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Cracked Membrane Model with fixed, interlocked cracks – Numerical implementation and validation

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Abstract

The Cracked Membrane Model with fixed, interlocked cracks (CMM-F), whose basic concepts were already outlined in this journal more than 20 years ago, is the most general approach for cracked reinforced concrete members subjected to in-plane stresses as long as one set of uniformly-spaced cracks is considered and steel and bond stresses are modelled by equivalent, uniformly-distributed stresses. However, the CMM-F was not implemented so far, due to the numerical intricacy of the general solution procedure. In this paper, these issues are overcome by determining the variation of steel and concrete stresses and strains between cracks using the Tension Chord Model (TCM), rather than by iteratively integrating over a crack element. After a discussion of the compression field approaches, the TCM and the CMM-F, including the constitutive relationships of concrete and reinforcement and aggregate-interlock models, response predictions obtained from the CMM-F are validated against experimental data. While the agreement is generally good, the predicted stresses at the cracks and the crack kinematics differ significantly between the different aggregate-interlock relationships. These values should be measured in future experiments, using appropriate instrumentation, in order to validate the aggregate-interlock models.

Author keywords: In-plane shear; Plane stress; Compression field approach; Cracked Membrane Model; Aggregate interlock; Fixed-crack model; Load-deformation analysis.

Introduction

The Cracked Membrane Model with fixed, interlocked cracks (CMM-F) (Kaufmann 1998; Kaufmann and Marti 1998) is the most general approach for cracked reinforced concrete members subjected to in-plane stresses as long as (i) only one set of uniformly-spaced cracks is considered, (ii) steel and bond stresses are modelled by equivalent stresses uniformly-distributed over the thickness and in the transverse direction between the reinforcing bars and (iii) dowel action of the reinforcement is neglected. However, the implementation of this general model is numerically intricate even for monotonic loading, particularly because
steel and concrete stresses need to be integrated over the crack element as part of the iterative solution process for each load step, and since the required aggregate-interlock relationships are highly nonlinear and sensitive to very small crack displacements (Kaufmann 1998). Furthermore, currently available aggregate-interlock relationships (e.g., (Gambarova and Karakoç 1983; Li et al. 1989; Walraven 1981) are based on simple, small push-off tests on typically notched or pre-cracked specimens and might not be representative of the stress transfer across cracks in larger concrete members. Therefore, primarily a simplified version of the Cracked Membrane Model considering rotating, stress-free cracks (CMM-R) was used in the past (Kaufmann 1998; Kaufmann and Martí 1998; Seelhofer 2009, Kaufmann and Mata-Falcón 2017), and also implemented in FE-codes by several researchers (e.g., Foster and Marti 2003; Pimentel and Figueiras 2010; Thoma et al. 2014). Using a sandwich model approach (e.g., Karagiannis and Kaufmann 2018; Seelhofer 2009), it has also been extended to the analysis of shell elements subjected to arbitrary loading and recently also implemented as user material in a FE code (Thoma 2018).

The CMM-R yields reliable response predictions for members provided with a minimum amount of reinforcement in (at least) two different directions and is therefore suitable for most practical applications. However, it is less suitable for elements with very little reinforcement (below minimum reinforcement) and, more importantly, cannot be applied to uniaxially reinforced elements subjected to shear (except for cases where a direct strut action is possible): Cracks may occur in any direction but except in special cases, equilibrium at a stress-free crack can only be satisfied with reinforcement stresses in two (or more) directions. The same drawbacks are common to limit analysis design methods neglecting the tensile strength of the concrete and other so-called rotating-crack models (e.g., Baumann 1972; Hsu 1988; Mitchell and Collins 1974), with the exception of the Modified Compression Field Theory (MCFT) (Vecchio and Collins 1986), which considers average tensile stresses in the concrete.

The shear behavior of uniaxially reinforced members can be examined on a mechanically-consistent basis by using models like the CMM-F, considering cracks of fixed direction capable of transferring shear stresses by aggregate interlock and possibly by the dowel action of the reinforcement (which is often neglected at ultimate load). Several such fixed-crack models were developed in the past (e.g., Kupfer et al. 1983; Dei Poli et al. 1987; Rots 1988; Dei Poli et al. 1990; Reineck and Hardjasaputra 1990; Kupfer and Bulicek 1992; Pang and Hsu 1996; Belletti et al. 2001). Furthermore, combinations of rotating and fixed crack models also exist (e.g., Vecchio 2000; Carbone et al. 2001). More recent works on the shear stress transfer in RC panels have been conducted e.g. by Ruggiero et al. (2016) and Calvi et al. (2017). However, all these models typically use simplified aggregate-interlock relationships and semi-empirical parameters calibrated on experiments, which limits their applicability. This also applies to Maekawa et al. 2001, whose fixed crack model was developed with the intention to be used in connection with finite elements or Pimentel et al. 2010, who essentially implemented the CMM with fixed crack inclination based on the simplified approach proposed in (Kaufmann 1998), yet using an empirical softening factor with the adopted aggregate-interlock relationship.
Hence, the implementation of a mechanically consistent model like the CMM-F would be very useful for analyzing the behavior of elements with little or no reinforcement in one direction, which is particularly relevant in the assessment of existing structures. As outlined in the following, this goal was achieved by determining the variation of steel and concrete stresses and strains between cracks using the Tension Chord Model (TCM) (Marti et al. 1998), which greatly reduces the numerical intricacies of the general solution procedure proposed in (Kaufmann and Marti 1998). This paper (i) reviews the basic principles of compression field approaches; (ii) summarizes the basic assumptions of the CMM-F; (iii) presents its numerical implementation; (iv) discusses the relevant parameters, particularly regarding aggregate interlock; and (iv) compares predictions obtained from the CMM-F with experimental evidence and results obtained considering fictitious, rotating cracks.

**Compression Field Approaches**

Compression-field approaches, developed several decades ago (Baumann 1972; Mitchell and Collins 1974), are a powerful tool for analyzing the load-deformation behavior of diagonally cracked concrete members in plane stress. The basic concepts of such models are shown in Fig. 1 for the special case of orthogonal reinforcement. Noting that the sum of forces in the concrete and reinforcement must correspond to the applied loads [Fig. 1(c) and (d)], the following two equivalent sets of equilibrium conditions are obtained

\[
\begin{align*}
\sigma_x &= \sigma_{cx} + \sigma_{sz} = \sigma_{c3} \cos^2 \theta_{ac} + \sigma_{c1} \sin^2 \theta_{ac} + \rho_x \sigma_{sx} & \rho_x \sigma_{sx} &= \sigma_x + \tau_{cs} \cot \theta_{ac} - \sigma_{c1} \\
\sigma_z &= \sigma_{cz} + \sigma_{cz} = \sigma_{c3} \sin^2 \theta_{ac} + \sigma_{c1} \cos^2 \theta_{ac} + \rho_z \sigma_{sz} & \rho_z \sigma_{sz} &= \sigma_z + \tau_{cs} \tan \theta_{ac} - \sigma_{c1} \\
\tau_{xz} &= \tau_{cxs} + \tau_{cxs} = (\sigma_{c1} - \sigma_{c3}) \sin \theta_{ac} \cos \theta_{ac} \quad \sigma_{c3} = -\tau_{cs} \left( \tan \theta_{ac} + \cot \theta_{ac} \right) + \sigma_{c1}
\end{align*}
\]

(1)

if the reinforcing bars are assumed to carry only uniaxial stresses in their direction, i.e. neglecting dowel action, \( \tau_{cxs} = 0 \).

Note that these equations are obtained by formulating equilibrium on suitable free bodies (similar as shown in Fig. 4 for the CMM-F, but with stress-free cracks), but also directly follow from Mohr’s circles of stresses, Fig. 1(d). Here, \( \rho_x, \rho_z = \) reinforcement ratios, \( \sigma_{cx}, \sigma_{cz} = \) steel stresses, \( \sigma_{c1}, \sigma_{c3}, \tau_{cxs} = \) concrete stresses, \( \sigma_{c1}, \sigma_{c3} = \) principal concrete stresses, and \( \theta_{ac} = \) principal direction of concrete compressive stresses. Compatibility requires that the strains [Fig. 1(b)] are related by

\[
\cot^2 \theta_z = \frac{\varepsilon_x - \varepsilon_3}{\varepsilon_x - \varepsilon_1} = \frac{\varepsilon_x - \varepsilon_1}{\varepsilon_x - \varepsilon_3} = \frac{\varepsilon_x - \varepsilon_3}{\varepsilon_x - \varepsilon_1}
\]

(2)

i.e., the state of strain is fully determined by three independent unknowns. Expressing the steel and concrete stresses in terms of any 3 non-collinear strains, e.g. \( (\varepsilon_x, \varepsilon_z, \varepsilon_1) \), and using constitutive relationships relating stresses and strains of concrete and reinforcement, respectively, the states of stress and strain can be determined iteratively by solving Eq. (1) for the 3 unknown strains if the principal directions of concrete stresses and strains are assumed to coincide \( (\theta_{ac} = \theta_z) \). The latter condition is satisfied if fictitious, rotating and orthogonally opening cracks are considered, that are stress-free and parallel to the principal compressive direction of concrete stresses, leading to \( \theta_{ac} = \theta_x = \theta_z = \theta \).
Neglecting tensile stresses in the concrete [Fig. 1(c)], the second set of Eq. (1) simplifies to

\[
\begin{align*}
\rho \sigma_{ss} &= \sigma_s + \tau_s \cot \theta \\
\rho \sigma_{sc} &= \sigma_s + \tau_s \tan \theta \\
\sigma_{r3} &= -\tau_3 \left( \tan \theta + \cot \theta \right)
\end{align*}
\]

(3)

Response predictions obtained from Eq. (3) using the stress-strain characteristics of bare reinforcing steel and concrete in uniaxial compression, as done in early compression field models (Baumann 1972; Mitchell and Collins 1974), are too soft and may markedly overestimate failure loads in case of concrete crushing (Kaufmann 1998). In order to overcome these weaknesses, Vecchio and Collins (1986) proposed the MCFT. In this model, compression softening of the concrete is considered, and tension stiffening is implicitly taken into account by formulating equilibrium in terms of average stresses between the cracks, including average tensile stresses in the concrete [Fig. 1(d)]; thereby, good predictions of the overall load-deformation response are obtained. However, by treating the average tensile stresses in the concrete as a material property, depending only on the principal tensile strain, the respective empirical constitutive equations disregard the fact that tension stiffening strongly depends on the reinforcement ratios. Furthermore, no direct information on the maximum stresses at the cracks nor on crack spacing or crack widths is obtained. These remarks apply to most other models formulating equilibrium in terms of average stresses between cracks, such as the Softened Truss Model (Hsu 1988), with an exception being the Disturbed Stress Field Model proposed by Vecchio (2000), which explicitly considers crack induced effects.

A mechanically consistent way of accounting for tension stiffening in compression field approaches, while still considering rotating, stress-free cracks, was proposed by the CMM-R. In the CMM-R, equilibrium is expressed in terms of stresses at the cracks, i.e. using Eq. (3) replacing \( \theta \) with \( \theta_c \) = crack direction and \( \sigma_s, \sigma_{sc} \) with \( \sigma_{ss}, \sigma_{sp} \) = steel stresses at cracks. Stress-strain relationships of the reinforcement accounting for tension stiffening, expressing the steel stresses at the cracks in terms of the average strains in the direction of the respective reinforcement (\( \varepsilon_m \)) are used. The latter could basically be obtained using any mechanically based model for reinforced concrete tension members; in the CMM-R as well as in the CMM-F, see below, they are obtained from the Tension Chord Model (TCM) (Marti et al. 1998). The TCM is also used to determine crack spacings and variations of the stresses and strains in concrete and reinforcement between the cracks on a mechanically consistent basis.

The TCM assumes a stepped, rigid-perfectly plastic bond shear stress-slip relationship [Fig. 2(b)]. Combined with a bilinear idealization of the stress-strain relationship of the bare reinforcing steel [Fig. 2(a)], this yields the following constitutive equations for a uniaxially loaded tension chord
\[
\sigma_u = E_s \varepsilon_u + \frac{\tau_{50} \delta_s}{\Phi} \sqrt{\left( f_s - E_s \varepsilon_u \right) \frac{s_{sh}^2}{\tau_{sh}} + \frac{E_s}{E_{sh}} \frac{\tau_{50} \delta_s}{\Phi}} \quad \text{for } \sigma_u \leq f_s
\]

\[
\sigma_u = f_s + 2 \frac{f_s - f_g}{E_{sh}} \left( \frac{f_s - f_g}{f_s} \right) + \frac{s_{sh}^2}{\tau_{sh}} \quad \text{for } f_s \leq \sigma_u \leq \left( f_s + \frac{2 \tau_{50} \delta_s}{\Phi} \right)
\]

\[
\sigma_u = f_s + E_{sh} \left( \varepsilon_u - \frac{f_w}{E_s} \right) + \frac{\tau_{50} \delta_s}{\Phi} \quad \text{for } \left( f_s + \frac{2 \tau_{50} \delta_s}{\Phi} \right) \leq \sigma_u \leq f_s
\]

where \( \Phi = \) bar diameter, \( s_r = \) crack spacing, \( f_s, f_g = \) yield and ultimate strength of reinforcement, \( \varepsilon_u = \) strain of bare reinforcement at peak stress \( f_s \), \( E_{sh} = \) \( (f_s - f_g) / (\varepsilon_u - f_g / f_s) \) = hardening modulus of reinforcement, and \( \tau_{50}, \tau_{sh} = \) bond shear stresses for elastic and yielding reinforcement, respectively, with \( \tau_{50} = 2 \tau_{sh} = 2 f_s = 0.6 f_c^{2/3} \) (\( f_c = \) concrete cylinder strength in MPa). Note that Eq. (4) implicitly assume positive slip and bond stresses over the entire length between the crack face and the center between cracks, which is always satisfied in uniaxial tension for applied loads above the cracking load, \( \sigma_u \geq f_s \left( 1 - \rho n \rho \right) / \rho \), and crack spacings smaller than the theoretical maximum crack spacing discussed below.

The bilinear idealization of bare reinforcing steel is suitable for design purposes due to its simplicity, being fully defined by the reinforcement parameters specified by codes, i.e. \( E_s, f_s, f_g \) and \( \varepsilon_u \), respectively, and convenient for numerical analyses due to the closed form analytical expressions [Eq. (4)]. However, more realistic stress-strain characteristics of the bare reinforcement, representing e.g. cold-formed or quenched, self-tempered reinforcing bars, can also be used with the TCM (Alvarez 1998).

The assumed rigid-perfectly plastic bond shear stress-slip relationship in the TCM is of course a strong simplification of the actual bond behavior, where interface stresses depend on the slip to a certain degree. However, in spite of its simplicity, the TCM typically yields as reliable predictions as more refined models (Alvarez 1998; Burns 2012), but without having to numerically integrate the 2nd order ordinary differential equation of bond, making it computationally much more efficient.

The crack spacing for uniaxial tension in the reinforcement direction in Eq. (4) is obtained observing that while \( \sigma_{sh} = 0 \) at the cracks, tensile stresses are transferred to the concrete between the cracks by bond shear stresses [Figs. 2(d) and (e)], and that the tensile stresses in the concrete cannot exceed the concrete tensile strength \( f_c \) at the center between two cracks. Assuming that minimum reinforcement is provided (i.e., the reinforcement does not yield at cracking), the theoretical maximum crack spacings for uniaxial tension in the reinforcement directions are given by

\[
s_{ct} = \frac{\Phi f_c \left( 1 - \rho u \right)}{2 \tau_{50} \rho u}, \quad s_{sh} = \frac{\Phi f_c \left( 1 - \rho u \right)}{2 \tau_{50} \rho c}
\]
Note that the minimum theoretical crack spacing is half the maximum crack spacing, since at the limit, a new crack may form when $\sigma_{ct} = f_{ct}$ is attained at the center between two cracks. Hence, the crack spacing can only be determined by an upper and a lower bound, i.e., $s_{\lambda} = \lambda s_{r0}$ and $s_{r0} = 0.5 \times 1.0 s_{r0}$, with $\lambda = 0.5\ldots1.0$. This uncertainty cannot be avoided, even with more complex bond shear stress-slip relationships.

Considering an orthogonally reinforced panel, Fig. 2(c), both reinforcement directions $(x, z)$ can be treated as tension chords, Figs. 2(d) and (e), and the distribution of bond shear, steel and concrete stresses and hence the stress and strain distribution between two cracks can be determined for any given value of the maximum steel stresses (or strains) at the cracks, Fig. 2(f). However, the crack spacings in the reinforcement directions are now geometrically related by $s_r = s_{\lambda} \sin \theta = s_{r0} \cos \theta$, Fig. 2(c), where $s_r$ denotes the diagonal crack spacing. In order to determine the maximum diagonal crack spacing $s_{r0}$, the stresses in the concrete between the cracks can be determined by superimposing (i) the concrete stresses at the crack obtained from Eq. (3) (uniaxial compression $\sigma_{ct} = 0$, $\sigma_{c3} = -\sigma_{ct} (\tan \theta + \cot \theta)$), with (ii) the stresses transferred to the concrete between the cracks by bond shear stresses from the reinforcing bars in their respective directions. Noting that as for uniaxial tension, the maximum crack spacing $s_{r0}$ is attained when $\sigma_{ct} = f_{ct}$ at the center between two cracks, $s_{r0}$ can be determined analytically (Kaufmann 1998; Kaufmann and Marti 1998). A good upper bound approximation for $s_{r0}$ is given by:

$$s_{r0} = \frac{1}{\sin \theta + \cos \theta} \frac{s_{r0}}{s_{r0}}$$

For uniaxially reinforced elements and elements with reinforcement ratios below minimum reinforcement in one direction (i.e., reinforcement yields at cracking), the maximum diagonal crack spacing in the corresponding direction is theoretically infinite, but Eq. (6) remains valid, simplifying to $s_{r0} = s_{r0} / \sin \theta$ or $s_{r0} = s_{r0} / \cos \theta$, respectively.

Like in uniaxial tension, for $s_r = s_{r0}$ a new crack may form or not at the center between two cracks and consequently, the diagonal crack spacing may also vary by a factor of two, i.e. $s_r = \lambda s_{r0}$, with $\lambda = 0.5\ldots1.0$. However, except for tension in the reinforcement directions, crack spacings in the reinforcement directions are not limited by the conditions $s_{r0} = (0.5\ldots1.0) s_{r0}$ and $s_{r0} = (0.5\ldots1.0) s_{r0}$ as in uniaxial tension. Therefore, the implicit assumption of positive slip and bond stresses over the entire crack element underlying Eq. (4) may not be satisfied at low load levels even after diagonal cracking, leading to unrealistic compressive steel stresses at the center between cracks and an overestimation of the stiffness (Seelhoffer 2009).

This potential inconsistency can be avoided (Seelhoffer 2009) by modifying Eq. (4) at small steel stresses, corresponding essentially to considering an initially cracked element and modelling a pullout behavior until bond stresses are activated over the entire length of the crack element (Fig. 3). Accordingly, the steel stresses obtained from Eq. (4) need to be limited to the values
\[
\begin{align*}
\sigma_{\n} & \leq x_1 \frac{4\tau_{01}}{\Theta} (1 + n\rho) \quad \text{for } \sigma_{\n} \leq f_{\n} \land 0 \leq x_1 \leq \frac{s_1}{2} \\
\sigma_{\n} & \leq f_{\n} + x_{2p} \frac{4\tau_{01}}{\Theta} \quad \text{for } f_{\n} \leq \sigma_{\n} \land 0 \leq (x_{2p} + x_{2p}) \leq \frac{s_1}{2} 
\end{align*}
\]

(7)

where \( n = E_{\n} / E_{\varepsilon} \) is the modular ratio and

\[
\begin{align*}
x_1 & = \frac{s_1}{2} \sqrt{n^2 \rho^2 + \frac{E_{\varepsilon} \Theta}{\tau_{01}} - n\rho} \\
x_{2p} & = \frac{\Theta f_{\n} E_{\n}}{4\tau_{01} (1 + n\rho) E_{\varepsilon}} \left( \frac{s_{1} \tau_{01}}{f_{\n}} \left( \frac{1 + n\rho}{f_{\n} E_{\n}} - n\rho \right) \right) + 1 - 1 \quad \bigg( 0 \leq x_{2p} \leq \frac{s_1}{2} \bigg)
\end{align*}
\]

are the lengths over which bond stresses \( \tau_{01} \) and \( \tau_{02} \), respectively, are activated as illustrated in Fig. 3(a) and (b). The expressions in Eq. (7) can also be used to model the behavior of initially cracked elements in uniaxial tension at low load levels. The resulting stress-strain characteristics are illustrated in Fig. 3(c) for typical values of the parameter \( s_1 \). It can be seen that the reinforcement ratio greatly affects the response, confirming that tension stiffening cannot be treated consistently as a material property of the concrete.

Compression softening of the concrete is accounted for in the CMM-R by using the following stress-strain relationship of the concrete (Kaufmann 1998; Kaufmann and Marti 1998)

\[
\sigma_{\varepsilon} = \frac{\left( 2\varepsilon_{c3} \left( \frac{\varepsilon_{c3}}{\varepsilon_{c0}} \right)^2 \right) f_{\varepsilon}^{2/3}}{0.4 + 30\varepsilon_{c0}} \leq f_{\varepsilon} \quad \text{(9)}
\]

with \( f_{\varepsilon} \) = concrete cylinder strength in MPa and \( \varepsilon_{c0} \approx -0.002 = \) strain at the peak compressive stress

\[
f_{\varepsilon,\text{end}} = f_{\varepsilon}^{2/3} \left( \frac{0.4 + 30\varepsilon_{c0}}{0.4 + 30\varepsilon_{c0}} \right).
\]

Knowing the crack spacing (or rather, selecting a value between its theoretical upper and lower boundaries), setting \( \Theta = \Theta \), and \( \sigma_{\n} = \sigma_{\n} \), \( \sigma_{\varepsilon} = \sigma_{\varepsilon} \) in Eqs. (2) and (3), and using the stress-strain relationships defined by Eq. (9) for the concrete and Eqs. (4)-(8) for the reinforcement, respectively, the response according to the CMM-R is obtained just as outlined above for classic compression field models, i.e. solving Eq. (3) to obtain the state of strain (3 unknowns) for any given set of applied stresses.

**Cracked Membrane Model with Fixed, Interlocked Cracks**

The CMM-F is essentially a compression field approach, satisfying equilibrium as well as compatibility requirements. However, fixed, interlocked rather than rotating, stress-free cracks are considered.

Equilibrium of the stresses at the cracks in an orthogonally reinforced concrete panel with a set of parallel, uniformly-spaced cracks, Fig. 4, requires
\begin{align}
\sigma_x &= \rho \sigma_{eyr} + \sigma_{ey} \sin^2 \theta + \sigma_{ey} \cos^2 \theta - 2\tau_{eyr} \sin \theta \cos \theta, \\
\sigma_y &= \rho \sigma_{eyr} + \sigma_{ey} \cos^2 \theta + \sigma_{ey} \sin^2 \theta + 2\tau_{eyr} \sin \theta \cos \theta, \\
\tau_{xy} &= (\sigma_{eyr} - \sigma_{ey}) \sin \theta \cos \theta - \tau_{eyr} \left( \cos^2 \theta - \sin^2 \theta \right)
\end{align}

where \( n \) and \( t \) are the coordinates aligned with the crack direction \( \theta_c \). As in the CMM-R, steel and bond shear stresses (acting between concrete and reinforcement) are replaced by equivalent stresses, and homogeneous material properties are assumed. Hence, the displacements at the cracks as well as the strains in the concrete between the cracks are independent of the coordinate \( t \). Fig. 4(b), implying that \( \varepsilon_{cr} = \text{constant} \) (Kaufmann and Marti 1998) and that consideration of the conditions along the \( n \)-axis \( (t = 0) \) is sufficient. If the crack inclination \( \theta_c \) and the crack spacing \( s_c \) are known, it is therefore possible to determine the complete states of stress and strain for given constitutive equations, i.e. stress-strain relationships for the concrete and the reinforcing steel as well as a bond shear stress-slip relationship and an aggregate-interlock model

\begin{align}
\sigma_{eyr} &= \sigma_{eyr} (\delta, \delta) \\
\tau_{eyr} &= \tau_{eyr} (\delta, \delta)
\end{align}

relating the crack opening and sliding displacements, \( \delta_s \) and \( \delta_i \), and the normal and shear stresses \( \sigma_{eyr} \) and \( \tau_{eyr} \) acting on the crack surface. Note that for \( \tau_{eyr} \neq 0 \) and \( \delta_i \neq 0 \), the principal direction of concrete stresses at the cracks, \( \theta_{eyr} \), and the principal strain direction, \( \theta_e \), respectively, deviate from the crack direction \( \theta_c \), i.e. \( \theta_{eyr} \neq \theta_c \neq \theta_e \).

A suitable solution procedure involving the steel stresses \( \sigma_{eyr} \) and \( \sigma_{ey} \) and the concrete stress component \( \sigma_{eyr} \) at the cracks, the crack opening \( \delta_s \) and the crack slip \( \delta_i \) as well as the concrete displacement components \( u_{ey} \) and \( v_{ey} \) at the cracks as the primary unknowns was presented in Kaufmann and Marti (1998). This algorithm could be implemented in a computer program; however, the solution is numerically intricate as mentioned in the introduction; it essentially involves simultaneously solving a set of seven highly nonlinear equations for the primary unknowns.

An alternative, simplified solution procedure was proposed in Kaufmann (1998). This approach is based on a subdivision of the total strains \( \varepsilon = \varepsilon^{(nx)} \) into strains \( \varepsilon^{(c)} \) due to the deformations of the concrete between the cracks and strains \( \varepsilon^{(r)} \) caused by crack kinematics (as proposed by Rots (1988)), i.e.

\begin{align}
\varepsilon = \varepsilon^{(nx)} = \varepsilon^{(c)} + \varepsilon^{(r)}
\end{align}

as illustrated in Fig. 5; note that the strains \( \varepsilon_c \), \( \varepsilon^{(r)} \) and \( \varepsilon^{(r)} \) are average values over a crack element. Considering a crack opening at an arbitrary angle \( \alpha_c \) [Fig. 5(b)], the strains caused by crack kinematics \( \varepsilon^{(r)} \) are given by

\begin{align}
\varepsilon_{ex}^{(r)} = \frac{\delta}{s_c}, \quad \varepsilon_{ey}^{(r)} = 0, \quad \gamma_{ex}^{(r)} = \frac{\delta}{s_c}
\end{align}
i.e., they contain only two unknowns if \( \theta_n \) (defining the directions \( n \) and \( t \)) and \( s_n \) are known. Steel stresses at the cracks can then be determined from Eqs. (4)-(8), relating the components of the total strains \( \varepsilon \) in the reinforcement directions to the steel stresses at the cracks. In order to avoid the numerical intricacies of the general algorithm, Kaufmann (1998) proposed to neglect local variations of the concrete stresses and strains between the cracks and use stress-strain relationships relating the concrete stresses at the cracks to the strains \( \varepsilon^{(c)} \) (and potentially also \( \varepsilon^{(r)} \), e.g. to account for compression softening effects governed by the total strain component \( \varepsilon_t \)). Knowing the crack inclination \( \theta_c \), e.g. from an uncracked analysis, and estimating the crack spacing from Eq. (6), the strains \( \varepsilon^{(r)} \) and \( \varepsilon^{(c)} \) – and thus the total strains \( \varepsilon = \varepsilon^{(c)} + \varepsilon^{(r)} \) – can then be determined by solving the equilibrium conditions (10) and the aggregate-interlock relationship (11), i.e., 5 equations for five unknowns, as implemented by Pimentel et al. (2010).

Neglecting the variations of concrete stresses and strains between the cracks is, strictly speaking, inconsistent, but appears acceptable at first glance since typically, \( |\varepsilon^{(r)}| \ll |\varepsilon^{(c)}| \) and hence, small errors in \( \varepsilon^{(r)} \) should have a minor influence on the overall response. However, aggregate-interlock relationships, discussed in the following section, are sensitive to small changes in crack openings and hence, even minor differences in the concrete strains between the cracks may significantly influence the overall response.

These inconsistencies can be avoided if instead of the concrete stresses at the cracks \( \sigma_c \), characteristic concrete stresses \( \sigma_{ck} \), essentially corresponding to the average concrete stresses between the cracks and therefore representative for the deformations of the concrete between the cracks, are used to determine the concrete strains \( \varepsilon^{(c)} \). The characteristic concrete stresses \( \sigma_{ck} \) can be determined by superimposing the concrete stresses at the cracks, \( \sigma_{cr} \), with the average value of stresses transferred to the concrete by bond shear stresses over the entire crack element in each reinforcement direction, just like in the determination of the maximum crack spacing in the CMM-R. For reinforcing bars that are either elastic or yielding over the entire crack element, \( \sigma_{ck} \) are the concrete stresses at the quarter points between the cracks, similar as assumed in the analytical solution for cracked-elastic behavior in Kaufmann (1998) and Kaufmann and Mata-Falcón (2017). Furthermore, concrete stresses \( \sigma_{cr} \) at the center between cracks can be evaluated in order to check whether the tensile strength of the concrete is exceeded between the original cracks, meaning that a second set of cracks would have to be considered in subsequent load stages (which is, however, beyond the scope of this paper). No additional unknowns are introduced by \( \sigma_{ck} \) (nor \( \sigma_{cr} \)) in addition to \( \sigma_{cr} \) since the differences between these three sets of stresses, illustrated in Fig. 6, can be obtained using the TCM, noting that the steel and concrete stress components in the reinforcement directions are related by equilibrium:
\[
\frac{d\sigma_{z}}{dx} = \frac{1 - \rho_s d\sigma_{zs}}{\rho_s} = \frac{4r_s}{\Theta_s} \quad \text{i.e.} \quad \rho_s\sigma_{zs} + (1 - \rho_s)\sigma_{zs} = \text{const}
\]
\[
\frac{d\sigma_{x}}{dx} = -\frac{1 - \rho_c d\sigma_{xc}}{\rho_c} = \frac{4r_c}{\Theta_c} \quad \text{i.e.} \quad \rho_c\sigma_{xc} + (1 - \rho_c)\sigma_{xc} = \text{const}
\] (14)

Note that different reinforcement ratios and bond characteristics of the horizontal and vertical reinforcement result in rotations of the principal stress directions between the cracks, i.e., \( \theta_{ovr} \neq \theta_{ovc} \neq \theta_{ovc} \). The most direct way of calculating \( \sigma_{zk} \) consists in determining the average value \( \sigma_{zm} \) of steel stresses over a crack element (in each reinforcement direction) and multiplying the difference \( \sigma_{zm} - \sigma_{zs} \) by \( \rho/(1 - \rho) \), see Eq. (14).

The characteristic concrete stresses \( \sigma_{ck} \) and the average strains of the concrete between the cracks \( \varepsilon^{(c)} \) are related by the constitutive equations of concrete loaded in biaxial tension-compression. In the direction of the larger principal strain \( \varepsilon_{1}^{(c)} \), relatively small concrete stresses are expected and hence, elastic behavior is assumed, i.e.

\[
\sigma_{ck} = E_c \varepsilon_{1}^{(c)} \quad (\sigma_{ck} \leq f_{ct})
\] (15)

Note that \( \varepsilon_{1}^{(c)} \) and \( \sigma_{ck} \) may be tensile or compressive, depending on the aggregate-interlock stresses at the crack (where usually \( \sigma_{con} < 0 \)), the crack spacing, and the bond characteristics of the reinforcement.

In the direction of the smaller principal strain \( \varepsilon_{2}^{(c)} \), which is always compressive, selecting a suitable stress-strain relationship of the concrete is not straightforward. On one hand, transverse tensile strains imposed to the concrete by the reinforcement disturb the compressive stress field in the concrete, just as in rotating crack models where this is considered by using stress-strain relationships accounting for compression softening, such as Eq. (9). On the other hand, the latter relationships predict a very pronounced compression softening particularly at high transverse strains, which implicitly accounts for failures by crack sliding (Beck and Kaufmann 2018; Kaufmann et al. 2018), that otherwise cannot be captured in rotating crack models.

Since such failures are explicitly covered in the CMM-F (by the aggregate-interlock relationships), using a less pronounced compression softening relationship appears appropriate. Nonetheless, in the current implementation of the CMM-F, Eq. (9) is used without changes; note that as long as no failures by concrete crushing are predicted (as typical in elements with low transverse reinforcement ratios), this choice does not significantly affect the response predictions.

**Aggregate-Interlock Relationships**

Aggregate-interlock relationships, describing the transfer of stresses across cracks and the associated crack kinematics, are a key element of models considering cracks with fixed directions. Generally speaking, aggregate interlock is relevant in any situation where cracks are not aligned with the principal stress directions; such cracks may have nucleated by previous, non-proportional loading (e.g. restrained shrinkage) or initial flexural cracks in webs of girders. More specifically, in an
orthotropically reinforced element, the principal direction of concrete compressive stresses typically rotates towards the direction of the stronger reinforcement, particularly after the onset of yielding of the weaker reinforcement (i.e., becoming “flatter” in webs of girders once the stirrups yield). While rotating crack models account for this by adjusting the direction of the fictitious cracks accordingly, stresses have to be transferred across cracks formed at an earlier load-stage if the crack inclination remains fixed after cracking.

As illustrated in Fig. 7(a), aggregate-interlock stresses consist of a combination of normal and shear stresses, \( \sigma_{cnr} \) and \( \tau_{cnr} \), acting on the crack surface. These stresses may reach values in the order of the concrete compressive strength, but they are limited to the strength of uncracked concrete in plane stress (\(- f_c \leq \sigma_c \leq f_c\) in any direction), i.e. \( |\sigma_{cnr}| \leq f_c \) and \( |\tau_{cnr}| \leq (f_c + f_u)/2 \) for zero crack opening. Most existing models for aggregate interlock postulate unique relationships relating the crack opening and sliding displacements, \( \delta_n \) and \( \delta_t \), to \( \sigma_{cnr} \) and \( \tau_{cnr} \) and thus neglect the fact that the behavior is path dependent and therefore, an appropriate constitutive description of aggregate interlock would require an incremental formulation (Kaufmann 1998). This type of aggregate-interlock relationship is also used in the CMM-F, see Eq. (11), where it should be kept in mind that the quantities \( \delta_n \), \( \delta_t \), \( \sigma_{cnr} \), and \( \tau_{cnr} \) are average values over large crack areas because of the irregular nature of the crack surfaces.

Three aggregate-interlock relationships are discussed here, and implemented in the CMM-F: (i) the Two-Phase Model (TPM) (Walraven 1979, 1980, 1981); (ii) the Rough Crack Model (RCM) (Bažant and Gambarova 1980), later refined by Gambarova and Karakoç (1983) and (iii) the Contact Density Model (CDM) (Li and Maekawa 1987), later refined by (Li et al. 1989).

In the TPM [Fig. 7(b)], aggregate-interlock stresses are determined assuming randomly distributed spherical, rigid aggregate particles penetrating a rigid-perfectly plastic mortar matrix, yielding the expressions

\[
\sigma_{cnr}(\delta_n, \delta_t) = -\sigma_{pu}(A_l - \mu \cdot A_n)
\]
\[
\tau_{cnr}(\delta_n, \delta_t) = \sigma_{pu}(A_n + \mu \cdot A_t)
\]

where \( \mu \) is the coefficient of friction and \( \sigma_{pu} \) is denoted as the matrix yielding stress. Note, however, that the assumed combination of normal stresses, \( -\sigma_{pu} \), and shear stresses, \( \mu \sigma_{pu} \), corresponds to principal compressive stresses exceeding \( \sigma_{pu} \) at the aggregate-matrix interface Kaufmann et al. (2019). The values \( A_l \) and \( A_n \) are statistically evaluated integrals of the projections of the aggregate-matrix contact surfaces. They depend on the crack displacements, \( \delta_n \) and \( \delta_t \), on the maximum
aggregate size $D_{\text{max}}$, the grading curve of the aggregates and the volume fraction of aggregates per unit volume of concrete. Walraven (1981) proposes the empirical values $\mu = 0.4$ and $\sigma_{\text{cu}} = 6.39 f_{\text{c}}^{0.56}$, where $f_{\text{c}}$ is the cube strength of concrete in MPa, based on the evaluation of numerous tests. In high strength concrete, cracks tend to pass through the aggregate particles, as opposed to around them, leading to smoother crack surfaces with less aggregate-interlock capacity. In these cases, Walraven and Stroband (1994) proposed a reduction factor of 0.35 applied on $\sigma_{\text{cr}}$ and $\tau_{\text{cr}}$ which was again calibrated on tests. This reduction factor has been adopted by the fib Model Code 2010 (International Federation for Structural Concrete 2013), where simplified linear equations for aggregate interlock based on Walraven (1981) are proposed. However, these simplified relationships deviate substantially from Eq. (16) for small as well as large crack openings or slip. Therefore, the general relationships are used in the CMM-F.

In the original RCM (Bažant and Gambarova 1980), Bažant and Gambarova idealized the crack surface as a regular array of trapezoidal asperities. They proposed aggregate-interlock relationships based on this geometrical assumption and general, theoretical considerations of the crack interface regarding crack kinematics and stress state, and they calibrated the resulting model on tests with constant crack opening. The primary influencing parameter for the shear stresses in their model is the ratio $\delta_b / \delta_a$. Gambarova and Karakoç (1983) later modified the RCM, introducing the effect of the maximum aggregate size $D_{\text{max}}$ and re-calibrating the relationships on tests with a constant confinement stress as they considered this boundary condition more representative of real cracks than a constant crack opening. The resulting aggregate-interlock relationships are given as

$$\sigma_{\text{cr}}(\delta_a, \delta_b) = -0.62 \frac{\delta_b}{\sqrt{\delta_a^2 + \delta_b^2}} \tau_{\text{cr}}(\delta_a, \delta_b) \quad (\delta_a, \delta_b \text{ in mm})$$

$$\tau_{\text{cr}}(\delta_a, \delta_b) = \tau_0 \left( 1 - \frac{2.45 \delta_a}{D_{\text{max}}} \right) \frac{2.44 \left(1 - \frac{4 \text{MPa}}{\tau_0} \right)}{1 + 2.44 \left(1 - \frac{4 \text{MPa}}{\tau_0} \right)} \left( \frac{\delta_b}{\delta_a} \right)^4$$

with $\tau_0 = 0.25 \ldots 0.30 f_{\text{c}}$.

In the CDM (Li and Maekawa 1987; Li et al. 1989) [Fig. 7(c)], the asperity of the crack surface is divided into infinitely small potential contact planes, the contact units $dA_{\text{n}}$ with inclination $\theta$ with respect to the $n$-axis. The contact area is determined using a contact density function $\rho(\theta)$. Li and Maekawa (1987) claimed that frictional forces on the contact planes would lead to discontinuities in the orientation of aggregate-interlock stresses at slip reversals, which they did not observe in experiments, and thus, they assumed the contact stresses to act perpendicularly to the individual contact planes. Assuming a rigid-perfectly plastic behavior of the contact stresses, the contact forces can then be computed by integration, and after simplification for monotonic loading, the aggregate-interlock stresses are given as
\[
\sigma_{\text{cr}} (\delta_x, \delta_y) = -\tau_{\text{lim}} (\delta_y) \left( \frac{\mu}{2} - \tan^{-1} \frac{\delta_x}{\delta_y} - \frac{\delta_x \delta_y}{\delta_x^2 + \delta_y^2} \right)
\]

\[
\tau_{\text{cr}} (\delta_x, \delta_y) = \tau_{\text{lim}} (\delta_y) \cdot \frac{\delta_x^2}{\delta_x^2 + \delta_y^2}
\]

with \( \tau_{\text{lim}} (\delta_y) = K(\delta_y) \cdot 3.83 \left( \frac{\mu}{\text{MPa}} \right)^{1/3} \text{MPa} \) \( K(\delta_y) = 1 - \exp(1 - D_{\text{max}} / 2 \delta_y) \) determined based on experimental results as \( K(\delta_y) = 1 - \exp(1 - D_{\text{max}} / 2 \delta_y) \) according to Li et al. (1989), which is used in the response predictions shown in this paper.

The three aggregate-interlock models Eqs. (16)-(18) are compared in Fig. 7(d). They show a qualitatively similar behavior, but there are substantial quantitative differences in the predicted normal and shear stresses.

### Numerical implementation

Knowing the constitutive relationships of concrete and reinforcement, the solution procedure proposed by Kaufmann (1998), i.e. determining the five primary unknowns \( \epsilon^{(v)} \) and \( \epsilon^{(e)} \) using equilibrium conditions, kinematic compatibility the TCM and one of the aggregate-interlock models discussed above, can essentially be applied. A suitable algorithm for a shear test, with constant axial stresses \( \sigma_x \) and \( \sigma_z \), on an initially cracked element is shown in Fig. 8.

### Comparison with Experimental Results

Fig. 9 illustrates the comparison of experimental results of five panels with variable reinforcement ratios, tested at the University of Toronto under pure membrane shear loading, with response predictions obtained from the CMM-F and CMM-R:

Three orthogonally reinforced specimens (Kirschner and Collins 1986) with identical, strong longitudinal reinforcement \( \rho_x = 2.93\% \) but different transverse reinforcement ratios of \( \rho_z = 2.93\% \) (SE5), \( \rho_z = 0.98\% \) (SE1), and \( \rho_z = 0.33\% \) (SE6), respectively, shown in Fig. 9(a); and two uniaxially reinforced specimens \( \rho_z = 0\) (Bhide and Collins 1989), with different longitudinal reinforcement ratios of \( \rho_x = 1.08\% \) (PB12) and \( \rho_x = 2.02\% \) (PB18), respectively, shown in Fig. 9(b). The predictions were carried out for maximum \( \lambda = 1\) (solid lines) and minimum \( \lambda = 0.5\) (dashed lines) theoretical crack spacings.

For the CMM-F, all three aggregate-interlock relationships discussed in this paper were used for comparison; the corresponding aggregate-interlock stresses and crack kinematics are also shown in Fig. 9.

In the CMM-F analyses, the direction of the fixed cracks was chosen perpendicular to the principal tensile stresses according to the uncracked elastic behavior, i.e., approximately 45° for pure shear. This corresponds well with the experimentally observed crack directions in the orthogonally reinforced panels (SE1, SE5, and SE6). In the uniaxially reinforced specimens (PB12 and PB18), cracks forming close to the ultimate load tended to propagate along the reinforcing bars (Bhide and Collins 1989),
particularly in Specimen PB18 (presumably due to the reduced spacing of the reinforcing bars), resulting in an average crack inclination closer to the reinforcement directions.

In the isotropically reinforced (\(\rho_s = \rho_z\)) Specimen SE5, the principal stress and strain directions remain constant, and cracks open orthogonally. Consequently, the CMM-F predicts vanishing aggregate-interlock stresses, and response predictions obtained using any of the three aggregate-interlock relationships are identical. These predictions are slightly stiffer than the one obtained from the CMM-R since the same constitutive equations are used for the concrete, but concrete stresses are accounted for at different locations, i.e. characteristic concrete stresses in the CMM-F, but maximum stresses at the cracks in the CMM-R.

In the orthotropically reinforced (\(\rho_s > \rho_z\)) Specimens SE1 and SE6, the lower amount of reinforcement in the z-direction results in skew crack openings and correspondingly increasing aggregate-interlock contributions in the CMM-F, while progressively flatter inclinations of the fictitious cracks are obtained in the CMM-R. These effects increase the ductility, but drastically reduce the shear strength of the panels, particularly in the case of Specimen SE6. The crack spacing (parameter \(\lambda\)) primarily affects the predicted deformation capacity, having a more pronounced effect for smaller transverse reinforcement ratios. Furthermore, it can be seen from Fig. 9(a) that the CMM-R tends to overestimate the ultimate load and ductility in the panels with low reinforcement ratios, and that the CMM-F using the CDM (Li et al. 1989) best predicts the experimental results in the orthogonally reinforced panels considered here.

For Specimens PB12 and PB18, only predictions obtained from the CMM-F are shown in Fig. 9(b), since the CMM-R cannot be used for uniaxially reinforced panels. It can be seen that the choice of the aggregate-interlock model has a much more pronounced influence on the response predictions than in the orthotropically reinforced panels, despite the similar magnitude of aggregate-interlock stresses. In the uniaxially reinforced panels, the ultimate load is overestimated when using the CDM (Li et al. 1989), particularly in Specimen PB18. This could be explained by the fact that the flatter cracks following the reinforcing bars observed near ultimate load (see above) are not accounted for in the predictions; considering a flatter crack inclination would significantly reduce the predicted ultimate load. On the other hand, however, predictions obtained from the TPM (Walraven 1981) and RCM (Gambarova and Karakoç 1983) closely match the experimentally observed response of Specimen PB18. Unfortunately, the significant differences between the predictions obtained using different aggregate-interlock models cannot be examined in more detail since neither crack kinematics nor stresses at cracks were measured in any of the experiments.

**Conclusion**

In this paper, (i) existing compression field approaches are summarized; (ii) the relevant features – i.e., constitutive relationships of reinforcement and concrete, aggregate-interlock models – of the Cracked Membrane Model with fixed, interlocked cracks (CMM-F) are discussed; and (iii) its numerical implementation is outlined. A main advantage of the CMM-F lies in its rigorous mechanical formulation, allowing the use of constitutive equations and aggregate-interlock relationships.
obtained from pertinent, specific tests, rather than relying on empirically calibrated “smeared” constitutive relationships. Furthermore, concrete stresses and strains are determined both at the cracks (where equilibrium is formulated) as well as between cracks, yet without introducing additional unknowns since all stresses can be obtained from the concrete stresses at the cracks using the Tension Chord Model. This eliminates potential errors in crack kinematics and in the overall response when neglecting the variations of concrete stresses and strains between the cracks.

Response predictions obtained from the CMM-F using three different aggregate-interlock models are compared to representative experimental data and predictions obtained from the Cracked Membrane Model considering fictitious, rotating cracks (CMM-R). The comparison with experimental results reveals that the agreement is generally good, and that the CMM-F yields more accurate response predictions for panels with low transverse reinforcement than the CMM-R. Furthermore, the CMM-F is capable of predicting the behavior of uniaxially reinforced panels, where the CMM-R cannot be applied. However, response predictions obtained from the CMM-F for elements with low transverse reinforcement ratios, and even more so in the uniaxially reinforced panels, differ significantly, depending on the crack spacing and the aggregate-interlock relationship used. Unfortunately, since neither crack kinematics nor stresses at cracks were measured in the tests, these differences cannot be examined in more detail based on the existing experimental evidence. These values should therefore be measured in future experiments, using appropriate instrumentation, in order to validate and, if applicable, update the aggregate-interlock models. An in-depth experimental validation of several aspects of the CMM-F, with respect to both new own experiments (currently under way) as well as tests documented in the literature, including non-proportional loading, is currently in progress.
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