_{N\to\infty} P^N\Big[\sup_{x\in E} \sum_{k\geq 1} N_k^x\ 1\big\{R_k^x \leq N^{2d-\delta}\big\} \geq c_0(\log N)\Big] = 0\,. \tag{2.11}$$

Proof. Note that for $x \in E$, $k \geq 1$, $D_{k+1}^x - D_1^x$ has the same distribution under P^N as $D_{k+1} - D_1$ in (1.18). Hence for large N, for any $x \in E$, using the strong Markov property for the random walk Y of (1.16) at the entrance times in k, we find

$$\begin{aligned}
&P^N\big[R^x_{[N^{(d-1)(1-\gamma)}]} \leq N^{2d-\delta}\big] \leq P^Y\big[H_{N([N^{(d-1)(1-\gamma)}]-2)} \leq N^{2d-\delta}\big] \\
&\leq P^Y[H_1 \leq N^{2d-\delta}]^{N([N^{(d-1)(1-\gamma)}]-2)} \leq (1-cN^{-(d-\delta/2)})^{\frac{1}{2}N^{1+(d-1)(1-\gamma)}} \\
&\leq \exp\big\{-cN^{1+(d-1)(1-\gamma)-d+\delta/2}\big\} \leq \exp\{-cN^{\delta/6}\big\}\,,
\end{aligned} \tag{2.12}$$

thanks to our assumption on γ, as well as (3.4) of Chapter 3 of Durrett [9], and the remark below (1.16).

Then observe that P^N-a.s., up to time $N^{2d-\delta}$, the $\mathbb{Z}$-component of $X_.$ remains bounded in absolute value by $N + N^{2d-\delta}$. Hence, for large N, the sum inside the probability in (2.11) vanishes for any $x \in E$ with $|x^{d+1}| \geq N^{2d}$. From this remark and (2.12), we see that the claim (2.11) follows from:

$$\lim_{N\to\infty} P^N\Big[\sup_{x:|x^{d+1}|\leq N^{2d}} \sum_{1\leq k\leq N^{(d-1)(1-\gamma)}} N_k^x \geq c_0(\log N)\Big] = 0\,. \tag{2.13}$$

The proof of (2.13) will rely on the next

Lemma 2.3. $(d \geq 2,\ 0 < \gamma < 1)$
There are positive constants $c_1, c_2 > 0$ *such that for large* N, $0 \leq \lambda \leq c_1$, *and* $x \in E$*:*

i) P^N*-a.s., for all* $k \geq 2$, $E_{X_{R_k^x}}[e^{\lambda N_1^x}] \leq 1 + c_2\,\lambda N^{-(d-1)(1-\gamma)}$

ii) $E^N[e^{\lambda N_1^x}] \leq 1 + c_2\,\lambda N^{-(d-1)(1-\gamma)}$. (2.14)

Proof. First observe that for $k \geq 1$, $\lambda \geq 0$, $x \in E$, dropping the superscripts from the stopping times (as mentioned before), we find:

$$E_{X_{R_k}}[e^{\lambda N_1^x}] = 1 + (e^\lambda - 1) \sum_{m\geq 0} e^{\lambda m}\ P_{X_{R_k}}[N_1^x > m]\,, \tag{2.15}$$

and with N large, for $x \in E$, $m, k \geq 1$:

$$\begin{aligned}
P_{X_{R_k}}[N_1^x > m] &\overset{(2.10)}{=} P_{X_{R_k}}[D'_{m+1} < D_1] \overset{(2.9)}{=} P_{X_{R_k}}[R'_{m+1} < D_1] \\
&= E_{X_{R_k}}\big[D'_m < D_1,\ P_{X_{D'_m}}[H_{C(x)} < D_1]\big]\,,
\end{aligned} \tag{2.16}$$

using the strong Markov property at D'_m in the last step.

Then note that the simple random walk on $\mathbb{Z}^{d+1} \simeq \mathbb{Z}^d \times \mathbb{Z}$, for large N, when starting at $y \in \partial B_\infty(0, 2[N^\gamma])$ has a probability bigger than $c > 0$ of first reaching $B_\infty(0, [\frac{N}{4}])^c$ without entering $B_\infty(0, [N^\gamma])$ and then exiting $\mathbb{Z}^d \times [-2N, 2N]$ without entering $B_\infty(0, [N^\gamma]) + N\mathbb{Z}^d \times \{0\}$, as follows from instance for the invariance principle and standard estimates on the Green function. Hence for large N, and any $x \in E$,

$$\forall y \in \partial \widetilde{C}(x),\ \ P_y[H_{C(x)} < D_1] \le (1-c)\,.$$

Inserting this inequality in the last line of (2.16), we find that for large N, $m, k \ge 1$:

$$\begin{aligned} P_{X_{R_k}}[N_1^x > m] &= P_{X_{R_k}}[D'_{m+1} < D_1] \le (1-c)\, P_{X_{R_k}}[D'_m < D_1] \\ &\overset{\text{induction}}{\le} (1-c)^m P_{X_{R_k}}[D'_1 < D_1] = (1-c)^m P_{X_{R_k}}[N_1^x > 0]\,. \end{aligned}$$

Coming back to (2.15), we see that when $e^\lambda(1-c) < 1$, for large N, any $x \in E$, and $k \ge 1$:

$$P^N\text{-a.s.},\ \ E_{X_{R_k}}[e^{\lambda N_1^x}] \le 1 + \frac{(e^\lambda - 1)}{1 - e^\lambda(1-c)}\, P_{X_{R_k}}[N_1^x > 0]\,. \tag{2.17}$$

Note that when $k \ge 2$, P^N-a.s., $X_{R_k} \in \partial(B(x)^c)$, and

$$\begin{aligned} P_{X_{R_k}}[N_1^x > 0] &\le \sup_{y \in \partial(B(x)^c)} P_y\big[H_{C(x)} < T_{\widetilde{B}(x)}\big] \\ &\underset{\text{translation invariance}}{\overset{\text{strong Markov}}{\le}} \sup_{y \in \partial B^c} E_y\Big[\sum_{n=0}^{T_{\widetilde{B}}-1} 1\{X_n \in C(0)\}\Big] \Big/ \inf_{z \in C(0)} E_z\Big[\sum_{n=0}^{T_{\widetilde{B}}-1} 1\{X_n \in C(0)\}\Big]\,. \end{aligned} \tag{2.18}$$

Using now estimates on the Green function of simple random walk in a strip $V = \mathbb{Z}^d \times \{-2N+1, \dots, 2N-1\}$, cf. (2.14) of [14], to bound the numerator from above, cf. the term before the multiplication sign in (2.19) below, and for the denominator a similar inequality as (1.11) to bound $g_V(\cdot,\cdot)$ from below by $cg(\cdot,\cdot)$ on $B_\infty(0, [N^\gamma]) \times B_\infty(0, [N^\gamma])$, cf. the term after the multiplication sign in (2.19) below, we see that for large N, for $k \ge 2$, and any $x \in E$:

$$P^N\text{-a.s.},\ \ P_{X_{R_k}}[N_1^x > 0] \le c\,\frac{N^{\gamma(d+1)}}{N^{(d-1)}} \times N^{-2\gamma} = cN^{-(d-1)(1-\gamma)}\,. \tag{2.19}$$

Inserting this bound in (2.17), the claim (2.14) i) readily follows. As for (2.14) ii), noting that X_0 is uniformly distributed over B under P^N, we see that

$$\begin{aligned} P^N[N_1^x > 0] &= P^N\big[X_0 \notin B(x),\, H_{C(x)} \circ \theta_{R_1^x} < T_{\widetilde{B}(x)} \circ \theta_{R_1^x}\big] \\ &\quad + P^N[X_0 \in B(x),\, H_{C(x)} < T_{\widetilde{B}(x)}] \overset{(2.18),(2.19)}{\le} cN^{-(d-1)(1-\gamma)} \\ &\quad + |B|^{-1} \sum_{y \in B(0) \cap B(x)} E_y\Big[\sum_{n=0}^{T_{\widetilde{B}(x)}-1} 1\{X_n \in C(x)\}\Big] \Big/ cN^{2\gamma}\,, \end{aligned}$$

where we have used once again the same lower bound on the denominator of the last expression in (2.18), as explained above (2.19). From the reversibility of the walk on E with respect to the counting measure, the Green function of the walk killed outside $\widetilde{B}(x)$, (that is defined analogously to (1.9)), is symmetric in its arguments. We hence find that

$$\begin{aligned}&P^N[N_1^x > 0] \le cN^{-(d-1)(1-\gamma)}\\&\quad + cN^{-(d+1+2\gamma)} \sum_{z\in C(x)} E_z\Big[\sum_{n=0}^{T_{\widetilde{B}(x)}-1} 1\{X_n \in B(0)\cap B(x)\}\Big] \le cN^{-(d-1)(1-\gamma)}\\&\quad + cN^{-(d+1+2\gamma)} N^{(d+1)\gamma} \sup_{z\in C(x)} E_z[T_{\widetilde{B}(x)}] \overset{(1.20)}{\le} cN^{-(d-1)(1-\gamma)}.\end{aligned}$$

Coming back to (2.17), with $k = 1$, and integrating over the distribution of X_{R_1}, we readily obtain (2.14) ii). □

We will now prove (2.13) and thus conclude the proof of Proposition 2.2. To this end we choose $0 < \lambda \le c_1$, cf. (2.14), and using the strong Markov property at R_k^x we find that for large N, any $x \in E$,

$$E^N\Big[\exp\Big\{\lambda \sum_{1\le k\le N^{(d-1)(1-\gamma)}} N_k^x\Big\}\Big] \le \Big(1 + \frac{c_2\lambda}{N^{(d-1)(1-\gamma)}}\Big)^{N^{(d-1)(1-\gamma)}} \le \exp\{c_2\lambda\}\,.$$

Hence for large N, the probability in (2.13) is smaller than

$$cN^{2d+d}\exp\{-\lambda c_0(\log N) + c_2\lambda\}\,,$$

and choosing $\lambda = c_1$, c_0 large enough we obtain (2.13). □

We now come back to the proof of Theorem 2.1. For $x \in E$, we define the successive returns to $C(x)$ and departures from $\widetilde{C}(x)$ of the walk:

$$\begin{aligned}&\widetilde{R}_1^x = H_{C(x)},\ \widetilde{D}_1^x = T_{\widetilde{C}(x)} \circ \theta_{\widetilde{R}_1^x} + \widetilde{R}_1^x,\ \text{and for } m \ge 1\,,\\&\widetilde{R}_{m+1}^x = H_{C(x)} \circ \theta_{\widetilde{D}_m^x} + \widetilde{D}_m^x,\ \widetilde{D}_{m+1}^x = T_{\widetilde{C}(x)} \circ \theta_{\widetilde{R}_{m+1}^x} + \widetilde{R}_{m+1}^x\,.\end{aligned} \tag{2.20}$$

Again for simplicity, we write $\widetilde{R}_m$, $\widetilde{D}_m$ in place of $\widetilde{R}_m^x$, $\widetilde{D}_m^x$ when this causes no confusion. With (2.8) we also find:

$$\begin{aligned}&\text{for large } N,\ P^N\text{-a.s., for all } x \in E,\ \sup\{m \ge 1;\ \widetilde{R}_m^x \le N^{2d-\delta}\}\\&\quad\le \sum_{k\ge 1} N_k^x\, 1\{R_k^x \le N^{2d-\delta}\}\,.\end{aligned} \tag{2.21}$$

From now on we assume that γ, cf. (2.3), satisfies

$$0 < \gamma \le \frac{\delta'}{(d-1)}\,. \tag{2.22}$$

From (2.11), using the fact that visits to $C(x)$ only occur during the time intervals $[\widetilde{R}_m, \widetilde{D}_m - 1]$, we find that:

$$\lim_{N\to\infty} P^N\Big[\forall x \in E,\ \sum_{n=0}^{N^{2d-\delta}} 1\{X_n \in C(x)\} \le \sum_{m=1}^{c_0(\log N)} T_{\widetilde{C}(x)} \circ \theta_{\widetilde{R}^x_m}\Big] = 1\,. \tag{2.23}$$

Analogously to (1.19), we also have for $N \ge 1$:

$$\sup_{x,y\in E} E_y\Big[\exp\Big\{\frac{c}{N^{2\gamma}}\, T_{\widetilde{C}(x)}\Big\}\Big] \le c'\,. \tag{2.24}$$

Note that for large N, P^N-a.s., the first sum in the probability in (2.23) vanishes for all $x \in E$ with $|x^{d+1}| \ge N^{2d}$. We hence find that:

$$\begin{aligned}
&\overline{\lim_{N\to\infty}}\ P^N\Big[\text{for some } x \in E,\ \sum_{n=0}^{N^{2d-\delta}} 1\{X_n \in C(x)\} \ge c_3(\log N)N^{2\gamma}\Big] \\
&\le \overline{\lim_{N\to\infty}}\ cN^{2d+d} \sup_{x\in E} P^N\Big[\sum_{1\le m\le c_0(\log N)} T_{\widetilde{C}(x)} \circ \theta_{\widetilde{R}^x_m} \ge c_3(\log N)N^{2\gamma}\Big] \\
&\overset{(2.24)}{\le} \overline{\lim_{N\to\infty}}\ cN^{3d}\ \exp\{-cc_3(\log N)\}\, c'^{\,c_0(\log N)} = 0\,,
\end{aligned} \tag{2.25}$$

if c_3 is chosen large enough. In other words, when γ fulfills (2.22), we see that:

$$\lim_{N\to\infty} P^N\Big[\text{for all } x \in E,\ \sum_{n=0}^{N^{2d-\delta}} 1\{X_n \in C(x)\} \le c_3(\log N)N^{2\gamma}\Big] = 1\,. \tag{2.26}$$

To conclude the proof of Theorem 2.1 for $d \ge 3$, we will use the next geometric lemma that holds true for $d \ge 1$ and general $0 < \gamma < 1$. We refer to the end of the Introduction for our convention concerning constants.

Lemma 2.4. $(d \ge 1,\ 0 < \gamma < 1)$

There is a positive constant $c(\gamma)$ such that for $N \ge c(\gamma)$, whenever $S \subseteq E$ disconnects E, there is an $x \in E$ such that

$$|C(x) \cap S| \ge cN^{d\gamma}, \ \textit{(cf. (2.6) for the notation)}\,. \tag{2.27}$$

Proof. Assume S disconnects E, and denote with $\mathcal{T}op$ the connected component of $E\backslash S$ containing $(\mathbb{Z}/N\mathbb{Z})^d \times [M,\infty)$, when M is large. We can define the function:

$$t(x) = \frac{1}{|C(x)|} \sum_{y\in C(x)} 1\{y \in \mathcal{T}op\} = \frac{|\mathcal{T}op \cap C(x)|}{|C(x)|}\,,\ x \in E\,. \tag{2.28}$$

Note that

$$t(x) = 1, \text{ for large } x^{d+1},\ t(x) = 0, \text{ for large negative } x^{d+1}\,.$$

Moreover when $|x - x'| = 1$, (with Δ standing for the symmetric difference)

$$|t(x) - t(x')| \le \frac{|C(x)\Delta C(x')|}{|C(0)|} \le \frac{c}{N^\gamma}\,.$$

Thus for $N \ge c(\gamma)$, there is at least one $x_* \in E$ such that:

$$\Big|t(x_*) - \frac{1}{2}\Big| \le \frac{c}{N^\gamma} < \frac{1}{4}\,. \tag{2.29}$$

Then for $A \subseteq C(x_*)$ we define the relative boundary of A:

$$\partial_{C(x_*)} A = \{y \in C(x_*) \backslash A; \exists\, x \in A \text{ such that } |y - x| = 1\}. \tag{2.30}$$

Observe that:

$$\partial_{C(x_*)}(\mathcal{T}op \cap C(x_*)) \subseteq S \cap C(x_*), \tag{2.31}$$

indeed any point in $C(x_*)$ neighbor of a point in $\mathcal{T}op \cap C(x_*)$ has to belong to S if it is not in $\mathcal{T}op \cap C(x_*)$. Moreover from the isoperimetric controls in (A.3), p. 480 of [8],

$$|\partial_{C(x_*)}(\mathcal{T}op \cap C(x_*))| \geq c\, |\mathcal{T}op \cap C(x_*)|^{\frac{d}{d+1}} \overset{(2.29)}{\geq} c\, |C(x_*)|^{\frac{d}{d+1}} \geq c N^{d\gamma}.$$

This and (2.31) proves (2.27). □

Assuming $d \geq 3$, and (2.22), we see that (2.26) and (2.27) imply that

$$P^N\big[X_{[0,N^{2d-\delta}]} \text{ disconnects } E\big] \underset{N \to \infty}{\longrightarrow} 0, \tag{2.32}$$

which is the statement of Theorem 2.1.

Considering now the case of $d = 2$, we need a few additional notations, starting with the grids in $C(x)$:

$$\mathbb{L}_x \overset{\text{def}}{=} \big(\ell \mathbb{Z}^3 \cap B_\infty(0, L)\big) + x \subseteq \mathcal{L}_x \overset{\text{def}}{=} \Big(\frac{\ell}{1000}\, \mathbb{Z}^3 \cap B_\infty(0, L)\Big) + x \subseteq C(x). \tag{2.33}$$

For $i \in \{1, 2, 3\}$, we denote with π^i the respective projections $E \to \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}$, when $i = 1, 2$, or $(\mathbb{Z}/N\mathbb{Z})^2$, when $i = 3$, obtained by omitting the ith component of $x = (x^1, x^2, x^3) \in E = (\mathbb{Z}/N\mathbb{Z})^2 \times \mathbb{Z}$, and replace Lemma 2.4 with the following geometric lemma.

Lemma 2.5. *In case $d = 2$, there is a positive constant $c(\gamma, \alpha)$ such that for $N \geq c(\gamma, \alpha)$, whenever $S \subseteq E$ is a finite subset disconnecting E, one can find $x_* \in E$, $i_* \in \{1, 2, 3\}$, and $\mathcal{E}_* \subseteq \mathcal{L}_{x_*}$ with*

$$|\mathcal{E}_*| = [c(L/\ell)^2], \text{ and } y \neq y' \text{ in } \mathcal{E}_* \implies |y - y'|_\infty \geq 10\ell, \tag{2.34}$$

and

$$|\pi^{i*}(S \cap D(y))| \geq c\ell^2, \qquad \text{for each } y \in \mathcal{E}_*. \tag{2.35}$$

Proof. Similarly to (2.28) we define the function

$$u(y) = \frac{|\mathcal{T}op \cap D(y)|}{|D(y)|}, \quad y \in E, \tag{2.36}$$

(see (2.7) for the definition of $D(y)$), and for $x \in E$, let

$$\tau(x) = \sum_{y \in \mathbf{L}_x} 1\Big\{|y - x|_\infty \leq \frac{1}{2}L,\ u(y) \geq \frac{1}{2}\Big\} \Big/ \sum_{y \in \mathbf{L}_x} 1\Big\{|y - x|_\infty \leq \frac{1}{2}L\Big\}. \tag{2.37}$$

If $x, x' \in E$ are such that $|x^3 - (x')^3| = \ell$, then

$$|\tau(x) - \tau(x')| \leq c\frac{\ell}{L}.$$

Note that when the "vertical" component x^3 of x is large positive $\tau(x) = 1$, whereas when it is large negative $\tau(x) = 0$. Hence when $N \geq c(\gamma, \alpha)$, we can find $x_* \in E$ such that

$$|\tau(x_*) - \frac{1}{2}| \leq \frac{1}{4}. \tag{2.38}$$

Consider the discrete box $\mathcal{B}_* = \{y \in \mathbb{L}_{x_*};\ |y - x_*|_\infty \leq \frac{1}{2}L\}$ and its subset $\mathcal{A}_* = \{y \in \mathcal{B}_*;\ u(y) \geq \frac{1}{2}\}$. In view of (2.38) and the isoperimetric controls in (A.3) of Deuschel-Pisztora [8] (here $d = 2$), we have that

$$|\partial_{\mathcal{B}_*}\mathcal{A}_*| \geq c|\mathcal{B}_*|^{2/3} \geq c\Big(\frac{L}{\ell}\Big)^2. \tag{2.39}$$

Observe that for each $y \in \partial_{\mathcal{B}_*}\mathcal{A}_*$ there is a $y' \in \mathcal{B}_*$ which is a neighbor of y in $\mathbb{L}_{x_*}$, such that $y' \in \mathcal{A}_*$, whereas $y \notin \mathcal{A}_*$. We then look at $z \in \mathcal{L}_{x_*} \cap [y, y']$, where $[y, y']$ denotes the "segment" $\{\mu y + (1 - \mu)y';\ 0 \leq \mu \leq 1\}$. Observe that when N is large (i.e. $\geq c(\gamma, \alpha)$), whenever z and z' are neighbors in $\mathcal{L}_{x_*}$:

$$|u(z) - u(z')| \leq \frac{|D(z)\Delta D(z')|}{|D(z)|} \leq \frac{2\ell}{1000}\ \frac{(2\ell-1)^2}{(2\ell-1)^3} \leq 10^{-2}.$$

Since $u(y') \geq \frac{1}{2}$ and $u(y) < \frac{1}{2}$, we can choose a $z(y) \in \mathcal{L}_{x_*} \cap [y, y']$ for each $y \in \partial_{\mathcal{B}_*}\mathcal{A}_*$ with $|u(z) - \frac{1}{2}| \leq 10^{-2}$, and naturally $|z - y|_\infty \leq \ell$.

Then with a similar argument as in (2.31) and (A.6) of Deuschel-Pisztora [8], we see that for some $i(z) \in \{1, 2, 3\}$,

$$|\pi^{i(z)}\big(S \cap D(z)\big)| \geq |\pi^{i(z)}(\partial_{D(z)}\mathcal{T}op \cap D(z))| \geq c\ell^2. \tag{2.40}$$

Therefore, looking at the restriction of $\partial_{\mathcal{B}_*}\mathcal{A}_*$ to the sub-grids $\big((100\ell\mathbb{Z}^3 + \ell v) \cap B_\infty(0, L)\big) + x_* \subseteq C(x_*)$ of $\mathbb{L}_{x_*}$, for $v \in \{0, \ldots, 99\}^3$, one can make sure that one such restriction has at least cardinality $c(L/\ell)^2$. By further considering for y in this restriction the $z(y)$ with same $i(z(y))$, we can then find a subset $\mathcal{E}_*$ of $\mathcal{L}_{x_*}$ and an $i_* \in \{1, 2, 3\}$ so that both (2.34) and (2.35) hold. □

Recall the definition for $x \in E$, of the successive returns $\widetilde{R}_k^x$, $k \geq 1$, to $C(x)$ and of the departures from $\widetilde{C}(x)$, $\widetilde{D}_k^x$, $k \geq 1$, cf. (2.20). Assuming (2.22), we know from (2.21), (2.11), that

$$\lim_{N\to\infty} P^N[\inf_{x\in E} \widetilde{R}^x_{[c_0' \log N]} \leq N^{2d-\delta}] = 0, \tag{2.41}$$

for a suitable constant $c_0' \geq 1$. We thus see that

$$\overline{\lim_{N\to\infty}}\, P^N[T_N \leq N^{4-\delta}] \leq \overline{\lim_{N\to\infty}}\, P^N[X_{[0, \inf\{\widetilde{R}^x_{[c_0' \log N]} : x\in E\} \wedge N^{4-\delta}]} \text{ disconnects } E] \tag{2.42}$$

and using Lemma 2.5

$$\leq \overline{\lim_{N\to\infty}} \sum_{x_*,\mathcal{E}_*,i_*} P^N\Big[\forall z \in \mathcal{E}_*, |\pi^{i*}(X_{[0,\widetilde{R}^{x_*}_{[c'_0 \log N]}]} \cap D(z))| \geq c\ell^2\Big],$$

where in the above sum $x_* \in E$ is such that $|x_*^3| \leq N^4$, $i_* \in \{1, 2, 3\}$ and $\mathcal{E}_* \subseteq \mathcal{L}_{x_*}$ satisfies (2.34). In this sum we consider cN^6 points $x_* \in E$, and since $|\mathcal{L}_{x_*}| \leq c'(L/\ell)^3$, for each x_* there are at most $\exp[c(L/\ell)^2 \log(cL/\ell)]$ subsets $\mathcal{E}_* \subset \mathcal{L}_{x_*}$ of size $[c(L/\ell)^2]$ to consider. As $L/\ell = 10^{-3}(\log N)^\alpha$, to conclude the proof of the theorem it thus suffices to show that for some $\beta > 2\alpha \vee 1$, some $\rho > 0$, all N large enough and any x_*, $\mathcal{E}_*$, i_* as above,

$$\begin{aligned} q_{x_*,\mathcal{E}_*,i_*} &= P^N[\forall y \in \mathcal{E}_*, |\pi^{i_*}(X_{[0,\widetilde{R}^{x_*}_{[c'_0 \log N]}]} \cap D(y))| \geq c\ell^2] \\ &\leq \exp(-\rho(\log N)^\beta)\,. \end{aligned} \tag{2.43}$$

Fixing now N and x_*, $\mathcal{E}_*$, i_* as above, for each $y \in E$ we denote by $\mathcal{S}(y)$ the collection of $(2\ell-1)^2$ disjoint discrete segments $I = \{z' \in D(y);\ \pi^{i*}(z') = \pi^{i*}(z)\}$ of length $(2\ell-1)$ each, that partition $D(y)$, and let $\mathcal{S}_*$ be the union of the collections $\mathcal{S}(y)$ for $y \in \mathcal{E}_*$.

Since for any $y \in E$ and (possibly random) time t,

$$|\pi^{i_*}(X_{[0,t]} \cap D(y))| = \sum_{I\in\mathcal{S}(y)} 1\{H_I < t\}\,,$$

it follows that for $N \geq c(\gamma, \alpha)$, and any $\lambda > 0$ we have:

$$q_{x_*,\mathcal{E}_*,i_*} \leq \exp\{-\lambda|\mathcal{E}_*|c\ell^2\} E^N\Big[\exp\big\{\lambda \sum_{I\in\mathcal{S}_*} 1\{H_I < \widetilde{D}^{x_*}_{[c'_0 \log N]}\}\big\}\Big], \tag{2.44}$$

where for $U \subseteq E$, the notations H_U and T_U respectively denote the entrance and exit times of $X_.$ in or from U, cf. (1.2). Note that P^N-a.s., for $k \geq 1$,

$$\sum_{I\in\mathcal{S}_*} 1\{H_I < \widetilde{D}^{x_*}_{k+1}\} \leq \sum_{I\in\mathcal{S}_*} 1\{H_I < \widetilde{D}^{x_*}_k\} + \Big(\sum_{I\in\mathcal{S}_*} 1\{H_I < T_{\widetilde{C}(x_*)}\}\Big) \circ \theta_{\widetilde{R}^{x_*}_k}\,,$$

and hence using the strong Markov property at times $\widetilde{R}^{x_*}_k$ we find that

$$E^N\left[\exp\big\{\lambda \sum_{I\in\mathcal{S}_*} 1\{H_I < \widetilde{D}^{x_*}_{[c'_0 \log N]}\}\big\}\right] \leq \Big(\sup_{z\in C(x_*)} E_z[\exp\{\lambda V_*\}]\Big)^{c'_0 \log N}, \tag{2.45}$$

where $V_* = \sum_{I \in \mathcal{S}_*} 1\{H_I < T_{\widetilde{C}(x_*)}\}$. Further for $z \in C(x_*)$, we have

$$\begin{aligned}
E_z\Big[\exp\{\lambda V_*\}\Big] &= \sum_{m\geq 0} \frac{\lambda^m}{m!} E_z\Big[V_*^m\Big] \\
&\leq \sum_{m\geq 0} \frac{\lambda^m}{m!} \sum_{\sigma \in \mathcal{S}_m} \sum_{I_1,\dots,I_m \in \mathcal{S}_*} P_z[H_{I_{\sigma(1)}} \leq \dots \leq H_{I_{\sigma(m)}} < T_{\widetilde{C}(x_*)}] \\
&= \sum_{m\geq 0} \lambda^m \sum_{I_1,\dots,I_m \in \mathcal{S}_*} P_z[H_{I_1} \leq \dots \leq H_{I_m} < T_{\widetilde{C}(x_*)}] \\
&\leq 1 + \sum_{m\geq 1} \lambda^m \sum_{I_1,\dots,I_{m-1} \in \mathcal{S}_*} E_z\Big[H_{I_1} \leq \dots \leq H_{I_{m-1}} < T_{\widetilde{C}(x_*)}, E_{X_{H_{I_{m-1}}}}(V_*)\Big] \\
&\leq \sum_{m\geq 0} \lambda^m \Big(\sup_{z\in C(x_*)} E_z(V_*)\Big)^m. \qquad (2.46)
\end{aligned}$$

Moreover, for $z \in C(x_*)$, by using the strong Markov property at the stopping time $H_{D(y)}$, we have that

$$\begin{aligned}
E_z(V_*) &= \sum_{y\in\mathcal{E}_*} \sum_{I\in\mathcal{S}(y)} P_z[H_{D(y)} < T_{\widetilde{C}(x_*)}, P_{X_{H_{D(y)}}}[H_I < T_{\widetilde{C}(x_*)}]] \\
&\leq \sum_{y\in\mathcal{E}_*} P_z[H_{D(y)} < T_{\widetilde{C}(x_*)}] \times \sup_{y\in C(x_*), v\in D(y)} \\
&\quad \cdot \sum_{I\in\mathcal{S}(y)} P_v[H_I < T_{\widetilde{C}(x_*)}] \stackrel{\text{def}}{=} h_1(z,\mathcal{E}_*) h_2(\ell).
\end{aligned}$$

Of the $(2\ell-1)^2$ segments $I \in \mathcal{S}(y)$, at most ck are of distance $k = 1, \dots, c\ell$ from $v \in D(y)$, so using classical estimates on hitting probability for a simple random walk on $\mathbb{Z}^2$, we have that

$$h_2(\ell) \leq \sum_{k=1}^{c\ell} ck \frac{\log\left(\frac{c\ell}{k}\right)}{\log c\ell} \leq c_4 \frac{\ell^2}{\log \ell}\,.$$

Similarly, using classical estimates on hitting probabilities for a simple random walk on $\mathbb{Z}^3$, and considering the worst case choice of $z \in C(x_*)$ and $\mathcal{E}_* \subset \mathcal{L}_{x_*}$ of a given size, we have that

$$h_1(z,\mathcal{E}_*) \leq \sum_{y\in\mathcal{E}_*} \Big(\frac{c}{|\frac{y-z}{\ell}|}\Big) \wedge 1 \leq \sum_{k=1}^{c|\mathcal{E}_*|^{1/3}} ck^2 \frac{1}{k} \leq c_5 |\mathcal{E}_*|^{2/3}\,.$$

We can now choose

$$\lambda = \frac{1}{2c_4c_5} |\mathcal{E}_*|^{-2/3} \frac{\log \ell}{\ell^2}, \qquad (2.47)$$

so that coming back to (2.46) we see that $E_z(\exp(\lambda V_*)) \le 2$. Substituting this in (2.44) and (2.45), we find that

$$q_{x_*,\mathcal{E}_*,i_*} \le \exp\{-\lambda|\mathcal{E}_*|c\ell^2\} 2^{c_0' \log N} .$$

Note that by (2.34) and (2.4), for $\beta = 1 + 2\alpha/3$, some $c_6(\gamma, \alpha) > 0$ and all N sufficiently large,

$$\lambda|\mathcal{E}_*|c\ell^2 \ge c(L/\ell)^{2/3} \log \ell \ge c_6(\log N)^\beta .$$

Since $(2\alpha \vee 1) < 1 + 2\alpha/3$ in view of the choice of $\alpha < 3/4$ in (2.3), this proves (2.43), and therefore the theorem. □

Remark 2.6. When $d = 1$, it is immediate to prove that (2.1) holds. This together with Remark 1.4 shows that Theorem 1 holds when $d = 1$ as well. However as already mentioned in the discussion below (0.6) when $d \ge 2$, there is a substantial discrepancy between the disconnection time T_N and $\widetilde{C}_N$ the cover time of the box by the projection of $X_.$, whereas for $d = 1$, both $\log T_N / \log N$ and $\log \widetilde{C}_N / \log N$ tend to 2 in P-probability. The results of Section 3 will amplify the qualitative difference between the two cases.

3. Clogging at time T_N

The main objective of this section is to prove Theorem 2 of the Introduction, and thus show that when $d \ge 2$; for any $\epsilon > 0$, for large N, the truncated cylinder

$$B_\epsilon = \{x \in E;\ |x^{d+1}| \le N^{d-\epsilon}\} , \tag{3.1}$$

with high P_0-probability is pretty much "clogged" by the trajectory $X_{[0,T_N]}$. This effect ought to be contrasted with what happens when $d = 1$, cf. Remark 3.2.

Theorem 3.1. $(d \ge 2)$

$$\textit{For } \epsilon, \eta \in (0, 1),\ \max_{x \in B_\epsilon} \frac{d(x, X_{[0,T_N]})}{N^\eta} \underset{N\to\infty}{\longrightarrow} 0, \textit{ in } P_0\textit{-probability} , \tag{3.2}$$

(cf. above (0.7) *for the notation).*

Proof. We introduce the sequence τ_k, $k \ge 0$, of $(\mathcal{F}_n)$-stopping times describing the successive displacements of the $\mathbb{Z}$-component $X^{d+1}_.$ of X, at distance $2N$:

$$\begin{cases} \tau_0 = 0,\ \tau_1 = \inf\{n \ge 0,\ |X_n^{d+1} - X_0^{d+1}| \ge 2N\}, \text{ and for } k \ge 1 , \\ \tau_{k+1} = \tau_1 \circ \theta_{\tau_k} + \tau_k . \end{cases} \tag{3.3}$$

On an auxiliary probability space $(\sum, \mathcal{A}, P)$, we consider a simple random walk on $\mathbb{Z}$, starting at 0, $(Z_k)_{k\ge0}$. From the strong Markov property applied at times $(\tau_k)_{k\ge0}$, we find that

$$\text{under } P_0,\ (Z_k^N)_{k\ge0} \overset{\text{def}}{=} \Big(\frac{1}{2N}\, X_{\tau_k}^{d+1}\Big)_{k\ge0} \text{ has same law as } (Z_k)_{k\ge0} \text{ under } P . \tag{3.4}$$

We will be interested in the local time processes

$$L_N(z,k) = \sum_{m=0}^{k} 1_{\{Z_m^N = z\}}, \quad L(z,k) = \sum_{m=0}^{k} 1_{\{Z_m = z\}}\,, \text{ with } k \geq 0, z \in \mathbb{Z}\,. \quad (3.5)$$

We then choose:

$$\epsilon, \eta \in (0,1) \text{ and } 0 < \delta < \frac{1}{4}\,(\eta \wedge \epsilon)\,. \quad (3.6)$$

We first observe that

$$\lim_{N\to\infty} P_0[\tau_{[N^{2d-2-\delta}]} \geq T_N] = 0\,. \quad (3.7)$$

Indeed the above probability is smaller than

$$P_0[T_N < N^{2d-\delta/2}] + P_0[\tau_{[N^{2d-2-\delta}]} \geq N^{2d-\delta/2}]\,.$$

In view of Theorem 2.1, the first term tends to 0 as N goes to infinity. As for the second term, it follows from (1.19) and the strong Markov property at τ_k that

$$P_0[\tau_{[N^{2d-2-\delta}]} \geq N^{2d-\delta/2}] \leq \exp\Big\{-\frac{c}{N^2}\,N^{2d-\delta/2}\Big\}\,c'^{[N^{2d-2-\delta}]} \underset{N\to\infty}{\longrightarrow} 0\,,$$

whence (3.7). As a result Theorem 3.1 will be proved once we show that:

$$A \overset{\text{def}}{=} P_0\big[\max_{x\in B_\epsilon} d(x, X_{[0,\tau_{[N^{2d-2-\delta}]}]}) > N^\eta\big] \underset{N\to\infty}{\longrightarrow} 0\,. \quad (3.8)$$

To this end we observe that

$$\begin{aligned} A\ &\leq A_1 + A_2, \text{ where} \\ A_1 &= P_0\big[\max_{x\in B_\epsilon} d(x, X_{[0,\tau_{[N^{2d-2-\delta}]}]}) > N^\eta, \text{ and} \\ &\qquad \inf_{|z|\leq N^{d-1-\epsilon}} L_N(z,[N^{2d-2-\delta}]) \geq N^{d-1-2\delta}\big], \\ A_2 &= P_0\big[\inf_{|z|\leq N^{d-1-\epsilon}} L_N(z,[N^{2d-2-\delta}]) < N^{d-1-2\delta}\big]. \end{aligned} \quad (3.9)$$

We first bound A_1. For $x \in E$, we denote with $B_{x,\eta}$ the ball $B_\infty(x, \frac{N^\eta}{d+1}) \subseteq E$, see the beginning of Section 1 for the notation, and note that standard Green function estimates imply that for large N:

$$\inf_{|y^{d+1}-x^{d+1}|\leq N} P_y[H_{B_{x,\eta}} < \tau_1] \geq c\,N^{-(d-1)(1-\eta)} \geq c\,N^{-(d-1-\eta)}\,. \quad (3.10)$$

Now for $x \in B_\epsilon$, denote with z some integer such that $|2zN - x^{d+1}| \leq N$, and $|z| \leq N^{d-1-\epsilon}$, (such a z exists for all $x \in B_\epsilon$, when N is large). Let $H_m^z, m \geq 1$, stand for the successive times $\tau_k, k \geq 0$, when X_{τ_k} has a $\mathbb{Z}$-component equal to

$2zN$. The strong Markov property of $X_\cdot$ at H^z_m shows that when N is large, we have:

$$\begin{aligned} A_1 &\le |B_\epsilon| \max_{x \in B_\epsilon} P_0\big[\text{for } 1 \le m < N^{d-1-2\delta}, H_{B_{z,\eta}} \circ \theta_{H^z_m} > \tau_1 \circ \theta_{H^z_m}\big] \\ &\overset{(3.10)}{\le} |B_\epsilon|\big(1 - cN^{-(d-1-\eta)}\big)^{\frac{1}{2}N^{d-1-2\delta}} \underset{N\to\infty}{\overset{(3.6)}{\longrightarrow}} 0\,. \end{aligned} \tag{3.11}$$

We now bound A_2 in (3.9). With (1.20) of [5], one can construct on some probability space $(\widetilde{\Sigma}, \widetilde{A}, \widetilde{P})$ a one-dimensional Brownian motion $(\widetilde{B}_t)_{t\ge 0}$, and a simple random walk on $\mathbb{Z}$ starting from 0, so that denoting by $\widetilde{L}(x, t)$, $x \in \mathbb{R}$, $t \ge 0$, a jointly continuous version of the local time of $\widetilde{B}_\cdot$, and by $L(x, k)$, $x \in \mathbb{Z}$, $k \ge 0$, with an abuse of notation, the local time of the simple random walk, as in (3.5), one has:

$$\widetilde{P}\text{-a.s., for all } \rho > 0,\ \lim_{n\to\infty} n^{-\frac{1}{4}-\rho} \sup_{x\in\mathbb{Z}} |\widetilde{L}(x, n) - L(x, n)| = 0\,. \tag{3.12}$$

In view of (3.4) and the above one finds

$$\begin{aligned} A_2 &\le B_1 + B_2, \text{ with} \\ B_1 &= \widetilde{P}\big[\sup_{x\in\mathbb{Z}} |\widetilde{L}(x, [N^{2d-2-\delta}]) - L(x, [N^{2d-2-\delta}])| \ge N^{d-1-2\delta}\big], \text{ and} \\ B_2 &= \widetilde{P}\big[\inf_{|z|\le N^{d-1-\epsilon}} \widetilde{L}(z, [N^{2d-2-\delta}]) < 2N^{d-1-2\delta}\big]. \end{aligned} \tag{3.13}$$

Now from (3.6) we see that $d - 1 - 2\delta > \frac{1}{4}\,(2d - 2 - \delta)$, and it follows from (3.12) that

$$B_1 \underset{N\to\infty}{\longrightarrow} 0\,. \tag{3.14}$$

Let us now bound B_2. For $\lambda > 0$, $\widetilde{L}(\lambda\cdot, \lambda^2\cdot)$ and $\lambda\widetilde{L}(\cdot, \cdot)$ have same law under $\widetilde{P}$, as a result of Brownian scaling. We thus see that for large N

$$\begin{aligned} B_2 &\le \widetilde{P}\big[\inf_{|z|\le N^{d-1-\epsilon}} \widetilde{L}(z, N^{2d-2-2\delta}) < 2N^{d-1-2\delta}\big] \\ &\overset{\text{scaling}}{=} \widetilde{P}\big[\inf_{|y|\le N^{\delta-\epsilon}} \widetilde{L}(y, 1) < 2N^{-\delta}\big] \underset{N\to\infty}{\longrightarrow} \widetilde{P}[\widetilde{L}(0, 1) = 0] = 0\,, \end{aligned} \tag{3.15}$$

where we used monotone convergence, continuity of the local time and $\delta < \epsilon$, cf. (3.6), in the calculation of the limit. With (3.11) this concludes the proof of (3.8) and hence of Theorem 3.1. □

Remark 3.2. When $d = 1$, the "clogging" effect mentioned in (3.2) does not take place, and with non-vanishing probability, as N tends to infinity, there are points in B_ϵ at distance of order N from $X_{[0,T_N]}$. This fact is a straightforward consequence of the invariance principle and the support theorem for Wiener measure.

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