Diss. ETH No. 26098

ON BOUNDED-COHOMOLOGICAL STABILITY FOR CLASSICAL GROUPS

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES OF ETH ZÜRICH
(Dr. sc. ETH Zürich)

presented by

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2019

Abstract

Bounded cohomology—introduced independently and in different contexts by Johnson, Trauber, and Gromov in the late seventies and early eighties—is a very rich invariant of groups that can detect some of their coarse geometric features. Due to a lack of general computational tools, bounded cohomology remains in general obscure as of today.

In the late nineties, Burger and Monod introduced the notion of *continuous* bounded cohomology, a suitable generalization for topological groups. Apart from having several interesting applications, this theory has remarkably shed light on the bounded cohomology of higher-rank lattices, which is determined to some extent by the continuous bounded cohomology of their ambient semisimple Lie groups. The relationship can be fully exploited in degree two: Being isomorphic to their second continuous cohomology, the second continuous bounded cohomology of any connected semisimple Lie group with finite center (or more generally, of any connected, simply connected, semisimple algebraic group over a local field) is completely understood. This fact and a few examples of groups in this class for which the isomorphism holds in degrees three or four, gave rise to the so-called *isomorphism conjecture*. Usually attributed to Dupont and Monod, it states that the isomorphism known in degree two should also hold in every degree.

The ultimate goal of this thesis is to add further evidence to this conjectural picture: We prove the isomorphism conjecture in degree three for the family of complex symplectic groups. This will be obtained as corollary of the *bounded-cohomological stability* along said family.

One says that continuous bounded cohomology is *stable* along an infinite nested sequence of topological groups if it is eventually constant in every degree. We develop here a machinery that gives bounded-cohomological stability along any sequence of locally compact, second-countable groups, provided that there exists a sequence of complexes on which the respective groups act. It is based on an original idea of Quillen in the setting of group homology, and on an *ad hoc* treatment in continuous bounded cohomology by Monod for the families of general and special linear groups. Our method improves Monod's stability range in degree three for special linear groups over non-Archimedean fields.

Upon constructing a family of complexes that serves as an input to the aforementioned machinery—the so-called *symplectic Stiefel complexes*—we then prove stability of continuous bounded cohomology along the families of real and complex symplectic groups. While the stability range produced is insufficient to prove the isomorphism conjecture in degree three, we complete its proof in the complex case via a bootstrapping procedure. Based on a computation by Bucher–Burger–Iozzi, we moreover determine the Gromov norm of a generating class of the third continuous bounded cohomology.

Inspired on the situation in continuous cohomology, continuous bounded cohomology is expected to be stable along all families of classical split groups over local fields (indexed by the rank). We conclude by explaining how our methods should extend to other families.

Zusammenfassung

Das Endziel dieser Dissertation ist es, Evidenz für folgende Vermutung zu liefern: Für halbeinfache algebraische Gruppen über lokalen Körpern ist die stetige beschränkte Kohomologie (sbK) im Sinne von Burger–Monod isomorph zur stetigen Kohomologie. Wir beweisen das Isomorphismus in Grad drei für die Familie der komplexen symplektischen Gruppen als Korollar ihrer *sbK-Stabilität*.

Man sagt, die sbK sei *stabil* entlang einer unendlichen Folge verschachtelter topologischer Gruppen, falls sie in jedem Grad stationär ist. Wir entwickeln hier ein Kriterium für die Stabilität entlang einer beliebigen Folge lokal kompakter, zweitabzählbarer Gruppen mit der Eigenschaft, dass eine Folge von Komplexen existiert, auf denen die Gruppen entsprechend wirken. Im Falle der speziellen linearen Gruppen über nicht-archimedischen Körpern verbessern wir dank unserem Kriterium ein früheres Resultat von Monod.

Wir zeigen auch die Stabilität entlang der Folgen der reellen und komplexen symplektischen Gruppen. Dies basiert sich auf der Konstruktion der sogenannten *symplektischen Stiefel-Komplexe*. Obwohl der erhaltene Stabilitätsbereich unzureichend ist, um die Vermutung zu beweisen, vervollständigen wir im komplexen Fall das Argument durch ein „Bootstrapping-Verfahren“. Ausserdem benutzen wir eine bestehende Berechnung von Bucher–Burger–Iozzi, um die Gromov-Norm einer erzeugenden Klasse der dritten Kohomologie zu bestimmen.

Resumen

El objetivo último de esta tesis es el de proporcionar evidencia para la siguiente conjetura: para grupos algebraicos semisimples sobre cuerpos locales, la *cohomología continua acotada* (CCA) en el sentido de Burger–Monod es isomorfa a la cohomología continua. Se demuestra el isomorfismo en grado tres para las familia de grupos simplécticos complejos. Dicho resultado es obtenido como corolario de la *estabilidad* de la CCA a lo largo de dicha familia.

Se dice que la CCA es *estable* a lo largo de una sucesión infinita de grupos topológicos encajados si ésta alcanza en cada grado en algún momento un valor estacionario. Primeramente, desarrollamos un criterio de estabilidad para cualquier sucesión de grupos localmente compactos y segundo-enumerables que admita una sucesión de complejos sobre los cuales cada grupo actúe respectivamente. En el caso de los grupos especiales lineales sobre cuerpos no arquimedianos, este criterio permite mejorar un resultado de Monod.

Se demuestra también la estabilidad para las sucesiones de grupos simplécticos reales y complejos a partir de la construcción de los denominados *complejos de Stiefel simplécticos*. A pesar de que el rango de estabilidad obtenido es insuficiente para demostrar la conjetura en grado tres para estos grupos, completamos el argumento en el caso complejo mediante un procedimiento de “bootstrapping”. Aparte de esto, utilizando un cálculo previo de Bucher–Burger–Iozzi, determinamos la norma de Gromov de una clase generadora del tercer grupo de cohomología.

Acknowledgments

First of all, I warmly thank my thesis director, Marc Burger, for transmitting me through countless anecdotes and insights his enthusiasm for mathematics all along this PhD project. I also thank him for granting me the freedom to dive into the realm of bounded cohomology independently for the first time in my life.

This mathematical adventure would not have been possible without the encouragement and support of Alessandra Iozzi (allegedly even before knowing her!). To her I owe my love for Lie groups and symmetric spaces, topics that are central to my research and that I first encountered in one of her courses at ETH Zürich.

Halfway through, with no end in sight, the support of my collaborator, Tobias Hartnick, played a key role in the completion of my doctoral goals. Not only do I thank him for teaching me about buildings or for working side by side with me in marathon sessions of mathematics all over the world, but also for keeping my hopes up every time that the solution to a problem was elusive. As a matter of fact, several of the results exposed in this thesis are the product of our joint work, and for this reason I am more than indebted to him.

I thank my co-referee, Nicolas Monod, for his interest in my work, and for accepting to review this manuscript. There could not be a more adequate choice of a co-referee, since his doctoral dissertation was my *vade mecum* throughout these years, and many of the ideas in this thesis were inspired in his work.

This margin is probably too narrow to contain a complete list of all of those dear friends, whether in Zurich or not, who have shared my joys or filled my "emotional tank" in various occasions during the last years. To all of them, my gratitude is boundless. Especially relevant were those inside ETH, the extended Burger–Iozzi crowd—the old ones and the new—to whom I am thankful for the occasional bQm or VR-Party, for the lunchtime coziness, for all sorts of random (mathematical) discussions, and for their irreplaceable friendship.

Mis últimas líneas de agradecimiento son para mi gran familia, y muy en especial, para mis padres y hermanas. Ni el más mínimo de mis logros habría sido posible, ni ninguna de mis empresas tendría sentido de no contar con su apoyo y amor incondicional. Éste, como todos, se lo debo a ustedes.

Zurich, 21.5.2019

Contents

Introduction

Several deep and interesting questions on the geometry of manifolds, one-dimensional dynamics, and geometric group theory, among others, can be approached via the *bounded cohomology of groups* $\mathrm{H}_\mathrm{b}^\bullet$, first considered by Johnson [49] in the context of Banach algebras, by Phillip Trauber in a group-theoretic setting (unpublished), and further developed by Gromov in his celebrated 1983 paper [37], Brooks [8], Ivanov [46, 47], and Noskov [63]. The following bullet points cover just few of these striking connections.

- It was established by Gromov [37] (see also Brooks [8]) that the infimum of the volumes of a smooth manifold with respect to appropriately normalized Riemannian metrics can be estimated via the "norm" of a certain bounded cohomology class of its fundamental group.
- Ghys [34] showed that the actions by orientation-preserving homeomorphisms of a group Γ on the circle are classified by their *bounded Euler class*, an invariant with values on the bounded cohomology group $\mathrm{H}_\mathrm{b}^2(\Gamma; \mathbb{Z})$.
- Bounded cohomology is successful in detecting "coarse negative curvature" in finitely generated groups (regarded with the metric structure given by their Cayley graphs). An exemplification of this claim is the fact that the free group on two generators $\mathbb{F}_2$ has an extremely rich bounded cohomology in degrees two and three [37, 8, 57]. On the other hand, it was first shown by Trauber that bounded cohomology vanishes in all positive degrees for "non-negatively curved" groups, such as abelian or finite groups. See also [56] for a bounded-cohomological characterization of the Gromov hyperbolicity of finitely generated groups.

Presumably, however, the power of bounded cohomology remains unexploited due to the lack of general computational tools that would be at hand for other invariants of cohomological nature, such as an excision theorem. As a matter of fact, apart from complete vanishing results, there is no single group whose bounded cohomology is totally understood up to the date of submission of this manuscript.

In 1999, Burger and Monod [59, 14] introduced *continuous* bounded cohomology, a version for topological groups of the aforementioned notion that incorporates their topology. The continuous bounded cohomology $\mathrm{H}_\mathrm{cb}^\bullet(G)$ of a locally compact group G (with trivial $\mathbb{R}$-coefficients) can be defined as the cohomology of the homogeneous cochain complex

$$0 \to \mathrm{C}_\mathrm{b}(G)^G \to \mathrm{C}_\mathrm{b}(G^2)^G \to \mathrm{C}_\mathrm{b}(G^3)^G \to \cdots \tag{1}$$

of G-invariant, continuous, bounded functions $G^{k+1} \to \mathbb{R}$; see Definition 1.2. One retrieves from this definition the bounded cohomology $\mathrm{H}_{\mathrm{b}}^{\bullet}(G)$ of G if G is a discrete group. For every n, $\mathrm{H}_{\mathrm{cb}}^{n}(G)$ is a topological vector space, which notably, as a quotient of a Banach space, is endowed with a canonical seminorm. The seminorm information—not available in (continuous) group cohomology—makes (continuous) bounded cohomology a more refined invariant of groups. A homological-algebraic treatment of this theory that incorporates this analytic piece of data was further developed by Bühler [12] at the beginning of the present decade.

The initial motivation for investigating continuous bounded cohomology was the hope that, at least for some locally compact groups, this one would (i) contain a great deal of information on the bounded cohomology of their lattices, and (ii) be more easily computable than its discrete analogue. In the case of connected Lie groups (or more generally, of connected algebraic $\mathscr{k}$-groups over local fields $\mathscr{k}$), which ultimately reduces to the (semi)simple ones, the point (i) is a fulfilled hope. A sample theorem is the next one.

Theorem 0.1 (see [59, Proposition 8.6.2] and [60, Corollary 4.8])**.** *Let G be a connected simple Lie group of real rank r, and let $\Gamma < G$ be a lattice. Then for every $q < 2r$, the inclusion $\Gamma \hookrightarrow G$ induces an isometric isomorphism*

$$\mathrm{H}_{\mathrm{cb}}^{q}(G) \longrightarrow \mathrm{H}_{\mathrm{b}}^{q}(\Gamma). \qquad \square$$

There has been some success in the computational matter raised in (ii). Indeed, the theory is completely understood in degree less than or equal to two. This was exploited by Burger–Monod [14] to obtain a proof of a remarkable rigidity result on actions of higher-rank lattices on the circle by C^1-diffeomorphisms (cf. Ghys [33]). However, the point (ii) remains unsolved to a general extent, the following being the main conjecture in this respect. It is known to hold only in degree two [14, 13], and for certain few examples of groups in higher degrees; see Subsection 1.5.3 below. In all of those cases the existing proofs rely on very diverse techniques.

Conjecture 0.2 (see [58, Problem A] or [16])**.** *Let G be a connected, simple Lie group with finite center (or more generally, $G = \mathbf{G}(\mathscr{k})$,where $\mathbf{G}$ is a connected, simply connected, $\mathscr{k}$-isotropic, simple algebraic group over a local field $\mathscr{k}$.) Then the* comparison map

$$c^{\bullet} : \mathrm{H}_{\mathrm{cb}}^{\bullet}(G) \longrightarrow \mathrm{H}_{\mathrm{c}}^{\bullet}(G),$$

is an isomorphism, where $\mathrm{H}_{\mathrm{c}}^{\bullet}(G)$ denotes the continuous cohomology *of G.*

Here, the *continuous cohomology* $\mathrm{H}_{\mathrm{c}}^{\bullet}(G)$ with trivial $\mathbb{R}$-coefficients is defined as the homology of the complex obtained from (1) after removing the condition of boundedness of cochains, and the *comparison map* $c^{\bullet}$ is defined at the level of cochains by the inclusion (see Section 1.5). The continuous cohomology of simple Lie groups with finite center can be computed explicitly by virtue of theorems by van Est [77] and É. Cartan (see [19]), which relate it to the cohomology of the associated compact symmetric space; see also the monograph [6]. In turn, the cohomology of symmetric spaces is a classical subject, studied in the 20th century by É. Cartan, Koszul, Ehresmann, Iwamoto, among others. A good reference on the subject is [36]. In the non-Archimedean setting above, continuous cohomology with trivial $\mathbb{R}$-coefficients vanishes [6].

The question of the surjectivity of the comparison map was posed originally by Dupont [27], and is actually the more interesting one in terms of applications. While it seems that a proof of Conjecture 0.2 in full generality is currently out of reach, the main goal of the present thesis is to provide further evidence to support its veracity.

Our results

The lettered theorems in this section constitute the main results of the present thesis, around the *bounded-cohomological stability* of continuous bounded cohomology along the family of symplectic groups Sp_{2r}. Theorems A to D are joint work with T. Hartnick, contained in [22].

Consider the usual group cohomology functor $\mathrm{H}^\bullet$ with trivial coefficients, and let $(G_r)_{r\in\mathbb{N}}$ be a sequence of groups. For example, G_r could be the symmetric group $\mathfrak{S}_r$, or $\mathrm{SL}(r,\mathbb{Z})$.

Definition 0.3 (see Definition 6.1)**.** Given a sequence $\mathrm{H}^\bullet = (\mathrm{H}^q : \mathcal{C} \to \mathcal{C}')_{q\geq 0}$ of functors, one says that $\mathrm{H}^\bullet$ is *stable along a sequence* $(G_r)_{r\in\mathbb{N}}$ of objects in $\mathcal{C}$ provided that

$$\forall\, q \geq 0 \quad \exists\, r_0 = r_0(q): \qquad \mathrm{H}^q(G_{r_0}) \cong \mathrm{H}^q(G_{r_0+1}) \cong \mathrm{H}^q(G_{r_0+2}) \cong \cdots \tag{2}$$

Any function $q \mapsto r_0(q)$ satisfying this property will be called a *stability range* for $\mathrm{H}^\bullet$ along the sequence $(G_r)_{r\in\mathbb{N}}$.

The dual phenomenon of homological stability has been an active area of research since the sixties, and has been established for several sequences of groups, with diverse coefficients and stability ranges; we refer the reader to the non-exhaustive list of references [1, 39, 43, 68, 76, 72]. Cohomological stability with field coefficients follows then from a universal coefficient theorem for group cohomology. Stability along a sequence of groups with inclusions

$$G_0 \subset G_1 \subset G_2 \subset \cdots \tag{3}$$

facilitates the description of the cohomology of its direct limit.

The analogous stability question in continuous bounded cohomology $\mathrm{H}^\bullet_{\mathrm{cb}}$ (with trivial coefficients) is very pertinent. First of all, because a bounded-cohomological stability theorem can be used as a computational tool. We elaborate: Say the sequence $(G_r)_{r\in\mathbb{N}}$ corresponds to a classical familiy of complex or split simple Lie groups, with the real rank as an index and block inclusions. Most of what is known about the continuous bounded cohomology beyond degree two for groups in these families is limited to low ranks. Thus, a stability result with a sufficiently *good* stability range (i.e. one that assigns a low value to a certain high degree) could allow us to push the available information to higher terms in the sequence. Second, note that the stability of $\mathrm{H}^\bullet_{\mathrm{cb}}$ along families of classical simple Lie groups constitutes evidence towards Conjecture 0.2: in fact, continuous cohomology is known to stabilize, being an exterior algebra over Borel classes (see Section 1.5).

The first stability theorems for $\mathrm{H}^\bullet_{\mathrm{cb}}$ are due to Monod [61] and concern the sequences of general linear groups $(\mathrm{GL}_r(\Bbbk))_{r\in\mathbb{N}}$ and special linear groups $(\mathrm{SL}_r(\Bbbk))_{r\in\mathbb{N}}$ over any local field

$\Bbbk$. Another related result by Pieters [65], establishing injections in (2) instead of isomorphisms for the rank-one families $(\mathrm{SO}(r,1))_{r\geq 1}$ and $(\mathrm{SU}(r,1))_{r\geq 1}$. The only other stability theorem in the literature in the sense of Definition 0.3 was proven by T. Hartnick and the author of this thesis in [22], and constitutes its main new result.

Theorem A (see Theorem 6.26)**.** $\mathrm{H}^{\bullet}_{\mathrm{cb}}$ *is stable along the sequence* $(\mathrm{Sp}_{2r}(\Bbbk))_{r\in\mathbb{N}}$ *of symplectic groups over the field* $\Bbbk \in \{\mathbb{R},\mathbb{C}\}$.

Theorem 6.26 gives also an explicit stability range, which we believe to be far from optimal. The techniques used in our proof of this theorem should generalize to the other classical families of simple algebraic groups over general local fields, which arise as automorphism groups of sesquilinear forms. The obstacles for a direct generalization are of measure-theoretic nature and are discussed in Section 7.6 below. An immediate corollary of Theorem A and Theorem 0.1 is the following stability result for lattices in symplectic groups.

Corollary. *Fix* $\Bbbk \in \{\mathbb{R},\mathbb{C}\}$ *and let* $\Gamma_r < \mathrm{Sp}_{2r}(\Bbbk)$ *be a lattice for every* $r\in\mathbb{N}$. *Then* $\mathrm{H}^{\bullet}_{\mathrm{b}}$ *is stable along* $(\Gamma_r)_{r\in\mathbb{N}}$. *For example,* $\mathrm{H}^{\bullet}_{\mathrm{b}}$ *is stable along* $(\mathrm{Sp}_{2r}(\mathbb{Z}))_{r\in\mathbb{N}}$ *and* $(\mathrm{Sp}_{2r}(\mathbb{Z}[i]))_{r\in\mathbb{N}}$. □

Our proof of Theorem A relies on an abstraction of Monod's *ad hoc* argument for the families of general and special linear groups, in the spirit of the so-called *Quillen's stability method*, a general procedure that yields homological stability in the setting of usual group homology. Quillen's original description of his method is contained in unpublished notes that are now available online [67]; for more recent accounts, see [3] or [68]. Let us describe how it works: Let $(G_r)_{r\in\mathbb{N}}$ be a sequence of groups with inclusions as in (3), and suppose that for every index r, one produces a *semi-simplicial set*[1] $X_{r,\bullet}$ with the following properties:

(Q1) $X_{r,\bullet}$ is *increasingly connected* in function of r—i.e. the reduced simplicial homology of $X_{r,\bullet}$ vanishes up to a certain degree $\gamma(r)$, and $\gamma(r)\to\infty$ as $r\to\infty$;

(Q2) $X_{r,\bullet}$ is *increasingly transitive* in function of r—i.e. G_r acts transitively on skeleta of $X_{r,\bullet}$ up to a certain dimension $\tau(r)$, and $\tau(r)\to\infty$ as $r\to\infty$; and

(Q3) $(G_r)_r$ and $(X_{r,\bullet})_r$ are *compatible* in the following sense: for every r and $k\in\{0,\dots,\tau(r)\}$, the point stabilizer $H_{r,k}$ of the transitive G_r-action on the k-skeleton $X_{r,k}$ is isomorphic to G_{r-k-1} (plus a more technical assumption on the stabilizers.)

Then $\mathrm{H}_{\bullet}$ is stable along $(G_r)_r$, and the stability range depends on how fast γ and τ tend to infinity. We call the function γ (resp. τ) a *connectivity range* (resp. a *transitivity range*) of the family $(X_{r,\bullet})_r$ of semi-simplicial sets.

We propose a *bounded-cohomological Quillen's method* for ascending sequences of locally compact, second countable (lcsc) groups. Suppose that the groups in the sequence $(G_r)_{r\in\mathbb{N}}$, with

[1] A *semi-simplicial set* (also known as Δ-*set*) $X_{\bullet}$ is a sequence of sets $(X_k)_{k=0}^{\infty}$, the *skeleta* of X, together with *face maps* $\delta_{i,k} : X_{k+1}\to X_k$ for all k and $i\in\{0,\dots,k\}$. This is the minimal setting that enables the definition of the *simplicial (co)homology* of $X_{\bullet}$. In a first reading, they can be thought of as simplicial complexes. For the precise definition of a semi-simplicial object in a category, see the first paragraph of Section 6.1.

inclusions as in (3), are lcsc. We first isolate the adequate analogues of the objects $X_{r,\bullet}$ and of the conditions (Q1)-(Q3) above. The nature of the modifications are a reflection of the strong infusion of functional analysis in the theory of continuous bounded cohomology. For example, each $X_{r,\bullet}$ will be required to be a so-called *measured semi-simplicial G_r-complex*, i.e. a semi-simplicial object in the category Reg_{G_r} of *regular G_r-spaces*[2]; see Definition 5.1. This allows the consideration of complexes

$$0 \to L^\infty(X_{r,0}) \to L^\infty(X_{r,1}) \to L^\infty(X_{r,2}) \to \cdots \tag{4}$$

of L^∞-functions on the skeleta $X_{r,k}$. An additional technical assumption on this complex is that the coboundary operators be weak-$*$ continuous. The analogue of property (Q1) is the increasing *measurable connectivity* of $X_{r,\bullet}$, i.e. that the cohomology of (4) vanishes up to a high degree, increasing with r, for all r. The property (Q2) transfers directly to the bounded-cohomological setting. The analogue of property (Q3) requires the point stabilizers to be groups of lower index in the sequence only up to an amenable factor.

The recollection of these adapted requirements leads to our definition of a *measured Quillen family* $(G_r, X_r)_r$; see Definition 6.8. Based on it, we obtain our bounded-cohomological Quillen's method:

Theorem B (see Theorem 6.9)**.** *Let $(G_r)_{r\in\mathbb{N}}$ be an infinite ascending sequence of lcsc groups, and assume that there exists a measured Quillen family $(G_r, X_{r,\bullet})_{r\in\mathbb{N}}$ with parameters (γ, τ). If $\gamma(r) \to \infty$ and $\tau(r) \to \infty$ as $r \to \infty$, then $\mathrm{H}^\bullet_{\mathrm{cb}}$ stabilizes along $(G_r)_{r\in\mathbb{N}}$.*

A quantitative version of this theorem (that is, with an explicit stability range) is stated below as Theorem 6.10; it also allows the consideration of finite sequences of lcsc groups.

With this machinery at our disposal, we can re-prove the stability theorems for $(\mathrm{GL}_r(\Bbbk))_r$ and $(\mathrm{SL}_r(\Bbbk))_r$ from [61]. We point out that there is an inaccuracy[3] in an induction step in [61] that accounted for stability ranges $r_0(q) \sim q$ as $q \to \infty$. Our amendment produces stability ranges of “slope two”, that is, of the form $r_0(q) \sim 2q$ as $q \to \infty$. While in general these bounds are worse than the original ones, the corrected stability theorems do yield all the corollaries claimed up to this date in degree three, and even allow us to make a new contribution, based on more recent work of Bucher–Monod [11] on the bounded cohomology of $\mathrm{SL}_2(\Bbbk)$ with $\Bbbk$ non-Archimedean:

Theorem C (see Corollary 6.22)**.** *For any non-Archimedean local field $\Bbbk$, $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_3(\Bbbk))$ vanishes.*

The same holds for $\mathrm{SL}_n(\Bbbk)$ with $n > 3$ as a consequence of [61] after the rank one result in [11].

[2]For a lcsc group G, the category Reg_G has as objects pairs (X, μ), where X is a Polish space with a Borel G-action, and μ a G-quasi-invariant probability measure. Its morphisms are Borel, G-equivariant, measure-class preserving maps. See Definition 1.10.

[3]This is mentioned by Monod in the Note before Lemma 10 in [62].

Theorem A follows from Theorem B after constructing an adequate family of measured semi-simplicial complexes for $(\mathrm{Sp}_{2r}(\mathscr{k}))_r$, the so-called *symplectic Stiefel complexes* $X_{r,\bullet}$.

Definition 0.4 (see Definition 7.1)**.** Let (V,ω) be a symplectic vector space of dimension $2r$, and for any integer $0 \leq k \leq r-1$, let $\mathcal{G}_k$ denote the space of isotropic $(k+1)$-dimensional Grassmannians of (V,ω). Set also $\mathcal{P} := \mathcal{G}_0 = \mathbb{P}(V)$. For any $0 \leq k \leq r-1$, we define the set

$$X_k := \{(p_0, \dots, p_k) \in \mathcal{P}^{k+1} \mid \mathrm{span}(p_0, \dots, p_k) \in \mathcal{G}_k\},$$

and call it the *k-Stiefel variety* of (V,ω). Note that the symplectic group $G := \mathrm{Sp}(V,\omega)$ acts on every X_k diagonally by multiplication.

The use of the name *Stiefel variety* is in analogy with the classical objects, introduced by Stiefel[4] in [74], whose points are orthonormal frames of a vector space. An elementary, yet crucial observation is that the group G acts transitively on every X_k. A choice of a probability measure μ_k on X_k within its unique G-invariant measure class turns (X_k, μ_k) into a regular G-space.

Definition 0.5 (see Definition 7.3)**.** We define the *symplectic Stiefel complex* $(X_\bullet, \mu_\bullet)$ of (V,ω) as the semi-simplicial set with skeleta X_k, endowed with the obvious face operators that arise from restricting the i-th deletion maps $\mathcal{P}^{k+1} \to \mathcal{P}^k$. Finally, we let $X_{r,\bullet}$ be the symplectic Stiefel complex of the *standard* symplectic vector space $(\mathscr{k}^{2r}, \omega)$ (we refer to Section 2.1 for the definition of *standard*.)

While the analogue of properties (Q2) and (Q3) are not hard to prove for symplectic Stiefel complexes, the increasing measurable connectivity of X_r, and the weak-$*$-continuity of the differentials in (4) are significantly more delicate. We prove:

Theorem D (see Theorem 6.27)**.** *For $\mathscr{k} \in \{\mathbb{R}, \mathbb{C}\}$, the collection $(\mathrm{Sp}_{2r}(\mathscr{k}), X_{r,\bullet})_{r\in\mathbb{N}}$ is a measured Quillen family with parameters $\gamma(r) \sim \log_2 r$ as $r \to \infty$, and $\tau(r) = r-1$.*

The proof of Theorem D requires the development of several probabilistic techniques, potentially interesting on their own right. A crucial ingredient in the construction of explicit probability measures μ_k on the skeleta X_k of the Stiefel complexes is the notion of *perpendicular measures*, which we introduce in Section 7.3: a *symplectic perpendicular measure* is a "consistent" choice

$$\mathcal{P}^{k+1} \ni (p_0, \dots, p_k) \mapsto \nu_{(p_0,\dots,p_k)} \in \mathrm{Prob}(\mathcal{P})$$

of probability measures $\nu_{(p_0,\dots,p_k)}$ on $\mathcal{P}$, supported on the symplectic complement of the space $\mathrm{span}(p_0, \dots, p_k)$. To illustrate the construction of the measures μ_k on X_k based on the perpendicular measures, we consider the case $k = 1$: the measure μ_1 on the closure $\bar{X}_1$ of $X_1 \subset \mathcal{P}^2$ could be defined by either the formula

$$\int_{\mathcal{P}} \int_{\mathcal{P}} f(p_0, p_1)\, \mathrm{d}\nu_{p_0}(p_1)\, \mathrm{d}\mu_0(p_0) \quad \text{or} \quad \int_{\mathcal{P}} \int_{\mathcal{P}} f(p_0, p_1)\, \mathrm{d}\nu_{p_1}(p_0)\, \mathrm{d}\mu_0(p_1),$$

[4] Interestingly, a former professor at ETH Zürich, and founder of the Institute of Applied Mathematics (today known as SAM) at the same institution in 1948 [70].

where $f \in C(\bar{X}_1)$ and $\mu_0 \in \mathrm{Prob}(\mathcal{P})$ is a uniform probability measure on $\mathcal{P}$. Despite the apparent lack of symmetry of both constructions, we show that they correspond to one another. More generally, we prove the following Fubini-like result:

Proposition (see Lemma 7.13)**.** *The measure μ_k on X_k is symmetric, i.e. invariant under the action of the symmetric group $\mathfrak{S}_{k+1}$ on X_k by permuting the coordinates.*

The weak-$*$ continuity of the maps in the L^∞-complex (4) relies on this proposition. Its proof makes use of the *co-area formula* from geometric measure theory.

Thanks to the perpendicular measures, we are also able to prove the measurable connectivity of the Stiefel complex. In this direction, we introduce a tool for constructing contracting homotopies of our L^∞-complexes, the so-called *random chainings*, described in Section 7.5. They are stochastic processes consisting of random points $(t_I \in \mathcal{P})_I$—indexed by finite subsets $I \subset \mathcal{P}$—that are almost surely perpendicular to each other (with respect to ω) depending on their indices, and that are distributed with a certain G-equivariance property.

The stability range given by the quantitative version of Theorem A does not suffice for computational purposes—not even in degree three—because of the fast (exponential) growth of the stability range in function of the degree. Strikingly, as a corollary of the stability theorem for the sequence $(\mathrm{Sp}_{2r}(\mathbb{C}))_{r\in\mathbb{N}}$, we manage to improve the range optimally in degree three for that family after studying the action of the group $\mathrm{Sp}_{2r}(\mathbb{C})$ on the *product measured semi-simplicial complex* $(\mathcal{P}^{\bullet+1}, \mu^{\otimes\bullet+1})$; see Section 5.3 for their definition.

Theorem E (see Theorem 8.1)**.** *The inclusions* $\mathrm{Sp}_{2r}(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2r+2}(\mathbb{C})$ *induce the following sequence of maps in continuous bounded cohomology:*

$$\cdots \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{20}(\mathbb{C})) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{18}(\mathbb{C})) \hookrightarrow \cdots \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2}(\mathbb{C})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_{2}(\mathbb{C})) \cong \mathbb{R}$$

Because the proof of Theorem E builds on the non-optimal stabilty given in Theorem A, we say that the argument is of a *bootstrapping* nature. A key element of the bootstrapping procedure is a parametrization of the measure-theoretic quotients $\mathrm{Sp}_{2r}(\mathbb{C}) \backslash\!\backslash \mathcal{P}^p$, where $r \geq 2$ and $p = 4, 5$. Said parametrization is achieved via so-called symplectic cross-ratios, direct generalizations of the classical cross-ratio on the complex projective line $\mathbb{CP}^1$ that are introduced in Subsection 8.1.2. A curious corollary of this enterprise is the inexistence of non-trivial solutions $f \in L^\infty(\mathbb{C}^2)$ of the functional equation

$$f(\alpha_1, \alpha_2) - f(\beta_1, \beta_2) + f\left(\frac{\beta_1}{\alpha_1}, \gamma_2\right) - f\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}, \frac{\gamma_2}{\beta_2}\right) + f\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1\gamma_2}{\beta_2}\right) = 0.$$

with $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_2 \in \mathbb{C}$; see Corollary 8.13 below. It can be regarded as a higher-dimensional analogue of the Spence–Abel 5-term functional equation,

$$f(\alpha) - f(\beta) + f\left(\frac{\beta}{\alpha}\right) - f\left(\frac{1-\beta}{1-\alpha}\right) + f\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right) = 0$$

with $\alpha, \beta \in \mathbb{C}$, whose unique solution $f \in L^\infty(\mathbb{C})$ (up to a multiplicative constant) is the Bloch–Wigner dilogarithm (see [15], or the survey [78].)

Theorem E implies that $\dim \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) \leq 1$. Combining this with the one-dimensionality of $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$, $r \geq 2$, as in [10, Theorem 2], we establish the equality above. We also determine as a corollary of [10, Theorem 2] the Gromov norm of a generating class of $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C}))$.

Theorem F (see Corollary 8.16)**.** *For any $r \geq 1$, the inclusion $\mathrm{Sp}_{2r}(\mathbb{C}) \hookrightarrow \mathrm{SL}_{2r}(\mathbb{C})$ induces an isometric isomorphism in degree three. In particular, the restriction $\widetilde{\beta}^{\mathrm{b}}_{2r}$ of the bounded Borel class $\beta^{\mathrm{b}}_{2r} \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_{2r}(\mathbb{C}))$ from [10] generates $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C}))$ and has Gromov norm*

$$\|\widetilde{\beta}^{\mathrm{b}}_{2r}\| = \frac{r(4r^2-1)}{3}\, v_3, \tag{5}$$

with v_3 being the maximal volume of an ideal tetrahedron in $\mathbb{H}^3$. Moreover, the comparison map

$$c^3 : \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) \to \mathrm{H}^3_{\mathrm{c}}(\mathrm{Sp}_{2r}(\mathbb{C}))$$

is an isomorphism for every $r \geq 1$.

As before, an immediate corollary of this theorem and Theorem 0.1 is the determination of the third bounded cohomology groups of all lattices in higher-rank complex symplectic groups.

Corollary. *For any $r \geq 2$, let $\Gamma < \mathrm{Sp}_{2r}(\mathbb{C})$ be a lattice. Then $\mathrm{H}^3_{\mathrm{b}}(\Gamma)$ is one-dimensional.* □

The fact the isomorphism in Theorem 0.1 is isometric tells us that the Gromov norm of the restriction of the class in $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C}))$ from Theorem F equals the right-hand side in (5).

Apart from Theorem A in general degree, Theorems C and E are concrete evidence in favor of Conjecture 0.2 in degree three, which can be formulated as below in virtue of the next theorem.

Theorem G (see Theorem 1.38 and Appendix A)**.** *Let G be a connected, simple Lie group with finite center. Then $\mathrm{H}^3_{\mathrm{c}}(G) \neq 0$ if and only if G is a complex Lie group. If so, $\dim \mathrm{H}^3_{\mathrm{c}}(G) = 1$.*

Conjecture 0.6. *Let $G = \mathbf{G}(\mathscr{k})$, where $\mathbf{G}$ is a connected, simply connected, $\mathscr{k}$-isotropic, simple algebraic group over a local field $\mathscr{k}$. Then $\mathrm{H}^3_{\mathrm{cb}}(G) \neq 0$ if and only if $\mathscr{k} = \mathbb{C}$, and in that case, $\dim \mathrm{H}^3_{\mathrm{cb}}(G) = 1$.*

Theorem G can be derived from van Est's theorem and classical computations of the cohomology of symmetric spaces, found, for example, in [55, §3, §7]. A proof of the same fact was given by the author of this thesis in [21]; it has the advantage of being comparatively low in prerrequisites and independent of the classification of simple Lie groups. The proof is included in this thesis, however, left as an Appendix due to of the difference in flavor with the rest of our results.

Structure of the thesis

This thesis consists of two parts. Part I is entitled "Background Material": Chapters 1 and 3 contain the irreducible essentials to this thesis on the theory of continuous bounded cohomology, and on spectral sequences, respectively. In the former, we will present our favorite examples of coefficient modules, relevant cohomological tools, and the relationship of continuous bounded cohomology to continuous cohomology in the case of (semi)simple Lie groups. The ultimate goal of Chapter 3 is to explain the origin of the two spectral sequences associated to a double complex. No proofs are included. Chapter 2 gives us a tour through some basic results on sesquilinear forms on vector spaces, and on their isotropic Grassmannians. A possible novelty of our approach in this chapter is that we introduce the usual topology on Grassmannians via the Chabauty topology and not, as usually, via a Lie group action. This allows for the simplification of the proofs of some continuity arguments. Finally, Chapter 4 is dedicated to establish the measurability of certain parameter-dependent integrals, an aspect that is necessary for the construction of Stiefel complexes.

Our main results are exposed in Part II, called "Measured Semi-Simplicial Complexes and Stability." Its first chapter, Chapter 5, contains the definition of these complexes, first examples, and the derivation of a spectral sequence associated to them. By means of this spectral sequence, in Chapter 6, we prove Theorem B, our bounded-cohomological Quillen's method, and then collect our first stability statements for classical groups. The proof of Theorem A is given modulo Theorem D, in which we postulate the existence of a symplectic measured Quillen family and whose proof is the object of Chapter 7. There, we define the Stiefel complex associated to any vector space with a sesquilinear form, and give a proof of its transitivity, as well as of the analogue of property (Q3). We then focus on the symplectic case, in which we are able to define the so-called perpendicular measures, which gives the basis to the proofs of admissibility and measurable connectivity. We then finish the chapter with a discussion of the technical difficulties when considering other sesquilinear forms. Part II concludes with 8, in which we present the bootstrapping procedure that proves Theorems E and F.

The Appendix A contains complete proofs of two facts about the continuous cohomology in degree three of connected simple Lie groups: In the finite-center case, this one does not vanish precisely if the group admits a complex structure; in the infinite-center case, its non-vanishing characterizes the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. The former is Theorem G in this Introduction.

Notation and Conventions

We adopt the convention that $0 \in \mathbb{N}$. For $R \in \mathbb{N}$, we denote by $[R]$ the subset $\{0, 1, \ldots, R\}$ of $\mathbb{N}$, and let $[\infty]$ be the whole of $\mathbb{N}$. We use the notation $\mathfrak{S}_k$ for the symmetric group on k letters.

The word "space" without any adjectives will be employed to refer to a *topological* space. We will assume locally compact spaces to be Hausdorff, and use the shorthand "*lcsc* space" for "*locally compact, second-countable* space." An action of a topological group G on a space (resp. on a Borel space) X is said to be continuous (resp. Borel) if the action map $G \times X \to X$ is continuous (resp. Borel). All spaces will be automatically endowed with their Borel σ-algebra.

If not specified, all of our function spaces (e.g. the space of continuous functions $C(X)$ of a space X, or its subspace $C_c(X) \subset C(X)$ of compactly supported functions) will consist of *real-valued* functions, and will be hence vector spaces over $\mathbb{R}$.

Two measures μ and ν on a measurable space are said to be equivalent if their collections of null sets coincide; we will write $\mu \sim \nu$ in that case, and $[\mu]$ for the equivalence class of μ. On a space X, we will only consider Radon measures, i.e. measures defined on the Borel σ-algebra of X that are inner and outer regular, and locally finite. For economy, we will always omit the adjective "Radon" and call them simply "measures". Relying on Riesz–Markov theorem, we will make loose the identification of measures on a locally compact space X and positive bounded functionals on $C_c(X)$.

Given a measure space (X, μ) and $1 \leq p < \infty$, we denote by $\mathscr{L}^p(X, \mu)$ the semi-normed space of Borel functions $f : X \to \mathbb{R}$ such that $\mu(|f|^p) < \infty$. For $p = \infty$, $\mathscr{L}^\infty(X)$ denotes the Banach space of bounded Borel functions on X. For distinction, the corresponding Banach spaces of function classes will be denoted by $L^p(X, \mu)$ for $1 \leq p \leq \infty$.

If X is a lcsc space, we write $\mathrm{Prob}(X)$ for the set of probability measures on X, topologized as follows: If X is compact, then we regard $\mathrm{Prob}(X)$ as a subspace of the (unit ball in the operator norm of the) dual space $C(X)^*$, where the latter is equipped with its weak-$*$-topology. As such, $\mathrm{Prob}(X)$ is compact and metrizable, hence Hausdorff and second-countable. On the other hand, if X is not compact, then for every Hausdorff second-countable compactification $\iota : X \hookrightarrow \bar{X}$ (e.g. the one-point compactification), there exists an injection given by the push-forward map $\iota_* : \mathrm{Prob}(X) \hookrightarrow \mathrm{Prob}(\bar{X})$. We then regard $\mathrm{Prob}(X)$ as a subspace of $\mathrm{Prob}(\bar{X})$.

PART I

BACKGROUND MATERIAL

Chapter 1

Continuous Bounded Cohomology

This chapter aims to summarize content from Monod's treatment of continuous bounded cohomology that is indispensable to this thesis. Beyond self-containment, its central purpose is to fix notational conventions. Throughout the chapter, G will be a locally compact group, and from Section 1.2 on, also second-countable. We shall assume all vector spaces to be over $\mathbb{R}$.

Chapter 1 is divided into five sections, the first four of which are almost completely extracted from [59]. We begin Section 1.1 by presenting the category of Banach G-modules, which underlies the theory, and by setting the necessary homological formalism. Relying on that language, we then give a first definition of the continuous bounded cohomology of a locally compact group, via continuous bounded functions on Cartesian products of G.

Section 1.2 treats the category of coefficient G-modules, whose objects are, roughly speaking, dual Banach G-modules. The prime example of these will be L^∞-spaces. They play a fundamental role in the theory of continuous bounded cohomology, for they allow for more flexibility in our choice of spaces of cochains: in fact, bounded classes of G can be represented by G-invariant L^∞-function classes on products of any amenable G-space. The notion of amenability and a statement that formalizes the previous sentence will be exposed in Section 1.3, an additional reference for this section being [80].

An aspect of L^∞-spaces from which we will profit is the fact that they enable an Eckmann–Shapiro induction lemma for continuous bounded cohomology. Induction in group cohomology is a tool to relate the cohomology of a group to the one of a larger ambient group, with twisted coefficients. In Section 1.4, we give short account of the version developed in [59].

We finish with Section 1.5, on the isomorphism conjecture, which was already presented as Conjecture 0.2 in the Introduction. We first define continuous cohomology, and state a theorem, after work of van Est and Cartan, that connects the continuous cohomology of semisimple Lie groups with the de Rham cohomology of an associated compact symmetric space. Then, we expose motivation for the statement of the conjecture. In the last section, we present the most important pieces of evidence in its favor existing up to today.

1.1 First definitions

1.1.1 Banach G-modules and complexes. A *Banach G-module* is a Banach space E equipped with a G-action by linear isometries. A morphism of Banach G-modules $E \to F$, or simply a *G-morphism*, is a G-equivariant, bounded linear map. We denote by E^G the closed subspace of E of G-invariant elements, i.e.

$$E^G := \{x \in E \mid g \cdot x = x \text{ for all } g \in G\}.$$

We say that a Banach G-module E is *separable* if the Banach space E is; we say it is *continuous* (resp. *trivial*) if the underlying G-action on E is.

A *complex* of Banach G-modules $(E^\bullet, \mathrm{d})$, or simply a *complex*, is a pair consisting of a $\mathbb{Z}$-indexed sequence $E^\bullet = (E^n)_n$ of Banach G-modules, and corresponding G-morphisms

$$\cdots \to E^{n-1} \xrightarrow{\mathrm{d}^{n-1}} E^n \xrightarrow{\mathrm{d}^n} E^{n+1} \to \cdots$$

such that $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ for every integer n. An element of E^n will be called an *n-cochain*. As customary in homological algebra, we will often omit the index n in the differential d^n, and write just d; also, the pair $(E^\bullet, \mathrm{d})$ will be often denoted simply by $E^\bullet$. A *right* complex is one for which $E^n = 0$ for every $n < 0$. A *morphism of complexes* is a map $\alpha^\bullet : E^\bullet \to F^\bullet$ that restricts to a bounded linear map $\alpha^n : E^n \to F^n$ for every integer n, and is such that the diagram

$$\begin{array}{ccc} E^n & \xrightarrow{\mathrm{d}} & E^{n+1} \\ {\scriptstyle \alpha^n}\downarrow & & \downarrow{\scriptstyle \alpha^{n+1}} \\ F^n & \xrightarrow{\mathrm{d}} & F^{n+1} \end{array}$$

commutes for every $n \in \mathbb{Z}$. If, in addition, α^n is a G-morphism for every integer n, then we say $\alpha^\bullet$ is a *G-morphism of complexes*. We denote by $(E^\bullet)^G$ the complex of G-invariants of $E^\bullet$, with differential given by the restriction $(E^n)^G \to (E^{n+1})^G$ of d at every degree $n \in \mathbb{Z}$. Abusively, we continue to denote this restriction by d. If $\alpha^\bullet : E^\bullet \to F^\bullet$ is a G-morphism of complexes, then it induces a morphism $(E^\bullet)^G \to (F^\bullet)^G$ of the complexes of G-invariants that we will also call α.

For $n \in \mathbb{Z}$, the *cohomology* of a complex $E^\bullet$ in degree n is the topological vector space

$$\mathrm{H}^n(E^\bullet) := \ker \mathrm{d}^n \,/\, \operatorname{im} \mathrm{d}^{n-1},$$

where $\operatorname{im} \mathrm{d}^{n-1} < \ker \mathrm{d}^n$ because $\mathrm{d}^n \circ \mathrm{d}^{n-1} = 0$. The elements of $\ker \mathrm{d}^n$ are called *n-cocycles*, the ones of $\operatorname{im} \mathrm{d}^{n-1}$, *$n$-coboundaries*, and the ones of $\mathrm{H}^n(E^\bullet)$, *n-classes*. We will write $\mathrm{H}^\bullet(E^\bullet)$ to refer to the graded vector space $\bigoplus_n \mathrm{H}^n(E^\bullet)$. For every n, we will regard $\mathrm{H}^n(-)$ as a covariant functor from the category of complexes to the category **Vect** of vector spaces and linear maps.

Let $E^\bullet$ be a complex, and $k, l \in \mathbb{N} \cup \{\pm\infty\}$ with $k \leq l$. A *contracting homotopy* of $E^\bullet$ from degree $k-1$ to l is a collection of bounded linear maps $h^n : E^{n+1} \to E^n$, for $n \in [k-1, l] \cap \mathbb{Z}$, that satisfy the identity $\mathrm{d}^{n-1} h^{n-1} + h^n \mathrm{d}^n = \mathrm{id}$ for every $n \in [k, l] \cap \mathbb{Z}$. If $k = -\infty$ and $l = \infty$, then we just say $h^\bullet$ is a contracting homotopy. A contracting homotopy is a tool to show vanishing of the cohomology of a complex. It gives a way of assigning to an n-cocycle $f \in E^n$ a *primitive*, that is, an $(n-1)$-cochain $\varphi \in E^{n-1}$ such that $\mathrm{d}\varphi = f$. Indeed, if h is a contracting homotopy of $E^\bullet$ from degree $k-1$ to l, and $n \in [k, l] \cap \mathbb{Z}$, then with $\varphi := h^{n-1} f$ one has $\mathrm{d}^{n-1}\varphi = f - h^n \mathrm{d}^n f = f$. This proves the following useful lemma.

Lemma 1.1. *Let $k, l \in \mathbb{N}$ with $k \leq l$, and $E^\bullet$ be a complex admitting a contracting homotopy from degree $k-1$ to l. Then the cohomology $\mathrm{H}^n(E^\bullet) = 0$ for every $n \in [k, l] \cap \mathbb{Z}$.* □

1.1.2 The definition of continuous bounded cohomology. For any space X, and any Banach space E, we denote by $C_b(X; E)$ the Banach space of continuous, bounded functions $X \to E$, endowed with the supremum norm; we write $C_b(X)$ if $E = \mathbb{R}$. If, furthermore, G acts on X by homeomorphisms, and E is a Banach G-module, we give $\mathrm{C_b}(X; E)$ the structure of a Banach G-module by the *left-regular* action

$$(g \cdot \varphi)(x) := g \cdot \varphi(g^{-1} \cdot x), \quad g \in G,\ \varphi \in \mathrm{C_b}(X; E),\ x \in E. \tag{1.1}$$

If G acts trivially on E, we call the action defined by this formula the *left-translation* action.

Let now E be a Banach G-module. For every $n \in \mathbb{N}$, consider the Banach G-module $C_b(G^{n+1}; E)$, where G is equipped with the action by left multiplication, and the Cartesian product G^{n+1} with the corresponding diagonal action. Out of these, we assemble the right complex

$$0 \to C_b(G; E) \xrightarrow{\mathrm{d}} C_b(G^2; E) \xrightarrow{\mathrm{d}} C_b(G^3; E) \to \cdots \tag{1.2}$$

where the differential d is defined as

$$\mathrm{d}\varphi(g_0, \dots, g_{n+1}) := \sum_{i=0}^{n+1} (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_{n+1}), \tag{1.3}$$

for any $n \in \mathbb{N}$, $\varphi \in C_b(G^{n+1}; E)$, and $(g_0, \dots, g_{n+1}) \in G^{n+1}$, where as usual the hat $\widehat{\cdot}$ represents as usual the omission of the argument underneath. The map d in (1.3) is called the *homogeneous differential*, and the complex $C_b(G^{\bullet+1}; E)$ from (1.2) is called the *continuous homogeneous complex*.

Definition 1.2. The *n-th continuous bounded cohomology group of G with E-coefficients* is defined as the n-th cohomology

$$\mathrm{H}^n_{\mathrm{cb}}(G; E) := \mathrm{H}^n(C_b(G^{\bullet+1}; E)^G)$$

of the complex $C_b(G^{\bullet+1}; E)^G$ of invariants of (1.2).

Remark 1.3. Being a quotient of a Banach space, $\mathrm{H}^n_{\mathrm{cb}}(G; E)$ comes with topology induced by a canonical semi-norm—the so-called *Gromov norm*—which is defined as the infimum of the operator norms of all representatives of a class. Keeping track of continuous bounded cohomology groups as semi-normed spaces is important in most of the applications of this theory. As a matter of fact, this piece of information makes it a more refined invariant than continuous cohomology, which a priori lacks this geometric aspect (see Section 1.5 for the definition of continuous cohomology.) Considering that the main scope of this thesis is to validate the isomorphism of Conjecture 0.2 from the Introduction, we choose to omit all the topological and geometric data and regard the cohomology groups $\mathrm{H}^n_{\mathrm{cb}}(G;E)$ simply as objects of the underlying category **Vect**. Nevertheless, the functorial approach developed by Monod in [59] for continuous bounded cohomology makes most of these statements hold in the more refined categories.

Example 1.4. If G is the trivial group and E is any Banach space, then $C_b(G^{n+1};E) \cong E$ for every $n \in \mathbb{N}$, and the complex $C_b(G^{\bullet+1};E)^G$ is of the form

$$0 \to E \xrightarrow{0} E \xrightarrow{\sim} E \xrightarrow{0} E \xrightarrow{\sim} E \xrightarrow{0} \cdots$$

Thus, $\mathrm{H}^0_{\mathrm{cb}}(G;E) = E$. In every positive degree, its continuous bounded cohomology vanishes.

1.1.3 Notational remarks on the functoriality of $\mathrm{H}^\bullet_{\mathrm{cb}}$.

If E and F are Banach G-modules, and $\alpha : E \to F$ is a G-morphism, we will write α_* for the G-morphism of complexes

$$\alpha_* : C_b(G^{\bullet+1};E) \to C_b(G^{\bullet+1};F)$$

defined by postcomposition of α. This induces a morphism of the complexes of G-invariants, and this one, in turn a linear map $\mathrm{H}^\bullet(\alpha_*)$ in cohomology. We will write

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G;\alpha) : \mathrm{H}^\bullet_{\mathrm{cb}}(G;E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(G;F) \tag{1.4}$$

instead of $\mathrm{H}^\bullet(\alpha_*)$ for that map. It is a simple verification then that $\mathrm{H}^\bullet_{\mathrm{cb}}(G;-)$ is a covariant functor from the category of Banach G-modules and G-morphisms to **Vect**.

On the other hand, if H is a locally compact group, and $\psi : H \to G$ is a homomorphism, then E is also a Banach H-module with the action given by $h \cdot x := \psi(h)x$ for every $h \in H$ and $x \in E$. We denote ψ^* the morphism of complexes

$$\psi^* : C_b(G^{\bullet+1};E) \to C_b(H^{\bullet+1};E).$$

that results from precomposing the diagonal application of ψ. Note that for all n, $C_b(G^{n+1};E)$ is a Banach H-module by the associated left-regular action of $\psi(H)$, and that then ψ^* is an H-morphism of complexes. Moreover, we have the inclusion $C_b(G^{\bullet+1};E)^G \subset C_b(G^{\bullet+1};E)^H$, which composed with ψ^* gives rise to a morphism of complexes

$$C_b(G^{\bullet+1};E)^G \overset{\iota}{\hookrightarrow} C_b(G^{\bullet+1};E)^H \xrightarrow{\psi^*} C_b(H^{\bullet+1};E)^H.$$

We denote the induced linear map $\mathrm{H}^\bullet(\psi^* \circ \iota)$ by

$$\mathrm{H}^\bullet_{\mathrm{cb}}(\psi;E) : \mathrm{H}^\bullet_{\mathrm{cb}}(G;E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(H;E). \tag{1.5}$$

One verifies then that $\mathrm{H}^\bullet_{\mathrm{cb}}(-;E)$ is a contravariant functor from the category of locally compact groups to **Vect**. If E is the trivial module $\mathbb{R}$, then we will write $\mathrm{H}^\bullet_{\mathrm{cb}}(-)$ instead of $\mathrm{H}^\bullet_{\mathrm{cb}}(-;\mathbb{R})$.

Both the covariance and contravariance of $\mathrm{H}^\bullet_{\mathrm{cb}}$ can be considered simultaneously. If $\alpha : E \to F$ is a morphism of Banach G-modules and $\psi : H \to G$ is a homomorphism, then the compositions $\mathrm{H}^\bullet_{\mathrm{cb}}(\psi; F)\circ\mathrm{H}^\bullet_{\mathrm{cb}}(G;\alpha)$ and $\mathrm{H}^\bullet_{\mathrm{cb}}(H;\alpha)\circ\mathrm{H}^\bullet_{\mathrm{cb}}(\psi;E)$ are the same map

$$\mathrm{H}^\bullet_{\mathrm{cb}}(G;E) \to \mathrm{H}^\bullet_{\mathrm{cb}}(H;F),$$

which we denote $\mathrm{H}^\bullet_{\mathrm{cb}}(\psi;\alpha)$.

1.1.4 Degrees zero and one. Although basic, the conclusions of the next two examples will be used recurrently in the Chapters 6 and 8 of this thesis.

Example 1.5 (Degree zero)**.** Let E be a Banach G-module. One sees immediately that the kernel of $\mathrm{d}^0 : C_b(G;E) \to C_b(G^2;E)$ is isomorphic to E, and thus $\mathrm{H}^0_{\mathrm{cb}}(G;E) = E^G$.

Example 1.6 (Degree one)**.** Consider a trivial G-module E and the right complex

$$0 \to \mathbb{R} \xrightarrow{\mathrm{d}'^0} C_b(G;E) \xrightarrow{\mathrm{d}'^1} C_b(G^2;E) \to \cdots$$

with differential d' given by the formula

$$\mathrm{d}'\varphi(g_1,\dots,g_{n+1}) = \varphi(g_2,\dots,g_{n+1}) + \sum_{i=1}^{n}(-1)^i\varphi(g_1,\dots,g_ig_{i+1},\dots,g_{n+1}) + \\ + (-1)^{n+1}\varphi(g_1,\dots,g_n) \tag{1.6}$$

for any $n \in \mathbb{N}$ and $\varphi \in C_b(G^n;E)$. This is usually known as the *continuous inhomogeneous complex*, and d' as the *inhomogeneous differential*. The following formula defines an (isometric) isomorphism $A^\bullet : (C_b(G^{\bullet+1};E)^G, \mathrm{d}) \xrightarrow{\sim} (C_b(G^\bullet;E),\mathrm{d}')$ of complexes, and thus $\mathrm{H}^n_{\mathrm{cb}}(G;E) \cong \mathrm{H}^n(C_b(G^\bullet;E),\mathrm{d}')$:

$$A^n(\varphi)(g_1,\dots,g_n) = \varphi(1,g_1,g_1g_2,\dots,g_1\cdots g_n), \quad \varphi \in C_b(G^{n+1};E)^G, (g_1,\dots,g_n) \in G^n.$$

In degree one, the formula (1.6) reduces to $\mathrm{d}'\varphi(g_1,g_2) = \varphi(g_2) - \varphi(g_1g_2) + \varphi(g_1)$. Thus, 1-cocycles are continuous bounded homomorphisms $G \to E$, which have to be identically zero. Consequently, $\mathrm{H}^1_{\mathrm{cb}}(G;E) = 0$.

Remark 1.7. If the Banach G-module E is not trivial, then the first summand of the inhomogeneous differential (1.6) should be replaced by $g_1 \cdot \varphi(g_2,\dots,g_{n+1})$, and the morphism $A^\bullet$ ranges in a proper subcomplex of $C_b(G^\bullet;E)$; see [59, Proposition 7.4.12]. However, the fact from Example 1.6 that $\mathrm{H}^1_{\mathrm{cb}}(G;E)$ vanishes does hold for a larger class of Banach G-modules than just trivial ones; see e.g. Proposition 6.2.1 and Corollary 11.4.2 in [59].

1.2 The category of coefficient G-modules

Recall that from now on G is a lcsc group. We denote by $\mathbf{Ban}_G$ the category of separable, continuous Banach G-modules, and G-morphisms. The *category of coefficient G-modules* is the opposite category $\mathbf{Ban}_G^{\mathrm{op}}$. We will use the concrete model for $\mathbf{Ban}_G^{\mathrm{op}}$.

Definition 1.8. Given a Banach space E, we denote by E^* the dual Banach space of E, i.e. the space of bounded, linear functionals $E \to \mathbb{R}$, with the operator norm. If E is a Banach G-module, then so is E^*, equipped with the *contragredient* G-action

$$(g \cdot \varphi)(v) := \varphi(g^{-1}v), \quad g \in G,\ \varphi \in E^*,\ v \in E.$$

A pair (E, E^*) of a separable, continuous Banach G-module E and its dual E^* is called a *coefficient G-module*. We will refer to E^* as the *underlying* Banach G-module of (E, E^*).

Let $(E^\flat, E)$ and $(F^\flat, F)$ be two coefficient G-modules. A *morphism of coefficient G-modules* $(\varphi^\flat, \varphi) : (E^\flat, E) \to (F^\flat, F)$ consists of a bounded linear map $\varphi^\flat : F^\flat \to E^\flat$ and a G-morphism $\varphi : E \to F$, such that φ is dual to $\varphi^\flat$, i.e. such that

$$\langle \varphi(f) \mid v \rangle_{F,F^\flat} = \langle f \mid \varphi^\flat(v) \rangle_{E,E^\flat} \quad \text{for all } f \in E,\ v \in F^\flat,$$

where $\langle - \mid - \rangle_{E,E^\flat} : E \times E^\flat \to \mathbb{R}$ denotes the dual pairing of E and $E^\flat$, and analogously for F and $F^\flat$. A G-morphism $\varphi : E \to F$ is said to be a *dual morphism* if there exists a bounded linear map $\varphi^\flat : F^\flat \to E^\flat$ such that $(\varphi^\flat, \varphi)$ is a morphism of coefficient G-modules. Equivalently, φ is a dual morphism if it is weak-$*$ continuous, where the weak-$*$ topologies on E resp. F come from their pre-duals $E^\flat$ resp. $F^\flat$.

Remark 1.9. Given a coefficient G-module $(E^\flat, E)$, note that the underlying module E needs not be an object of $\mathbf{Ban}_G$, even if that is the case for $E^\flat$. Apart from the operator topology, we will very often consider the weak-$*$ topology on E. However, we emphasize that the weak-$*$ topology on a dual space depends on the specific choice of a pre-dual.

If (X, μ) and (Y, ν) are two Borel measure spaces, and $T : X \to Y$ is a Borel map, we say that T is *measure-class-preserving*, or shortened *mcp*, if $T_*\mu \sim \nu$. If in addition X admits a Borel G-action, we say that the measure μ is *G-quasi-invariant* if every $g \in G$ is mcp when regarded as the left-multiplication Borel map $g : X \to X$. A Borel measure class $\mathfrak{M}$ on X is said to be *G-invariant* if $g_*\mu \in \mathfrak{M}$ for every $g \in G$ and $\mu \in \mathfrak{M}$.

The following concept will allow us to produce the examples of coefficient modules that are relevant to the present thesis.

Definition 1.10. Let X be a *standard* Borel space (i.e. the Borel space associated to a completely metrizable and separable topology), and μ be a probability measure on X. We will say that (X, μ) is a *regular G-space* if X admits a Borel G-action, and μ is a G-quasi-invariant measure. We denote by $\mathbf{Reg}_G$ the category whose objects are regular G-spaces and morphisms are G-equivariant mcp Borel maps.

Remark 1.11. A Borel G-action on a standard Borel space X has the feature that orbits are Borel subsets of X, and point stabilizers are closed subgroups of G. See [80, Corollary 2.1.20].

Example 1.12. If $H < G$ is a closed subgroup, then G/H admits a unique G-invariant measure class that comes from the Haar measure on G. The homogeneous space G/H is lcsc, thus σ-

compact. This ensures the existence of a probability measure in the invariant measure class on G/H. If μ is any such measure, then $(G/H, \mu)$ is a regular G-space.

If (X, μ) is a regular G-space, then all of the measures in $\{g_*\mu \mid g \in G\}$ belong to the measure class $[\mu]$. We will use the abbreviation

$$r_\mu(g, x) := \frac{\mathrm{d}(g_*^{-1}\mu)}{\mathrm{d}\mu}(x), \tag{1.7}$$

for the *Radon-Nikodym derivative* on the right-hand side, i.e. the unique function class $r_\mu(g, \cdot) \in L^1(X, \mu)$ determined by the identity

$$\int_X \varphi(gx)\,\mathrm{d}\mu(x) = \int_X \varphi(x)\, r_\mu(g^{-1}, x)\,\mathrm{d}\mu(x), \quad \text{for all } \varphi \in L^1(X, \mu).$$

Recall that μ-almost everywhere $r_\mu(g, \cdot) > 0$, and that $r_\mu : G \times X \to \mathbb{R}_{\geq 0}$ is a cocycle (see [80, Definition 4.2.1]), i.e. it satisfies the identity $r_\mu(gh, x) = r_\mu(g, hx) \cdot r_\mu(h, x)$ for every $g, h \in G$ and μ-almost every $x \in X$. The map r_μ is known as the *Radon-Nikodym cocycle*.

Example 1.13. If (X, μ) is a regular G-space, then $L^1(X, \mu)$ and $L^\infty(X, \mu)$ are Banach G-modules, where $L^\infty(X, \mu)$ is equipped with the left-translation G-action (which descends to an action on function classes), and G acts on $L^1(X, \mu)$ by the formula

$$(g \cdot \varphi)(x) := r_\mu(g, x)\, \varphi(g^{-1}x), \quad \varphi \in L^1(X, \mu),$$

for μ-almost every $x \in X$. Moreover, $L^1(X, \mu)$ is a continuous, separable Banach G-module, and hence, the pair $(L^1(X, \mu), L^\infty(X, \mu))$ is a coefficient G-module. See Appendix D in [12] for a proof of the continuity statement.

Note that if μ and ν are equivalent measures on a Borel space X, then $L^\infty(X, \mu) = L^\infty(X, \nu)$. Thus, it makes sense to write $L^\infty(X, \mathfrak{M})$ if $\mathfrak{M}$ is a measure class on X, or even simply $L^\infty(X)$ if, as in Example 1.12, the measure class $\mathfrak{M}$ on X is understood from the context. Furthermore, if X admits a Borel G-action and a G-invariant measure class $\mathfrak{M}$, then $L^\infty(X, \mathfrak{M})$ is a Banach G-module exactly as Example 1.13, without having to specify a probability measure in $\mathfrak{M}$. We will refer at a later point to the following remark related to this notation and the previous example.

Remark 1.14. It needs not be the case that different measures on a space, even within the same measure class, give rise to isomorphic L^1-spaces. Thus, when considering a coefficient G-module $(L^1(X, \mu), L^\infty(X, \mathfrak{M}))$, the choice of a particular $\mu \in \mathfrak{M}$ is important because it determines the weak-$*$ topology on $L^\infty(X, \mathfrak{M})$; see also Remark 1.9.

We will need the next generalization of Example 1.13, explained in §2.3 of [59].

Example 1.15. Let (X, μ) be a regular G-space, let $(B^\flat, B)$ be a coefficient G-module and define

$$L^\infty(X; B) := \{\phi : X \to B \mid \phi \text{ is weak-}*\text{ Borel and bounded}\}/\sim,$$

where "weak-$*$ Borel" means that the Borel structure on B is taken from its weak-$*$ topology, and $\sim$ denotes μ-almost everywhere equality. Equipped with the essential supremum norm, this is a Banach space. We endow it with the left-regular action as in (1.1). On the other hand, there is an isomorphism

$$L^\infty(X;B) \cong L^1(X,\mu;B^\flat)^*,$$

of Banach G-modules, where $L^1(X,\mu;B^\flat)$ is an object of $\mathbf{Ban}_G$. In consequence, the pair $(L^1(X,\mu;B^\flat), L^\infty(X;B))$ is a coefficient G-module.

Remark 1.16. Once the choice of a measure μ on X is fixed, we refer just to $L^\infty(X;B)$ as the coefficient G-module, instead of using the unwieldy notation $(L^1(X,\mu;B^\flat), L^\infty(X;B))$.

The L^∞-modules satisfy the so-called exponential law, which is given as [59, Corollary 2.3.3] and stated here.

Lemma 1.17. *For regular G-spaces (X_1,μ_1) and (X_2,μ_2) and a coefficient G-module $(B^\flat, B)$, there is an isomorphism $L^\infty(X_1\times X_2;B)\cong L^\infty(X_1;L^\infty(X_2;B))$ of coefficient G-modules.* □

We have the following notion of exactness of sequences of coefficient modules.

Definition 1.18. We say that a sequence

$$0 \to ((B^0)^\flat, B^0) \to ((B^1)^\flat, B^1) \to ((B^2)^\flat, B^2) \to \cdots \tag{1.8}$$

of coefficient G-modules and morphisms of coefficient G-modules is a *complex*, resp. *exact*, if the underlying sequence

$$0 \to B^0 \to B^1 \to B^2 \to \dots \tag{1.9}$$

of vector spaces has the corresponding property. In this case, in order to avoid an overloaded notation, we will omit mention of the preduals and refer simply to (1.9) and not to (1.8) as the complex, resp. exact sequence of coefficient G-modules whenever it is affordable.

The next lemma is one of our main technical tools; it is [59, Lemma 8.2.5].

Lemma 1.19. *The functor $\mathbf{Ban}_G^{\mathrm{op}} \to \mathbf{Vect}$ given by $(A^\flat, A) \mapsto L^\infty(G^n, A)^G$ is exact, i.e. if $0\to A\to B\to C\to 0$ is a short exact sequence of coefficient G-modules, then*

$$0 \to L^\infty(G^n;A)^G \to L^\infty(G^n;B)^G \to L^\infty(G^n;C)^G \to 0$$

is an exact sequence of vector spaces. □

We point out that the fact that the G-morphisms in the sequence $0\to A\to B\to C\to 0$ of the proposition above are dual morphisms is highly exploited in the proof of Lemma 1.19.

1.3 Amenability and continuous bounded cohomology

1.3.1 Amenable groups and actions. The notion of amenability for both lcsc groups and their actions play a key role in the theory of continuous bounded cohomology. For groups, it was introduced by von Neumann in the context of the Banach–Tarski paradox; see Chapter 0 in

[69]. We give below the definition of an amenable group as appearing in Chapter 4 of [80], and mention the main examples and non-examples without proofs.

Definition 1.20. A lcsc group G is said to be *amenable* if every compact metrizable space X with a continuous G-action admits a G-invariant probability measure, or equivalently, if the corresponding G-action on $\mathrm{Prob}(X)$ has a fixed point.

Example 1.21. Examples of amenable groups are compact and topologically solvable groups, in particular, Abelian ones. Combining this two classes yields, too, amenable groups: compact extensions of topologically solvable groups are amenable. Among connected Lie groups, that property actually characterizes amenability; in particular, a connected semisimple Lie group is amenable if and only if it is compact. Non-Abelian free groups are not amenable.

The concept of an amenable action was introduced by Zimmer in [79], and generalizes that of an amenable group in the sense that every amenable group acts amenably on a one-point space. Again, we follow Chapter 4 of [80], and [59] for definition and examples.

Definition 1.22. A lcsc group G is said to act *amenably* on a regular G-space (X, μ) if for every separable Banach space E and every Borel (right) cocycle $\alpha : X \times G \to \mathrm{Isom}(E)$, the dual α^*-twisted action on E^* satisfies the following property: any α^*-invariant Borel field $\{A_x\}_{x\in X}$ of non-empty, convex, weak-$*$ compact subsets A_x of the unit ball in E^* admits an α^*-invariant Borel section. If the G-action on (X, μ) is amenable, we also say that (X, μ) is an *amenable G-space*.

Example 1.23. • Any lcsc group G acts amenably on itself equipped with the Haar measure.

- A group G is amenable if and only if it acts amenably on a one-point space.
- If (X, μ) is an amenable G-space and $H < G$ is a closed subgroup, then (X, μ) is also an amenable H-space.
- If $P < G$ is a closed subgroup, then the action of G on G/P (equipped with a measure in the G-invariant measure class) is amenable if and only if P is amenable. For example, the group $G := \mathrm{SL}_r(\mathbb{R})$ acts transitively on the space of flags

  $$\mathcal{F}_{i_0<\cdots<i_k} = \{F_{i_0} \subset \cdots \subset F_{i_k} \mid \forall n \in [k] \,:\, F_{i_n} \text{ is a subspace of } \mathbb{R}^r \text{ of } \dim F_{i_n} = i_n\}$$

  of type $1 \le i_0 < \cdots < i_n \le r-1$, for every type. Thus, it is a homogeneous space G/P, where P is a point stabilizer. The closed subgroup $P < G$ is amenable if and only if the flags are complete, i.e. of type $1 < 2 < \cdots < r-1$, in which case it is isomorphic to the subgroup of G of upper-triangular matrices. More generally, if G is a non-compact semisimple Lie group and $P < G$ is a parabolic subgroup (i.e. the stabilizer subgroup of a point in the visual boundary of its associated symmetric space), then G acts amenably on G/P if and only if P is minimal parabolic. The space G/P, with P minimal parabolic is known as the maximal Furstenberg boundary of G; in rank one, it coincides with the whole of the visual boundary of the group.
- A Gromov hyperbolic group Γ acts amenably on its visual boundary $\partial_\infty\Gamma$ if this one is equipped with a Γ-quasi-invariant measure.

1.3.2 The role of amenability. Let (X, μ) be a regular G-space, and $(B^\flat, B)$ be a coefficient G-module. With the diagonal action, for any $n \in \mathbb{N}$, the product space $(X^n, \mu^{\otimes n})$ is also a regular G-space. In virtue of the Example 1.15, one may consider the *L^∞-homogeneous complex*

$$0 \to L^\infty(X; B) \xrightarrow{\mathrm{d}} L^\infty(X^2; B) \xrightarrow{\mathrm{d}} L^\infty(X^3; B) \to \cdots$$

where the differentials are defined exactly as the homogeneous differentials of the formula (1.3). It underlies a complex of coefficient G-modules. The next theorem shows the important connection of amenable spaces to the theory of continuous bounded cohomology. It says, in other words, that we may *realize* continuous bounded cohomology n-classes of G as G-invariant bounded weak-$*$-Borel function classes $X^{n+1} \to E$ on the $(n+1)$-fold product of any amenable G-space X. It is [59, Proposition 7.5.1].

Theorem 1.24. *Let G' be a lcsc group, $G < G'$ a closed subgroup, (X, μ) an amenable G'-space, and $(B^\flat, B)$ a coefficient G'-module. Then there exists a canonical isomorphism*

$$\mathrm{H}^n_{\mathrm{cb}}(G; B) \cong \mathrm{H}^n(L^\infty(X^{\bullet+1}; B)^G)$$

for every $n \in \mathbb{N}$. □

In the light of Example 1.23, a first corollary of Theorem 1.24 is that the continuous bounded cohomology of any amenable group G with values in a coefficient G-module vanishes in every positive degree. It also allows the realization of the continuous bounded cohomology classes of non-compact semisimple Lie groups on their Furstenberg boundaries, or of classes of hyperbolic groups on their visual boundaries.

Moreover, the canonical isomorphism of Theorem 1.24 behaves well in regards to the functoriality properties of $\mathrm{H}^\bullet_{\mathrm{cb}}$ from Subsection 1.1.3. Indeed, if $(C^\flat, C)$ is another coefficient G-module and $\alpha : B \to C$ is a dual morphism, then it induces a dual morphism

$$\alpha_* : L^\infty(X^{\bullet+1}; B) \to L^\infty(X^{\bullet+1}; C), \tag{1.10}$$

and the corresponding linear map $\mathrm{H}^\bullet(\alpha_*)$ in cohomology makes the diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(G; B) & \xrightarrow{\mathrm{H}^\bullet_{\mathrm{cb}}(G;\alpha)} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; C) \\ \cong \updownarrow & & \updownarrow \cong \\ \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; B)^G) & \xrightarrow{\mathrm{H}^\bullet(\alpha_*)} & \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; C)^G) \end{array}$$

commute, where the vertical isomorphisms are the ones from Theorem 1.24. In view of this, we will use the notation $\mathrm{H}^\bullet_{\mathrm{cb}}(G; \alpha)$ for the map $\mathrm{H}^\bullet(\alpha_*)$ given here as well.

Analogously, a homomorphism $\psi : H \to G$ of lcsc groups turns $L^\infty(X^{\bullet+1}; B)$ into a coefficient H-module via the action $h \cdot \varphi := \psi(h) \cdot \varphi$ for $h \in H$, where the right-hand side denotes the left-regular action of G. This action, as observed in Example 1.23, is also amenable.

The map $\mathrm{H}^\bullet(\iota)$ induced by the inclusion $\iota : L^\infty(X^{\bullet+1}; B)^G \hookrightarrow L^\infty(X^{\bullet+1}; B)^H$ makes the diagram

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(G; B) & \xrightarrow{\mathrm{H}^\bullet_{\mathrm{cb}}(\psi; B)} & \mathrm{H}^\bullet_{\mathrm{cb}}(H; B) \\ \cong \updownarrow & & \updownarrow \cong \\ \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; B)^G) & \xrightarrow{\mathrm{H}^\bullet(\iota)} & \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; B)^H) \end{array}$$

commute. We shall write $\mathrm{H}^\bullet_{\mathrm{cb}}(\psi; B)$ for $\mathrm{H}^\bullet(\iota)$. The combined notation $\mathrm{H}^\bullet_{\mathrm{cb}}(\psi; \alpha)$ from Section 1.2 will also be used in this setting. For the proofs of these functorial properties, we refer the reader to Propositions 8.1.1 and 8.4.2 of [59].

The following is a corollary of Theorem 1.24 and of the functorial properties of $\mathrm{H}^\bullet_{\mathrm{cb}}$. See [59, Corollary 7.5.10 and Corollary 8.5.2].

Corollary 1.25. *Let G be a lcsc group, $(B^\flat, B)$ a coefficient G-module, and $N \lhd G$ a closed normal subgroup. If N is amenable, then the map*

$$\mathrm{H}^\bullet_{\mathrm{cb}}(p; \iota) : \mathrm{H}^n_{\mathrm{cb}}(G/N; B^N) \to \mathrm{H}^n_{\mathrm{cb}}(G; B)$$

is an isomorphism in all degrees, where $p : G \twoheadrightarrow G/N$ is the quotient homomorphism, and $\iota : B^N \hookrightarrow B$ the inclusion. □

1.4 Eckmann–Shapiro induction

Let $H < G$ be a closed subgroup, and $(B^\flat, B)$ be a coefficient G-module. Since G/H is a regular G-space after a choice of a G-quasi-invariant probability measure on it, the coefficient G-module $\mathrm{Ind}_H^G B := L^\infty(G/H; B)$ with the left-regular action (1.1). It is known as the *induction module*.[1]

Remark 1.26 (Functorial properties of Ind). If $(C^\flat, C)$ is another coefficient G-module, and $\alpha : B \to C$ is a dual morphism, we write $\alpha_* : \mathrm{Ind}_H^G B \to \mathrm{Ind}_H^G C$ for the induced dual morphism, as in (1.10). On the other hand, let G' be another lcsc group, $H' < G'$ a closed subgroup, $(B^\flat, B)$ a coefficient G'-module, and $\psi : G \to G'$ a homomorphism with $\psi(H) < H'$. Then

- $(B^\flat, B)$ is also a coefficient G-module with the left-regular action of $\psi(G)$,
- G'/H' is both a regular G'- and G-space with the action by left-multiplication of $\psi(G)$, and thus,
- $\mathrm{Ind}_{H'}^{G'} B$ is both a coefficient G'- and G-module, and
- $\mathrm{Ind}_H^G B$ is a coefficient G-module.

Also, ψ gives rise to a map $\bar{\psi} : G/H \to G'/H'$, which induces a dual morphism

$$\psi^* : \mathrm{Ind}_{H'}^{G'} B \to \mathrm{Ind}_H^G B \tag{1.11}$$

underlying a morphism of coefficient G-modules.

[1] With a slightly different definition, one only needs to assume that $(B^\flat, B)$ is a coefficient H-module for $\mathrm{Ind}_H^G B$ to be a coefficient G-module. For the purposes of this thesis, however, we will only need the version stated here.

Now, let (X, μ) be an amenable G-space, and consider the bounded linear map

$$\mathrm{Ind}_X^n : L^\infty(X^{n+1}; B)^H \to L^\infty(X^{n+1}; \mathrm{Ind}_H^G B)^G$$

defined by the formula

$$\mathrm{Ind}_X^n(f)(\mathbf{x})(gH) := g \cdot f(g^{-1}\mathbf{x}), \quad f \in L^\infty(X^{n+1}; B)^H,\ \mathbf{x} \in X^{n+1},\ g \in G.$$

Then the following proposition holds. It is [59, Proposition 10.1.3].

Proposition 1.27. *With the notation fixed in this section, the map* Ind_X^n *induces an isomorphism*

$$\mathrm{Ind}^n : \mathrm{H}^n_{\mathrm{cb}}(H; B) \xrightarrow{\sim} \mathrm{H}^n_{\mathrm{cb}}(G; \mathrm{Ind}_H^G B) \tag{1.12}$$

at every degree $n \in \mathbb{N}$, *which is independent of the choice of* X *in the following sense: If* Y *is another amenable* G*-space, then the diagram*

$$\begin{array}{ccccccc}
 & & \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; B)^H) & \xrightarrow{\mathrm{H}^\bullet(\mathrm{Ind}_X^\bullet)} & \mathrm{H}^\bullet(L^\infty(X^{\bullet+1}; \mathrm{Ind}_H^G B)^G) & & \\
 & \cong\swarrow & \updownarrow\cong & & \updownarrow\cong & \searrow\cong & \\
\mathrm{H}^\bullet_{\mathrm{cb}}(H; B) & & & & & & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}_H^G B) \\
 & \cong\searrow & & & & \swarrow\cong & \\
 & & \mathrm{H}^\bullet(L^\infty(Y^{\bullet+1}; B)^H) & \xrightarrow{\mathrm{H}^\bullet(\mathrm{Ind}_Y^\bullet)} & \mathrm{H}^\bullet(L^\infty(Y^{\bullet+1}; \mathrm{Ind}_H^G B)^G) & &
\end{array}$$

commutes, where the isomorphisms in the diagonal arrows are given by the functorial characterization of continuous bounded cohomology. Furthermore, the map (1.12) *is natural in the following ways:*

(i) *If* $\alpha : B \to C$ *is a dual morphism underlying a morphism of coefficient* G*-modules, then the diagram below commutes:*

$$\begin{array}{ccc}
\mathrm{H}^\bullet_{\mathrm{cb}}(H; B) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}_H^G B) \\
\downarrow{\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(H;\alpha)} & & \downarrow{\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(G;\alpha_*)} \\
\mathrm{H}^\bullet_{\mathrm{cb}}(H; C) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}_H^G C)
\end{array}$$

(ii) *Let* $\psi : G \to G'$ *be a homomorphism to another lcsc group* G', *let* $H' < G$ *be a closed subgroup containing* $\psi(H)$, *and let* B *be a coefficient* G'*-module. Then the diagram below commutes, where the diagonal arrow corresponds to the map* (1.11).

$$\begin{array}{ccccc}
\mathrm{H}^\bullet_{\mathrm{cb}}(H'; B) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G'; \mathrm{Ind}_{H'}^{G'} B) & \xrightarrow{\mathrm{H}^\bullet_{\mathrm{cb}}(\psi;\mathrm{Ind}_{H'}^{G'} B)} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}_{H'}^{G'} B) \\
\downarrow{\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(\psi|_H;B)} & & & \swarrow{\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(G;\psi^*)} & \\
\mathrm{H}^\bullet_{\mathrm{cb}}(H; B) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}_H^G B) & &
\end{array}$$

□

Remark 1.28. We will apply the following particular case of the naturality statement in (iii). Keeping the notation, set $G' := G$ and $\psi := \mathrm{id} : G \to G$. Thus $H < H'$, and the commutative diagram from (iii) reduces to

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{cb}}(H'; B) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}^G_{H'} B) \\ \downarrow {\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(j;B)} & & \downarrow {\scriptstyle \mathrm{H}^\bullet_{\mathrm{cb}}(G;\mathrm{id}^*)} \\ \mathrm{H}^\bullet_{\mathrm{cb}}(H; B) & \xrightarrow{\mathrm{Ind}^\bullet} & \mathrm{H}^\bullet_{\mathrm{cb}}(G; \mathrm{Ind}^G_{H} B) \end{array}$$

where $j : H \hookrightarrow H'$ is the inclusion.

1.5 Some aspects of the isomorphism conjecture

Probably the most important open problem around the subject of continuous bounded cohomology is the question of whether the one of connected, semisimple Lie groups with finite center (or more generally, of the groups of $\Bbbk$-points of connected, simply connected, semisimple algebraic $\Bbbk$-groups) coincides with their *continuous cohomology*. A positive answer is predicted by the so-called *isomorphism conjecture*. This section treats some aspects of that conjectural connection that will be relevant to us in the development of this work.

1.5.1 Continuous cohomology and the comparison map. Given a locally compact space X and a topological vector space E, we will write $C(X; E)$ to denote the space of continuous functions $C(X; E)$ endowed with the compact-open topology. If E is a *topological G-module*, that is, if it additionally admits a continuous G-action, then so is $C(G; E)$ by the same left-regular action defined in (1.1). If furthermore E is a Banach space, then there exists an inclusion

$$C_b(X; E) \subset C(X; E). \tag{1.13}$$

Again if $E = \mathbb{R}$ is a Banach space, we will just write $C(X)$ instead of $C(X; \mathbb{R})$.

Definition 1.29. The *continuous cohomology* $\mathrm{H}^n_{\mathrm{c}}(G; E)$ of G in degree n and with coefficients in a topological G-module E is defined as the n-th cohomology of the right complex

$$0 \to C(G; E)^G \to C(G^2; E)^G \to C(G^3; E)^G \to \cdots, \tag{1.14}$$

of G-invariants, where the coboundary operators are the homogeneous differentials from (1.3). With trivial coefficients $E = \mathbb{R}$, we write $\mathrm{H}^n_{\mathrm{c}}(G)$ instead of $\mathrm{H}^n_{\mathrm{c}}(G; \mathbb{R})$.

Remark 1.30. Again, $\mathrm{H}^n_{\mathrm{c}}(G; E)$ is a topological vector space (however with no seminorm), but we will treat it simply as an object of **Vect** for all our purposes.

The inclusion (1.13) gives rise to a linear map in cohomology that is known as the *comparison map*:

$$c^\bullet : \mathrm{H}^\bullet_{\mathrm{cb}}(G) \longrightarrow \mathrm{H}^\bullet_{\mathrm{c}}(G). \tag{1.15}$$

Using the complex of G-invariants of (1.2) and the one in (1.14), the surjectivity of the comparison map means that any continuous cocycle should have a continuous *bounded* representative.

Its injectivity, in turn, means that any continuous bounded cocycle with a continuous primitive should also admit a continuous bounded primitive.

In degree zero, $\mathrm{H}^0_{\mathrm{cb}}(G) = \mathrm{H}^0_{\mathrm{c}}(G) = \mathbb{R}$. In degree one, the injectivity of the comparison map holds for any lcsc group G. This is because $\mathrm{H}^1_{\mathrm{cb}}(G) = 0$, consisting of continuous bounded homomorphisms $G \to \mathbb{R}$, while $\mathrm{H}^1_{\mathrm{c}}(G)$ can be identified with the space of continuous homomorphisms $G \to \mathbb{R}$. If G is a compact group, then one observes immediately that $\mathrm{H}^1_{\mathrm{c}}(G) = 0 = \mathrm{H}^1_{\mathrm{cb}}(G)$. The same holds if G is a connected semisimple Lie group: Indeed, if G is such a group, and $\alpha : G \to \mathbb{R}$ is a continuous homomorphism, then it is also smooth. If $\mathfrak{g}$ is the Lie algebra of G, then

$$D_1\alpha(\mathfrak{g}) = D_1\alpha([\mathfrak{g},\mathfrak{g}]) = [D_1\alpha(\mathfrak{g}), D_1\alpha(\mathfrak{g})] = 0$$

since $\mathbb{R}$ is abelian. By connectedness, $\alpha \equiv 0$, and because α is arbitrary, we have $\mathrm{H}^1_{\mathrm{c}}(G) = 0$. However, in general, the comparison map needs not be injective nor surjective.

Example 1.31. A counterexample for both the injectivity and surjectivity of the comparison map is given by the free group $F_2 = \langle a, b\rangle$ in two generators:

- The space $\mathrm{H}^1_c(F_2) = \mathrm{H}^1(F_2)$ is two-dimensional, as homomorphisms $F_2 \to \mathbb{R}$ are determined by the two choices of images of the generators a and b. Thus, the comparison map in degree one cannot be surjective.
- Since $\mathbb{S}^1 \vee \mathbb{S}^1$ is a $K(F_2, 1)$, we have $\mathrm{H}^2_c(F_2) = \mathrm{H}^2(F_2) \cong \mathrm{H}^2(\mathbb{S}^1 \vee \mathbb{S}^1) = 0$. On the other hand, $\mathrm{H}^2_{\mathrm{cb}}(F_2) = \mathrm{H}^2_{\mathrm{b}}(F_2)$ is an infinite-dimensional vector space by work of Brooks [8].

A non-discrete counterexample is the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$, for which the comparison map is not injective in degree two, and not surjective in degree three. Even though it is a connected semisimple Lie group, it has an *infinite* center, which is the cause of these discrepancies. For a reference, see [59, Example 9.3.11] or the Appendix A.

Let G be a connected, simple[2] Lie group with finite center, and $K < G$ a maximal compact subgroup. Then, equipped with any G-invariant Riemannian metric, the space G/K is a symmetric space of non-compact type (see [44]). The following theorem is the most powerful tool for determining the continuous cohomology of G. It follows after combining work of van Est and Cartan, among others. We point the reader to the Appendix A and the references contained therein for more robust versions of this statement.

Theorem 1.32. *Let $\mathfrak{k} \subset \mathfrak{g}$ be the Lie algebras of $K < G$, let G_u be the only 1-connected compact simple Lie group with Lie algebra $\mathfrak{g}$ up to complexification, and K_u its corresponding connected subgroup with Lie algebra $\mathfrak{k}$. Then*

$$\mathrm{H}^\bullet_{\mathrm{c}}(G) \cong \Omega^\bullet(G/K)^G \cong \mathrm{H}^\bullet(G_u/K_u), \tag{1.16}$$

where $\Omega^\bullet(G/K)^G$ denotes the space of G-invariant forms on the associated symmetric space G/K, and $\mathrm{H}^\bullet(G_u/K_u)$ is the de Rham cohomology of the homogeneous space G_u/K_u, which is a symmetric space of compact type. □

[2] For simplicity, instead of semisimple.

The first isomorphism in (1.16) is a version of van Est's isomorphism [77] adapted to our setting; the second one is collected in the paper [45] by Hochschild–Mostow, and attributed to Cartan. Dupont [28] showed that the first isomorphism can be realized at the level of cochains by integration over simplices, on which we expand in the next paragraph. This realization of van Est's isomorphism motivated also a question related to the surjectivity of the comparison map for groups like G.

Fix a k-form $\omega \in \Omega^k(G/K)^G$ and a base point $o \in G/K$. For any k-tuple $(g_0, \dots, g_k) \in G^k$, consider the *geodesic coning k-simplex* $\Delta(g_0, \dots, g_k) \subset G/K$, defined inductively as follows: let $\Delta(g_0) := \{g_0 \cdot o\}$, and for $k > 0$, set $\Delta(g_0, \dots, g_k)$ to be the geodesic cone with apex $g_k \cdot o$ and base $\Delta(g_0, \dots, g_{k-1})$. It is not hard to verify that the expression

$$I_\omega(g_0, \dots, g_k) := \int_{\Delta(g_0,\dots,g_k)} \omega \tag{1.17}$$

is a well-defined G-invariant continuous k-cocycle; the cocycle identity follows from Stokes' theorem and the lemma by Cartan that G-invariant forms on a symmetric space G/K are automatically closed. This gives an assignment

$$I : \Omega^\bullet(G/K)^G \to \mathrm{H}^\bullet_c(G)$$

that is independent of the choice of base point o. It was proven by Dupont that I is an isomorphism; see [28, Prop. 1.5]. He also posed the question (see [27, Remark 3]) of whether cocycles obtained in this fashion were always bounded. A positive answer would imply that the comparison map for G is surjective.

1.5.2 Degree two and the conjecture. Dupont's question can be inquired for a specific differential form. For example, if G is non-compact and *of Hermitian type* (i.e. such that the symmetric space G/K is Hermitian; see Chapter VIII and in [44], and §X.6.3 of the same reference for the classification of irreducible ones), then G admits a Kähler form $\omega \in \Omega^2(G/K)^G$, and hence has non-trivial $\mathrm{H}^2_c(G)$. As a matter of fact, such a form is the only possible source of continuous cohomology in degree two for G:

Theorem 1.33. *Let G be a connected, simple Lie group with finite center. Then the following are equivalent:*

(i) $\mathrm{H}^2_c(G) \neq 0$.
(ii) $\dim \mathrm{H}^2_c(G) = 1$.
(iii) *G is non-compact and of Hermitian type.*

About the proof. See Guichardet–Wigner [38], where an explicit cocycle is given without using (1.17). □

One can readily observe that the cocycle from [38] is bounded. By Theorem 1.33, this proves the surjectivity of the comparison map in degree two for all groups that satisfy the equivalent conditions. Domic–Toledo [26] and Clerc–Ørsted [20] use the cocycle I_ω to obtain an upper bound of the Gromov norm of the continuous bounded cohomology class, which they prove to be effective. If the minimal holomorphic sectional curvature of G/K is normalized to be -1, and $0 \neq \kappa = [I_\omega] \in \mathrm{H}^2_{\mathrm{cb}}(G)$ denotes the Kähler class, then $\|\kappa\| = \pi r$, where $r := \mathrm{rank}_{\mathbb{R}}(G)$.

On the other hand, the injectivity of the comparison map for G in degree two was derived by Burger–Monod [13] by exploiting the relationship between $\mathrm{H}^2_{\mathrm{cb}}$ and continuous quasi-morphisms. The results can be recollected in the next theorem.

Theorem 1.34. *Let $G = \mathbf{G}(\mathscr{k})$, where $\mathbf{G}$ is a connected, simply connected, $\mathscr{k}$-isotropic, simple algebraic group over a local field $\mathscr{k}$. Then the comparison map $c^2 : \mathrm{H}^2_{\mathrm{cb}}(G) \to \mathrm{H}^2_{\mathrm{c}}(G)$ in degree two is an isomorphism. In particular, $\mathrm{H}^2_{\mathrm{cb}}(G)$ is one-dimensional if $\mathscr{k} = \mathbb{R}$ and $\mathbf{G}$ is of Hermitian type, and it vanishes otherwise.* □

Remark 1.35. Taking $\mathscr{k}$ to be either $\mathbb{R}$ or $\mathbb{C}$ in the statement of the theorem coincides with the assumption that G is a non-compact, connected, simple Lie group with finite center. Surjectivity in the case of a non-Archimedean $\mathscr{k}$ (i.e. $\mathscr{k}$ is the field of Laurent series $\mathbb{F}_q((t))$ for a power q of a prime, or a finite extension of $\mathbb{Q}_p$ for some prime p) is covered by the next theorem. The injectivity for non-Archimedean fields was also shown by Burger–Monod [13].

Theorem 1.36. *Let $G = \mathbf{G}(\mathscr{k})$, as in Theorem 1.34, with $\mathscr{k}$ non-Archimedean. Then $\mathrm{H}^q_{\mathrm{c}}(G) = 0$ for every $q > 0$.*

About the proof. See [6, Theorem X.4.12]. □

Theorem 1.34 gave rise to the question of whether an analogue would hold in every degree for this class of groups, and then to the announced *isomorphism conjecture*; see also Conjecture 0.2 in the Introduction.

Conjecture 1.37 (see [58, Problem A] or [16])**.** *For any $G = \mathbf{G}(\mathscr{k})$ as in Theorem 1.34, the comparison map (1.15) is an isomorphism in every degree.*

1.5.3 Evidence in degree three and higher.

Beyond degree two, Conjecture 1.37 is known in full generality only for $\mathrm{SL}_2(\mathscr{k})$, with $\mathscr{k}$ a non-Archimedean local field, and in a few cases in degrees three and four. We list all existing results in the table below, and discuss afterwards the situation in degree three.

q	G		$\dim \mathrm{H}^q_{\mathrm{c(b)}}(G)$	Author
3	$\mathrm{SL}_2(\mathscr{k})$	$\mathscr{k} = \mathbb{R}$	0	Burger–Monod [15]
		$\mathscr{k} = \mathbb{C}$	1	Bloch [5]
		$\mathscr{k}$ non-Archim.	0	Bucher–Monod [11]
	$\mathrm{SL}_r(\mathscr{k})$	$\mathscr{k} = \mathbb{R}$, $r \geq 3$	0	Monod [61]
		$\mathscr{k} = \mathbb{C}$, $r \geq 3$	1	Monod [61], Goncharov [35]
		$\mathscr{k}$ non-Archim., $r \geq 4$	0	Monod [61]
	$\mathrm{SO}^+(r,1)$	$r \geq 3$	0	Pieters [65]
4	$\mathrm{SL}_2(\mathbb{R})$		0	Hartnick–Ott [40]
$q \geq 4$	$\mathrm{SL}_2(\mathscr{k})$, $\mathscr{k}$ non-Archim.		0	Bucher–Monod [11]

We begin the discussion in degree three with continuous cohomology. For $\mathrm{SL}_2(\Bbbk)$ with $\Bbbk$ a local field, it can be easily determined from Theorem 1.32 and Theorem 1.36:

$$\dim \mathrm{H}^3_c(\mathrm{SL}_2(\Bbbk)) = \begin{cases} \dim \mathrm{H}^3_c(\mathbb{S}^2) = 0 & \text{if } \Bbbk = \mathbb{R}, \\ \dim \mathrm{H}^3_c(\mathbb{S}^3) = 1 & \text{if } \Bbbk = \mathbb{C}, \\ 0 & \text{if } \Bbbk \text{ is non-Archimedean.} \end{cases}$$

And in fact, the observation that $\mathrm{H}^3_c(\mathrm{SL}_2(\Bbbk)) \neq 0$ if and only if $\Bbbk = \mathbb{C}$ is just an instance of a more general phenomenon:

Theorem 1.38 (see also Theorem G from Introduction)**.** *Let G be a connected, simple Lie group with finite center. Then the following are equivalent:*

(i) $\mathrm{H}^3_c(G) \neq 0$.
(ii) $\dim \mathrm{H}^3_c(G) = 1$.
(iii) *G admits the structure of a complex Lie group.*

With the notation from Theorem 1.32, the statement of Theorem 1.38 would follow from the Killing–Cartan classification of simple Lie groups and the determination of $\mathrm{H}^3(G_u/K_u)$ in each case. For instance: In the setting of a complex simple Lie group G, one can show that the symmetric space G_u/K_u is diffeomorphic to the maximal compact subgroup K. The next table lists the maximal compact subgroups in each of the classical families.

Complex simple Lie group	Maximal compact[3]
$\mathrm{SL}_r(\mathbb{C})$	$\mathrm{SU}(r)$
$\mathrm{SO}_{2r+1}(\mathbb{C})$	$\mathrm{SO}(2r+1)$
$\mathrm{Sp}_{2r}(\mathbb{C})$	$\mathrm{Sp}(r)$
$\mathrm{SO}_{2r}(\mathbb{C})$	$\mathrm{SO}(2r)$

Then, the next theorem [55, Theorem III.6.5] establishes non-vanishing in degree three.

Theorem 1.39. *The following compact Lie groups have as de Rham cohomology ring exterior algebras generated by* Borel classes *e_i and s_i of degree i:*

(i) $\mathrm{H}^\bullet(\mathrm{SU}(r)) = \Lambda(e_3, e_5, \dots, e_{2r-1}), \quad r \geq 2,$

(ii) $\mathrm{H}^\bullet(\mathrm{SO}(2r+1)) = \Lambda(e_3, e_7, \dots, e_{4r-1}),$

(iii) $\mathrm{H}^\bullet(\mathrm{Sp}(r)) = \Lambda(e_3, e_7, \dots, e_{4r-1}),$

(iv) $\mathrm{H}^\bullet(\mathrm{SO}(2r)) = \Lambda(e_3, e_7, \dots, e_{4r-5}, s_{2r-1}), \quad r \geq 2.$

Moreover, the homomorphism induced in cohomology by the inclusion $G(r) \hookrightarrow G(r+1)$ (for $G \in \{\mathrm{SU}, \mathrm{SO}, \mathrm{Sp}\}$) to a generator with the same symbol, or to zero if there is no such symbol. □

[3] The group $\mathrm{Sp}(r)$ is defined as the intersection $\mathrm{Sp}_{2r}(\mathbb{C}) \cap U(2r)$ and is known as the compact symplectic group.

For similar tables that would settle the non-vanishing direction, we refer the reader to [55] or to [36]. In the Appendix A, we give an argument for Theorem 1.38 that does not rely on results like Theorem 1.39 or the classification.

We turn now to continuous bounded cohomology. In the case of $\mathrm{SL}_2(\Bbbk)$, the vanishing statement was proven for $\Bbbk = \mathbb{R}$ in [15] with an argument of harmonic-analytic flavor, and for $\Bbbk$ non-Archimedean in [11] by studying an ℓ^∞-complex arising from configuration spaces of vertices in the Bruhat–Tits tree of $\mathrm{SL}_2(\Bbbk)$. For $\Bbbk = \mathbb{C}$, the volume form on the hyperbolic 3-space $\mathbb{H}^3$—the non-compact symmetric space associated to $\mathrm{SL}_2(\mathbb{C})$—produces a non-zero class $\beta_2 \in \mathrm{H}^3_{\mathrm{c}}(\mathrm{SL}_2(\mathbb{C}))$ by means of (1.17). The volume cocycle is bounded, and hence it represents a bounded class $\beta_2^{\mathrm{b}} \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{C}))$. The latter can be represented in the limit, in the spirit of Theorem 1.24, by a cocycle in $L^\infty((\partial\mathbb{H}^3)^4)^{\mathrm{SL}_2(\mathbb{C})}$. It is defined as the oriented volume Vol of ideal tetrahedra in $\mathbb{H}^3$ (i.e. tetrahedra in $\mathbb{H}^3$ with vertices on $\partial\mathbb{H}^3$.) The essential transitivity of $\mathrm{SL}_2(\mathbb{C})$ on 3-tuples of points in $\partial\mathbb{H}^3 \cong \mathbb{P}(\mathbb{C}^2)$ guarantees the lack of non-trivial 3-coboundaries. Then, it is a theorem of Bloch [5] that, up to scalar multiples, Vol is the only L^∞-function that satisfies the cocycle equation. This is precisely the injectivity of the comparison map for $\mathrm{SL}_2(\mathbb{C})$.

For $\mathrm{SL}_r(\Bbbk)$, the injectivity in degree three of the comparison map can be derived from a combination of the corresponding result for $\mathrm{SL}_2(\Bbbk)$ (or $\mathrm{GL}_2(\Bbbk)$), and of a bounded-cohomological *stability* theorem—in the sense of Definition 0.3 from the Introduction—with a sufficiently good range. For $\mathrm{SL}_r(\Bbbk)$ (and for $\mathrm{GL}_r(\Bbbk)$) over any local field, such a stability theorem was proven by Monod in [61]. The case of $\mathrm{SL}_3(\Bbbk)$ with $\Bbbk$ non-Archimedean is not covered by [61]; see also Remark 6.23 below. The case of $\mathrm{SO}^+(r,1)$ follows analogously by a stability result of Pieters [65]. Note that $\mathrm{SO}^+(2,1) \cong \mathrm{PSL}_2(\mathbb{R})$, so vanishing is covered by the result for $\mathrm{SL}_2(\mathbb{R})$.

Remark 1.40. Prior to this dissertation, the only such bounded-cohomological stability theorems in the literature were by Monod [61] for $\mathrm{GL}_r(\Bbbk)$ and $\mathrm{SL}_r(\Bbbk)$ over any local field $\Bbbk$, and by Pieters [65] for $\mathrm{SO}^+(r,1)$ and $\mathrm{SU}^+(r,1)$.

Among the groups mentioned here, the only one with non-vanishing $\mathrm{H}^3_{\mathrm{c}}$ is $\mathrm{SL}_r(\mathbb{C})$. Goncharov [35] defined a Borel bounded 3-cocycle on "generic" 4-tuples of the manifold $\mathcal{F}(\mathbb{C}^r)$ of complete flags of $\mathbb{C}^r$, which is the amenable Furstenberg boundary of $\mathrm{SL}_r(\mathbb{C})$. This cocycle was extended to all 4-tuples by Bucher–Burger–Iozzi [10], and the bounded 3-class $\beta_r^{\mathrm{b}} \in \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$ that it represents is non-zero. Bucher–Burger–Iozzi [10] compute the Gromov norm of the class:

Theorem 1.41 (see [10, Theorem 2])**.** *Let $r \geq 2$. The class β_r^{b} generates $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$, and it has Gromov norm*

$$\|\beta_r^{\mathrm{b}}\| = \frac{r(r^2-1)}{6} v_3,$$

where v_3 is the maximal volume of an ideal tetrahedron in $\partial\mathbb{H}^3$. □

We conclude this subsection mentioning some additional evidence towards Conjecture 1.37. The relaxed question of surjectivity has been established in several other cases. Relying on the fact by Gromov [37] that characteristic classes are bounded, Hartnick–Ott [41] showed surjectivity for all semisimple Lie groups of Hermitian type. Moreover, Dupont's problem of whether the cocycles obtained by (1.17) are bounded was answered, beyond degree two, in top degree by Lafont–Schmidt [50] and Bucher [9], and in low codimension for $G \notin \{\mathrm{SL}_3(\mathbb{R}), \mathrm{SL}_4(\mathbb{R})\}$, by Lafont–Wang [51]. In both cases, they do not use the geodesic coning simplices described above, but a more suitable kind that arises from a so-called barycentric straightening.

We point out that continuous cohomology stabilizes along classical families, and so, any bounded-cohomological stability result serves also as evidence for Conjecture 1.37. Theorem 1.39 allows us to write down the optimal stability ranges of continuous cohomology along the classical families of simple complex Lie groups:

- Along $(\mathrm{SL}_r(\mathbb{C}))_r$, in the non-vanishing degrees $q \equiv 1 (\mathrm{mod}\ 2)$, we have an optimal linear range $r_0(q) = (q+1)/2$.
- Along $(\mathrm{SO}_{2r+1}(\mathbb{C}))_r$ and $(\mathrm{Sp}_{2r}(\mathbb{C}))_r$, in the non-vanishing degrees $q \equiv -1 (\mathrm{mod}\ 4)$, it is $r_0(q) = (q+1)/4$.

Curiously, $\mathrm{H}_c^\bullet$ stabilizes along $(\mathrm{SO}_{2r}(\mathbb{C}))_r$, but the presence in this case of the classes $s_{2r-1} \in \mathrm{H}_c^{2r-1}(\mathrm{SO}_{2r}(\mathbb{C}))$ seems to worsen the stability range.

Chapter 2

Formed Spaces and Grassmannians

The object of this chapter is twofold. Firstly, we establish a common ground for treating simultaneously all classical families of Lie groups in the framework of the stability theorems of Part II. Every classical Lie group arises as a subgroup of the automorphisms of a *non-degenerate formed space* over $\mathbb{R}$ or $\mathbb{C}$, i.e. of a vector space equipped with a non-degenerate sesquilinear form. In Section 2.1, we define formed spaces and list some basic facts, among which we highlight the existence of so-called *adapted bases*. In the sesquilinear setting, they are the natural analogues of orthonormal bases for inner product spaces, thus allowing us to make sense of the notion of a *standard formed space* and to regard each classical family as an ascending infinite sequence of groups. At the end of Section 2.1, in Subsection 2.1.4, we prove a linear-algebraic lemma for projective points in formed spaces that will be useful in the construction of so-called generic random chainings in Chapter 7. Most of the statements in this section are classical and we omit their proofs. To that effect, we refer the reader to §7 of [75], or to §6–§8 of [25]. Another excellent reference is the recent preprint [71], which treats in a unified way formed spaces over general fields.

Secondly, in Section 2.2, we present the Grassmannians of a vector space as endowed with the Chabauty topology. Even though it coincides with the usual topology on Grassmannians (that is, when regarded as homogeneous spaces of a general linear group), this definition facilitates the proof of the continuity of polarities. After covering those statements, we include a brief discussion of the isotropic Grassmannians of formed spaces.

Through the whole of the present section, we let V be a vector space over $\mathscr{k} \in \{\mathbb{R}, \mathbb{C}\}$ with finite dimension n.

2.1 Formed spaces

2.1.1 Sesquilinear forms. Let $\sigma\colon \mathscr{k} \to \mathscr{k}$ be a continuous automorphism of $\mathscr{k}$, hence either the identity or complex conjugation $\bar{\cdot}$ in case $\mathscr{k} = \mathbb{C}$. An additive map $f : V_1 \to V_2$ between two $\mathscr{k}$-vector spaces is σ-linear if

$$f(av) = \sigma(a)f(v) \quad \text{for all } a \in \mathscr{k}, v \in V_1.$$

If $\sigma = \mathrm{id}$, we will just say f is linear. Let $\omega : V \times V \to \Bbbk$ be a *σ-sesquilinear form*, i.e. a function such that for every $v \in V$, the assignment $\omega(-, v)$ is a σ-linear map and $\omega(v, -)$ is a linear map, and that is reflexive in the sense that $\omega(v, w) = 0$ implies $\omega(w, v) = 0$ for every $v, w \in V$. We say in that case that (V, ω) is a *formed space*.

Remark 2.1. It is easy to see, thanks to the finite-dimensionality of V and the continuity of σ, that any σ-sesquilinear form is a continuous function $V \times V \to \Bbbk$.

The *radical* of the formed space (V, ω) is defined as the set

$$\mathrm{rad}(V, \omega) := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in V\}.$$

We say that the form ω (or the formed space (V, ω)) is *non-degenerate* if $\mathrm{rad}(V, \omega)$ is trivial. We assume the non-degeneracy of (V, ω) from now on in the rest of Section 2.1. The next theorem states that non-degenerate σ-sesquilinear forms on V come in three different flavors. It known as the Birkhoff–von Neumann theorem [4]; a proof can be found also in [25, §6].

Theorem 2.2. *A non-degenerate σ-sesquilinear form ω on a vector space V is either*

(a) alternating*: $\sigma = \mathrm{id}$, and $\omega(v, v) = 0$ for every $v \in V$;*

(b) symmetric*: $\sigma = \mathrm{id}$, and $\omega(v, w) = \omega(w, v)$ for every $v, w \in V$; or*

(c) Hermitian*: $\Bbbk = \mathbb{C}$, $\sigma = \bar{\cdot}$ is complex conjugation, and $\omega(v, w) = \overline{\omega(w, v)}$ for every $v, w \in V$.* □

Two vectors $v, w \in V$ are said to be *ω-perpendicular* if $\omega(v, w) = 0$. Two subsets $S, T \subset V$ are *ω-perpendicular* if every vector in S is ω-perpendicular to every vector in T. We also define the *ω-complement* S^ω of a subset $S \subset V$ as the set

$$S^\omega := \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in S\}.$$

This complement is always a linear subspace of V since $S^\omega = (\mathrm{span}(S))^\omega$. Moreover, for any linear subspaces $W, U < V$, we have

$$\dim W + \dim W^\omega = \dim V, \qquad W \subset U \text{ implies } U^\omega \subset W^\omega, \qquad (W^\omega)^\omega = W,$$
$$(W + U)^\omega = W^\omega \cap U^\omega, \qquad (W \cap U)^\omega = W^\omega + U^\omega.$$

A non-zero vector $v \in V$ is said to be *isotropic* if $\omega(v, v) = 0$. A subspace $W < V$ is said to be *totally isotropic* if $W < W^\omega$. On the other hand, W is said to be *non-degenerate* if $W \cap W^\omega = \{0\}$, or equivalently, if the restriction $\omega|_{W \times W}$ is non-degenerate; in this case, we have that $W \oplus W^\omega = V$. Finally, we say that a pair (e, f) of isotropic vectors in V is a *hyperbolic pair* if $\omega(e, f) = 1$. Note that if (e, f) is a hyperbolic pair, then the subspace $\mathrm{span}\{e, f\}$ is non-degenerate.

If (V_1, ω_1) and (V_2, ω_2) are two formed spaces, we say that a linear map $\varphi : V_1 \to V_2$ is a *linear isometry* if it is an injective linear map that satisfies the equation

$$\omega_2(\varphi(v), \varphi(w)) = \omega_1(v, w), \quad \text{for all } v, w \in V_1.$$

We denote by $G := \mathrm{Aut}(V,\omega) < \mathrm{GL}(V)$ the automorphism group of the formed space (V,ω), which consists of all bijective linear isometries $V \to V$. It is an algebraic subgroup of $\mathrm{GL}(V)$, closed under the Hausdorff topology that the latter group inherits from V.

The following theorem is a classical result on sesquilinear forms, known as Witt's lemma. It was proven by Witt in the case of symmetric and Hermitian forms. A proof of it can be found in [75], where it is stated as Theorem 7.4.

Theorem 2.3. *Let (V,ω) be a non-degenerate formed space, let W be a subspace of V regarded as a formed space with the restriction of ω, and $\varphi : W \to V$ be a linear isometry. Then there exists $g \in \mathrm{Aut}(V,\omega)$ such that $g|_W = \varphi$.* □

It is a consequence of this theorem that any two maximal totally isotropic subspaces of (V,ω) have the same dimension. This maximal dimension, denoted r from now on, is known as the *rank* of (V,ω). A second corollary of Theorem 2.3 is the following fact, which will be of relevance to us in Chapter 7. Here, we regard Cartesian products of V as endowed with the diagonal G-action.

Corollary 2.4. *Let (V,ω) be a non-degenerate formed space, and $k \in [r-1]$. If $(v_0,\dots,v_k)$ and $(w_0,\dots,w_k)$ are two ordered tuples of linearly independent vectors in V that span totally isotropic subspaces of V, then there exists $g \in \mathrm{Aut}(V,\omega)$ such that $g.(v_0,\dots,v_k) = (w_0,\dots,w_k)$.*

Proof. Let $U := \mathrm{span}\{v_0,\dots,v_k\}$. The linear map $U \to V$ defined by $v_i \mapsto w_i$ for every $i \in [k]$ is a linear isometry, and by Theorem 2.3, extends to $g \in \mathrm{Aut}(V,\omega)$. □

2.1.2 Adapted bases and the standard formed spaces.

The following proposition guarantees the existence of ordered bases of V that are "adapted" to the form ω, allowing for a unified treatment of the three types from Theorem 2.2 when it comes to computations in coordinates. It can be found within the section entitled "Flags and frames" of [75, §7].

Proposition 2.5. *Let (V,ω) be a non-degenerate formed space of rank r, and let U be a maximal isotropic subspace of V, with basis $\{e_1,\dots,e_r\}$. Then there exist a maximal isotropic subspace $U' = \mathrm{span}\{f_1,\dots,f_r\}$ of V and a subspace $W < V$ such that*

(i) *$(e_1,f_1),\dots,(e_r,f_r)$ are hyperbolic pairs that span pairwise ω-perpendicular subspaces,*

(ii) *W contains no isotropic vectors, and is ω-perpendicular to both U and U', and*

(iii) *$V = \bigoplus_{i=1}^{r} \mathrm{span}\{e_i,f_i\} \oplus W = U \oplus U' \oplus W$.* □

An *adapted basis* of (V,ω) will be defined as an ordered basis of V that contains the vectors $e_1,\dots,e_r, f_1,\dots,f_r$. What a "desirable" basis is for the remaining subspace W in Proposition 2.5 will depend on the type of ω. We re-classify now sesquilinear forms into three types—I, II, and III—and define adapted bases accordingly. Every such basis determines an isomorphism class of formed spaces; we will also select a natural representative in each class, that we will call the *standard formed space* of that type. Set $d := n - 2r = \dim W$.

Type I: ω is alternating. The subspace W is trivial, since all vectors in V are isotropic by definition. Hence, $d = 0$ and $n = 2r$. Because $\Bbbk$ is of characteristic not two, alternance is equivalent to the equality $\omega(v, w) = -\omega(w, v)$ holding for every $v, w \in V$. Thus, in the ordered basis $\mathcal{B} := (e_r, \dots, e_1, f_1, \dots, f_r)$, the form ω is represented by the matrix

$$J_r := \begin{pmatrix} & Q_r \\ -Q_r & \end{pmatrix}, \text{ where } Q_r = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

We say that $\mathcal{B}$ is a basis of V *adapted to ω*.

If (V', ω') is another non-degenerate formed space of rank r, ω' is alternating, and $\mathcal{B}' := (e'_r, \dots, e'_1, f'_1, \dots, f'_r)$ is an adapted basis, then there exists a unique isomorphism $\varphi : (V, \omega) \to (V', \omega')$ of formed spaces that maps $\mathcal{B}$ to $\mathcal{B}'$ (respecting the order). In particular, there is precisely one isomorphism class of non-degenerate formed spaces over $\Bbbk$ of rank r and with ω alternating. We denote by $(\Bbbk^{2r}, \omega_{\mathrm{I},0})$ the representative for which the standard basis is adapted, i.e. such that

$$\omega_{\mathrm{I},0}\big((\alpha_r, \dots, \alpha_1, \beta_1, \dots, \beta_r)^\top, (\alpha'_r, \dots, \alpha'_1, \beta'_1, \dots, \beta'_r)^\top\big) = \sum_{i=1}^{r} (\alpha_i \beta'_i - \beta_i \alpha'_i),$$

and call it the *standard formed space* of type I and rank r, or simply the *standard symplectic space* of rank r. The corresponding automorphism group

$$\mathrm{Aut}(\Bbbk^{2r}, \omega_{\mathrm{I},0}) = \{A \in \mathrm{GL}_{2r}(\Bbbk) \mid A^\top J_r A = J_r\},$$

is known as a *symplectic group over $\Bbbk$*, and denoted $\mathrm{Sp}_{2r}(\Bbbk)$.

Type II: ω is symmetric and $\Bbbk = \mathbb{C}$. We claim that $d \leq 1$, and hence, $n \in \{2r, 2r+1\}$. Indeed, assuming that $d > 2$, one can produce by Gram–Schmidt two (linearly independent) vectors $h_1, h_2 \in W$ with $\omega(h_i, h_j) = \delta_{ij}$. But then the non-zero vector $h_1 + ih_2 \in W$ would be isotropic, a contradiction. Note that if $d = 1$, there exists a non-zero vector $h \in W$ with $\omega(h, h) = 1$. In either case: in the ordered basis $\mathcal{B}_0 := (e_r, \dots, e_1, f_1, \dots, f_r)$ resp. $\mathcal{B}_1 := (e_r, \dots, e_1, h, f_1, \dots, f_r)$, the form ω is represented by the matrix Q_{2r} resp. Q_{2r+1}. Depending on the value of d (that is, on the parity of n), $\mathcal{B}_d$ is a basis adapted to ω.

We observe that there are two isomorphism classes of non-degenerate formed spaces over $\mathbb{C}$ with rank r and symmetric ω. We write $(\mathbb{C}^{d+2r}, \omega_{\mathrm{II},d})$ for the standard formed space of type II, rank r and parameter $d \in \{0, 1\}$. In either case, the form $\omega_{\mathrm{II},d}$ is given by

$$d = 0: \quad \omega_{\mathrm{II},0}\big((\alpha_r, \dots, \alpha_1, \beta_1, \dots, \beta_r)^\top, (\alpha'_r, \dots, \alpha'_1, \beta'_1, \dots, \beta'_r)^\top\big) = \sum_{i=1}^{r} (\alpha_i \beta'_i + \beta_i \alpha'_i),$$

$$d = 1: \quad \omega_{\mathrm{II},1}\big((\alpha_r, \dots, \alpha_1, \gamma, \beta_1, \dots, \beta_r)^\top, (\alpha'_r, \dots, \alpha'_1, \gamma', \beta'_1, \dots, \beta'_r)^\top\big) = \sum_{i=1}^{r} (\alpha_i \beta'_i + \beta_i \alpha'_i) + \gamma\gamma'.$$

The corresponding automorphism group

$$\mathrm{Aut}(\Bbbk^{d+2r}, \omega_{\mathrm{II},d}) = \{A \in \mathrm{GL}_{d+2r}(\Bbbk) \mid A^\top Q_{2r+d} A = Q_{2r+d}\}$$

is known as a *complex orthogonal group* and denoted $\mathrm{O}_{d+2r}(\mathbb{C})$.

Type III: ω is symmetric and $\Bbbk = \mathbb{R}$, or ω is Hermitian. In both cases, $\omega(v, v) \in \mathbb{R}$ for every $v \in V$. By the intermediate value theorem, the lack of isotropic vectors in W implies that the restriction $\omega|_{W\times W}$ is either positive- or negative-definite. Gram–Schmidt produces a basis $h_1, \dots, h_d$ of W such that $\omega(h_i, h_j) = \pm\delta_{ij}$, where the sign is always the same and equals the one of $\omega|_{W\times W}$. In sum, in the ordered basis $\mathcal{B}_d = (e_r, \dots, e_1, h_1, \dots, h_d, f_1, \dots, f_r)$, the form ω is represented by the matrix

$$K_{r,\pm d} := \begin{pmatrix} & & Q_r \\ & \pm I_d & \\ Q_r & & \end{pmatrix}.$$

We say that $\mathcal{B}_d$ is an adapted basis to ω.

For every $d \in \mathbb{N}$, there are at most two isomorphisms classes of non-degenerate formed spaces of type III with rank r, one for each sign of the middle block if $d > 0$, and just one if $d = 0$. We write $(\Bbbk^{d+2r}, \omega_{\mathrm{III},\varepsilon d})$ for the standard formed space of type III, rank r, parameter $d \in \mathbb{N}$ and sign $\varepsilon \in \{\pm 1\}$. The form is given by

$$\omega_{\mathrm{III},\varepsilon d}\big((\alpha_r, \dots, \alpha_1, \gamma_1, \dots, \gamma_d, \beta_1, \dots, \beta_r)^\top, (\alpha'_r, \dots, \alpha'_1, \gamma'_1, \dots, \gamma'_d, \beta'_1, \dots, \beta'_r)^\top\big) = \\ = \sum_{i=1}^{r} (\sigma(\alpha_i)\beta'_i + \sigma(\beta_i)\alpha'_i) + \varepsilon \sum_{j=1}^{d} \sigma(\gamma_j)\gamma'_j.$$

Extending componentwise the action of the automorphism $\sigma \in \mathrm{Aut}(\Bbbk)$ to matrices, the automorphism group is

$$\mathrm{Aut}(\Bbbk^{d+2r}, \omega_{\mathrm{III},\varepsilon d}) = \{A \in \mathrm{GL}_{d+2r}(\Bbbk) \mid \sigma(A)^\top K_{r,\varepsilon d} A = K_{\varepsilon d}\}$$

and will be denoted

$$\begin{array}{lllll} \mathrm{O}(r, d+r) & \text{for } \varepsilon = -1 & \text{or} & \mathrm{O}(d+r, r) & \text{for } \varepsilon = 1 \quad \text{if } \Bbbk = \mathbb{R}; \text{ or} \\ \mathrm{U}(r, d+r) & \text{for } \varepsilon = -1 & \text{or} & \mathrm{U}(d+r, r) & \text{for } \varepsilon = 1 \quad \text{if } \Bbbk = \mathbb{C}. \end{array}$$

The groups $\mathrm{O}(p, q)$ are known as *real orthogonal groups*, and $\mathrm{U}(p, q)$ as *unitary groups*.

Remark 2.6. The notations used for the automorphism groups of the standard formed spaces are the ones employed customarily for the families of classical groups.

2.1.3 Inclusions of standard formed spaces and of their automorphism groups. Let $T \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$, $d \in \mathbb{N}$, and $\varepsilon \in \{\pm 1\}$, and consider the standard formed space $(\Bbbk^{d+2r}, \omega_{\mathrm{T},\varepsilon d})$. Note that d or ε may be completely determined by the value of T. For an $r \geq 0$, the map between standard formed spaces

$$\begin{array}{rcl} \left(\Bbbk^{d+2r}, \omega_{T,\varepsilon d}\right) & \longrightarrow & \left(\Bbbk^{d+2(r+1)}, \omega_{T,\varepsilon d}\right) \\ v & \longmapsto & (0, v^\top, 0)^\top \end{array}$$

is a linear isometry. Varying r, we get the ascending sequence

$$\left(\Bbbk^{d}, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+2}, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+4}, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+6}, \omega_{T,\varepsilon d}\right) \hookrightarrow \dots \tag{2.1}$$

of formed spaces. Now, let $G_r := \operatorname{Aut}(\Bbbk^{d+2r}, \omega_{T,\varepsilon d})$ for every $r \in \mathbb{N}$. Then every $A \in G_r$ extends to an automorphism $\widetilde{A} \in G_{r+1}$ acting trivially on the subspace $\operatorname{span}\{e_{r+1}, f_{r+1}\}$ of $\Bbbk^{d+2(r+1)}$, where we use the notation established in the previous subsection for adapted bases. This defines embeddings

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots, \tag{2.2}$$

which on the level of matrices (with respect to the standard bases) are given by

$$A \mapsto \widetilde{A} = \begin{pmatrix} 1 & & \\ & A & \\ & & 1 \end{pmatrix}.$$

In particular, we have made sense of the following increasing sequences of automorphism groups of standard formed spaces:

- Complex Lie groups:

$$1 < \mathrm{Sp}_2(\mathbb{C}) < \mathrm{Sp}_4(\mathbb{C}) < \mathrm{Sp}_6(\mathbb{C}) < \cdots \tag{2.3}$$
$$\{\pm 1\} < \mathrm{O}_3(\mathbb{C}) < \mathrm{O}_5(\mathbb{C}) < \mathrm{O}_7(\mathbb{C}) < \cdots \tag{2.4}$$
$$1 < \mathrm{O}_2(\mathbb{C}) < \mathrm{O}_4(\mathbb{C}) < \mathrm{O}_6(\mathbb{C}) < \cdots \tag{2.5}$$

- Non-complex Lie groups:

$$1 < \mathrm{Sp}_2(\mathbb{R}) < \mathrm{Sp}_4(\mathbb{R}) < \mathrm{Sp}_6(\mathbb{R}) < \cdots \tag{2.6}$$
$$\mathrm{O}(d) < \mathrm{O}(d+1,1) < \mathrm{O}(d+2,2) < \mathrm{O}(d+3,3) < \cdots \tag{2.7}$$
$$\mathrm{O}(0,d) < \mathrm{O}(1,d+1) < \mathrm{O}(2,d+2) < \mathrm{O}(3,d+3) < \cdots$$
$$\mathrm{U}(d) < \mathrm{U}(d+1,1) < \mathrm{U}(d+2,2) < \mathrm{U}(d+3,3) < \cdots \tag{2.8}$$
$$\mathrm{U}(0,d) < \mathrm{U}(1,d+1) < \mathrm{U}(2,d+2) < \mathrm{U}(3,d+3) < \cdots$$

Here, we write $\mathrm{O}(d)$ in (2.7) (resp. $\mathrm{U}(d)$ in (2.8)) instead of $\mathrm{O}(d,0)$ (resp. $\mathrm{U}(d,0)$.) Because of symmetry, from now on we will not consider the two unnumbered sequences, which correspond to type III and $\varepsilon = -1$. Their treatment is completely analogous to the sequences (2.7) and (2.8), for which $\varepsilon = 1$.

2.1.4 A linear-algebraic lemma. The following lemma will be useful in Section 7.5. We denote by $\mathbb{P}(V) := \Bbbk^\times \backslash (V \smallsetminus \{0\})$ the projectivization of V. If $p, q \in \mathbb{P}(V)$ are two projective points, we say that $\omega(p,q) = 0$ (resp. $\omega(p,q) \neq 0$) whenever $\omega(v,w)$ equals zero (resp. does not equal zero) for two non-zero vectors $v \in p$ and $w \in q$. Note that this property is independent of the choices of v and w within p and q, respectively.

Lemma 2.7. *Let (V, ω) be a non-degenerate formed space, and let $p_0, \ldots, p_k, q_0, \ldots, q_k \in \mathbb{P}(V)$ be such that*

$$\omega(p_i, q_i) \neq 0 \textit{ for all } i \in [k], \quad \omega(p_i, p_j) = 0 \textit{ for all } i, j \in [k], \quad \text{and}$$
$$\omega(p_i, q_j) = 0 \textit{ for all } i, j \in [k], i \neq j.$$

Then $U := \operatorname{span}(\bigcup_i (p_i \cup q_i))$ *is a non-degenerate subspace of* V *of dimension* $2(k+1)$*. In particular,* $\dim V \geq 2(k+1)$.

Proof. We prove the lemma by induction on k, where the base case $k = 0$ is immediate. We take the statement of the lemma for $k-1$ as our induction hypothesis, and prove it for k. Thus, if $p_0, \dots, p_k, q_0, \dots, q_k$ are as in the assumptions, the induction hypothesis says that the subspace $W_0 := \operatorname{span}(\bigcup_{i=0}^{k-1} (p_i \cup q_i))$ is non-degenerate and of dimension $2k$. In particular, V splits as the direct sum $V = W_0 \oplus W_0^{\omega}$.

By assumption, $p_k \subset W_0^{\omega}$. Now let v_k and w_k be non-zero vectors in p_k and q_k, respectively, and let $w_{k,0} \in W_0$ and $w_{k,1} \in W_0^{\omega}$ be vectors such that $w_k = w_{k,0} + w_{k,1}$. Then

$$0 \neq \omega(v_k, w_k) = \omega(v_k, w_{k,0}) + \omega(v_k, w_{k,1}) = \omega(v_k, w_{k,1}).$$

Hence, $\operatorname{span}\{v_k, w_{k,1}\} \subset W_0^{\omega}$ is non-degenerate and of dimension two, and in consequence, $W = \operatorname{span}(W_0 \cup \{v_k, w_{k,1}\})$ is, too, non-degenerate and of dimension $2(k+1)$. □

2.2 Grassmannians

2.2.1 Chabauty convergence on Grassmannians. If X is a lcsc space, the set $\mathcal{C}(X)$ of closed subsets admits a compact metrizable topology, called the *Chabauty topology*; see §E.1 in [2]. The convergence of sequences in the Chabauty topology is characterized as follows: Given a sequence $(W_m)_m \subset \mathcal{C}(X)$, we have that $W_m \to W \in \mathcal{C}(X)$ as $m \to \infty$ if and only if the following two conditions hold:

(C1) If $(w_{m_l})_k$ is a sequence in X such that $w_{m_l} \in W_{m_l}$ for every l, and such that $w_{m_l} \to w \in X$ as $l \to \infty$, then $w \in W$.

(C2) If $w \in W$, then there exists a sequence $(w_m)_m \subset X$ with $w_m \in W_m$, and such that $w_m \to w$.

It is immediate from the characterization of convergence above that the set $\operatorname{Gr}(V)$ of all linear subspaces of V is a closed subset of $\mathcal{C}(V)$, on which the group $\operatorname{GL}(V)$ acts continuously.

Definition 2.8. The space $\operatorname{Gr}(V)$ equipped with the partial order given by inclusion is called the *projective geometry* of V.

Given $k \in [n]$, let us denote by $\operatorname{Gr}_k(V)$ the k-Grassmannian of V, i.e. the subset of $\operatorname{Gr}(V)$ of all k-dimensional linear subspaces, equipped with the restriction of the Chabauty topology. We observe, firstly, that $\operatorname{GL}(V)$ acts transitively on $\operatorname{Gr}_k(V)$ and, secondly, that $\operatorname{Gr}_k(V)$ is closed in $\operatorname{Gr}(V)$, hence compact. Moreover, since

$$\operatorname{Gr}(V) = \bigsqcup_{k=0}^{n} \operatorname{Gr}_k(V) \tag{2.9}$$

is a finite disjoint union, we deduce that all of the Grassmannians $\mathrm{Gr}_k(V)$ are clopen; in fact, they are precisely the connected components of $\mathrm{Gr}(V)$, as implied by the following lemma.

Lemma 2.9. *The Chabauty topology on* $\mathrm{Gr}_k(V)$ *coincides with the quotient topology with respect to the* $\mathrm{GL}(V)$*-action.*

Proof. This is an immediate consequence of the open mapping theorem for homogeneous spaces, which we state below. □

Theorem 2.10. *Let G be a Polish group, X a Polish space, and $x \in X$. If G acts continuously and transitively on X, then the orbit map $G \to X$, $g \mapsto gx$ is open. In particular, if $H < G$ denotes the stabilizer of x, then the continuous bijection $G/H \to X$ is a homeomorphism.*

About the proof. It is a particular case of Theorem 2.1 in [29]. □

The next one is a straightforward corollary of Theorem 2.10.

Corollary 2.11. *Let G be a Polish group, and X and Y be Polish spaces on which G acts continuously and transitively. Then any G-equivariant map $X \to Y$ is surjective, continuous, and open. In particular, it is a quotient map.* □

The following proposition constitutes the main advantage of equipping Grassmannians with the Chabauty topology.

Proposition 2.12. *For a fixed $k \in [n]$, let $(W_m)_m$ be a sequence in $\mathrm{Gr}_k(V)$. Then $W_m \to W \in \mathrm{Gr}_k(V)$ as $m \to \infty$ if and only if the property (C1) above holds.*

Proof. We only need to show the sufficiency of (C1); thus, assume that $(W_m)_m$ and W satisfy that property. Let $(W_{m_l})_l$ be a convergent subsequence of $(W_m)_m$ with limit W'. If $w' \in W'$, then by (C2) we find $w_{m_l} \in W_{m_l}$ with $w_{m_l} \to w'$. But then we have $w' \in W$ by our assumption of the property (C1). Since w' was chosen arbitrarily, we have shown that $W' \subset W$. However, given that W and W' are both of dimension k, we deduce that $W = W'$. The compactness of $\mathrm{Gr}_k(V)$ and the arbitrariness of (W_{m_k}) imply that $W_m \to W$. □

2.2.2 Three measurability lemmas.

We prove now three measurability statements that will be used in Chapter 7. Let us identify now the projectivization $\mathbb{P}(V)$ with the 1-Grassmannian $\mathrm{Gr}_1(V)$. For any $k \in \mathbb{N}$ and any tuple $(p_0, \ldots, p_k) \in \mathbb{P}(V)^{k+1}$, we set

$$\mathrm{span}(p_0, \ldots, p_k) := \mathrm{span}(p_0 \cup \cdots \cup p_k). \tag{2.10}$$

Regarded as a function of $(k+1)$-tuples, $\mathrm{span}(\cdot)$ is obviously $\mathfrak{S}_{k+1}$-invariant. We will say that $(p_0, \ldots, p_k)$ is linearly independent if $\dim \mathrm{span}(p_0, \ldots, p_k) = k + 1$.

Lemma 2.13. *Let $k \in [n-1]$. Then the map* $\operatorname{span} : \mathbb{P}(V)^{k+1} \to \operatorname{Gr}(V)$ *is piecewise continuous with respect to the Borel partition of* $\mathbb{P}(V)^{k+1}$ *given by the subsets*

$$S_{k,l} := \{(p_0, \dots, p_k) \in \mathbb{P}(V)^{k+1} \mid \dim \operatorname{span}(p_0, \dots, p_k) = l + 1\}, \quad l \in [k].$$

In particular, it is Borel.

Proof. Assume first that $l = k$. Note that the diagonal action of $\operatorname{GL}(V)$ on $\mathbb{P}(V)^{k+1}$ restricts to one on $S_{k,k}$. The latter is continuous and transitive, and $\operatorname{span} : S_{k,k} \to \operatorname{Gr}_{k+1}(V)$ is $\operatorname{GL}(V)$-equivariant. Thus, by Corollary 2.11, this map is continuous.

Now, let $l \in [k]$ be arbitrary, and assume that $(p_0^n, \dots, p_k^n) \to (p_0, \dots, p_k)$ in $S_{k,l}$ as $n \to \infty$. There exist indices $0 \le i_0 < \dots < i_l \le k$ such that $(p_{i_0}, \dots, p_{i_l})$ is linearly independent. Since linear independence is an open condition in $\mathbb{P}(V)^{k+1}$, we deduce that $(p_{i_0}^n, \dots, p_{i_l}^n)$ are linearly independent for all sufficiently large n, hence $(p_{i_0}^n, \dots, p_{i_l}^n) \to (p_{i_0}, \dots, p_{i_l})$ in $S_{l,l}$. From the previous case we thus deduce that

$$\operatorname{span}(p_0^n, \dots, p_k^n) = \operatorname{span}(p_{i_0}^n, \dots, p_{i_l}^n) \to \operatorname{span}(p_{i_0}, \dots, p_{i_l}) = \operatorname{span}(p_0, \dots, p_k),$$

as $n \to \infty$, which establishes the desired continuity. □

Lemma 2.14. *For every subspace $W < V$, the map $\mathbb{P}(V) \to \operatorname{Gr}(V)$, $p \mapsto W + p$ is Borel.*

Proof. Fix a subspace W and lines $p_0, \dots, p_k \in \mathbb{P}(V)$ such that $W = \operatorname{span}(p_0, \dots, p_k)$. Then the map in the statement is the composition

$$\begin{array}{ccccc} \mathbb{P}(V) & \longrightarrow & \mathbb{P}(V)^{k+2} & \xrightarrow{\operatorname{span}} & \operatorname{Gr}(V) \\ p & \longmapsto & (p_0, \dots, p_k, p) & \longmapsto & \operatorname{span}(p_0, \dots, p_k, p) = W + p, \end{array}$$

which is Borel by Lemma 2.13. □

Let us consider now the ω-perpendicular complement as a function

$$(-)^{\omega} : \operatorname{Gr}(V) \to \operatorname{Gr}(V), \quad W \mapsto W^{\omega}. \tag{2.11}$$

We call the map (2.11) the *polarity* associated to ω. It is a non-trivial involution, and it restricts to maps $\operatorname{Gr}_k(V) \to \operatorname{Gr}_{n-k}(V)$ of every Grassmannian. Furthermore:

Lemma 2.15. *The polarity* $(-)^{\omega} : \operatorname{Gr}(V) \to \operatorname{Gr}(V)$ *is continuous.*

Proof. It suffices to show that the restriction $(-)^{\omega} : \operatorname{Gr}_k(V) \to \operatorname{Gr}_{n-k}(V)$ is continuous for every $k \in [n]$. Let (W_m) be a sequence in $\operatorname{Gr}_k(V)$ converging to $W \in \operatorname{Gr}_k(V)$. We use Proposition 2.12 to show that $W_m^{\omega} \to W^{\omega}$ as $m \to \infty$. Let $x_{m_j} \in W_{m_j}^{\omega}$ and assume that $x_{m_j} \to x \in V$ as $j \to \infty$, and let $w \in W$. By the condition (C2) of Chabauty convergence, our assumption $W_m \to W$ implies that there exists a sequence $w_{m_j} \in W_{m_j}$ such that $w_{m_j} \to w$. But then

$$\omega(x, w) = \lim_{j \to \infty} \omega\left(x_{m_j}, w_{m_j}\right) = 0.$$

Since the choice of $w \in W$ was arbitrary, it follows that $x \in W^{\omega}$, which finishes the proof. □

2.2.3 Isotropic Grassmannians. Let (V, ω) be a non-degenerate formed space of rank r. For every $k \in [r-1]$, we define a subset $\mathcal{G}_k(V, \omega)$ of the Grassmannian $\mathrm{Gr}_{k+1}(V)$ by

$$\mathcal{G}_k(V, \omega) := \{W \in \mathrm{Gr}_{k+1}(V) \mid W \text{ is totally isotropic}\}. \tag{2.12}$$

We refer to $\mathcal{G}_k(V, \omega)$ as the *isotropic Grassmannian* of *type* k. We let also $\mathcal{G}(V, \omega) := \bigsqcup \mathcal{G}_k(V, \omega)$ and equip it with the subspace topology from the inclusion $\mathcal{G}(V, \omega) \subset \mathrm{Gr}(V)$. The space $\mathcal{G}(V, \omega)$ is known as the *polar geometry* of (V, ω). In the language of incidence geometry, the elements of $\mathcal{G}_0(V, \omega)$, $\mathcal{G}_1(V, \omega)$, etc. are the points, lines, etc. of the polar geometry of (V, ω), hence the shift in enumeration in (2.12). As a consequence of Corollary 2.4, the group $G = \mathrm{Aut}(V, \omega)$ acts transitively on $\mathcal{G}_k(V, \omega)$ for all $k \in [r-1]$.

Lemma 2.16. *(i) The polar geometry $\mathcal{G}(V, \omega)$ and all $\mathcal{G}_k(V, \omega)$ are compact.*

(ii) The isotropic Grassmannians $\mathcal{G}_k(V, \omega)$ are the connected components of $\mathcal{G}(V, \omega)$.

(iii) On each $\mathcal{G}_k(V, \omega)$, the subspace topology coincides with the one as homogeneous G-space.

Proof. Item (iii) follows again from Theorem 2.10, since the action of $G < \mathrm{GL}(V)$ is continuous and transitive. This, in turn, implies that the $\mathcal{G}_k(V, \omega)$ are connected, and being clopen, we deduce that they are the connected components of $\mathcal{G}(V, \omega)$. This completes the proof of (ii). For (i), it remains to show only that $\mathcal{G}(V, \omega)$ is closed in $\mathrm{Gr}(V)$. However, by Lemma 2.15, the polarity is continuous, and it follows that the condition $W \subset W^{\omega}$ is a closed condition. □

Chapter 3

Spectral Sequences

In this chapter, we collect all the concepts on spectral sequences that are relevant to this thesis, restricting ourselves to the category **Vect** of vector spaces. The main goal is to state the theorem that guarantees the existence of the two spectral sequences associated to a double complex, that is, Theorem 3.6 below. They constitute the base for the main theorems in Chapters 6 and 8. During this section, let $\Bbbk$ be a field. When we say "vector space", we always mean "$\Bbbk$-vector space."

A *bigraded cochain complex* is a pair $(E^{\bullet,\bullet}, \mathrm{d})$, where $E^{\bullet,\bullet} = \bigoplus_{(p,q)\in\mathbb{Z}^2} E^{p,q}$ is a bigraded vector space, and $\mathrm{d} : E^{\bullet,\bullet} \to E^{\bullet,\bullet}$ is a *differential* (i.e. a linear map with $\mathrm{d} \circ \mathrm{d} = 0$) that for some fixed $(h, k) \in \mathbb{Z}^2$ restricts to linear maps

$$\mathrm{d}^{p,q} : E^{p,q} \longrightarrow E^{p+h,q+k}$$

for every $(p, q) \in \mathbb{Z}^2$. The pair $(h, k) \in \mathbb{Z}^2$ is known as the *bi-degree* of the differential. If it does not lead to confusion, we will omit mention of the differential d and denote the pair $(E^{\bullet,\bullet}, \mathrm{d})$ by $E^{\bullet,\bullet}$. The usual graphical representation of a bigraded chain complex $E^{\bullet,\bullet}$ is on the integer lattice of the plane, associating to the integer point $(p, q) \in \mathbb{Z}^2$ the vector space $E^{p,q}$; the differential $\mathrm{d}^{\bullet,\bullet}$ is a collection of parallel, congruent arrows connecting pairs of integer points. See Figure 3.1.

Definition 3.1. A *cohomological spectral sequence* (or simply *spectral sequence*) is a sequence $\{(E_s^{\bullet,\bullet}, d_s) \mid s \in \mathbb{N}, s \geq s_0\}$ of bigraded cochain complexes (with $s_0 \in \mathbb{N}$ fixed) such that for every s:

(i) the complex $E_s^{\bullet,\bullet}$ is of bi-degree $(s, 1 - s)$, and

(ii) the space $E_{s+1}^{p,q}$ is isomorphic to the (p, q)-*cohomology group*

$$\mathrm{H}^{p,q}(E_s^{\bullet,\bullet}) := \ker \mathrm{d}^{p,q} \,/\, \operatorname{im} \mathrm{d}^{p-s,q-1+s}$$

for every $(p, q) \in \mathbb{Z}$.

It is customary to denote a spectral sequence simply by $E_\bullet^{\bullet,\bullet}$, omitting mention of differentials. Each complex $E_s^{\bullet,\bullet}$ is known as the s-th *page* of $E_\bullet^{\bullet,\bullet}$.

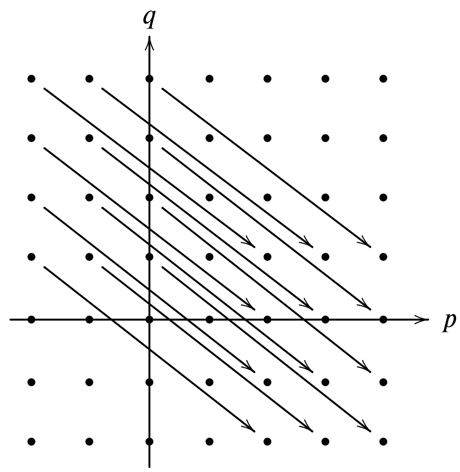


Figure 3.1: A bigraded cochain complex of bi-degree $(4, -3)$

We must think of a spectral sequence $E_\bullet^{\bullet,\bullet}$ as an iterative process, where the bigraded vector space underlying the $(s+1)$-th page $E_{s+1}^{\bullet,\bullet}$ is completely determined by the s-th page $E_s^{\bullet,\bullet}$ and its differentials d_s. After infinitely many iterations, we would like to speak about a "limit page" $E_\infty^{\bullet,\bullet}$. For a *first-quadrant* spectral sequence $E_\bullet^{\bullet,\bullet}$, that is, one such that $E_s^{p,q} = 0$ for all $s \geq s_0$ and whenever $p < 0$ or $q < 0$, it is very easy to make sense of this. Fix $(p,q) \in \mathbb{N}^2$ and set $s := \max\{p, q+1\} + 1$. Then there exists an infinite chain of isomorphisms

$$E_s^{p,q} \cong E_{s+1}^{p,q} \cong E_{s+2}^{p,q} \cong E_{s+3}^{p,q} \cong \cdots. \tag{3.1}$$

Indeed: If $s' > \max\{p, q+1\}$, then $E_{s'}^{p+s',q+1-s'} = 0$ and $E_{s'}^{p-s',q-1+s'} = 0$. In consequence, $\ker \mathrm{d}_{s'} = E_{s'}^{p,q}$ and $\operatorname{im} \mathrm{d}_{s'} = 0$, proving that $E_{s'+1}^{p,q} \cong E_{s'}^{p,q}$. Inductively, we obtain (3.1). Note also that (3.1) holds trivially for $p < 0$ or $q < 0$.

Definition 3.2. For a first-quadrant spectral sequence as above, and every $(p,q) \in \mathbb{Z}$, we set $E_\infty^{p,q} := E_s^{p,q}$ to be the stable term of (3.1), and define the *limit page* of $E_\bullet^{\bullet,\bullet}$ or E_∞*-page* $E_\infty := \bigoplus_{(p,q)\in\mathbb{Z}^2} E_\infty^{p,q}$. We also set $E_\infty^n := \bigoplus_{p+q=n} E_\infty^{p,q}$ for each $n \in \mathbb{Z}$.
The notion of limit page of a spectral sequence can be defined without the first-quadrant assumption. We refer the reader, for instance, to [52] or to [7].

Now, let $H^\bullet$ be a graded vector space. A *filtration* F of $H^\bullet = \bigoplus_{n\in\mathbb{Z}} H^n$ is a sequence of subspaces of $H^\bullet$ arranged in a descending chain

$$H^\bullet = F^0 H^\bullet \supset F^1 H^\bullet \supset F^2 H^\bullet \supset \cdots.$$

We will let the index p of the filtration take negative values by setting $F^p H^\bullet = H^\bullet$ for $p < 0$. Fixing a degree n and letting $F^p H^n := F^p H^\bullet \cap H^n$, we also obtain descending chains

$$H^n = F^0 H^n \supset F^1 H^n \supset F^2 H^n \supset \cdots. \tag{3.2}$$

We say that the filtration F is *bounded* if for every degree n, there exists $t(n) \in \mathbb{N}$ such that

$F^p H^n = 0$ for every $p \geq t(n)$. To any filtration F on $H^\bullet$, we can associate the bigraded vector space $E^{\bullet,\bullet}(H^\bullet, F)$ defined as

$$E^{p,q}(H^\bullet, F) := F^p H^{p+q} / F^{p+1} H^{p+q} \text{ for any } p, q \in \mathbb{Z}.$$

Note that if the filtration F on $H^\bullet$ is bounded, then the graded vector space $H^\bullet$ can be recovered up to isomorphism from $E^{\bullet,\bullet}(H^\bullet, F)$, since for each $n \in \mathbb{Z}$,

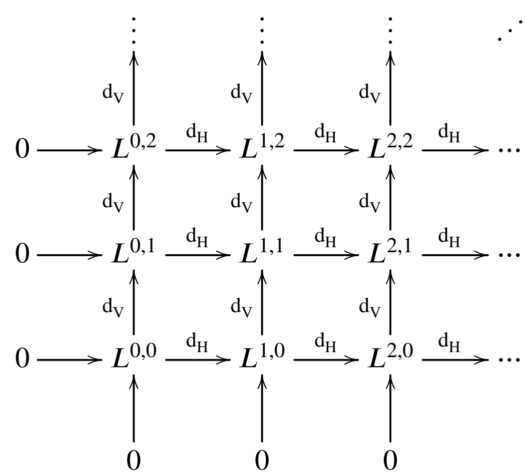
$$H^n \cong \bigoplus_{p+q=n} E^{p,q}(H^\bullet, F). \tag{3.3}$$

Definition 3.3. A spectral sequence $E_\bullet^{\bullet,\bullet}$ is said to *converge* to a graded vector space $H^\bullet$ if there exists a filtration F on $H^\bullet$ such that $E_\infty^{p,q} \cong E_0^{p,q}(H^\bullet, F)$ for every p, q. By (3.3), the convergence of $E_\bullet^{\bullet,\bullet}$ to $H^\bullet$ implies that $E_\infty^n \cong H^n$ for every n.

The spectral sequences that will appear below have their origin in a double complex of vector spaces.

Definition 3.4. A *double complex* is a triple $(L^{\bullet,\bullet}, \mathrm{d_H}, \mathrm{d_V})$ such that $L^{\bullet,\bullet} = \bigoplus_{(p,q)\in\mathbb{Z}} L^{p,q}$ is a bigraded vector space, and $\mathrm{d_H}, \mathrm{d_V} : L^{\bullet,\bullet} \to L^{\bullet,\bullet}$ are differentials of bi-degrees $(1, 0)$ and $(0, 1)$, respectively, that satisfy the identity

$$\mathrm{d_H} \circ \mathrm{d_V} + \mathrm{d_V} \circ \mathrm{d_H} = 0. \tag{3.4}$$



Now, for any $n \geq 0$, the *total complex* $T^\bullet := T(L^{\bullet,\bullet})^\bullet$ of $L^{\bullet,\bullet}$ is defined as the cochain complex

$$T^n := \bigoplus_{p+q=n} L^{p,q}$$

with differential $D := \mathrm{d_H} + \mathrm{d_V}$. The condition (3.4) guarantees that as a matter of fact $D \circ D = 0$. Furthermore, the graded vector space $T^\bullet$ is naturally endowed with two filtrations,

$$F_{\mathrm{I}}^k T^n = \bigoplus_{p\geq k} L^{p,n-p} \quad \text{and} \quad F_{\mathrm{II}}^k T^n = \bigoplus_{q\geq k} L^{n-q,q},$$

called respectively the *vertical* and *horizontal* filtration. They are compatible with the differential D, in the sense that $D(F_*^k T^\bullet) \subset F_*^k T^{\bullet+1}$ for every $k \in \mathbb{Z}$ and $* \in \{\mathrm{I}, \mathrm{II}\}$, and thus, we say that they are filtrations of the cochain complex $(T^\bullet, D)$. Filtrations of cochain complexes give rise to spectral sequences:

Theorem 3.5 (see [52, Theorem 2.6])**.** *Every filtration F of a cochain complex $(C^\bullet, \mathrm{d})$ determines a spectral sequence $E_\bullet^{\bullet,\bullet}$ with first page*

$$E_1^{p,q} \cong \mathrm{H}^{p+q}(F^p C / F^{p+1} C).$$

Furthermore, if the filtration is bounded, then the spectral sequence $E_\bullet^{\bullet,\bullet}$ converges to the cohomology $\mathrm{H}^\bullet(C^\bullet)$.

The following theorem will be the starting point of our proofs in Part II.

Theorem 3.6 (see [52, Theorem 2.15])**.** *Given a first-quadrant double complex $\left(L^{\bullet,\bullet}, \mathrm{d}_{\mathrm{H}}, \mathrm{d}_{\mathrm{V}}\right)$, there exist two first-quadrant spectral sequences, ${}^{\mathrm{I}}E_\bullet^{\bullet,\bullet}$ and ${}^{\mathrm{II}}E_\bullet^{\bullet,\bullet}$, with first pages and first-page differentials*

$$\begin{aligned} {}^{\mathrm{I}}E_1^{p,q} &= \mathrm{H}^{p,q}\left(L^{\bullet,\bullet}, \mathrm{d}_{\mathrm{V}}\right), & {}^{\mathrm{I}}\mathrm{d}_1^{p,q} &:= \left(\mathrm{d}_{\mathrm{H}}^{p,q}\right)_* : {}^{\mathrm{I}}E_1^{p,q} \to {}^{\mathrm{I}}E_1^{p+1,q}, \\ {}^{\mathrm{II}}E_1^{p,q} &= \mathrm{H}^{q,p}\left(L^{\bullet,\bullet}, \mathrm{d}_{\mathrm{H}}\right), & {}^{\mathrm{II}}\mathrm{d}_1^{p,q} &:= \left(\mathrm{d}_{\mathrm{V}}^{p,q}\right)_* : {}^{\mathrm{II}}E_1^{p,q} \to {}^{\mathrm{II}}E_1^{p+1,q}, \end{aligned}$$

for $p, q \geq 0$, both converging to the total complex $T^\bullet$. In particular, one has the (non-canonical) isomorphism ${}^{\mathrm{I}}E_\infty^n \cong {}^{\mathrm{II}}E_\infty^n$ for every $n \in \mathbb{N}$.

In virtue of Theorem 3.5, the filtration F_{I} can be easily seen to give rise to the spectral sequence ${}^{\mathrm{I}}E_\bullet^{\bullet,\bullet}$, which has a first page as above and which converges to the cohomology of the total complex $T^\bullet$. The fact that the first-page differential is given above is less trivial and can be derived by means of the equivalent approach to the spectral sequence associated to a filtration via exact couples; for a proof, see [52]. Transposing the complex $L^{\bullet,\bullet}$, the filtration F_{II} produces in the same way the spectral sequence ${}^{\mathrm{II}}E_\bullet^{\bullet,\bullet}$.

Chapter 4

Parameter-Dependent Integrals

This chapter is a technical prerrequisite in which we prove several measurability statements for a kind of parameter-dependent integrals that will appear in the construction of measures on Stiefel complexes and their contracting homotopies in Chapter 7.

Let X and Y be lcsc spaces, and if X is not compact, let $\iota : X \hookrightarrow \bar{X}$ be a Hausdorff second-countable compactification (e.g. the one-point compactification.) Let also

$$\nu : Y \to \mathrm{Prob}(X), \quad y \mapsto \nu_y$$

be a Borel map; we refer the reader to the section Notation and Conventions at the beginning of this thesis for the definition of the topology on $\mathrm{Prob}(X)$. We recall that $\mathrm{Prob}(X)$ is Hausdorff and second-countable. Let also $\mu_Y \in \mathrm{Prob}(Y)$. For any compactly supported continuous function $f : X \times Y \to \mathbb{R}$, we will consider the *parameter-dependent integral*

$$\widetilde{f} : Y \to \mathbb{R}, \quad \widetilde{f}(y) := \int_X f(x, y) \, \mathrm{d}\nu_y(x), \tag{4.1}$$

Clearly, $\widetilde{f}$ is bounded, with $|\widetilde{f}(y)| \leq \|f\|_\infty$ for every $y \in Y$. We prove:

Lemma 4.1. *For any $f \in C_c(X \times Y)$, the function $\widetilde{f} : Y \to \mathbb{R}$ is Borel.*

For its proof, we rely on the following version of Lusin's theorem.

Theorem 4.2 (see §2.7 in [48])**.** *Let (Z, μ) be a Hausdorff measure space, W be a second-countable space, and $f : Z \to W$ a map. Then the following are equivalent:*

(i) *f is Borel.*

(ii) *For every $\varepsilon > 0$ and every Borel subset $A \subset Z$ with $\mu(A) < \infty$, there exists a compact subset $C \subset A$ with $\mu(A \setminus C) < \varepsilon$ such that $f|_C$ is continuous.* □

Proof of Lemma 4.1. Let $\varepsilon > 0$ and $A \subset Y$ with $\mu_Y(A) < \infty$. By Lusin's theorem, there exists $C \subset Y$ with $\mu_Y(A \setminus C) < \varepsilon$ and such that $\nu|_C$ is continuous. Now, let $(y_n)_n \subset C$ be a sequence with $y_n \to y \in C$. By assumption, the functions $f_{y_n} \in C_c(X)$ defined by $f_{y_n}(x) := f(x, y_n)$ converge uniformly to $f_y \in C_c(X)$, $f_y(x) := f(x, y)$. Also, since $\nu_{y_n} \to \nu_y$ in the weak-$*$-topology, we have that $\nu_{y_n}(f_y) \to \nu_y(f_y)$ as $n \to \infty$ (this is the case also if X is not compact, as $C_c(X) \subset C(\bar{X})$.) Thus,

$$\begin{aligned}|\widetilde{f}(y_n) - \widetilde{f}(y)| &= |\nu_{y_n}(f_{y_n} - f_y) + (\nu_{y_n}(f_y) - \nu_y(f_y))| \\ &\leq \|\nu_{y_n}\| \cdot \|f_{y_n} - f_y\|_\infty + |\nu_{y_n}(f_y) - \nu_y(f_y)| \\ &\leq \|f_{y_n} - f_y\|_\infty + |\nu_{y_n}(f_y) - \nu_y(f_y)| \to 0\end{aligned}$$

as $n \to \infty$, and therefore $\widetilde{f}|_C$ is continuous. We conclude that $\widetilde{f}$ is Borel again by Lusin's theorem. □

The formula in (4.1) defines a linear map $\widetilde{\cdot} : C_c(X \times Y) \to \mathscr{L}^\infty(Y) \subset \mathscr{L}^1(Y, \mu_Y)$ by Lemma 4.1. Hence:

Lemma 4.3. *The assignment*

$$\mu_Y(\nu) : C_c(X \times Y) \to \mathbb{R}, \quad f \mapsto \mu_Y(\nu)(f) := \int_Y \left(\int_X f(x, y) \, \mathrm{d}\nu_y(x) \right) \mathrm{d}\mu_Y(y) = \int_Y \widetilde{f} \, \mathrm{d}\mu_Y,$$

defines a probability measure on $X \times Y$.

Proof. Let $f \in C_c(X \times Y)$. Since $\widetilde{f} \in \mathscr{L}^1(Y, \mu_Y)$, the right hand side is well defined. Because μ_Y is a probability measure, we have $|\mu_Y(\nu)(f)| = |\mu_Y(\widetilde{f})| \leq \|\widetilde{f}\|_\infty \leq \|f\|_\infty$. Since f was arbitrary, the linear functional $\mu_Y(\nu)$ is bounded. Its positivity follows from the positivity of the measures ν_y and of μ_Y. Moreover, we have

$$\mu_Y(\nu)(X \times Y) = \int_Y \nu_y(X) \, \mathrm{d}\mu_Y(y) = \int_Y \mathrm{d}\mu_Y(y) = 1,$$

which completes the proof. □

Composing the linear map $\widetilde{\cdot}$ with the projection $\mathscr{L}^1(Y, \mu_Y) \twoheadrightarrow L^1(Y, \mu_Y)$, we obtain an operator that we abusively denote by $\widetilde{\cdot} : C_c(X \times Y) \to L^1(Y, \mu_Y)$ again.

Lemma 4.4. *Equip $C_c(X \times Y)$ with the L^1-norm with respect to the measure $\mu_Y(\nu)$. Then the linear map $\widetilde{\cdot} : C_c(X \times Y) \to L^1(Y, \mu_Y)$ is bounded, and hence, there exists a unique bounded linear extension*

$$\widetilde{\cdot} : L^1(X \times Y, \mu_Y(\nu)) \to L^1(Y, \mu_Y). \tag{4.2}$$

Furthermore, the map in (4.2) restricts to a linear map

$$\widetilde{\cdot} : L^\infty(X \times Y, \mu_Y(\nu)) \to L^\infty(Y, \mu_Y) \tag{4.3}$$

that is bounded with respect to the essential supremum norms.

Proof. If $f \in C_c(X \times Y)$, then $\|\widetilde{f}\|_1 = \int_Y |\widetilde{f}| \, \mathrm{d}\mu_Y \leq \|f\|_1$. This establishes the boundedness of the linear map in (4.2). The density of $C_c(X \times Y)$ in $L^1(X \times Y, \mu_Y(\nu))$ yields the unique extension. Recall that for finite measures, the inclusion $L^\infty \subset L^1$ holds; this is the case for $\mu_Y(\nu)$ by Lemma 4.3. The statement after (4.1) implies that the image of the restriction of (4.2) to $L^\infty(X \times Y, \mu_Y(\nu))$ is contained in $L^\infty(Y, \mu_Y)$, as well as its boundedness. □

Remark 4.5. For $f \in L^1(X \times Y, \mu_Y(\nu)) \setminus C_c(X \times Y)$, we will also write $\int_X f(x,y)\, \mathrm{d}\nu_y(x) := \widetilde{f}(y)$, where the right-hand side is the extension in (4.2).

From now on, let $Z \subset X \times Y$ be the *support* of the measure $\mu_Y(\nu)$, i.e. the set of points $(x,y) \in X \times Y$ for which every open neighbourhood $U \ni (x,y)$ has positive measure $\mu_Y(\nu)(U)$. We will regard $\mu_y(\nu)$ as a probability measure on Z, and think of the linear maps in (4.2) and (4.3) as

$$\widetilde{\cdot} : L^1(Z, \mu_Y(\nu)) \to L^1(Y, \mu_Y) \quad \text{and} \quad \widetilde{\cdot} : L^\infty(Z, \mu_Y(\nu)) \to L^\infty(Y, \mu_Y). \tag{4.4}$$

As discussed in the paragraph before Remark 1.14, if $\mu_Z \in \mathrm{Prob}(Z)$ is equivalent to $\mu_Y(\nu)$, then $L^\infty(Z, \mu_Z) = L^\infty(Z, \mu_Y(\nu))$, and (4.3) makes sense as an operator

$$\widetilde{\cdot} : L^\infty(Z, \mu_Z) \to L^\infty(Y, \mu_Y).$$

Under presence of a group action, we will give sufficient conditions for the equivalence $\mu_Z \sim \mu_Y(\nu)$ to hold. This criterion will be useful to verify the well-definedness of certain homotopy operators in Chapter 7.

Thus, let G be a lcsc group with continuous actions by homeomorphism on Z and Y, and let μ_Z resp. μ_Y be G-quasi-invariant probability measures on Z resp. Y. In this way, in particular, (Z, μ_Z) and (Y, μ_Y) are regular G-spaces.

Definition 4.6. We say that the map $\nu : Y \to \mathrm{Prob}(X)$ is *G-quasi-equivariant* if $\nu_{gy} \sim g_* \nu_y$ for every $g \in G$ and $y \in Y$. We will write $r_\nu : G \times Y \times X \to \mathbb{R}_{>0}$ for the collection of Radon-Nikodym cocycles

$$r_\nu(g, y, x) := \frac{d(g^{-1}_* \nu_y)}{d\nu_{g^{-1}y}}(x)$$

for every $g \in G$, $y \in Y$ and ν_{gy}-almost every $x \in X$. This convention is similar to the one adopted in (1.7).

Lemma 4.7. *If G acts transitively on Z, and ν is G-quasi-equivariant, then $\mu_Z \sim \mu_Y(\nu)$.*

Proof. It suffices to show that $\mu_Y(\nu) \in \mathrm{Prob}(Z)$ is G-quasi-invariant. Indeed, since Z is a homogeneous G-space, there exists a unique G-invariant measure class on Z, and therefore any two G-quasi-invariant measures on Z are equivalent. In order to establish the G-quasi-invariance of $\mu_Y(\nu) \in \mathrm{Prob}(Z)$, let $A \subset Z$ be a Borel subset and let $g \in G$. Then we have

$$\begin{aligned} g_*\big(\mu_Y(\nu)\big)(A) &= \int_Y \int_X \mathbb{1}_{g^{-1}A}(x,y)\, \mathrm{d}\nu_y(x)\, \mathrm{d}\mu_Y(y) = \int_Y \int_X \mathbb{1}_A(gx, gy)\, \mathrm{d}\nu_y(x)\, \mathrm{d}\mu_Y(y) \\ &= \int_Y \int_X \mathbb{1}_A(x, gy) \cdot r_\nu(g^{-1}, y, x)\, \mathrm{d}\nu_{gy}(x)\, \mathrm{d}\mu_Y(y) \end{aligned}$$

$$= \int_Y \int_X \mathbb{1}_A(x, y) \cdot r_\nu(g^{-1}, g^{-1}y, x) \cdot r_{\mu_Y}(g^{-1}, y) \,\mathrm{d}\nu_y(x) \,\mathrm{d}\mu_Y(y).$$

Thus $g_*\big(\mu_Y(\nu)\big)(A) = 0$ if and only if the integrand in the last term of the last series of equalities is zero for $\mu_Y(\nu)$-almost every $(x, y) \in Z$. Since the two last factors of the integrand are positive, this is equivalent to

$$\int_Y \int_X \mathbb{1}_A(x, y) \,\mathrm{d}\nu_y(x) \,\mathrm{d}\mu_Y(y) = \mu_Y(\nu)(A) = 0.$$

This shows that $g_*\big(\mu_Y(\nu)\big)$ and $\mu_Y(\nu)$ have the same null sets, finishing the proof. □

PART II

MEASURED SEMI-SIMPLICIAL COMPLEXES AND STABILITY

Chapter 5

Measured Semi-Simplicial Complexes

Until the end of this chapter, fix a lcsc group G. We introduce here the so-called *measured semi-simplicial G-complexes*, which are semi-simplicial objects in the category of regular G-spaces (see Definition 1.10.) They carry the minimal structure that enables the consideration of complexes of L^∞ function classes over their skeleta—hence of their cohomology—and that allows us to approach bounded-cohomological stability à la Quillen. We also define the properties of admissibility, connectivity, and transitivity of measured semi-simplicial complexes.

In virtue of the formalism discussed in the Chapter 3 of the present work, in Section 5.2 we will associate to every admissible measured semi-simplicial G-complex a spectral sequence that carries complete information on the continuous bounded cohomology of G. It can be regarded as a generalization of Theorem 1.24, and will be the base of the proof of our bounded-cohomological Quillen's method in the next chapter.

To finish the chapter, in Section 5.3, we give one of the concrete examples of measured semi-simplicial complexes that motivated the abstraction: the *product complex* of a regular G-space. A second motivating example is given by the *measured* Tits building of a connected semisimple Lie group. They are admissible and highly connected in the sense presented in this chapter, but the corresponding action fails to be transitive, having finitely many orbits. For a treatment of these, we refer the reader to [60], where they are introduced with the structure of *sot complexes*.

5.1 Definitions and properties

A *semi-simplicial object* $X_\bullet$ in a category C is a sequence of objects $(X_k)_{k=0}^\infty$, together with morphisms $\delta_{i,k} : X_{k+1} \to X_k$ for all k and $i \in [k+1]$, called the *face maps* of the semi-simplicial object, such that

$$\delta_{i,k-1} \circ \delta_{j,k} = \delta_{j-1,k-1} \circ \delta_{i,k} \qquad \text{whenever } i < j. \tag{5.1}$$

If k is clear from the context, we denote the face map $\delta_{i,k}$ simply by δ_i. In the situations we are interested in, C will always be a concrete category[1]. Elements of the set underlying X_k will then

[1] A *concrete category* is a pair (C, T), where C is a category and $T : C \to$ **Sets** is a faithful functor. The functor T gives a way of assigning to each object (resp. morphism) in the category C an underlying set (resp. function). In many cases, given the nature of the category C, the choice of a functor T is clear, so one omits mention of it.

be referred to as *k-simplices* of the semi-simplicial object. If $X_k = \emptyset$ for all $k \geq n+1$, then we say that $X_\bullet$ is *n-dimensional* and write $\dim(X_\bullet) = n$.

We are going to consider semi-simplicial objects in the category Reg_G of regular G-spaces, as introduced in Definition 1.10.

Definition 5.1. We will call a semi-simplicial object $(X_\bullet, \mu_\bullet)$ in the category Reg_G a *measured semi-simplicial G-complex*, or simply a *measured G-complex*.

Let $(X_\bullet, \mu_\bullet)$ be an n-dimensional measured G-complex (with possibly $n = \infty$). By Example 1.13, the pairs $(L^1(X_k, \mu_k), L^\infty(X_k))$ are coefficient G-modules in the sense of Definition 1.8. Note that, being morphisms in the category $\mathbf{Reg}_G$, the face maps $\delta_i : X_{k+1} \to X_k$ induce G-morphisms

$$\delta^i : L^\infty(X_k) \to L^\infty(X_{k+1}), \quad \delta^i(\varphi) := \varphi \circ \delta_i,$$

for all $i \in [k+1]$. It follows from the face identities (5.1) that the G-morphisms $\mathrm{d}^k := \sum_{i=0}^{k+1} (-1)^i \delta^i$ satisfy the equality $\mathrm{d}^{k+1} \circ \mathrm{d}^k = 0$ for all $k \geq 0$. Thus the sequence

$$0 \to \mathbb{R} \xrightarrow{\mathrm{d}^{-1}} L^\infty(X_0) \xrightarrow{\mathrm{d}^0} L^\infty(X_1) \xrightarrow{\mathrm{d}^1} L^\infty(X_2) \xrightarrow{\mathrm{d}^2} \cdots \xrightarrow{\mathrm{d}^{n-1}} L^\infty(X_n) \tag{5.2}$$

is a complex of Banach G-modules, where $\mathrm{d}^{-1} : \mathbb{R} \to L^\infty(X_0)$ in (5.2) is the inclusion of constants. We point out that our numbering of the Banach G-modules in (5.2) is so that $\mathbb{R}$ is the (-1)-st term, and $L^\infty(X_k)$ the k-th one.

Definition 5.2. We call (5.2) the *augmented L^∞-complex* associated to $(X_\bullet, \mu_\bullet)$.

A priori there is no reason why (5.2) would be a complex of coefficient G-modules, since the differentials d^k need not be dual morphisms. In view of this, we introduce the concept of *admissibility*.

Definition 5.3. A measured G-complex is called *admissible* if its associated augmented L^∞-complex is a complex of coefficient G-modules, i.e. if each of the morphisms d^k is dual to a bounded linear map $L^1(X_{k+1}, \mu_{k+1}) \to L^1(X_k, \mu_k)$.

As noted in Remark 1.14, while the Banach spaces $L^\infty(X_k)$ depend only on the G-invariant measure class underlying the regular G-spaces (X_k, μ_k), the spaces $L^1(X_k, \mu_k)$ depend on the specific choice of a probability measure within that class. Thus, the notion of admissibility seems to be dependent on the choices of measures.

We will see two examples of admissible measured G-complexes below: *product complexes*, treated in Subsection 5.3 (for a proof of their admissibility, see Lemma 5.14), and the *symplectic Stiefel complexes*, which are the subject of Chapter 7 and whose admissibility is established after the construction of appropriate quasi-invariant probability measures. Our inability to exhibit any example of a non-admissible measured G-complex leaves the following question open:

Question 5.4. Is every measured G-complex $(X_\bullet, \mu_\bullet)$ automatically admissible? More concretely, do there always exist choices of G-quasi-invariant measures $\mu_\bullet$ on $X_\bullet$ that make the coboundary operators $\mathrm{d}^k : L^\infty(X_k) \to L^\infty(X_{k+1})$ in (5.2) dual to L^1-maps $L^1(X_{k+1}, \mu_{k+1}) \to L^1(X_k, \mu_k)$?

We introduce some other properties of measured G-complexes.

Definition 5.5. Let $(\bar{X}_\bullet, \mu_\bullet)$ be an n-dimensional measured G-complex and let $q \in [n]$ be an integer. We say that $(\bar{X}_\bullet, \mu_\bullet)$ is:

(i) *measurably q-connected* if the homology of its augmented L^∞-complex (5.2) vanishes up to degree q.

(ii) *q-transitive* if for all $k \in [q]$, the G-action on $\bar{X}_k$ is transitive. In other words, there exists only one orbit of k-simplices of $\bar{X}_\bullet$ for every $k \in [q]$.

(iii) *essentially q-transitive* if for all $k \in [q]$, the G-action on $\bar{X}_k$ has an orbit X_k of full measure and such that $\delta_i(X_k) \subset X_{k-1}$ for all $i \in [k]$ whenever $k \geq 1$.

We will say that a measured G-complex $(\bar{X}_\bullet, \mu_\bullet)$ is *measurably $(-\infty)$-connected* if it is not measurably q-connected for any $q \in [n]$. If $n = \infty$ and $(\bar{X}_\bullet, \mu_k)$ is measurably q-connected for all q, we also say that it is *measurably ∞-connected*.

Remark 5.6. As indicated in Remark 1.11, the G-orbits $X_k \subset \bar{X}_k$ from (iii) are Borel, hence also standard Borel spaces in their own right. Thus, if $\bar{X}_\bullet$ is measurably q_1-connected and essentially q_2-transitive, then we can pass from $\bar{X}_\bullet$ to $X_\bullet$ to obtain a measurably q_1-connected and q_2-transitive G-complex. In other words, actual transitivity (as opposed to essential transitivity) can be assumed without loss of generality.

5.2 The spectral sequence of an admissible measured semi-simplicial complex

Let $(X_\bullet, \mu_\bullet)$ be an admissible measured G-complex. Let also X_{-1} be a singleton, and thus $L^\infty(X_{-1}) = \mathbb{R}$. For all $p, q \geq 0$, we define the Banach spaces

$$L^{p,q} := L^\infty\left(G^{p+1} \times X_{q-1}\right)^G \cong L^\infty\left(G^{p+1}; L^\infty(X_{q-1})\right)^G, \tag{5.3}$$

(see Example 1.15 and Lemma 1.17), and the continuous linear maps

$$\begin{aligned} \mathrm{d}_{\mathrm{H}}^{p,q} : L^{p,q} \to L^{p+1,q}, \qquad & \mathrm{d}_{\mathrm{H}}^{p,q} f(g_0, \ldots, g_p) := \sum_{i=0}^{p} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_p), \\ \mathrm{d}_{\mathrm{V}}^{p,q} : L^{p,q} \to L^{p,q+1}, \qquad & \mathrm{d}_{\mathrm{V}}^{p,q} f(g_0, \ldots, g_{p-1}) := \mathrm{d}^q\left(f(g_0, \ldots, g_{p-1})\right), \end{aligned}$$

with d^q being the coboundary operators in (5.2). A routine computation, which we omit, yields the next lemma.

Lemma 5.7. $(L^{\bullet,\bullet}, \mathrm{d}_{\mathrm{H}}, \mathrm{d}_{\mathrm{V}})$ *is a first-quadrant double complex.* □

According to Theorem 3.6, let ${}^{\mathrm{I}}E_{\bullet}^{\bullet,\bullet}$ and ${}^{\mathrm{II}}E_{\bullet}^{\bullet,\bullet}$ be the spectral sequences associated with the horizontal and vertical filtrations of $L^{\bullet,\bullet}$, respectively, both of which converge to the cohomology of the total complex of $L^{\bullet,\bullet}$ and whose first-page terms and differentials are given by

$$\begin{aligned} {}^{\mathrm{I}}E_1^{p,q} &= \mathrm{H}^q(L^{p,\bullet}, \mathrm{d}_{\mathrm{V}}^{p,\bullet}), & {}^{\mathrm{I}}\mathrm{d}_1^{p,q} &= \mathrm{H}^q(\mathrm{d}_{\mathrm{H}}^{p,\bullet}) : {}^{\mathrm{I}}E_1^{p,q} \to {}^{\mathrm{I}}E_1^{p+1,q}, \\ {}^{\mathrm{II}}E_1^{p,q} &= \mathrm{H}^q(L^{\bullet,p}, \mathrm{d}_{\mathrm{H}}^{\bullet,p}), & {}^{\mathrm{II}}\mathrm{d}_1^{p,q} &= \mathrm{H}^q(\mathrm{d}_{\mathrm{V}}^{\bullet,p}) : {}^{\mathrm{II}}E_1^{p,q} \to {}^{\mathrm{II}}E_1^{p+1,q}. \end{aligned}$$

As an immediate consequence of Theorem 1.24, we can determine the terms and differentials of the first page ${}^{\mathrm{II}}E_1^{\bullet,\bullet}$.

Lemma 5.8. *For every $p, q \geq 0$, we have*

$$ {}^{\mathrm{II}}E_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(X_{p-1})) \quad \text{and} \quad {}^{\mathrm{II}}\mathrm{d}_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G; \mathrm{d}^{p-1}). \qquad \square$$

We remind the reader that the functorial notation for $\mathrm{H}^{\bullet}_{\mathrm{cb}}$ was defined in Subsection 1.1.3. On the other hand, the next proposition shows how the measurable connectivity of $X_{\bullet}$ has implications on the convergence of ${}^{\mathrm{I}}E_{\bullet}^{\bullet,\bullet}$.

Lemma 5.9. *Assume that $(X_{\bullet}, \mu_{\bullet})$ is measurably γ_0-connected for some $\gamma_0 \in \mathbb{N}$. Then for all $p \in [\gamma_0 + 1]$, the limit term ${}^{\mathrm{I}}E_\infty^p$ vanishes.*

Proof. By the measurable γ_0-connectivity, the augmented L^∞-complex (5.2) is exact up to degree γ_0. Applying the functor $L^\infty(G^{p+1}; -)^G$ to it, we obtain

$$0 \to L^{p,0} \to L^{p,1} \to \cdots \to L^{p,\gamma_0} \to L^{p,\gamma_0+1} \to \cdots,$$

which is exact up to degree $\gamma_0 + 1$ because of Lemma 1.19. This implies vanishing of the entries ${}^{\mathrm{I}}E_1^{p,q}$ for all $p \geq 0$ and $q \in [\gamma_0 + 1]$. $\square$

We can collect the results from Lemmas 5.8 and 5.9 in the next proposition.

Proposition 5.10. *Let G be a lcsc group, $(X_{\bullet}, \mu_{\bullet})$ an admissible measured G-complex, and let $\mathrm{d}^{\bullet}$ denote the differential of the L^∞-complex associated to $(X_{\bullet}, \mu_{\bullet})$. Then there exists a first-quadrant spectral sequence $E_{\bullet}^{\bullet,\bullet}$ with first page terms and differentials*

$$E_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(X_{p-1})) \quad \text{and} \quad \mathrm{d}_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G; \mathrm{d}^{p-1}) \quad \textit{for all } p, q \geq 0. \tag{5.4}$$

If in addition $(X_{\bullet}, \mu_{\bullet})$ is measurably γ_0-connected for some $\gamma_0 \in \mathbb{N}$, then we have limit terms $E_\infty^q = 0$ for every $q \in [\gamma_0 + 1]$. $\square$

Remark 5.11. A spectral sequence related to the one from Proposition 5.10 was the one considered by Monod in [59, §12.1] and [61]. It is recorded in Proposition 5.12 below; for its proof, see [59, Proposition 12.2.1]. In our language of measured semi-simplicial complexes, it would be the spectral sequence $F_{\bullet}^{\bullet,\bullet}$ associated to the double complex of spaces

$$M^{p,q} := L^\infty\left(G^{p+1} \times X_q\right)^G \cong L^\infty\left(G^{p+1}; L^\infty(X_q)\right)^G.$$

for $p, q \geq 0$. The difference between this one and the one in (5.3) is simply a shift by one in the parameter q. While the change seems innocent, the spectral sequence in Proposition 5.10 has the crucial advantage that it keeps track of an additional concrete map between the cohomology $E_1^{0,\bullet} = \mathrm{H}^\bullet_{\mathrm{cb}}(G)$ and the terms $E_1^{1,\bullet}$. This information is lost in Proposition 5.12: the relationship between $\mathrm{H}^\bullet_{\mathrm{cb}}(G)$ and the terms in the first page $F_1^{\bullet,\bullet}$ is realized only in the ∞-page, where all possible relations are not canonical anymore. In the sequel, we will profit heavily from this choice.

Proposition 5.12. *Let G be a lcsc group, $(X_\bullet, \mu_\bullet)$ an admissible measured G-complex, and let $\mathrm{d}^\bullet$ denote the differential of the L^∞-complex associated to $X_\bullet$. Then there exists a first-quadrant spectral sequence $F_\bullet^{\bullet,\bullet}$ with first page terms and differentials*

$$F_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(X_p)) \quad \text{and} \quad D_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G; \mathrm{d}^p) \quad \textit{for all } p, q \geq 0. \tag{5.5}$$

If in addition $(X_\bullet, \mu_\bullet)$ is measurably γ_0-connected for some $\gamma_0 \in \mathbb{N}$, then we have limit terms $F_\infty^q \cong \mathrm{H}^q_{\mathrm{cb}}(G)$ for every $q \in [\gamma_0]$.

5.3 Product complexes

Let (X, μ) be a regular G-space. Then for every $k \in \mathbb{N}$, the product space $(X^{k+1}, \mu^{\otimes k+1})$ is a regular G-space endowed with the diagonal action. For all $k \in \mathbb{N}$ and $i \in [k]$, let us denote by $\delta_i : X^{k+2} \to X^{k+1}$ the face map given by deletion of the i-th entry, i.e.

$$\delta_i(x_0, \dots, x_k) := (x_0, \dots, \widehat{x_i}, \dots, x_k).$$

The face maps are clearly morphisms of regular G-spaces, and a verification shows that they satisfy the identity (5.1).

Definition 5.13. The pair $(X^{\bullet+1}, \mu^{\otimes \bullet+1})$ is a measured G-complex, that we call the *product G-complex* of X.

The connection of product complexes with the continuous bounded cohomology of G in the case of an amenable G-space X is given by Theorem 1.24. Non-amenable product complexes were also considered by Monod in [61] to prove stability along the sequences of general and special linear groups. We establish the admissibility and measurable ∞-connectivity of product complexes below.

Lemma 5.14. *The product G-complex $(X^{\bullet+1}, \mu^{\otimes \bullet+1})$ is admissible.*

Proof. We claim that for any $k \in \mathbb{N}$ and $i \in [k+1]$, the formula

$$\partial_i \phi(x_0, \dots, x_k) := \int_X \phi(x_0, \dots, x_{i-1}, t, x_i, \dots, x_k) \, \mathrm{d}\mu(t)$$

defines a bounded linear map $\partial_i : L^1(X^{k+2}, \mu^{\otimes k+2}) \to L^1(X^{k+1}, \mu^{\otimes k+1})$ that is pre-dual to the face operator $\delta^i : L^\infty(X^{k+1}) \to L^\infty(X^{k+2})$ induced by δ_i. This implies that the coboundary operator $\mathrm{d} : L^\infty(X^{k+1}) \to L^\infty(X^{k+2})$ is a dual morphism, for the pair (∂, d) with $\partial = \sum(-1)^i \partial_i$ is a morphism of coefficient G-modules.

In order to verify the claim, let $\langle - \mid - \rangle$ be the dual pairing between L^∞ and L^1, and let $f \in L^\infty(X^{k+1})$ and $\phi \in L^1(X^{k+2}, \mu^{\otimes k+2})$. Then

$$\begin{aligned}\langle f \mid \partial_i \phi\rangle &= \int_{X^{k+1}} f(x_0, \dots, x_k) \left(\int_X \phi(x_0, \dots, x_{i-1}, t, x_i, \dots, x_k) \,\mathrm{d}\mu(t) \right) \mathrm{d}\mu^{\otimes k+1}(x_0, \dots, x_k) \\ &= \int_{X^{k+2}} f(x_0, \dots, x_k)\, \phi(x_0, \dots, x_{i-1}, t, x_i, \dots, x_k) \,\mathrm{d}\mu^{\otimes k+2}(x_0, \dots, x_{i-1}, t, x_i, \dots, x_k) \\ &= \int_{X^{k+2}} \delta^i f(x_0, \dots, x_{k+1})\, \phi(x_0, \dots, x_{k+1}) \,\mathrm{d}\mu^{\otimes k+2}(x_0, \dots, x_{k+1}) = \langle \delta^i f \mid \phi \rangle,\end{aligned}$$

where the second equality is Fubini's theorem. □

Lemma 5.15. *There exists a contracting homotopy for the augmented L^∞-complex*

$$0 \to \mathbb{R} \xrightarrow{\mathrm{d}^{-1}} L^\infty(X) \xrightarrow{\mathrm{d}^0} L^\infty(X^2) \xrightarrow{\mathrm{d}^1} L^\infty(X^3) \to \cdots$$

associated to $(X^{\bullet+1}, \mu^{\otimes \bullet+1})$. Thus, the product G-complex is measurably ∞-connected.

The existence of the contracting homotopy and Lemma 1.1 imply the conclusion of this lemma. Because it will be useful at a later stage, we describe the process of guessing a contracting homotopy: We are looking for bounded linear maps $h^k : L^\infty(X^{k+2}) \to L^\infty(X^{k+1})$ such that

$$\mathrm{d}^{k-1}\, h^{k-1} + h^k\, \mathrm{d}^k = \mathrm{id} \tag{5.6}$$

holds for every $k \in [-1, \infty] \cap \mathbb{Z}$. We have $\mathrm{d}^{-2} = 0$ and $h^{-2} = 0$. A canonical choice for a bounded functional $h^{-1} : L^\infty(X) \to \mathbb{R}$ is integration with respect to μ:

$$h^{-1} f := \int_X f(t) \,\mathrm{d}\mu(t), \quad \text{for } f \in L^\infty(X).$$

An easy check shows that it verifies the identity (5.6) for $k = -1$. One constructs then $h^0, h^1, \dots$ inductively: for example, if h^0 is assumed to satisfy (5.6) for $k = 0$, then

$$h^0(\mathrm{d}^0 \varphi)(x_0) \stackrel{!}{=} \varphi(x_0) - d^{-1} h^{-1} \varphi(x_0) = \int_X (\varphi(x_0) - \varphi(t)) \,\mathrm{d}\mu(t) = \int_X \mathrm{d}^0 \varphi(t, x_0) \,\mathrm{d}\mu(t),$$

for all $\varphi \in L^\infty(X)$, which suggests defining $h^0 : L^\infty(X^2) \to L^\infty(X)$ by

$$h^0 f(x_0) := \int_X f(t, x_0) \,\mathrm{d}\mu(t).$$

Recursively, one finds the bounded linear maps

$$h^k : L^\infty(X^{k+2}) \to L^\infty(X^{k+1}), \quad h^k f(x_0, \dots, x_k) = \int_X f(t, x_0, \dots, x_k) \,\mathrm{d}\mu(t), \tag{5.7}$$

for all $k \geq -1$. Note that given the inclusion $L^\infty \subset L^1$ for finite measure spaces, the map h^k equals the restriction of ∂_0 as in the proof of Lemma 5.14.

Proof of Lemma 5.15. For any integer $k \geq -1$, any function $f \in L^\infty(X^{k+2})$, and any tuple $(x_0, \dots, x_{k+1}) \in X^{k+2}$, we have

$$\mathrm{d}^k \circ h^k(f)(x_0, \dots, x_{k+1}) = \sum_{i=0}^{k+1} (-1)^i \int_X f(t, x_0, \dots, \widehat{x}_i, \dots, x_{k+1}) \, \mathrm{d}\mu(t);$$
$$h^{k+1} \circ \mathrm{d}^{k+1}(f)(x_0, \dots, x_{k+1}) = f(x_0, \dots, x_k) - \mathrm{d}^k \circ h^k(f)(x_0, \dots, x_k).$$

The second line translates into the identity (5.6). We conclude by Lemma 1.1 that the homology of the augmented L^∞-complex vanishes in every degree. □

In the light of Lemma 5.14 and Lemma 5.15, Proposition 5.10 implies:

Proposition 5.16. *Let G be a lcsc group, and X be a regular G-space. Then there exists a first-quadrant spectral sequence $E_\bullet^{\bullet,\bullet}$ with first page*

$$E_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(X^p)) \quad \text{and} \quad \mathrm{d}_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G; \mathrm{d}^{p-1}), \quad \textit{for all } p, q \geq 0$$

and limit terms $E_\infty^q = 0$ for all $q \in \mathbb{N}$. □

We conclude this subsection with a remark on the transitivity of product complexes.

Remark 5.17. It is standard to say a G-set X is k-transitive if the diagonal action of G on k-tuples of (pairwise distinct) elements of X is transitive. Compared to this one, our notation for transitivity of actions on products is shifted: (essential) k-transitivity of a product G-complex $(X^{\bullet+1}, \mu^{\otimes \bullet+1})$ means that the G-action on (a co-null subset of) the $(k+1)$-tuples of X is transitive. This shift is justified, for in the language of semi-simplicial objects, the k-skeleton of $(X^{\bullet+1}, \mu^{\otimes \bullet+1})$ consists of $(k+1)$-tuples.

Chapter 6

A Bounded-Cohomological Analogue of Quillen's Stability Method

In this chapter, we first present and prove our criterion for the stability of continuous bounded cohomology along ascending sequences of lcsc groups. It is a functional-analytic adaptation of a criterion for homological stability known as *Quillen's stability method* [67], briefly described in the Introduction, and of which account can be also found in [3] and [68, Chapter 3]. The functional-analytic component is inspired in Monod's *ad hoc* treatment in [61] of the families of general and special linear groups over any local field.

Our first goal is to formulate measure-theoretic versions of Quillen's conditions (Q1)–(Q3) from the Introduction, and of our criterion. We symbolize below the corresponding conditions by (MQ1)–(MQ3); based on them, we give the notion of a *measured Quillen family*. We state two versions of our stability method. The first one, Theorem 6.9, allows to establish stability along an infinite sequence of lcsc groups completely omitting any insight on a stability range, albeit with a very concise statement. It follows readily from the second one, Theorem 6.10, a *quantitative* version in whose proof we delve in Section 6.2.

We are then ready to harvest from our abstract machinery and prove $\mathrm{H}^\bullet_{\mathrm{cb}}$-stability results along some families of classical groups. In Sections 6.3 and 6.4, we revisit Monod's stability results along $(\mathrm{GL}_r(\mathscr{k}))_r$ and $(\mathrm{SL}_r(\mathscr{k}))_r$ over any local field $\mathscr{k}$, illustrating how they follow from Theorem 6.9 and Theorem 6.10. We seize the opportunity to record all corrected stability ranges, which, as previously mentioned, are a consequence of an inaccuracy in an induction step in [61]. In degree three, the existing stability theorems had as corollaries several vanishing results or produced upper bounds in the dimension of cohomology groups. We recover all of these also from the new ranges, with the notable addition that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_3(\mathscr{k})) = 0$ for any non-Archimedean local field $\mathscr{k}$.

Section 6.5 treats the case of the family $(\mathrm{Sp}_{2r}(\mathscr{k}))_r$ over $\mathscr{k} \in \{\mathbb{R}, \mathbb{C}\}$. After postulating the existence of a corresponding measured Quillen family, we prove a stability result in general degree, stated below as Theorem 6.26. The construction of the Quillen family is delayed to Chapter 7 of this thesis. Due to the fast growth of our stability ranges, we are not able to prove directly any concrete corollaries. An improvement in degree three is proven in Chapter 8.

6.1 Measured Quillen families and stability of $\mathrm{H}^\bullet_{\mathrm{cb}}$

Let us recall the concept of (cohomological) stability. It appeared previously as Definition 0.3 in the Introduction.

Definition 6.1. Let $\mathrm{H}^\bullet = (\mathrm{H}^q : \mathcal{C} \to \mathcal{C}')_{q\geq 0}$ be a sequence of functors, and $(G_r)_{r\in\mathbb{N}}$ a sequence of objects in $\mathcal{C}$. We say that $\mathrm{H}^\bullet$ is *stable along* $(G_r)_{r\in\mathbb{N}}$ provided that

$$\forall\, q \geq 0 \quad \exists\, r_0 = r_0(q): \qquad \mathrm{H}^q(G_{r_0}) \cong \mathrm{H}^q(G_{r_0+1}) \cong \mathrm{H}^q(G_{r_0+2}) \cong \cdots \tag{6.1}$$

Any function $q \mapsto r_0(q)$ satisfying this property will be called a *stability range* for $\mathrm{H}^\bullet$ along the sequence $(G_r)_{r\in\mathbb{N}}$.

Remark 6.2. We will consider two variations of the notion of stability above: one of them requires the existence of inclusions $G_r \hookrightarrow G_{r+1}$ and that the isomorphisms in (6.1) be induced by them; another one the first few isomorphisms of (6.1) to be replaced by injections, giving rise to sequences of the form

$$\mathrm{H}^q(G_{r_0}) \hookleftarrow \mathrm{H}^q(G_{r_0-1}) \hookleftarrow \cdots \hookleftarrow \mathrm{H}^q(G_{r_1}) \cong \mathrm{H}^q(G_{r_1+1}) \cong \mathrm{H}^q(G_{r_1+2}) \cong \cdots$$

A combination of the two variations will also appear in the present thesis. Moreover, if the property in (6.1) holds for a fixed q (not necessarily for all), we will say that $\mathrm{H}^\bullet$ is stable *in degree* q along $(G_r)_{r\in\mathbb{N}}$.

From now until the end of this section, let

$$G_0 \subset G_1 \subset G_2 \subset \cdots \subset G_R \tag{6.2}$$

be an ascending sequence of lcsc groups of length[1] $R \in \mathbb{N} \cup \{\infty\}$, with inclusions $\iota_r : G_r \hookrightarrow G_{r+1}$ for every $r \in [R-1]$. Also, for each $r \in [R]$, let $(X_{r,\bullet}, \mu_{r,\bullet})$ be an admissible measured G_r-complex (see Definitions 5.1 and 5.3.)

In the spirit of Quillen's condition (Q3) from the Introduction, we propose a measure-theoretic notion of compatibility of the sequence $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$.

Definition 6.3. Suppose that there exists a function $\tau : [R] \to \mathbb{N}$ such that $(X_{r,\bullet}, \mu_{r,\bullet})$ is $\tau(r)$-transitive for all r. We say that $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ has the *τ-descent property* if for every $r \in [R]$, there exist a point $o_{r,\tau(r)} \in X_{r,\tau(r)}$ and a collection

$$\left(\pi_{r,q} : H_{r,q} \twoheadrightarrow G_{r-q-1} \mid q \in [\tau(r)]\right)$$

of surjective homomorphisms with amenable kernel, where $H_{r,0} > H_{r,1} > \cdots > H_{r,\tau(r)}$ are closed subgroups of G_r defined as $H_{r,q} := \mathrm{stab}_{G_r}(o_{r,q})$ for all $q \in [\tau(r)]$, with

$$o_{r,q} := \delta_{q+1}(o_{r,q+1}) = \delta_{q+1} \circ \cdots \circ \delta_{\tau(r)}(o_r) \in X_{r,q} \tag{6.3}$$

for $q \in [\tau(r) - 1]$.

[1] We assume that the sequence (6.2) does not terminate if $R = \infty$.

Remark 6.4. In words, for a fixed r, the point $o_{r,q}$ is defined inductively over q by applying the last face map, i.e. the $(q+1)$-th, to the point $o_{r,q+1} \in X_{r,q+1}$. Note that these choices of base points give rise to canonical Borel identifications $X_{r,q} \cong G_r/H_{r,q}$. By G_r-equivariance, the face maps

$$\delta_i : G_r/H_{r,q+1} \cong X_{r,q+1} \to X_{r,q} \cong G_r/H_{r,q}, \quad i \in [q+1]$$

are determined by a choice of an element $w_{r,q,i} \in G_r$ such that $\delta_i(H_{r,q+1}) = w_{r,q,i}H_{r,q}$.

Definition 6.5. Let τ be as in Definition 6.3. We say that $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ is *τ-measurably compatible* if it has the τ-descent property with collections of closed subgroups $(H_{r,q})_{r,q}$ and of homomorphisms $(\pi_{r,q})_{r,q}$ such that the next two properties hold for all $r \in [R]$ and all $q < \tau(r)$:

(I) For every $i \in [q+1]$, there exists an element $w_{r,q,i}$ in the normalizer $\mathcal{N}_{G_r}(H_{r,q+1})$ such that $\delta_i(H_{r,q+1}) = w_{r,q,i}H_{r,q}$.

(II) There exists a section $\sigma_{r,q+1} : G_{r-q-2} \to H_{r,q+1}$ of $\pi_{r,q+1}$ such that the diagram below commutes, with the horizontal arrows being the inclusions:

$$\begin{array}{ccc} H_{r,q+1} & \hookrightarrow & H_{r,q} \\ {\scriptstyle \sigma_{r,q+1}}\uparrow & & \downarrow{\scriptstyle \pi_{r,q}} \\ G_{r-q-2} & \hookrightarrow & G_{r-q-1} \end{array} \tag{6.4}$$

Remark 6.6. In these two definitions, G_r is understood as the trivial group whenever $r < 0$.

Remark 6.7. It is possible to define the two notions above if G_r is merely *essentially* $\tau(r)$-transitive on $(\bar{X}_{r,\bullet}, \mu_{r,\bullet})$: in this case one has to require the base points to be in the co-null G_r-orbit $X_{r,q} \subset \bar{X}_{r,q}$ as in Definition 5.5 (iii).

We are ready to define a measured Quillen family, and then state the main theorems of the present chapter.

Definition 6.8. Let $\gamma : [R] \to \mathbb{N} \cup \{-\infty, \infty\}$ and $\tau : [R] \to \mathbb{N}$ be functions. We say that $(G_r, (X_{r,\bullet}, \mu_\bullet))_{r\in[R]}$ is a *measured Quillen family* of length R with parameters (γ, τ) if

(MQ1) $X_{r,\bullet}$ is measurably $\gamma(r)$-connected for every $r \in [R]$;

(MQ2) $X_{r,\bullet}$ is $\tau(r)$-transitive for every $r \in [R]$;

(MQ3) $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ is τ-measurably compatible.

Theorem 6.9. *Assume that $(G_r, (X_{r,\bullet}, \mu_\bullet))_{r\in\mathbb{N}}$ is a measured Quillen family of infinite length with parameters (γ, τ). If $\gamma(r) \to \infty$ and $\tau(r) \to \infty$ as $r \to \infty$, then $\mathrm{H}^\bullet_{\mathrm{cb}}$ stabilizes along $(G_r)_{r\in\mathbb{N}}$.*

As announced at the beginning of this chapter, Theorem 6.9 is a corollary of a *quantitative* stability criterion, which has an output an explicit stability range in terms of the parameters γ and τ. Moreover, it provides a sort of version of Theorem 6.9 for finite-length measured Quillen families, provided those parameters grow fast enough.

Theorem 6.10. *Assume that* $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ *is a measured Quillen family with parameters* (γ, τ)*. Let also* $q_0 \in \mathbb{N}_{>0}$ *be such that for every r, the map* $\mathrm{H}^q_{\mathrm{cb}}(\iota_r) : \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}) \to \mathrm{H}^q_{\mathrm{cb}}(G_r)$ *induced by the inclusion* $\iota_r : G_r \hookrightarrow G_{r+1}$ *is an isomorphism whenever* $q \le q_0$*. Finally, let us set for* $r \in [R-1]$ *and* $q \ge 0$ *the quantities*

$$\widetilde{\gamma}(q,r) := \min_{j=q_0}^{q} \left\{ \gamma\bigl(r+1-2(q-j)\bigr) - j \right\} \quad \text{and} \quad \widetilde{\tau}(q,r) := \min_{j=q_0}^{q} \left\{ \tau\bigl(r+1-2(q-j)\bigr) - j \right\},$$

(understanding that $\min \emptyset = \infty$*.) Then, for all such r and q, the inclusion* ι_r *induces isomorphism and injection*

$$\mathrm{H}^q_{\mathrm{cb}}(\iota_r) : \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}) \xrightarrow{\sim} \mathrm{H}^q_{\mathrm{cb}}(G_r) \quad \text{and} \quad \mathrm{H}^{q+1}_{\mathrm{cb}}(\iota_r) : \mathrm{H}^{q+1}_{\mathrm{cb}}(G_{r+1}) \hookrightarrow \mathrm{H}^{q+1}_{\mathrm{cb}}(G_r),$$

respectively, whenever $\min\{\widetilde{\gamma}(q,r), \widetilde{\tau}(q,r) - 1\} \ge 0$.

Remark 6.11. We refer to the number q_0 in Theorem 6.10 as the *initial condition*. We can always choose $q_0 = 1$, since for trivial $\mathbb{R}$-coefficients, $\mathrm{H}^0_{\mathrm{cb}} = \mathbb{R}$ (with inclusions inducing isomorphisms) and $\mathrm{H}^1_{\mathrm{cb}} = 0$; see Examples 1.5 and 1.6. If the G_r are all connected simple Lie groups and either all of Hermitian type or all of non-Hermitian type, then we can choose $q_0 = 2$ by Theorem 1.34. This is for instance the case for the family $(\mathrm{Sp}_{2r}(\mathscr{k}))_r$ with $\mathscr{k} \in \{\mathbb{R}, \mathbb{C}\}$.

Let us reassure ourselves that Theorem 6.10 implies Theorem 6.9:

Proof of Theorem 6.9. Let $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in\mathbb{N}}$ be an infinite measured Quillen family with parameters (γ, τ) and assume that $\gamma(r) \to \infty$ and $\tau(r) \to \infty$ as $r \to \infty$. By Remark 6.11, we may choose the initial condition $q_0 := 1$. Since $\gamma(r) \to \infty$ and $\tau(r) \to \infty$, we then find for every $q \ge q_0$ some $r(q) \in \mathbb{N}$ such that for all $j \in \{q_0+1, \dots, q\}$ and all $r \ge r(q)$ we have

$$\gamma\bigl(r+1-2(q-j)\bigr) \ge j \quad \text{and} \quad \tau\bigl(r+1-2(q-j)\bigr) \ge j+1.$$

Then Theorem 6.10 implies that $\mathrm{H}^q_{\mathrm{cb}}(G_{r(q)}) \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r(q)+1}) \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r(q)+2}) \cong \cdots$. □

6.2 Proof of the quantitative stability criterion

We break the proof of Theorem 6.10 into two steps. In the first step—accounted in Subsection 6.2.1—we use the properties (MQ2) and (MQ3) from Definition 6.8 to derive further information on the first-page terms and differentials of the spectral sequence associated to a pair $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))$ for a certain fixed r. More concretely, we show that the relevant terms are isomorphic to continuous bounded cohomology groups $\mathrm{H}^q_{\mathrm{cb}}(G_s)$, and that the relevant differentials are conjugated to either the identically zero map or induced by inclusions $G_s \hookrightarrow G_{s+1}$ with $s \le r$. Based on this fact and additionally assuming the property (MQ1), the second step of the proof is an argument by induction on the degree q, and is carried out in Subsection 6.2.2.

6.2.1 Description of first-page terms and differentials. We start with two general lemmas.

Lemma 6.12. *Let G, M be lcsc groups, $H < G$ be a closed subgroup, and $\pi : H \twoheadrightarrow M$ a surjective homomorphism with amenable kernel. Then for all $q \geq 0$, there exists an isomorphism $\mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(G/H)) \cong \mathrm{H}^q_{\mathrm{cb}}(M)$.*

Proof. By Proposition 1.27, there exists an isomorphism

$$\mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(G/H)) = \mathrm{H}^q_{\mathrm{cb}}(G; \mathrm{Ind}_H^G \mathbb{R}) \cong \mathrm{H}^q_{\mathrm{cb}}(H)$$

Then, relying on the amenability of the kernel of π, from Corollary 1.25 we conclude that π induces an isomorphism $\mathrm{H}^q_{\mathrm{cb}}(H) \cong \mathrm{H}^q_{\mathrm{cb}}(M)$. □

Lemma 6.13. *Let G be a lcsc group, $H < G$ be a closed subgroup, and $\mathrm{d}^{-1} : \mathbb{R} \to L^\infty(G/H)$ be the coefficient inclusion. Then for any $q \geq 0$, the map $\mathrm{Ind}^q : \mathrm{H}^q_{\mathrm{cb}}(H) \to \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(G/H))$ from Section 1.4 makes the diagram*

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{cb}}(G) & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(G;\mathrm{d}^{-1})} & \mathrm{H}^q_{\mathrm{cb}}(G; L^\infty(G/H)) \\ & \searrow^{\mathrm{H}^q_{\mathrm{cb}}(j_0)} & \cong \uparrow \mathrm{Ind}^q \\ & & \mathrm{H}^q_{\mathrm{cb}}(H_0) \end{array}$$

commute, with $j : H \hookrightarrow G$ being the inclusion.

Proof. In the notation of Section 1.4, d^{-1} equals the dual morphism $\mathrm{id}^* : \mathrm{Ind}_G^G \mathbb{R} \to \mathrm{Ind}_{H_0}^G \mathbb{R}$ induced by the identity $\mathrm{id} : G \to G$. The claim follows from the simplified version of Proposition 1.27 (iii) stated as Remark 1.28. □

Let us turn again to the sequence $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r \in [R]}$. Particularly, assume that it satisfies the property (MQ2) of Definition 6.8 for a function $\tau : [R] \to \mathbb{N}$, and that it has the τ-descent property with closed subgroups $(H_{r,q} = \mathrm{stab}_{G_r}(o_{r,q}))_{r,q}$ and homomorphisms $(\pi_{r,q})_{r,q}$ as in Definition 6.3. Let also $r \in [R]$ be fixed. Proposition 5.10 grants the existence of a spectral sequence $E^{\bullet,\bullet}_\bullet$ with first page

$$E_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G_r; L^\infty(X_{r,p-1})) \quad \text{and} \quad \mathrm{d}_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G_r; \mathrm{d}^{p-1}) \tag{6.5}$$

for all $p, q \geq 0$. The next lemma is then an immediate consequence of the τ-descent property and Lemma 6.12.

Lemma 6.14. *For all $p \in [\tau(r) + 1]$ and $q \geq 0$, there are isomorphisms $E_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r-p})$.* □

Let us abbreviate now $G := G_r$, and $X_p := X_{r,p}$, $H_p := H_{r,p}$ for all $p \in [\tau(r)]$, and $H_{-1} := G$. An assumption related to property (I) from Definition 6.5 will provide information on the differentials $\mathrm{d}_1^{p,q} = \sum_{i=0}^{p} (-1)^i \, \mathrm{H}^q_{\mathrm{cb}}(G; \delta^i)$ for $p > 0$.

Lemma 6.15. *Suppose that there exist elements $w_{p-1,i} \in \mathcal{N}_G(H_p)$ such that $\delta_i(H_p) = w_{p-1,i}H_{p-1}$ for all $p \in \{1,\dots,\tau(r)\}$ and $i \in [p]$. Also, for every $p \in [\tau(r)]$ and every $q \geq 0$, we define $\Delta^{p,q} : \mathrm{H}^q_{\mathrm{cb}}(H_{p-1}) \to \mathrm{H}^q_{\mathrm{cb}}(H_p)$ by*

$$\Delta^{p,q} := \begin{cases} \mathrm{H}^q_{\mathrm{cb}}(j_p) & \text{if } p \text{ is even,} \\ 0 & \text{if } p \text{ is odd,} \end{cases}$$

where $j_p : H_p \hookrightarrow H_{p-1}$ denotes the inclusion. Then there are isomorphisms $E_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(H_{p-1})$ for all $p \in [\tau(r)+1]$ and $q \geq 0$ that make the diagram

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{\mathrm{d}_1^{p,q}} & E_1^{p+1,q} \\ \updownarrow\cong & & \cong\updownarrow \\ \mathrm{H}^q_{\mathrm{cb}}(H_{p-1}) & \xrightarrow{\Delta^{p,q}} & \mathrm{H}^q_{\mathrm{cb}}(H_p) \end{array} \tag{6.6}$$

commute whenever $p \leq \tau(r)$.

Proof. The case $p = 0$ is Lemma 6.13. Fix then $p \in \{1,\dots,\tau(r)\}$, $i \in [p]$, $q \geq 0$, and set $w := w_{p-1,i}$ and $H'_{p-1} := wH_{p-1}w^{-1}$. Because w is in the normalizer, we have $H_p < H'_{p-1}$. Note also that $X_{p-1} \cong G/H'_{p-1}$, and the face map

$$\delta_i : G/H_p \cong X_p \to X_{p-1} \cong G/H'_{p-1}$$

is simply the canonical projection. In the notation of Section 1.4, the dual morphism

$$\mathrm{id}^* : \mathrm{Ind}^G_{H'_{p-1}} \mathbb{R} = L^\infty(G/H'_{p-1}) \to L^\infty(G/H_p) = \mathrm{Ind}^G_{H_p} \mathbb{R}$$

induced by the identity $\mathrm{id} : G \to G$ coincides with δ^i. Hence, $\mathrm{H}^q_{\mathrm{cb}}(G;\delta^i) = \mathrm{H}^q_{\mathrm{cb}}(G;\mathrm{id}^*)$. Now, if we write $j'_p : H_p \hookrightarrow H'_{p-1}$ for the inclusion, then by Remark 1.28, the diagram

$$\begin{array}{ccc} E_1^{p,q} & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(G;\delta^i) = \mathrm{H}^q_{\mathrm{cb}}(G;\mathrm{id}^*)} & E_1^{p+1,q} \\ \mathrm{Ind}^q \uparrow \cong & & \cong \uparrow \mathrm{Ind}^q \\ \mathrm{H}^q_{\mathrm{cb}}(H'_{p-1}) & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(j'_p)} & \mathrm{H}^q_{\mathrm{cb}}(H_p) \end{array} \tag{6.7}$$

commutes. If we let $c_w : G \to G$ denote conjugation by w, then the diagram on the left below commutes, where the H_{p-1}-action on $G^{\bullet+1}$ in the bottom-left term is defined in terms of the H'_{p-1}-action on the top-left one, by $h \cdot \mathbf{g} := c_w(h) \cdot \mathbf{g}$.

$$\begin{array}{ccc} L^\infty(G^{\bullet+1})^{H'_{p-1}} & \xrightarrow{(j'_p)^*} & L^\infty(G^{\bullet+1})^{H_p} \\ c_w^* \downarrow \cong & \nearrow j_p^* & \\ L^\infty(G^{\bullet+1})^{H_{p-1}} & & \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} \mathrm{H}^q_{\mathrm{cb}}(H'_{p-1}) & \xrightarrow{\mathrm{H}^q_{\mathrm{cb}}(j'_p)} & \mathrm{H}^q_{\mathrm{cb}}(H_p) \\ \mathrm{H}^q_{\mathrm{cb}}(c_w) \downarrow \cong & \nearrow \mathrm{H}^q_{\mathrm{cb}}(j_p) & \\ \mathrm{H}^q_{\mathrm{cb}}(H_{p-1}) & & \end{array} \tag{6.8}$$

By functoriality, so does the right diagram. The combination of (6.7) and (6.8) produces a commutative diagram

$$\begin{CD} E_1^{p,q} @>{\mathrm{H}^q_{\mathrm{cb}}(G;\delta^i)}>> E_1^{p+1,q} \\ @V{\cong}VV @V{\cong}VV \\ \mathrm{H}^q_{\mathrm{cb}}(H_{p-1}) @>{\mathrm{H}^q_{\mathrm{cb}}(j_p)}>> \mathrm{H}^q_{\mathrm{cb}}(H_p) \end{CD} \tag{6.9}$$

The differential $\mathrm{d}_1^{p,q}$ is therefore conjugated to the alternating sum $\sum_{i=0}^{p}(-1)^i\, \mathrm{H}^q_{\mathrm{cb}}(j_p) = \Delta^{p,q}$. □

Finally:

Proposition 6.16. *Suppose that* $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ *satisfies properties (MQ2) and (MQ3) for the function* $\tau : [R] \to \mathbb{N}$*. Then for every* $p \in [\tau(r)+1]$ *and* $q \geq 0$*, there exist isomorphisms* $E_1^{p,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r-p})$ *that make the diagram*

$$\begin{CD} E_1^{p,q} @>{\mathrm{d}_1^{p,q}}>> E_1^{p+1,q} \\ @V{\cong}VV @V{\cong}VV \\ \mathrm{H}^q_{\mathrm{cb}}(G_{r-p}) @>{D^{p,q}}>> \mathrm{H}^q_{\mathrm{cb}}(G_{r-p-1}) \end{CD}$$

commute whenever $p \leq \tau(r)$*, where* $D^{p,q} := \mathrm{H}^q_{\mathrm{cb}}(\iota_{r-p-1})$ *if* p *is even, and* $D^{p,q} := 0$ *if* p *is odd.*

Proof. Fix $r \in [R]$, and subgroups $(H_{r,q})_{q\in[\tau(r)]}$ and projections $(\pi_{r,q})_{q\in[\tau(r)]}$ as in Definition 6.5. Let also $H_{r,-1} = G_r$ and $\pi_{r,-1} : H_{r,-1} \to G_r$ be the identity. By Corollary 1.25, the projections $\pi_{r,p-1}$ with amenable kernel induce isomorphisms

$$\mathrm{H}^q_{\mathrm{cb}}(\pi_{r,p-1}) : \mathrm{H}^q_{\mathrm{cb}}(G_{r-p}) \xrightarrow{\sim} \mathrm{H}^q_{\mathrm{cb}}(H_{r,p-1})$$

for every $q \geq 0$ and every $p \in [\tau(r)+1]$. Let now $p \leq \tau(r)$, and consider the commutative diagram

$$\begin{CD} H_{r,p} @>{\hookrightarrow}>> H_{r,p-1} \\ @A{\sigma_{r,p}}AA @VV{\pi_{r,p-1}}V \\ G_{r-p-1} @>{\hookrightarrow}>> G_{r-p} \end{CD} \tag{6.10}$$

from property (II) of Definition 6.5, where $\sigma_{r,p}$ is the section of $\pi_{r,p} : H_{r,p} \twoheadrightarrow G_{r-p-1}$. Because

$$\mathrm{H}^q_{\mathrm{cb}}(\sigma_{r,p}) \circ \mathrm{H}^q_{\mathrm{cb}}(\pi_{r,p}) = \mathrm{id},$$

we deduce that $\mathrm{H}^q_{\mathrm{cb}}(\sigma_{r,p})$ is an isomorphism. By the naturality of $\mathrm{H}^q_{\mathrm{cb}}$ applied to the diagram (6.10) combined with the diagram (6.6) from Lemma 6.15, we complete the proof. □

6.2.2 The induction argument.

We put all the pieces together in order to prove Theorem 6.10. Let $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_{r\in[R]}$ be a measured Quillen family with parameters (γ, τ), let $r \in [R-1]$ be fixed, and let $q_0 \in \mathbb{N}_{>0}$ be an initial condition for Theorem 6.10 as defined in Remark 6.11. We introduce also the shorthand notations $\mathrm{H}^q_{r-p+1} := \mathrm{H}^q_{\mathrm{cb}}(G_{r-p+1})$ and $\mathrm{H}^q(\iota_{r-p}) := \mathrm{H}^q_{\mathrm{cb}}(\iota_{r-p})$ for all q and all suitable p. The conclusion of Theorem 6.10 can be then re-stated as follows:

Main Claim. *Let* $q \geq 0$ *be such that* $\min\{\widetilde{\gamma}(q,r), \widetilde{\tau}(q,r) - 1\} \geq 0$. *Then*

- $\mathrm{H}^q(\iota_r) : \mathrm{H}^q_{r+1} \to \mathrm{H}^q_r$ *is an isomorphism, and*
- $\mathrm{H}^{q+1}(\iota_r) : \mathrm{H}^{q+1}_{r+1} \to \mathrm{H}^{q+1}_r$ *is an injection.*

We are going to prove the main claim by induction on q. We take $q \in [q_0 - 1]$ as the base case of the induction, which is covered by the initial condition (and note that $\widetilde{\gamma}(q,r) = \widetilde{\tau}(q,r) = \infty$.)

For the induction step, take $q \geq q_0$ with $\min\{\widetilde{\gamma}(q,r), \widetilde{\tau}(q,r) - 1\} \geq 0$, and as induction hypothesis that the main claim holds for any $q' < q$. We consider the spectral sequence $E^{\bullet,\bullet}_{\bullet}$ that the Proposition 5.10 associates to the pair $(G_{r+1}, (X_{r+1,\bullet}, \mu_{r+1,\bullet}))$, i.e. the one with

$$\begin{array}{l} E_1^{p',q'} = \mathrm{H}^{q'}_{\mathrm{cb}}(G_{r+1}; L^\infty(X_{r+1,p'-1})),\ \mathrm{d}_1^{p',q'} = \mathrm{H}^{q'}_{\mathrm{cb}}(G_{r+1}; \mathrm{d}^{p'-1}),\ \text{ for } p', q' \geq 0 \quad \text{and} \\ E_\infty^{q'} = 0 \ \text{ for } q' \in [\gamma(r+1)]. \end{array} \tag{6.11}$$

By the definition of $\widetilde{\gamma}$ and $\widetilde{\tau}$, the induction hypothesis is equivalent to saying that

$$\gamma(r+1-2(q-j)) \geq j \quad \text{and} \quad \tau(r+1-2(q-j)) \geq j+1 \tag{6.12}$$

for all $j \in \{q_0+1, \ldots, q\}$. For the value $j = q$, the inequalities in (6.12) read $\gamma(r+1) \geq q$ and $\tau(r+1) \geq q+1$. Thus, Lemma 6.14 applied to (6.11) gives

$$\begin{array}{rcll} E_1^{p',q'} & \cong & \mathrm{H}^{q'}_{r-p'+1} & \text{for all } p' \in [q+2] \text{ and all } q' \geq 0 \quad \text{and} \\ E_\infty^{q'} & = & 0 & \text{for all } q' \in [q+1]. \end{array} \tag{6.13}$$

Figure 6.1 below displays the arrangement of the terms and arrows in $E_1^{\bullet,\bullet}$ that are relevant to verify the main claim for q. Indeed, according to Proposition 6.16, the maps between these terms

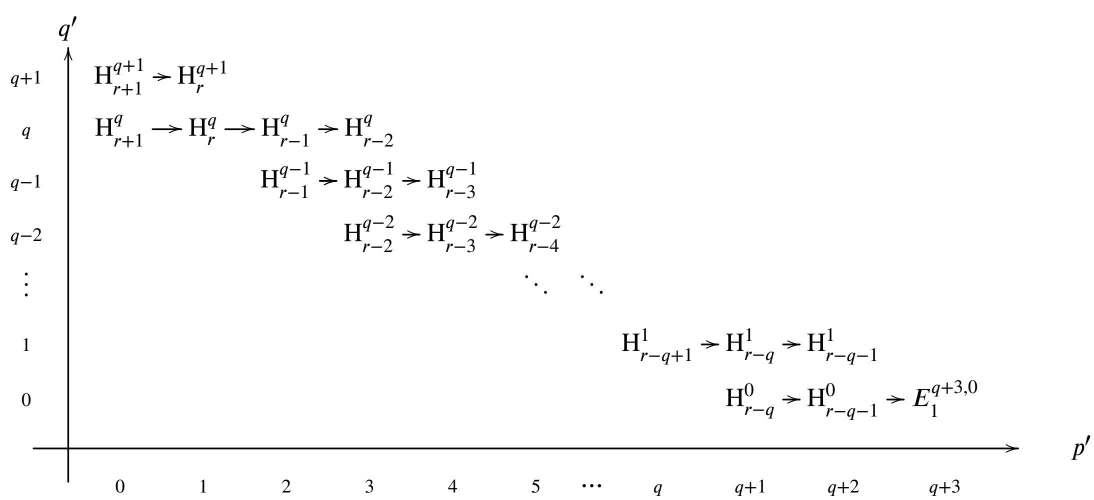


Figure 6.1: First page $E_1^{\bullet,\bullet}$.

are given as follows. The maps at the very left of the q-th and $(q+1)$-th rows are respectively

$$\mathrm{H}^q_{r+1} \xrightarrow{\mathrm{H}^q(\iota_r)} \mathrm{H}^q_r \quad \text{and} \quad \mathrm{H}^{q+1}_{r+1} \xrightarrow{\mathrm{H}^{q+1}(\iota_r)} \mathrm{H}^{q+1}_r, \tag{6.14}$$

that is, the very same maps we are interested in. For $q' \in \{1, \dots, q\}$, we consider the following two maps in the q'-th row:

$$E_1^{q-q'+1,q'} \to E_1^{q-q'+2,q'} \to E_1^{q-q'+3,q'} \tag{6.15}$$

If we set $p' := q - q' + 1$, then $p' \in \{1, \dots, q\}$ and our two relevant maps are given by

$$\begin{cases} \mathrm{H}^{q-p+1}_{r-p+1} \xrightarrow{0} \mathrm{H}^{q-p+1}_{r-p} \xrightarrow{\mathrm{H}^{q-p+1}(\iota_{r-p-1})} \mathrm{H}^{q-p+1}_{r-p-1} & \text{if } p \text{ is odd,} \\ \mathrm{H}^{q-p+1}_{r-p+1} \xrightarrow{\mathrm{H}^{q-p+1}(\iota_{r-p})} \mathrm{H}^{q-p+1}_{r-p} \xrightarrow{0} \mathrm{H}^{q-p+1}_{r-p-1} & \text{if } p \text{ is even.} \end{cases} \tag{6.16}$$

Finally consider the bottom row; here we have the maps

$$\begin{cases} \mathrm{H}^0_{r-q} \xrightarrow{\mathrm{H}^0(\iota_{r-q-1})} \mathrm{H}^0_{r-q-1} \to E_1^{q+3,0} & \text{if } q \text{ is odd,} \\ \mathrm{H}^0_{r-q} \xrightarrow{0} \mathrm{H}^0_{r-q-1} \to E_1^{q+3,0} & \text{if } q \text{ is even.} \end{cases} \tag{6.17}$$

From the arrangement of terms in Figure 6.1, (6.14), (6.16), and (6.17), one observes that the following assertions are sufficient to conclude the main claim for q:

(a) The map $\mathrm{H}^q(\iota_r)$ is injective,
(b) The map $\mathrm{H}^{q-p+1}(\iota_{r-p})$ is an isomorphism for $p \in \{2, \dots, q+1\}$ even,
(c) The map $\mathrm{H}^{q-p+1}(\iota_{r-p-1})$ is an injection for $p \in \{1, \dots, q\}$ odd, and
(d) The map $\mathrm{H}^0_{r-q-1} \cong E_1^{q+2,0} \to E_1^{q+3,0}$ is an injection for q even.

We will prove them below, making use of the induction hypothesis and re-writing the equations (6.12) adequately. Let us see how Theorem 6.10 follows from (a)-(d). If (b)-(d) hold true, then the second page has the form indicated in Figure 6.2 below. We included only the terms in the positions displayed in Figure 6.1 and placed asterisks at the positions among those that no longer carry relevant information to us. The zeros in the diagonal $p' + q' = q + 2$ are achieved in virtue of those three assertions. Figure 6.2 shows that the arrows emanating from the terms $E_2^{0,q+1}$ and $E_2^{1,q}$ end in two of those zeros, guaranteeing that $E_3^{0,q+1} \cong E_2^{0,q+1}$ and $E_3^{1,q} \cong E_2^{1,q}$. The rest of the zeros in that diagonal ensure that the same will happen at every further iteration, giving isomorphisms

$$E_\infty^{0,q+1} \cong \ker \mathrm{H}^{q+1}(\iota_r) \quad \text{and} \quad E_\infty^{1,q} \cong \mathrm{H}^q_r \,/\, \operatorname{im} \mathrm{H}^q(\iota_r).$$

Thus,

$$\ker \mathrm{H}^{q+1}(\iota_r) \oplus \mathrm{H}^q_r \,/\, \operatorname{im} \mathrm{H}^q(\iota_r) \hookrightarrow \bigoplus_{p'+q'=q+1} E_\infty^{p',q'} \cong E_\infty^{q+1} = 0,$$

where the last equality is in (6.13). This gives the injectivity of $\mathrm{H}^{q+1}(\iota_r)$ and the surjectivity of $\mathrm{H}^q(\iota_r)$; the injectivity of the latter is the assertion (a). We conclude with the proofs of the statements (a)-(d).

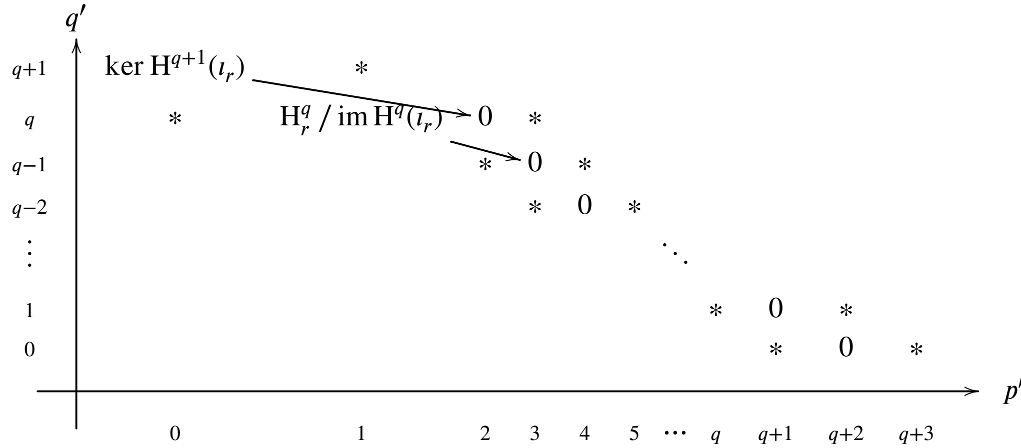


Figure 6.2: Second page $E_2^{\bullet,\bullet}$.

Proof of (a). This assertion will follow from the induction hypothesis once we show that $\min\{\widetilde{\gamma}(q-1,r), \widetilde{\tau}(q-1,r)-1\} \geq 0$, or equivalently, that

$$\gamma(r+1-2(q-1-j)) \geq j \quad \text{and} \quad \tau(r+1-2(q-1-j)) \geq j+1$$

for all $j \in \{q_0+1, \dots, q-1\}$ (if $q \leq q_0+1$, there is nothing to check.) Note, however, that for every such j we have by (6.12) that

$$\begin{aligned} \gamma(r+1-2(q-1-j)) &= \gamma(r+1-2(q-(j+1))) \geq j+1 > j \quad \text{and} \\ \tau(r+1-2(q-1-j)) &= \tau(r+1-2(q-(j+1))) \geq j+2 > j+1, \end{aligned}$$

since $j+1 \in \{q_0+2, \dots, q\} \subset \{q_0+1, \dots, q\}$.

Proof of (b). By the initial condition in Theorem 6.10, the map $H^{q-p+1}(\iota_{r-p})$ is an isomorphism for every $p \geq q-q_0+1$, so it remains to show the isomorphism for all even $p \in \{2, \dots, q-q_0\}$. This condition that $p \geq 2$ implies that $q-p+1$ is strictly less than q. Thus, the claim will follow from the induction hypothesis upon verifying that for every p as above, we have that $\min\{\widetilde{\gamma}(q-p+1, r-p), \widetilde{\tau}(q-p+1, r-p)-1\} \geq 0$, or in other words, that

$$\gamma(r-p+1-2(q-p+1-j)) \geq j \quad \text{and} \quad \tau(r-p+1-2(q-p+1-j)) \geq j+1$$

for all $j \in \{q_0+1, \dots, q-p+1\}$. Note that by (6.12) we have the inequalities

$$\begin{aligned} \gamma(r-p+1-2(q-p+1-j)) &= \gamma\left(r+1-2\big(q-(j-1+p/2)\big)\right) \geq j-1+p/2 \geq j, \\ \tau(r-p+1-2(q-p+1-j)) &= \tau\left(r+1-2\big(q-(j-1+p/2)\big)\right) \geq j+p/2 \geq j+1, \end{aligned}$$

given that $p/2$ is an integer and that

$$j-1+p/2 \in \{q_0+p/2, \dots, q-p/2\} \subset \{q_0+1, \dots, q\}.$$

Proof of (c). In virtue of the initial condition, as in the proof of (b), it suffices to show that the map $\mathrm{H}^{q-p+1}(\iota_{r-p-1})$ is injective for every odd $p \in \{1, \ldots, q-q_0\}$. Note that $q-p < q$ for all such p, so (c) follows from the induction hypothesis if $\min\{\widetilde{\gamma}(q-p, r-p-1), \widetilde{\tau}(q-p, r-p-1)-1\} \geq 0$ holds. This translates into the collection of inequalities

$$\gamma(r-p-2(q-p-j)) \geq j \quad \text{and} \quad \tau(r-p-2(q-p-j)) \geq j+1$$

with $j \in \{q_0+1, \ldots, q-p\}$. Again by (6.12), we obtain

$$\begin{aligned}\gamma(r-p-2(q-p-j)) &= \gamma\left(r+1-2\big(q-(j+(p-1)/2)\big)\right) \geq j+(p-1)/2 \geq j \quad \text{and}\\ \tau(r-p-2(q-p-j)) &= \tau\left(r+1-2\big(q-(j+(p-1)/2)\big)\right) \geq j+(p+1)/2 \geq j+1,\end{aligned}$$

given that $(p-1)/2$ is an integer and

$$j-(p-1)/2 \in \{q_0+(p+1)/2, \ldots, q-(p+1)/2\} \subset \{q_0+1, \ldots, q\}.$$

Proof of (d). This assertion requires a slightly different argument. Recall that by Proposition 5.10 and Example 1.5, for any $q' \geq 0$,

$$E_1^{q',0} \cong \mathrm{H}^0_{\mathrm{cb}}(G_{r+1}; L^\infty(X_{r+1,q'-1})) \cong L^\infty(X_{r+1,q'-1})^{G_{r+1}}. \tag{6.18}$$

Setting $j = q$ in (6.12), we have that $\tau(r+1) \geq q+1$, which implies that G_{r+1} acts transitively on $X_{r+1,q+1}$. Thus, for $q' = q+1$ the right-hand side of (6.18) is isomorphic to $\mathbb{R}$, and we have the commutative diagram

$$\begin{array}{ccc} E_1^{q+2,0} \cong L^\infty(X_{r+1,q+1})^{G_{r+1}} & \xrightarrow{\mathrm{H}^0_{\mathrm{cb}}(G_{r+1};\delta^i)=\delta^i} & L^\infty(X_{r+1,q+2})^{G_{r+1}} \cong E_1^{q+3,0} \\ \uparrow\cong & \nearrow & \\ \mathbb{R} & & \end{array}$$

for every $i \in [q+2]$, where both arrows emanating from $\mathbb{R}$ are the coefficient inclusion; the vertical arrow is an isomorphism due to the transitivity of the G_{r+1}-action on $X_{r+1,q+1}$. Thus, for q even, the differential $\mathrm{H}^0_{\mathrm{cb}}(G_{r+1}, \mathrm{d}^{q+1}) = \sum_{i=0}^{q+2}(-1)^i\delta^i$ is conjugated to the coefficient inclusion, and is therefore injective. □

6.3 Stability along $(\mathrm{GL}_r(\mathscr{k}))_r$

Consider the ascending infinite sequence of general linear groups over a local field $\mathscr{k}$

$$1 < \mathrm{GL}_1(\mathscr{k}) < \mathrm{GL}_2(\mathscr{k}) < \mathrm{GL}_3(\mathscr{k}) < \cdots$$

where $\iota_r : \mathrm{GL}_r(\mathscr{k}) \hookrightarrow \mathrm{GL}_{r+1}(\mathscr{k})$ is the block inclusion, say, of $\mathrm{GL}_r(\mathscr{k})$ in the bottom-right corner. We abbreviate during this subsection $G_r := \mathrm{GL}_r(\mathscr{k})$ for any $r \in \mathbb{Z}$, where we adopt the convention that $\mathrm{GL}_r(\mathscr{k})$ is trivial for $r \leq 0$. For every $r \in \mathbb{N}_{>0}$, let also $\mathcal{P}_r := \mathbb{P}(\mathscr{k}^r) = \mathbb{P}^{r-1}(\mathscr{k}) = (\mathscr{k}^r \smallsetminus \{0\})/\mathscr{k}^\times$ be the $(r-1)$-dimensional projective space, and let $\mathcal{P}_r$ be a one-point space for $r \leq 0$. The group G_r acts continuously and transitively on $\mathcal{P}_r$, so $\mathcal{P}_r$ is a homogeneous space. As such, it admits a unique G_r-invariant measure class that comes from the Haar measure on $\Bbbk$. Let μ_r be any probability measure in this class, so that $(\mathcal{P}_r, \mu_r)$ is a regular G_r-space.

Fix $r \in \mathbb{N}$. By Lemmas 5.14 and 5.15, the product G_r-complex $(\mathcal{P}_r^{\bullet+1}, \mu_r^{\otimes \bullet+1})$ is admissible and measurably ∞-connected. Furthermore, we define for $k \in \mathbb{N}$ the open subset

$$X_{r,k} := \{(p_0, \dots, p_k) \in \mathcal{P}_r^{k+1} \mid \dim \operatorname{span}(p_I) = \min\{|I|, r\} \text{ for all } I \subset [k]\}, \tag{6.19}$$

of $\mathcal{P}_r^{k+1}$, where for a subset of indices $I \subset [k]$, we set $p_I := \{p_i \mid i \in I\}$. Note that for $k \in [r-1]$, the space $X_{r,k}$ consists of $(k+1)$-tuples of linearly independent lines in $\Bbbk^r$. That $X_{r,k}$ has full measure in $\mathcal{P}_r^{k+1}$ for $k \in \mathbb{N}$, that linear independence is preserved by face maps δ_i, and that the group G_r acts transitively on $X_{r,k}$ for all $k \in [r-1]$, are all basic facts.

We determine the stabilizers for these transitive G_r-actions. Let $\mathcal{E} = \{e_1, \dots, e_r\}$ denote be the standard basis of $\Bbbk^r$, and for $k \in [r-1]$, let $o_{r,k} \in X_{r,k}$ be the tuple $([e_1], \dots, [e_{k+1}])$. If we set $H_{r,k} := \operatorname{stab}_{G_r}(o_{r,k})$, then

$$H_{r,k} = \left\{ \left(\begin{array}{c|c} D & U \\ \hline & A \end{array} \right) \in G_r \;\middle|\; \begin{array}{l} D \in G_{k+1} \text{ diagonal}, A \in G_{r-k-1}, \\ U \in M_{(k+1)\times(r-k-1)}(\Bbbk) \end{array} \right\}. \tag{6.20}$$

This implies, in particular, that the group G_r acts transitively on $X_{r,r}$, too. Indeed, note that any tuple $(p_0, \dots, p_r) \in X_{r,r}$ is in the G_r-orbit of $(o_{r,r-1}, p) \in X_{r,r}$, where the point $p \in \mathcal{P}_r$ is such that every one of its representatives has all coordinates non-zero because of the defining condition of $X_{r,r}$. The stabilizer $H_{r,r-1}$ equals the diagonal subgroup of G_r, and this one can send any such p to $[e_\Sigma]$, where $e_\Sigma = \sum_{i=1}^r e_i$. Let $o_{r,r} = (o_{r,r-1}, [e_\Sigma])$, and note that $H_{r,r} := \operatorname{stab}_{G_r}(o_{r,r})$ is

$$H_{r,r} = \{\lambda I_r \mid \lambda \in \Bbbk^\times\}. \tag{6.21}$$

We have shown that $(X_{r,\bullet}, \mu_r^{\otimes \bullet+1})$ is r-transitive, and so, that $(\mathcal{P}_r^{\bullet+1}, \mu_r^{\otimes \bullet+1})$ is essentially r-transitive. Note that our choices of $o_{r,k}$ are in concordance with the notation for base points from the Definition 6.3 of the τ-descent property, so the stabilizers in (6.20) and (6.21) are arranged in a descending chain

$$H_{r,0} > H_{r,1} > \cdots > H_{r,r}.$$

We will prove now:

Lemma 6.17. *The pair $(G_r, (X_{r,\bullet}, \mu_r^{\otimes \bullet+1}))_{r\in\mathbb{N}}$ is an infinite measured Quillen family with parameters $\gamma \equiv \infty$ and $\tau(r) := r$.*

Proof. It remains to be shown that $(G_r)_{r\in\mathbb{N}}$ and $(X_{r,\bullet}, \mu_r^{\otimes \bullet+1})_{r\in\mathbb{N}}$ are τ-measurably compatible. Set for all $r \in \mathbb{N}$ and $k \in [r]$ the projection $\pi_{r,k} : H_{r,k} \twoheadrightarrow G_{r-k-1}$,

$$\left(\begin{array}{c|c} D & U \\ \hline & A \end{array} \right) \mapsto A,$$

where G_{-1} is taken to be trivial. Each $\pi_{r,k}$ has solvable (hence amenable) kernel, consisting matrices where D and A are identity matrices. This establishes the τ-descent property. Fix now

$r \in \mathbb{N}$. For the property (I) of Definition 6.5, observe that for $k \in [r-2]$ and $i \in [k+1]$,

$$\delta_i(o_{r,k+1}) = ([e_1], \dots, \widehat{[e_{i+1}]}, \dots, [e_{k+2}]) = w_{r,k,i}\, o_{r,k},$$

with $w_{r,k,i} \in \mathcal{N}_{G_r}(H_{r,k+1})$ being the block matrix $I_i \oplus M_{k,i} \oplus I_{r-k-2}$, and

$$M_{k,i} := \left(\begin{array}{c|c} 0 & 1 \\ \hline I_{k-i} & 0 \end{array}\right) \in G_{k+2-i},$$

which is a permutation matrix. As for $k = r-1$, the normalizer $\mathcal{N}_{G_r}(H_{r,r})$ is the whole of G_r, so the existence of $w_{r,r,i}$ such that $\delta_i(o_{r,r}) = w_{r,r,i}\, o_{r,r}$ for every $i \in [r]$ is evident. After letting $\sigma_{r,k+1} : G_{r-k-2} \to H_{r,k+1}$ be defined as $\sigma_{r,k+1}(A) := I_{k+2} \oplus A$ for all $k \le r-1$, the fact that the diagram (6.4) in property (II) commutes is an easy check. □

Theorem 6.9 and the previous lemma imply that $\mathrm{H}^\bullet_{\mathrm{cb}}$ is stable along the sequence $(G_r)_{r\in\mathbb{N}}$. The stability range can be computed using Theorem 6.10. We have $\gamma(r) = \infty$, $\tau(r) = r$ and we can assume that $q_0 = 2$, given that $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{GL}_r(\Bbbk))$ vanishes for every $r \ge 0$ (see [13, Lemma 6.1].) In the notation of Theorem 6.10, $\widetilde{\gamma}(q,r) = \infty$ and $\widetilde{\tau}(q,r) = r - 2q + 3$ for $q \ge 2$. In consequence:

Theorem 6.18. *For $r \in \mathbb{N}$ and $q \ge 3$, the map $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_{r+1}(\Bbbk)) \to \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_r(\Bbbk))$ induced by the inclusion $\mathrm{GL}_r(\Bbbk) \hookrightarrow \mathrm{GL}_{r+1}(\Bbbk)$ is an injection if $r \ge 2q-4$, and an isomorphism if $r \ge 2q-2$.*

Proof. For $q' \ge 2$, we have $\min\{\widetilde{\gamma}(q',r), \widetilde{\tau}(q',r) - 1\} = r - 2q' + 2 \ge 0$ if and only if $r \ge 2q' - 2$. Theorem 6.10 applied to q' and r, and then a substitution $q := q' + 1$ finish the proof. □

In degree three, Theorem 6.18 can be used as a computational tool in the way discussed in the Introduction. In fact, it gives rise to the following sequence of induced maps:

$$\cdots \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_5(\Bbbk)) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_4(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_3(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk)). \tag{6.22}$$

The flawed stability range from [61] yielded almost the same sequence, with the sole difference that the penultimate map would have been an isomorphism. However, we still get the next corollary. It is part of [61, Theorem 1.2] in the case of $\Bbbk = \mathbb{R}$, and [11, Corollary 5] for non-Archimedean $\Bbbk$.

Corollary 6.19. *If $\Bbbk \ne \mathbb{C}$, then $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_r(\Bbbk))$ vanishes for all $r \ge 0$.*

Proof. It is a consequence of (6.22) and the fact that $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk)) = 0$, which is [15, Corollary 5.3] for $\Bbbk = \mathbb{R}$, and [11, Corollary 2] for non-Archimedean $\Bbbk$. □

6.4 Stability along $(\mathrm{SL}_r(\Bbbk))_r$

A first approach to prove stability for the family of special linear groups over a local field $\Bbbk$,

$$1 < \mathrm{SL}_1(\Bbbk) < \mathrm{SL}_2(\Bbbk) < \mathrm{SL}_3(\Bbbk) < \cdots$$

is to consider the block inclusions $\mathrm{SL}_r(\Bbbk) \hookrightarrow \mathrm{SL}_{r+1}(\Bbbk)$. While it is true that for every $r \in \mathbb{N}$, the pair $(X_{r,\bullet}, \mu_r^{\otimes \bullet+1})$ from Section 6.3 is a measurably ∞-connected, $(r-1)$-transitive, admissible measured $\mathrm{SL}_r(\mathscr{k})$-complex, the measurable compatibility of $(\mathrm{SL}_r(\mathscr{k}), (X_{r,\bullet}, \mu_{r,\bullet}))_r$ does not hold; this is due to the fact that the obvious diagrams (6.4) from property (II) do not commute.

The strategy in [61] is to consider rather for a fixed $r \in \mathbb{N}$ the finite ascending sequence

$$G_0 < G_1 < \cdots < G_{r-1} < G_r < G_{r+1},$$

with $G_s := \mathrm{GL}_s(\mathscr{k})$ for $s \in [r]$ and $G_{r+1} := \mathrm{SL}_{r+1}(\mathscr{k})$, where $\iota_r : G_r \hookrightarrow G_{r+1}$ is

$$A \mapsto \left(\begin{array}{c|c} (\det A)^{-1} & \\ \hline & A \end{array}\right)$$

and all the previous maps ι_s are the block inclusions. The following lemma implies the subsequent stability theorem.

Lemma 6.20. *The sequence $(G_s, (X_{s,\bullet}, \mu_s^{\otimes \bullet+1}))_{s\in[r+1]}$ is a finite measured Quillen family with parameters $\gamma \equiv \infty$ and $\tau : [r+1] \to \mathbb{N}$ given by*

$$\tau(s) = \begin{cases} s & \text{if } s \in [r] \\ r & \text{if } s = r+1 \end{cases}$$

Proof. Admissibility for every s, and the property (MQ1) of Definition 6.5 were proven in Subsection 5.3. The property (MQ2) for the function τ as above is elementary. For (MQ3), it suffices to make the same choice from Lemma 6.17 of stabilizers $H_{s,k} := \mathrm{stab}_{G_s}(o_{s,k})$, and projections $\pi_{s,k} : H_{s,k} \twoheadrightarrow G_{s-k-1}$ with sections $\sigma_{s,k}$, for $s \in [r+1]$, $k \in [\tau(s)]$, and for $s \in [r]$ and appropriate k and i, also of the group elements $w_{s,k,i}$. For $s = r+1$, $k \in [r]$, the stabilizers are of the form

$$H_{r+1,k} = \left\{ \left(\begin{array}{c|c} D & U \\ \hline & A \end{array}\right) \in \mathrm{GL}_{r+1}(\mathscr{k}) \;\middle|\; \begin{array}{l} D \in G_{k+1} \text{ diagonal, } A \in G_{r-k}, \\ U \in M_{(k+1)\times(r-k)}(\mathscr{k}),\ \det(D)\cdot\det(A) = 1 \end{array} \right\}.$$

In order to prove the property (I) from Definition 6.5 for $s = r+1$, we choose the elements $w_{r+1,k,i} \in N_{G_{r+1}}(H_{r+1,k+1})$ to be block matrices $I_i \oplus M'_{k,i} \oplus I_{s-k-2}$, where this time $M'_{k,i}$ is

$$\left(\begin{array}{c|c} 0 & (-1)^{k+1-i} \\ \hline I_{k-i} & 0 \end{array}\right) \in \mathrm{SL}_{k+2-i}(\mathscr{k}),$$

to ensure that $w_{r+1,k,i}$ has determinant one. □

Theorem 6.21. *For $r \geq 2$ and $q \geq 3$, the map $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_{r+1}(\mathscr{k})) \to \mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_r(\mathscr{k}))$ induced by the inclusion $\mathrm{GL}_r(\mathscr{k}) \hookrightarrow \mathrm{SL}_{r+1}(\mathscr{k})$ is an injection if $r \geq 2q-4$, and an isomorphism if $r \geq 2q-2$.*

Proof. We may apply Theorem 6.10 with initial condition $q_0 = 2$, given that $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{GL}_s(\mathscr{k})) = 0$ for all $s \in \mathbb{N}$ (see [13, Lemma 6.1]), and that $\mathrm{H}^2_{\mathrm{cb}}(\mathrm{SL}_{r+1}(\mathscr{k})) = 0$ for all $r \geq 2$, since they are all of non-Hermitian type (see Theorem 1.34 above.) The induced map in the statement of the theorem is an isomorphism if $\tau(r+1-2(q-j)) \geq j+1$ for every $j = 2, \ldots, q$. If $j < q$, this inequality is equivalent to $r \geq 2q - j$. If $j = q$, it is equivalent to $r \geq q + 1$. Thus, for the isomorphism to hold, we need

$$r \geq \max\left\{ \max_{j=2}^{q-1} (2q - j), q + 1 \right\} = \max\{2q - 2, q + 1\} = 2q - 2,$$

because $q \geq 3$. The induced map is an injection if $\tau(r + 1 - 2(q - 1 - j)) \geq j + 1$ for every $j = 2, \ldots, q - 1$. Analogously, observe that this is equivalent to the condition $r \geq 2q - 4$. □

Again in degree three, we have a corollary. Combining Theorem 6.21 and Theorem 6.18 for $q = 3$, we obtain that the inclusions induce the following maps in cohomology, and hence, the subsequent statement.

$$\begin{aligned}
&\text{For } r \geq 5: \quad \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\Bbbk)) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_4(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_3(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk)).\\
&\text{For } r = 4: \quad \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_4(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_3(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk)).\\
&\text{For } r = 3: \quad \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_3(\Bbbk)) \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk)).
\end{aligned} \tag{6.23}$$

Corollary 6.22. *Let* $r \in \mathbb{N}$. *Then* $\dim \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C})) \leq 1$. *Furthermore, for* $\Bbbk \neq \mathbb{C}$, *the cohomology group* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_r(\Bbbk))$ *vanishes.*

Proof. The statements for $r \in \{0, 1\}$ are trivial. For $r = 2$, see the references given in Subsection 1.5.3. If $r \geq 3$, then the isomorphisms or inclusions in (6.23) complete the proof, since

(i) the inclusion $\mathrm{SL}_2(\mathbb{C}) \hookrightarrow \mathrm{GL}_2(\mathbb{C})$ induces an isomorphism $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\mathbb{C})) \cong \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_2(\mathbb{C}))$ (see [59, Corollary 8.5.5]), and the latter cohomology group is one-dimensional; and

(ii) $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{GL}_2(\Bbbk))$ vanishes for any local field $\Bbbk \neq \mathbb{C}$ (see Corollary 6.19 above). □

Remark 6.23. For $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$, we have recovered Theorem 1.2 and Remark 3.5 from [61]. The non-Archimedean setting is covered only for $r \geq 4$ by the stability results in [61], for the injection for $r = 3$ as in (6.23) requires assuming that $\Bbbk^\times/(\Bbbk^\times)^3 = 1$. We have removed this assumption. The gain is ultimately a consequence of the use of the spectral sequence from Proposition 5.10 to prove Theorem 6.10; see Remark 5.11.

In order to conclude that $\mathrm{H}^\bullet_{\mathrm{cb}}$ stabilizes along $(\mathrm{SL}_r(\Bbbk))_r$, one could consider the inclusions $\mathrm{SL}_r(\Bbbk) \hookleftarrow \mathrm{GL}_{r-1}(\Bbbk) \hookrightarrow \mathrm{GL}_r(\Bbbk) \hookrightarrow \mathrm{SL}_{r+1}(\Bbbk)$ from the previous and current sections. Combining Theorem 6.18 and Theorem 6.21, one easily derives:

Corollary 6.24. *For* $r \geq 2$ *and* $q \geq 3$, *there is an isomorphism* $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_{r+1}(\Bbbk)) \cong \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_r(\Bbbk))$ *whenever* $r \geq 2q - 1$. □

Corollary 6.24 sacrifices some of the range information from Theorem 6.21, and the fact that the isomorphisms are induced by inclusions. In the particular case of $\Bbbk = \mathbb{C}$, however, we do have:

Corollary 6.25. *For $r \in \mathbb{N}$ and $q \geq 3$, the map $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_{r+1}(\mathbb{C})) \to \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$ induced by $\mathrm{SL}_r(\mathbb{C}) \hookrightarrow \mathrm{SL}_{r+1}(\mathbb{C})$ is an injection if $r \geq 2q-4$, and an isomorphism if $r \geq 2q-2$.*

Proof. The diagram of inclusions

$$\begin{array}{ccc} \mathrm{GL}_r(\mathbb{C}) & \overset{\iota_r}{\hookrightarrow} & \mathrm{SL}_{r+1}(\mathbb{C}) \\ \uparrow & \nearrow & \\ \mathrm{SL}_r(\mathbb{C}) & & \end{array}$$

commutes. The vertical arrow induces an isomorphism $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{GL}_r(\mathbb{C})) \cong \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$ for every q; this is [59, Corollary 8.5.5], which relies itself on the fact that $\mathbb{C}^\times/(\mathbb{C}^\times)^r = 1$ for all r. It follows from this fact and Theorem 6.18 that the induced map $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_{r+1}(\mathbb{C})) \to \mathrm{H}^q_{\mathrm{cb}}(\mathrm{SL}_r(\mathbb{C}))$ is an isomorphism for all q and all r as in the statement. □

6.5 Stability along $(\mathrm{Sp}_{2r}(\Bbbk))_r$ for $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$

Let $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$, and recall the infinite ascending sequence (2.3)

$$1 < \mathrm{Sp}_2(\Bbbk) < \mathrm{Sp}_4(\Bbbk) < \mathrm{Sp}_6(\Bbbk) < \cdots$$

from Subsection 2.1.2, which are the inclusions $\iota_r : \mathrm{Sp}_{2r}(\Bbbk) \hookrightarrow \mathrm{Sp}_{2r+2}(\Bbbk)$ are defined by

$$A \mapsto \left(\begin{array}{c|c|c} 1 & & \\ \hline & A & \\ \hline & & 1 \end{array}\right). \tag{6.24}$$

We prove the following bounded-cohomological stability result, which is a quantitative version of Theorem A from the Introduction. Let us abbreviate $\widehat{\gamma}(q) := 2^q + \lceil (q+1)/2 \rceil$ for all $q \in \mathbb{N}$.

Theorem 6.26. *For $r \geq 0$ and $q \geq 3$, the map $\mathrm{H}^q_{\mathrm{cb}}(\mathrm{Sp}_{2r+2}(\Bbbk)) \to \mathrm{H}^q_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\Bbbk))$ induced by the inclusion $\mathrm{Sp}_{2r}(\Bbbk) \hookrightarrow \mathrm{Sp}_{2r+2}(\Bbbk)$ is a injection if $r \geq \widehat{\gamma}(q-1)-1$, and an isomorphism whenever $r \geq \widehat{\gamma}(q)-1$.*

It is a consequence of the next theorem.

Theorem 6.27. *For $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$ there exists a measured Quillen family $(\mathrm{Sp}_{2r}(\Bbbk), (X_{r,\bullet}, \mu_{r,\bullet}))$ with parameters $\tau(r) = r-1$ and $\gamma(r) = \sup\{q \in \mathbb{N} \mid \widehat{\gamma}(q) \leq r\}$.*

The measured semi-simplicial complexes that we construct and fulfill the statement of the theorem above are called *measured Stiefel complexes*, and will be the center of Chapter 7. Theorem A from the Introduction follows immediately from Theorem 6.9 and Theorem 6.27.

Remark 6.28. The function $\widehat{\gamma} : \mathbb{N} \to \mathbb{N}$ is strictly increasing. Moreover, the equivalence

$$\widehat{\gamma}(q) \leq r \quad \Longleftrightarrow \quad q \leq \gamma(r)$$

holds: the sufficiency of the left-hand side is just the definition of γ; its necessity follows from the monotonicity of $\widehat{\gamma} : \mathbb{N} \to \mathbb{N}$. Note also that $\gamma : \mathbb{N} \to \mathbb{N} \cup \{-\infty\}$ is an increasing function, yet not strictly, and that $\gamma(r) < r$ for all $r \in \mathbb{N}$. Asymptotically, $\gamma(r) \sim \log_2 r$ as $r \to \infty$.

Proof of Theorem 6.26 modulo Theorem 6.27. As explained in Remark 6.11, we may take $q_0 = 2$ in Theorem 6.10. Fix $q \geq 3$, and observe that $\widetilde{\tau}(q, r) = r - 2q + 2$. Therefore, we have

$$\widetilde{\tau}(q, r) - 1 \geq 0 \quad \Longleftrightarrow \quad r \geq 2q - 1. \tag{6.25}$$

On the other hand, by Remark 6.28, for $j \in \{2, \dots, q\}$,

$$\gamma\big(r + 1 - 2(q - j)\big) - j \geq 0 \quad \Longleftrightarrow \quad r \geq 2q - 1 + (\widehat{\gamma}(j) - 2j)$$

Thus,

$$\widetilde{\gamma}(q, r) \geq 0 \quad \Longleftrightarrow \quad r \geq 2q - 1 + \max_{j=2}^{q} (\widehat{\gamma}(j) - 2j). \tag{6.26}$$

Since the function $\{2, \dots, q\} \ni j \mapsto \widehat{\gamma}(j) - 2j$ is strictly increasing, we have $\max_{j=2}^{q} (\widehat{\gamma}(j) - 2j) = \widehat{\gamma}(q) - 2q$. As a result of the equivalences (6.25) and (6.26), we deduce from Theorem 6.10 that the desired isomorphism holds if $r \geq \max\{2q - 1, \widehat{\gamma}(q) - 1\} = \widehat{\gamma}(q) - 1$. Analogously, observe that the injectivity statement holds if $r \geq \max\{2q - 3, \widehat{\gamma}(q - 1) - 1\} = \widehat{\gamma}(q - 1) - 1$. □

Chapter 7

Stiefel Complexes

Let (V, ω) be an n-dimensional formed space over the field $\Bbbk \in \{\mathbb{R}, \mathbb{C}\}$; assume further that ω is indefinite if it is either orthogonal with $\Bbbk = \mathbb{R}$ or unitary with $\Bbbk = \mathbb{C}$. As observed in Subsection 2.1.3, the automorphism group $G := \mathrm{Aut}(V, \omega)$ is a non-compact classical Lie group: either a symplectic, an orthogonal, or a unitary one. In Section 7.1, we will introduce the so-called *Stiefel complex* $(X_\bullet, \mu_\bullet)$ associated to (V, ω) and endow it with the structure of a measured G-object, however, with a non-canonical choice of probability measures on the skeleta.

In Section 7.2, we will consider the inclusions

$$\left(\Bbbk^d, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+2}, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+4}, \omega_{T,\varepsilon d}\right) \hookrightarrow \left(\Bbbk^{d+6}, \omega_{T,\varepsilon d}\right) \hookrightarrow \dots \tag{7.1}$$

of standard formed spaces as in (2.1), and of their automorphism groups, say

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \cdots . \tag{7.2}$$

Let $(X_{r,\bullet}, \mu_{r,\bullet})$ denote the Stiefel complex of $(\Bbbk^{d+2r}, \omega_{T,\varepsilon d})$. Our final goal is to prove stability of continuous bounded cohomology along the family $(G_r)_r$. This would follow from Theorem 6.9 if we showed that $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_r$ is a measured Quillen family. The goal of Section 7.2 is to establish the properties (MQ2) and (MQ3) for $(G_r, X_{r,\bullet})_r$ from Definition 6.8, which correspond to the increasing transitivity and measurable compatibility of $(G_r, (X_{r,\bullet}, \mu_{r,\bullet}))_r$.

From Section 7.3 to 7.5, we go back to studying a single formed space (V, ω) and the complex $X_\bullet$ under the special assumption that ω is an alternating bilinear form. In Section 7.4, we will construct a concrete family of probability measures $\mu_\bullet$ on the skeleta of $X_\bullet$, so that $(X_\bullet, \mu_\bullet)$ is an admissible measured G-object. We will then complete the proof of Theorem 6.27, by establishing the measurable connectivity statement. Section 7.5 concerns the proof of the measurable connectivity of $(X_\bullet, \mu_\bullet)$, and consequently, of property (MQ1) for the family $(G_r, X_{r,\bullet})_r$. In both proofs of admissibility and of measurable connectivity, a central element will be the so-called *symplectic perpendicular measures*, which we introduce in Section 7.3.

Finally, in Section 7.6, we will comment on the difficulties in generalizing the concepts introduced in the symplectic concepts to spaces with general sesquilinear forms.

7.1 Stiefel varieties and Stiefel complexes

We recall that for $l \in [r-1]$ the *isotropic Grassmannian* of type l is defined by

$$\mathcal{G}_l(V,\omega) := \{W \in \mathrm{Gr}_{l+1}(V) \mid W \text{ is totally isotropic}\}, \tag{7.3}$$

and is a compact homogeneous G-space. We abbreviate $\mathcal{G}_l := \mathcal{G}_l(V,\omega)$, and set $\mathcal{P} := \mathcal{G}_0$ and $\mathcal{G} := \bigsqcup_l \mathcal{G}_l$. For any $k \in [r-1]$, we define the spaces

$$\bar{X}_k := \{(p_0,\dots,p_k) \in \mathcal{P}^{k+1} \mid \mathrm{span}(p_0,\dots,p_k) \in \mathcal{G}\}, \quad \text{and} \tag{7.4}$$

$$X_k := \{(p_0,\dots,p_k) \in \mathcal{P}^{k+1} \mid \mathrm{span}(p_0,\dots,p_k) \in \mathcal{G}_k\}. \tag{7.5}$$

The space $\bar{X}_k$ is a closed (hence compact) subspace of the product space $\mathcal{P}^{k+1}$ for every $k \in \mathbb{N}$. Furthermore, X_k is an open, dense subset of $\bar{X}_k$ for all $k \in [r-1]$. Note that both $\bar{X}_k$ and X_k admit continuous G-actions by restricting the diagonal action on $\mathcal{P}^{k+1}$, and obvious $\mathfrak{S}_{k+1}$-actions. By Corollary 2.4, the G-action on X_k is transitive for all $k \in [r-1]$. Also, by Corollary 2.11, the map

$$\mathrm{span} : X_k \to \mathcal{G}_k, \quad (p_0,\dots,p_k) \mapsto \mathrm{span}(p_0,\dots,p_k)$$

is a continuous, G-equivariant, $\mathfrak{S}_{k+1}$-invariant surjection. In fact, X_k is a fiber bundle over $\mathcal{G}_k$ via this map. In analogy to the classical Stiefel varieties, we introduce the following terminology:

Definition 7.1. For any $k \in [r-1]$, the homogeneous space X_k is called the *k-Stiefel variety* of (V,ω), and $\bar{X}_k$ is called the *compactified* k-Stiefel variety of (V,ω).

Remark 7.2. Compactified Stiefel varieties make sense also for integers $k \geq r$ exactly as defined above. We could define Stiefel varieties as well for these values of k by considering tuples of lines in V such that every subcollection of r lines spans a maximal isotropic subspace of V. We will not need to consider this extension for the purposes of this thesis.

For a fixed k and any $i \in [k]$, the face maps $\delta_i : \mathcal{P}^{k+1} \to \mathcal{P}^k$ that delete the i-th component in a $(k+1)$-tuple restrict to continuous, G-equivariant maps $\bar{X}_{k+1} \to \bar{X}_k$ and $X_{k+1} \to X_k$ that we also denote by δ_i. Equipped with these face maps, $\bar{X}_\bullet$ and $X_\bullet$ are semi-simplicial objects in the category of topological G-spaces. Note that $X_\bullet$ may be also regarded as a measured G-object in the sense of Definition 5.1: Indeed, for any $k \in [r-1]$, the Stiefel variety X_k is a homogeneous G-space, and as such, it is equipped with a unique G-invariant Borel measure class; choose an arbitrary probability measure μ_k on X_k in that measure class.

Definition 7.3. We call a measured G-object $(X_\bullet, \mu_\bullet)$ a *Stiefel complex* of (V,ω).

We stress the usage of the indefinite article in the definition of *a* Stiefel complex, which is supposed to hint on its dependence of the choices of probability measures on each of its skeleta. We will see below that an instance in which that choice becomes relevant is in the matter of the admissibility.

7.2 Transitivity and measurable compatibility

Consider the inclusions (7.1) of standard formed spaces $(\Bbbk^{d+2r}, \omega_{T,\varepsilon d})$, and the corresponding ones (7.2) of their automorphism groups G_r. Given $r \geq 0$, we denote by $X_{r,\bullet}$ a Stiefel complex associated to $(\Bbbk^{d+2r}, \omega_{T,\varepsilon d})$. By Corollary 2.4, this G_r-object is $(r-1)$-transitive. We claim that $(G_r)_r$ and $(X_{r,\bullet})_r$ are τ-measurably compatible with $\tau(r) = r - 1$.

Indeed, consider the action of G_r on $X_{r,k}$ for $r \geq 1$ and $k \in [r-1]$. Fix an adapted basis $(e_r, \dots, e_1, h_1, \dots, h_d, f_1, \dots, f_r)$ of $(\Bbbk^{d+2r}, \omega_{T,\varepsilon d})$ (see Subsection 2.1.2). We can choose as a base point in $X_{r,k}$ the tuple $o_{r,k} = ([e_r], \dots, [e_{r-k}])$. A computation shows that the stabilizer of $o_{r,k}$ is given by

$$H_{r,k} = \left\{ \begin{pmatrix} D & * & * \\ & A & * \\ & & Q_{k+1} D^{-1} Q_{k+1} \end{pmatrix} \;\middle|\; \begin{array}{l} D = \operatorname{diag}(\lambda_r, \dots, \lambda_{r-k}) \in \mathrm{GL}_{k+1}(\Bbbk) \\ A \in G_{r-k-1} \end{array} \right\} < G_r,$$

where Q_{k+1} is the $(k+1) \times (k+1)$-matrix with only 1's on its antidiagonal, and the asterisks correspond to entries conditioned so that the matrix is in G_r. It follows from the form of the matrices that $H_{r,k+1} < H_{r,k}$ for $k < r - 1$. Moreover, we have surjective homomorphisms $H_{r,k} \twoheadrightarrow G_{r-k-1}$,

$$\begin{pmatrix} D & * & * \\ & A & * \\ & & Q_{k+1} D^{-1} Q_{k+1} \end{pmatrix} \mapsto A.$$

with solvable (hence amenable) kernel. This establishes the τ-descent property. Property (II) of Definition 6.5 is obvious. For property (I), observe that if $k \in [r-2]$ and $i \in [k+1]$, then

$$\delta_i(o_{r,k+1}) = ([e_r], \dots, \widehat{[e_{r-i}]}, \dots, [e_{r-k-1}]) = w_{r,k,i}\, o_{r,k},$$

with $w_{r,k,i} \in N_{G_r}(H_{r,k+1})$ being the block matrix $R_{r,k,i} \oplus I_{d+2(r-k-2)} \oplus Q_{k+2} R_{r,k,i}^{-1} Q_{k+2}$, with

$$R_{r,k,i} := I_i \oplus M_{k,i} \quad \text{and} \quad M_{k,i} := \left(\begin{array}{c|c} 0 & 1 \\ \hline I_{k-i} & 0 \end{array} \right) \in \mathrm{GL}_{k+2-i}(\Bbbk).$$

At this point we have established Theorem 6.27 for all families of Stiefel complexes of formed spaces, except for their measurable connectivity (Q1). We will prove now the fact that the Stiefel complex associated with a *symplectic vector space* of rank r is $\gamma(r)$-connected.

7.3 Perpendicular measures and their existence in the symplectic case

From now until the end of the Section 7.5, we let ω be a non-degenerate, alternating bilinear form on V, so that the pair (V, ω) is of type I, according to our notation of Subsection 2.1.2. Note that in this case $n = 2r$. We have that $G = \mathrm{Aut}(V, \omega) \cong \mathrm{Sp}(2r, \Bbbk)$ is the symplectic group. We also fix an inner product $\langle \cdot, \cdot \rangle$ on V and denote by $K(V)$ the maximal compact subgroup of

$\mathrm{GL}(V)$ that preserves $\langle\cdot,\cdot\rangle$. We denote the induced norm and metric by $\|\cdot\|$ and d_V respectively.

Given a non-zero linear subspace $W \subset V$, we denote by B_W the intersection of W with the unit ball with respect to d_V. We denote by $\mathcal{L}_W$ the unique multiple of the Lebesgue measure on W normalized to $\mathcal{L}_W(B_W) = 1$. Finally we denote by $\mathbb{P}(W)$ the associated projective space and by $\pi_W : W \smallsetminus \{0\} \to \mathbb{P}(W)$ the quotient map. We then define a probability measure λ_W on $\mathbb{P}(W)$ by

$$\int_{\mathbb{P}(W)} f \,\mathrm{d}\lambda_W := \int_{B_W} \pi_W^* f \,\mathrm{d}\mathcal{L}_W, \quad f \in C(\mathbb{P}(W)).$$

We will consider λ_W as a measure on $\mathcal{P} = \mathbb{P}(V)$ supported on the compact subset $\mathbb{P}(W)$; in other words, we will identify the measure on $\mathbb{P}(W)$ defined by the formula above with its push-forward under the inclusion map $\mathbb{P}(W) \hookrightarrow \mathbb{P}(V)$. In this way, we obtain an assignment

$$\lambda : \mathrm{Gr}(V) \smallsetminus \big\{\{0\}\big\} \to \mathrm{Prob}(\mathcal{P}), \quad W \mapsto \lambda_W$$

where the space $\mathrm{Gr}(V) := \{W \mid W \text{ is a subspace of } V\}$ is equipped with the Chabauty topology, as in (2.9).

Remark 7.4. In group-theoretical terms, we can characterize the measures λ_W as follows. If $W \subset V$ is a linear subspace and $W^\perp$ denotes its orthogonal complement with respect to $\langle\cdot,\cdot\rangle$, then $V = W \oplus W^\perp$. Hence, we can extend every $g \in \mathrm{GL}(W)$ to an automorphism of V by acting identically on $W^\perp$, and thereby define an embedding $\mathrm{GL}(W) \hookrightarrow \mathrm{GL}(V)$. Under this embedding, the group $K(W) := \mathrm{GL}(W) \cap K(V)$ is the unique maximal compact subgroup of $\mathrm{GL}(W)$ preserving the inner product $\langle\cdot,\cdot\rangle|_{W\times W}$. Moreover, if $P(V,W) := \mathrm{Stab}_{\mathrm{GL}(V)}(W)$ denotes the set-stabilizer of W in $\mathrm{GL}(V)$, then

$$\mathrm{Stab}_{K(V)}(W) = P(V,W) \cap K(V) \cong K(W) \times K(W^\perp).$$

Now, $\mathbb{P}(W)$ is a homogeneous space of both $\mathrm{GL}(W)$ and $K(W)$, and λ_W is invariant under $K(W)$ by construction. It is thus the unique $K(W)$-invariant probability measure on $\mathbb{P}(V)$ that is supported on $\mathbb{P}(W)$; its measure class $[\lambda_W]$ is the unique $\mathrm{GL}(W)$-invariant measure class on $\mathbb{P}(W)$, and also invariant under $P(V,W)$.

Using Remark 7.4, we can establish the following properties of the map λ.

Lemma 7.5. *The map λ is continuous with respect to the weak-$*$-topology on* $\mathrm{Prob}(\mathcal{P})$*. Moreover, it is $K(V)$-equivariant and* $\mathrm{GL}(V)$-quasi-equivariant*, i.e. for every $g \in \mathrm{GL}(V)$ and every $W \in \mathrm{Gr}(V)$, the probability measures $g_*\lambda_W$ and λ_{gW} are mutually absolutely continuous.*

Proof. We first establish the $K(V)$-equivariance and $\mathrm{GL}(V)$-quasi-equivariance of λ. Let $W \in \mathrm{Gr}(V)$. For $k \in K(V)$, the probability measure $k_*\lambda_W$ is supported on kW and invariant under $kK(W)k^{-1} = K(kW)$, hence coincides with λ_{kW} by uniqueness. This shows that λ is $K(V)$-equivariant. Now, if $g \in \mathrm{GL}(V)$, there exists $k \in K(V)$ such that $gW = kW$ and hence $gk^{-1} \in P(V,kW)$, so $g_*\lambda_W = (gk^{-1})_* k_* \lambda_W = (gk^{-1})_*\lambda_{kW}$. Since λ_{kW} is $P(V,kW)$-quasi-invariant, because of the last equality, $g_*\lambda_W$ is equivalent to $\lambda_{kW} = \lambda_{gW}$, showing that the action is $\mathrm{GL}(V)$-quasi-equivariant.

For the continuity of λ, in view of (2.9), it suffices to establish continuity for each of the restrictions $\lambda_i := \lambda|_{\mathrm{Gr}_{i+1}(V)}$, where $i \in [n-1]$. For this purpose, fix i and let $(W_n)_n$ be a sequence in $\mathrm{Gr}_{i+1}(V)$ converging to a subspace $W \in \mathrm{Gr}_{i+1}(V)$. Since $K(V)$ acts transitively on $\mathrm{Gr}_{i+1}(V)$, we find $k_n \in K(V)$ with $W_n = k_n W$, and we may assume by passing to a subsequence that k_n converges to some $k \in K(V)$. By continuity of the action, we then have $kW = \lim k_n W = W$, i.e. $k \in \mathrm{Stab}_{K(V)}(W) = K(W) \times K(W^\perp)$, and hence $k_* \lambda_W = \lambda_W$. We deduce that

$$\lim_{n\to\infty} \lambda_{W_n} = \lim_{n\to\infty} \lambda_{k_n W} = \lim_{n\to\infty} (k_n)_* \lambda_W = k_* \lambda_W = \lambda_W,$$

where the second-to-last equality follows from the continuity of the $\mathrm{GL}(V)$-action on $\mathrm{Prob}(\mathcal{P})$. $\square$

In probabilistic language, λ_W is the distribution of a *random point* $p \in \mathcal{P}$ subject to the condition that $p \in \mathbb{P}(W)$. Similarly, the following definition describes the distribution of a “random symplectic perpendicular”, i.e. a random point that is perpendicular with respect to ω to a given finite set of points in $\mathcal{P}$. As in Subsection 2.1.1, given a subset $S \subset V$, we write S^ω for the symplectic complement of S with respect to ω.

Definition 7.6. Given an integer $k \in [2r-2]$, we define

$$\nu^k : \mathcal{P}^{k+1} \to \mathrm{Prob}(\mathcal{P}), \quad (p_0, \dots, p_k) \mapsto \nu^k_{(p_0,\dots,p_k)} := \lambda_{\mathrm{span}(p_0,\dots,p_k)^\omega}.$$

The measure $\nu^k_{(p_0,\dots,p_k)}$ is called *perpendicular measure to* $(p_0, \dots, p_k)$.

The restriction of the definition to $k \le 2r-2$ is important, since it may happen for for $k > 2r-2$ that $\mathrm{span}(p_0, \dots, p_k) = V$ and hence $\mathrm{span}(p_0, \dots, p_k)^\omega = \{0\}$. The following proposition summarizes basic properties of the map ν^k. Here we recall our abbreviation $G := \mathrm{Sp}(V, \omega)$, and let $K := G \cap K(V)$, a maximal compact subgroup of G.

Proposition 7.7. *For every $k \in [2r-2]$, the map ν^k is an $\mathfrak{S}_{k+1}$-invariant, K-equivariant and G-quasi-equivariant Borel map. In particular, for every $l \in [k+1]$, it is continuous on the subset $\mathcal{P}^{k+1}_l := \{(p_0, \dots, p_k) \mid \dim \mathrm{span}\{p_0, \dots, p_k\} = l\} \subset \mathcal{P}^{k+1}$.*

Proof. Observe that ν^k can be written as the composition

$$\mathcal{P}^{k+1} \xrightarrow{\mathrm{span}} \mathrm{Gr}(V) \setminus \{V\} \xrightarrow{(-)^\omega} \mathrm{Gr}(V) \setminus \big\{\{0\}\big\} \xrightarrow{\lambda} \mathrm{Prob}(\mathcal{P}). \tag{7.6}$$

Now, the map $\mathrm{span} : \mathcal{P}^{k+1} \to \mathrm{Gr}(V)$ is Borel (and continuous on each $\mathcal{P}^{k+1}_l$) by Lemma 2.13, and clearly $\mathfrak{S}_{k+1}$-invariant and $\mathrm{GL}(V)$-equivariant; the symplectic polarity $(-)^\omega : \mathrm{Gr}(V) \to \mathrm{Gr}(V)$ is G-equivariant and continuous by Lemma 2.15; and λ is continuous, K-invariant and G-equivariant by Lemma 7.5. $\square$

Remark 7.8. From now on, unless specificity is necessary, we will avoid making the index k explicit and refer to all the maps ν^k simply as ν and write $\nu_{p_0,\dots,p_n} := \nu^k_{(p_0,\dots,p_k)}$. Note that the latter only depends on the set $\{p_0, \ldots, p_k\}$. By construction, $\nu_{p_0,\ldots,p_k}$ is supported on the subspace $\mathbb{P}(\operatorname{span}(p_0, \ldots, p_k)^\omega)$ of $\mathcal{P}$ and thus

$$\omega(p, p_j) = 0 \quad \text{for all } j \in \{0, \ldots, k\} \text{ and } \nu_{p_0,\ldots,p_k}\text{-almost all } p \in \mathcal{P}. \tag{7.7}$$

Because of this, we refer to a random variable distributed according to the measure $\nu_{p_0,\ldots,p_n}$ as a *random perpendicular* to $p_0, \ldots, p_n$. The following lemma ensures that generically a random perpendicular to $(p_0, \ldots, p_k)$ is linearly independent of $(p_0, \ldots, p_k)$.

Lemma 7.9. *For all $k \in [r-2]$ and all $(p_0, \ldots, p_k) \in X_k$, the Borel set $\{p \in \mathcal{P} \mid (p_0, \ldots, p_k, p) \in X_{k+1}\}$ has full $\nu_{p_0,\ldots,p_k}$-measure.*

Proof. Let $W := \operatorname{span}(p_0, \ldots, p_k)$ and let $A := \{p \in \mathcal{P} \mid (p_0, \ldots, p_k, p) \in X_{k+1}\}$. Then we have that $W \subsetneq W^\omega$, since equality holds only in the Lagrangian case. Moreover, the equality $A = \mathbb{P}(W^\omega) \setminus \mathbb{P}(W)$ holds, and as a positive-codimension, closed embedded submanifold of $\mathbb{P}(W^\omega)$, the projective subspace $\mathbb{P}(W)$ is Borel and a Lebesgue null set. Hence,

$$\nu_{p_0,\ldots,p_k}(A) = \lambda_{W^\omega}(\mathbb{P}(W^\omega) \setminus \mathbb{P}(W)) = 1. \qquad \square$$

7.4 Measures for symplectic Stiefel complexes

We are going to define recursively probability measures μ_k on the compactified Stiefel varieties $\bar{X}_k$ for all $k \in [r-1]$, using perpendicular measures.

Definition 7.10. For $k = 0$, we define $\mu_0 := \lambda_V$, a probability measure on $\bar{X}_0 = X_0 = \mathbb{P}(V) = \mathcal{P}$. Given $k = 1, \ldots, r-1$, we set

$$\int_{\bar{X}_{k+1}} f(\mathbf{p}, p_{k+1})\, \mathrm{d}\mu_{k+1}(\mathbf{p}, p_{k+1}) := \int_{\bar{X}_k} \left(\int_{\mathcal{P}} f(\mathbf{p}, p_{k+1})\, d\nu_{\mathbf{p}}(p_{k+1}) \right) \mathrm{d}\mu_k(\mathbf{p}) \quad (f \in C(\bar{X}_{k+1})).$$

The formula above defines a measure μ_{k+1} on $\bar{X}_{k+1}$, since the inner integral defines a Borel function $\bar{X}_k \to \mathbb{R}$ by Proposition 7.7 and Lemma 4.1.

Proposition 7.11. *For every $k \in [r-1]$, the measure μ_k restricts to a G-quasi-invariant probability measure on X_k.*

Proof. Since μ_0 is G-quasi-invariant and ν is G-quasi-equivariant by Proposition 7.7, it follows by induction that the measures $\mu_1, \ldots, \mu_{r-1}$ are all G-quasi-invariant. Now, to show that $\mu_k(X_k) = 1$, we argue by induction on k. For $k = 0$, the lemma follows from $\bar{X}_0 = X_0$. For $k \in \{1, \ldots, r-1\}$, we assume as an induction hypothesis that $\mu_k(X_k) = 1$. Then

$$\mu_{k+1}(X_{k+1}) = \int_{X_k} \nu_{\mathbf{p}}\big(\{p_{k+1} \in \mathcal{P} \mid (\mathbf{p}, p_{k+1}) \in X_{k+1}\}\big)\, \mathrm{d}\mu_k(\mathbf{p}).$$

By Lemma 7.9, the integrand equals to 1 for every fixed $(p_0, \ldots, p_k) \in X_k$, and $\mu_{k+1}(X_{k+1}) = \mu_k(X_k) = 1$ as claimed. $\square$

In the sequel, we will say that *the* Stiefel complex associated to the symplectic vector space (V,ω) is the measured G-object $(X_\bullet, \mu_\bullet)$, where the G-quasi-invariant probability measures μ_k are the ones constructed above.

7.4.1 Admissibility of Stiefel complexes.

The purpose of this subsection is to establish admissibility of the symplectic Stiefel complexes:

Proposition 7.12. *The Stiefel complex $(X_\bullet, \mu_\bullet)$ associated to a symplectic vector space (V,ω) is an admissible G-object.*
We rely on the symmetry of the measures μ_k as in the next lemma.

Lemma 7.13. *The measure μ_k is symmetric, i.e. invariant under the action of the symmetric group $\mathfrak{S}_{k+1}$ on X_k by permuting the coordinates.*

In turn, Lemma 7.13 is a consequence of the co-area formula, Theorem 7.14, and we delay its proof to Subsection 7.4.2.

Proof of Proposition 7.12 modulo Lemma 7.13. In order to prove the admissibility of the Stiefel complex $X_\bullet$, we show that for every $i \in [k]$, the maps $\delta^i : L^\infty(X_k, \mu_k) \to L^\infty(X_{k+1}, \mu_{k+1})$ induced by the face maps δ_i are weak-$*$ continuous. Since $L^\infty(X_k, \mu_k) = L^\infty(\bar{X}_k, \mu_k)$, we can as well work with the compactified Stiefel varieties $\bar{X}_k$. For any function $\phi \in \mathscr{L}^1(\bar{X}_{k+1}, \mu_{k+1})$, define

$$\partial_i \phi\,(p_0, \dots, p_k) := \int_{\mathcal{P}} \phi(p_0, \dots, p_{i-1}, p, p_i, \dots, p_k)\, \mathrm{d}\nu_{p_0,\dots,p_k}(p) = \nu_{p_0,\dots,p_k}\big(\phi \circ \tau_i(p_0, \dots, p_k, \cdot)\big)$$

where τ_i is the cycle $(k, k-1, \dots, i+1, i) \in \mathfrak{S}_{k+2}$. Then $\partial_i \phi \in \mathscr{L}^1(\bar{X}_k, \mu_k)$: Indeed, $\partial_i\phi$ is measurable by Proposition 7.7 and Lemma 4.1, and

$$\begin{aligned}\|\partial_i\phi\|_1 &= \int_{\bar{X}_k} |\partial_i\phi|\, \mathrm{d}\mu_k \le \int_{\bar{X}_k}\int_{\mathcal{P}} |\phi\circ\tau_i(p_0,\dots,p_k,p)|\, \mathrm{d}\nu_{p_0,\dots,p_k}(p)\, \mathrm{d}\mu_k(p_0,\dots,p_k)\\ &= \int_{\bar{X}_{k+1}} |\phi\circ\tau_i|\, \mathrm{d}\mu_{k+1} = \int_{\bar{X}_{k+1}} |\phi|\, \mathrm{d}\mu_{k+1} = \|\phi\|_1 < \infty,\end{aligned}$$

where the second-to-last equality is Lemma 7.13. Clearly $\partial_i : \mathscr{L}^1(\bar{X}_{k+1}, \mu_{k+1}) \to \mathscr{L}^1(\bar{X}_k, \mu_k)$ is a linear map, and it descends to a bounded linear operator $\partial_i : L^1(\bar{X}_{k+1}, \mu_{k+1}) \to L^1(\bar{X}_k, \mu_k)$ by a computation similar to the one above.

We show that ∂_i is pre-dual to δ^i. Indeed, if $f \in L^\infty(\bar{X}_k, \mu_k)$, $\phi \in L^1(\bar{X}_{k+1}, \mu_{k+1})$, and $(-|-)$ denotes the dual pairing, then

$$\begin{aligned}(f \mid \partial_i\phi) &= \int_{\bar{X}_k} f(p_0,\dots,p_k)\cdot\left(\int_{\mathcal{P}} \phi(p_0,\dots,p_{i-1},p,p_i,\dots,p_k)\, \mathrm{d}\nu_{p_0,\dots,p_k}(p)\right) \mathrm{d}\mu_k(p_0,\dots,p_k)\\ &= \int_{\bar{X}_k}\int_{\mathcal{P}} \big(((f\circ\delta_i)\cdot\phi)\circ\tau_i\big)(p_0,\dots,p_k,p)\, \mathrm{d}\nu_{p_0,\dots,p_k}(p)\, \mathrm{d}\mu_k(p_0,\dots,p_k)\\ &= \int_{\bar{X}_{k+1}} \big(\delta^i f\cdot\phi\big)\circ\tau_i\, \mathrm{d}\mu_{k+1} = \int_{\bar{X}_{k+1}} \delta^i f\cdot\phi\, \mathrm{d}\mu_{k+1} = (\delta^i f \mid \phi),\end{aligned}$$

where the second-to-last equality holds due to Lemma 7.13 once again. In conclusion, we have that the coboundary $\mathrm{d} : L^\infty(X_k, \mu_k) \to L^\infty(X_{k+1}, \mu_k)$ is dual to the morphism $\partial = \sum_{i=0}^{k}(-1)^i \partial_i$, hence weak-$*$ continuous. This establishes the admissibility of the Stiefel complex. □

7.4.2 Proof of Lemma 7.13. We state now the version of the co-area formula that will suffice for our purposes, and then derive from it the missing Lemma 7.13. It is a simplified version of [30, Theorem 3.2.22].

Theorem 7.14. *Let $W \subset \mathbb{R}^{d_1}$ resp. $Z \subset \mathbb{R}^{d_2}$ be C^1-submanifolds of dimension m resp. n, with $m \geq n$, and let $f : W \to Z$ be a C^1-map. Then for almost every $z \in Z$, the level set $f^{-1}(z)$ is a C^1-submanifold of W of dimension $m - n$, and if $g : W \to \mathbb{R}$ is a Borel function, then*

$$\int_W g \cdot J_m f \; \mathrm{d}\mathcal{H}^m = \int_Z \int_{f^{-1}(z)} g \; \mathrm{d}\mathcal{H}^{m-n} \; \mathrm{d}\mathcal{H}^n(z).$$

Here W (resp. Z) is equipped with the restricted Euclidean metric from $\mathbb{R}^{d_1}$ (resp. from $\mathbb{R}^{d_2}$.), and the corresponding Hausdorff measures. The notation $J_m f$ is used for the *generalized Jacobian*, as in [30, Section 3.2]: It is defined as

$$(J_m f)(w) := \|\Lambda^m D_w f\|,$$

where $D_w f : T_w W \to T_{f(w)} Z$ is the derivative of f at w, the map $\Lambda^m D_w f$ is the extension to m-th exterior products (see [30, Section 1.3]), and $\|\cdot\|$ the ℓ_2-norm. Given any matrix representative of $D_w f$, the Jacobian can be computed as the ℓ^2-norm of the vector consisting of all maximal minors of the matrix.

Now, let us fix a subspace W of our symplectic vector space (V, ω) of dimension $d \in \{1, \dots, 2r\}$. Let $\mathrm{Rad}(W) := W \cap W^\omega$ denote the radical of $\omega|_{W \times W}$. It is a positive-codimension subspace of W unless W is an isotropic subspace of V, in which case $\mathrm{Rad}(W) = W$. For every $p \in \mathbb{P}(W)$ we have $p \subset W \cap p^\omega$, hence $W \cap p^\omega \neq \{0\}$. Thus, we may define maps

$$s_W : \mathbb{P}(W) \to \mathrm{Gr}(V) \setminus \{\{0\}\}, \; p \mapsto W \cap p^\omega, \quad \text{and} \quad t_W : \mathbb{P}(W) \longrightarrow \mathrm{Prob}(\mathcal{P}), \; p \mapsto \lambda_{W \cap p^\omega}.$$

Note that the map s_W is continuous by Lemma 2.14 and Lemma 2.15, and hence $t_W = \lambda \circ s_W$ is continuous by Lemma 7.5. Consider now the subspace $X_W \subset \mathbb{P}(W)^2$ given by

$$X_W := \{(p, q) \in \mathbb{P}(W)^2 \mid \mathrm{span}(p, q) \in \mathcal{G}\}.$$

We define two probability measures μ_1, μ_2 on X_W by

$$\mu_1(f) := \int_{\mathcal{P}} \int_{\mathcal{P}} f(p, q) \; \mathrm{d}\lambda_{W \cap p^\omega}(q) \; \mathrm{d}\lambda_W(p) \text{ and } \mu_2(f) := \int_{\mathcal{P}} \int_{\mathcal{P}} f(p, q) \; \mathrm{d}\lambda_{W \cap q^\omega}(p) \; \mathrm{d}\lambda_W(q).$$

To see that these are well-defined, one has to check that the inner integrals are Borel functions of their corresponding free variables. For this we observe that the map $p \mapsto W \cap p^\omega$ is continuous by Lemma 2.14 and Lemma 2.15, and hence the inner integrals are measurable by Proposition 7.7 and Lemma 4.1. We are going to show:

Lemma 7.15. *The two probability measures μ_1 and μ_2 on X_W coincide.*

Note that if $W \subset V$ is isotropic, then $X_W = \mathbb{P}(W)^2$ and $\lambda_{W\cap p^\omega} = \lambda_W$, hence Lemma 7.15 reduces to Fubini's theorem. The non-trivial case of Lemma 7.15 is thus the case where W is non-isotropic, and hence $\mathrm{Rad}(W) \subsetneq W$. In this case we are going to use the following Fubini-type theorem, which follows from the co-area formula. We denote by $\pi_1, \pi_2 : W \times W \to W$ the canonical projections, and given a subset $E \subset W \times W$ and $w \in W$ we denote by $E_w^{(j)}$ the π_j-fiber or E over w.

Lemma 7.16. *Let $W^0 \subset W$ be an open subset, and let E be a C^1 codimension-one submanifold of $W^0 \times W^0$ such that E projects surjectively onto both factors, and such that for all $(v, w) \in E$, the tangent space $T_{(v,w)}E \subset W \times W$ projects surjectively onto both factors. Then for all $h \in C_c(E)$ one has*

$$\int_{W^0} \int_{E_w^{(1)}} h(v,w)\, \mathrm{d}\mathcal{H}^{d-1}(v)\, \mathrm{d}\mathcal{H}^d(w) = \int_{W^0} \int_{E_v^{(2)}} h(v,w)\, \mathrm{d}\mathcal{H}^{d-1}(w)\, \mathrm{d}\mathcal{H}^d(v)$$

Proof. For all $(v, w) \in W \times W$, there exist bases of $T_{(v,w)}(W \times W)$, $T_v W$ and $T_w W$ such that $D\pi_j(v, w) = (I_d\, 0_d)$. The assumption on E implies that for all $(v, w) \in E$ one has $D(\pi_j|_E) = (\mathbf{1}_d\, 0_{d-1})$, and thus, $J_d(\pi_j|_E) = \|\Lambda^d D(\pi_j|_E)(v, w))\| = 1$. Then, the co-area formula implies that both sides are equal to the integral of h against $\mathcal{H}^{2d-1}$ on E, given that the Hausdorff measure is invariant under the flip $(v, w) \mapsto (w, v)$, which is an isometry. □

Proof of Lemma 7.15. We may assume that W is non-isotropic. This implies that $W^0 := W \setminus \mathrm{Rad}(W)$ is a dense open subset of W, and in particular $(W^0, \mathcal{H}^d) \cong (W, \mathcal{L}_W)$ as measure spaces. We now consider

$$E := \{(v, w) \in W^0 \times W^0 \mid \omega(v, w) = 0\} \subset W^0 \times W^0.$$

If $v \in W^0$, then $v \notin W^\omega$, hence $\omega(v, w) \neq 0$ for w in a dense subset of W; in particular, we can choose $w \in W^0$. This shows that $\pi_1(E) = W^0$ and similary $\pi_2(E) = W^0$.

If $(v, w) \in E$, then $D\omega(v, w)(X, Y) = \omega(v, Y) - \omega(w, X)$, and since $v, w \notin W^\omega$, neither this map nor either of its summands is zero. This implies, firstly, that $D\omega(v, w)$ has full rank; hence, $E \subset W \times W$ is a smooth codimension-one submanifold. Secondly, for all $(v, w) \in E$, the tangent space

$$T_{(v,w)}E = \{(X, Y) \in W \times W \mid \omega(v, Y) = \omega(w, X)\}$$

projects surjectively onto both factors. We deduce that Lemma 7.16 applies. Given $(v, w) \in E$, the coresponding fibers are given by

$$E_w^{(1)} = (W \cap w^\omega) \setminus W^\omega \quad \text{and} \quad E_v^{(2)} = (W \cap v^\omega) \setminus W^\omega$$

Since E has dimension $2d - 1$, these fibers are $(d - 1)$-dimensional. On the other hand, since $v, w \notin W^\omega$, the vector spaces $W \cap w^\omega$ and $W \cap v^\omega$ are proper linear subspaces of W, hence of dimension $d-1$. It follows that W^ω intersects these vector spaces in positive codimension, and hence

$$(E^{(1)}_w, \mathcal{H}^{d-1}) = (W \cap w^\omega, \mathcal{L}_{W\cap w^\omega}) \quad \text{and} \quad (E^{(1)}_v, \mathcal{H}^{d-1}) = (W \cap v^\omega, \mathcal{L}_{W\cap v^\omega}).$$

We conclude that for all $h \in C_c(E)$ one has

$$\int_W \int_{W\cap w^\omega} h(v,w)\,\mathrm{d}\,\mathcal{L}_{W\cap w^\omega}(v)\,\mathrm{d}\,\mathcal{L}_W(w) = \int_W \int_{W\cap v^\omega} h(v,w)\,\mathrm{d}\,\mathcal{H}_{W\cap v^\omega}(w)\,\mathrm{d}\,\mathcal{L}_W(v).$$

If we denote by $\pi : W \setminus \{0\} \to \mathbb{P}(W)$ the canonical projection and choose $h(v,w) := \chi_{B_W}(v)\chi_{B_W}(w) f(\pi(v),\pi(w))$ for some $f \in C(X_W)$, then unravelling definitions, we see that the left-hand side equals $\mu_2(f)$ and the right-hand side equals $\mu_1(f)$. □

We can now finish the proof of Lemma 7.13, and thereby of Proposition 7.12.

Proof of Lemma 7.13. We argue by induction on k. For $k = 0$, there is nothing to show; for $k = 1$, we can apply Lemma 7.15 with $W = V$ to obtain for every $f \in C(\bar{X}_1)$ that

$$\begin{aligned}\int_{\bar{X}_1} f(p_0,p_1)\,\mathrm{d}\mu_1(p_0,p_1) &= \int_{\bar{X}_0}\int_{\mathcal{P}} f(p_0,p_1)\,\mathrm{d}\nu_{p_0}(p_1)\,\mathrm{d}\mu_0(p_0)\\ &= \int_{\bar{X}_0}\int_{\mathcal{P}} f(p_0,p_1)\,\mathrm{d}\nu_{p_1}(p_0)\,\mathrm{d}\mu_0(p_1) \quad = \quad \int_{\bar{X}_1} f(p_0,p_1)\,\mathrm{d}\mu_1(p_1,p_0).\end{aligned}$$

Now assume that $k \geq 2$ and let $W := \operatorname{span}(p_0,\dots,p_{k-2})^\omega$. Then we have for $f \in C(\bar{X}_k)$ that

$$\int_{\bar{X}_k} f(p_0,\dots,p_k)\,\mathrm{d}\mu_k(p_0,\dots,p_k) = \int_{\bar{X}_{k-1}}\int_{\mathcal{P}} f(p_0,\dots,p_k)\,\mathrm{d}\nu_{p_0,\dots,p_{k-1}}(p_k)\,\mathrm{d}\mu_{k-1}(p_0,\dots,p_{k-1});$$

by the induction hypothesis, μ_k is invariant under all permutations of the variables $p_0,\dots,p_{k-1}$, and by Proposition 7.7, so is ν. Moreover, we have the chain of equalities

$$\begin{aligned}\int_{\bar{X}_k} f(p_0,\dots,p_k)\,\mathrm{d}\mu_k(p_0,\dots,p_k) &= \int_{\bar{X}_{k-1}}\int_{\mathcal{P}} f(p_0,\dots,p_k)\,\mathrm{d}\nu_{p_0,\dots,p_{k-1}}(p_k)\,\mathrm{d}\mu_{k-1}(p_0,\dots,p_{k-1})\\ &= \int_{\bar{X}_{k-2}}\int_{\mathcal{P}}\int_{\mathcal{P}} f(p_0,\dots,p_k)\,\mathrm{d}\nu_{p_0,\dots,p_{k-1}}(p_k)\,\mathrm{d}\nu_{p_0,\dots,p_{k-2}}(p_{k-1})\,\mathrm{d}\mu_{k-2}(p_0,\dots,p_{k-2})\\ &= \int_{\bar{X}_{k-2}}\int_{\mathcal{P}}\int_{\mathcal{P}} f(p_0,\dots,p_k)\,\mathrm{d}\lambda_{W\cap(p_{k-1})^\omega}(p_k)\,\mathrm{d}\lambda_W(p_{k-1})\,\mathrm{d}\mu_{k-2}(p_0,\dots,p_{k-2})\\ &= \int_{\bar{X}_{k-2}}\int_{\mathcal{P}}\int_{\mathcal{P}} f(p_0,\dots,p_k)\,\mathrm{d}\lambda_{W\cap(p_k)^\omega}(p_{k-1})\,\mathrm{d}\lambda_W(p_k)\,\mathrm{d}\mu_{k-2}(p_0,\dots,p_{k-2})\\ &= \int_{\bar{X}_k} f(p_0,\dots,p_k)\,\mathrm{d}\mu_k(p_0,\dots,p_{k-2},p_k,p_{k-1}),\end{aligned}$$

where the second-to-last one follows from Lemma 7.15. This shows that μ_k is invariant under the transposition $(k-1,k)$. Since this transposition and the copy of $\mathfrak{S}_k$ in $\mathfrak{S}_{k+1}$ that corresponds to permutations of the variables $p_0,\dots,p_{k-1}$ generate the symmetric group $\mathfrak{S}_{k+1}$, the conclusion follows. □

7.5 Measurable connectivity of the symplectic Stiefel complexes

In this section, we complete the proof of Theorem 6.27 by showing that the symplectic Stiefel complex $X_\bullet$ is measurably $\gamma(r)$-connected, i.e. that the cohomology of the L^∞-complex

$$0 \to \mathbb{R} \to L^\infty(X_0) \to L^\infty(X_1) \to \cdots \to L^\infty(X_{r-2}) \to L^\infty(X_{r-1}) \tag{7.8}$$

associated to the Stiefel complex vanishes at every degree $k \in [\gamma(r)]$. Set $q := \gamma(r)$; we will first consider the complex

$$0 \to \mathbb{R} \xrightarrow{\mathrm{d}^{-1}} \mathscr{L}^\infty(X_0) \xrightarrow{\mathrm{d}^0} \mathscr{L}^\infty(X_1) \to \cdots \to \mathscr{L}^\infty(X_q) \xrightarrow{\mathrm{d}^q} \mathscr{L}^\infty(X_{q+1}), \tag{7.9}$$

where $\mathrm{d}^{-1} : \mathbb{R} \to \mathscr{L}^\infty(X_1)$ denotes the inclusion of constants. We adopt the conventions that $\mathscr{L}^\infty(X_{-1}) := \mathbb{R}$, and $\mathscr{L}^\infty(X_{-k}) = 0$ and $\mathrm{d}^{-k} := 0$ for $k \geq 2$. We are going to construct bounded linear maps

$$h^k : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k)$$

for all $k \in \{-1, \dots, q\}$ in Definition 7.34 below. For $k \geq 2$, we set $h^{-k} := 0$.

Theorem 7.17. (i) *The maps $h^k : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k)$ from Definition 7.34 below satisfy the identity*

$$h^k\, \mathrm{d}^k + \mathrm{d}^{k-1}\, h^{k-1} = \mathrm{id}\,. \tag{7.10}$$

for all $k \in \{-1, \dots, q\}$. In particular, the complex (7.9) has trivial cohomology up to degree q.

(ii) *The bounded linear map h^k descends to $h^k : L^\infty(X_{k+1}) \to L^\infty(X_k)$ for all $k \in \{-1, \dots, q\}$. In particular, the complex (7.8) has trivial cohomology up to degree q.*

Recall that $\mathcal{P} = X_0 = \mathbb{P}(V)$ is the underlying set of points of the symplectic Stiefel complex $X_\bullet$. The following lemma reveals the reason why our argument for Theorem 7.17 works only for degrees $k \leq q$.

Lemma 7.18. *Let $I \subset \mathcal{P}$ be a subset of size at most $q+1$, and for every $J \subsetneq I$, let $t_J \in \mathcal{P}$ be given. Then the symplectic complement $(I \cup \{t_J \mid J \subsetneq I\})^\omega$ is non-trivial.*

Proof. If $i := |I| \leq q+1$, then the set $I \cup \{t_J \mid J \subsetneq I\}$ has cardinality at most $i+2^i-1 \leq q+2^{q+1}$. In particular, we have

$$\dim \operatorname{span}(I \cup (t_J \mid J \subsetneq I\})) \leq q + 2^{q+1} \leq 2\hat{\gamma}(q) - 1 \leq 2r - 1,$$

where the last inequality follows from the fact that $\hat{\gamma}(q) \leq r$; see Remark 6.28. We deduce that $\dim(I \cup \{t_J \mid J \subsetneq I\})^\omega \geq 2r - (2r-1) = 1$. □

7.5.1 Finding a formula for h^k. We have to guess a formula for homotopies $h^k : \mathscr{L}^\infty(X_k) \to \mathscr{L}^\infty(X_{k-1})$. To get some inspiration, we first consider a toy case.

Example 7.19. Let (X, μ) be a probability space and consider

$$0 \to \mathbb{R} \xrightarrow{d^{-1}} \mathscr{L}^\infty(X) \xrightarrow{d^0} \mathscr{L}^\infty(X^2) \xrightarrow{d^1} \mathscr{L}^\infty(X^3) \to \dots$$

To interpret these formulas geometrically, think of elements of X^k as $(k-1)$-simplices. If $\Delta \in X^k$ is such a $(k-1)$-simplex and $f \in \mathscr{L}^\infty(X^{k+1})$ is a function on k-simplices, then $h^k f(\Delta)$ is the expected value of the random variable $f(t(\Delta))$, where $t(\Delta)$ is a random k-simplex subject to the condition that Δ is the 0-th face of $t(\Delta)$. The distribution of the k-simplex $t(\Delta)$ is given by the product measure $\mu \otimes \delta_\Delta$, where $\delta_\Delta \in \mathrm{Prob}(X^k)$ is the Dirac measure at Δ.

We now try to argue similarly in our case of interest: the case of the Stiefel complex $X_\bullet$.

Remark 7.20 (Degree 0). We may choose again $h^{-1} : \mathscr{L}^\infty(X_0) \to \mathbb{R}$ to be the integral with respect to μ_0. The condition we derive for $h^0 : \mathscr{L}^\infty(X_1) \to \mathscr{L}^\infty(X_0)$ is

$$h^0(\mathrm{d}^0\varphi)(p_0) \overset{!}{=} \varphi(p_0) - d^{-1}h^{-1}\varphi(p_0) = \int_{X_0} \varphi(p_0) - \varphi(t)\ \mathrm{d}\mu_0(t).$$

The first attempt would be to rewrite the integrand $\varphi(p_0)-\varphi(t)$ as $\mathrm{d}^0\varphi(t, p_0)$, but this rewriting is illegal, since (t, p_0) is generically not an element of X_1. The correct way to rewrite the integrand is to observe that for every t_0 perpendicular to both t and p_0 we have

$$\varphi(p_0) - \varphi(t) = \varphi(p_0) - \varphi(t_0) + \varphi(t_0) - \varphi(t) = \mathrm{d}^0\varphi(t_0, p_0) - \mathrm{d}^0\varphi(t_0, t).$$

In particular, we can choose t_0 to be an auxiliary random perpendicular to t and p_0. Passing to the expectation then yields the condition

$$h^0(\mathrm{d}^0\,\varphi)(p_0) \overset{!}{=} \int_{X_0} \left(\int_{\mathcal{P}} \left(\mathrm{d}^0\varphi(t_0, p_0) - \mathrm{d}^0\varphi(t_0, t) \right) \mathrm{d}\nu_{t,p_0}(t_0) \right) \mathrm{d}\mu_0(t),$$

where ν_{t,p_0} denotes the perpendicular measure from Definition 7.6. Note that the function of t in brackets above is Borel measurable by Lemma 4.1, hence integrable. We may thus choose

$$h^0 f(p_0) := \int_{X_0} \int_{\mathcal{P}} \left(f(t_0, p_0) - f(t_0, t) \right) \mathrm{d}\nu_{t,p_0}(t_0)\ \mathrm{d}\mu_0(t).$$

If we consider t and t_0 as (dependent) random variables, then we can write this formula as $h^0 f(p_0) = \mathbb{E}(f(t_0, p_0) - f(t_0, t))$. If we continue to higher degrees, we have to choose more and more (mutually dependent) auxiliary random variables, and we need to introduce some form of bookkeeping device to keep track of the dependencies among these auxiliary random variable. This will lead us to the notion of a random chaining in the next subsection.

7.5.2 Random chainings.

Given $q \in \mathbb{N}$, we denote by $\mathcal{P}^{[\leq q+1]}$ the collection of all finite subsets of $\mathcal{P}$ of cardinality at most $q+1$.

Definition 7.21. A *random q-chaining* of $\mathcal{P}$ is a collection $t = \{t_I \mid I \in \mathcal{P}^{[\leq q+1]}\}$ of (mutually dependent) $\mathcal{P}$-valued random variables with the following properties:

(i) The variable $t_\emptyset$ is distributed according to μ_0.

(ii) If $I \in \mathcal{P}^{[\leq q+1]}$ and $p \in I$, then $\omega(t_I, p) = 0$ almost surely.

(iii) If $I \in \mathcal{P}^{[\leq q+1]}$ and $J \subseteq I$, then $\omega(t_I, t_J) = 0$ almost surely.

(iv) If $g \in G$ and $I \in \mathcal{P}^{[\leq q+1]}$, then the distributions of $t_{g.I}$ and $g.t_I$ are mutually absolutely continuous.

The existence of chainings on $\mathcal{P}$ will be treated in Proposition 7.32. Using a random q-chaining we can define $\bar{X}_n$-valued random values for every $n \in [q]$, as follows: Assume we are given a tuple $(p_0, \dots, p_n) \in \bar{X}_n$, and set $t_0 := t_{\{p_0\}}, t_{01} := t_{\{p_0,p_1\}}, \dots, t_{01\dots n} := t_{\{p_0,\dots,p_n\}}$. Then expressions like

$$(t_{01\dots n}, \dots, t_{01}, t_0, t_\emptyset),\ (t_{01\dots n}, \dots, t_{01}, t_0, p_0),\ (t_{01\dots n}, \dots, t_{01}, p_0, p_1),\ \text{etc.}$$

define $\bar{X}_{n+1}$-valued random variables. We will formalize this idea in Lemma 7.25 after setting up some notation.

Definition 7.22. Let $k \in \mathbb{N}$. Given $0 \leq n \leq k+1$, a *descending k-chain* of *length n* is a sequence of the form

$$C = (C_0 \supset C_1 \supset \cdots \supset C_n) \tag{7.11}$$

with $C_i \subset [k]$ and $|C_i| = k+1-i$. The sets C_j is called the j-th *component* of C and C_n is called its *final component*. We also define its *ordered final component* to be $\mathrm{ord}(C_n) := (i_0, \dots, i_{k-n})$ if $C_n = \{i_0, \dots, i_{k-n}\}$ and $i_0 < \cdots < i_{k-n}$.

In the sequel, we will denote by $\mathfrak{C}_k$, $\mathfrak{C}_k^\emptyset$ and $\mathfrak{C}_k^+$ the collection of all descending k-chains, all descending k-chains of length $k+1$ (i.e. of maximal length) and all descending k-chains of length at most k respectively.

Example 7.23. (i) Let $k = 0$. Then $\mathfrak{C}_0^+ = \{(\{0\})\}$ and $\mathfrak{C}_0^\emptyset = \{(\{0\} \supset \emptyset)\}$.

(ii) Let $k = 1$. Then $\mathfrak{C}_1^+$ contains the three chains $(\{0,1\})$, $(\{0,1\} \supset \{1\})$ and $(\{0,1\} \supset \{0\})$, and the last two can be extended uniquely into chains in $\mathfrak{C}_1^\emptyset$.

(iii) Let $k = 2$. Then $\mathfrak{C}_k^+$ consists of 10 chains given by

$$\begin{gathered}(\{0,1,2]),\ (\{0,1,2\} \supset \{1,2\}),\ (\{0,1,2\} \supset \{0,2\}),\ (\{0,1,2\} \supset \{0,1\}),\\ (\{0,1,2\} \supset \{1,2\} \supset 2\}),\ (\{0,1,2\} \supset \{1,2\} \supset \{1\}),\ (\{0,1,2\} \supset \{0,2\} \supset \{2\}),\\ (\{0,1,2\} \supset \{0,2\} \supset \{0\}),\ (\{0,1,2\} \supset \{0,1\} \supset \{1\}),\ (\{0,1,2\} \supset \{0,1\} \supset \{0\})\end{gathered}$$

Each of the last six chains can be uniquely prolonged into a chain in $\mathfrak{C}_k^\emptyset$.

Definition 7.24. Assume we are given a q-chaining t of $\mathcal{P}$ and an element $p = (p_0, \dots, p_k) \in \mathcal{P}^{k+1}$ for some $k \leq q$. Given $C \in \mathfrak{C}_k$ of length n with ordered final component $(i_0, \dots, i_{k-n})$, we define a $\mathcal{P}^{k+2}$-valued random variable $t(p, C)$ by

$$t(p, C) := (t_{p_{C_0}}, t_{p_{C_1}} \dots, t_{p_{C_n}}, p_{i_0}, \dots, p_{i_{k-n}}), \tag{7.12}$$

where for a subset $S \subset [k]$ we write $p_S := \{p_s \mid s \in S\}$.

Note that if $n = k+1$, then $C_n = C_{k+1} = \emptyset$, and (7.12) has to be understood as $t(p, C) := (t_{p_{C_0}}, \dots, t_{p_{C_n}})$.

Lemma 7.25. *For every $p = (p_0, \dots, p_k) \in \bar{X}_k$ and every k-chain C, we have $t(p, C) \in \bar{X}_{k+1}$ almost surely.*

Proof. We observe that $i_0, \dots, i_{k-n}$ are contained in each of the sets C_j, hence $p_{i_0}, \dots, p_{i_{k-n}}$ are perpendicular with respect to ω to every $t_{p_{C_j}}$ almost surely by property (iii) in Definition 7.21. Moreover, if $m < n$, then $C_n \subsetneq C_m$ and hence $\omega(t_{p_{C_n}}, t_{p_{C_m}}) = 0$ almost surely by property (ii). □

Definition 7.26. A random q-chaining is called *generic* if for all $k \le q$ and $(p_0, \dots, p_k) \in X_k$ and all k-chains C, we have $t(p, C) \in X_{k+1} \subset \bar{X}_{k+1}$ almost surely.

Remark 7.27 (Geometric interpretation). Assume that t is a generic random q-chaining and let $k \le q$. If we think of $p = (p_0, \dots, p_k) \in X_k$ as a k-simplex, then every $C \in \mathfrak{C}_k$ defines a random $(k+1)$-simplex $t(p, C) \in X_{k+1}$, and the following bullet points hold:

- If $C = ([k])$ has length 0, then $t(p, C)$ is a random $(k+1)$-simplex in $X_\bullet$ with base given by the k-simplex p and tip $t_{\{p_0, \dots, p_k\}}$.
- If we prolong a given chain $C = (C_0 \supset \cdots \supset C_n)$ to a chain $C' = (C_0 \supset \cdots \supset C_n \supset C_{n+1})$, then the random $(k+1)$-simplices $t(p, C)$ and $t(p, C')$ have a common face.

Example 7.28. Based on the previous remark, we expand on the geometric interpretation for a low value of k. Assume that t is a generic random q-chaining of $\mathcal{P}$ for $q \ge 1$. Fix $k = 1$ and let $p = (p_0, p_1) \in X_1$. There are five chains in $\mathfrak{C}_1$, which were listed in Example 7.23 (ii). Evaluating them in $t(p, \cdot)$ gives rise to the following five random variables in X_2:

$$t(p, (\{0,1\})) = (t_{\{p_0,p_1\}}, p_0, p_1)\,; \qquad t(p, (\{0,1\} \supset \{0\})) = (t_{\{p_0,p_1\}}, t_{\{p_0\}}, p_0)\,;$$
$$t(p, (\{0,1\} \supset \{1\})) = (t_{\{p_0,p_1\}}, t_{\{p_1\}}, p_1)\,; \qquad t(p, (\{0,1\} \supset \{0\} \supset \emptyset)) = (t_{\{p_0,p_1\}}, t_{\{p_0\}}, t_\emptyset)\,;$$
$$t(p, (\{0,1\} \supset \{1\} \supset \emptyset)) = (t_{\{p_0,p_1\}}, t_{\{p_1\}}, t_\emptyset)$$

Set $t_i := t_{\{p_i\}}$ for $i \in [1]$, and $t_{01} = t_{\{p_0,p_1\}}$. Geometrically, each one of them corresponds to a random 2-simplex of $X_\bullet$. We can visualize their arrangement as the simplicial complex in Figure 7.1.

The vertices in this complex are the points $p_0, p_1 \in \mathcal{P}$ and all the $\mathcal{P}$-valued random variables t_I for every subset $I \subset \{p_0, p_1\}$. We place an edge between two of these vertices, say u_0 and u_1, if and only if (u_0, u_1) lies in X_1 almost surely. Similarly, we place a 2-simplex joining vertices u_1, u_2 and u_3 if and only if $(u_0, u_1, u_2) \in X_2$ almost surely. The genericity assumption on the chaining guarantees that the complex above is "non-degenerate", for instance, in the sense that all of its cells are distinct and that apart from the ones in Figure 7.1, there exist no further edges or 2-cells in the arrangement.

Remark 7.29. So far, the necessity of property (iv) in Definition 7.21 and of the genericity of a random chaining have not yet become evident. We anticipate that these features will be highly exploited only in Section 7.5.5.

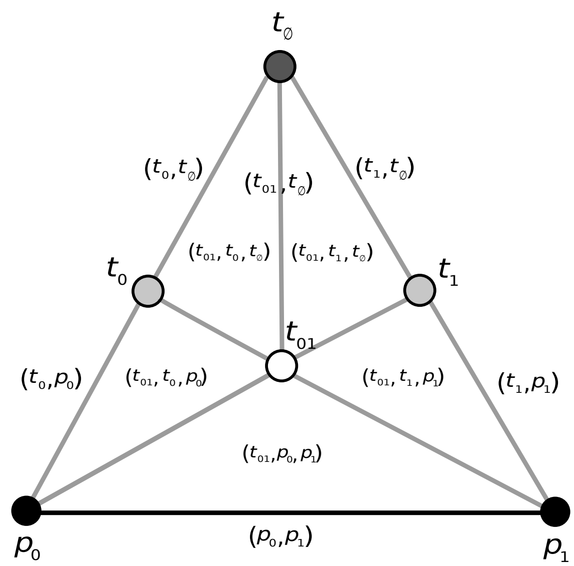


Figure 7.1: Geometric interpretation of a generic random chaining, and the images of 1-chains.

7.5.3 Construction of a generic random chaining on $\mathcal{P}$. Set again $q := \gamma(r)$. We are going to construct a generic random q-chaining on $\mathcal{P}$. For this, we must define a random variable t_I for every $I \in \mathcal{P}^{[\leq q+1]}$. We do so by induction on $|I|$, according to the next rules. The subsequent example should clarify the generating procedure.

(a) If $I = \emptyset$, then we let $t_\emptyset$ be the $\mathcal{P}$-valued random variable with distribution μ_0.

(b) Assume that t_J has been defined for all J of cardinality at most $n \in [q]$.

(i) Let $I \in \mathcal{P}^{[\leq q+1]}$ be such that $|I| = n + 1$. We define t_I as the $\mathcal{P}$-valued random variable with "regular conditional probability" $\nu_{I \cup \{t_J \mid J \subsetneq I\}}$ given $\{t_J \mid J \subsetneq I\}$; here ν denotes the assignment of perpendicular measures as in Definition 7.6. In other words, if μ denotes the joint probability distribution of the random variables $\{t_J \mid J \subsetneq I\}$, then the expression

$$\sigma_I(f) := \int_{\mathcal{P}^{2^I \setminus \{I\}}} \left(\int_{\mathcal{P}} f(t_J \mid J \subset I)\, \mathrm{d}\nu_{I \cup \{t_J \mid J \subsetneq I\}}(t_I) \right) \mathrm{d}\mu(t_J \mid J \subsetneq I), \qquad (f \in C(\mathcal{P}^{2^I})) \tag{7.13}$$

defines the joint probability distribution of the random variables $\{t_J \mid J \subset I\}$. In particular, the distribution μ_I of t_I is given by the pushforward measure $(\pi_I)_* \sigma_I$, where $\pi_I : \mathcal{P}^{2^I} \to \mathcal{P}$ is the projection

$$(x_J \mid J \subset I) \mapsto x_I.$$

(ii) If $I_0, I_1 \in \mathcal{P}^{[\leq q+1]}$ are two distinct sets of cardinality $n + 1$, then we declare the corresponding random variables t_{I_0} and t_{I_1} to be independent given

$$\{t_J \mid J \subsetneq I_0\} \cup \{t_J \mid J \subsetneq I_1\}.$$

Remark 7.30. The assumption that $\widehat{\gamma}(q) \leq r$ and Lemma 7.18 guarantee that the support of the measure $\nu_{I \cup \{t_J \mid J \subsetneq I\}}$ in (b)(i) is non-empty for any $I \in \mathcal{P}^{[\leq q+1]}$. On the other hand, for a fixed $f \in C(\mathcal{P}^{2^I})$, the function

$$(t_J \mid J \subsetneq I) \mapsto \int_{\mathcal{P}} f(t_J \mid J \subset I) \, \mathrm{d}\nu_{I \cup \{t_J \mid J \subsetneq I\}}(t_I)$$

defined by the inner integral in (b)(i) is bounded and Borel measurable by Proposition 7.7 and Lemma 4.1, hence integrable.

Example 7.31. Let $I = \{p_0, p_1\}$ be a subset of $\mathcal{P}$ consisting of two distinct points. We give the joint distribution of the random variables $\{t_J \mid J \subset I\} = \{t_\emptyset, t_0, t_1, t_{01}\}$ according to the rules above, where, as in Example 7.28, we let $t_i := t_{\{p_i\}}$ for $i \in [1]$, and $t_{01} = t_{\{p_0,p_1\}}$. By the rule (a), $t_\emptyset$ is distributed according to μ_0; by (b)(i), for $i \in [1]$, the joint probability distribution of $\{t_\emptyset, t_i\}$ is given by the expression

$$\sigma_{p_i}(f) = \int_{\mathcal{P}} \left(\int_{\mathcal{P}} f(t_\emptyset, t_i) \, \mathrm{d}\nu_{p_i,t_\emptyset}(t_i) \right) \mathrm{d}\mu_0(t_\emptyset), \quad f \in C(\mathcal{P}^2).$$

In order to obtain the joint distribution of $\{t_\emptyset, t_0, t_1, t_{01}\}$, we need first the one of the variables $\{t_\emptyset, t_0, t_1\}$. If we denote it by μ, then, by the rule (b)(ii), we have

$$\mu(f) = \int_{\mathcal{P}} \left(\int_{\mathcal{P}^2} f(t_\emptyset, t_0, t_1) \, \mathrm{d}(\nu_{p_0,t_\emptyset} \otimes \nu_{p_1,t_\emptyset})(t_0, t_1) \right) \mathrm{d}\mu_0(t_\emptyset), \quad f \in C(\mathcal{P}^2),$$

and again by (b)(i), we obtain that the variables $\{t_\emptyset, t_0, t_1, t_{01}\}$ are distributed according to the law

$$\sigma_{p_0,p_1}(f) = \int_{\mathcal{P}} \left(\int_{\mathcal{P}^2} \left(\int_{\mathcal{P}} f(t_\emptyset, t_0, t_1, t_{01}) \, \mathrm{d}\nu_{p_0,p_1,t_\emptyset,t_0,t_1}(t_{01}) \right) \mathrm{d}(\nu_{p_0,t_\emptyset} \otimes \nu_{p_1,t_\emptyset})(t_0, t_1) \right) \mathrm{d}\mu_0(t_\emptyset),$$

for $f \in C(\mathcal{P}^4)$. More generally, if $I = \{p_0, \ldots, p_n\}$, then the joint distribution of the random variables $\{t_J \mid J \subset I\}$ is given by

$$\prod_{J \subset I} \mathrm{d}t_J = \left(\prod_{\emptyset \neq J \subset I} \mathrm{d}\nu_{J \cup \{t_{J'} \mid J' \subset J\}}(t_J) \right) \mathrm{d}\mu_0(t_\emptyset).$$

Here the terms in both products are arranged according to the cardinality of J from largest to smallest as in the example of $I = \{p_0, p_1\}$.

We will prove now:

Proposition 7.32. *The collection $\{t_I \mid I \in \mathcal{P}^{[\leq q+1]}\}$ of random variables defines a generic random q-chaining t on $\mathcal{P}$.*

Proof. Property (i) of Definition 7.21 holds by definition, and (ii) and (iii) follow from (7.7) and (iv) follows from Proposition 7.7. It remains to show that the chaining is generic.

For fixed $k \in [q]$, let $p = (p_0, \ldots, p_k) \in X_k$ be a k-simplex and $C \in \mathfrak{C}_k$ be a k-chain of length n, say $C = (C_0 \supset C_1 \supset \cdots \supset C_n)$. If $C \in \mathfrak{C}_k^+$, let $\mathrm{ord}(C_n) = (i_0, \ldots, i_{k-n})$ be the ordered final component of C; otherwise, $C \in \mathfrak{C}_k^{\emptyset}$ and $C_n = C_{k+1} = \emptyset$. By Lemma 7.25, the random variable $t(p, C)$ as in (7.12) lies in $\bar{X}_k$ almost surely. Thus, in order to complete the proof of the genericity of t, we need to show that $t(p, C) \in X_k$ almost surely, or in other words, that the set

$$\{t_{p_{C_0}}, t_{p_{C_1}}, \dots, t_{p_{C_n}}\} \cup p_{C_n}$$

is *linearly independent* in V almost surely, i.e. consists of linearly independent 1-dimensional subspaces of V almost surely. Here p_{C_i} is as in Definition 7.24. In turn, that statement is the case $m = n$ of the next claim, which we prove by induction.

Claim. *For all $m \in \{-1, 0, \dots, n\}$, the set*

$$T_m := \{t_{p_{C_{n-m}}}, \dots, t_{p_{C_n}}\} \cup p_{C_n}$$

is almost surely linearly independent in V.

The case $m = -1$ corresponds to $T_{-1} = p_{C_n}$, which is a (deterministic) linearly independent set since $p \in X_k$. For the induction step, assume that T_{m-1} is almost surely linearly independent in V for an integer $m \in [n]$. Then, the statement follows immediately after showing that $t_{p_{C_{n-m}}} \notin \mathbb{P}(\mathrm{span}(T_{m-1}))$ almost surely. To see this, set first

$$\begin{aligned}
a_0 &:= p_{i_0}, & b_0 &:= t_{p_{C_n} \smallsetminus \{p_{i_0}\}},\\
a_1 &:= p_{i_1}, & b_1 &:= t_{p_{C_n} \smallsetminus \{p_{i_1}\}},\\
&\vdots & &\vdots\\
a_{k-n} &:= p_{i_{k-n}}, & b_{k-n} &:= t_{p_{C_n} \smallsetminus \{p_{i_{k-n}}\}},\\
a_{k-(n-1)} &:= t_{p_{C_n}}, & b_{k-(n-1)} &:= p_{C_{n-1} \smallsetminus C_n},\\
a_{k-(n-2)} &:= t_{p_{C_{n-1}}}, & b_{k-(n-2)} &:= t_{p_{C_{n-2} \smallsetminus (C_{n-1} \smallsetminus C_n)}},\\
&\vdots & &\vdots\\
a_{k-(n-m)} &:= t_{p_{C_{n-m+1}}}, & b_{k-(n-m)} &:= t_{p_{C_{n-m} \smallsetminus (C_{n-m+1} \smallsetminus C_{n-m+2})}},
\end{aligned}$$

and observe that by definition of the random variables, the following identities hold almost surely:

$$\begin{aligned}
&\omega(a_i, b_i) \neq 0 \ \text{ for all } i \in [k], \quad \omega(a_i, a_j) = 0 \ \text{ for all } i, j \in [k], \quad \text{and}\\
&\omega(a_i, b_j) = 0 \text{ for all } i, j \in [k], i \neq j.
\end{aligned}$$

Thus, by Lemma 2.7, the subspace $\mathrm{span}(T_{m-1})$ of V is, with probability one, symplectic. Now, by definition the random variable $t_{p_{C_{n-m}}}$ is distributed uniformly in the projective subspace $\mathbb{P}(\mathrm{span}(p_{C_{n-m}} \cup \{t_J \mid J \subsetneq C_{n-m}\})^{\omega})$ of $\mathbb{P}(V)$. But

$$\mathrm{span}(p_{C_{n-m}} \cup \{t_J \mid J \subsetneq p_{C_{n-m}}\})^{\omega} \cap \mathrm{span}(T_{m-1}) \subset \mathrm{span}(T_{m-1})^{\omega} \cap \mathrm{span}(T_{m-1}) = (0)$$

with probability one. Therefore, $t_{C_{n-m}} \notin \mathbb{P}(\mathrm{span}(T_{m-1}))$ almost surely, as asserted above. This completes the proof of the induction step. □

7.5.4 The contracting homotopy. Let t be a generic random q-chaining on $\mathcal{P}$.

Definition 7.33. Given $0 \le k \le q$ and a chain $C \in \mathfrak{C}_k$ we define the associated *partial homotopy* by

$$h_C : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k), \quad h_C f(p_0, \dots, p_k) = \mathbb{E}[f(t(p, C))].$$

We now construct the desired homotopy $h^\bullet$ as follows.

Definition 7.34. We define $h^{-1} : \mathscr{L}^\infty(X_0) \to \mathbb{R}$ by $h^{-1} f := \mathbb{E}[f(t_\emptyset)]$, and for every $k \in \{0, \dots, q\}$ we set

$$h^k : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k), \quad h^k = \sum_{C \in \mathfrak{C}_k} \operatorname{sgn}(C) \cdot h_C, \tag{7.14}$$

where $\operatorname{sgn} : \mathfrak{C}_k \to \{\pm 1\}$ is defined by the following convention.

Remark 7.35 (Sign convention for chains). If $I = \{i_0, \dots, i_l\} \subset [k]$ is such that $i_0 < \cdots < i_l$, we define the operators

$$\partial_a(I) := I \setminus \{i_a\} \quad \text{for any } a \in [l].$$

Now, assume first that $C \in \mathfrak{C}_k^+$ has length $n \le k$ so that $C_n \neq \emptyset$. There exist unique integers $j_0, \dots, j_{n-1} \in \{0, \dots, k\}$ such that the components of C are given as

$$C_m = \partial_{j_m} \circ \ldots \circ \partial_{j_0}([k]), \tag{7.15}$$

and we define the *sign* of C by

$$\operatorname{sgn}(C) := (-1)^{n + j_0 + \cdots + j_{n-1}}.$$

If $C' \in \mathfrak{C}_k^\emptyset$ has length $k+1$, then we have to modify this definition as follows. In this case, $C_k = \{j\}$ is a singleton, and we define

$$\operatorname{sgn}(C') = (-1)^{k+1+j}.$$

This ensures that if C is a chain of length k and C' is its unique extension to a chain of length $k+1$, then C and C' have opposite signs.

Example 7.36. Let $C = (\{0,1,2\} \supset \{0,2\} \supset \{2\})$ and $C' = (\{0,1,2\} \supset \{0,2\} \supset \{2\} \supset \emptyset)$ be two chains in $\mathfrak{C}_2$. Then $C = \partial_1 \circ \partial_1([2])$, and thus

$$\operatorname{sgn}(C) = (-1)^{2+1+1} = 1 \quad \text{and} \quad \operatorname{sgn}(C') = (-1)^{2+1+2} = -1.$$

Example 7.37. The following are the expressions of h^k for small values of k.

(i) For $k = 0$, we recover the formula from Remark 7.20:

$$h^0 f(p_0) = \mathbb{E}[f(t_{\{p_0\}}, p_0) - f(t_{\{p_0\}}, t_\emptyset)].$$

(ii) For $k = 1$:

$$\begin{aligned} h^1 f(p_0, p_1) \;=\; \mathbb{E}\Big[& f(t_{\{p_0,p_1\}}, p_0, p_1) - f(t_{\{p_0,p_1\}}, t_{\{p_1\}}, p_1) + f(t_{\{p_0,p_1\}}, t_{\{p_0\}}, p_0) \\ & + f(t_{\{p_0,p_1\}}, t_{\{p_1\}}, t_\emptyset) - f(t_{\{p_0,p_1\}}, t_{\{p_0\}}, t_\emptyset)\Big]. \end{aligned}$$

(iii) For $k = 2$:

$$\begin{aligned} & h^2 f(p_0, p_1, p_2) = \\ = \mathbb{E}\big[& f(t_{012}, p_0, p_1, p_2) - f(t_{012}, t_{12}, p_1, p_2) + f(t_{012}, t_{02}, p_0, p_2) - f(t_{012}, t_{01}, p_0, p_1) \\ & + f(t_{012}, t_{12}, t_2, p_2) - f(t_{012}, t_{12}, t_1, p_1) - f(t_{012}, t_{02}, t_2, p_2) + f(t_{012}, t_{02}, t_0, p_0) \\ & + f(t_{012}, t_{01}, t_1, p_1) - f(t_{012}, t_{01}, t_0, p_0) - f(t_{012}, t_{12}, t_2, t_\emptyset) + f(t_{012}, t_{12}, t_1, t_\emptyset) \\ & + f(t_{012}, t_{02}, t_2, t_\emptyset) - f(t_{012}, t_{02}, t_0, t_\emptyset) - f(t_{012}, t_{01}, t_1, t_\emptyset) + f(t_{012}, t_{01}, t_0, t_\emptyset)\big], \end{aligned}$$

where we have used the shorthand notations $t_i := t_{\{p_i\}}, t_{ij} := t_{\{p_i, p_j\}}$ and $t_{012} := t_{\{p_0, p_1, p_2\}}$.

For larger values of k, writing out h^k explicitly gets quite tedious.

Proof of Theorem 7.17 (i). We fix $k \le q$, $p = (p_0, \dots, p_k) \in X_k$ and $f \in \mathscr{L}^\infty(X_k)$. We have to show that

$$h^k \,\mathrm{d}^k f(p) + \mathrm{d}^{k-1} h^{k-1} f(p) = f(p). \tag{7.16}$$

The cases $k = -1$ and $k = 0$ are immediate from the formulas above, hence we will assume $k \ge 1$. Since p will be fixed throughout our discussion, we will use the shorthand notations $t_A := t_{p_A}$ for $A \subset [k]$ and $t(C) := t(p, C)$ for $C \in \mathfrak{C}_k$. Now, by definition,

$$h^k \,\mathrm{d}^k f(p) = \mathbb{E}\left[\sum_{C \in \mathfrak{C}_k} \mathrm{sgn}(C)\, \mathrm{d}^k f(t(C))\right] = \mathbb{E}\left[\sum_{j=0}^{k+1} \sum_{C \in \mathfrak{C}_k} (-1)^j \mathrm{sgn}(C) \delta^j f(t(C))\right].$$

Let us first deal with the summand $j = 0$. We distinguish two cases: First we consider the length 0 chain $C = ([k])$. In this case we have

$$(-1)^j \mathrm{sgn}(C) \delta^j f(t(C)) = \delta^0 f(t_{\{0,\dots,k\}}, p_0, \dots, p_k) = f(p_0, \dots, p_k).$$

Secondly, let C be a chain of length $n \ge 1$. Assume that $C_1 = [k] \smallsetminus \{i\}$ and $\mathrm{ord}(C_n) = (i_0, \dots, i_{k-n})$. Then

$$\begin{aligned} (-1)^j \mathrm{sgn}(C) \delta^j f(t(C)) &= \mathrm{sgn}(C) \delta^0 f(t_{[k]}, t_{[k]\smallsetminus\{i\}}, \dots, t_{\{i_0,\dots,i_{k-n}\}}, p_{i_0}, \dots, p_{i_{k-n}}) \\ &= \mathrm{sgn}(C) f(t_{[k]\smallsetminus\{i\}}, \dots, t_{\{i_0,\dots,i_{k-n}\}}, p_{i_0}, \dots, p_{i_{k-n}}) \end{aligned}$$

Using this identity, it is not hard to see that for a fixed $i \in [k]$,

$$\sum_{\substack{C \in \mathfrak{C}_k \smallsetminus \{[k]\} \\ C_1 = [k] \smallsetminus \{i\}}} \mathrm{sgn}(C)\, \delta^0 f(t(C)) = (-1)^{i+1} \delta^i h^{k-1} f(p),$$

and the sum over all $i \in [k]$ of the right-hand side equals $-d^{k-1} h^{k-1} f(p)$. Hence we obtain

$$h^k \,\mathrm{d}^k f(p) + \mathrm{d}^{k-1} h^{k-1} f(p) = f(p) + \mathbb{E}\left[\sum_{C \in \mathfrak{C}_k} \sum_{j=1}^{k+1} (-1)^j \mathrm{sgn}(C) \delta^j f(t(C))\right].$$

Now let C be a chain of length $n \ge 1$ and let $1 \le j \le n - 1$. Let $C_{j-1} \smallsetminus C_j = \{a\}$ and

$C_j \setminus C_{j+1} = \{b\}$ and set

$$C' := (C_0 \supset \cdots \supset C_{j-1} \supset C_{j-1} \setminus \{b\} \supset C_{j+1} \supset \dots C_n)$$

Then

$$\delta^j f(t(C)) = \delta^j(t(C')),$$

but C and C' have opposite signs, hence these two terms cancel in the sum above. We thus obtain

$$h^k\, \mathrm{d}^k\, f(p) + \mathrm{d}^{k-1}\, h^{k-1} f(p) - f(p) \;=\; \mathbb{E}\left[\sum_{C\in\mathfrak{C}_k} \sum_{j=\mathrm{length}(C)}^{k+1} (-1)^j \mathrm{sgn}(C) \delta^j f(t(C))\right]. \tag{7.17}$$

Now assume that C has length k and $C' := (C_0 \supset \cdots \supset C_k \supset \emptyset) \in \mathfrak{C}_k^{\emptyset}$ is the unique extension of C to length $k+1$. On the one hand we observe that

$$\delta^{k+1} f(t(C)) = \delta^{k+1} f(t(C')),$$

and since C and C' have opposite signs, these terms cancel each other.[1] Let us abbreviate by $\mathfrak{C}_k^k \subset \mathfrak{C}_k^+$ the subset of all chains of length k. Observe that for any $C \in \mathfrak{C}_k^{\emptyset}$, the inner sum in the right-hand side of (7.17) consists of a single term, corresponding to $j = k+1 = \mathrm{length}(C)$; for $C \in \mathfrak{C}_k^k$, the sum has two terms, namely $j = k = \mathrm{length}(C)$ and $j = k+1$. We thus obtain

$$\begin{aligned}
&h^k\, \mathrm{d}^k\, f(p) + \mathrm{d}^{k-1}\, h^{k-1} f(p) - f(p) \\
=&\mathbb{E}\left[\sum_{C\in\mathfrak{C}_k^+} \sum_{j=\mathrm{length}(C)}^{k+1} (-1)^j \mathrm{sgn}(C)\delta^j f(t(C)) - \sum_{C\in\mathfrak{C}_k^k} (-1)^{k+1}\mathrm{sgn}(C)\delta^{k+1} f(t(C))\right] \\
=&\mathbb{E}\left[\sum_{C\in\mathfrak{C}_k^+} \mathrm{sgn}(C)(-1)^{\mathrm{length}(C)}\delta^{\mathrm{length}(C)} f(t(C)) + \sum_{C'\in\mathfrak{C}_k^+\setminus\mathfrak{C}_k^k} \sum_{j=\mathrm{length}(C')+1}^{k+1} \mathrm{sgn}(C')(-1)^j \delta^j f(t(C'))\right].
\end{aligned}$$

Each term which appears in the first sum also appears in the second sum with the opposite sign and vice versa. Indeed, if $C \in \mathfrak{C}_k^+$ has length n, then we can define

$$C' := (C_0 \supset \cdots \supset C_{n-1}).$$

Then $C' \in \mathfrak{C}_k^+ \setminus \mathfrak{C}_k^k$ (since C' is shorter than C) and

$$\delta^{\mathrm{length}(C)} f(t(C)) = \delta^j f((t(C'))$$

for a unique $j \geq \mathrm{length}(C') + 1 = n$. Upon checking that the signs in front of these terms are opposite, this finishes the proof. □

Remark 7.38. What we have actually proved is that if $\mathcal{P}$ admits a generic random q-chaining, then the statement of Theorem 7.17 (i) holds. It is a consequence of the rather crude estimate from Lemma 7.18 that such a chaining exists if $\widehat{\gamma}(q) \leq r$. If one were able to obtain a more

[1]This is the reason why we had to define the sign differently for chains of maximal length.

efficient random chaining, then one would obtain a better bound in Theorem 7.17, which would result in a better stability range.

7.5.5 From $\mathscr{L}^\infty$ to L^∞. The purpose of this subsection is to establish part (ii) of Theorem 7.17 which says that the maps $h^k : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k)$ descend to maps $h^k : L^\infty(X_{k+1}) \to L^\infty(X_k)$ for all $k = -1, \dots, q$. This means: if $f_0, f_1 \in \mathscr{L}^\infty(X_{k+1})$ agree μ_{k+1}-almost everywhere, then $h^k f_0$ and $h^k f_1$ agree μ_k-almost everywhere. This is obvious for $h^{-1} : L^\infty(X_0) \to \mathbb{R}$, since if f_0 and f_1 agree μ_0-almost everywhere then

$$h^{-1}(f_1) = \int_{X_0} f_1(t_\emptyset)\, \mathrm{d}\mu_0(t_\emptyset) = \int_{X_0} f_2(t_\emptyset)\, \mathrm{d}\mu_0(t_\emptyset) = h^{-1}(f_2).$$

Now if $k \in 0, \dots, q$, then by definition

$$h^k = \sum_{C \in \mathfrak{C}_k} \mathrm{sgn}(C) \cdot h_C,$$

hence the proof of Theorem 7.17 (ii) reduces immediately to the following lemma.

Lemma 7.39. *Assume that $\mathcal{P}$ admits a generic q-chaining (which holds e.g. if $\hat{\gamma}(q) \leq r$) and let $k \leq q$. Then for every $C \in \mathfrak{C}_k$ the map $h_C : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k)$ descends to a map $h_C : L^\infty(X_{k+1}) \to L^\infty(X_k)$.*

From now on we fix q and k as in Lemma 7.39 and a chain $C \in \mathfrak{C}_k$. We recall that the map $h_C : \mathscr{L}^\infty(X_{k+1}) \to \mathscr{L}^\infty(X_k)$ is given by $h_C(f) = \mathbb{E}[f(t(p, C))]$. For every $p \in X_k$, recall from (7.13) that σ_p denotes the distribution of the random variable $t(p, C) \in X_{k+1}$. This defines a map

$$\sigma : X_k \to \mathrm{Prob}(X_{k+1}),\ p \mapsto \sigma_p, \quad \text{such that} \quad h_C(f)(p) = \int_{X_k} f \, \mathrm{d}\sigma_p. \tag{7.18}$$

Note that the measures σ_p are probability measures on the non-compact space $X_{k+1} \subset \bar{X}_{k+1}$ by Lemma 4.3.

Proposition 7.40. *The map σ from (7.18) is Borel and G-quasi-equivariant, i.e. for every $\mathbf{p} \in X_k$ and every $g \in G$, the measures $g_*\sigma_{\mathbf{p}}$ and $\sigma_{g\mathbf{p}}$ are mutually absolutely continuous.*

Proof. The assignment σ is G-quasi-equivariant, since μ_0 is G-quasi-invariant and ν is G-quasi-equivariant by Proposition 7.7. To see that it is Borel (with respect to the weak-$*$-topology on X^+_{k+1}) we have to show that for every $f \in C_0(X_{k+1})$ the map $\mathbf{p} \mapsto \int_{X_{k+1}} f\, \mathrm{d}\sigma_{\mathbf{p}}$ is Borel. For this, we first observe that the map $X_m \to \mathrm{Prob}(\mathcal{P})$, $(q_0, \dots, q_m) \mapsto \nu_{q_0,\dots,q_m}$ is Borel for every m by Proposition 7.7. In view of this observation, that $\int_{X_{k+1}} f\, \mathrm{d}\sigma_{\mathbf{p}}$ is Borel in $\mathbf{p}$ follows from the explicit formula by iterated application of Lemma 4.1; here we use that the chaining is generic, so that at each integration step the random points in the index of ν are linearly independent almost surely. □

From Proposition 7.40 and Lemma 4.7, we deduce Lemma 7.39, and hence Theorem 7.17 (ii).

7.6 The symplectic case vs. other formed spaces

We would like to extend our proof of stability from the symplectic family to any other family of automorphism groups of standard formed spaces of a fixed type. Until Section 7.2, our results are proven for any classical family, which means: if $(X_{r,\bullet}, \mu_{r,\bullet})_r$ is the sequence of Stiefel complex of standard formed spaces of a classical family, and $(G_r)_r$ is the sequence of automorphism groups, then $(G_r, (X_r, \mu_r))_r$ satisfies the axioms (MQ2) and (MQ3) of the definition of a measured Quillen family. One must still prove admissibility and (MQ1).

Let us consider again the setting in which (V, ω) is any formed space of rank r and dimension n, not necessarily of symplectic, and assume further that it contains isotropic vectors, so that the space $X_0 = \mathcal{P}$ is non-empty. The obstruction to proving the missing properties of the Stiefel complexes $(X_\bullet, \mu_\bullet)$ of (V, ω) is the lack of *perpendicular measures* in this setting. The following is the axiomatic definition of perpendicular measures that would make our argument for the symplectic families go through.

Definition 7.41. Let $m \leq n$. A perpendicular measure ν on $\mathcal{P}$ is a collection of maps

$$\nu^k : \mathcal{P}^{k+1} \longrightarrow \mathrm{Prob}(\mathcal{P})$$

for $k \in [m]$ with the following properties:

(i) The support of $\nu_{(p_0,\dots,p_k)}$ equals $\mathbb{P}(\mathrm{span}(p_0, \dots, p_k)^\omega) \cap \mathcal{P}$.

(ii) ν^k is Borel and G-quasi-equivariant,

(iii) ν^k is $\mathfrak{S}_{k+1}$-invariant and with the Fubini-like property

$$\int_{\mathcal{P}} \int_{\mathcal{P}} f(p,q)\ \mathrm{d}\nu_{\mathbf{p},p}(q)\ \mathrm{d}\nu_{\mathbf{p}}(p) = \int_{\mathcal{P}} \int_{\mathcal{P}} f(p,q)\ \mathrm{d}\nu_{\mathbf{p},q}(p)\ \mathrm{d}\nu_{\mathbf{p}}(q)$$

for every $f \in C(\mathcal{P}^2)$ and every $\mathbf{p} \in \mathcal{P}^k$.

(iv) $\nu_{\mathbf{p}}\big(\{p \in \mathcal{P} \mid (\mathbf{p}, p) \in X_k\}\big) = 1$ for all $\mathbf{p} \in X_{k-1}$.

The key difference between the symplectic and non-symplectic settings is the following: In the former, all vectors in V are isotropic, and thus, $\mathcal{P} = \mathbb{P}(V)$ is (modulo projectivization) a linear space; in the latter, neither $\mathcal{P}$ nor the intersection of $\mathcal{P}$ with subspaces of V is a linear space. This deprives us of the choice of Lebesgue measures for orthogonal complexes of subspaces. We would like to define them by means of a composition

$$\mathcal{P}^{k+1} \xrightarrow{\mathrm{span}} \mathrm{Gr}(V) \smallsetminus \{V\} \xrightarrow{(-)^\omega} \mathrm{Gr}(V) \smallsetminus \big\{\{0\}\big\} \xrightarrow{\lambda} \mathrm{Prob}(\mathcal{P}).$$

as in (7.6) above, where now λ_W is supposed to have support as in (i). Because the subspaces spanned by isotropic points do not necessarily satisfy any isotropy condition, it is not possible to make use of the G-action to define our measures. The use of the co-area formula for the proof of the property (ii) hints that in general the Lebesgue measures should be replaced by Hausdorff measures.

Chapter 8

The Isomorphism Conjecture for Complex Symplectic Groups

In degree three, the stability range from Theorem 6.26 delivers the following sequence of maps induced by inclusions $\mathrm{Sp}_{2r}(\Bbbk) \hookrightarrow \mathrm{Sp}_{2r+2}(\Bbbk)$:

$$\cdots \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{20}(\Bbbk)) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{18}(\Bbbk)) \hookrightarrow \cdots \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{10}(\Bbbk)). \tag{8.1}$$

Unfortunately, this sequence does not suffice for computational purposes, for there is no information available about the dimension of $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\Bbbk))$ with $r > 1$. Despite this insufficiency, we present in this section an argument that improves the stability range to an optimal one in degree three for $\Bbbk = \mathbb{C}$:

Theorem 8.1 (see Theorem E from Introduction)**.** *The inclusions* $\iota_r : \mathrm{Sp}_{2r}(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2r+2}(\mathbb{C})$ *for* $r \geq 1$ *induce the following sequence of maps in continuous bounded cohomology:*

$$\cdots \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{20}(\mathbb{C})) \xrightarrow{\sim} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{18}(\mathbb{C})) \hookrightarrow \cdots \hookrightarrow \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2}(\mathbb{C})) = \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_{2}(\mathbb{C})). \tag{8.2}$$

Given that the proof of the sequence (8.2) relies itself on the existence of the sequence (8.1), we say that our argument for Theorem 8.1 is of a *bootstrapping* nature. Its proof is based on the study of a product complex associated to the sequence of complex symplectic groups, and is given in Section 8.1. As a corollary, in Section 8.2 we prove the validity of the isomorphism conjecture in degree three for these groups, stated in the Introduction as Theorem F.

8.1 The bootstrapping procedure

Recall the definition of the standard symplectic vector space $(\mathbb{C}^{2r}, \omega_{\mathrm{I},0})$ of rank r from Subsection 2.1.2; the matrix representative of $\omega_{\mathrm{I},0}$ is called J_r. For abbreviation purposes, let us write from now on ω instead of $\omega_{\mathrm{I},0}$. We denote by $(e_r, \ldots, e_1, f_1, \ldots, f_r)$ the ordered standard basis of $(\mathbb{C}^{2r}, \omega)$; we also adopt the convention that $e_k = f_k = 0$ for every $k \leq 0$. For every $r \in \mathbb{N}$, we let $G_r := \mathrm{Sp}_{2r}(\mathbb{C})$, write $\iota_r : G_r \hookrightarrow G_{r+1}$ for the inclusions, and $\mathcal{P}_r$ for the $(2r-1)$-dimensional complex projective space $\mathbb{P}^{2r-1}(\mathbb{C})$ if $r > 0$ or for a singleton if $r \leq 0$. By Witt's theorem (Theorem 2.3 above), the group G_r acts continuously and transitively on $\mathcal{P}_r$, so as a homogeneous space, it acquires a unique G_r-invariant measure class. We fix a probability measure in that class, say μ_r, so that $(\mathcal{P}_r, \mu_r)$ is a regular G_r-space, and then consider the family $(\mathcal{P}_r^{\bullet+1}, \mu_r^{\otimes \bullet+1})_r$ of product semi-simplicial complexes. For non-zero vectors $v_0, \dots, v_k \in \mathbb{C}^{2r}$, we shall write $[v_0, \dots, v_k]$ for the tuple $([v_0], \dots, [v_k]) \in \mathcal{P}_r^{k+1}$.

8.1.1 Essential transitivity of the product complex, and the spectral sequence. For every r and $k \in \mathbb{N}$, let $X_{r,k}$ be the open, co-null submanifold of $\mathcal{P}_r^{k+1}$ as defined in (6.19), and let $Y_{r,k}$ be the open subset of $X_{r,k}$ defined as

$$Y_{r,k} := \{(p_0, \dots, p_k) \in X_{r,k} \mid \omega(p_i, p_j) \neq 0 \text{ for } i \neq j\},$$

Its complement in $X_{r,k}$, being a closed, positive-codimension submanifold, has zero measure, so $Y_{r,k}$ is open and co-null in $\mathcal{P}_r^{k+1}$. Clearly, $Y_{r,k}$ is also G_r-invariant, and the face maps $\delta_i : \mathcal{P}_r^{k+2} \to \mathcal{P}_r^{k+1}$ restrict to $\delta_i : Y_{r,k+1} \to Y_{r,k}$. We show:

Lemma 8.2. *For every $r \in \mathbb{N}$, the pair $(Y_{r,\bullet}, \mu_r^{\otimes \bullet+1})$ is a measurable ∞-connected, 2-transitive, admissible measured G_r-complex.*

Proof. Fix $r \geq 1$. By Lemma 5.14 and Lemma 5.15, the product G_r-complex $(\mathcal{P}_r^{\bullet+1}, \mu_r^{\otimes \bullet+1})$ is admissible and measurably ∞-connected. These properties are then inherited by $(Y_{r,\bullet}, \mu_r^{\otimes \bullet+1})$. We show now that G_r acts transitively on $Y_{r,k}$ for all $k \in [2]$. For $k = 1$, let $[v_0, v_1] \in Y_{r,1}$ where without loss of generality (v_0, v_1) is a hyperbolic pair. By Witt's theorem, the linear isometry $\operatorname{span}(v_0, v_1) \to \mathbb{C}^{2r}$ with $v_0 \mapsto e_r$ and $v_1 \mapsto f_r$ extends to an element of $\mathrm{Sp}_{2r}(\mathbb{C})$, proving transitivity in this case. The stabilizers $H_{r,0} := \operatorname{stab}_{G_r}[e_r]$ and $H_{r,1} := \operatorname{stab}_{G_r}[e_r, f_r]$ are

$$H_{r,0} = \left\{ \underbrace{\left(\begin{array}{c|c|c} \lambda & * & \alpha \\ \hline & A & v \\ \hline & & \lambda^{-1} \end{array} \right)}_{=: M_r(\lambda, A, \alpha, v)} \;\middle|\; \begin{array}{l} \lambda \in \mathbb{C}^\times, \ A \in G_{r-1}, \\ \alpha \in \mathbb{C}, \ v \in \mathbb{C}^{2r-2} \end{array} \right\} > \left\{ \left(\begin{array}{c|c|c} \lambda & & \\ \hline & A & \\ \hline & & \lambda^{-1} \end{array} \right) \right\} = H_{r,1}, \quad (8.3)$$

where the asterisk in the matrix $M_r(\lambda, A, \alpha, v)$ corresponds to a vector in $\mathbb{C}^{2r-2}$ that is completely determined by the other parameters and the fact that the matrix is symplectic. For the transitivity of the G_r-action on $Y_{r,2}$, note that for $v = (\alpha, w, \beta)^\top \in \mathbb{C}^{2r}$ with $\alpha, \beta \in \mathbb{C}$ and $w \in \mathbb{C}^{2r-2}$, the tuple $[e_r, f_r, v]$ belong to $Y_{r,2}$ if and only if

$$\alpha \neq 0, \quad \beta \neq 0, \quad \text{and} \quad w \neq 0.$$

Since $\mathbb{C}^\times/(\mathbb{C}^\times)^2 = 1$, there exist scalars $\lambda, \mu \in \mathbb{C}^\times$ and a matrix $A \in G_{r-1}$ such that

$$\lambda\alpha = \lambda^{-1}\beta = \mu \quad \text{and} \quad (1 \oplus A \oplus 1) \cdot (0, w, 0)^\top = \mu e_{r-1},$$

and therefore, the matrix $\lambda \oplus A \oplus \lambda^{-1} \in H_{r,1}$ maps $[e_r, f_r, v]$ to $[e_r, f_r, e_r + e_{r-1} + f_r]$. □

Remark 8.3. We point out that the transitivity of the G_r-action on $Y_{r,2}$ does not hold once we replace $\mathbb{C}$ by any other local field. This is because it relies on the existence of all square roots;

in the non-Archimedean setting, this requires consideration of infinite extensions, which are no longer local.

It is not hard to check computationally that the stabilizer $H_{r,2} := \mathrm{stab}_{G_r}[e_r, f_r, e_r+e_{r-1}+f_r]$ is explicitly given as

$$H_{r,2} = \left\{\varepsilon \oplus M_{r-1}(\varepsilon, A, \alpha, v) \oplus \varepsilon \mid \varepsilon \in \{\pm 1\},\ A \in G_{r-2},\ \alpha \in \mathbb{C},\ v \in \mathbb{C}^{2r-4}\right\}. \tag{8.4}$$

We also have the following surjective homomorphisms with amenable kernel:

$$\begin{array}{lll} \pi_{r,0} : H_{r,0} \to G_{r-1} & \pi_{r,1} : H_{r,1} \to G_{r-1} & \pi_{r,2} : H_{r,2} \to G_{r-2} \\ \pi_{r,0}(M_r(\lambda, A, \alpha, v)) := A, & \pi_{r,1} := \pi_{r,0}\,|_{H_{r,1}}, & \pi_{r,2}(\varepsilon \oplus M_{r-1}(\varepsilon, A, \alpha, v) \oplus \varepsilon) := A. \end{array}$$

Now, in virtue of Proposition 5.16, there exists for every $r \geq 1$ a spectral sequence ${}_{r+1}E_\bullet^{\bullet,\bullet}$ with first page

$${}_{r+1}E_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}; L^\infty(\mathcal{P}^p_{r+1})) \quad \text{and} \quad {}_{r+1}\mathrm{d}_1^{p,q} = \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}; \mathrm{d}^p), \quad \text{for all } p, q \geq 0,$$

and limit page ${}_{r+1}E_\infty^q = 0$ for all $q \in \mathbb{N}$.

Lemma 8.4. *For any $r \geq 1$ and $q \geq 0$. There exist isomorphisms*

$${}_{r+1}E_1^{1,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G_r) \cong {}_{r+1}E_1^{2,q}, \quad \text{and} \quad {}_{r+1}E_1^{3,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r-1}), \tag{8.5}$$

of which the first one make the next diagram commute:

$$\begin{array}{ccc} {}_{r+1}E_1^{0,q} & \xrightarrow{\ {}_{r+1}\mathrm{d}_1^{p,q}\ } & {}_{r+1}E_1^{1,q} \\ \Big\| & & \cong \big\updownarrow \\ \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}) & \xrightarrow{\ \mathrm{H}^q_{\mathrm{cb}}(\iota_r)\ } & \mathrm{H}^q_{\mathrm{cb}}(G_r) \end{array} \tag{8.6}$$

Proof. For $p \in [3]$, the isomorphisms from (8.5) follow from Lemma 6.12 with $G = G_r$, subgroups $H = H_{r+1,p}$ and projections $\pi = \pi_{r+1,p}$. To prove the commutativity of (8.6), note that by Lemma 6.13 with $G = G_{r+1}$ and $H = H_{r+1,0}$, the upper triangle in the diagram below commutes, where $j_0 : H_{r+1,0} \hookrightarrow G_{r+1}$ is the inclusion. The lower triangle commutes as well, since $\iota_r \circ \pi_{r+1,0} = j_0$.

$$\begin{array}{ccc} \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}) = {}_{r+1}E_1^{0,q} & \xrightarrow{\ {}_{r+1}\mathrm{d}_1^{0,q}\ } & {}_{r+1}E_1^{1,q} \cong \mathrm{H}^q_{\mathrm{cb}}(G_{r+1}; L^\infty(G_{r+1}/H_{r+1,0})) \\ & \searrow^{\mathrm{H}^q_{\mathrm{cb}}(j_0)} & \cong \big\uparrow \mathrm{Ind}^q \\ & & \mathrm{H}^q_{\mathrm{cb}}(H_{r+1,0}) \\ & \searrow_{\mathrm{H}^q_{\mathrm{cb}}(\iota_r)} & \cong \big\uparrow \mathrm{H}^q_{\mathrm{cb}}(\pi_{r+1,0}) \\ & & \mathrm{H}^q_{\mathrm{cb}}(G_r) \end{array}$$

□

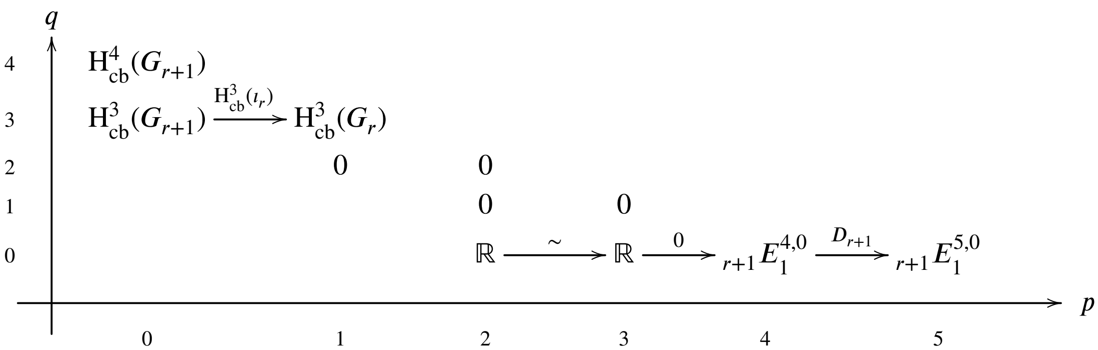


Figure 8.1: First page ${}_{r+1}E_1^{\bullet,\bullet}$.

We are ready to fill out the first page of our spectral sequence. Note first that according to the previous lemma, the zeroth and first row are given by

$$ {}_{r+1}E_1^{p,0} \cong \mathbb{R} \quad \text{and} \quad {}_{r+1}E_1^{p,1} = 0 $$

for $p \in [3]$ (see also Subsection 1.1.4.) Under these isomorphisms, the differentials ${}_{r+1}\,\mathrm{d}_1^{2,0}$ and ${}_{r+1}\,\mathrm{d}_1^{2,0}$ are conjugated respectively to the identity and the zero map; this follows from an argument identical to the one used in the proof of the assertion (d) within the proof of Theorem 6.10. Moreover, since simple complex Lie groups are of non-Hermitian type, by Theorem 1.34,

$$ {}_{r+1}E_1^{p,2} = 0 \quad \text{for } p \in [3]. $$

Let us also write from now on D_{r+1} for the differential ${}_{r+1}\,\mathrm{d}_1^{4,0}$. Figure 8.1 above shows the data in the bottom-left corner of the first page ${}_{r+1}E_1^{\bullet,\bullet}$ that is relevant to the proof of Theorem 8.1.

Lemma 8.5. *For $r \geq 1$ sufficiently large, the map D_r is injective.*

Proof. In virtue of Theorem 6.26, fix r large enough for the induced map $\mathrm{H}^3_{\mathrm{cb}}(\iota_r)$ to be injective (according to (8.1), any $r \geq 5$ fulfills our purposes.) Then, Figure 8.2 shows the terms in the second page ${}_{r+1}E_2^{\bullet,\bullet}$ for the positions displayed in Figure 8.1; we put asterisks in all those that are irrelevant to us at this stage. Thanks to the injectivity of $\mathrm{H}^3_{\mathrm{cb}}(\iota_r)$, we are certain that every term ${}_{r+1}E_2^{p,q}$ in the diagonal $p+q=3$ vanishes, and thus ${}_{r+1}E_s^{p,q} = 0$ for all $p+q=3$ and $s \geq 2$. Note that ${}_{r+1}E_2^{4,0} = \ker D_{r+1} \cong \ker {}_{r+1}\,\mathrm{d}_2^{4,0}$, and that the arrow ${}_{r+1}\,\mathrm{d}_2^{2,1}$ is the one to have that term as a target in this iteration. However, because ${}_{r+1}E_2^{2,1} = 0$, we observe that

$$ {}_{r+1}E_3^{4,0} \cong \ker D_{r+1} / \operatorname{im} {}_{r+1}\,\mathrm{d}_2^{2,1} \cong \ker D_{r+1}. $$

Since all the arrows that will have the position $(4,0)$ as a target in subsequent pages emanate from terms in the diagonal $p+q=3$, we may repeat the previous argument a couple of times to conclude that ${}_{r+1}E_\infty^{4,0} \cong \ker D_{r+1}$. But now

$$ \ker D_{r+1} \hookrightarrow \bigoplus_{p+q=4} {}_{r+1}E_\infty^{p,q} \cong {}_{r+1}E_\infty^4 = 0. $$

□

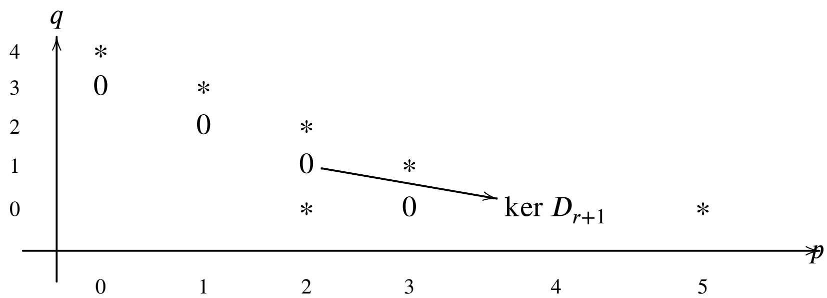


Figure 8.2: Second page ${}_{r+1}E_2^{\bullet,\bullet}$ as in the proof of Lemma 8.5.

The key step of our proof of Theorem 8.1 is the next proposition, whose proof is contained in Subsection 8.1.2. The subsequent corollary follows immediately from the proposition and Lemma 8.5.

Proposition 8.6. *For any two natural numbers $r, r' \geq 2$, the map D_r is injective if and only if $D_{r'}$ is injective.*

Corollary 8.7. *For all $r \geq 1$, the map D_{r+1} is injective.* □

Proof of Theorem 8.1. Let $r \geq 1$. Proving that $\mathrm{H}^3_{\mathrm{cb}}(\iota_r)$ is injective proves the theorem by induction. Note that we are actually interested in $r < 5$, since otherwise, the injectivity of $\mathrm{H}^3_{\mathrm{cb}}(\iota_r)$ is already known to hold by Theorem 6.26. By Corollary 8.7, the entry ${}_{r+1}E_2^{4,0}$ in the second page ${}_{r+1}E_2^{\bullet,\bullet}$ vanishes. The useful terms are displayed in Figure 8.3. Arguing as in the proof of Theorem 6.10, the zeros in the diagonal $p + q = 4$ imply that ${}_{r+1}E_\infty^{0,3} \cong \ker \mathrm{H}^3_{\mathrm{cb}}(\iota_r)$. Again, we conclude by observing that

$$\ker \mathrm{H}^3_{\mathrm{cb}}(\iota_r) \hookrightarrow \bigoplus_{p+q=3} {}_{r+1}E_\infty^{p,q} \cong {}_{r+1}E_\infty^3 = 0.$$

□

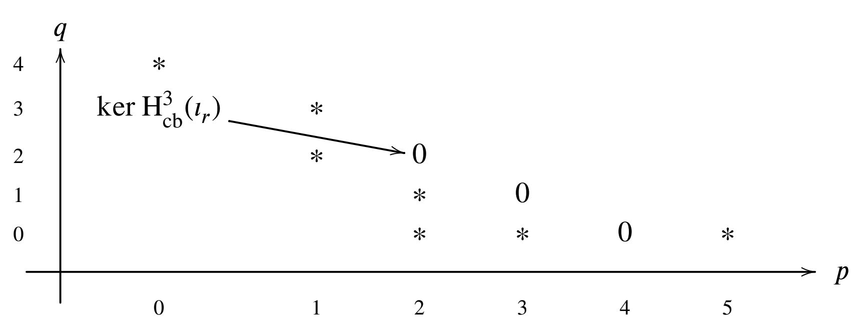


Figure 8.3: Second page ${}_{r+1}E_2^{\bullet,\bullet}$ as in the proof of Theorem 8.1.

8.1.2 Symplectic cross-ratios and the functional equation. Let $r \geq 1$. We now take a closer look at the map

$$D_r : L^\infty(Y_{r,3})^{G_r} \longrightarrow L^\infty(Y_{r,4})^{G_r}.$$

The idea is to parametrize the orbit spaces $G_r \setminus Y_{r,p}$ for $p = 3, 4$, and write an expression of D_r in the new parameters. For that purpose, we resort to a Gram–Schmidt-like procedure.

Lemma 8.8. *For all $r \geq 1$ and $p = 3, 4$, each element of $Y_{r,p}$ lies in the orbit of a unique tuple*

$$\left[e_r,\ f_r,\ e_r+f_r+e_{r-1},\quad e_r+\alpha_1 f_r+A f_{r-1},\quad e_r+\beta_1 f_r+C e_{r-1}+B f_{r-1}+e_{r-2}\right] \in Y_{r,p}, \tag{8.7}$$

with $\alpha_1, \beta_1, A, B, C \in \mathbb{C}^\times$, where we omit the last projective point in the tuple if $p = 3$.

Remark 8.9. If $r = 1$ resp. $r = 2$, the tuple from (8.7) reduces to

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & \alpha_1 & \beta_1 \end{bmatrix}, \quad \text{resp. to} \quad \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & C \\ 0 & 0 & 0 & A & B \\ 0 & 1 & 1 & \alpha_1 & \beta_1 \end{bmatrix}.$$

The constants in the statement of Lemma 8.8 will be defined in terms of the following cross-ratios.

Definition 8.10. For any $[v_0, v_1, v_2, v_3] \in Y_{r,3}$, we define

$$\mathrm{cr}_1[v_0, v_1, v_2, v_3] := \frac{\omega(v_0, v_3) \cdot \omega(v_1, v_2)}{\omega(v_0, v_2) \cdot \omega(v_1, v_3)} \quad \text{and} \quad \mathrm{cr}_2[v_0, v_1, v_2, v_3] := \frac{\omega(v_0, v_1) \cdot \omega(v_2, v_3)}{\omega(v_0, v_2) \cdot \omega(v_1, v_3)}.$$

Being independent of the choice of a vector representing each of the four projective points $[v_i]$, both quantities are well defined, non-zero complex numbers, that we call the *complex symplectic cross-ratios* of the tuple $[v_0, v_1, v_2, v_3] \in Y_{r,3}$. The functions

$$\mathrm{cr}_1 : Y_{r,3} \to \mathbb{C} \quad \text{and} \quad \mathrm{cr}_2 : Y_{r,3} \to \mathbb{C}$$

are Borel and mcp (they are indeed C^1 in their whole domain, arising as a product of C^1 functions), and G_r-invariant, thus descending to maps on the orbit spaces $G_r \setminus Y_{r,p}$.

Proof of Lemma 8.8. Let $[v_0, \ldots, v_p] \in Y_{r,p}$, and define

$$x_r := v_1/\omega(v_1, v_0) \quad \text{and} \quad y_r := v_0.$$

If $r \geq 2$, set also

$$v_i' := v_i - \frac{\omega(v_1, v_i)}{\omega(v_1, v_0)} v_0 - \frac{\omega(v_0, v_i)}{\omega(v_0, v_1)} v_1 \quad \text{for } i = 2, 3.$$

Then

$$\omega(v_i', v_0) = \omega(v_i', v_1) = \omega(v_i', v_i) = 0 \text{ for } i = 2, 3, \text{ and } \omega(v_2', v_3) = \omega(v_2, v_3') = \omega(v_2', v_3'). \tag{8.8}$$

By the definition of $Y_{r,3}$, the vectors $v_0, \ldots, v_3$ must be linearly independent, so by the non-degeneracy of ω, we have $\omega(v_2', v_3) \neq 0$. This allows us to define

$$x_{r-1} := v_3'/\omega(v_3', v_2') \quad \text{and} \quad y_{r-1} := v_2',$$

Note that the ordered 4-tuple $(x_r, x_{r-1}, y_{r-1}, y_r)$ is an adapted basis of $\operatorname{span}\{v_0, \dots, v_3\}$. By Witt's theorem, we may extend it to an adapted basis $(x_r, \dots, x_1, y_1, \dots, y_r)$ of the whole standard formed space $(\mathbb{C}^{2r}, \omega)$. For $r = 1$, we define the matrix $X := (x_r\ y_r)$; otherwise, we let $X := (x_r\ \cdots\ x_1\ y_1\ \cdots\ y_r)$. In any of the two cases, X is an element of G_r. Now, also $X^\top J_r \in G_r$, and moreover, for each $i \in [p]$,

$$(X^\top J_r)v_i = \big(\omega(x_r, v_i), \dots, \omega(x_1, v_i), \omega(y_1, v_i), \dots, \omega(y_r, v_i)\big)^\top.$$

Thus, the image $(X^\top J_r) \cdot [v_0, \dots, v_4]$ equals

$$\begin{bmatrix} 1 & 0 & \omega(x_r, v_2) & \omega(x_r, v_3) & \omega(x_r, v_4) \\ 0 & 0 & 1 & 0 & \omega(x_{r-1}, v_4) \\ 0 & 0 & 0 & 0 & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & \omega(y_{r-1}, v_3) & \omega(y_{r-1}, v_4) \\ 0 & 1 & \omega(y_r, v_2) & \omega(y_r, v_3) & \omega(y_r, v_4) \end{bmatrix},$$

where the asterisk in the fifth column denotes a vector in $\mathbb{C}^{2r-4}$, which must be non-zero if $r \geq 3$ so as to maintain linear independence. Now, since $\omega(x_r, v_2) \neq 0$ and $\omega(y_r, v_2) \neq 0$, we send the third line in the tuple to $[e_r + f_r + e_{r-1}]$ by multiplying by the matrix $D \oplus I_{2r-2} \oplus Q_2 D^{-1} Q_2$ in the stabilizer $H_{r,1}$ as in (8.3), where Q_2 is the (2×2)-matrix with 1's on the anti-diagonal,

$$D = \operatorname{diag}(\lambda, \mu), \quad \lambda = (\omega(y_r, v_2)/\omega(x_r, v_2))^{1/2}, \quad \text{and}$$
$$\mu = \lambda \cdot \omega(x_r, v_2) = \lambda^{-1} \cdot \omega(y_r, v_2).$$

After normalizing the fourth and fifth columns to have both 1 as a first entry, we are left with a tuple of the form

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & C \\ 0 & 0 & 0 & 0 & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & A & B \\ 0 & 1 & 1 & \alpha_1 & \beta_1 \end{bmatrix}, \tag{8.9}$$

with constants $\alpha_1, \beta_1, A, B, C \in \mathbb{C}^\times$ given as follows:

$$\begin{aligned} A &:= 1 - \alpha_1 - \alpha_2, & B &:= 1 - \beta_1 - \beta_2, & C &:= \frac{-\alpha_1(1 - \gamma_1 - \gamma_2)}{1 - \alpha_1 - \alpha_2}, \\ \alpha_i &:= \operatorname{cr}_i[v_0, v_1, v_2, v_3], & \beta_i &:= \operatorname{cr}_i[v_0, v_1, v_2, v_4] & \gamma_i &:= \operatorname{cr}_i[v_0, v_1, v_3, v_4], \end{aligned} \tag{8.10}$$

with $i = 1, 2$. It is clear that $\alpha_i, \beta_i, \gamma_i \neq 0$. Recall that, according to Remark 8.9, the constants A, B, C appear only if $r \geq 2$. We argue why they do not vanish if $r \geq 2$: if A were zero, then

$$\omega(v_0, v_1)\omega(v_2, v_3) + \omega(v_0, v_3)\omega(v_1, v_2) + \omega(v_0, v_2)\omega(v_1, v_3) = 0,$$

which would imply that $\omega(v_2', v_3) = 0$, contradicting by (8.8) the non-degeneracy of ω. That $B, C \neq 0$ can be proven analogously. Finally, if $r \geq 3$, we can use the embedded $G_{r-2} < G_r$ to move the non-zero vector corresponding to the asterisks in the fifth column of (8.9) to $(1, 0, \ldots, 0) \in \mathbb{C}^{2r-4}$. This proves the existence of a representative as in (8.7).

For uniqueness, an easy check shows the setwise stabilizer $H_{r,3}$ in $H_{r,2} = \operatorname{stab}[e_r, f_r, e_r + f_r + e_{r-1}]$ of the space of lines

$$\left\{[1 : 0 : \cdots : 0 : A : \alpha_1]^\top \mid \alpha_1, A \in \mathbb{C}^\times\right\},$$

as the fourth one, equals the subgroup $\{\pm I_2\} \oplus G_{r-2} \oplus \{\pm I_2\} < G_r$, hence fixing it also pointwise. Also, the setwise stabilizer $H_{r,4}$ in $H_{r,3}$ of

$$\left\{[1 : C : 1 : \cdots : 0 : B : \alpha_1]^\top \mid \beta_1, B, C \in \mathbb{C}^\times\right\}$$

fixes it pointwise. □

Remark 8.11. Note that the following "cocycle identity" holds for $[v_0, \ldots, v_4] \in Y_{r,4}$:

$$\operatorname{cr}_1[v_0, v_1, v_2, v_4] = \operatorname{cr}_1[v_0, v_1, v_2, v_3] \cdot \operatorname{cr}_1[v_0, v_1, v_3, v_4].$$

Thus, in the notation of (8.10), we have the equality $\beta_1 = \alpha_1\gamma_1$.

Lemma 8.12. *For $r \geq 2$ and $p = 3, 4$, the maps Φ_p, defined as*

$$\begin{aligned} \Phi_3 &: Y_{r,3} \to \mathbb{C}^2, & \Phi_3 &:= (\operatorname{cr}_1, \operatorname{cr}_2),\\ \Phi_4 &: Y_{r,4} \to \mathbb{C}^5, & \Phi_4 &:= (\operatorname{cr}_2 \circ \delta_2, \Phi_3 \circ \delta_3, \Phi_3 \circ \delta_4), \end{aligned}$$

are Borel and mcp. Moreover, the descents

$$\widehat{\Phi}_3 : G_r \setminus Y_{r,3} \to \mathbb{C}^2 \quad \text{and} \quad \widehat{\Phi}_4 : G_r \setminus Y_{r,4} \to \mathbb{C}^5$$

have a Borel, mcp essential inverse. In particular, they induce respective isomorphisms

$$L^\infty(Y_{r,3})^{G_r} \cong L^\infty(\mathbb{C}^2) \quad \text{and} \quad L^\infty(Y_{r,4})^{G_r} \cong L^\infty(\mathbb{C}^5).$$

Proof. That the maps Φ_3 and Φ_4 are Borel and mcp follows clearly from the fact that $\operatorname{cr}_i : Y_{r,3} \to \mathbb{C}$ and $\delta_j : Y_{r,4} \to Y_{r,3}$ are. The same holds for the descents $\widehat{\Phi}_3$ and $\widehat{\Phi}_4$. Inspired by Lemma 8.8 and Remark 8.11, we define now the open, co-null subset

$$S_4 := \left\{(\gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2) \in (\mathbb{C}^\times)^5 \mid \alpha_1 + \alpha_2 \neq 1, \beta_1 + \beta_2 \neq 1, \beta_1 + \alpha_1\gamma_2 \neq \alpha_1\right\} \subset \mathbb{C}^5$$

and the map $\Psi_4 : S_4 \to G_r \setminus Y_{r,4}$ by

$$\Psi_4(\gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2) := G_r \cdot \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & \frac{\beta_1 + \alpha_1\gamma_2 - \alpha_1}{1 - \alpha_1 - \alpha_2} \\ 0 & 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 - \alpha_1 - \alpha_2 & 1 - \beta_1 - \beta_2 \\ 0 & 1 & 1 & \alpha_1 & \beta_1 \end{bmatrix},$$

where the middle $2r - 4$ rows are deleted if $r = 2$. We also define

$$S_3 := \{(\alpha_1, \alpha_2) \in (\mathbb{C}^\times)^2 \mid \alpha_1 + \alpha_2 \neq 1\} \subset \mathbb{C}^2$$

and $\Psi_3 : S_3 \to G_r \setminus Y_{r,3}$ by letting $\Psi_2(\alpha_1, \alpha_2)$ to be the orbit of the 4-tuple remaining when deleting the last column of $\Psi_4(\gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2)$. Both maps are seen to be well defined after computing all pairwise cross-ratios. The fact that $\widehat{\Phi}_3 \circ \Psi_3$ and $\widehat{\Phi}_4 \circ \Psi_4$ equal the identity follows from a simple computation. In turn, after computing the compositions $\Psi_3 \circ \widehat{\Phi}_3$ and $\Psi_4 \circ \widehat{\Phi}_4$, it follows from the uniqueness statement in Lemma 8.8 that they are the identity. □

Proof of Proposition 8.6. Let $r \geq 2$, and notice that the map $T : L^\infty(\mathbb{C}^2) \to L^\infty(\mathbb{C}^5)$ defined by the commutative diagram

$$\begin{array}{ccc} L^\infty(Y_{r,3})^{G_r} & \xrightarrow{\quad D_r \quad} & L^\infty(Y_{r,4})^{G_r} \\ \widehat{\Phi}_3^* \uparrow\cong \;\; \cong\downarrow \Psi_3^* & & \widehat{\Phi}_4^* \uparrow\cong \;\; \cong\downarrow \Psi_4^* \\ L^\infty(\mathbb{C}^2) & \overset{T}{\dashrightarrow} & L^\infty(\mathbb{C}^5) \end{array} \tag{8.11}$$

is independent of r. Indeed, it is defined almost everywhere by

$$T f(\gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2) := f(\alpha_1, \alpha_2) - f(\beta_1, \beta_2) + f\left(\frac{\beta_1}{\alpha_1}, \gamma_2\right) - f\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}, \frac{\gamma_2}{\beta_2}\right) + \\ + f\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1\gamma_2}{\beta_2}\right) = 0.$$

Thus, $\ker D_r = 0$ if and only if $\ker T = 0$. The arbitrariness of $r \geq 2$ finishes the proof. □

Curiously, we have recovered from Lemma 8.5—and ultimately from our stability result from Theorem 6.26—the inexistence of solutions of a functional equation:

Corollary 8.13. *The only $f \in L^\infty(\mathbb{C}^2)$ that satisfies the functional equation*

$$f(\alpha_1, \alpha_2) - f(\beta_1, \beta_2) + f\left(\frac{\beta_1}{\alpha_1}, \gamma_2\right) - f\left(\frac{\alpha_2\beta_1}{\alpha_1\beta_2}, \frac{\gamma_2}{\beta_2}\right) + f\left(\frac{\alpha_2}{\beta_2}, \frac{\alpha_1\gamma_2}{\beta_2}\right) = 0 \tag{8.12}$$

for almost every $(\gamma_2, \beta_1, \beta_2, \alpha_1, \alpha_2) \in \mathbb{C}^5$ is the almost everywhere identically zero function. □

Remark 8.14. If $r = 1$, the procedure from the proof of Proposition 8.6 gives rise to the function $T : L^\infty(\mathbb{C}) \to L^\infty(\mathbb{C}^2)$,

$$T f(\alpha, \beta) := f(\alpha) - f(\beta) + f\left(\frac{\beta}{\alpha}\right) - f\left(\frac{1-\beta}{1-\alpha}\right) + f\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right).$$

The functional equation $Tf = 0$ for $f \in L^\infty(\mathbb{C})$ is the Spence–Abel 5-term functional equation, whose only solution up to a multiplicative constant is the Bloch–Wigner dilogarithm (see [15], or the survey [78] and references therein for a similar statement with stronger regularity assumptions.)

8.2 End of the proof

Let $n \geq 2$ be a natural number. Recall from Subsection 1.5.3 the *bounded Borel class* β_n^{b}, for which a cocycle representative is given in [35, 10] and which generates $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_n(\mathbb{C}))$. According to Theorem 1.41 above, quoted from [10], its Gromov norm is

$$\|\beta_n^{\mathrm{b}}\| = \frac{n(n^2-1)}{6}\, v_3,$$

where v_3 is the maximal volume of an ideal tetrahedron in $\mathbb{H}^3$. In addition, let $\iota_n : \mathrm{SL}_n(\mathbb{C}) \hookrightarrow \mathrm{SL}_{n+1}(\mathbb{C})$ denote the block inclusion, and let $\pi_n : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$ denote the unique (up to isomorphism) complex irreducible n-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$. With this notation, the following is part of one of the main two theorems in [10].

Theorem 8.15 (see [10, Theorem 2])**.** *Let $n \geq 2$. Then the induced map $\mathrm{H}^3_{\mathrm{cb}}(\iota_{n+1})$ is an isomorphism that sends the bounded Borel class β_{n+1}^{b} to β_n^{b}, and $\mathrm{H}^3_{\mathrm{cb}}(\pi_n)$ is an isometric isomorphism.* □

Now, for any $r \geq 1$, we let $j_{2r} : \mathrm{Sp}_{2r}(\mathbb{C}) \hookrightarrow \mathrm{Sp}_{2r+2}(\mathbb{C})$ be the block inclusion defined as in (6.24). Also, for any such r, the irreducible complex $2r$-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$ preserves an alternating non-degenerate bilinear form on $\mathbb{C}^{2r}$ (see [53, Lemma 1.2].) After conjugation by a symplectic isometry, we let $\widetilde{\pi}_{2r} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{Sp}_{2r}(\mathbb{C})$ denote the irreducible representation that preserves our choice of standard symplectic form from Subsection 2.1.2, and π_{2r} be the composition

$$\mathrm{SL}_2(\mathbb{C}) \xrightarrow{\widetilde{\pi}_{2r}} \mathrm{Sp}_{2r}(\mathbb{C}) \hookrightarrow \mathrm{SL}_{2r}(\mathbb{C}).$$

Consider the commutative diagram of inclusions

$$\begin{array}{ccc} \mathrm{SL}_{2r+2}(\mathbb{C}) & \hookleftarrow & \mathrm{Sp}_{2r+2}(\mathbb{C}) \\ {\scriptstyle \iota_{2r-1}\circ\iota_{2r}} \uparrow & & \uparrow {\scriptstyle j_{2r}} \\ \mathrm{SL}_{2r}(\mathbb{C}) & \hookleftarrow & \mathrm{Sp}_{2r}(\mathbb{C}) \end{array}$$

By functoriality, the diagram below commutes as well, where we adopt the labels next to the arrows to denote the respective induced maps in cohomology:

$$\begin{array}{ccc} \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_{2r+2}(\mathbb{C})) & \xrightarrow{\mathrm{res}_{2r+2}} & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r+2}(\mathbb{C})) \\ {\scriptstyle \mathrm{I}_{2r}} \downarrow {\scriptstyle \cong} & & \downarrow {\scriptstyle \mathrm{J}_{2r}} \\ \mathrm{H}^3_{\mathrm{cb}}(\mathrm{SL}_{2r}(\mathbb{C})) & \xrightarrow[\mathrm{res}_{2r}]{} & \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) \end{array} \tag{8.13}$$

That I_{2r} is an isomorphism is Theorem 8.15, and that J_{2r} is an injection is Theorem 8.1. Note also that $\mathrm{H}^3_{\mathrm{cb}}(\pi_{2r}) = \mathrm{H}^3_{\mathrm{cb}}(\widetilde{\pi}_{2r}) \circ \mathrm{res}_{2r}$. Let us set $\widetilde{\beta}_{2r}^{\mathrm{b}} := \mathrm{res}_{2r}(\beta_{2r}^{\mathrm{b}})$ for every $r \geq 1$. We will prove:

Corollary 8.16. *Let $r \geq 1$. Then:*

(i) *The map* res_{2r} *is an isometric isomorphism. In particular, the bounded class* $\widetilde{\beta}^{\mathrm{b}}_{2r}$ *generates* $\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C}))$ *and has Gromov norm*

$$\|\widetilde{\beta}^{\mathrm{b}}_{2r}\| = \frac{r(4r^2-1)}{3}\, v_3.$$

(ii) J_{2r} *is an isomorphism that maps* $\widetilde{\beta}^{\mathrm{b}}_{2r+2}$ *to* $\widetilde{\beta}^{\mathrm{b}}_{2r}$*.*

(iii) *The comparison map* $c^3 : \mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) \to \mathrm{H}^3_{\mathrm{c}}(\mathrm{Sp}_{2r}(\mathbb{C}))$ *is an isomorphism.*

Proof. The commutativity of (8.13) and Theorem 8.15 imply that

$$\mathrm{J}_{2r}\big(\widetilde{\beta}^{\mathrm{b}}_{2r+2}\big) = \widetilde{\beta}^{\mathrm{b}}_{2r} \tag{8.14}$$

for every $r \geq 1$. Note also that if res_{2r} is an isomorphism for some r, then so is J_{2r}: indeed, the assumption would imply that

$$\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) = \mathrm{span}\left(\widetilde{\beta}^{\mathrm{b}}_{2r}\right),$$

and because of (8.14), the map J_{2r} would be surjective. Its injectivity is Theorem 8.1.

The claim (ii) follows after showing that res_{2r} is an isomorphism for any $r \geq 1$. For this we argue by induction over r. For $r = 1$, the equality $\mathrm{Sp}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})$ holds, and res_2 is the identity. If we assume as induction hypothesis that res_{2r} is an isomorphism, then so is J_{2r}. By the commutativity of (8.13), the map res_{2r+2} is an isomorphism, too, completing the induction. Proving that res_{2r} is isometric for every r finishes the proof of (i). This is true, since

$$\|\beta^{\mathrm{b}}_{2r}\| = \|\mathrm{H}^3_{\mathrm{cb}}(\pi_{2r})\big(\beta^{\mathrm{b}}_{2r}\big)\| = \|\mathrm{H}^3_{\mathrm{cb}}(\widetilde{\pi}_{2r})\big(\widetilde{\beta}^{\mathrm{b}}_{2r}\big)\| \leq \|\widetilde{\beta}^{\mathrm{b}}_{2r}\| \leq \|\beta^{\mathrm{b}}_{2r}\|,$$

where the first equality is Theorem 8.15.

The claim (iii) follows similarly by induction over $r \geq 1$, where the base case $r = 1$ is Bloch's theorem [5] for $\mathrm{SL}_2(\mathbb{C})$ (cf. [15, Theorem 1.2]), and the induction step is a consequence of the commutativity of the diagram

$$\begin{array}{ccc}
\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r+2}(\mathbb{C})) & \xrightarrow{\;c^3\;} & \mathrm{H}^3_{\mathrm{c}}(\mathrm{Sp}_{2r+2}(\mathbb{C})) \\
{\scriptstyle \mathrm{J}_{2r}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow \\
\mathrm{H}^3_{\mathrm{cb}}(\mathrm{Sp}_{2r}(\mathbb{C})) & \xrightarrow{\;c^3\;} & \mathrm{H}^3_{\mathrm{c}}(\mathrm{Sp}_{2r}(\mathbb{C}))
\end{array}$$

where the fact that the inclusion j_{2r} induces an isomorphism in continuous cohomology is a combination of van Est's isomorphism (see Theorem 1.32) and Theorem 1.39 above. □

APPENDIX

Appendix A

Third Continuous Cohomology of Simple Lie Groups

The main purpose of this appendix is to give a proof of Theorem 1.38, on a characterization of complex Lie groups as those among connected simple Lie groups with finite center whose continuous cohomology in degree three does not vanish. The result was stated also as Theorem G in the Introduction. Recall that a complex structure on a Lie algebra $\mathfrak{g}$ is a linear map $J \in \operatorname{End}(\mathfrak{g})$ that satisfies the identity $J^2 = -\operatorname{id}$ and that commutes with the adjoint representation of $\mathfrak{g}$.

Theorem 1.38. *Let G be a connected, simple Lie group with finite center, and let $\mathfrak{g}$ be its Lie algebra. Then the following are equivalent:*

(A) $\mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) \neq 0$.
(B) $\dim \mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) = 1$.
(C) $\mathfrak{g}$ *admits a complex structure.*
(D) G *admits the structure of a complex Lie group.*

En passant, we will show that removing the hypothesis of finite center results in the addition of only one Lie group to the collection of those whose $\mathrm{H}^3_{\mathrm{c}}$ does not vanish.

Theorem A.1. *For a connected, simple Lie group G of infinite center, $\mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) \neq 0$ if and only if G is isomorphic to $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal cover of $\mathrm{SL}(2,\mathbb{R})$. In that case, $\dim \mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) = 1$.*

This appendix relies heavily on Lie-theoretic results contained in the textbooks [36, 44, 64].

A.1 An explicit continuous 3-cocycle for complex Lie groups

Before proceeding to the proofs of the theorems, we comment on the question of explicit 3-cocycles in the setting of Theorem 1.38. Thus, fix G as in Theorem 1.38, and assume that the equivalent conditions hold. Let J be a complex structure on the Lie algebra $\mathfrak{g}$ of G, and regard it as a complex Lie algebra. Let

- $\mathfrak{k} \subset \mathfrak{g}$ be a compact real form of $\mathfrak{g}$,
- $B_{\mathfrak{g}}$ be the Killing form of $\mathfrak{g}$, and
- K the connected subgroup of G with Lie algebra $\mathfrak{k}$.

Then $\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k}$ is a Cartan decomposition, and $\mathfrak{k}$ is simple; see Theorem A.11 below for a reference. The subgroup K is maximal compact in G, and G/K is a symmetric space of non-compact type with $\mathrm{T}_K(G/K) \cong \mathfrak{g}/\mathfrak{k} \cong J\mathfrak{k}$. Let $(\Lambda^\bullet(\mathfrak{g}/\mathfrak{k})^*)^{\mathfrak{k}}$ denote the complex of $\mathfrak{k}$-invariant, alternating, multilinear forms on $\mathfrak{g}/\mathfrak{k}$, and $\Omega^\bullet(G/K)^G$ be the complex of G-invariant differential forms on G/K. Left-translation of an element of the former gives rise of an element of the latter, and this assignment is an isomorphism.

It is a consequence of van Est's theorem that there is an isomorphism $\mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) \cong (\Lambda^\bullet(J\mathfrak{k})^*)^{\mathfrak{k}}$; use Corollary A.9 with $\mathfrak{p} = J\mathfrak{k}$. The formula

$$\omega(X, Y, Z) := B_{\mathfrak{g}}(X, J[Y, Z]), \qquad X, Y, Z \in J\mathfrak{k}, \tag{A.1}$$

defines a non-zero element $\omega \in (\Lambda^3(J\mathfrak{k})^*)^{\mathfrak{k}}$. Let $\tilde{\omega} \in \Omega^3(G/K)^G$ be corresponding 3-form. This one integrated over "3-simplices" produces a non-trivial, G-invariant continuous 3-cocycle $I_\omega : G^4 \to \mathbb{R}$.

More precisely: Fix a base point $o \in G/K$. For any k-tuple $(g_0, \dots, g_k) \in G^k$, consider the *geodesic k-simplex* $\Delta(g_0, \dots, g_k) \subset G/K$, defined inductively as follows: let $\Delta(g_0) := \{g_0 \cdot o\}$, and for $k > 0$, set $\Delta(g_0, \dots, g_k)$ to be the union of the geodesics connecting $g_k \cdot o$ to each point in $\Delta(g_0, \dots, g_{k-1})$. It is not hard to verify that the expression

$$I_\omega(g_0, \dots, g_3) := \int_{\Delta(g_0, \dots, g_3)} \tilde{\omega} \tag{A.2}$$

is a well-defined G-invariant continuous 3-cocycle. Finally, its non-triviality follows from the fact, by Dupont [28], that integration over simplices realizes van Est's isomorphism at the level of cochains.

Remark A.2. Concerning Theorem A.1, the group $G := \widetilde{\mathrm{SL}(2,\mathbb{R})}$ has trivial maximal compact subgroup M. Thus, van Est's theorem (Theorem A.4 below) yields an isomorphism $\mathrm{H}^3_{\mathrm{c}}(G;\mathbb{R}) \cong \mathrm{H}^3(\Omega^\bullet(G)^G)$. An obvious G-invariant 3-form of G is its volume form. Arguing as with (A.2), we conclude that the volume of 3-simplices is a non-trivial invariant continuous 3-cocycle of G.

Remark A.3. An interesting question is the one of the boundedness of the cocycle in (A.2) upon choosing appropriate filling of tetrahedra in G/K. A satisfactory observation in a positive direction is that the form $\tilde{\omega}$ vanishes along flats of G/K.

We point out that the expression obtained by removing the J in (A.1) is known to define a generating class of $\mathrm{H}^3(\mathfrak{k};\mathbb{R})$ and of the de Rham cohomology group $\mathrm{H}^3(K;\mathbb{R})$; see the reference [36]. However, we have found no account in the literature of the formula (A.1) nor of the statements of our two theorems.

A.2 Notation and background

Notation. Whenever there is no explicit mention of coefficients when using any notion of cohomology, it should be understood for the rest of this appendix that they are trivial $\mathbb{R}$-coefficients.

The functor $\mathrm{H}_{\mathrm{c}}^{\bullet}$ refers to the continuous cohomology of topological groups. On the other hand, when applied to manifolds (including Lie groups), $\mathrm{H}^{\bullet}$ denotes their cohomology as spaces.

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{m} \subset \mathfrak{g}$ is any subalgebra, then $(\Lambda^{\bullet}(\mathfrak{g}/\mathfrak{m})^*)^{\mathfrak{m}}$ will denote the complex of $\mathfrak{m}$-invariant, alternating, multilinear forms on $\mathfrak{g}/\mathfrak{m}$; for the definition of the $\mathfrak{m}$-action on $\mathfrak{g}/\mathfrak{m}$ and of the coboundary operator, we refer the reader to Chapter 1 of [6]. The cohomology of this complex, denoted by $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{m})$ is known as the *Lie algebra cohomology of* $\mathfrak{g}$ *relative to* $\mathfrak{m}$. If $\mathfrak{m}$ is trivial, we will write $\mathrm{H}^{\bullet}(\mathfrak{g})$ instead of $\mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{m})$.

If G is a connected Lie group and $M < G$ is a closed subgroup, then $\Omega^{\bullet}(G/M)^G$ denotes the complex of G-invariant differential forms on G/M, where the coboundary operator is the usual differential for forms, and the G-action on forms on G/M is by left-translation. We denote the cohomology of this complex by $\mathrm{H}_G^{\bullet}(G/M)$; it is known as the *$G$-invariant de Rham cohomology of G/M*.

Background on continuous cohomology of Lie groups. A powerful tool for computing the continuous cohomology of a connected Lie group G is *van Est's theorem*; see the original references by van Est [77] and Hochschild–Mostow [45]. We quote it from [6, Corollary IX.5.6].

Theorem A.4 (van Est, Hochschild–Mostow)**.** *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$, and M be a maximal compact (connected) subgroup of G with Lie algebra $\mathfrak{m} \subset \mathfrak{g}$. Then*

$$\mathrm{H}_{\mathrm{c}}^{\bullet}(G) \cong \mathrm{H}_G^{\bullet}(G/M) \cong \mathrm{H}^{\bullet}(\mathfrak{g}, \mathfrak{m}). \tag{A.3}$$

For the rest of this section, let G be a non-compact, connected, semisimple[1] Lie group with Lie algebra $\mathfrak{g}$, and let $M < G$ be a maximal compact subgroup with Lie algebra $\mathfrak{m}$. Let us fix:

- a maximal compactly embedded subalgebra[2] $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{m} \subset \mathfrak{k}$,
- an associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$,
- the connected subgroup K of G with Lie algebra $\mathfrak{k}$,
- the compact dual $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}$,
- the *compact dual* G_u of G, i.e. the 1-connected, compact, semisimple Lie group with Lie algebra $\mathfrak{g}_u$, and
- the connected subgroup M_u of G_u with Lie algebra $\mathfrak{m}$.
- the connected subgroup K_u of G_u with Lie algebra $\mathfrak{k}$.

Remark A.5. Any subgroup $K' < G$ with Lie algebra $\mathfrak{k}$ is automatically connected and closed in G, and contains the center of G. Furthermore, K' is compact if and only if the center of G is finite. If that is the case, K' is a maximal compact subgroup of G, and G/K' is a symmetric space of non-compact type. This is the content of Theorem VI.1.1 in [44].

[1] For background in semisimple Lie groups and Lie algebras, we refer the reader to [44].

[2] We recall the definition of a compactly embedded subalgebra $\mathfrak{k}$ of a Lie algebra $\mathfrak{g}$, which can be found after the statement of Corollary II.5.3 in [44]. Let $\mathrm{Int}(\mathfrak{g})$ denote the connected subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{g})$. We say that a subalgebra $\mathfrak{k}$ is *compactly embedded* in $\mathfrak{g}$ if the connected subgroup K^* of $\mathrm{Int}(\mathfrak{g})$ with Lie algebra $\mathrm{ad}(\mathfrak{k}) \subset \mathrm{ad}(\mathfrak{g})$ is compact. Maximal compactly embedded subalgebras of semisimple Lie algebras give rise to Cartan decompositions.

Remark A.6. The connected subgroup $K_u < G_u$ above is necessarily closed in G_u. This and the 1-connectedness of G_u imply that G_u/K_u is a symmetric space of compact type. This follows from Proposition IV.3.6 in [44].

A second computational tool is the next theorem of Chevalley–Eilenberg [19], whose main ideas they attribute to Cartan.

Theorem A.7. *If $M_u < G_u$ is closed, then* $\mathrm{H}^\bullet(\mathfrak{g}_u, \mathfrak{m}) \cong \mathrm{H}^\bullet(G_u/M_u)$.

It is not hard to observe that there is an isomorphism $\mathrm{H}^\bullet(\mathfrak{g}, \mathfrak{m}) \cong \mathrm{H}^\bullet(\mathfrak{g}_u, \mathfrak{m})$. Combining it with (A.3), we obtain:

Corollary A.8. *If $M_u < G_u$ is closed, then* $\mathrm{H}^\bullet_c(G) \cong \mathrm{H}^\bullet(G_u/M_u)$.

The case of finite center. We impose now the additional assumption that G has a finite center. Then, by Remark A.5, the subgroup $K < G$ fixed above is maximal compact. In particular,

$$M = K, \quad \mathfrak{m} = \mathfrak{k} \quad \text{and} \quad M_u = K_u. \tag{A.4}$$

Moreover, by the same remark, G/K is a symmetric space of non-compact type. It is a theorem by Cartan that invariant differential forms on a symmetric space are automatically closed. Hence:

Corollary A.9. *Under the assumption of finite center of G, there exist isomorphisms*

$$\mathrm{H}^\bullet_c(G) \cong \Omega^\bullet(G/K)^G \cong (\Lambda^\bullet \mathfrak{p}^*)^{\mathfrak{k}} \cong (\Lambda^\bullet (i\mathfrak{p})^*)^{\mathfrak{k}} \cong \mathrm{H}^\bullet(G_u/K_u).$$

A.3 Finite center

The goal of this section is proving Theorem 1.38. The equivalence (C) ⇔ (D) is a classical fact that holds for any connected Lie group, and goes back to Newlander–Nirenberg. Thus, we will only prove the equivalence (A) ⇔ (B) ⇔ (C). A first reduction is provided by the following lemma.

Lemma A.10. *If G is a compact, connected, simple Lie group (hence with finite center), then none of the statements* (A)-(D) *in Theorem 1.38 holds.*

Proof. Assume first that G is a compact, connected Lie group that satisfies (D), and let us regard it as a complex Lie group. If $\mathfrak{g}$ denotes the complex Lie algebra of G, then the adjoint representation $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is holomorphic, and thus it is constant. After differentiating, this implies that the Lie bracket vanishes identically, which in turn shows that G is Abelian. In particular, G cannot be simple. On the other hand, properties (A) and (B) fail for any compact, connected Lie group G because by Theorem A.4, $\mathrm{H}^k_c(G) = 0$ for every $k > 0$. □

Hence, from now on in this section, let G be a *non-compact*, connected, simple Lie group with finite center, and let $\mathfrak{g}$ be its Lie algebra. We argue now in the order (C) ⇒ (B) ⇒ (A) ⇒ (C), where (B) ⇒ (A) is evident.

Towards the implication (C) $\Rightarrow$ (B), let us assume that $\mathfrak{g}$ admits a complex structure J. Then:

Theorem A.11. *Any compact real form $\mathfrak{k}$ of $\mathfrak{g}$ is simple, maximal compactly embedded in $\mathfrak{g}$, and*

$$\mathfrak{g} = \mathfrak{k} \oplus J\mathfrak{k} \tag{A.5}$$

is a Cartan decomposition of $\mathfrak{g}$.

About the proof. The fact that $\mathfrak{k}$ is maximal compactly embedded and that the decomposition above is a Cartan decomposition is Corollary III.7.5 of [44]. The simplicity of $\mathfrak{k}$ is part of the proof of Theorem VIII.5.4 in [44]. □

Fix a compact real form $\mathfrak{k}$ of $\mathfrak{g}$. By Corollary A.9 (with $\mathfrak{p} = J\mathfrak{k}$), there exists isomorphisms $\mathrm{H}^3_{\mathrm{c}}(G) \cong (\Lambda^3(J\mathfrak{k})^*)^{\mathfrak{k}} \cong (\Lambda^3\mathfrak{k}^*)^{\mathfrak{k}}$. Let $(\mathrm{V}^2\mathfrak{k}^*)^{\mathfrak{k}}$ denote the space of $\mathfrak{k}$-invariant, symmetric bilinear forms on $\mathfrak{k}$. We make use of the following fact:

Proposition A.12. *The assignment $\Phi : (\mathrm{V}^2\mathfrak{k}^*)^{\mathfrak{k}} \to (\Lambda^3\mathfrak{k}^*)^{\mathfrak{k}}$, defined by*

$$\Phi_B(X, Y, Z) = B(X, [Y, Z]), \quad \textit{for } B \in (\mathrm{V}^2\mathfrak{k}^*)^{\mathfrak{k}} \textit{ and } X, Y, Z \in \mathfrak{k},$$

is a linear isomorphism.

About the proof. This is Proposition I in Section 5.7 of [36]. The statement holds if $\mathfrak{k}$ is any compact, simple Lie algebra. □

We conclude the proof of the implication by pointing out that the space $(\mathrm{V}^2\mathfrak{k}^*)^{\mathfrak{k}}$ has dimension one because of the simplicity of $\mathfrak{k}$; a generator of it is the Killing form of $\mathfrak{k}$. A proof of this fact can be found, for example, in the discussion at the beginning of VIII.§5 of [44].

We devote the rest of this section to the proof of (A) $\Rightarrow$ (C). Assume now that $\mathfrak{g}$ does not admit a complex structure. We adopt the same notation of (A.4) and Corollary A.9, and fix:

- a maximal compact subgroup $K < G$,
- its Lie algebra $\mathfrak{k} \subset \mathfrak{g}$,
- an associated Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$,
- the compact dual $\mathfrak{g}_u := \mathfrak{k} \oplus i\mathfrak{p}$ of $\mathfrak{g}$,
- the compact dual G_u of G, and
- the connected subgroup K_u of G_u with Lie algebra $\mathfrak{k}$.

The starting point for the proof of this implication is the following theorem.

Theorem A.13. (i) *The compact dual $\mathfrak{g}_u$ is simple.*

(ii) *The symmetric space of compact type G_u/K_u is irreducible. In particular, the action of $\mathrm{Ad}(K_u)$ on the vector space $i\mathfrak{p}$ is irreducible.*

About the proof. Part (i) is a combination of Theorem VIII.5.3 and Theorem V.2.4 of [44]. Part (ii) is a consequence of Theorem VIII.5.3 of [44]; see also the Definition at the beginning of VIII.§5 of [44]. □

Remark A.14. Note that (i) does not hold for Lie algebras that do admit a complex structure. For example, the compact dual of $\mathfrak{sl}(2,\mathbb{C})$ is the Lie algebra $\mathfrak{su}(2) \oplus \mathfrak{su}(2) \cong \mathfrak{so}(4,\mathbb{R})$.

By Corollary A.9, we have $\mathrm{H}^3_{\mathrm{c}}(G) \cong \mathrm{H}^3(G_u/K_u)$. Thus it suffices to show that $\mathrm{H}^3(G_u/K_u)$ vanishes. We distinguish two cases:

Case 1: $\mathfrak{k}$ is Abelian. The following proposition establishes the claim.

Proposition A.15. *The symmetric space G_u/K_u is diffeomorphic to the 2-dimensional sphere S^2.*

Proof. The group K_u is Abelian; because it is a non-trivial torus, it contains an element j of order four. The image of j under the adjoint representation $\mathrm{Ad}_{G_u}(j)|_{i\mathfrak{p}}$ is a complex structure on the real vector space $i\mathfrak{p}$. Thus, we regard $i\mathfrak{p}$ now as a $\mathbb{C}$-vector space. By the commutativity of K_u, the $\mathrm{Ad}(K_u)$-action on $i\mathfrak{p}$ is $\mathbb{C}$-linear. By (ii) of Theorem A.13 and Schur's lemma, we obtain that $\dim_{\mathbb{C}} i\mathfrak{p} = 1$. In consequence,

$$\dim(G_u/K_u) = \dim_{\mathbb{R}} i\mathfrak{p} = 2\dim_{\mathbb{C}} i\mathfrak{p} = 2.$$

Since G_u is 1-connected and K_u is connected, the space G_u/K_u is 1-connected, and we conclude by the classification of surfaces. □

Case 2: $\mathfrak{k}$ is non-Abelian. The claim follows from the next proposition with $U = G_u$ and $L = K_u$.

Proposition A.16. *Let U be a 1-connected, compact, simple Lie group, and $L < U$ be a connected, closed, non-Abelian subgroup. Then* $\mathrm{H}^3(U/L) = 0$.

We point out that U/L needs not be a symmetric space. We give a proof of this proposition, making use of three facts about the topology of Lie groups.

Theorem A.17 (Weyl)**.** *The universal covering group of a compact semisimple Lie group is compact. In particular, any Lie group with a compact, semisimple Lie algebra is compact.*

About the proof. A proof of this theorem is found in [44] as Theorem II.6.9. □

Theorem A.18 (Bott)**.** *For any connected Lie group H, the second homotopy group $\pi_2(H)$ vanishes. If, moreover, H is simple, then $\pi_3(H) \cong \mathbb{Z}$.*

About the proof. It is a consequence of the fact, due to Bott, that the loop space of a compact, 1-connected Lie group has the homotopy type of a CW-complex with no odd-dimensional cells and finitely many cells of each even dimension. A proof of this fact is completely Morse-theoretical; see Theorem 21.6 in [54]. Based on it, one can conclude as indicated in [23]. □

Before quoting the third fact, we give the definition of the *Dynkin index* of a homomorphism between two compact, connected, simple Lie groups, as found in [64]:

Definition. Let G_1 and G_2 be compact, connected, simple Lie groups with Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, respectively, and let $\alpha_i \in \mathfrak{h}_i^*$ be a root of maximal length of $\mathfrak{g}_i$ with respect to a Cartan subalgebra $\mathfrak{h}_i$ ($i = 1, 2$). Moreover, let $B_i \in (\mathrm{V}^2\mathfrak{g}_i^*)^{\mathfrak{g}_i}$ be a negative-definite, $\mathfrak{g}_i$-invariant bilinear form[3] on $\mathfrak{g}_i$ normalized in such a way that the square of the length of the root α_i with respect to the associated inner product on $\mathfrak{h}_i^*$ equals two. If $\varphi : G_1 \to G_2$ is a homomorphism, then there exists a non-negative real number j_φ such that $B_2 = j_\varphi \cdot B_1$, called the *Dynkin index* of φ.

Theorem A.19. *Let $\varphi : G_1 \to G_2$ be a homomorphism between two compact, connected, simple Lie groups G_1 and G_2. Then:*

(i) The Dynkin index j_φ is a non-negative integer. It is equal to zero if and only if φ is the homomorphism that maps every element to the identity.

(ii) If $\pi_3(G_i) = \langle \epsilon_i \rangle$ for $i = 1, 2$ (see Theorem A.18), then $\varphi_\# \epsilon_1 = \pm j_\varphi \epsilon_2$.

About the proof. Part (i) is Proposition 11, §3 of [64], and (ii) is Theorem 2, §17 of the same reference. □

Proof of Proposition A.16. We will prove that the integral homology group $\mathrm{H}_3(U/L;\mathbb{Z})$ is finite and therefore by the universal coefficient theorem $\mathrm{H}^3(U/L) = 0$. Because U is 1-connected and L is connected, we have that U/L is 1-connected. Therefore, the Hurewicz homomorphism $h : \pi_3(U/L) \to H_3(U/L;\mathbb{Z})$ in degree three is surjective. The claim follows after proving finiteness of $\pi_3(U/L)$.

Consider now the long exact sequence in homotopy

$$\cdots \to \pi_3(L) \xrightarrow{\iota_\#} \pi_3(U) \xrightarrow{\pi_\#} \pi_3(U/L) \to \pi_2(L) \to \cdots$$

of the fibration $L \overset{\iota}{\hookrightarrow} U \overset{\pi}{\twoheadrightarrow} U/L$. By Theorem A.18 and exactness, the homomorphism $\pi_\#$ is surjective and $\pi_3(U/L) \cong \pi_3(U)/\mathrm{im}\,\iota_\# \cong \mathbb{Z}/\mathrm{im}\,\iota_\#$. Therefore, $\pi_3(U/L)$ is finite if and only if the image of $\iota_\#$ is non-trivial.

Let $\mathfrak{k}$ be the Lie algebra of L, hence compact and non-Abelian. In particular, $\mathfrak{k}$ splits as the direct sum of its center and a non-trivial semisimple ideal. Thus, let $\mathfrak{k}_1$ be a simple ideal of $\mathfrak{k}$, and let L_1 be the 1-connected, simple Lie group with Lie algebra $\mathfrak{k}_1$, which is compact by Theorem A.17. Now let $\phi : L_1 \to L$ be the unique Lie group homomorphism whose derivative is the inclusion $\mathfrak{k}_1 \hookrightarrow \mathfrak{k}$. Again by Theorem A.18, we know that $\pi_3(K_1) \cong \mathbb{Z}$. We obtain the following diagram in homotopy:

$$\begin{array}{ccc} \pi_3(L) & \xrightarrow{\iota_\#} & \pi_3(U) \cong \mathbb{Z} \\ \phi_\# \uparrow & \overset{\iota_\# \circ \phi_\# =: \psi_\#}{\nearrow} & \\ \mathbb{Z} \cong \pi_3(L_1) & & \end{array}$$

[3] As mentioned in the proof of the implication (3) ⇒ (2) of Theorem 1.38, $\dim(\mathrm{V}^2\mathfrak{g}_i^*)^{\mathfrak{g}_i} = 1$, so the forms B_i are uniquely determined up to a positive constant.

Set $\psi := \iota \circ \phi$. Obviously, the image of $\psi_\# = \iota_\# \circ \phi_\#$ is contained in the image of $\iota_\#$. Hence, it suffices to show that the former one is non-trivial in order to prove that so is the latter.

To conclude, note that ψ is an immersion, since its derivative is injective. In particular, it is not the homomorphism that maps every element of L_1 to the identity of U. Thus, the Dynkin index j_ψ of ψ is not zero by Theorem A.19 (i). Furthermore, by Theorem A.19 (ii), the generator ϵ_{L_1} gets mapped by $\psi_\#$ to $\pm j_\psi \, \epsilon_U$, so $\operatorname{im} \psi_\# \cong j_\psi \mathbb{Z} \neq \{0\}$. □

A.4 Infinite center

We prove now Theorem A.1. Let G be a connected, simple Lie group with infinite center Z and Lie algebra $\mathfrak{g}$. We set $G_0 := G/Z$, which is a connected, center-free Lie group with Lie algebra $\mathfrak{g}$, and let $p : G \twoheadrightarrow G_0$ be the canonical projection. In addition, let

- $\mathfrak{k}$ be a maximal compactly embedded subalgebra of $\mathfrak{g}$, which splits as the direct sum $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{m}$ of its center $\mathfrak{z}(\mathfrak{k})$ and a semisimple or trivial ideal $\mathfrak{m}$;
- $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$,
- $\mathfrak{g}_u = \mathfrak{k} \oplus i\mathfrak{p}$ be the compact dual of $\mathfrak{g}$,
- $K_0 < G_0$ be the connected subgroup with Lie algebra $\mathfrak{k}$,
- $K := p^{-1}(K_0)$,
- $G_{0,u}$ denote the compact dual of G_0,
- $K_{0,u}$ be the connected subgroup of $G_{0,u}$ with Lie algebra $\mathfrak{k}$, and
- G_u be the compact dual of G.

Note first that $\mathfrak{g}$ does not admit a complex structure, because complex Lie groups cannot have infinite center. In consequence, by Theorem A.13, the Lie algebra $\mathfrak{g}_u$ and the compact dual $G_{0,u}$ are simple. Furthermore, being a cover of K_0, the Lie group K has also Lie algebra $\mathfrak{k}$. By Remark A.5, K is connected, closed and non-compact in G. The non-compactness of K implies that $\mathfrak{z}(\mathfrak{k})$ cannot be trivial: if that were the case, then $\mathfrak{k} = \mathfrak{m} \neq 0$, and K would be semisimple, contradicting Theorem A.17.

We distinguish again two cases: $\mathfrak{k}$ Abelian and $\mathfrak{k}$ non-Abelian. We will show that the former assumption corresponds to the situation in which G is isomorphic to $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, and then show that $\mathrm{H}^3_\mathrm{c}(G)$ is one-dimensional in that case. Then, we show that the latter implies vanishing of $\mathrm{H}^3_\mathrm{c}(G)$.

Case 1: $\mathfrak{k}$ is Abelian. We are exactly in the situation of Proposition A.15. Thus, it follows that the symmetric space of non-compact type $G_{0,u}/K_{0,u}$ is diffeomorphic to S^2. Then, via duality, G_0/K_0 is diffeomorphic to the hyperbolic plane H^2, $G_0 \cong \mathrm{PSL}(2,\mathbb{R})$, and $G \cong \widetilde{\mathrm{SL}(2,\mathbb{R})}$, being the only infinite cover of G_0.

Before showing that the dimension of $\mathrm{H}^3_c(G)$ equals one, we prove the following lemma, which will also be useful in the case of $\mathfrak{k}$ non-Abelian.

Lemma A.20. *If* $\dim \mathfrak{z}(\mathfrak{k}) = 1$*, then the connected subgroup* $M < G$ *with Lie algebra* $\mathfrak{m}$ *is maximal compact in* G*.*

Proof. By the semisimplicity of $\mathfrak{m}$ and Theorem A.17, the Lie subgroup $M < G$ is compact. Since K is non-compact, the connected Lie subgroup $R < K$ with Lie algebra $\mathfrak{z}(\mathfrak{k})$ must be non-compact as well. The dimension assumption on $\mathfrak{z}(\mathfrak{k})$ implies that R is isomorphic to $\mathbb{R}$.

Note that the intersection $R \cap M$ is a compact subgroup of $R \cong \mathbb{R}$, hence trivial. From the decomposition $\mathfrak{k} = \mathfrak{z}(\mathfrak{k}) \oplus \mathfrak{m}$ and the previous fact, we obtain an isomorphism $K \cong R \times M$. This implies that M is the unique maximal compact subgroup of K. It is in fact a maximal compact subgroup of G: If $L < G$ is a compact subgroup containing M, then there exists an element $g \in G$ such that $gLg^{-1} < K$. Consequently, $gMg^{-1} < gLg^{-1} < M$. The equalities hold because $gMg^{-1} < M$ and the two are connected subgroups of K with the same Lie algebra. □

Note that in our case $\mathfrak{z}(\mathfrak{k}) = \mathfrak{k} \cong \mathfrak{so}(2, \mathbb{R})$, which is one-dimensional, and $\mathfrak{m}$ is trivial. Hence, the connected subgroup M of G with Lie algebra $\mathfrak{m}$ is therefore trivial, and by the previous lemma, maximal compact in G. Moreover, it is well known that the compact dual of $\mathfrak{g}$ is $\mathfrak{g}_u = \mathfrak{su}(2)$. Thus, G has as compact dual the Lie group $G_u = \mathrm{SU}(2) \cong S^3$. From Corollary A.8 with M_u trivial, we have an isomorphism

$$\mathrm{H}^3_c(G) \cong \mathrm{H}^3(G_u) = \mathrm{H}^3(S^3),$$

and the last one is clearly one-dimensional.

Case 2: $\mathfrak{k}$ is non-Abelian. This assumption means that both $\mathfrak{z}(\mathfrak{k})$ and $\mathfrak{m}$ are non-trivial. By the non-triviality of $\mathfrak{z}(\mathfrak{k})$ and the simplicity of G_0, it follows from Theorems VIII.6.1 and VIII.6.2 of [44] that G_0/K_0 is an irreducible Hermitian symmetric space of non-compact type, and that the center $Z(K_0)$ of K_0 is isomorphic to S^1. In particular, $\dim \mathfrak{z}(\mathfrak{k}) = 1$.

The connected subgroup $M < G$ with Lie algebra $\mathfrak{m} \subset \mathfrak{g}$ is non-trivial and semisimple. By Lemma A.20, it is also a maximal compact subgroup. Thus, by Theorem A.4, $\mathrm{H}^3_c(G) \cong \mathrm{H}^3(\mathfrak{g}, \mathfrak{m})$. On the other hand, by Theorem A.17, the connected Lie subgroup $M_u < G_u$ with Lie algebra $\mathfrak{m}$ is closed. We conclude now by Corollary A.8 and Proposition A.16:

$$\mathrm{H}^3_c(G) \cong \mathrm{H}^3(\mathfrak{g}, \mathfrak{m}) \cong \mathrm{H}^3(G_u/M_u) = 0.$$

□

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