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Fixed point and Lipschitz extension theorems for barycentric metric spaces

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So many people have come and gone
Their faces fade as the years go by
Yet I still recall as I wander on
As clear as the sun in the summer sky

Boston – *More than a feeling*

Abstract

The subject of this doctoral thesis is the class of barycentric metric spaces, which encompasses both Banach spaces and complete CAT(0) spaces. Encouraged by known results as well as open questions in the context of CAT(0) spaces, we study similar objectives in the framework of barycentric metric spaces. For example, we show that certain fixed point properties, which are given in CAT(0) spaces, do not hold for some barycentric metric spaces, and prove two fixed point results adapted to the new situation. These results are phrased for the class of metric spaces that allow a conical bicombing; this is no restriction, since the class of barycentric metric spaces agrees with this class. This equality leads to a variety of questions regarding the existence and uniqueness of certain classes of conical bicombings. In particular, we consider conical bicombings on open subsets of normed vector spaces and show that these bicombings are locally given by linear segments. This result implies that any open convex subset in a large class of Banach spaces possesses a unique consistent conical bicombing.

Besides this, we consider various Lipschitz extension problems, where in some cases any complete barycentric metric space may appear as target space. One such Lipschitz extension problem involves the extension of a Lipschitz function to finitely many additional points. Our contribution consists of finding upper bounds for the distortion of the Lipschitz constant, and we construct examples which demonstrate that we found the best possible bounds in the case of an extension to one additional point. Many Lipschitz extension constants may be computed by solving an associated linear extension problem, which is why, in the last part, we turn our attention to absolute linear projection constants of real Banach spaces. We succeeded in finding a formula for the maximal linear projection constant amongst n-dimensional Banach spaces. By means of this formula, we give another proof of the Grünbaum conjecture, which was first proven by Chalmers and Lewicki in 2010.

Zusammenfassung

Der Gegenstand dieser Doktorarbeit ist die Klasse der baryzentrischen metrischen Räume, die sowohl Banachräume wie auch vollständige CAT(0)-Räume umfasst. Motiviert durch bekannte Sätze und offene Fragen im Kontext der CAT(0)-Räume untersuchen wir Ähnliches im Rahmen der baryzentrischen Räume. Beispielsweise zeigen wir, dass gewisse Fixpunkteigenschaften, welche in CAT(0)-Räumen gegeben sind, für manche baryzentrische Räume nicht mehr gelten, und beweisen, angepasst an die neue Situation, zwei Fixpunktsätze. Diese Sätze sind für die Klasse der metrischen Räume, die ein konisches Bicombing zulassen, formuliert; dies ist keine Einschränkung, da die Klasse der baryzentrischen Räume mit dieser identisch ist. Diese Gleichheit öffnet die Tür für verschiedene Fragestellungen, welche die Eindeutigkeit und Existenz gewisser Klassen von konischen Bicombings betreffen. Insbesondere betrachten wir konische Bicombings auf offenen Teilmengen von normierten Vektorräumen und zeigen, dass diese Bicombings lokal durch lineare Segmente gegeben sind. Dieses Resultat hat zur Folge, dass offene konvexe Mengen in einer grossen Klasse von Banachräumen ein eindeutiges konsistentes konisches Bicombing besitzen.

Unabhängig davon betrachten wir verschiedene Lipschitz Erweiterungsprobleme, bei denen teilweise jeder vollständige baryzentrische Raum als Zielraum zugelassen ist. Eine von uns untersuchte Problemstellung beinhaltet die Erweiterung einer Lipschitz Funktion auf endlich viele zusätzliche Punkte. Unser Beitrag besteht darin, obere Schranken für die Verzerrung der Lipschitz Konstante anzugeben und wir konstruieren Beispiele, die aufzeigen, dass unsere Schranken im Falle der Erweiterung auf einen Punkt bestmöglich sind. Viele Lipschitz Erweiterungskonstanten lassen sich berechnen, indem man ein assoziiertes lineares Erweiterungsproblem löst, weswegen wir uns zuletzt der absoluten linearen Projektionskonstante eines reellen Banachraums zuwenden. Es ist uns gelungen, eine Formel für die maximale Projektionskonstante n-dimensionaler Banachräume herzuleiten. Mittels dieser Formel geben wir einen weiteren Beweis der Grünbaumschen Vermutung, welche erstmals 2010 von Chalmers und Lewicki bewiesen wurde.

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Contents

1 Overview

Be it by accident or not, some of the methods and results from the realm of Banach spaces transfer readily to CAT(0) spaces. Often, this transfer happens almost verbatim. Moreover, several notions in the theory of CAT(0) spaces were motivated by a linear role model. In view of these connections, the search for a reasonable definition of "space" that includes both Banach and CAT(0) spaces seems natural. Barycentric metric spaces form a class of metric spaces that achieves this objective, and they are the main objects of study of the present doctoral thesis. They appear in two ways: first, we study questions regarding their geometry, and secondly, they serve as target spaces in some of the Lipschitz extension problems that are considered.

The family of barycentric metric spaces possesses useful structural properties: it is closed under ultralimits and 1-Lipschitz projections. Furthermore, the lesser-studied complete Busemann spaces and the injective metric spaces are barycentric metric spaces. Due to the many members of the class of barycentric metric spaces and its structural properties, one may wonder if a unified treatment of these spaces could be something worth pursuing. Luckily, the presence of non-positive curvature, in the sense that every barycentric metric space admits a conical geodesic bicombing, does indeed lead to many interesting geometric questions. Some of these questions are answered in the first part of the thesis.

The second part is more analytic in nature, as we study various Lipschitz extension problems, of which some allow any complete barycentric metric space as target space. For instance, we consider the problem of extending such Lipschitz maps defined on certain F-transforms of a Hilbert space to finitely many additional points. The classical linear projection constants of real Banach spaces are also studied in detail. We recall their close connection to several well-known non-linear Lipschitz extension moduli and derive a formula for the maximal linear projection constant amongst n-dimensional Banach spaces. Using this formula we give an alternative proof of the Grünbaum conjecture, which was first proven by Chalmers and Lewicki in 2010.

The bulk of this thesis is based on the articles [Bas18a; Bas18b; BM19] and [Bas19]. We proceed by presenting our results.

1.1 The geometry of barycentric metric spaces

1.1.1 – Following Sturm, cf. [Stu03, Remark 6.4], a 1-Lipschitz map $\beta\colon P_1(X) \to X$ with $\beta(\delta_x) = x$ for all $x \in X$ is called a *contracting barycenter map.* Here, (X, d_X) is a metric space and $P_1(X)$ denotes the set of all Radon probability measures on X with finite first moment. We equip $P_1(X)$ with the 1-Wasserstein distance W_1. A *barycentric metric space* is a metric space (X, d_X) that admits a contracting barycenter map. Occasionally, we denote barycentric metric spaces by (X, d_X, β) to emphasize the contracting barycenter map.

Every complete CAT(0) space admits a contracting barycenter map. Indeed, the Cartan barycenter map is a contracting barycenter map, cf. [LPS00; Stu03]. Moreover, Navas established that every complete Busemann space is a barycentric metric space, cf. [Nav13]; see [Des16] for a streamlined proof thereof. A contracting barycenter map β distinguishes a family $\{\sigma_{xy}(\cdot)\}_{x,y\in X}$ of geodesics of X. Throughout the thesis, a *geodesic* is a map $\sigma\colon [0,1] \to X$ such that $d(\sigma(s), \sigma(t)) = |s-t|\, d(\sigma(0), \sigma(1))$ for all $0 \leq s, t \leq 1$. For $x, y \in X$ we define the geodesic $\sigma_{xy}(\cdot)$ via

$$\sigma_{xy}(t) := \beta((1-t)\delta_x + t\delta_y), \qquad \text{for all } t \in [0,1]. \tag{1.1}$$

It is not hard to check that the map $\sigma\colon X \times X \times [0,1] \to X$ given by $(x, y, t) \mapsto \sigma_{xy}(t)$ satisfies the following weak, but non-coarse, global non-positive curvature condition:

$$d_X(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t)d_X(x, x') + t d_X(y, y'), \tag{1.2}$$

for all points $x, y, x', y' \in X$ and all real numbers $t \in [0,1]$. Thus, σ is a *conical geodesic bicombing* in the terminology of [DL15], see Section 2.1. Conversely, a complete metric space with a conical geodesic bicombing also admits a contracting barycenter map:

Theorem 1.1. *Let (X, d_X) be a complete metric space. The following are equivalent:*

1. *X is a barycentric metric space.*
2. *X admits a conical geodesic bicombing.*

The proof of Theorem 1.1 is given in Section 2.3. The key component in the proof is a 1-Lipschitz barycenter construction that traces back to A. Es-Sahib and H. Heinich, cf. [ESH99], and A. Navas, cf. [Nav13]. Moreover, we use a result due to Miesch which allows us to pass to a reversible conical bicombing starting from a conical geodesic bicombing, cf. [Mie17a, p.87]. The class of complete CAT(0) spaces is closed under ultralimits and

1-Lipschitz retractions. Due to Theorem 1.1, one may readily verify that the class of complete barycentric metric spaces enjoys the same properties. Recently, other classical results from the theory of CAT(0) spaces have been transferred to barycentric metric spaces, cf. [Des16; DL16; Mie17b; Kel19].

1.1.2 – It is well-known that if (X, d_X) is a complete CAT(0) space, then every subgroup of the isometry group of X with bounded orbits has a non-empty fixed point set, cf. [BH99, Corollary II.2.8]. Analogous results hold for a wide variety of metric spaces. For example, the above statement holds if the metric space (X, d_X) is an L-embedded Banach space or an injective metric space, cf. [BGM12, Theorem A] and [Lan13, Proposition 1.2]. Further results can be found in [KL10; Ede64]. It turns out that if (X, d_X) is a complete Busemann space instead of a complete CAT(0) space, then there exists a fixed point free isometry with bounded orbits. This is discussed in Section 2.7.

Let $\varphi\colon X \to X$ be an isometry of (X, d_X) and let $\sigma\colon X \times X \times [0,1] \to X$ be a conical geodesic bicombing. We say that *σ is φ-equivariant* if $\varphi \circ \sigma_{xy} = \sigma_{\varphi(x)\varphi(y)}$ for all points x, y in X. Let Σ be a subsemigroup of the isometry group of X. We say that *σ is Σ-equivariant* if σ is s-equivariant for every isometry $s \in \Sigma$.

Let $\sigma\colon X \times X \times [0,1] \to X$ be a conical geodesic bicombing and let $A \subset X$ be a subset. The *σ-convex hull of A* is the set $\operatorname{conv}_\sigma(A) := \bigcup_{k \geq 1} A_k$, where the sequence $(A_k)_{k\geq 1}$ of subsets of X is given by the recursive rule

$$A_1 := A \qquad \text{and} \qquad A_{k+1} := \left\{ \sigma_{xy}(t) : x, y \in A_k, t \in [0,1] \right\}, \quad \text{for all } k \geq 1.$$

We use $\overline{\operatorname{conv}}_\sigma(A)$ to denote the closure of the convex hull of A.

The main result of this paragraph reads as follows:

Theorem 1.2. *Let (X, d_X) denote a complete metric space, let Σ be a subsemigroup of the isometry group of X, and let $\sigma\colon X \times X \times [0,1] \to X$ be a Σ-equivariant conical geodesic bicombing. If there is a non-empty compact subset $K \subset X$ such that $s(K) = K$ for all $s \in \Sigma$, then there is a point $x_\star$ in the closed σ-convex hull $\overline{\operatorname{conv}_\sigma}(K)$ such that $s(x_\star) = x_\star$ for all $s \in \Sigma$.*

In [Nav13, p. 620], Navas introduced a simple geometric argument that implies Theorem 1.2 if one requires additionally that the closed σ-convex hull of K is compact. Unfortunately, Navas's method seems not to work without this additional assumption. In [Gro93, p. 86], Gromov stated the following question: "When is the closed convex hull of a compact subset of a complete CAT(0) space compact?" To the author's knowledge, Gromov's question is still completely open, even in the setting of complete barycentric metric space.

The proof of Theorem 1.2 is given in Section 2.9. The proof strategy may be roughly described as follows: We use Ryll-Nardzewski's fixed point theorem to construct an invariant Radon probability measure first, and then we use the equivariant contracting barycenter map from Theorem 2.10 to obtain a fixed point.

Note that the assumption in Theorem 1.2 of the metric space (X, d_X) having a conical geodesic bicombing is necessary, as for instance the unit circle $S^1 \subset \mathbb{R}^2$ clearly admits isometries without fixed points. How restrictive is the assumption in Theorem 1.2 that σ is Σ-equivariant? Clearly, the unique geodesic bicombing of a Busemann space X is $\mathrm{Iso}(X)$-equivariant. Moreover, Proposition 3.8 in [Lan13] asserts that every injective metric space (X, d_X) admits a conical geodesic bicombing σ that is $\mathrm{Iso}(X)$-equivariant. Furthermore, it follows from a generalised version of the Mazur-Ulam Theorem that for every isometry φ of an open convex subset of a normed vector space the conical geodesic bicombing σ given by the linear geodesics is φ-equivariant, cf. [Man72, p. 368].

The subsequent result is a strengthened version of Theorem 1.2 for when the subsemigroup Σ is generated by a single isometry.

Theorem 1.3. *Let (X, d_X) denote a complete metric space, let $\varphi\colon X \to X$ be an isometry and let $\sigma\colon X \times X \times [0,1] \to X$ be a φ-equivariant conical geodesic bicombing. If there is a point x_0 in X and a compact subset $K \subset X$ such that the strict inequality*

$$\limsup_{k \to +\infty} \left(\sup_{l \geq 0} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}_K(\varphi^{i+l}(x_0)) \right) > 0 \tag{1.3}$$

holds, then there is a point $x_\star$ in $\overline{\mathrm{conv}_\sigma}\left(\left\{\varphi^k(x_0) : k \geq 0\right\}\right)$ such that $\varphi(x_\star) = x_\star$.

The function $\mathbb{1}_K\colon X \to \{0,1\}$ in Theorem 1.3 denotes the indicator function of the subset $K \subset X$.

Note that the left hand side of (1.3) is equal to the *upper Banach density*, cf. [Fur14, Definition 3.7], of the set $D := \left\{k \geq 0 : \mathbb{1}_{K_0}(\varphi^k(x_0)) = 1\right\}$. This fact allows us to invoke a basic result from combinatorial number theory in order to show that the orbits of the isometry φ are bounded, see Lemma 2.31. One key ingredient in the proof of Theorem 1.3 is a generalisation of a classical existence result for invariant Radon measures, see Theorem 2.28; this result may be of independent interest.

1.1.3 – It is a direct consequence of a result of Gähler and Murphy that the only conical geodesic bicombing on a normed vector space is the one that consists of the linear geodesics, cf. [GM81, Theorem 1]. With a mild geometric assumption on the norm, we show in Section 2.5 that already a conical geodesic bicombing on an open subset of a normed vector space locally consists of linear geodesics. More generally, we get the following result:

Theorem 1.4. *Let $(V, \|\cdot\|)$ be a normed vector space such that its closed unit ball is the closed convex hull of its extreme points. Suppose that $A \subset V$ is a subset that admits a conical geodesic bicombing $\sigma \colon A \times A \times [0,1] \to A$ and let $p_0 \in A$ be a point. If $r \geq 0$ is a real number such that the closed ball $B_{2r}(p_0)$ is contained in A, then $\sigma(p,q,t) = (1-t)p + tq$ for all points $p, q \in B_r(p_0)$ and all $t \in [0,1]$.*

We do not know if Theorem 1.4 remains true if we drop the assumption of the normed vector space $(V, \|\cdot\|)$ having the property that its closed unit ball is the closed convex hull of its extreme points. But how common is this property?

By invoking the Banach-Alaoğlu theorem and the Kreĭn-Mil'man theorem one may show that the closed unit ball of a dual Banach space has this property. Consequently, we obtain in particular that Theorem 1.4 is valid in every reflexive Banach space. Moreover, using a classification result due to Nachbin, Goodner, and Kelley, cf. [Kel52], and a result of Goodner, cf. [Goo50, Theorem 6.4], it is readily verified that Theorem 1.4 also holds for every injective Banach space.

Note that the classical Mazur-Ulam Theorem is a direct consequence of Theorem 1.4, as every isometric isomorphism between two normed vector spaces extends to an isometric isomorphism between their linear injective hulls, which by the above satisfy the assumptions of Theorem 1.4.

We proceed with another application of Theorem 1.4. In [Mie17b], Miesch generalized the classical Cartan-Hadamard theorem to metric spaces that locally admit a consistent convex geodesic bicombing. A geodesic bicombing $\sigma \colon X \times X \times [0,1] \to X$ is *consistent* if for all points p, q in X it holds that $\operatorname{im}(\sigma_{p'q'}) \subset \operatorname{im}(\sigma_{pq})$ whenever $p' = \sigma_{pq}(s)$ and $q' = \sigma_{pq}(t)$ with $0 \leq s \leq t \leq 1$. For instance, the geodesic bicombing given by the linear segments of a convex subset of a Banach space is consistent. Consistent geodesic bicombings appear also in [FL08] and [HL07]. With Theorem 1.4 at hand, it is possible to use Miesch's generalized Cartan-Hadamard Theorem to obtain the following uniqueness result:

Theorem 1.5. *Let $(E, \|\cdot\|)$ be a Banach space such that its closed unit ball is the closed convex hull of its extreme points. Suppose that $C \subset E$ is a closed convex subset with non-empty interior. If $\sigma \colon C \times C \times [0,1] \to C$ is a consistent conical geodesic bicombing, then $\sigma(p,q,t) = (1-t)p + tq$ for all points $p, q \in C$ and all $t \in [0,1]$.*

Hence, for subsets $C \subset E$ as in Theorem 1.5 the geodesic bicombing given by the linear segements of C is the only consistent conical geodesic on C. The proof of Theorem 1.5 is given in Section 2.6. In Example 2.20 we use a non-affine isometry originally introduced by Schechtman to construct two distinct consistent conical geodesic bicombings on a closed convex subset $B \subset L^1([0,1])$ with empty interior. As it is possible to consider B as a subset of the injective hull of $L^1([0,1])$, it follows that the assumption in Theorem 1.5 of C having non-empty interior is necessary.

Moreover, Theorem 1.5 is false if one considers only conical geodesic bicombings. A counterexample is discussed in Section 2.6, see Example 2.21. This answers Question 1.6 from [BM19].

1.2 Lipschitz extensions for barycentric target spaces

1.2.1 – Lipschitz maps are generally considered as an indispensable tool in the study of metric spaces. The need for a Lipschitz extension of a given Lipschitz map often presents itself naturally. Deep extension results have been obtained by Johnson, Lindenstrauss, and Schechtman [JLS86], Ball [Bal92], Lee and Naor [LN05], and Lang and Schlichenmaier [LS05]. The literature surrounding Lipschitz extension problems is vast, for a recent monograph on the subject see [BB11; BB12] and the references therein. Before we explain our results in detail, we start with a short presentation of what we will call the Lipschitz extension problem.

Let (X, ρ_X) be a *quasi-metric space*, that is, the function $\rho_X \colon X \times X \to \mathbb{R}$ is non-negative, symmetric and vanishes on the diagonal, cf. [Sch38, p. 827]. Unfortunately, the term "quasi-metric space" has several different meanings in the mathematical literature. In this thesis, we stick to the definition given above. Let $S \subset X$ be a subset and let (Y, ρ_Y) be a quasi-metric space. A *Lipschitz map* is a map $f \colon S \to Y$ such that the quantity

$$\operatorname{Lip}(f) := \inf \{L \geq 0 : \text{for all points } x, x' \in S \colon \rho_Y(f(x), f(x')) \leq L \rho_X(x, x')\}$$

is finite. We use the convention $\inf \varnothing = +\infty$. We consider the following Lipschitz extension problem:

Question 1.6. *Let (X, ρ_X) and (Y, ρ_Y) be a quasi-metric spaces, and suppose that $S \subset X$ is a subset of X. Under what conditions on S, X and Y is there a real number $D \geq 1$ such that every Lipschitz map $f \colon S \to Y$ has a Lipschitz extension $\overline{f} \colon X \to Y$ with $\operatorname{Lip}\left(\overline{f}\right) \leq D \operatorname{Lip}(f)$?*

Let $\mathsf{e}(S; X, Y)$ denote the infimum of the D's satisfying the desired property in the "Lipschitz extension problem". Given integers $n, m \geq 1$, we define

$$\mathsf{e}_n(X, Y) := \sup \big\{\mathsf{e}(S; X, Y) : S \subset X,\ |S| \leq n\big\},$$
$$\mathsf{e}^m(X, Y) := \sup \big\{\mathsf{e}(S; S \cup T, Y) : S, T \subset X,\ S \text{ closed},\ |T| \leq m\big\}.$$

We use $|\cdot|$ or $\operatorname{card}(\cdot)$ to denote the cardinality of a set. We equip (X, ρ_X) with the smallest topology that contains the sets $\{x \in X : \rho_X(x, x_0) < \epsilon\}$ for all $x_0 \in X$ and $\epsilon > 0$.

The Lipschitz extension modulus $\mathsf{e}_n(X, Y)$ has been studied intensively in various settings. Nevertheless, many important questions surrounding $\mathsf{e}_n(X, Y)$ are still open, cf. [NR17] for a recent overview.

We are interested in an upper bound for $\mathsf{e}^m(X, Y)$. We get the following result.

Theorem 1.7. *Let (X, d_X) be a metric space and let (Y, ρ_Y) be a quasi-metric space. If $m \geq 1$ is an integer, then*

$$\mathsf{e}^m(X, Y) \leq m + 1. \tag{1.4}$$

A constructive proof of Theorem 1.1 is given in Section 3.5. The estimate (1.4) is optimal. This follows from the following simple example. We set $P_{m+1} := \{0, 1, \ldots, m, m+1\} \subset \mathbb{R}$ and we consider the subset $S := Y := \{0, m+1\} \subset P_{m+1}$ and the map $f\colon S \to Y$ given by $x \mapsto x$. Suppose that $F\colon P_{m+1} \to Y$ is a Lipschitz extension of f to P_{m+1}. Without effort it is verified that $\operatorname{Lip}(F) = (m+1)\operatorname{Lip}(f)$; hence, it follows that (1.4) is sharp. The sharpness of Theorem 1.7 allows us to obtain a lower bound for the parameter $\alpha(\omega)$ of the dichotomy theorem for metric transforms [MN11, Theorem 1], see Corollary 3.4.

If the condition that the subset $S \subset X$ has to be closed is removed in the definition of $\mathsf{e}^m(X, Y)$, then Theorem 1.7 is not valid. Indeed, if (X, d_X) is not complete and $z \in \overline{X}$ is a point contained in the completion $\overline{X}$ of X such that $z \notin X$, then the identity map $\mathsf{id}_X\colon X \to X$ does not extend to a Lipschitz map $\overline{\mathsf{id}_X}\colon X \cup \{z\} \to X$ if we equip $X \cup \{z\} \subset \overline{X}$ with the subspace metric. This is a well-known obstruction. As pointed out by Mendel and Naor, there is the following upper bound of $\mathsf{e}^m(X, Y)$ in terms of $\mathsf{e}_m(X, Y)$.

Lemma 1.8 (Claim 1 in [MN17]). *Let (X, d_X) and (Y, d_Y) be two metric spaces. If $m \geq 1$ is an integer, then*

$$\mathsf{e}^m(X, Y) \leq \mathsf{e}_m(X, Y) + 2.$$

By the use of Lemma 1.8 and [LN05, Theorem 1.10], one can deduce that if (X, d_X) is a metric space and $(E, \|\cdot\|_E)$ is a Banach space, then

$$\mathsf{e}^m(X, E) \lesssim \frac{\log(m)}{\log\big(\log(m)\big)}$$

for all integers $m \geq 3$, where the notation $A \lesssim B$ means $A \leq CB$ for some universal constant $C \in (0, +\infty)$. As a result, for sufficiently large integers $m \geq 3$ the estimate in Theorem 1.7 is not optimal if we restrict the target spaces to the class of Banach spaces.

In Section 3.1, we present an example that shows that for Banach space targets the estimate (1.4) is sharp if $m = 1$. As a byproduct of the construction in Section 3.1, we obtain the lower bound

$$\mathsf{e}(\ell_2, \ell_1) \geq \sqrt{2}, \tag{1.5}$$

where $\mathsf{e}(\ell_2, \ell_1) := \sup\big\{\, \mathsf{e}(S; \ell_2, \ell_1) : S \subset \ell_2 \big\}$. It is unknown if $\mathsf{e}(\ell_2, \ell_1)$ is finite or infinite. This question has been raised by Ball, cf. [Bal92].

1.2.2 – In this paragraph, we are interested in extending Lipschitz maps with values in complete barycentric metric spaces to finitely many additional points. Given a quasi-metric space (X, ρ_X) and a subset $S \subset X$, we define

$$\mathsf{e}_{\text{bar}}(S; X) := \sup \big\{ \mathsf{e}(S; X, Z) : Z \text{ complete barycentric metric space } \big\}.$$

We are mainly interested in quasi-metric spaces of the following form: Let $F\colon [0, +\infty) \to [0, +\infty)$ be a map with $F(0) = 0$; The *F-transform of X*, denoted by $F[X]$, is by definition the quasi-metric space $(X, F \circ \rho_X)$. F-transforms of Hilbert spaces have been studied in detail by Schoenberg in the 1930's, cf. [Sch38]. Now, the main result of this paragraph can be stated as follows:

Theorem 1.9. *Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space and let $F\colon [0, +\infty) \to [0, +\infty)$ be a map such that the composition $F(\sqrt{\cdot})$ is a strictly-increasing concave function with $F(0) = 0$. If $S \subset X \subset F[H]$ are finite subsets, then*

$$\mathsf{e}_{bar}(S; X) \leq \sup_{x > 0} \frac{F(\sqrt{m+1}\, x)}{F(x)},$$

where $m := \operatorname{card}(X \setminus S)$.

Theorem 1.9 is optimal if $m = 1$ and $F(t) = t$, see Proposition 3.1. Via this sharpness result we obtain that certain F-transforms of ℓ_p, for $p > 2$, do not isometrically embed into ℓ_2, see Corollary 3.3.

Suppose that $F\colon [0, +\infty) \to [0, +\infty)$ is a strictly-increasing continuous function such that the F-transform of ℓ_2 embeds isometrically into a Hilbert space. By a celebrated result of Schoenberg $F(\sqrt{\cdot})^2$ is a Bernstein function, cf. [Sch38, Theorem 6′]; thus, the function $F(\sqrt{\cdot})$ is concave and therefore satisfies the assumptions on F in Theorem 1.9. This provides a natural class of examples for which Theorem 1.9 may be applied. For instance, by considering the function $F(t) = t^\alpha$, with $0 < \alpha \leq 1$, we obtain the following direct corollary of Theorem 1.9.

Corollary 1.10. *Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space, let (Z, d_Z) be a complete barycentric metric space and let $0 < \alpha \leq 1$ and $L \geq 0$ be real numbers. If $X \subset H$ is a finite subset, $S \subset X$, and $f\colon S \to Z$ is an (α, L)-Hölder map, then there is an extension $\overline{f}\colon X \to Z$ of f such that $\overline{f}$ is an $(\alpha, \overline{L})$-Hölder map with*

$$\overline{L} \leq \left(\sqrt{m+1}\right)^{\alpha} L,$$

where $m := \operatorname{card}(X \setminus S)$.

In Corollary 1.10, an (α, L)-*Hölder map* is a map $f\colon X \to Y$ such that

$$d_Y(f(x), f(x')) \le L d_X(x, x')^\alpha$$

for all points $x, x' \in X$.

Along the lines of the proof of Claim 1 in [MN17] one can show that if (X, d_X) and (Y, d_Y) are metric spaces, then for all integers $m \ge 1$ we have

$$\mathsf{e}^m(X, Y) \le \sup_{n \ge 1} \mathsf{e}_n^m(X, Y) + 2,$$

where

$$\mathsf{e}_n^m(X, Y) := \sup\big\{\mathsf{e}(S; S \cup T, Y) : S, T \subset X,\ |S| \le n,\ |T| \le m\big\}.$$

Thus, by the use of Theorem 1.9, we may deduce that if H is a Hilbert space and E is a Banach space, then

$$\mathsf{e}^m(H, E) \le \sqrt{m+1} + 2 \tag{1.6}$$

for all integers $m \ge 1$. In [LN05, Theorem 1.12], Lee and Naor demonstrate that $\mathsf{e}_n(H, E) \lesssim \sqrt{\log(n)}$ for all integers $n \ge 2$. Thus, via this estimate (and Lemma 1.8) it is possible to obtain the upper bound

$$\mathsf{e}^m(H, E) \lesssim \sqrt{\log(m)}$$

that has a better asymptotic behaviour than estimate (1.6). However, since Lee and Naor use different (probabilistic) methods, we believe that our approach has its own interesting aspects.

1.2.3 – For a quasi-metric space (X, ρ_X) and a subset $S \subset X$, we define

$$\mathsf{e}_{\text{fin}}(S; X) := \sup\big\{\mathsf{e}(S; X, E) : E \text{ finite-dimensional real Banach space}\big\}.$$

For every finite metric space (S, d_S) we let

$$\text{æ}(S) := \sup\big\{\mathsf{e}_{\text{fin}}(S; X) : X \text{ metric space with } S \subset X\big\}$$

denote the *absolute Lipschitz extendability constant* of S.

Naor and Rabani, cf. [NR17], and Lee and Naor, cf. [LN05], have shown that

$$\sqrt{\log(n)} \lesssim \text{æ}(n) := \sup\big\{\text{æ}(S) : |S| = n\big\} \lesssim \frac{\log(n)}{\log\big(\log(n)\big)} \tag{1.7}$$

provided that $n \geq 3$. The main result of this paragraph, see Theorem 1.11, is a formula for $\text{æ}(S)$ using only *linear* Lipschitz extension moduli. Let $(E, \|\cdot\|)$ be a real Banach space and let $F \subset E$ denote a finite-dimensional linear subspace. The number

$$\Pi(F, E) := \inf \left\{ \|P\| \mid P \colon E \to F \text{ bounded surjective linear map with } P^2 = P \right\}$$

is called the *relative projection constant* of F with respect to E. We get the following connection from the non-linear to the linear world.

Theorem 1.11. *Let (S, d_S) denote a finite metric space. Then*

$$\text{æ}(S) = \sup \left\{ \Pi(\mathcal{F}(S), \mathcal{F}(X)) \ : X \textit{ finite metric space such that } S \subset X \right\};$$

in particular,

$$\text{æ}(S) \leq \Pi(\mathcal{F}(S), \ell_\infty(\mathbb{N})). \tag{1.8}$$

We use $\mathcal{F}(X)$ to denote the Lipschitz-free space of a metric space X. Lipschitz-free spaces have been introduced by Arens and Eells in the 1950s, cf. [AE56], and the term "Lipschitz-free space" has been coined by Godefroy and Kalton, cf. [GK03]. We recall the construction of Lipschitz-free spaces in Section 3.6, cf. [Ost13] or [Wea99] for further information. The proof of Theorem 1.11, given in Section 3.6, is a variant of the proof of Theorem 1.2 in [BB07], due to Brudnyi and Brudnyi.

The result from Section 3.9 tells us that for $|S| = 3$, the right hand side of (1.8) is bounded by $\frac{4}{3}$; in Example 3.18, we construct a metric space S consisting of three points such that $\text{æ}(S) \geq \frac{4}{3}$. For that reason, we obtain:

Corollary 1.12.

$$\text{æ}(3) = \frac{4}{3}.$$

However, for large $n \geq 1$ the inequality (1.8) is not sharp. Indeed, for a finite weighted tree T, Godard, cf. [God10, Corollary 3.6], proved that $\mathcal{F}(T) \cong \ell_1^n$, for $n := |T| - 1$; thus a result of Grünbaum, cf. [Grü60], tells us that for such a weighted tree T with $n + 1 \in 2\,\mathbb{Z}$ vertices, the right hand side of (1.8) equals

$$\frac{n\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)} \sim \sqrt{\frac{2n}{\pi}}.$$

For $n \geq 1$ large enough, this is strictly greater than the upper bound (1.7) due to Lee and Naor. Hence, (1.8) cannot be sharp for $n \geq 1$ large enough.

The last two sections of this chapter are devoted to linear projection constants.

1.2.4 – As a consequence of ideas developed by Lindenstrauss, cf. [Lin64], for a finite-dimensional Banach space $E \subset \ell_\infty(\mathbb{N})$ the smallest constant $C \in [0, +\infty)$ such that E is an C-absolute Lipschitz retract is completely determined by the linear theory of E. Indeed, Rieffel, cf. [Rie06], established that it is equal to the *linear projection constant* of E, which is the number $\Pi(E) \in [0, +\infty]$ defined as

$$\inf \left\{ \|P\| \mid P\colon \ell_\infty(\mathbb{N}) \to E \text{ bounded surjective linear map with } P^2 = P \right\}.$$

Linear projections have been the object of study of many researchers and the literature can be traced back to the classical book by Banach, cf. [Ban32, p.244-245]. The question about the maximal value Π_n of the linear projection constants of n-dimensional Banach spaces has persisted and is a notoriously difficult one. We establish a formula that relates Π_n with eigenvalues of certain two-graphs. This reduces the problem (in principle) to the classification of certain two-graphs and thus allows the introduction of tools from graph theory. Following this approach, we present an alternative proof of $\Pi_2 = \frac{4}{3}$, see Section 3.9, and we establish that the maximal relative projection constants of codimension n in ℓ_∞^d converge to $1 + \Pi_n$ as $d \to +\infty$, see Corollary 1.16. In the remainder of this overview, we summarize the current state of the theory.

For $n \geq 1$, define Ban_n to be the set of linear isometry classes of n-dimensional Banach spaces over the real numbers. The set Ban_n equipped with the Banach-Mazur distance is a compact metric space, cf. [TJ89]. Thus, the map $\log \circ \Pi\colon \mathrm{Ban}_n \to [0, +\infty)$ is 1-Lipschitz and consequently for all $n \geq 1$ the maximal projection constant of order n,

$$\Pi_n := \max \left\{ \Pi(X) : X \in \mathrm{Ban}_n \right\},$$

is a well-defined real number. Apart from $\Pi_1 = 1$, the only known value is $\Pi_2 = \frac{4}{3}$, due to Chalmers and Lewicki, cf. [CL10]. There is numerical evidence indicating that $\Pi_3 = (1 + \sqrt{5})/2$, cf. [FS17, Appendix B], but to the author's knowledge, there is no known candidate for Π_n for all $n \geq 4$. From a result of Kadets and Snobar, cf. [KS71],

$$\Pi_n \leq \sqrt{n}.$$

The above estimate has independently been obtained by Gromov, cf. [Gro83, Proposition 2.1.A]. Moreover, König, cf. [Kön85], has shown that this estimate is asymptotically the best possible. Indeed, there exists a sequence $(X_{n_k})_{k \geq 1}$ of finite-dimensional real Banach spaces such that $\dim(X_{n_k}) = n_k$, where $n_k \to +\infty$ for $k \to +\infty$, and

$$\lim_{k \to +\infty} \frac{\Pi(X_{n_k})}{\sqrt{n_k}} = 1.$$

There are many non-isometric maximizers of the function $\Pi_n(\cdot)$, cf. [KTJ03].

A finite-dimensional Banach space is called *polyhedral* if its unit ball is a polytope. Equivalently, a finite-dimensional Banach space $(E, \|\cdot\|)$ is polyhedral if there exists an integer $d \geq 1$ such that $(E, \|\cdot\|)$ admits a linear isometric embedding into ℓ_∞^d. Using a result of Klee, cf. [Kle60, Proposition 4.7], and elementary functional analysis, we show that there exist maximizers of $\Pi_n(\cdot)$ that are polyhedral, see Theorem 1.17. In the 1960s, Grünbaum, cf. [Grü60], calculated $\Pi(\ell_1^n)$, $\Pi(\ell_2^n)$ and $\Pi(E_{_{hex}})$, where $E_{_{hex}}$ is the 2-plane with the hexagonal norm. In particular, $\Pi(E_{_{hex}}) = \frac{4}{3}$, which Grünbaum conjectured to be the maximal value of $\Pi(\cdot)$ amongst 2-dimensional Banach spaces. In 2010, Chalmers and Lewicki presented an intricate proof of Grünbaum's conjecture employing the implicit function theorem and Lagrange multipliers, cf. [CL10].

Our main result, see Theorem 1.14, provides a characterization of the number Π_n in terms of certain maximal sums of eigenvalues of two-graphs that are K_{n+2}-free. In [FF84], Frankl and Füredi give a full description of two-graphs that are K_4-free. Via this description and Theorem 1.14 we can derive from first principles that $\Pi_2 = \frac{4}{3}$. This is done in Section 3.9.

Next, we introduce the necessary notions from the theory of two-graphs that are needed to properly state our main result.

The subsequent definition of a two-graph via cohomology follows Taylor [Tay77], and Higman [Hig73]; see also [Sei91, Remark 4.10]. Let V denote a finite set. For each integer $n \geq 0$ we set

$$E_n(V) := \big\{ B \subset V : |B| = n \big\} \quad \text{and} \quad \mathcal{E}_n(V) := \big\{ f \colon E_n(V) \to \mathbb{F}_2 \big\},$$

where $\mathbb{F}_2$ denotes the field with two elements. Elements of $\mathcal{E}_2(V)$ are finite simple graphs. If n is strictly greater than the cardinality of V, then $\mathcal{E}_n(V)$ consists only of the empty function $\varnothing \to \mathbb{F}_2$. For each $f \in \mathcal{E}_n(V)$ the map $\delta f \in \mathcal{E}_{n+1}(V)$ is given by

$$B \mapsto \sum_{v \in B} f(B \setminus \{v\}).$$

Clearly, it holds that $\delta \circ \delta = 0$, where 0 denotes the neutral element of the group $\mathcal{E}_{n+2}(V)$. Two-graphs can be defined as follows.

Definition 1.13 (two-graph). *A two-graph is a tuple $T = (V, \varDelta)$, where V and $\varDelta$ are finite sets and there exists a map $f_T \in \mathcal{E}_3(V)$ such that $\delta f_T = 0$ and $\varDelta = f_T^{-1}(1)$. The cardinality of V is called the order of T.*

Among other things, two-graphs naturally occur in the study of systems of equiangular lines and 2-transitive permutation groups; authoritative surveys are [Sei91; Sei92]. Given a two-graph $T = (V, \varDelta)$, the following set is always non-empty:

$$[T] := \{ f \colon E_2(V) \to \mathbb{F}_2 : \delta f = f_T \}.$$

Each $f \in [T]$ gives rise to a graph $G_f := (V, f^{-1}(1))$. The *Seidel adjacency matrix* of a graph $G = (V, E)$ is the matrix $S(G)$, which is the symmetric $|V| \times |V|$-matrix given by

$$S(G)_{ij} = \begin{cases} 0 & \text{if } i = j \\ -1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 1 & \text{otherwise.} \end{cases}$$

For each choice $f_1, f_2 \in [T]$ the matrices $S(G_{f_1})$ and $S(G_{f_2})$ have the same spectrum. By definition, the *eigenvalues of* $T = (V, \varDelta)$ are the real numbers

$$\lambda_1(T) \geq \ldots \geq \lambda_{|V|}(T)$$

that are the eigenvalues of $S(G_f)$ for $f \in [T]$ (counted with multiplicity). This definition is independent of $f \in [T]$.

We say that a two-graph $T = (V, \varDelta)$ is K_n-free if there is no injective map $\varphi \colon \{1, \ldots, n\} \to V$ such that $\{\varphi(v_1), \varphi(v_2), \varphi(v_3)\} \in \varDelta$ for all distinct points $v_1, v_2, v_3 \in V_n$.

Our main result reads as follows:

Theorem 1.14. *If $n \geq 1$ is an integer, then*

$$\Pi_n = \sup_{d \geq 1} \max \left\{ \frac{n}{d} + \frac{1}{d} \sum_{k=1}^{n} \lambda_k(T) \ : \ T \textit{ is a } K_{n+2}\textit{-free two-graph of order } d \right\}.$$

To prove Theorem 1.14, we invoke a simple trick, see Lemma 3.20, that allows us to greatly narrow down the matrices that need to be considered. This is done in Section 3.7.

1.2.5 – The following question has first been systematically addressed by König, Lewis, and Lin in [KLL83]:

Question 1.15. *Let $n, d \geq 0$ be integers. What is*

$$\Pi(n, d) := \sup \{ \Pi(E) : E \subset \ell_\infty^d \textit{ is an } n\textit{-dimensional Banach space} \}?$$

By definition, $\sup \varnothing = -\infty$. Clearly, $\Pi(d,d) = 1$ and it is a direct consequence of the classical Hahn-Banach theorem that $\Pi(1,d) = 1$ for all integers $d \geq 1$. The quantity $\Pi(d-1,d)$ has been examined by Bohnenblust, cf. [Boh38], where it is shown that $\Pi(d-1,d) \leq 2 - \frac{2}{d}$. In [CL09], Chalmers and Lewicki determined the exact value of $\Pi(3,5)$. In [KLL83], König, Lewis, and Lin established the general upper bound

$$\Pi(n,d) \leq \frac{n}{d} + \sqrt{(d-1)\frac{n}{d}(1-\frac{n}{d})}$$

with equality if and only if $\mathbb{R}^n$ admits a system of d distinct equiangular lines. Thereby, as $\mathbb{R}^3$ admits a system of six equiangular lines, cf. [LS73, p. 496], it holds that

$$\Pi(3,6) = \frac{1+\sqrt{5}}{2}.$$

In light of

$$\Pi(4,6) = \frac{5}{3},$$

which we demonstrate in [Bas19, Section 4.2], up to $d = 6$ all exact values of $\Pi(n,d)$ for $1 \leq n \leq d$ are now computed. It is well-known that

$$\Pi(n,d) \leq \Pi(n,d+1) \quad \text{and} \quad \Pi(n,d) \leq \Pi(n+1,d+1)$$

for all $1 \leq n \leq d$, cf. [CL09]. Via Theorem 1.14, we infer the following asymptotic relation between these two increasing sequences:

Corollary 1.16. *For each integer $n \geq 1$ we have*

$$1 + \Pi_n = \lim_{d \to +\infty} \Pi\left(d-n, d\right).$$

A proof of Corollary 1.16 is given in Section 3.7. If $n = 1$, then Corollary 1.16 follows directly from the fact that Bohnenblust's upper bound of $\Pi(d-1,d)$ is sharp, cf. [CL09, Lemma 2.6]. Recently, the special case $n = 2$ has been considered by Sokołowski in [Sok17]. The upper bound

$$\Pi\left(d-n, d\right) \leq 1 + \sqrt{n}$$

for $d \geq n$ has been obtained by Garling and Gordon, cf. [GG71], by the use of John's Theorem.

Recall that $\Pi(1,d) = \Pi(1,1) = 1$ for all $d \geq 1$. The proof of Grünbaum's conjecture, cf. [CL10], shows that

$$\Pi(2,d) = \Pi(2,3) = \frac{4}{3}, \quad \text{for all } d \geq 3.$$

Numerical experiments, cf. [FS17, Appendix B], suggest that if $d \in \{6, \ldots, 10\}$, then $\Pi(3,d) = \Pi(3,6)$. Since $\Pi_n(\cdot)$ admits a polyhedral maximizer, the sequence $\Pi(n,\cdot)$ stabilizes eventually. This is the content of the subsequent theorem:

Theorem 1.17. *Let $n \geq 1$ be an integer. There exists a polyhedral n-dimensional Banach space $(F_n, \|\cdot\|)$ such that*

$$\Pi(F_n) = \Pi_n.$$

As a result, there is an integer $D \geq 1$ such that

$$\Pi(n, d) = \Pi(n, D)$$

for all $d \geq D$.

A proof of Theorem 1.17 can be found in Section 3.8. Unfortunately, the proof of Theorem 1.17 is not constructive. Obtaining an explicit upper bound for the quantity D seems out of reach at the moment.

2 The geometry of barycentric metric spaces

2.1 Preliminaries

2.1.1 – In this section, we collect some facts from the theory of optimal transportation. Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space. We denote by $\mathcal{B}_X$ the Borel σ-algebra of $(X, \mathcal{T}_X)$ and by $\mathcal{K}_X$ the set that consists of all compact subsets of $(X, \mathcal{T}_X)$. A non-negative Borel measure $\mu\colon \mathcal{B}_X \to [0, +\infty]$ is called a *Radon measure* if $\mu(K) < +\infty$ for all compact subsets K of $(X, \mathcal{T}_X)$ and

$$\mu(B) = \sup\left\{\mu(K) : K \subset B, K \in \mathcal{K}_X\right\}$$

for all Borel measurable sets B of $(X, \mathcal{T}_X)$. A signed finite Borel measure $\mu\colon \mathcal{B}_X \to \mathbb{R}$ is called a signed finite Radon measure if the total variation $|\mu| : \mathcal{B}_X \to [0, +\infty)$ is a Radon measure. Let $P(X)$ denote the set that consists of all non-negative Borel measures on $(X, \mathcal{B}_X)$ that are Radon probability measures. Suppose that $f\colon X \to X$ is a continuous map. We define the map

$$\begin{aligned} f_*\colon P(X) &\to P(X) \\ \mu &\mapsto \begin{cases} f_*\mu\colon \mathcal{B}_X \to [0,1] \\ B \mapsto \mu(f^{-1}(B)). \end{cases} \end{aligned}$$

The map f_* is well-defined and for every μ in $P(X)$ the measure $f_*\mu$ is called the *pushforward of μ by f*. A measure μ in $P(X)$ is called *f-invariant* if $f_*\mu = \mu$.

For the rest of this subsection let (X, d_X) be a metric space. Suppose that the Borel measure $\mu\colon \mathcal{B}_X \to [0,1]$ is a Radon probability measure. The subset $\operatorname{spt}(\mu)$ of all points x in X such that $\mu(U) > 0$ for all open neighborhoods of x is called the *support* of μ. We say that μ has a *finite first moment* if there is a point x_0 in X such that

$$\int_X d(x, x_0)\, \mu(dx) < +\infty.$$

We let $P_1(X)$ be the set that consists of all measures of $(X, \mathcal{B}_X)$ that are Radon probability measures with a finite first moment. We denote by $W^1\colon P_1(X) \times P_1(X) \to \mathbb{R}$ the first Wasserstein distance on $P_1(X)$. Due to the Kantorovich-Rubinstein Duality Theorem the first Wasserstein distance W_1 is given by

$$W_1(\mu, \nu) = \sup \left\{ \int_X f\, d\mu - \int_X f\, d\nu \;:\; f\colon X \to \mathbb{R} \text{ is 1-Lipschitz} \right\}$$

and thus defines a metric on $P_1(X)$, cf. [Edw11]. We define

$$P_{\mathbb{Q}}(X) := \left\{ \sum_{k=1}^{n} \alpha_k \delta_{x_k} \;:\; n \geq 1, \sum_{k=1}^{n} \alpha_k = 1, \alpha_k \in \mathbb{Q}_{\geq 0}, x_k \in X \right\}.$$

On $P_{\mathbb{Q}}(X)$ there is an explicit formula for the first Wasserstein distance.

Proposition 2.1. *Let (X, d_X) denote a metric space. If $n \geq 1$ is an integer and x_i, y_i for $i = 1, \ldots, n$ are points in X, then we have*

$$W_1\left(\frac{1}{n}\left(\delta_{x_1} + \cdots + \delta_{x_n}\right), \frac{1}{n}\left(\delta_{y_1} + \cdots + \delta_{y_n}\right) \right) = \frac{1}{n} \min_{\tau \in S_n} \sum_{k=1}^{n} d(x_k, y_{\tau(k)}).$$

Proof. See [Vil03, p. 5]. □

It turns out that the set $P_{\mathbb{Q}}(X)$ is W_1-dense in $P_1(X)$. This is the content of the following proposition.

Proposition 2.2. *Let (X, d_X) be a metric space and let $\epsilon > 0$ be a positive real number. If the Borel measure $\mu\colon \mathcal{B}_X \to [0, 1]$ is a Radon probability measure contained in $P_1(X)$, then there exists a measure ν_ϵ contained in $P_{\mathbb{Q}}(X)$ with $spt(\nu_\epsilon) \subset spt(\mu)$ such that $W_1(\mu, \nu_\epsilon) < \epsilon$.*

Proof. See Theorem 6.1 in [Edw11] and Theorem 6.18 in [Vil09]. □

Suppose that the map $\varphi\colon X \to X$ is 1-Lipschitz. By the use of the Kantorovich-Rubinstein Duality Theorem it is readily verified that the map $\varphi_*\colon P_1(X) \to P_1(X)$ is well-defined and 1-Lipschitz as well. This functorial property gives rise to the subsequent Lipschitz extension property.

Lemma 2.3. *Let (B, d_B) be a metric space, $A \subset B$ be a subset and let $L \geq 0$ be a real number. If there exists a L-Lipschitz map $\iota\colon B \to P_1(A)$ such that $\iota(a) = \delta_a$ for all $a \in A$, then every 1-Lipschitz map from A to a barycentric metric space extends to an L-Lipschitz map on B.*

Proof. Let (Z, d_Z) be a barycentric metric space, let $\varphi\colon A \to Z$ be a 1-Lipschitz map and let $\beta\colon P_1(Z) \to Z$ be a contracting barycenter map. We set

$$\overline{\varphi}(b) := \beta(\varphi_*(\iota(b))) \qquad (b \in B),$$

where $\varphi_*(\iota(b))$ denotes the pushforward of the measure $\iota(b)$ by φ. It holds that

$$d_Z(\overline{\varphi}(b), \overline{\varphi}(b')) \leq W_1(\varphi_*(\iota(b)), \varphi_*(\iota(b'))) \leq W_1(\iota(b), \iota(b')) \leq L d_B(b, b')$$

for all $b, b' \in B$. The second inequality follows directly from the Kantorovich-Rubinstein duality theorem. By construction, $\overline{\varphi}(a) = \varphi(a)$ for all $a \in A$. This completes the proof. □

2.1.2 – In what follows, we briefly introduce injective metric spaces and injective hulls of metric spaces. A metric space (X, d_X) is *injective* if for every metric space (B, d) and every 1-Lipschitz map $f\colon A \to X$ defined on a subset $A \subset B$, there exists a 1-Lipschitz map $\overline{f}\colon B \to X$ such that $\overline{f}|_A = f$. Basic examples of injective metric spaces are the real line with the standard metric, $\ell_\infty(I)$ for any index set I and all complete $\mathbb{R}$-trees. In [DP17], an explicit characterization of all injective subsets of $\ell_\infty(I)$ is obtained.

An injective hull of the metric space (X, d) is a pair (Y, e), where (Y, d) is an injective metric space and $e\colon X \to Y$ is an isometric embedding with the property that if there is a metric space (Z, d) and a 1-Lipschitz map $f\colon Y \to Z$ such that the composite map $f \circ e$ is an isometric embedding, then the map f is an isometric embedding. Isbell showed that every metric space possesses an essentially unique injective hull, cf. [Isb64]. We denote the injective hull of (X, d_X) by $(E(X), e)$. Isbell construction has been rediscovered several times. We refer to [Lan13] for a short overview.

The subsequent lemma tells us that every contracting barycenter map on a metric space (X, d_X) is induced by a contracting barycenter map on its injective hull $(E(X), e)$.

Lemma 2.4. *Let (X, d_X) denote a metric space and let $(E(X), e)$ the injective hull of X. Let $S \subset P_1(X)$ be a subset such that $\delta_x \in S$ for all $x \in X$. If $\beta\colon S \to X$ is 1-Lipschitz and $\beta(\delta_x) = x$ for all $x \in X$, then there exists a contracting barycenter map $\beta_E\colon P_1(E(X)) \to E(X)$ such that $e(\beta(s)) = \beta_E(e_*(s))$ for all $s \in S$.*

Proof. The map $e \circ \beta$ is a 1-Lipschitz map and the push-forward map e_* is an isometric embedding. Therefore, there exists a 1-Lipschitz map $\beta_E\colon P_1(E(X)) \to E(X)$ such that $e \circ \beta = \beta_E \circ e_*|_S$, as the metric space $(E(X), d_E)$ is injective. Thus, we are left to show that $\beta_E(\delta_z) = z$ for all points z contained in $E(X)$. Let $i\colon E(X) \to P_1(E(X))$ denote the canonical isometric embedding given by the assignment $z \mapsto \delta_z$. The map $\varphi := \beta_E \circ i$ is a 1-Lipschitz map from $E(X)$ to $E(X)$. By construction, we have $\varphi(e(x)) = e(x)$ for all points x in X. Using [Lan13, Theorem 3.3 (1)] we get that $\varphi = \mathrm{id}_{E(X)}$; thus, we infer $\beta_E(\delta_z) = z$ for all points z in $E(X)$. This completes the proof. □

Lemma 2.4 is the key component in the construction in Example 2.21.

2.1.3 – Here we collect all notions related to geodesic bicombings. We follow [DL15] and define the notion of a geodesic bicombing on a metric space as follows.

Definition 2.5 (geodesic bicombing). *Let (X, d_X) denote a metric space. A map $\sigma\colon X \times X \times [0,1] \to X$ is called a geodesic bicombing, if for all points $x, y \in X$ and for all $s, t \in [0,1]$:*

$$d(\sigma(x,y,s), \sigma(x,y,t)) = |s-t|\, d(x,y), \tag{2.1}$$

and $\sigma(x,y,0) = x, \sigma(x,y,1) = y$.

The term bicombing was coined by D. Epstein and W. Thurston in the context of combinatorial group theory, cf. [Eps+92, p. 84]. Note that if σ is a geodesic bicombing on a metric space (X, d_X), then we have $\sigma(x,x,t) = x$ for all points x in X and all t in the interval $[0,1]$. We often use the notation $\sigma_{xy}(t)$ to denote the point $\sigma(x,y,t)$. A map $\sigma_{xy}\colon [0,1] \to X$ with $\sigma_{xy}(0) = x$ and $\sigma_{xy}(1) = y$ that satisfies (2.1) is called *geodesic from x to y*.

Definition 2.6 (classes of geodesic bicombings). *Let (X, d_X) denote a metric space and let $\sigma\colon X \times X \times [0,1] \to X$ be a geodesic bicombing. We use the following terminology:*

1. *We say that σ is conical if for all points $x, y, x', y' \in X$ and for all $t \in [0,1]$:*

$$d(\sigma_{xy}(t), \sigma_{x'y'}(t)) \leq (1-t) d(x,x') + t d(y,y'). \tag{2.2}$$

   This inequality is called the conical inequality.

2. *We call σ convex if we have for all points $x, y, x', y' \in X$ that the function $[0,1] \to \mathbb{R}$ given by the assignment $t \mapsto d(\sigma_{xy}(t), \sigma_{x'y'}(t))$ is convex.*

3. *Assume that*

$$\sigma_{pq}(\lambda) = \sigma_{xy}((1-\lambda)s + \lambda t), \tag{2.3}$$

   whenever $x, y \in X$, $0 \leq s \leq t \leq 1$, $p := \sigma_{xy}(s)$, $q := \sigma_{xy}(t)$, and $\lambda \in [0,1]$. Then we call σ consistent.

4. *If we have for all points x, y in X that*

$$\sigma_{xy}(t) = \sigma_{yx}(1-t) \quad \text{for all } t \in [0,1], \tag{2.4}$$

then we say that σ is reversible.

5. *We say that σ has the midpoint property if*

$$\sigma_{xy}(\tfrac{1}{2}) = \sigma_{yx}(\tfrac{1}{2}) \tag{2.5}$$

for all points x, y in X.

To the author's knowledge, conical geodesic bicombings have first been considered by Itoh under the name W-convexity mappings that satisfy condition (III), cf. [Ito79]. It is immediate that a consistent and conical geodesic bicombing is convex. But it is not known that whenever the metric space (X, d_X) admits a conical geodesic bicombing, then X admits also a convex geodesic bicombing. For (X, d_X) proper, this has been established by Descombes and Lang, cf. [DL15, Theorem 1.1]. Note that if a geodesic bicombing is reversible, then it has the midpoint property. In Section 2.4, we construct a geodesic bicombing that has the midpoint property but is not reversible.

Basic examples of convex geodesic bicombings are the unique geodesics $\sigma_{xy} \colon [0,1] \to X$ in a CAT(0) space or in a non-positively curved global Busemann space. Another example for convex geodesic bicombings are the linear geodesics $\lambda_{xy}(t) = (1-t)x + ty$ in a normed vector space $(V, \|\cdot\|)$. Moreover, the family of geodesic $\tau_{\mu\nu}(t) = (1-t)\mu + t\nu$ on $P_1(X)$ constitute a consistent conical geodesic bicombing. As pointed out in [Duc18, Example 2.11], the metric space $\mathrm{GL}(H)/\mathrm{O}(H)$ for any Hilbert space H admits a convex geodesic bicombing.

The subsequent lemma that tells us that the conical inequality implies in fact a slightly stronger inequality.

Lemma 2.7. *Let (X, d_X) be a metric space, let $A \subset X$ be a subset and let $\{\sigma_{xy}(\cdot)\}_{x,y \in A}$ be a collection of geodesics $\sigma_{xy} \colon [0,1] \to X$ such that $\sigma_{xy}(0) = x, \sigma_{xy}(1) = y$ and $\sigma_{xy}(t) = \sigma_{yx}(1-t)$ for all $t \in [0,1]$ and $x, y \in A$. If*

$$d_X(\sigma_{xy}(t), \sigma_{xz}(t)) \leq t d_X(y, z)$$

for all $x, y, z \in A$, then

$$d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \leq W_1\big((1-t)\delta_{x_1} + t\delta_{x_2}, (1-t)\delta_{y_1} + t\delta_{y_2}\big)$$

for all $x_1, x_2, y_1, y_2 \in A$.

Proof. Without loss of generality, we may suppose that $t \in [\frac{1}{2}, 1]$. Proposition 2.1 tells us that

$$\begin{aligned}
&W_1\big((1-t)\delta_{x_1} + t\delta_{x_2}, (1-t)\delta_{y_1} + t\delta_{y_2}\big) \\
&= \min_{\epsilon \in [0,1-t]} \big(\epsilon(d_X(x_1, y_2) + d_X(y_1, x_2)) + (t-\epsilon) d_X(x_2, y_2) + ((1-t)-\epsilon) d_X(x_1, y_1)\big).
\end{aligned}$$

On the one hand, we compute

$$d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \leq d_X(\sigma_{x_1x_2}(t), \sigma_{x_1y_2}(t)) + d_X(\sigma_{y_2x_1}(1-t), \sigma_{y_2y_1}(1-t)),$$

thus we get

$$d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \leq (1-t) d_X(x_1, y_1) + t d_X(x_2, y_2),$$

but on the other hand, we estimate

$$\begin{aligned}
&d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \\
&\leq d_X(\sigma_{x_2x_1}(1-t), \sigma_{x_2y_2}(1-t)) + d_X(\sigma_{x_2y_2}(1-t), \sigma_{x_2y_2}(t)) + d_X(\sigma_{x_2y_2}(t), \sigma_{y_1y_2}(t))
\end{aligned}$$

and thus

$$\begin{aligned}
&d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \\
&\leq (1-t) d_X(x_1, y_2) + (2t-1) d_X(x_2, y_2) + (1-t) d_X(x_2, y_1).
\end{aligned}$$

Consequently, by putting everything together we conclude

$$d_X(\sigma_{x_1x_2}(t), \sigma_{y_1y_2}(t)) \leq W_1\big((1-t)\delta_{x_1} + t\delta_{x_2}, (1-t)\delta_{y_1} + t\delta_{y_2}\big),$$

as desired. □

A direct consequence of Lemma 2.4 and Lemma 2.7 is that every reversible conical geodesic bicombing on X lifts to a reversible concial geodesic bicombing on $E(X)$.

Proposition 2.8. *Let (X, d_X) be a metric space and let $\sigma\colon X \times X \times [0,1] \to X$ be a conical geodesic bicombing. If σ is reversible, then there exists a reversible conical geodesic bicombing $\overline{\sigma}\colon E(X) \times E(X) \times [0,1] \to E(X)$ on $E(X)$ such that*

$$\sigma_{xy}(t) = \overline{\sigma}_{xy}(t)$$

for all $x, y \in X$ and $t \in [0,1]$.

Proof. We set

$$S := \{(1-t)\delta_x + t\delta_y \colon x, y \in X, t \in [0,1]\}.$$

Because of Lemma 2.7 we have that the map

$$\begin{aligned}\beta \colon S &\to X, \\ (1-t)\delta_x + t\delta_y &\mapsto \sigma_{xy}(t)\end{aligned}$$

is 1-Lipschitz. Therefore, Lemma 2.4 tells us that there exists a contracting barycenter map $\beta_E \colon P_1(E(X)) \to E(X)$ such that $\beta_E(e_*(s)) = e(\beta(s))$ for all $s \in S$. Thus, the conical geodesic bicombing $\overline{\sigma}$ induced by β_E has the desired property. This completes the proof. $\square$

2.2 A 1-Lipschitz barycenter construction

In [ESH99], Es-Sahib and Heinich developed a barycenter construction for non-empty finite subsets of separable complete Busemann spaces. Es-Sahib and Heinich's barycenter construction translates with no effort to complete metric spaces that admit a conical geodesic bicombing. This is the content of the subsequent proposition.

Proposition 2.9. *Let (X, d_X) be a complete metric space. If X admits a conical geodesic bicombing $\sigma \colon X \times X \times [0,1] \to X$, then there exists a collection $\{b_n \colon X^n \to X\}_{n \geq 1}$ of maps that satisfies the following four conditions:*

1. *(Locality) For all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ in X^n we have that the point $b_n(\mathbf{x})$ is contained in $\overline{conv_\sigma}\big(\{x_1, \ldots, x_n\}\big)$.*
2. *(Recursion) For all integers $n \geq 3$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ in X^n we have*
$$b_n(\mathbf{x}) = b_n(b_{n-1}(\mathbf{x}_1), \ldots, b_{n-1}(\mathbf{x}_n)),$$
where $\mathbf{x}_k := (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$ for all integers $1 \leq k \leq n$.
3. *(Nonexpansiveness) For all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \ldots, x_n)$ and $\mathbf{y} := (y_1, \ldots, y_n)$ in X^n it holds*
$$d(b_n(\mathbf{x}), b_n(\mathbf{y})) \leq \frac{1}{n} \sum_{k=1}^{n} d(x_k, y_k).$$

4. (W_1-Nonexpansiveness) If σ has the midpoint property, then we have that

$$d(b_n(\mathbf{x}), b_n(\mathbf{y})) \leq \frac{1}{n} \min_{\tau \in S_n} \sum_{k=1}^{n} d(x_k, y_{\tau(k)})$$

for all integers $n \geq 1$ and all points $\mathbf{x} := (x_1, \dots, x_n)$ and $\mathbf{y} := (y_1, \dots, y_n)$ in X^n.

Proof. Let b_1 denote the identity map of X and define the map $b_2 \colon X^2 \to X$ through the assignment $(x, y) \mapsto \sigma_{xy}(1/2)$. It is straightforward to show that the map b_2 satisfies all four conditions. Now, we proceed by induction. Let $n \geq 3$ be an integer and suppose that b_{n-1} is defined and satisfies all four conditions. Let $\mathbf{x} := (x_1, \dots, x_n)$ be a point in X^n. We define the sequence $(\mathbf{x}^{(k)})_{k \geq 0} \subset X^n$ via the recursive rule

$$\mathbf{x}^{(0)} := \mathbf{x}, \quad \mathbf{x}^{(k+1)} := (b_{n-1}(\mathbf{x}_1^{(k)}), \dots, b_{n-1}(\mathbf{x}_n^{(k)})),$$

where for each integer $k \geq 0$ and each integer $1 \leq l \leq n$ the $(n-1)$-tuple $\mathbf{x}_l^{(k)}$ is obtained from the n-tuple $\mathbf{x}^{(k)}$ by deleting the l-th entry. From now on, let $x_l^{(k)}$ denote the l-th entry of the n-tuple $\mathbf{x}^{(k)}$. For every integer $k \geq 0$ we set $A_k := \overline{\operatorname{conv}_\sigma}\left(\left\{x_1^{(k)}, \dots, x_n^{(k)}\right\}\right)$ and $D_k := \operatorname{diam}(A_k)$. The definition of the closed σ-convex hull $\overline{\operatorname{conv}_\sigma}$ is given in the introduction. Note that the sequence $(D_k)_{k \geq 0}$ is non-increasing. We claim that $D_{2k} \leq \frac{1}{(n-1)^k} D_0$ for all integers $k \geq 1$. Let $k \geq 1$ be an integer and suppose that $1 \leq l < l' \leq n$. We compute

$$\begin{aligned} d\left(x_l^{(2k)}, x_{l'}^{(2k)}\right) &= d\left(b_{n-1}(\mathbf{x}_l^{(2k-1)}), b_{n-1}(\mathbf{x}_{l'}^{(2k-1)})\right) \\ &\leq \frac{1}{n-1} \sum_{i=l}^{l'-1} d\left(b_{n-1}(\mathbf{x}_i^{(2k-2))}), b_{n-1}(\mathbf{x}_{i+1}^{(2k-2))})\right) \leq \frac{1}{n-1} D_{2(k-1)}. \end{aligned}$$

Since taking the σ-convex hull of a subset does not increase the diameter, we have shown that $D_{2k} \leq \frac{1}{n-1} D_{2(k-1)}$. Hence, it follows that $D_{2k} \leq \frac{1}{(n-1)^k} D_0$ for all integers $k \geq 1$. As a result, we obtain that the intersection $\bigcap_{k \geq 0} A_k$ consists precisely of one point which we call x^∞. For later use, observe that for each integer $1 \leq l \leq n$ we have that $x_l^{(k)} \to x^\infty$ as $k \to +\infty$. We define $b_n(x) := x^\infty$. It is readily verified that the map b_n satisfies the first two conditions. Next, we show that b_n satisfies the nonexpansiveness condition. Suppose that $\mathbf{x} := (x_1, \dots, x_n)$ and $\mathbf{y} := (y_1, \dots, y_n)$ are points in X^n. For each integer $k \geq 1$, we compute

$$\begin{aligned} d\left(x_l^{(k)}, y_l^{(k)}\right) &= d\left(b_{n-1}(\mathbf{x}_l^{(k-1)}), b_{n-1}(\mathbf{y}_l^{(k-1)})\right) \\ &\leq \frac{1}{n-1} \sum_{i=1, i \neq l}^{n} d\left(x_i^{(k-1)}, y_i^{(k-1)}\right) \end{aligned} \tag{2.6}$$

for all integers $1 \leq l \leq n$. By the use of (2.6) we obtain

$$\sum_{l=1}^{n} d\left(x_l^{(k)}, y_l^{(k)}\right) \leq \sum_{l=1}^{n} d\left(x_l^{(k-1)}, y_l^{(k-1)}\right)$$

for all integers $k \geq 1$. Hence, by passing to the limit $k \to +\infty$ we conclude that

$$n \cdot d\left(x^{\infty}, y^{\infty}\right) \leq \sum_{l=1}^{n} d(x_l, y_l),$$

as desired. Now, we are left to show that the map b_n satisfies the W_1-nonexpansiveness condition. Suppose that σ has the midpoint property. Let $\tau \in S_{n-1}$ be a permutation. Due to the W_1-nonexpansiveness of b_{n-1} we have that

$$\begin{aligned} &d\left(b_{n-1}(x_1, \ldots, x_{n-1}), b_{n-1}(x_{\tau(1)}, \ldots, x_{\tau(n-1)})\right) \\ &\leq \frac{1}{n-1} \min_{\rho \in S_{n-1}} \sum_{k=1}^{n-1} d\left(x_k, x_{\tau(\rho(k))}\right) = 0 \end{aligned}$$

for all points $x_1, \ldots, x_n$ in X. Consequently, we obtain that b_{n-1} is invariant under permutations, that is, $b_{n-1}(x_1, \ldots, x_{n-1}) = b_{n-1}(x_{\tau(1)}, \ldots, x_{\tau(n-1)})$ for all permutations τ in S_{n-1}. Hence, it follows from the second condition that b_n is invariant under permutations as well. Thus, the fourth condition is a consequence of the third condition and the permutation invariance of b_n. The proposition follows. $\square$

If the complete metric space (X, d_X) in Proposition 2.9 is a Banach space and the conical geodesic bicombing $\sigma\colon X \times X \times [0,1] \to X$ is given by $(x, y, t) \mapsto (1-t)x + ty$, then it is readily verified that the collection $\{\mathrm{bar}_n\colon X^n \to X\}_{n \geq 1}$, where for each integer $n \geq 1$ the map bar_n is given by the assignment

$$(x_1, \ldots, x_n) \mapsto \frac{1}{n}\left(x_1 + \cdots + x_n\right),$$

satisfies all four conditions of Proposition 2.9.

Let (X, d) denote a complete metric space. Suppose that (X, d_X) admits a conical geodesic bicombing $\sigma\colon X \times X \times [0,1] \to X$ that has the midpoint property. Let $\{b_n\colon X^n \to X\}_{n \geq 1}$ denote the collection of maps that we have constructed in Proposition 2.9, let $n \geq 1$ be an integer and let $\mathbf{x}$ be a point in X^n. For every integer $k \geq 1$ we denote by $Q^k(\mathbf{x})$ the element in X^{kn} that is equal to $(\mathbf{x}, \ldots, \mathbf{x})$. It is tempting to assume that

$$b_{nk_1}(Q^{k_1}(\mathbf{x})) = b_{nk_2}(Q^{k_2}(\mathbf{x})) \quad \text{for all } k_1, k_2 \geq 1. \tag{2.7}$$

However, this is not necessarily true. A counterexample can be found on page 614 in [Nav13]. Since the equality in (2.7) does not hold in general, one might ask: does at least the limit

$$\lim_{k\to+\infty} b_{nk}(Q^k(\mathbf{x})) \tag{2.8}$$

exist? Navas showed that the limit (2.8) exists for all integers $n \geq 1$ and all points $\mathbf{x}$ in X^n if X is a complete separable Busemann space, cf. [Nav13, Proposition 1.2]. As Navas's proof relies solely on the fact that the collection $\{b_n : X^n \to X\}_{n\geq 1}$ satisfies the recursion- and the W_1-nonexpansiveness condition, Navas's proof translates verbatim to collections $\{b_n : X^n \to X\}_{n\geq 1}$ that arose from complete metric spaces that admit a conical geodesic bicombing with the midpoint property.

A streamlined version of Navas's proof can be found in [Des16] (or the authors master thesis). If X satisfies a weak local compactness assumption, then it is possible to draw the conclusion that the limit in (2.8) exists via a martingale convergence theorem, cf. [ESH99, Theorem 2]. Navas used the existence of the limit (2.8) to construct a contracting barycenter map for every complete separable Busemann space, cf. [Nav13]. Essentially the same construction yields a contracting barycenter map for every complete metric space that admits a conical geodesic bicombing that has the midpoint property.

Theorem 2.10. *Let (X, d_X) be a complete metric space and let $\sigma \colon X \times X \times [0,1] \to X$ denote a conical geodesic bicombing. If σ has the midpoint property, then the map $\beta_\sigma \colon P_{\mathbb{Q}}(X) \to X$ given by the assignment*

$$\mu = \frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n}) \mapsto \lim_{k\to+\infty} b_{nk}(Q^k((x_1, \dots, x_n)) \tag{2.9}$$

is well-defined and extends uniquely to a contracting barycenter map $\beta_\sigma \colon P_1(X) \to X$ that has the following properties:

1. *(Locality) For all measures μ in $P_1(X)$ we have that the point $\beta_\sigma(\mu)$ is contained in $\overline{conv_\sigma}(spt(\mu))$.*
2. *(Equivariance) If $\varphi \colon X \to X$ is a 1-Lipschitz map and σ is φ-equivariant, then we have that β_σ is φ-equivariant, that is, it holds $\varphi \circ \beta_\sigma = \beta_\sigma \circ \varphi_*$.*

Proof. It is readily verified that the map $\beta_\sigma \colon P_{\mathbb{Q}}(X) \to X$ is well-defined, that is, the assignment (2.9) does not depend on the representation of μ. Let μ and ν denote two elements of $P_{\mathbb{Q}}(X)$. Note that there is an integer $n \geq 1$ and points $(x_1, \dots, x_n)$ and

$(y_1, \ldots, y_n)$ in X^n such that $\mu = \frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n})$ and $\nu = \frac{1}{n}(\delta_{y_1} + \cdots + \delta_{y_n})$. Due to Equation (2.9), the W_1-nonexpansiveness condition and Proposition 2.1 we have

$$d(\beta_\sigma(\mu), \beta_\sigma(\nu)) \leq \frac{1}{n} \min_{\tau \in S_n} \sum_{k=1}^{n} d(x_k, y_{\tau(k)}) = W_1(\mu, \nu);$$

hence, the map $\beta_\sigma \colon P_{\mathbb{Q}}(X) \to X$ is 1-Lipschitz. Proposition 2.2 tells us that $P_{\mathbb{Q}}(X)$ is dense in $(P_1(X), W_1)$; thus, as X is complete the map $\beta_\sigma \colon P_{\mathbb{Q}}(X) \to X$ extends uniquely to the whole space $P_1(X)$. We denote this map again by β_σ. Note that the extended map β_σ is 1-Lipschitz by construction and we have $\beta_\sigma(\delta_x) = x$ for all points x in X; hence, the map β_σ is a contracting barycenter map on (X, d_X).

The fact that the point $\beta_\sigma(\mu)$ is contained in $\overline{\mathrm{conv}_\sigma}(\mathrm{spt}(\mu))$ for all measures μ in $P_1(X)$ is a direct consequence of Proposition 2.2.

To conclude the proof we show that if $\varphi \colon X \to X$ is a 1-Lipschitz and σ is φ-equivariant, then we have that β_σ is φ-equivariant. As σ is φ-equivariant, we obtain that $\varphi(b_2(x, y)) = b_2(\varphi(x), \varphi(y))$ for all points x, y in X. A straightforward induction shows for all integers $n \geq 2$ and all points $\mathbf{x}$ in X^n that

$$\varphi(b_n(\mathbf{x})) = b_n(\boldsymbol{\varphi}(\mathbf{x})), \tag{2.10}$$

where the map $\boldsymbol{\varphi} \colon X^n \to X^n$ is given by the assignment $(x_1, \ldots, x_n) \mapsto (\varphi(x_1), \ldots, \varphi(x_n))$. Suppose that μ is a measure in $P_{\mathbb{Q}}(X)$. There is an integer $n \geq 1$ and a point $(x_1, \ldots, x_n)$ in X^n such that $\mu = \frac{1}{n}(\delta_{x_1} + \cdots + \delta_{x_n})$. Note that $\varphi_* \mu = \frac{1}{n}\left(\delta_{\varphi(x_1)} + \cdots + \delta_{\varphi(x_n)}\right)$. We compute

$$\varphi(\beta_\sigma(\mu)) = \lim_{k \to +\infty} \varphi\left(b_{nk}(Q^k(\mathbf{x}))\right) \overset{(2.10)}{=} \lim_{k \to +\infty} b_{nk}(Q^k(\boldsymbol{\varphi}(\mathbf{x}))) = \beta_\sigma(\varphi_* \mu).$$

Since the two 1-Lipschitz maps $\varphi \circ \beta_\sigma$ and $\beta_\sigma \circ \varphi_*$ agree on the W_1-dense subset $P_{\mathbb{Q}}(X) \subset P_1(X)$, we obtain that they coincide on the whole space $P_1(X)$. The theorem follows. $\square$

We call the map β_σ from Theorem 2.10 the *contracting barycenter map associated to* σ. The rest of this section is devoted to contracting barycenter maps on Banach spaces. In the subsequent proposition we show that there is precisely one contracting barycenter map on a Banach space.

Proposition 2.11. *Let $(E, \|\cdot\|)$ be a Banach space, let λ be the conical geodesic bicombing on E that consists of the linear geodesics and let $\beta_\lambda \colon P_1(E) \to E$ denote the contracting barycenter map associated to λ. It holds that the map $\beta_\lambda \colon P_1(E) \to E$ is given through the assignment*

$$\mu \mapsto \int_E x \, d\mu(x) \tag{2.11}$$

and that the map β_λ is the only contracting barycenter map on $(E, \|\cdot\|)$.

Proof. Suppose that $\beta \colon P_1(E) \to E$ is a contracting barycenter map on $(E, \|\cdot\|)$. Let μ be a measure contained in $P_1(E)$. The point $\beta(\mu)$ satisfies

$$\|\beta(\mu) - y\| \leq W_1(\mu, \delta_y) = \int_E \|x - y\| \, d\mu(x) \quad \text{for all } y \in E. \tag{2.12}$$

It is well-known that $\operatorname{spt}(\mu)$ is separable and that $\mu(E \setminus \operatorname{spt}(\mu)) = 0$; hence, the identity map $\mathrm{id} \colon (E, \mathcal{B}_E) \to (E, \mathcal{B}_E)$ is μ-essentially separably valued. Now, Pettis Measurability Theorem tells us that the identity map id is μ-measurable. Hence, we can use the definition of $P_1(E)$ and Bochner's criterion for integrability to deduce that the identity map id is Bochner integrable with respect to the measure μ. Thus, as the point $\beta(\mu)$ satisfies the inequality (2.12), Theorem 3.6 in [Mol06], which is a direct consequence of the strong law of large numbers, tells us that

$$\beta(\mu) = \int_E \mathrm{id}(x) \, d\mu(x). \tag{2.13}$$

Since the map β_λ is a contracting barycenter map, we have shown that β_λ is given through the assignment (2.13). Furthermore, as the contracting barycenter map β was arbitrary, we have also shown that β_λ is the unique contracting barycenter map on $(E, \|\cdot\|)$. □

Having Theorem 2.10 and Proposition 2.11 on hand we can deduce the following corollary.

Corollary 2.12. *Let $(E, \|\cdot\|)$ be a Banach space. If μ is a measure in $P_1(E)$, then the Bochner integral $\int_E x \, d\mu(x)$ is contained in the closure of the convex hull of spt(μ).*

Proof. This is a consequence of Theorem 2.10 and Proposition 2.11. □

2.3 Proof of Theorem 1.1

In [Des16], Descombes established that every proper metric space with a conical geodesic bicombing admits a reversible conical geodesic bicombing. Miesch generalized this result to arbitrary complete metric spaces.

Proposition 2.13 (p. 87 in [Mie17a]). *Let (X, d_X) be a complete metric space with a conical geodesic bicombing. Then X also admits a reversible, conical geodesic bicombing.*

We construct an example of a non-reversible conical geodesic bicombing in Section 2.4. Now, we have everything at hand to prove Theorem 1.1.

Proof of Theorem 1.1. (1.) $\Longrightarrow$ (2.). The map $\sigma\colon X \times X \times [0,1] \to X$ given by

$$(x, y, t) \mapsto \beta((1-t)\delta_x + t\delta_y)$$

is a geodesic bicombing. Indeed, for $0 \le s \le t \le 1$ we compute

$$\begin{aligned}
&d_X(x,y)\\
&\le d_X(x, \sigma_{xy}(s)) + d_X(\sigma_{xy}(s), \sigma_{xy}(t)) + d_X(\sigma_{xy}(t), y)\\
&\le s d_X(x,y) + W_1((1-s)\delta_x + s\delta_y, (1-t)\delta_x + t\delta_y) + (1-t)d_X(x,y).
\end{aligned}$$

By the use of the Kantorovich-Rubinstein Duality Theorem, we obtain

$$W_1((1-s)\delta_x + s\delta_y, (1-t)\delta_x + t\delta_y) = (t-s)W_1(\delta_x, \delta_y);$$

hence, by the estimate above it follows that $\sigma_{xy}(\cdot)$ is a geodesic from x to y.

Next, we show the conical inequality. Let $t \in [0,1]$ be a real number and let $(t_k)_{k\ge 1} \subset [0,1] \cap \mathbb{Q}$ be a sequence of rational numbers such that $t_k \to t$ for $k \to +\infty$. Using Proposition 2.1 we get

$$W_1\big((1-t_k)\delta_x + t_k\delta_y, (1-t_k)\delta_{x'} + t_k\delta_{y'}\big) \le (1-t_k)d_X(p,p') + t_k d_X(q,q')$$

for all points $x, y, x', y' \in X$. Hence, the map $\sigma\colon X \times X \times [0,1] \to X$ given by $(x,y,t) \mapsto \beta((1-t)\delta_x + t\delta_y)$ satisfies inequality (1.2), as desired.

(2.) $\Longrightarrow$ (1.). By employing Proposition 2.13, we get that X admits a reversible conical geodesic bicombing σ. Now, the map β_σ from Theorem 2.10 is a contracting barycenter map. Hence, X is a barycentric metric space, as was to be shown.

□

2.4 Reversibility of conical geodesic bicombings

In this section, we construct a non-reversible conical geodesic bicombing. Afterwards, we modify this non-reversible conical geodesic bicombing to satisfy the midpoint property. Let $s\colon \ell^2_\infty \to \ell^2_\infty$ denote the map given by $(x, y) \mapsto (x, -y)$. We define

$$X_1 := \big\{(x,y) \in \ell^2_\infty : x \in [-2,1] \text{ and } |x| - 1 \leq y \leq ||x| - 1| \big\},$$
$$A_1 := \big\{(x,y) \in \ell^2_\infty : |x+1| \leq y \leq 1\big\}.$$

and $X_2 := s(X_1)$, $A_2 := s(A_1)$. The set $X_1 \cup X_2$ is depicted in Figure 2.1. It is readily verified that the map $f\colon X_2 \to X_1$ given by

$$(x,y) \mapsto \begin{cases} (x,y), & \text{if } x \in [-1,1], \\ s(x,y), & \text{if } x \in [-2,-1] \end{cases}$$

is an isometry. Let $\bar{f}\colon X_1 \cup X_2 \to X_1$ be the map that is equal to id_{X_1} on X_1 and equal to f on X_2. Observe that the map $\bar{f}$ is 1-Lipschitz. We set $Y_k := X_k \cup A_k$ for $k \in \{1,2\}$.

Further, we define the map $\pi\colon Y_1 \cup Y_2 \to X_1 \cup X_2$ through the assignment

$$(x,y) \mapsto \big(x, \operatorname{sgn}(y) \min\{\, |y|, ||x|-1| \,\}\big).$$

Observe that π is a 1-Lipschitz retraction that maps Y_k to X_k for each $k \in \{1,2\}$. Let $\lambda\colon \mathbb{R}^2 \times \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$ be the conical geodesic bicombing on $\mathbb{R}^2$ that is given by the linear geodesics.

Lemma 2.14. *The map $\sigma\colon X_1 \times X_1 \times [0,1] \to X_1$ given by*

$$(p,q,t) \mapsto \begin{cases} \pi \circ \lambda(p,q,t), & \text{if } p_x \leq q_x, \\ f \circ \pi \circ \lambda\left(f^{-1}(p), f^{-1}(q), t\right), & \text{if } q_x \leq p_x. \end{cases}$$

is a non-reversible conical geodesic bicombing on $(X_1, \|\cdot\|_\infty)$.

Proof. Observe that both maps

$$\sigma^{(1)} := \pi \circ \lambda \text{ and } \sigma^{(2)} := f \circ \pi \circ \lambda \circ \left(f^{-1} \times f^{-1} \times \mathsf{Id}_{[0,1]}\right)$$

define conical geodesic bicombings on X_1. Thus, it follows that $\sigma\colon X_1 \times X_1 \times [0,1] \to X_1$ is a geodesic bicombing.

In the following, we show that σ is conical. Let $p, q, p', q' \in X_1$ be points. As both maps $\sigma^{(1)}$ and $\sigma^{(2)}$ are conical geodesic bicombings on X_1 with $\sigma^{(1)}_{pq} = \sigma^{(2)}_{pq}$ if $p_x, q_x \leq -1$

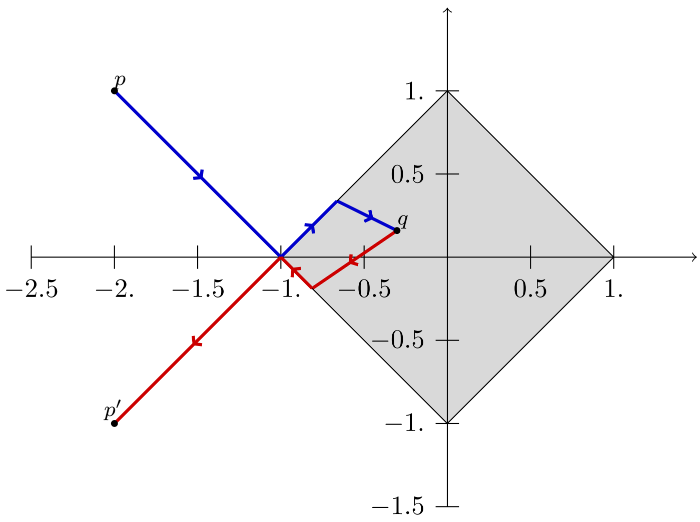

Figure 2.1: The blue line corresponds to σ_{pq} and the red line corresponds to the image of σ_{qp} under the isometry f^{-1}.

or $p_x, q_x \geq -1$, it remains to check inequality (1.2) if ($p_x, q'_x \leq -1$ and $q_x, p'_x \geq -1$) or ($p'_x, q_x \leq -1$ and $q'_x, p_x \geq -1$).

Now, suppose that $p_x, q'_x \leq -1$ and $q_x, p'_x \geq -1$. The other case is treated analogously. Since the map $\bar{f} \circ \pi$ is 1-Lipschitz, we compute

$$\begin{aligned} \|\sigma_{pq}(t) - \sigma_{p'q'}(t)\|_\infty &= \|\bar{f} \circ \pi \circ \lambda(p, q, t) - \bar{f} \circ \pi \circ \lambda(f^{-1}(p'), f^{-1}(q'), t)\|_\infty \\ &\leq (1-t)\|p - f^{-1}(p')\|_\infty + t\|q - f^{-1}(q')\|_\infty \end{aligned}$$

for all $t \in [0,1]$. By our assumptions on the points p, q, p', q', it follows that

$$\begin{aligned} &\|p - f^{-1}(p')\|_\infty = \|p - p'\|_\infty, \\ &\|q - f^{-1}(q')\|_\infty = \|f^{-1}(q) - f^{-1}(q')\|_\infty = \|q - q'\|_\infty. \end{aligned}$$

By putting everything together, we obtain that σ is a conical geodesic bicombing on X_1. By construction, it follows that σ is non-reversible; see Figure 2.1. $\square$

Now, we use the conical geodesic bicombing from Lemma 2.14 to construct a non-reversible conical geodesic bicombing that has the midpoint property.

Lemma 2.15. *Let $\sigma\colon X_1 \times X_1 \times [0,1] \to X_1$ denote the map from Lemma 2.14. The map $\tau\colon X_1 \times X_1 \times [0,1] \to X_1$ given by the assignment*

$$(p, q, t) \mapsto \begin{cases} \sigma\left(p, \frac{1}{2}\left(\sigma(p, q, \frac{1}{2}) + \sigma(q, p, \frac{1}{2})\right), 2t\right), & \text{if } t \in [0, \frac{1}{2}], \\ \sigma\left(\frac{1}{2}\left(\sigma(p, q, \frac{1}{2}) + \sigma(q, p, \frac{1}{2})\right), q, 2t - 1\right), & \text{if } t \in [\frac{1}{2}, 1], \end{cases}$$

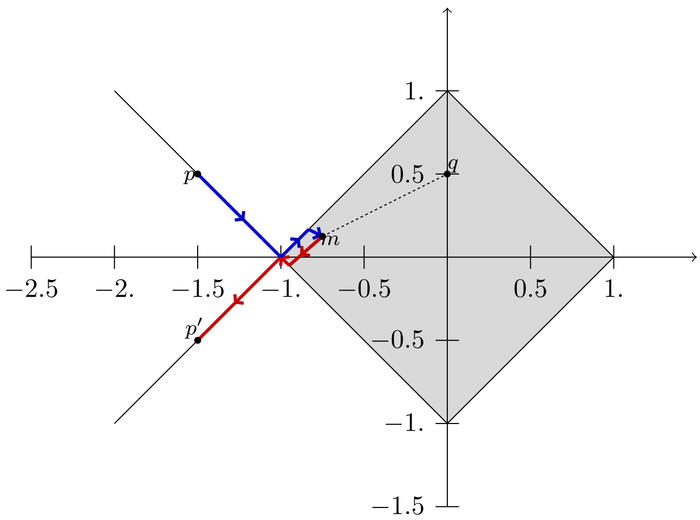

Figure 2.2: The blue line corresponds to $\tau_{pq}|_{[0,\frac{1}{2}]}$ and the red line corresponds to the image of $\tau_{qp}|_{[\frac{1}{2},1]}$ under the isometry f^{-1}. The point m is equal to $\frac{1}{2}\big(\sigma_{pq}(\frac{1}{2}) + \sigma_{qp}(\frac{1}{2})\big)$.

is a non-reversible conical geodesic bicombing on $(X_1, \|\cdot\|_\infty)$ that has the midpoint property.

Proof. It is readily verified that τ is a conical geodesic bicombing with the midpoint property. To see that τ is non-reversible, take for instance $p := (-\frac{3}{2}, \frac{1}{2})$, $q := (0, \frac{1}{2})$ and observe that $\tau(p, q, \frac{5}{12}) = (-\frac{7}{8}, \frac{1}{8}) \neq (-\frac{7}{8}, \frac{1}{48}) = \tau(q, p, \frac{7}{12})$; compare Figure 2.2. $\square$

2.5 Local behavior of conical geodesic bicombings

Let $(V, \|\cdot\|)$ be a normed vector space, let $p_0 \in V$ be a point and let $r \geq 0$ be a real number. We set

$$U_r(p_0) := \{z \in V : \|p_0 - z\| < r\},$$
$$B_r(p_0) := \{z \in V : \|p_0 - z\| \leq r\},$$
$$S_r(p_0) := \{z \in V : \|p_0 - z\| = r\}.$$

To ease notation, we abbreviate $B_r := B_r(0)$ and $S_r := S_r(0)$. The goal of this section is to establish the following rigidity result.

Theorem 2.16. *Let $(V, \|\cdot\|)$ be a normed vector space. Suppose that $A \subset V$ is a subset of V that admits a conical geodesic bicombing $\sigma\colon A \times A \times [0,1] \to A$ and let p, q be points of A. If there are points $e_1, \dots, e_n \in B_1$ that are extreme points of B_1 and a tuple $(\lambda_1, \dots, \lambda_n) \in [0,1]^n$ with $\sum_{k=1}^n \lambda_k = 1$ such that*

$$\frac{p-q}{2} = \frac{\|p-q\|}{2} \sum_{k=1}^{n} \lambda_k e_k \text{ and} \tag{2.14}$$

$$\frac{p+q}{2} + \frac{\|p-q\|}{2} \left\{ \sum_{k=1}^{n} (-1)^{\epsilon_k} \lambda_k e_k : (\epsilon_1, \dots, \epsilon_n) \in \{0,1\}^n \right\} \subset A, \tag{2.15}$$

then it follows that $\sigma(p,q,t) = (1-t)p + tq$ for all $t \in [0,1]$.

Theorem 1.4 then is a direct consequence.

Proof of Theorem 1.4. Let $p, q \in B_r(p_0)$ be two points. As $\frac{p+q}{2} \in B_r(p_0)$ and $\frac{\|p-q\|}{2} \leq r$, the ball $B_{\frac{\|p-q\|}{2}}(\frac{p+q}{2})$ is contained in A. Hence, since the unit ball of V is the closed convex hull of its extreme points, it follows that $\sigma(p,q,t) = (1-t)p + tq$ for all $t \in [0,1]$ by Theorem 2.16 and a straightforward limit argument. □

We will derive Theorem 2.16 via induction on the number of extreme points. For this induction, we need some preparatory lemmas and definitions. We define the map $\lambda\colon V \times V \times [0,1] \to V$ via the assignment

$$(p, q, t) \mapsto (1-t)p + tq.$$

It is readily verified that λ is a conical geodesic bicombing. Let $t \in [0,1]$ be a real number and let p, q be points in V. We define

$$M^{(t)}(p,q) := \left\{ z \in V : \|z-p\| = t\|p-q\|, \|z-q\| = (1-t)\|p-q\| \right\}.$$

Clearly, $\sigma(p,q,t) \in M^{(t)}(p,q)$ for every geodesic bicombing σ. Thus, if $M^{(t)}(p,q)$ is a singleton, then $\sigma(p,q,t) = \lambda(p,q,t)$. The first lemma of this section gives a sufficient condition for the set $M^{(t)}(p,q)$ to be a singleton.

Lemma 2.17. *Let $(V, \|\cdot\|)$ be a normed vector space and let $p \in V$ be a point. If p is an extreme point of $B_{\|p\|}$, then $M^{(t)}(p,-p) = \{(1-2t)p\}$ for all $t \in [0,1]$.*

Proof. By construction, we have

$$M^{(t)}(p,-p) = \left(S_{2t\|p\|} + p\right) \cap \left(S_{(1-t)2\|p\|} - p\right);$$

hence,

$$\frac{1}{2t}\left(p - M^{(t)}(p,-p)\right) = S_{\|p\|} \cap \left(\frac{1}{t}p - \frac{1-t}{t}S_{\|p\|}\right), \tag{2.16}$$

provided that $t \in (0,1]$. For each $t \in (0,1]$ we define the map $E^{(t)} \colon V \to \mathcal{P}(V)$ via the assignment

$$p \mapsto S_{\|p\|} \cap \left(\frac{1}{t}p - \frac{1-t}{t}S_{\|p\|}\right).$$

Note that $\mathcal{P}(V)$ denotes the power set of V. By the use of the identity (2.16) $M^{(t)}(p,-p) = \{(1-2t)p\}$ if and only if $E^{(t)}(p) = \{p\}$. Thus, we are left to show that if p is an extreme point of $B_{\|p\|}$, then $E^{(t)}(p) = \{p\}$ for all $t \in (0,1)$. We argue by contraposition. Suppose that there is a real number $t \in (0,1)$ and a point $p' \in E^{(t)}(p)$ with $p' \neq p$. As $p' \in E^{(t)}(p)$, it follows that $p' \in S_{\|p\|}$ and that there is a point $q \in S_{\|p\|}$ such that $p' = \frac{1}{t}p - \frac{1-t}{t}q$. Observe that $q \neq p$ and

$$(1-t)q + tp' = (1-t)q + t\left(\frac{1}{t}p - \frac{1-t}{t}q\right) = p.$$

Hence the point p is not extreme in $B_{\|p\|}$, as desired. By putting everything together, the lemma follows. $\square$

Lemma 2.17 will serve as base case for the induction in the proof of Theorem 2.16. The subsequent lemma is the key component for the inductive step in the proof of Theorem 2.16.

Lemma 2.18. *Let $(V, \|\cdot\|)$ be a normed vector space and let $A \subset V$ be a subset that admits a conical geodesic bicombing $\sigma \colon A \times A \times [0,1] \to A$. Let p be a point in A such that $-p \in A$. If there is a point z in V such that the points $2z-p$ and $p-2z$ are contained in A and such that $\sigma(p, p-2z, \cdot) = \lambda(p, p-2z, \cdot)$ and $\sigma(2z-p, -p, \cdot) = \lambda(2z-p, -p, \cdot)$, then we have that*

$$\sigma(p,-p,t) \in \left((1-2t)z + M^{(t)}\left(p-z, z-p\right)\right).$$

for all real numbers $t \in [0,1]$.

Proof. Let $t \in [0,1]$ be a real number. Using that σ is conical, we compute

$$\begin{aligned} \|\sigma(p,-p,t) - \lambda(p,p-2z,t)\| &\leq 2t\|p-z\| \\ \|\sigma(p,-p,t) - \lambda(2z-p,-p,t)\| &\leq 2(1-t)\|p-z\|. \end{aligned}$$

Note that $\|\lambda(p, p-2z, t) - \lambda(2z-p, -p, t)\| = 2\|p-z\|$. Therefore, it follows that

$$\sigma(p, -p, t) \in M^{(t)}\left(\lambda(p, p-2z, t), \lambda(2z-p, -p, t)\right).$$

It is readily verified that $M^{(t)}(u+h, v+h) = h + M^{(t)}(u, v)$ for all t in $[0,1]$ and $u, v, h \in V$. Consequently, we obtain that

$$M^{(t)}\left(\lambda(p, p-2z, t), \lambda(2z-p, -p, t)\right) = (1-2t)z + M^{(t)}\left(p-z, z-p\right).$$

Thus, the lemma follows. □

Suppose that A is a subset of a normed vector space $(V, \|\cdot\|)$ and assume that A admits a conical geodesic bicombing $\sigma\colon A \times A \times [0,1] \to A$. The translation $T_z\colon A \to T_z(A)$ about the vector $z \in V$ given by the assignment $x \mapsto x + z$ is an isometry and the map $(T_z)_*\sigma\colon T_z(A) \times T_z(A) \times [0,1] \to T_z(A)$ given by

$$(x, y, t) \mapsto T_z(\sigma(T_{-z}(x), T_{-z}(y), t)) \tag{2.17}$$

is a conical geodesic bicombing on $T_z(A)$. Now, we have everything on hand to prove Theorem 2.16.

Proof of Theorem 2.16. We proceed by induction on $n \geq 1$. If $n = 1$, then Lemma 2.17 tells us that

$$\left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(\frac{p-q}{2}, -\frac{p-q}{2}, t\right) = (1-2t)\frac{p-q}{2}$$

for all $t \in [0,1]$. Thus, we obtain that $\sigma(p, q, t) = (1-t)p + tq$ for all $t \in [0,1]$.

Suppose now that $n > 1$ and that the statement holds for $n-1$. We may assume that $\lambda_1 \in (0,1)$. We define $(\lambda'_1, \ldots, \lambda'_{n-1}) := \frac{1}{1-\lambda_1}(\lambda_2, \ldots, \lambda_n)$ and $(e'_1, \ldots, e'_{n-1}) := (e_2, \ldots, e_n)$. Observe that

$$\sum_{k=1}^{n} \lambda_k e_k = \lambda_1 e_1 + (1-\lambda_1)\sum_{k=1}^{n-1} \lambda'_k e'_k. \tag{2.18}$$

Further, note that

$$\|\sum_{k=1}^{n-1} \lambda'_k e'_k\| = 1, \text{ as otherwise (2.18) implies } \|\sum_{k=1}^{n} \lambda_k e_k\| < 1, \tag{2.19}$$

which is not possible due to (2.14). We abbreviate $r := \frac{\|p-q\|}{2}$ and we set

$$z := r(1-\lambda_1)\sum_{k=1}^{n-1} \lambda'_k e'_k, \qquad p' := \frac{p-q}{2}, \qquad q' := p' - 2z.$$

Note that

$$\frac{p'-q'}{2} = r(1-\lambda_1)\sum_{k=1}^{n-1}\lambda'_k e'_k.$$

Hence, by the use of (2.19) it follows that

$$\frac{\|p'-q'\|}{2} = r(1-\lambda_1). \tag{2.20}$$

We have that

$$\frac{p'+q'}{2} = \frac{p-q}{2} - z \overset{(2.14)}{=} r\sum_{k=1}^{n}\lambda_k e_k - r(1-\lambda_1)\sum_{k=1}^{n-1}\lambda'_k e'_k \overset{(2.18)}{=} r\lambda_1 e_1$$

and therefore

$$\begin{aligned}&\frac{p'+q'}{2} + \frac{\|p'-q'\|}{2}\left\{\sum_{k=1}^{n-1}(-1)^{\epsilon_k}\lambda'_k e'_k : (\epsilon_1,\ldots,\epsilon_{n-1}) \in \{0,1\}^{n-1}\right\}\\ &\overset{(2.20)}{=} r\left\{\lambda_1 e_1 + \sum_{k=2}^{n}(-1)^{\epsilon_k}\lambda_k e_k : (\epsilon_2\ldots,\epsilon_n) \in \{0,1\}^{n-1}\right\} \overset{(2.15)}{\subset} T_{-\frac{p+q}{2}}(A).\end{aligned}$$

Thus, we can apply the induction hypothesis to $p', q' \in T_{-\frac{p+q}{2}}(A)$ and obtain that

$$\left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(p', p'-2z, \cdot\right) = \lambda\left(p', p'-2z, \cdot\right).$$

Similarly, we obtain

$$\left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(2z-p', -p', \cdot\right) = \lambda\left(2z-p', -p', \cdot\right).$$

Now, by the use of Lemma 2.18 it follows that

$$\left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(p', -p', t\right) \in \left((1-2t)z + M^{(t)}\left(p'-z, z-p'\right)\right)$$

for all real numbers $t \in [0,1]$; consequently, we get

$$\left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(p', -p', t\right) = (1-2t)p',$$

since $p'-z = r\lambda_1 e_1$ is an extreme point in $B_{r\lambda_1}$ and thus we can use Lemma 2.17 to deduce that $M^{(t)}(p'-z, z-p') = \{(1-2t)(p'-z)\}$. Hence, we have

$$\sigma(p,q,t) = \left(T_{-\frac{p+q}{2}}\right)_* \sigma\left(p', -p', t\right) + \frac{p+q}{2} = (1-t)p + tq,$$

as desired. $\square$

2.6 Proof of Theorem 1.5

Before we start with the proof of Theorem 1.5, we recall some notions from [Mie17b]. Let (X, d_X) be a metric space, let $p \in X$ be a point and let $r > 0$ be a real number. We set $U_r(p) := \{q \in X : d(p,q) < r\}$. Let $U \subset X \times X \times [0,1]$ be a subset. A map $\sigma\colon U \to X$ is a *convex local geodesic bicombing* if for every point $p \in X$ there is a real number $r_p > 0$ such that

$$U = \bigcup_{p \in X} \mathsf{D}(U_{r_p}(p)), \qquad \text{where} \quad \mathsf{D}(U_{r_p}(p)) := U_{r_p}(p) \times U_{r_p}(p) \times [0,1]$$

and if the restriction $\sigma|_{\mathsf{D}(U_{r_p}(p))}\colon \mathsf{D}(U_{r_p}(p)) \to X$ is a consistent conical geodesic bicombing for each point $p \in X$. Furthermore, we say that a geodesic $c\colon [0,1] \to X$ is *consistent* with the convex local geodesic bicombing σ if for each choice of real numbers $0 \leq s_1 \leq s_2 \leq 1$ with $(c(s_1), c(s_2)) \in U_{r_p}(p) \times U_{r_p}(p)$ for some point $p \in X$, it holds

$$c((1-t)s_1 + ts_2) = \sigma(c(s_1), c(s_2), t)$$

for all $t \in [0,1]$. Consistent geodesics are uniquely determined by the local geodesic bicombing, compare [Mie17b, Theorem 1.1] and the proof thereof:

Theorem 2.19. *Let X be a complete, simply-connected metric space with a convex local geodesic bicombing σ. If we equip X with the length metric, then for every two points $p, q \in X$ there is a unique geodesic from p to q which is consistent with σ and the collection of all such geodesics is a convex geodesic bicombing.*

With Theorem 2.19 on hand it is possible to derive Theorem 1.5 by the use of Theorem 1.4.

Proof of Theorem 1.5. Let $\mathrm{int}(C)$ denote the interior of C and let p, q be two points in $\mathrm{int}(C)$. We abbreviate

$$[p,q] := \big\{(1-t)p + tq : t \in [0,1]\big\}.$$

As $\mathrm{int}(C)$ is convex, we have that $[p,q] \subset \mathrm{int}(C)$. For each point $z \in C$ we set

$$r_z := \begin{cases} \min\{\|z-w\| : w \in [p,q]\} & \text{if } z \in C \setminus \mathrm{int}(C) \\ \frac{1}{2}\inf\{\|z-w\| : w \in C \setminus \mathrm{int}(C)\} & \text{if } z \in \mathrm{int}(C). \end{cases}$$

Note that $r_z > 0$ for all points $z \in C$ and we have that $U_{r_z}(z) \cap [p,q] = \varnothing$ if $z \in C \setminus \mathrm{int}(C)$. Further, for every point $z \in \mathrm{int}(C)$ it follows that $B_{2r_z}(z) \subset C$; thus, we may invoke

Theorem 1.4 to deduce that if $z \in \operatorname{int}(C)$, then $\sigma_{z_1 z_2}(t) = (1-t)z_1 + tz_2$ for all points $z_1, z_2 \in B_{r_z}(z)$ and all real numbers $t \in [0,1]$. We define

$$U := \bigcup_{z \in C} \mathsf{D}(U_{r_z}(z)).$$

Note that the map $\sigma^{\mathrm{loc}} := \sigma|_U$ defines a convex local bicombing on C. The geodesic $\sigma_{pq}(\cdot)$ and the linear geodesic from p to q are both consistent with the local bicombing σ^{loc}. Hence, by Theorem 2.19, we conclude that $\sigma_{pq}(\cdot)$ must be equal to the linear geodesic from p to q, that is, we have $\sigma_{pq}(t) = (1-t)p + tq$ for all real numbers $t \in [0,1]$.

Now, suppose that $p, q \in C$. As C is convex, it is well-known that $C = \overline{\operatorname{int}(C)}$, cf. [AB06, Lemma 5.28]. Let $(p_k)_{k \geq 1}, (q_k)_{k \geq 1} \subset \operatorname{int}(C)$ be two sequences such that $p_k \to p$ and $q_k \to q$ with $k \to +\infty$. It is readily verified that $\sigma_{p_k q_k}(\cdot) \to \sigma_{pq}(\cdot)$ with $k \to +\infty$, since σ is a conical geodesic bicombing. As a result, we obtain that the geodesic $\sigma_{pq}(\cdot)$ is equal to the linear geodesic from p to q, as desired. □

We conclude this section with two examples that show that the assumptions in Theorem 1.5 cannot be dropped in general.

Example 2.20. The following construction is inspired by a similar construction due to Schechtman. We define the set

$$A := \big\{ f \colon [0,1] \to [0,1] : f(0) = 0, f(1) = 1, f \text{ is continuous and strictly increasing} \big\}.$$

We claim that the metric space $(A, \|\cdot\|_1)$ admits two distinct consistent conical geodesic bicombings. Clearly, as A is convex, the map $\lambda \colon A \times A \times [0,1] \to A$ given by $(f, g, t) \mapsto (1-t)f + tg$ is a consistent conical geodesic bicombing on $(A, \|\cdot\|_1)$. Let $\varphi \colon A \to A$ denote the map given by $f \mapsto f^{-1}$. The map φ is an isometry of $(A, \|\cdot\|_1)$. This is a simple consequence of the identity

$$\|f - g\|_1 = \operatorname{vol}_2 \big(\{ (x,y) \in [0,1]^2 : \min\{f(x), g(x)\} \leq y \leq \max\{f(x), g(x)\} \} \big)$$

which holds true for all $f, g \in A$ and where vol_2 denotes the two dimensional Lebesgue measure.

Let $\tau \colon A \times A \times [0,1] \to A$ be the map where each map $\tau_{fg}(\cdot)$ is given by the horizontal interpolation of the functions $f, g \in A$, that is, the map τ is given by the assignment $(f, g, t) \mapsto \varphi((1-t)\varphi(f) + t\varphi(g))$. As the map φ is an isometry, it follows that τ is a consistent conical geodesic bicombing. Indeed, it holds that $\tau = \varphi_* \lambda$, here we use the notation introduced in (2.17). Furthermore, if $f(x) := \sqrt{x}$ and $g(x) := x$, then we have that the map $\tau(f, g, t)\colon [0,1] \to [0,1]$ is given by

$$x \mapsto \frac{-t + \sqrt{4(1-t)x + t^2}}{2(1-t)}$$

for all $t \in [0,1]$, which is distinct from $\lambda(f, g, t) = (1-t)f + tg$ for all $t \in (0,1)$. Hence, the metric space $(A, \|\cdot\|_1)$ admits two distinct consistent conical geodesic bicombings. Let B denote the closure of $A \subset L^1([0,1])$. Note that λ and τ extend naturally to consistent conical geodesic bicombings on B. Hence, we have found a closed convex subset of a Banach space that admits two distinct consistent conical geodesic bicombings. It is readily verified that B has empty interior.

Example 2.21. We consider the normed vector space $(\mathbb{R}^2, \|\cdot\|_\infty)$, where $\|\cdot\|_\infty$ denotes the maximum norm. Recall that $\|\cdot\|_\infty\colon \mathbb{R}^2 \to \mathbb{R}$ is given by the assignment $p = (s,t) \mapsto |s| \vee |t|$. Throughout this example, we use $a \vee b$ to denote the maximum of the two quantities a and b. We define the set $C := \{(s,t) \in \mathbb{R}^2 : t \geq 0\}$. The goal of this example is to show that C admits two distinct conical geodesic bicombings. To begin, we define the points $x_1 := (-1, 0), x_2 := (1, 0)$ and $b := (0, 1)$ and we claim that

$$\|b - p\|_\infty \leq \frac{1}{2}\big(\|x_1 - p\|_\infty + \|x_2 - p\|_\infty\big) \tag{2.21}$$

for all points p in C. In order to show that the inequality in (2.21) is true for all points $p \in C$ we introduce some auxiliary functions first. We define the functions

$$\begin{aligned} f_{-1}\colon \mathbb{R} &\to \mathbb{R} & \qquad f_1\colon \mathbb{R} &\to \mathbb{R} \\ s &\mapsto |1+s| & s &\mapsto |1-s|\,. \end{aligned}$$

Furthermore, we define the sets

$$\begin{aligned} C^{\uparrow\uparrow} &:= \{p \in C : t \geq f_{-1}(s),\, t \geq f_1(s)\}, & C^{\downarrow\downarrow} &:= \{p \in C : t \leq f_{-1}(s),\, t \leq f_1(s)\}, \\ C^{\uparrow\downarrow} &:= \{p \in C : t \geq f_{-1}(s),\, t \leq f_1(s)\}, & C^{\downarrow\uparrow} &:= \{p \in C : t \leq f_{-1}(s),\, t \geq f_1(s)\}. \end{aligned}$$

Now, we distinguish three cases:

1. First, we suppose that the point $p := (s,t)$ is contained in the set $C^{\uparrow\uparrow}$. We compute $\|x_1 - p\|_\infty = |t|\,, \|x_1 - p\|_\infty = |t|$ and $\|b - p\|_\infty = |1 - t|$. Since we have $t \geq 1$, we obtain that the point p satisfies (2.21).

2. Second, we suppose that the point $p := (s,t)$ is contained in the set $C^{\downarrow\downarrow}$. We compute $\|x_1 - p\|_\infty = |1+s|\,, \|x_2 - p\|_\infty = |1-s|$. Note that $\frac{1}{2}\left(|1+s| + |1-s|\right) \geq 1 \vee |s|$. Thus, the point p satisfies (2.21), as $\|b - p\|_\infty \leq 1 \vee |s|$.

3. Third, we suppose that the point $p := (s,t)$ is contained in the union $C^{\uparrow\downarrow} \cup C^{\downarrow\uparrow}$. We compute $\|b-p\|_\infty = |s|$. Since we have that $|s| \leq \frac{1}{2}(|1+s| + |1-s|) \leq \frac{1}{2}(\|x_1 - p\|_\infty + \|x_2 - p\|_\infty)$, we obtain that the point p satisfies (2.21).

Consequently, we may conclude that the estimate (2.21) is true for all points p in C. Let $\sigma_{x_1x_2} \colon [0,1] \to C$ denote the geodesic given by

$$t \mapsto \begin{cases} (1-2t)x_1 + 2tb & \text{if } t \in [0, \frac{1}{2}] \\ 2(1-t)b + (2t-1)x_2 & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

Clearly, $\sigma_{x_1x_2}(\cdot)$ is a geodesic from x_1 to x_2. We claim that

$$\|\sigma_{x_1x_2}(t) - p\|_\infty \leq (1-t)\|x_1 - p\|_\infty + t\|x_2 - p\|_\infty \tag{2.22}$$

for all $p \in C$. Due to symmetry reasons, it suffices to consider the case $t \in [0, \frac{1}{2}]$. We compute

$$\|((1-2t)x_1 + 2tb) - p\|_\infty \leq (1-2t)\|x_1 - p\|_\infty + 2t\|b - p\|_\infty;$$

hence, by the use of (2.21) we get

$$\|\sigma_{x_1x_2}(t) - p\|_\infty \leq (1-t)\|x_1 - p\|_\infty + t\|x_2 - p\|_\infty,$$

as claimed. Now, we set

$$S = \big\{\delta_p : p \in C\big\} \cup \big\{(1-t)\delta_{x_1} + t\delta_{x_2} : t \in [0,1]\big\}$$

and we define the map $\beta \colon S \to C$ via

$$s \mapsto \begin{cases} p & \text{if } s = \delta_p \\ \sigma_{x_1x_2}(t) & \text{if } s = (1-t)\delta_{x_1} + t\delta_{x_2}. \end{cases}$$

Because of (2.22), we deduce that β is 1-Lipschitz if we equip S with W_1, Moreover, by construction $\beta(\delta_p) = p$. Therefore, since the metric space $(C, \|\cdot\|_\infty)$ is injective there exists a contracting barycenter map $\overline{\beta} \colon P_1(C) \to C$ that extends β. Let σ denote the conical geodesic bicombing induced by $\overline{\beta}$. By construction, σ is not equal to the geodesic bicombing given by the linear segments. So C admits two distinct conical geodesic bicombings.

2.7 A fixed-point free isometry of a Busemann space

In this section, we construct a bounded complete Busemann space that admits an isometry without fixed points. As usual, let $\ell^1(\mathbb{Z}) \subset \mathbb{R}^{\mathbb{Z}}$ denote the linear subspace of $\mathbb{R}^{\mathbb{Z}}$ that consists of all sequences $x := (x_k)_{k\in\mathbb{Z}}$ such that

$$\|x\|_1 := \sum_{k\in\mathbb{Z}} |x_k| < +\infty.$$

Now, we use a standard technique, cf. [JL01, p. 786], to renorm the Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_1)$ into a strictly convex Banach space. We define the map $\|\cdot\|_\star \colon \ell^1(\mathbb{Z}) \to \mathbb{R}$ through the assignment

$$(x_k)_{k\in\mathbb{Z}} \mapsto \sqrt{\left(\sum_{k\in\mathbb{Z}} |x_k|\right)^2 + \sum_{k\in\mathbb{Z}} |x_k|^2}.$$

It is straightforward to show that the map $\|\cdot\|_\star$ defines a norm on $\ell^1(\mathbb{Z})$. Elementary estimates show that

$$\frac{1}{\sqrt{2}}\|\cdot\|_\star \leq \|\cdot\|_1 \leq \|\cdot\|_\star;$$

hence, the norms $\|\cdot\|_1$ and $\|\cdot\|_\star$ are equivalent. It follows that $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is a Banach space. Recall that a normed vector space $(V, \|\cdot\|_V)$ is said to be *strictly convex* if for all distinct points x, y in V with $\|x\|_V = \|y\|_V = 1$ and for all λ in $(0,1)$ we have the strict inequality $\|(1-\lambda)x + \lambda y\|_V < 1$.

Lemma 2.22. *The Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is strictly convex.*

Proof. Let x and y denote two distinct points of $\ell^1(\mathbb{Z})$ that satisfy $\|x\|_\star = \|y\|_\star = 1$ and let λ in $(0,1)$ be a real number. Since x and y are distinct, there is an integer k_0 such that $x_{k_0} \neq y_{k_0}$. It follows that

$$\big((1-\lambda)x_{k_0} + \lambda y_{k_0}\big)^2 < (1-\lambda)x_{k_0}^2 + \lambda y_{k_0}^2, \tag{2.23}$$

as the real valued function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is strictly convex. Now, elementary estimates and the strict inequality in (2.23) imply that $\|(1-\lambda)x + \lambda y\|_\star^2 < (1-\lambda)\|x\|_\star^2 + \lambda\|y\|_\star^2 = 1$; hence, the Banach space $(\ell^1(\mathbb{Z}), \|\cdot\|_\star)$ is strictly convex, as was to be shown. □

The shift map $T\colon \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ given by the assignment

$$(x_k)_{k\in\mathbb{Z}} \mapsto (x_{k-1})_{k\in\mathbb{Z}}$$

is a linear map and an isometry of $(X, \|\cdot\|_\star)$. Note that the zero sequence is the only fixed point of T. Let $x_0 \in \ell^1(\mathbb{Z})$ be the sequence that is equal to one if $k = 0$ and equal to zero if $k \neq 0$. We define the set $A := \{T^k(x_0) : k \in \mathbb{Z}\}$. Let conv$(A)$ denote the convex hull of A.

Lemma 2.23. *If x is an element of conv(A), then we have $1 \leq \|x\|_\star \leq \sqrt{2}$.*

Proof. Let x be an element of conv(A). By the definition of conv(A), there is an integer $n \geq 0$, an element $(\alpha_0, \dots, \alpha_n)$ in the n-dimensional standard simplex $\Delta^n \subset \mathbb{R}^{n+1}$ and $n+1$ distinct integers $l_0, \dots, l_n$ such that $x = \alpha_0 T^{l_0}(x_0) + \cdots + \alpha_n T^{l_n}(x_0)$. We have for every integer $0 \leq i \leq n$ that the sequence $T^{l_i}(x_0) \in \ell^1(\mathbb{Z})$ is equal to one if $k = l_i$ and equal to zero if $k \neq l_i$. Therefore, we compute

$$\left(\sum_{k \in \mathbb{Z}} |x_k| \right)^2 = \left(\sum_{i=0}^n |\alpha_i| \right)^2 = 1$$

and

$$\sum_{k \in \mathbb{Z}} |x_k|^2 = \sum_{i=0}^n \alpha_i^2.$$

As a result, we obtain that

$$1 \leq \|x\|_\star = \sqrt{1 + \sum_{i=0}^n \alpha_i^2} \leq \sqrt{2},$$

since we have $\alpha_i^2 \leq \alpha_i$ for all integers $0 \leq i \leq n$. □

Let $\overline{\text{conv}}(A)$ denote the closure of conv(A). By the use of Lemma 2.23 we obtain that $1 \leq \|x\|_\star \leq \sqrt{2}$ for all points x in $\overline{\text{conv}}(A)$. Thus, we have in particular that the zero sequence is not an element of $\overline{\text{conv}}(A)$. A straightforward calculation shows that $T(\overline{\text{conv}}(A)) = \overline{\text{conv}}(A)$; hence, the map T is an isometry of the bounded metric space $(\overline{\text{conv}}(A), \|\cdot\|_\star)$ without fixed points. We claim that $(\overline{\text{conv}}(A), \|\cdot\|_\star)$ is a complete Busemann space. It is well-known that every convex subset of a strictly convex normed vector space is a Busemann space, cf. Proposition 8.1.6 and Proposition 8.1.5 in [Pap14]. Hence, it follows that $(\overline{\text{conv}}(A), \|\cdot\|_\star)$ is a Busemann space, as $\overline{\text{conv}}(A)$ is a convex subset of $\ell^1(\mathbb{Z})$. Note that $(\overline{\text{conv}}(A), \|\cdot\|_\star)$ is complete. Thus, we have constructed a complete bounded Busemann space that admits an isometry without fixed points.

2.8 Existence of invariant measures

The primary result of this section is Theorem 2.28. Some of the results below are needed in the proofs of Theorem 1.2 and Theorem 1.3.

2.8.1 – Let Σ denote a countable semigroup. A sequence $(\Sigma_k)_{k\geq 1}$ of non-empty finite subsets of Σ is a *Følner sequence* if

$$\lim_{k\to+\infty} \frac{|s\Sigma_k \,\Delta\, \Sigma_k|}{|\Sigma_k|} = 0$$

for all s in Σ. Here the symbol Δ denotes the symmetric difference of two sets. Recall that the sequence $(\{0,\ldots,k-1\})_{k\geq 1}$ is a Følner sequence of the semigroup of the non-negative integers.

Definition 2.24 (generalised limit). *Let Σ denote a countable semigroup. A generalised limit is a positive linear functional $\Theta\colon \ell^\infty(\Sigma) \to \mathbb{R}$ such that $\Theta((1)_{s\in\Sigma}) = 1$ and $\Theta((x_s)_{s\in\Sigma}) = \Theta((x_{s_0 s})_{s\in\Sigma})$ for all s_0 in Σ and x in $\ell^\infty(\Sigma)$.*

For convenience, we use the notation $\mathbf{1} := (1)_{s\in\Sigma}$ and $s_0.x := (x_{s_0 s})_{s\in\Sigma}$ for all s_0 in Σ and for all x in $\ell^\infty(\Sigma)$. The subsequent lemma is an extension of Theorem 1 in [Suc64].

Lemma 2.25. *Let Σ be a countable semigroup and let $(\Sigma_k)_{k\geq 1}$ denote a Følner sequence of Σ. Suppose that $\Theta\colon \ell^\infty(\Sigma) \to \mathbb{R}$ is a linear functional. Then the following statements are equivalent:*

1. *The linear functional $\Theta\colon \ell^\infty(\Sigma) \to \mathbb{R}$ is positive and a generalised limit.*
2. *For all points x in $\ell^\infty(\Sigma)$ we have*

$$\Theta(x) \leq \liminf_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs} \right).$$

3. *For all points x in $\ell^\infty(\Sigma)$ we have*

$$\Theta(x) \leq \limsup_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs} \right).$$

Proof. We show that (1.) $\Longrightarrow$ (2.) $\Longrightarrow$ (3.) $\Longrightarrow$ (1.).

(1.) $\Longrightarrow$ (2.). Let x in $\ell^\infty(\Sigma)$ be a point and let $k \geq 1$ be an integer. For every h in Σ_k we have that $\Theta(h.x) = \Theta(x)$; hence, it follows that

$$\Theta(x) = \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} \Theta(h.x) \leq \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x_{hs} \right) \Theta(\mathbf{1})$$

Since $\Theta(\mathbf{1}) = 1$, we have shown the desired inequality.

(2.) $\Longrightarrow$ (3.). This is trivial.

(3.) $\Longrightarrow$ (1.). To begin, we show that Θ is positive. Suppose that x in $\ell^\infty(\Sigma)$ is a point with $x \geq 0$. We have

$$\Theta(-x) \leq \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} -x_{hs} \right) \leq 0;$$

hence, it follows that $\Theta(x) \geq 0$. Next, we show that $\Theta(\mathbf{1}) = 1$. Since $\Theta(\mathbf{1}) \leq 1$ and $\Theta(-\mathbf{1}) \leq -1$, we obtain that $\Theta(\mathbf{1}) = 1$, as desired. To conclude, we show that Θ is left Σ-invariant. Let x in $\ell^\infty(\Sigma)$ be a point and let s_0 be an element of Σ. We define the point $y := x - s_0.x$. Note that y is contained in $\ell^\infty(\Sigma)$. We claim that $\Theta(y) = 0$. We have that

$$\begin{aligned} |\Theta(y)| &\leq \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \left| \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} (x_{hs} - x_{s_0 hs}) \right| \right) \\ &\leq \|x\|_\infty \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{|(s_0 \Sigma_k s) \mathbin{\Delta} \Sigma_k s|}{|\Sigma_k|} \right). \end{aligned} \tag{2.24}$$

Let s be an element of Σ. Observe that since $(s_0\Sigma_k \cup \Sigma_k)s = s_0\Sigma_k s \cup \Sigma_k s$ and $(s_0\Sigma_k \cap \Sigma_k)s \subset s_0\Sigma_k s \cap \Sigma_k s$, it follows that $s_0\Sigma_k s \mathbin{\Delta} \Sigma_k s \subset (s_0\Sigma_k \mathbin{\Delta} \Sigma_k)s$. As $|(s_0\Sigma_k \mathbin{\Delta} \Sigma_k)s| \leq |s_0\Sigma_k \mathbin{\Delta} \Sigma_k|$, we obtain $|(s_0\Sigma_k s) \mathbin{\Delta} \Sigma_k s| \leq |s_0\Sigma_k \mathbin{\Delta} \Sigma_k|$. Now, inequality (2.24) implies that $\Theta(y) = 0$, since $(\Sigma_k)_{k \geq 1}$ is a Følner sequence. Thus, we have shown that $\Theta(s_0.x) = \Theta(x)$, as desired. $\square$

We proceed with two immediate corollaries of Lemma 2.25.

Corollary 2.26. *Let Σ denote a countable semigroup. If Σ admits a Følner sequence $\boldsymbol{\Sigma} := (\Sigma_k)_{k \geq 1}$, then for every point $x^\star$ in $\ell^\infty(\Sigma)$ there is a generalised limit $\Theta_\star^{\boldsymbol{\Sigma}} \colon \ell^\infty(\Sigma) \to \mathbb{R}$ such that*

$$\Theta_\star^{\boldsymbol{\Sigma}}(x^\star) = \limsup_{k \to +\infty} \left(\sup_{s \in \Sigma} \frac{1}{|\Sigma_k|} \sum_{h \in \Sigma_k} x^\star_{hs} \right).$$

Proof. Fix a vector $x^\star$ in $\ell^\infty(\Sigma)$. Let $U \subset \ell^\infty(\Sigma)$ denote the linear span of the vector $x^\star$ and let $f\colon U \to \mathbb{R}$ denote the unique linear functional such that

$$f(x^\star) = \limsup_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x^\star_{hs} \right).$$

The Hahn-Banach Theorem tells us that there is a linear map $\Theta^{\boldsymbol{\Sigma}}_\star\colon \ell^\infty(\Sigma) \to \mathbb{R}$ such that $\Theta^{\boldsymbol{\Sigma}}_\star|_U = f$ and such that

$$\Theta^{\boldsymbol{\Sigma}}_\star(x) \leq \limsup_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs} \right)$$

for all points $x \in \ell^\infty(\Sigma)$. Due to Lemma 2.25 the map $\Theta^{\boldsymbol{\Sigma}}_\star$ is a generalised limit, hence the corollary follows. □

Corollary 2.27. *Let Σ denote a countable semigroup. If $\boldsymbol{\Sigma} := (\Sigma_k)_{k\geq1}$ is a Følner sequence of Σ, then we have that the map $\mathcal{L}_{\boldsymbol{\Sigma}}\colon \ell^\infty(\Sigma) \to \mathbb{R}$ given by the assignment*

$$x \mapsto \lim_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} x_{hs} \right)$$

is well-defined. Moreover, the equality $\mathcal{L}_{\boldsymbol{\Sigma}} = \mathcal{L}_{\boldsymbol{\Sigma}'}$ holds for all Følner sequences $\boldsymbol{\Sigma} := (\Sigma_k)_{k\geq1}$ and $\boldsymbol{\Sigma}' := (\Sigma'_k)_{k\geq1}$.

Proof. This is a direct consequence of Lemma 2.25 and Corollary 2.26.

□

2.8.2 – Let (X, d_X) be a complete separable metric space and let $T\colon X \to X$ be a homeomorphism of X. In [OU39], J. Oxtoby and S. Ulam showed that if there is a point x_0 in X and a compact subset $K_0 \subset X$ such that

$$\limsup_{k\to+\infty} \left(\frac{1}{k} \sum_{i=1}^{k} \mathbb{1}_{K_0}(T^i(x_0)) \right) > 0,$$

then there exists a T-invariant Radon probability measure $\mu\colon \mathcal{B}_X \to [0,1]$ such that $\mu(K_0) > 0$. In Oxtoby and Ulam's proof, the measure μ is obtained by the use of Carathéodory's extension theorem from a T-invariant metric outer measure, which is constructed by the means of generalised limits. In [Ada89], Adamski used the well-known construction of Radon measures via inner approximation due to Kisyński and Topsøe to generalise the result of Oxtoby and Ulam to Hausdorff topological spaces. In the following we use Adamski's approach to prove a further generalisation of Oxtoby and Ulam's result.

Theorem 2.28. *Let $(X, \mathcal{T}_X)$ be a Hausdorff topological space and let Σ be a countable subsemigroup of the semigroup of continuous self-maps of $(X, \mathcal{T}_X)$. Suppose that Σ admits a Følner sequence $(\Sigma_k)_{k\geq 1}$. If there is a point x_0 in X and a compact subset $K_0 \subset X$ such that*

$$\limsup_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k}\mathbb{1}_{K_0}(h\circ s(x_0))\right) > 0,$$

then there exists a Σ-invariant Radon probability measure $\mu\colon \mathcal{B}_X \to [0,1]$ such that $\mu(K_0) > 0$.

Proof. We define the sequence $x_0 := \left(\mathbb{1}_{K_0}(s(x_0))\right)_{s\in\Sigma}$. By the virtue of Corollary 2.26 there exists a generalised limit $\Theta\colon \ell^\infty(\Sigma) \to \mathbb{R}$ such that

$$\Theta(x_0) = \limsup_{k\to+\infty}\left(\sup_{s\in\Sigma}\frac{1}{|\Sigma_k|}\sum_{h\in\Sigma_k}\mathbb{1}_{K_0}(h\circ s(x_0))\right).$$

The set function $\beta\colon \mathcal{T}_X \to [0,1]$ given by the assignment

$$U \mapsto \Theta\left((\mathbb{1}_U(s(x_0)))_{s\in\Sigma}\right)$$

satisfies $\beta(\varnothing) = 0$ and $\beta(U\cap V) + \beta(U\cap V) = \beta(U) + \beta(V)$ for all U, V in $\mathcal{T}_X$. Moreover, we have for all U, V in $\mathcal{T}_X$ that $\beta(U) \leq \beta(V)$, whenever $U \subset V$. Thus, Theorem 2 in [Top70] asserts that the map $\mu\colon \mathcal{B}_X \to [0,1]$ given by the assignment

$$B \mapsto \sup_{\substack{K\subset B,\\ K\in\mathcal{K}_X}}\ \inf_{\substack{K\subset U,\\ U\in\mathcal{T}_X}} \beta(U)$$

is a Radon measure. Note that $\mu(U) \leq \beta(U)$ for all U in $\mathcal{T}_X$. Let s be an element of Σ. We claim that $s_*\mu = \mu$. Note that $\beta(s^{-1}(U)) = \beta(U)$ for all U in $\mathcal{T}_X$. Let K be a compact subset of $(X, \mathcal{T}_X)$. We compute

$$\mu(s^{-1}(K)) \leq \inf_{\substack{K\subset U,\\ U\in\mathcal{T}_X}} \mu(s^{-1}(U)) \leq \inf_{\substack{K\subset U,\\ U\in\mathcal{T}_X}} \beta(s^{-1}(U)) = \inf_{\substack{K\subset U,\\ U\in\mathcal{T}_X}} \beta(U) = \mu(K).$$

As a result, we obtain that $s_*\mu \leq \mu$, as μ is a Radon measure. We have

$$\mu(X) = s_*\mu(X) = s_*\mu(B) + s_*\mu(X\setminus B) \leq \mu(B) + \mu(X\setminus B) = \mu(X)$$

for all B in $\mathcal{B}_X$. Hence, it follows that $s_*\mu = \mu$, as claimed. By construction, we have $\mu(K_0) > 0$. Thus, by rescaling μ if necessary we obtain a Σ-invariant Radon probability measure on $(X, \mathcal{T}_X)$ such that $\mu(K_0) > 0$, as desired. $\square$

2.9 Proofs of Theorem 1.2 and Theorem 1.3

We start with a simple lemma that will be used several times in this section.

Lemma 2.29. *Let (X, d_X) be a complete metric space and let Σ be a subsemigroup of the isometry group of X. If σ is a Σ-equivariant conical geodesic bicombing on X, then there exists a Σ-equivariant reversible conical geodesic bicombing τ on X such that $\overline{conv_\tau}(A) \subset \overline{conv_\sigma}(A)$ for all subsets $A \subset X$.*

Proof. Let τ be the reversible conical geodesic bicombing obtained from σ by the construction of the proof of Proposition 1.1. in [Mie17a, p. 87]. Now, it follows readily from the definition of τ that it has the desired properties. □

We proceed with the proof of Theorem 1.2.

Proof of Theorem 1.2. Throughout the following proof we employ the notation from Section 2.1. Fix a measure μ_0 in $P(K)$. Let $M^0(K)$ denote the vector space of all signed finite Radon measures $\mu\colon \mathcal{B}_K \to \mathbb{R}$ such that $\mu(K) = 0$. The map $\|\cdot\|_0\colon M^0(K) \to \mathbb{R}$ given by the assignment

$$\mu \mapsto \sup\left\{ \int_K f\, d\mu \ :\ f\colon X \to \mathbb{R} \text{ is 1-Lipschitz} \right\}$$

defines a norm on $M^0(K)$, cf. [Edw11, Theorem 4.4]. Due to Theorem 4.1 in [Edw11] we have that $W_1(\mu, \nu) = \|\mu - \nu\|_0$ for all measures μ and ν in $P(K)$; hence, the map $\varphi\colon P(K) \to M^0(K)$ given by $\mu \mapsto \mu - \mu_0$ is an isometric embedding. It is well-known that the metric space $(P(K), W_1)$ is compact, cf. [Vil09, Remark 6.19]. As a result, the set $\varphi(P(K))$ is a non-empty compact convex subset of $M^0(K)$. Note that the restriction map $s|_K\colon K \to K$ is an isometry of K. For each s in Σ we define the map $\overline{s}\colon \varphi(P(K)) \to \varphi(P(K))$ through the assignment $\mu - \mu_0 \mapsto (s|_K)_*\mu - \mu_0$. Observe that $\overline{s}$ is an affine isometry of $\varphi(K)$. Ryll-Nardzewski's fixed-point theorem, cf. [RN67], asserts that there is a point $\mu_\star - \mu_0$ in $\varphi(P(K))$ such that $\overline{s}(\mu_\star - \mu_0) = \mu_\star - \mu_0$ for all s in Σ. Hence, the probability measure $\mu_\star\colon \mathcal{B}_K \to [0,1]$ is $s|_K$-invariant for all s in Σ. Let $i\colon K \hookrightarrow X$ denote the inclusion map. It is readily verified that the probability measure $i_*\mu_\star\colon \mathcal{B}_X \to [0,1]$ is contained in $P_1(X)$. Furthermore, the measure $i_*\mu_\star$ is Σ-invariant.

Lemma 2.29 tells us that there is a Σ-equivariant reversible conical geodesic bicombing τ on X such that $\overline{\mathrm{conv}_\tau}(K) \subset \overline{\mathrm{conv}_\sigma}(K)$. Let $\beta_\tau\colon P_1(X) \to X$ denote the contracting barycenter map associated to τ. We define the point $x_\star := \beta_\tau(i_*\mu_\star)$. Clearly, as $\mathrm{spt}(i_*\mu_\star)$ is a subset of K, Theorem 2.10 tells us that the point $x_\star$ is contained in $\overline{\mathrm{conv}_\tau}(K)$. Hence,

$x_\star \in \overline{\mathrm{conv}_\sigma}(K)$. Furthermore, we compute $s(x_\star) = \beta_\tau(s_* i_* \mu_\star) = \beta_\tau(i_* \mu_\star) = x_\star$ for all s in Σ, since τ is Σ-equivariant and $i_* \mu_\star$ is Σ-invariant. The theorem follows. □

In order to derive Theorem 1.3 we establish two results, Theorem 2.30 and Lemma 2.31, whose combination will directly imply Theorem 1.3.

Theorem 2.30. *Let (X, d_X) denote a complete metric space and let $\sigma\colon X \times X \times [0,1] \to X$ be a conical geodesic bicombing that has the midpoint property. Suppose that Σ is a countable subsemigroup of the semigroup of 1-Lipschitz self-maps of (X, d_X) and that σ is Σ-equivariant. Suppose that Σ admits a Følner sequence $(\Sigma_k)_{k\geq 1}$. If there is a point x_0 in X and a compact subset $K_0 \subset X$ such that the set $A := \{s(x_0) : s \in \Sigma\}$ is bounded and the inequality*

$$\limsup_{k\to+\infty} \left(\sup_{s\in\Sigma} \frac{1}{|\Sigma_k|} \sum_{h\in\Sigma_k} \mathbb{1}_{K_0}(h \circ s(x_0)) \right) > 0 \tag{2.25}$$

holds, then there is a point $x_\star$ in $\overline{\mathrm{conv}_\sigma}(A)$ such that $s(x_\star) = x_\star$ for all s in Σ.

Proof. The intersection $\overline{A} \cap K_0$ is a compact subset of X. Theorem 2.28 tells us that there is a Σ-invariant Radon probability measure $\mu\colon \mathcal{B}_X \to [0,1]$ such that $\mu(\overline{A} \cap K_0) > 0$. It is readily verified that the Borel measure

$$\begin{aligned} \mu_\star\colon \mathcal{B}_X &\to [0,1] \\ B &\mapsto \frac{1}{\mu(\overline{A})} \mu(\overline{A} \cap B) \end{aligned}$$

is a Radon probability measure. Note that $\overline{A} \subset s^{-1}(\overline{A})$ for all $s \in \Sigma$. Since μ is Σ-invariant, it follows that $\mu(s^{-1}(\overline{A}) \cap \overline{A}^c) = 0$ for all s in Σ. Now, it is straightforward to show that $\mu_\star$ is Σ-invariant. By construction, the support $\mathrm{spt}(\mu_\star)$ is a subset of $\overline{A}$. Since the subset $\overline{A}$ is bounded, we obtain that the measure $\mu_\star$ has a finite first moment and is thus contained in $P_1(X)$. Let $\beta_\sigma\colon P_1(X) \to X$ denote the contracting barycenter map associated to σ. We define the point $x_\star := \beta_\sigma(\mu_\star)$. Clearly, as $\mathrm{spt}(\mu_\star)$ is a subset of $\overline{A}$, Theorem 2.10 tells us that the point $x_\star$ is contained in $\overline{\mathrm{conv}_\sigma}(\overline{A}) = \overline{\mathrm{conv}_\sigma}(A)$. Furthermore, we compute $s(x_\star) = \beta_\sigma(s_*\mu_\star) = \beta_\sigma(\mu_\star) = x_\star$ for all s in Σ, since σ is Σ-equivariant and $\mu_\star$ is Σ-invariant. The theorem follows. □

Note that Corollary 2.27 asserts that the limit (2.25) does not depend on the Følner sequence $(\Sigma_k)_{k\geq 1}$.

Lemma 2.31. *Let (X, d_X) be a metric space and let $\varphi\colon X \to X$ be an isometry of X. If there is a point x_0 in X and a bounded subset $B \subset X$ such that*

$$\limsup_{k\to+\infty}\left(\sup_{l\geq 0}\frac{1}{k}\sum_{i=0}^{k-1}\mathbb{1}_B(\varphi^{i+l}(x_0))\right) > 0,$$

then φ has bounded orbits.

Proof. We define the set $A := \{\varphi^k(x_0) : k \geq 0\}$. Note that it suffices to show that $\operatorname{diam}(A) < +\infty$. We define the set $D := \{k \geq 0 : \mathbb{1}_{K_0}(\varphi^k(x_0)) = 1\}$. Theorem 3.19 (a) in [Fur14] asserts that there is an integer $k_0 \geq 1$ such that for every integer $k \geq 0$ at least one of the integers $k, k+1, \dots, k+k_0$ is contained in the set $D - D := \{d - d' : d, d' \in D, d \geq d'\}$. We define the real number $C := \max\{d(x_0, \varphi^k(x_0)) : 0 \leq k \leq k_0\}$. We claim that $\operatorname{diam}(A) \leq \operatorname{diam}(B) + C$. Let $k \geq 0$ be an integer. By the above there is an integer $0 \leq l \leq k_0$ such that the integer $k + l$ is contained in $D - D$. We compute

$$d(x_0, \varphi^k(x_0)) \leq d(x_0, \varphi^{k+l}(x_0)) + d(\varphi^{k+l}(x_0), \varphi^k(x_0)) \leq \operatorname{diam}(B) + C.$$

This concludes the proof, since $\operatorname{diam}(A) \leq \sup\{d(x_0, \varphi^k(x_0)) : k \geq 0\}$. □

Proof of Theorem 1.3. Since the sequence $(\{0, \dots, k-1\})_{k\geq 1}$ is a Følner sequence of the semigroup of the non-negative integers, so Theorem 1.3 is a direct consequence of Lemma 2.29, Theorem 2.30 and Lemma 2.31. □

3 Lipschitz extensions for barycentric target spaces

3.1 Lower bounds for one point extensions of Banach space valued maps

The collection of examples that we construct in this section is inspired by [Grü60]. We define the sequence $\{W_k\}_{k\geq 0}$ of matrices via the recursive rule

$$\begin{aligned} W_0 &:= 1, \\ W_{k+1} &:= \begin{pmatrix} W_k & W_k \\ W_k & -W_k \end{pmatrix}. \end{aligned}$$

The matrices W_k are commonly known as *Walsh matrices.* For each integer $k \geq 1$ let W_k' denote the $(2^k - 1) \times 2^k$ matrix that is obtained from W_k by deleting the first row of W_k. Further, for each integer $k \geq 1$ and each integer $\ell \in \{1, \ldots, 2^k\}$ we set

$$v_\ell^{(k)} := \ell\text{-th column of the matrix } W_k'. \tag{3.1}$$

By construction, $v_\ell^{(k)} \in \mathbb{R}^{2^k-1}$ for all $k \geq 1$ and $\ell \in \{1, \ldots, 2^k\}$. Clearly, $v_\ell^{(k)} \in \ell_p$ for all $p \in [1, +\infty]$ via the canonical embedding. The goal of this section is to prove the following proposition.

Proposition 3.1. *Let $p \in [1, +\infty]$ be an element of the extended real numbers and let $k \geq 1$ be an integer. If $F\colon \left(\{v_1^{(k)}, \ldots, v_{2^k}^{(k)}\} \cup \{0\}, \|\cdot\|_p\right) \to (\ell_1, \|\cdot\|_1)$ is a Lipschitz extension of the function*

$$\begin{aligned} f\colon & \left(\{v_1^{(k)}, \ldots, v_{2^k}^{(k)}\}, \|\cdot\|_p\right) \to (\ell_1, \|\cdot\|_1) \\ & v_\ell^{(k)} \mapsto v_\ell^{(k)}, \end{aligned}$$

then it holds that

$$\operatorname{Lip}(F) \geq \left(2 - \frac{1}{2^{k-1}}\right)^{\frac{1}{p_\star}} \operatorname{Lip}(f),$$

where $1/p_\star := 1 - 1/p$ if $p \neq +\infty$ and $1/p_\star := 1$ otherwise.

Note that Proposition 3.1 implies in particular that $e(\ell_2, \ell_1) \geq \sqrt{2}$. The key component in the proof of Proposition 3.1 is the following geometric lemma.

Lemma 3.2. *Let $k \geq 1$ be an integer and suppose that $w \in \mathbb{R}^{2^k-1}$ is a vector such that*

$$\|v_\ell^{(k)} - w\|_1 \leq \|v_\ell^{(k)}\|_1 \textit{ for all } \ell \in \{1, \ldots, 2^k\}, \tag{3.2}$$

then it holds that $w = 0$.

Proof. By the use of a simple induction it is straightforward to show that

$$\sum_{\ell=1}^{2^k} v_\ell^{(k)} = 0. \tag{3.3}$$

Moreover, since $v_\ell^{(k)}$ is a ± 1 vector, inequality (3.2) implies that

$$\langle w, v_\ell^{(k)} \rangle_{\mathbb{R}^{2^k-1}} \leq 0.$$

Equality (3.3) implies that none of these inequalities can be strict; thus, as (for instance) the vectors $v_2^{(k)}, \ldots, v_{2^k}^{(k)}$ form a basis of $\mathbb{R}^{2^k-1}$, we obtain $w = 0$, as desired. $\square$

Having Lemma 3.2 at our disposal, Proposition 3.1 can readily be verified.

Proof of Proposition 3.1. To begin, we compute $\mathrm{Lip}(f)$. We claim that

$$\mathrm{Lip}(f) = \left(2^{k-1}\right)^{\frac{1}{p_\star}}. \tag{3.4}$$

First, suppose that $p \in [1, +\infty)$. A simple induction implies that two distinct columns of W_k are orthogonal to each other. Since the entries of W_k consist only of plus and minus one, we obtain that

$$\|v_i^{(k)} - v_j^{(k)}\|_p^p = 2^p \operatorname{card}\left(\left\{\ell \in \{1, \ldots, 2^k - 1\} : (v_i^{(k)})_\ell \neq (v_j^{(k)})_\ell\right\}\right) = 2^p 2^{k-1},$$

where we use $\operatorname{card}(\cdot)$ to denote the cardinality of a set. Hence, if $p \in [1, +\infty)$, then the identity (3.4) follows. Since the p-norms $\|\cdot\|_p$ converge pointwise to the maximum norm $\|\cdot\|_\infty$ if $p \to +\infty$, the identity (3.4) follows also in the case $p = +\infty$, as was left to show.

By considering the contraposition of the statement in Lemma 3.2, we may deduce that there is an index $\ell \in \{1, \ldots, 2^k\}$ such that

$$\|v_\ell^{(k)} - F(0)\|_1 \geq \|v_\ell^{(k)}\|_1.$$

As a result, we obtain that

$$\operatorname{Lip}(F) \geq \frac{\|v_\ell^{(k)} - F(0)\|_1}{\|v_\ell^{(k)}\|_p} \geq \frac{\|v_\ell^{(k)}\|_1}{\|v_\ell^{(k)}\|_p} = (2^k - 1)^{\frac{1}{p_\star}}.$$

Hence, it follows that

$$\frac{\operatorname{Lip}(F)}{\operatorname{Lip}(f)} \geq \frac{(2^k-1)^{\frac{1}{p_\star}}}{(2^{k-1})^{\frac{1}{p_\star}}} = \left(2 - \frac{1}{2^{k-1}}\right)^{\frac{1}{p_\star}};$$

as desired. □

3.2 Embeddings and indices of $F-$transforms

In this section we collect some applications of our main theorems. Let (X, ρ_X) and (Y, ρ_Y) be quasi-metric spaces and let $f\colon X \to Y$ be an injective map. We set $\mathsf{dist}(f) := \operatorname{Lip}(f)\operatorname{Lip}(f^{-1})$ and

$$\mathsf{c}_Y(X) := \inf\left\{\mathsf{dist}(f) : f\colon X \to Y \text{ injective }\right\}.$$

The sharpness of 1.9 if $m = 1$ allows us to derive a necessary condition for an F-transform of an ℓ_p-space to embed into a Hilbert space.

Corollary 3.3. *Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space and suppose that $F\colon [0,+\infty) \to [0,+\infty)$ is a function such that $F(0) = 0$ and*

$$\sup_{x>0} \frac{F(x)}{x} < +\infty.$$

If $p \in [1,+\infty]$ is an extended real number and

$$\sup\left\{\mathsf{c}_H(A) : A \subset F[\ell_p],\ A \text{ finite }\right\} \leq 2^\epsilon, \qquad \text{where } \epsilon \in \Big[0, \frac{1}{2}\Big),$$

then $p \leq \left(\frac{1}{2} - \epsilon\right)^{-1}$.

Proof. We retain the notation from Section 3.1. Let $k \geq 1$ be an integer and let

$$g_F\colon \left(\{v_1^{(k)}, \ldots, v_{2^k}^{(k)}\}, F \circ \|\cdot\|_p\right) \to (\ell_1, \|\cdot\|_1)$$

denote the map such that $v_i^{(k)} \mapsto v_i^{(k)}$. The vectors $v_i^{(k)}$ are given as in (3.1) and interpreted as elements of ℓ_p via the canonical embedding. It is readily verified that

$$\operatorname{Lip}(g_F) = \frac{A}{F(A)}\operatorname{Lip}(g_{\mathsf{id}}),$$

where $A := \|v_i^{(k)} - v_j^{(k)}\|_p$. Now, let $\delta > 0$ be a real number. Using the assumptions in Corollary 3.3 and Theorem 1.9 (for the map $F(t) = t$) it follows that there is a map $G_F\colon \left(\{v_1^{(k)}, \dots, v_{2^k}^{(k)}\} \cup \{0\}, F \circ \|\cdot\|_p\right) \to (\ell_1, \|\cdot\|_1)$ that extends g_F such that

$$\mathrm{Lip}\,(G_F) \leq (1+\delta)\, 2^{\epsilon} \sqrt{2}\; \mathrm{Lip}\,(g_F)\,.$$

We define the map $T\colon \left(\{v_1^{(k)}, \dots, v_{2^k}^{(k)}\} \cup \{0\}, \|\cdot\|_p\right) \to (\ell_1, \|\cdot\|_1)$ via $x \mapsto G_F(x)$. We calculate

$$\mathrm{Lip}(T) \leq (1+\delta)\, 2^{\epsilon} \sqrt{2}\, \max\left\{\frac{F(A)}{A}, \frac{F(B)}{B}\right\} \mathrm{Lip}\,(g_F)\,,$$

where $B := \|v_i^{(k)} - 0\|_p$. Since the map T is a Lipschitz extension of g_{id}, Proposition 3.1 tells us that

$$\mathrm{Lip}(T) \geq \left(2 - \frac{1}{2^{k-1}}\right)^{\frac{1}{q}} \mathrm{Lip}\,(g_{\mathsf{id}}) = \frac{A}{B}\left(1 - \frac{1}{2^k}\right) \mathrm{Lip}\,(g_{\mathsf{id}})\,,$$

where $1/q := 1 - 1/p$ if $p \neq +\infty$ and $1/q := 1$ otherwise. We set $\gamma := \frac{A}{B}$. Thus, by putting everything together and via a simple scaling argument, we obtain for all $x > 0$

$$\gamma\left(1 - \frac{1}{2^k}\right)\frac{F(\gamma x)}{\gamma x} \leq (1+\delta) 2^{\epsilon}\sqrt{2}\, \max\left\{\frac{F(x)}{x}, \frac{F(\gamma x)}{\gamma x}\right\}.$$

Thus, since

$$\sup_{x>0} \frac{F(x)}{x} < +\infty$$

we obtain

$$\frac{\sqrt[q]{2}}{\sqrt[p]{1 - \frac{1}{2^k}}}\left(1 - \frac{1}{2^k}\right) = \gamma\left(1 - \frac{1}{2^k}\right) \leq (1+\delta) 2^{\epsilon}\sqrt{2}.$$

Consequently, as $k \geq 1$ and $\delta > 0$ are arbitrary, we deduce $p \leq \left(\frac{1}{2} - \epsilon\right)^{-1}$. This completes the proof. $\square$

If $2 < p < +\infty$ is a real number and the F-transform $F[\ell_p]$ embeds isometrically into a Hilbert space, then

$$F(x) = F_a(x) = \begin{cases} 0 & x = 0 \\ a & x > 0 \end{cases} \qquad \text{where } a \geq 0;$$

this follows essentially by combining a result of Kuelbs [Kue73, Corollary 3.1] with a classical result that relates isometric embeddings to positive definite functions, cf. for example [WW75, Theorem 4.5]. Furthermore, by a result of Johnson and Randrianarivony, ℓ_p with $p > 2$ does not admit a coarse embedding into ℓ_2, cf. [JR06; MN08].

Very recently, Eskenazis, Mendel, and Naor have shown that ℓ_p with $p > 2$ does not coarsely embed into any complete CAT(0) space, cf. [EMN19].

We proceed with an application of Theorem 1.7. Let $F\colon [0, +\infty) \to [0, +\infty)$ be a function with $F(0) = 0$. Suppose that F is subadditive and strictly increasing. We define

$$\mathsf{D}_F(\alpha) = \sup_{x>0} \frac{F(\alpha x)}{F(x)}$$

for all $\alpha \geq 0$. Clearly, the function $\mathsf{D}_F\colon [0, +\infty) \to [0, +\infty)$ is finite, submutliplicative and non-decreasing. Moreover,

$$F(\alpha x) \leq \mathsf{D}_F(\alpha) F(x)$$

for all real numbers $x, \alpha \geq 0$. The *upper index* of F is defined by

$$\beta(F) = \lim_{\alpha\to+\infty} \frac{\log(\mathsf{D}_F(\alpha))}{\log(\alpha)}. \tag{3.5}$$

The existence of the limit (3.5) may be deduced via the general theory of subadditive functions, since D_F is submultiplicative and non-decreasing, cf. [Mal85, Remark 1.3 (b)]. We have $0 \leq \beta(F) \leq 1$, for F is subadditive.

If (X, d_X) is a metric space, we set

$$\mathsf{c}_F(X) := \inf \left\{ \mathsf{c}_{F[Y]}(X) : (Y, d_Y) \text{ metric space } \right\}.$$

In [MN11, Theorem 1], Mendel and Naor obtained a dichotomy theorem for the quantity $\mathsf{c}_F(X)$, if F is concave and non-decreasing. The upper index of F allows us to obtain lower bounds for the rate of growth of $\mathsf{c}_F(P_n)$, where $P_n := \{0, 1, \dots, n\} \subset \mathbb{R}$.

Corollary 3.4. *Let $F\colon [0, +\infty) \to [0, +\infty)$ be a strictly-increasing subadditive function with $F(0) = 0$. If $0 \leq \alpha < 1 - \beta(F)$ is a real number, then there exists an integer $N \geq 1$ such that*

$$n^\alpha \leq \mathsf{c}_F(P_n)$$

for all $n \geq N$.

Proof. We may assume that $\beta(F) < 1$. Let (Y, ρ_Y) be a quasi-metric space and let (X, d_X) be a metric space. We may employ Theorem 1.7 to conclude that

$$\mathsf{e}^m\big(F[X], Y\big) \leq \sup_{x>0} \frac{F\big((m+1)x\big)}{F(x)}, \tag{3.6}$$

for all integers $m \geq 0$. We set $Y_m := \{0, m\} \subset P_m$. Since

$$\mathsf{e}^{m-1}\big(P_m, Y_m\big) = m,$$

inequality (3.6) asserts that

$$m \leq \sup_{x>0} \frac{F(mx)}{F(x)} \mathsf{c}_F(P_m) = \mathsf{D}_F(m)\mathsf{c}_F(P_m) \tag{3.7}$$

for all $m \geq 1$. Let $\epsilon > 0$ be a real number such that $\alpha < 1 - \beta(F) - \epsilon$. By the virtue of Theorem 1.2 in [Mal85] there exists a real number $C \geq 0$ such that

$$\mathsf{D}_F(\alpha) \leq \alpha^{\beta(F)+\epsilon}$$

for all $\alpha \geq C$. Consequently, by the use of (3.7) we obtain for all $n \geq N := \lceil C \rceil$ that

$$n^{\alpha} \leq n^{1-\beta(F)-\epsilon} \leq \mathsf{c}_F(P_n),$$

as desired. □

As a consequence of Corollary 3.4, we conclude that if $\beta(F) < 1$, then the second possibility of the dichotomy [MN11, Theorem 1] holds. Thus, there is the following natural question: If $\beta(F) = 1$, is it true that, then $\mathsf{c}_F(X) = 1$ for all finite metric spaces (X, d_X)?

3.3 Minimum value of a certain quadratic form in Hilbert space

Let $(H, \langle\cdot,\cdot\rangle_H)$ be a Hilbert space, let I denote a finite set and let $\mathbf{x}\colon I \to H$ be a map. Suppose that $\boldsymbol{\lambda}\colon I \times I \to \mathbb{R}$ is a symmetric, non-negative function. Further, assume that $G\colon [0,+\infty) \to [0,+\infty)$ is a convex, non-decreasing function with $G(0) = 0$. We define

$$\varPhi(\mathbf{x}, \boldsymbol{\lambda}, G) := \sum_{(k,\ell)\in I\times I} \boldsymbol{\lambda}\big(k,\ell\big)\, G\big(\|\mathbf{x}(k) - \mathbf{x}(\ell)\|_H^2\big)$$

and for each subset $J \subset I$ we set

$$\mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, G, J) := \inf\big\{\varPhi(\mathbf{z}, \boldsymbol{\lambda}, G) : \text{ where } \mathbf{z}\colon I \to H \text{ is a map such that } \mathbf{z}|_{J^c} = \mathbf{x}|_{J^c}\big\}.$$

The remainder of this section is devoted to calculate the quantity $\mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, \mathsf{id}, J)$.

Let $J \subset I$ be a proper subset. We may suppose that $J = \{1, \dots, m\}$, where $m := \operatorname{card}(J)$. To ease notation, we set $\lambda_{k\ell} := \boldsymbol{\lambda}(k, \ell)$ and we define the matrix

$$M(\boldsymbol{\lambda}, J) := \begin{bmatrix} \sum_{k \in J^c} \lambda_{1k} + \sum_{j=1}^{m} \lambda_{1j} & -\lambda_{12} & \dots & -\lambda_{1m} \\ -\lambda_{21} & \sum_{k \in J^c} \lambda_{2k} + \sum_{j=1}^{m} \lambda_{2j} & \dots & -\lambda_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{m1} & -\lambda_{m2} & \dots & \sum_{k \in J^c} \lambda_{mk} + \sum_{j=1}^{m} \lambda_{mj} \end{bmatrix}. \tag{3.8}$$

The matrices $M(\boldsymbol{\lambda}, J)$ appear naturally in the proof of Theorem 1.9.

If the symmetric matrix $M := M(\boldsymbol{\lambda}, J)$ is strictly diagonally dominant, that is, for each integer $1 \leq i \leq m$, it holds

$$|m_{ii}| > \sum_{j \neq i}^{m} |m_{ij}|,$$

it follows via Gershgorin's circle theorem that M is positive definite. As a result, the matrix $M(\boldsymbol{\lambda}, J)$ is non-singular if

$$\sum_{k \in J^c} \lambda_{ik} > 0 \quad \text{for all } 1 \leq i \leq m.$$

Next, we deduce the minimum value of $\mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, \mathsf{id}, J)$. The following proposition has been stated without a proof in [Bal92].

Proposition 3.5. *Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space, let I be a finite set and let $\mathbf{x} \colon I \to H$ be a map. Suppose that $\boldsymbol{\lambda} \colon I \times I \to \mathbb{R}$ is a symmetric, non-negative function and let $J \subset I$ be a proper subset. If the matrix $M := M(\boldsymbol{\lambda}, J)$ given by* (3.8) *is strictly diagonally dominant and $\lambda_{k\ell} = 0$ for all $k, \ell \in J^c$, then*

$$\mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, \mathsf{id}, J) = \sum_{i \in J} \sum_{j \in J} \sum_{k \in J^c} \sum_{\ell \in J^c} \lambda_{ik} c_{ij} \lambda_{j\ell} \|\mathbf{x}(k) - \mathbf{x}(\ell)\|_H^2 \tag{3.9}$$

where $C := M^{-1}$. Moreover,

$$\sum_{j=1}^{|J|} c_{ij} \sum_{k \in J^c} \lambda_{jk} = 1 \tag{3.10}$$

for all integers $1 \leq i \leq |J|$.

Proof. We set $m := |J|$. We may suppose that $J = \{1, \ldots, m\}$. Since $D^{-1}M\boldsymbol{j} = \boldsymbol{j}$, where $\boldsymbol{j} := (1, \ldots, 1) \in \mathbb{R}^m$ and $D := (d_{ij})_{1 \le i,j \le m}$ is a diagonal matrix with

$$d_{ii} := \sum_{k \in J^c} \lambda_{ik}, \qquad \text{for all } 1 \le i \le m,$$

we obtain $CD\boldsymbol{j} = \boldsymbol{j}$, that is,

$$\sum_{j=1}^{m} c_{ij} \sum_{k \in J^c} \lambda_{jk} = 1 \tag{3.11}$$

for all $1 \le i \le m$. Thus, (3.10) follows. Let the map $\Phi \colon H^m \to \mathbb{R}$ be given by the assignment

$$(z_1, \ldots, z_m) \mapsto \sum_{i=1}^{m} \sum_{k \in J^c} \lambda_{ik} \|z_i - \mathbf{x}(k)\|_H^2 + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \lambda_{ij} \|z_i - z_j\|_H^2.$$

Note that

$$2 \inf \Phi = \mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, G, J).$$

Thus, to conclude the proof we calculate the minimum value of the map Φ. Let $U \subset H$ denote the span of the vectors $\big(\mathbf{x}(k)\big)_{k \in J^c}$. Clearly, $\inf \Phi|_U = \inf \Phi$. In the following, we compute the minimal value of $\Phi|_U$.

The subset $U \subset H$ is linearly isometric to $(\mathbb{R}^d, \|\cdot\|_2)$ for some integer $1 \le d \le \operatorname{card}(J^c)$. Consequently, we may suppose (by abuse of notation) for all $k \in J^c$ that $\mathbf{x}(k) \in \mathbb{R}^d$, say $\mathbf{x}(k) = (x_{k1}, \ldots, x_{kd})$, and that the function $\Phi|_U \colon (\mathbb{R}^d)^m \to \mathbb{R}$ is given by the assignment

$$(p_1, \ldots, p_m) \mapsto \sum_{t=1}^{d} \left(\sum_{i=1}^{m} \sum_{j=1}^{m} p_{it} m_{ij} p_{jt} - 2 \sum_{i=1}^{m} p_{it} \sum_{k \in J^c} \lambda_{ik} x_{rk} + \sum_{i=1}^{m} \sum_{k \in J^c} \lambda_{ik} x_{kt}^2 \right),$$

where $p_i := (p_{i1}, \ldots, p_{id})$ for all integers $1 \le i \le m$. Using elementary analysis, one can deduce that the minimum value of $\Phi|_U$ is equal to

$$\sum_{t=1}^{d} \left(- \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{r=1}^{n} \sum_{s=1}^{n} \lambda_{js} c_{ij} \lambda_{ir} x_{st} x_{rt} + \sum_{i=1}^{m} \sum_{r=1}^{n} \lambda_{ir} x_{rt}^2 \right). \tag{3.12}$$

Thus, via (3.12) and (3.11) we conclude that the minimum value of Φ is equal to

$$\begin{aligned}
&\sum_{t=1}^{d} \left(\sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k \in J^c} \sum_{\ell \in J^c} \lambda_{j\ell} c_{ij} \lambda_{ik} \left(-x_{\ell t} x_{kt} + x_{kt}^2 \right) \right) \\
&= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k \in J^c} \sum_{\ell \in J^c} \lambda_{j\ell} c_{ij} \lambda_{ik} \left(\sum_{t=1}^{d} (x_{\ell t} - x_{kt})^2 \right) \\
&= \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \sum_{k \in J^c} \sum_{\ell \in J^c} \lambda_{j\ell} c_{ij} \lambda_{ik} \|\mathbf{x}(\ell) - \mathbf{x}(k)\|_H^2,
\end{aligned}$$

as claimed. This completes the proof. $\square$

3.4 An inequality involving the entries of an M-matrix and its inverse

A matrix $M \in \mathrm{Mat}(m \times m; \mathbb{R})$ with non-positive off-diagonal elements is said to be an *M-matrix* if M is non-singular and each entry of M^{-1} is non-negative, cf. [Mar72, Definition 1.1]. There are several equivalent definitions of an M-matrix, cf. [FP62]. M-matrices and their matrix inverses are generally well understood, cf. [PB74; Joh82] for a survey of the theory.

A primary example of M-matrices are matrices $M := M(\boldsymbol{\lambda}, J)$. Indeed, such matrices are strictly diagonally dominant (thus non-singular) and via Gauss elimination it is straightforward to show that each entry of the inverse of $M(\boldsymbol{\lambda}, J)$ is non-negative.

It is worth to point out that a matrix $M \in \mathrm{Mat}(m \times m; \mathbb{R})$ with non-positive off-diagonal elements is an M-matrix if and only if there are matrices $W, D \in \mathrm{Mat}(m \times m; \mathbb{R})$ such that W is a strictly diagonally dominant M-matrix, D is a diagonal matrix with positive diagonal elements and $M = WD$. This is a classical result of Fiedler and Pták, cf. [FP62, Theorem 4.3].

The following result will play a major role in the proof of Theorem 1.9.

Theorem 3.6. *Let $m \geq 2$ and let $M \in \mathrm{Mat}(m \times m; \mathbb{R})$ be a symmetric invertible matrix with non-positive off-diagonal elements. We set $C := M^{-1}$. If M is an M-matrix, then*

$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} |m_{ij}| \, |c_{ik} c_{j\ell} - c_{jk} c_{i\ell}| \leq (m-1) c_{k\ell} \tag{3.13}$$

for all integers $1 \leq k, \ell \leq m$ with $k \neq \ell$.

The estimate in Theorem 3.6 is sharp. This is the content of the following example.

Example 3.7. *Let $m \geq 2$ be an integer and let $M \in \mathrm{Mat}(m \times m; \mathbb{R})$ be the tridiagonal matrix given by*

$$m_{ij} := \begin{cases} 3 & \text{if } i = j \\ -1 & \text{if } i = j - 1 \\ -1 & \text{if } i = j + 1 \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, M is a symmetric M-matrix. As usual, we set $C := M^{-1}$. Since $\det(M) C = \mathrm{adj}(M)$, *where* $\mathrm{adj}(M)$ *is the adjugate matrix of M, it follows*

$$c_{1m} = \frac{1}{\det M}. \tag{3.14}$$

Furthermore, via Jacobi's equality [Jac41], see (3.19), *we get*

$$|c_{i1}c_{jm} - c_{j1}c_{im}| = \frac{\big|\det M\big[[m]\setminus\{1,m\},[m]\setminus\{i,i+1\}\big]\big|}{\det M} = \frac{1}{\det M} \tag{3.15}$$

for all pairs of integers (i,j) *with* $i = j-1$. *By virtue of* (3.14) *and* (3.15) *we obtain*

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}|m_{ij}\,(c_{i1}c_{jm} - c_{j1}c_{im})| = \frac{m-1}{\det M} = (m-1)c_{1m}.$$

Consequently, the estimate (3.13) *is best possible.*

This section is structured as follows. To begin, we gather some information that is needed to prove Theorem 3.6. At the end of the section, we establish Theorem 3.6.

We start with a lemma that calculates the sum in (3.13) if the absolute values from the 2×2-minors are removed.

Lemma 3.8. *Let* $m \geq 2$ *and let* $M \in \mathrm{Mat}(m\times m;\mathbb{R})$ *be an M-matrix. We set* $C := M^{-1}$. *If* $1 \leq k,\ell \leq m$ *are distinct integers, then*

$$\sum_{j=1}^{m}|m_{kj}|\,(c_{kk}c_{j\ell} - c_{jk}c_{k\ell}) = c_{k\ell}, \tag{3.16}$$

and for all integers $1 \leq i \leq m$ *with* $i \neq k,\ell$,

$$\sum_{j=1}^{m}|m_{ij}|\,(c_{ik}c_{j\ell} - c_{jk}c_{i\ell}) = 0. \tag{3.17}$$

Proof. Since C is the matrix inverse of M, we compute

$$\sum_{j=1}^{m} m_{ij}c_{ik}c_{j\ell} = \delta_{i\ell}c_{ik},$$
$$\sum_{j=1}^{m} m_{ij}c_{jk}c_{i\ell} = \delta_{ik}c_{i\ell}$$

for all $1 \leq i \leq m$. As a result, we obtain

$$\sum_{j=1}^{m} m_{ij}(c_{ik}c_{j\ell} - c_{jk}c_{i\ell}) = \delta_{i\ell}c_{ik} - \delta_{ik}c_{i\ell}.$$

Therefore, the desired equalities follow, since $m_{ij} \leq 0$ for all distinct integers $1 \leq i,j \leq m$. $\square$

We proceed with the following corollary.

Corollary 3.9 (zero pattern of inverse M-matrices). *Let $m \geq 2$ and let $M \in \mathrm{Mat}(m \times m; \mathbb{R})$ be an invertible matrix with non-positive off-diagonal elements. We set $C := M^{-1}$. If M is an M-matrix and $k, \ell \in \{1, \ldots, m\}$ are two distinct integers such that $c_{k\ell} = 0$, then*

1. *for all integers $i \in \{1, \ldots, m\}$, $m_{ki} = 0$ or $c_{i\ell} = 0$. In particular, $m_{k\ell} = 0$.*
2. *for all integers $i \in \{1, \ldots, m\}$, $m_{ki} = 0$ or $m_{i\ell} = 0$.*
3. *the matrix M has at least $m-1$ zero entries.*

Proof. Clearly, item 2 is a direct consequence of item 1 and item 3 is a direct consequence of item 2. To conclude the proof we establish item 1. Lemma 3.8 tells us that

$$\sum_{i=1}^{m} |m_{ki}| \left(c_{kk} c_{i\ell} - c_{ik} c_{k\ell}\right) = 0.$$

Thus, we obtain

$$|m_{ki}| \, c_{kk} c_{i\ell} = 0 \tag{3.18}$$

for all integers $1 \leq i \leq m$. Since each principal submatrix of C is the inverse matrix of an M-matrix, cf. [Joh82, Corollary 3], it follows $c_{kk} \neq 0$. Thus, via Equation (3.18) we obtain $m_{ki} = 0$ or $c_{i\ell} = 0$ for all $i \in \{1, \ldots, m\}$, as desired. □

Theorem 3.6 will be established via a density argument. As it turns out, it will be beneficial to approximate C by matrices with non-vanishing minors. To this end, we need the following genericity condition.

Definition 3.10 (generic matrix). *Let $m \geq 1$ be an integer and let $A \in \mathrm{Mat}(m \times m; \mathbb{R})$ be a matrix. Suppose that $1 \leq k \leq m$ is an integer and let $I, J \subset \{1, \ldots, m\}$ be two subsets such that $\mathrm{card}\,(I) = \mathrm{card}\,(J) = k$.*

We use the notation $A[I, J] \in \mathrm{Mat}(k \times k; \mathbb{R})$ to denote the matrix that is obtained from A by keeping the rows of A that belong to I and the columns of A that belong to J. We say that A is generic if

$$\det(A[I, J]) \neq 0$$

for all non-empty subsets $I, J \subset \{1, \ldots, m\}$ with $\mathrm{card}\,(I) = \mathrm{card}\,(J)$.

The subsequent lemma demonstrates that being generic is a 'generic property' as used in the context of algebraic geometry.

Lemma 3.11. *Let $m \geq 1$ be an integer and let $A \in \mathrm{Mat}(m \times m; \mathbb{R})$ be a matrix. The following holds*

1. *if A is generic, then A^{-1} is generic as well.*

2. *the set of generic matrices is open and dense in $\mathrm{Mat}(m \times m; \mathbb{R})$.*

Proof. The first item is a direct consequence of Jacobi's equality, cf. [Jac41],

$$\left|\det(A^{-1}[I, J]) \det(A)\right| = \left|\det\left(A\big[[m] \setminus J, [m] \setminus I\big]\right)\right|, \tag{3.19}$$

where $I, J \subset [m] := \{1, \ldots, m\}$ with $\mathrm{card}\,(I) = \mathrm{card}\,(J)$ and $A[\varnothing, \varnothing]$ is by definition equal to the identity matrix. Next, we establish the second item. A matrix $A \in \mathrm{Mat}(m \times m; \mathbb{R})$ is generic if and only if

$$p(A) := \prod_{I,J \subset [m], |I|=|J|} \det(A[I, J]) \neq 0.$$

Clearly, p is a non-zero polynomial in the entries of A. It is straightforward to show that the complement of the zero set of a non-zero polynomial $q \colon \mathbb{R}^N \to \mathbb{R}$ is an open and dense subset of $\mathbb{R}^N$, for all $N \geq 1$. Therefore, the set of generic matrices is an open and dense subset of $\mathrm{Mat}(m \times m; \mathbb{R})$, as was to be shown. $\square$

We proceed with the following lemma, which is the key component in the proof of Theorem 3.6.

Lemma 3.12. *Let $m \geq 2$ and let $A \in \mathrm{Mat}(m \times m; \mathbb{R})$ be a non-negative matrix. If A is a generic matrix, then for all distinct integers $1 \leq k, \ell \leq m$ the skew-symmetric matrix $A^{(k,\ell)} \in \mathrm{Mat}(m \times m; \mathbb{R})$ given by*

$$a_{ij}^{(k,\ell)} := a_{ik}a_{j\ell} - a_{jk}a_{i\ell},$$

has the property that each two rows of $A^{(k,\ell)}$ have a distinct number of positive entries.

Proof. We fix two distinct integers $1 \leq k, \ell \leq m$. If $m = 2$, then each two rows of $A^{(k,\ell)}$ have a distinct number of positive entries, since A is generic. Now, suppose that $m = 3$. The matrix $A^{(k,\ell)}$ is skew-symmetric; hence, as A is generic we obtain that $A^{(k,\ell)}$ can have 2^3 different sign patterns. If

$$a_{12}^{(k,\ell)}, a_{23}^{(k,\ell)}, a_{31}^{(k,\ell)} > 0 \quad \text{or} \quad a_{12}^{(k,\ell)}, a_{23}^{(k,\ell)}, a_{31}^{(k,\ell)} < 0, \tag{3.20}$$

then each row of $A^{(k,\ell)}$ has the same number of positive entries and the statement does not hold. For the other 6 sign patterns it is straightforward to check that each row of $A^{(k,\ell)}$ has a different number of positive entries.

In the following, we show that (3.20) cannot occur. For the sake of a contradiction, we suppose $a_{12}^{(k,\ell)}, a_{23}^{(k,\ell)}, a_{31}^{(k,\ell)} > 0$. Since $a_{12}^{(k,\ell)} > 0$, we obtain

$$a_{1k} > \frac{a_{2k}a_{1\ell}}{a_{2\ell}}. \tag{3.21}$$

Since $a_{31}^{(k,\ell)} > 0$, we estimate via (3.21)

$$a_{3k}a_{1\ell} > a_{1k}a_{3\ell} > \frac{a_{2k}a_{1\ell}}{a_{2\ell}}a_{3\ell}. \tag{3.22}$$

Thus, (3.22) tells us that

$$a_{3k}a_{2\ell} > a_{2k}a_{3\ell};$$

which contradicts $a_{23}^{(k,\ell)} > 0$. Hence, the case $a_{12}^{(k,\ell)}, a_{23}^{(k,\ell)}, a_{31}^{(k,\ell)} > 0$ cannot occur. The other invalid sign pattern can be treated analogously . Therefore, (3.20) cannot occur, as claimed. By putting everything together, we conclude that the statement is valid if $m = 3$.

We proceed by induction. Let $m \geq 4$ be an integer and suppose that the statement is valid for all $2 \leq m' < m$.

Before we proceed with the proof we introduce some notation. For every matrix $B \in \mathrm{Mat}(m \times m; \mathbb{R})$ we denote by $B_{ij} \in \mathrm{Mat}((m-1) \times (m-1); \mathbb{R})$ the matrix that is obtained from B by deleting the i-th row and the j-th column of B. Moreover, for all integers $1 \leq i, j \leq m$ with $i \neq j$ we set

$$\begin{aligned} n_i^+(B) &:= \text{number of positive entries of the i-th row of } B, \\ n_{i,j}^+(B) &:= \text{number of positive entries of } (b_{i1}, \ldots, \widehat{b_{ij}}, \ldots, b_{im}). \end{aligned}$$

We use $\widehat{b_{ij}}$ to indicate that the entry b_{ij} is omitted.

Since the non-negative $(m-1) \times (m-1)$-matrix A_{ij} is generic for all $1 \leq i, j \leq m$, we obtain via the induction hypothesis that each row of $(A^{(k,\ell)})_{ii}$ has a different number of positive entries for all $1 \leq i \leq m$.

For simplicity of notation, we abbreviate $B := A^{(k,\ell)}$ for the rest of this proof. We have to show that each two rows of B have a distinct number of positive entries.

Let $p \in \{1, \ldots, m\} \setminus \{m\}$ denote the unique integer such that $n_{p,m}^+(B) = (m-1) - 1$, that is, the p-th row of B_{mm} has the most positive entries.

Suppose that $b_{pm} > 0$. This implies $n_p^+(B) = m - 1$. Consequently, the p-th column of B has no positive entries; hence, as each two rows of B_{pp} have a distinct number of positive entries and the number of positive entries of each row of B_{pp} is strictly smaller than $m - 1$, we obtain that all rows of B have a distinct number of positive entries. Hence, the statement follows if $b_{pm} > 0$.

Now, we suppose that $b_{pm} < 0$. This implies $n_p^+(B) = m - 2$. There is precisely one integer $q \in \{1, \dots, m\} \setminus \{p\}$ such that $n_{q,p}^+(B) = (m-1) - 1$.

Suppose that $q = m$. Since $b_{mp} > 0$, we obtain that $n_m^+(B) = m - 1$. Thus, we obtain as before via the induction hypothesis that all rows of B have a distinct number of positive entries. Therefore, the statement follows if $q = m$.

We are left with the case $b_{pm} < 0$ and $q \neq m$. Note that in this case

$$n_p^+(B) = n_q^+(B) = m - 2 \text{ and } b_{qp} < 0. \tag{3.23}$$

As a result, for each integer $r \in \{1, \dots, m\} \setminus \{p, q, m\}$ both entries b_{pr} and b_{qr} are positive. But via (3.23) this implies

$$n_{p,r}^+(B) = n_{q,r}^+(B) = m - 3,$$

for all $r \in \{1, \dots, m\} \setminus \{p, q, m\}$ which is not possible due to the induction hypothesis. Therefore, the case $b_{pm} < 0$ and $q \neq m$ cannot occur.

We have considered all cases and thus the statement follows by induction. The lemma follows. $\square$

We conclude this section with the proof of Theorem 3.6.

Proof of Theorem 3.6. Fix $k, \ell \in \{1, \dots, m\}$ with $k \neq \ell$. Lemma 3.11 and a diagonal sequence argument tell us that there is a sequence $\{C_r\}_{r \geq 1}$, where $C_r := (c_{ij}^{(r)})_{1 \leq i,j \leq m}$, of non-negative generic matrices such that $C_r \to C$ with $r \to +\infty$. By passing to a subsequence (if necessary) we may assume that the matrices $C_r^{(k,\ell)}$, defined in Lemma 3.12, all have the same sign pattern. For each integer $r \geq 1$ let $T_r \in \mathrm{Mat}(m \times m; \mathbb{R})$ be the matrix given by

$$t_{ij}^{(r)} := |m_{ij}^{(r)}| \left(c_{ik}^{(r)} c_{j\ell}^{(r)} - c_{jk}^{(r)} c_{i\ell}^{(r)} \right),$$

where $M_r := C_r^{-1}$. Due to the first item in Lemma 3.11, it follows that $m_{ij}^{(r)} \neq 0$. Thus, the matrices T_r and $C_r^{(k,\ell)}$ have the same sign pattern.

Therefore, by the virtue of Lemma 3.12, each row of T_r has a distinct number of positive entries. Fix an integer $r \geq 1$. For each integer $1 \leq p \leq m$ let $c(p)$ be the unique integer such that the $c(p)$-th row of T_r has exactly $m-p$ positive entries. Since all matrices T_r have the same sign pattern, the definition of c is independent of the integer $r \geq 1$. The map $c\colon \{1,\dots,m\} \to \{1,\dots,m\}$ is a bijection and

$$\begin{cases} t^{(r)}_{c(p)j} < 0 & \text{if} \quad j \in \{c(1),\dots,c(p-1)\} \\ t^{(r)}_{c(p)j} > 0 & \text{if} \quad j \in \{c(p+1),\dots,c(m)\} \end{cases}$$

for all integers $r \geq 1$. Let $T \in \mathrm{Mat}(m \times m; \mathbb{R})$ be the matrix given by

$$t_{ij} := |m_{ij}| \, (c_{ik}c_{j\ell} - c_{jk}c_{i\ell}) \, .$$

Clearly, $T_r \to T$ with $r \to +\infty$. As a result,

$$\begin{cases} t_{c(p)j} \leq 0 & \text{if} \quad j \in \{c(1),\dots,c(p-1)\} \\ t_{c(p)j} \geq 0 & \text{if} \quad j \in \{c(p+1),\dots,c(m)\}. \end{cases} \tag{3.24}$$

By Lemma 3.8 and (3.24) we obtain that

$$\sum_{j=1}^{p-1} t_{c(j)c(p)} = \sum_{j=p+1}^{m} t_{c(p)c(j)} \tag{3.25}$$

for all integers $1 \leq p \leq m$ with $c(p) \neq k, \ell$, since T is skew-symmetric.

In [Mar72, Theorem 3.1], Markham established that every almost principal minor of C is non-negative. Hence,

$$|m_{kj}| \, (c_{kk}c_{j\ell} - c_{jk}c_{k\ell}) \geq 0 \quad \text{and} \quad |m_{\ell j}| \, (c_{\ell k}c_{j\ell} - c_{jk}c_{\ell\ell}) \leq 0$$

for all integers $1 \leq j \leq m$. Consequently, we obtain that $c(1) = k$ and $c(m) = \ell$. For each integer $2 \leq h \leq m-1$ we compute via (3.25),

$$\begin{aligned} \sum_{p=2}^{h} \sum_{j=p+1}^{m} t_{c(p)c(j)} &= \sum_{p=2}^{h} \sum_{j=1}^{p-1} t_{c(j)c(p)} \\ &= \sum_{j=2}^{h} t_{c(1)c(j)} + \sum_{j=2}^{h-1} \sum_{p=j+1}^{h} t_{c(j)c(p)} \\ &\leq \sum_{j=2}^{h} t_{c(1)c(j)} + \sum_{p=2}^{h-1} \sum_{j=p+1}^{m} t_{c(p)c(j)}. \end{aligned} \tag{3.26}$$

Note that

$$\frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} |m_{ij} \, (c_{ik}c_{jl} - c_{jk}c_{il})| = \sum_{p=1}^{m} \sum_{j=p+1}^{m} t_{c(p)c(j)}.$$

Therefore, by the use of (3.26) we obtain

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\left|m_{ij}\left(c_{ik}c_{jl}-c_{jk}c_{il}\right)\right|$$
$$\leq \sum_{h=2}^{m}\sum_{j=2}^{h} t_{c(1)c(j)} \leq (m-1)\sum_{j=1}^{m} t_{c(1)c(j)}.$$

Lemma 3.8 tells us that

$$\sum_{j=1}^{m} t_{c(1)c(j)} = c_{k\ell};$$

therefore, the theorem follows. □

3.5 Proofs of Theorem 1.7 and Theorem 1.9

3.5.1 – We begin with the proof of Theorem 1.7.

Proof of Theorem 1.7. Let $S \subset X$ be a closed subset and let $T \subset X$ be a finite subset such that $S \cap T = \varnothing$ and $|T| \leq m$. Let $f\colon S \to Y$ be a Lipschitz map. In what follows we construct for each $\epsilon > 0$ a map $F_\epsilon\colon S \cup T \to Y$ that is a Lipschitz extension of f to $S \cup T$ such that $\mathrm{Lip}(F_\epsilon) \leq ((1+\epsilon)m + 1)\,\mathrm{Lip}(f)$.

We start with a few definitions. Fix $\epsilon > 0$. Let $F \subset S$ be a finite subset such that for each point $z \in T$ there is a point $x \in F$ with

$$d_X(z, x) \leq (1+\epsilon) d_X(z, S). \tag{3.27}$$

Since S is closed and T is finite, such a set F clearly exists. We set

$$E := \big\{\{u, v\} : u \neq v \text{ with } \ (u, v \in T) \ \text{ or } \ (u \in T, v \in F)\,\big\}.$$

Let $G := (V, E)$ denote the graph with vertex set $V := F \cup T$ and edge set E. We say that a subset $E' \subset E$ is *admissible* if the graph $G' := (V, E')$ contains no cycles and has the property that if $v, v' \in F$ are distinct, then there is no path in G' connecting them.

For each edge $\{u, v\} \in E$ we set $\omega(\{u, v\}) := d_X(u, v)$. Furthermore, let $N \geq 0$ denote the cardinality of E. Let $e\colon \{1, \dots, N\} \to E$ be a bijective map such that the composition $\omega \circ e$ is a non-decreasing function. We construct the sequence $\{E_\ell\}_{\ell=0}^{N}$ of subsets of E via the following recursive rule:

$$E_0 := \varnothing, \quad E_\ell := \begin{cases} \{e(\ell)\} \cup E_{\ell-1} & \text{if } \{e(\ell)\} \cup E_{\ell-1} \text{ is admissible} \\ E_{\ell-1} & \text{otherwise.} \end{cases} \tag{3.28}$$

We claim that for each point $z \in T$ there exists an integer $L_z \geq 1$ and a unique injective path $\gamma_z \colon \{1, \dots, L_z\} \to E_N$ connecting z to a point x_z in F. Indeed, the uniqueness part of the claim follows directly, as E_N is admissible. Now, we show the existence part. Let $z \in T$ be a point. Choose an arbitrary point $x \in F$. If the edge $\{x, z\}$ is contained in E_N, then an injective path γ_z with the desired property surely exists. Suppose now that $\{x, z\} \notin E_N$. It follows from the recursive construction of E_N that in this case there either exists a path in E_N from z to x of length greater than or equal to two or there exists a path in E_N from z to a point $x' \in F$ distinct from x. Thus, in any case an injective path γ_z with the desired properties exists.

We define the map $F_\epsilon \colon S \cup T \to Y$ as follows

$$\begin{aligned} F_\epsilon(x) &:= f(x) && \text{for all } x \in S \\ F_\epsilon(z) &:= f(x_z) && \text{for all } z \in T. \end{aligned}$$

In other words, $F_\epsilon = f \circ R_\epsilon$, where $R_\epsilon \colon S \cup T \to S$ is the retraction that maps $z \in T$ to $x_z \in S$. In what follows, we show that R_ϵ has Lipschitz constant smaller than or equal to $(1 + \epsilon)m + 1$. This is the reason that enables us to put so low requirements onto 'distance' in Y .

Now, let $z \in T$ and $x \in S$ be points. By the use of the triangle inequality, we compute

$$\begin{aligned} &\rho_Y(F_\epsilon(x), F_\epsilon(z)) = \rho_Y(f(x), f(x_z)) \leq \mathrm{Lip}(f) d_X(x, x_z) \\ &\leq \mathrm{Lip}(f) \left(d_X(x, z) + \sum_{\ell=1}^{L_z} \omega(\gamma_z(\ell)) \right). \end{aligned} \tag{3.29}$$

Let $x' \in F$ be a point such that the pair (z, x') satisfies the estimate (3.27). By the recursive construction of E_N, it follows that $\omega(\gamma_z(\ell)) \leq d(x', z)$ for all $\ell \in \{1, \dots, L_z\}$, since the function $\omega \circ e$ is non-decreasing. Hence, by the use of (3.29) we obtain

$$\begin{aligned} &\rho_Y(F_\epsilon(x), F_\epsilon(z)) \\ &\leq \mathrm{Lip}(f) \left(d_X(x, z) + L_z d_X(x', z) \right) \\ &\leq \mathrm{Lip}(f) \left(1 + L_z(1 + \epsilon) \right) d_X(x, z) \\ &\leq \mathrm{Lip}(f) \left((1 + \epsilon)m + 1 \right) d_X(x, z). \end{aligned}$$

Now, let $z, z' \in T$ be points. If $x_z = x_{z'}$, then $F_\epsilon(z) = F_\epsilon(z')$, by construction. Suppose now that $x_z \neq x_{z'}$. We compute

$$\begin{aligned} &\rho_Y(F_\epsilon(z), F_\epsilon(z')) = \rho_Y(f(x_z), f(x_{z'})) \leq \mathrm{Lip}(f) d_X(x_z, x_{z'}) \\ &\leq \mathrm{Lip}(f) \left(\sum_{\ell=1}^{L_z} \omega(\gamma_z(\ell)) + d_X(z, z') + \sum_{\ell=1}^{L_{z'}} \omega(\gamma_{z'}(\ell)) \right). \end{aligned} \tag{3.30}$$

The edge $\{z, z'\}$ is not contained in E_N; thus, by the recursive construction of E_N we obtain that $\omega(\gamma_z(\ell)) \leq \omega(\{z, z'\})$ for all $\ell \in \{1, \ldots, L_z\}$ and $\omega(\gamma_{z'}(\ell)) \leq \omega(\{z, z'\})$ for all for all $\ell \in \{1, \ldots, L_{z'}\}$. By virtue of (3.30) we deduce

$$\begin{aligned}
&\rho_Y(F_\epsilon(z), F_\epsilon(z')) \\
&\leq \operatorname{Lip}(f)\,(L_z + 1 + L_{z'})\, d_X(z, z') \\
&\leq \operatorname{Lip}(f)(m+1) d_X(z, z').
\end{aligned}$$

The last inequality follows, since E_N is admissible and the paths $\gamma_z, \gamma_{z'}$ are injective; thus, $L_z + L_{z'} \leq m$. We have considered all possible cases and we have established that

$$\operatorname{Lip}(F_\epsilon) \leq ((1+\epsilon)m + 1) \operatorname{Lip}(f),$$

as desired. This completes the proof. $\square$

3.5.2 – In this paragraph, prove a simple lemma that allows us, in order to prove Theorem 1.9, to restrict our attention to closed convex subsets of Banach spaces.

Given a quasi-metric space (X, ρ_X), a subset $S \subset X$ and a Lipschitz map $f\colon S \to E$ into a Banach space we use $\mathsf{e}_{\mathrm{conv}}(S; X, E, f)$ to denote the infimum over those $D \geq 1$ such that there exists Lipschitz map $\overline{f}\colon X \to \overline{\mathrm{conv}}(f(S))$ that extends f and satisfies $\operatorname{Lip}(\overline{f}) \leq D \operatorname{Lip}(f)$. Accordingly, we set

$$\mathsf{e}_{\mathrm{conv}}(S; X) = \sup\big\{\mathsf{e}_{\mathrm{conv}}(S; X, E, f) : E \text{ Banach space, } f\colon S \to E \text{ Lipschitz}\big\}.$$

It turns out that $\mathsf{e}_{\mathrm{conv}}(S; X)$ coincides with $\mathsf{e}_{\mathrm{bar}}(S; X)$.

Lemma 3.13. *Let (X, ρ_X) be a quasi-metric space and let $S \subset X$ be a subset. Then*

$$\mathsf{e}_{conv}(S; X) = \mathsf{e}_{bar}(S; X).$$

Proof. Clearly, $\mathsf{e}_{\mathrm{conv}}(S; X) \leq \mathsf{e}_{\mathrm{bar}}(S; X)$. In what follows, we show the reversed inequality. To this end, we suppose that $\mathsf{e}_{\mathrm{conv}}(S; X) < +\infty$. Let (Z, d_Z) be a complete barycentric metric space and let $f\colon S \to Z$ be a Lipschitz map. Let $\Phi\colon Z \to \ell_\infty(Z)$ denote the Kuratowski embedding. Choose a point $z_0 \in \ell_\infty(Z)$ such that $d(z_0, \Phi(Z)) > 0$ and abbreviate $Z_0 := Z \cup \{z\}$. The map $\iota\colon Z \mapsto M^0(Z_0)$ given by $z \mapsto \delta_z - \delta_{z_0}$ is an isometric embedding. There exists a map $\overline{f}\colon X \to \overline{\mathrm{conv}}(\iota(Z))$ that extends the map $\iota \circ f$ such that $\operatorname{Lip}(\overline{f}) \leq \mathsf{e}_{\mathrm{conv}}(S; X) \operatorname{Lip}(f)$. Now, employing Proposition 2.2 and using the fact that $P_1(Z)$ is complete, we may deduce that

$$\overline{\mathrm{conv}}(\iota(Z)) = \big\{\mu - \delta_{z_0} : \mu \in P_1(Z)\big\} \cong P_1(Z).$$

Thus, we get $\mathsf{e}_{\mathrm{bar}}(S; X) \leq \mathsf{e}_{\mathrm{conv}}(S; X)$, as was left to show. $\square$

3.5.3 – We proceed with the proof of Theorem 1.9.

Proof of Theorem 1.9. Let $X := S \cup T$, with $S \cap T = \varnothing$, be a finite subset of $F[H]$ with $\mathrm{card}(T) = m$. Due to Lemma 3.13 it suffices to consider Lipschitz maps $f\colon S \to E$, where $(E, \|\cdot\|_E)$ is a Banach space. Without loss of generality we may assume (by scaling) that $\mathrm{Lip}(f) = 1$. We set $I := X$ and let the map $\mathbf{x}\colon I \to H$ be given by the identity.

Let $G\colon [0, +\infty) \to [0, +\infty)$ denote the function such that $x = F(\sqrt{G(x)})$ for all real numbers $x \in [0, +\infty)$. Observe that the function G is convex, strictly-increasing and $G(0) = 0$. We say that $\boldsymbol{\xi}\colon I \times I \to \mathbb{R}$ *lies above* f if there is a map $\overline{f}\colon X \to \overline{\mathrm{conv}}(f(S))$ such that $\overline{f}(s) = f(s)$ for all $s \in S$ and

$$G\left(\|\overline{f}\big(\mathbf{x}(i)\big) - \overline{f}\big(\mathbf{x}(j)\big)\|_{H}\right) \leq \boldsymbol{\xi}(i,j) \quad \text{for all } i, j \in I.$$

We use $\overline{\mathrm{conv}}$ to denote the closed convex hull. Let $E_f \subset \mathbb{R}^{I \times I}$ be the set of all $\boldsymbol{\xi} \in \mathbb{R}^{I \times I}$ that lie above f. Moreover, let $\boldsymbol{v}\colon I \times I \to \mathbb{R}$ be the map given by

$$\boldsymbol{v}(i,j) := \|\mathbf{x}(i) - \mathbf{x}(j)\|_{H}^{2}. \tag{3.31}$$

Suppose that $L \in [1, +\infty)$ is a real number. If $L\boldsymbol{v} \in E_f$, then the map f admits a Lipschitz extension $\overline{f}\colon X \to \overline{\mathrm{conv}}(f(S))$ such that

$$\mathrm{Lip}(\overline{f}) \leq \sup_{x>0} \frac{F(\sqrt{L}x)}{F(x)}.$$

Indeed, if $L\boldsymbol{v} \in E_f$, then (by definition) there exists a function $\overline{f}\colon X \to \overline{\mathrm{conv}}(f(S))$ such that

$$G\left(\|\overline{f}\big(\mathbf{x}(i)\big) - \overline{f}\big(\mathbf{x}(j)\big)\|_{H}\right) \leq L\boldsymbol{v}(i,j) \quad \text{for all } i, j \in I;$$

consequently, by applying the function $F\left(\sqrt{\cdot}\right)$ on both sides, we obtain

$$\|\overline{f}\big(\mathbf{x}(i)\big) - \overline{f}\big(\mathbf{x}(j)\big)\|_{H} \leq F\left(\sqrt{\left(L\|\mathbf{x}(i) - \mathbf{x}(j)\|_{H}^{2}\right)}\right) \leq \sup_{x>0} \frac{F(\sqrt{L}x)}{F(x)} F\big(\|\mathbf{x}(i) - \mathbf{x}(j)\|_{H}\big)$$

for all $i, j \in I$. Since $X \subset F[H]$ the map $\overline{f}$ is a Lipschitz extension of f such that $\mathrm{Lip}(\overline{f})$ has the desired upper bound. Thus, to prove the theorem it suffices to show that if $L \geq (m+1)$, then $L\boldsymbol{v} \in E_f$.

To this end, we suppose that $L\boldsymbol{v} \notin E_f$ and we show that $L < (m+1)$. Since the function G is strictly-increasing and convex, the set E_f is closed and convex; thus, by the hyperplane separation theorem we obtain a real number $\epsilon > 0$ and a non-zero vector $\boldsymbol{\lambda} \in \mathbb{R}^{I\times I}$ such that

$$\langle L\boldsymbol{v}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}} + \epsilon < \langle \boldsymbol{\xi}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}} \quad \text{for all } \boldsymbol{\xi} \in E_f. \tag{3.32}$$

We claim that each entry of $\boldsymbol{\lambda}$ is non-negative. Indeed, if $\boldsymbol{\xi} \in E_f$, then the point $(\xi_1, \dots, \xi_{k-1}, c\xi_k, \xi_{k+1}, \dots, \xi_N)$, where $N := \operatorname{card}(I\times I)$, is contained in E_f for all integers $1 \le k \le N$ and real numbers $c \in [1, +\infty)$. Hence, a simple scaling argument implies that the k-th entry of $\boldsymbol{\lambda}$ is non-negative for each integer $1 \le k \le N$, as claimed.

In the following, we estimate $\langle L\boldsymbol{v}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}}$ from below. We may assume that $\boldsymbol{\lambda}$ is symmetric. By adjusting $\epsilon > 0$ if necessary, we may assume that $\sum_{k\in S} \lambda_{ik} \neq 0$ for all $i \in T$. Let the matrix $M := M(\boldsymbol{\lambda}, T)$ be given as in (3.8). Since each entry of the vector $\boldsymbol{\lambda}$ is non-negative and $\sum_{k\in S} \lambda_{ik} \neq 0$ for all $i \in T$, the matrix $M(\boldsymbol{\lambda}, T)$ is non-singular. We set $C := M^{-1}$. Proposition 3.5 tells us that

$$\mathsf{m} := \mathsf{m}(\mathbf{x}, \boldsymbol{\lambda}, \mathsf{id}, T) = \sum_{r\in S}\sum_{s\in S} \boldsymbol{\eta}(r,s)\|\mathbf{x}(r) - \mathbf{x}(s)\|_H^2, \tag{3.33}$$

where $\boldsymbol{\eta}\colon I\times I \to \mathbb{R}$ is given by

$$\boldsymbol{\eta}(r,s) := \lambda_{rs} + \sum_{i\in T}\sum_{j\in T} \lambda_{ir} c_{ij} \lambda_{js}.$$

Clearly,

$$L\mathsf{m} \le \langle L\boldsymbol{v}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}}. \tag{3.34}$$

Next, we estimate $\langle L\boldsymbol{v}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}}$ from above. We set

$$\bar{\boldsymbol{\lambda}}_i := \frac{1}{\|\boldsymbol{\lambda}_i\|_1}\boldsymbol{\lambda}_i \in \Delta^{\operatorname{card}(S)-1}$$

for each $i \in T$, where $\boldsymbol{\lambda}_i := (\lambda_{ik})_{k\in S}$. By (3.10),

$$\sum_{j\in T} c_{ij}\|\boldsymbol{\lambda}_i\|_1 = \sum_{j\in T} c_{ij}\sum_{k\in S}\lambda_{jk} = 1 \tag{3.35}$$

for all $i \in T$. For each $i \in T$ we define

$$w_i := \sum_{j\in T} c_{ij}\left(\sum_{k\in S}\lambda_{jk}\right) y_{\bar{\boldsymbol{\lambda}}_j}, \quad \text{where } y_{\bar{\boldsymbol{\lambda}}_j} = \sum_{r\in S}\bar{\lambda}_{jr} f(r).$$

Using (3.35) we obtain $w_i \in \overline{\operatorname{conv}}(f(S))$ for all $i \in T$. Equation (3.32) tells us that

$$\langle L\boldsymbol{v}, \boldsymbol{\lambda}\rangle_{\mathbb{R}^{I\times I}} < A + B + C; \tag{3.36}$$

where,

$$A := 2\sum_{i\in T}\sum_{r\in S}\lambda_{ir}G\left(\|f(r)-w_i\|_{E}\right),$$
$$B := \sum_{i\in T}\sum_{j\in T}\lambda_{ij}G\left(\|w_i-w_j\|_{E}\right),$$
$$C := \sum_{r\in S}\sum_{s\in S}\lambda_{rs}G\left(\|f(r)-f(s)\|_{E}\right).$$

By convexity of the strictly-increasing function G and the use of (3.35), we estimate

$$\begin{aligned}&A+C\\ &\le 2\sum_{i\in T}\sum_{r\in S}\sum_{j\in T}\lambda_{ir}c_{ij}\|\boldsymbol{\lambda}_j\|_1\, G\left(\|f(r)-y_{\bar{\boldsymbol{\lambda}}_j}\|_{E}\right)+C\\ &\le 2\sum_{r\in S}\sum_{s\in S}\boldsymbol{\eta}(r,s)\, G\left(\|f(r)-f(s)\|_{E}\right).\end{aligned}$$

Thus, if

$$B=\sum_{i\in T}\sum_{j\in T}\lambda_{ij}G\left(\|w_i-w_j\|_{E}\right)\le (m-1)\sum_{r\in S}\sum_{s\in S}\boldsymbol{\eta}(r,s)G\left(\|f(r)-f(s)\|_{E}\right), \quad (3.37)$$

then we obtain via (3.36) and (3.34) that

$$L\mathsf{m} < (m+1)\sum_{r\in S}\sum_{s\in S}\boldsymbol{\eta}(r,s)G\left(\|f(r)-f(s)\|_{E}\right).$$

Since

$$\|f(r)-f(s)\|_{E}\le F\left(\sqrt{\|r-s\|_{H}^{2}}\right)\qquad \text{for all } r,s\in S,$$

it follows

$$G(\|f(r)-f(s)\|_{E})\le \|r-s\|_{H}^{2}\qquad \text{for all } r,s\in S;$$

as a result, we obtain

$$L\mathsf{m} < (m+1)\mathsf{m}.$$

By virtue of Corollary 3.9 every entry of the matrix C is positive, hence $\mathsf{m}>0$ and consequently $L<m+1$. Thus, to conclude the proof we are left to establish the estimate (3.37). It is readily verified that

$$w_i-w_j=\frac{1}{2}\sum_{k\in T}\sum_{\ell\in T}\|\boldsymbol{\lambda}_k\|_1\|\boldsymbol{\lambda}_\ell\|_1\left(c_{j\ell}c_{ik}-c_{i\ell}c_{jk}\right)\left(y_{\bar{\boldsymbol{\lambda}}_k}-y_{\bar{\boldsymbol{\lambda}}_\ell}\right).$$

Since

$$\frac{1}{2}\sum_{k\in T}\sum_{\ell\in T}\|\boldsymbol{\lambda}_k\|_1\|\boldsymbol{\lambda}_\ell\|_1\left|c_{j\ell}c_{ik}-c_{i\ell}c_{jk}\right|\le\sum_{k\in T}|c_{ik}|\,\|\boldsymbol{\lambda}_k\|_1\sum_{\ell\in T}|c_{j\ell}|\,\|\boldsymbol{\lambda}_\ell\|_1=1,$$

we can use the triangle inequality, the convexity of the strictly-increasing map G and $G(0) = 0$ to estimate

$$
\begin{aligned}
B &= \sum_{i\in T}\sum_{j\in T}\lambda_{ij} G\left(\|w_i - w_j\|_E\right) \\
&\leq \sum_{i\in T}\sum_{j\in T}\lambda_{ij}\,\frac{1}{2}\sum_{k\in T}\sum_{\ell\in T}\|\boldsymbol{\lambda}_k\|_1\|\boldsymbol{\lambda}_\ell\|_1\ |c_{j\ell}c_{ik} - c_{i\ell}c_{jk}|\ \ G\left(\|y_{\bar{\boldsymbol{\lambda}}_k} - y_{\bar{\boldsymbol{\lambda}}_\ell}\|_E\right) \\
&= \sum_{k\in T}\sum_{\ell\in T}\|\boldsymbol{\lambda}_k\|_1\|\boldsymbol{\lambda}_\ell\|_1 \left(\frac{1}{2}\sum_{i\in T}\sum_{j\in T}\lambda_{ij}\,|c_{ik}c_{j\ell} - c_{jk}c_{i\ell}|\right) G\left(\|y_{\bar{\boldsymbol{\lambda}}_k} - y_{\bar{\boldsymbol{\lambda}}_\ell}\|_E\right).
\end{aligned}
\tag{3.38}
$$

As pointed out in the beginning of Section 3.4, $M(\boldsymbol{\lambda}, T)$ is a symmetric M-matrix. Hence, we may invoke Theorem 3.6 and obtain

$$
\frac{1}{2}\sum_{i\in T}\sum_{j\in T}\lambda_{ij}\,|c_{ik}c_{j\ell} - c_{jk}c_{i\ell}| \leq (m-1)c_{k\ell}
$$

for all distinct $k, \ell \in T$. Using (3.38) we deduce

$$
\begin{aligned}
&\sum_{i\in T}\sum_{j\in T}\lambda_{ij} G\left(\|w_i - w_j\|_E\right) \\
&\leq (m-1)\sum_{k\in T}\sum_{\ell\in T}\|\boldsymbol{\lambda}_k\|_1\|\boldsymbol{\lambda}_\ell\|_1 c_{k\ell}\ G\left(\|y_{\bar{\boldsymbol{\lambda}}_k} - y_{\bar{\boldsymbol{\lambda}}_\ell}\|_E\right).
\end{aligned}
$$

By convexity,

$$
G\left(\|y_{\bar{\boldsymbol{\lambda}}_k} - y_{\bar{\boldsymbol{\lambda}}_\ell}\|_E\right) \leq \sum_{r\in S}\sum_{s\in S}\bar{\lambda}_{kr}\bar{\lambda}_{\ell r} G\big(\|f(r) - f(s)\|_E\big);
$$

thereby, the desired estimate (3.37) follows, as was left to show. This completes the proof. □

3.6 Linear and non-linear Lipschitz extension moduli

3.6.1 – The following lemma is well established. Variants of it appear at various places in the mathematical literature, cf. [Bal92, Lemma 1.1] and [Lin64, Theorem 5].

Lemma 3.14. *Let (X, ρ_X) be a quasi-metric space and let $S \subset X$ be a finite subset. Then*

$$
\mathsf{e}_{\mathrm{fin}}(S; X) = \sup\left\{\, \mathsf{e}_{\mathrm{fin}}(S; X') : X' \subset X \text{ finite and } S \subset X' \right\}.
$$

Proof. We follow closely the proof given in [Bal92, Lemma 1.1]. Another approach is sketched in [MN13, p. 168]. We abbreviate

$$K := \sup\big\{\mathsf{e}_{\text{fin}}(S, X') : X' \subset X \text{ finite and } S \subset X'\big\}.$$

Let $(E, \|\cdot\|_E)$ be a finite-dimensional Banach space, let $x_0 \in S$ be a point and let $f\colon S \to E$ be a 1-Lipschitz map. Without loss of generality, we may suppose that $f(x_0) = 0$. For each point $x \in X$ we define

$$B_x := \big\{y \in E : \|y\| \leq K\rho_X(x, x_0)\big\}$$

and we set

$$B := \prod_{x\in X} B_x.$$

For each finite subset $X' \subset X$ that contains S there exists an extension $\overline{f}_{X'}\colon X' \to E$ of the map f such that $\operatorname{Lip}(\overline{f}_{X'}) \leq K$. We define the the point $z_{X'} \in B$ via

$$(z_{X'})_x = \begin{cases} \overline{f}_{X'}(x) & \text{if } x \in X', \\ 0 & \text{otherwise.} \end{cases}$$

Now, Tychonoff's Theorem tells us that the net $(z_{X'})$, where $X' \subset X$ is a finite subset that contains S, has a subnet that converges to a point $z \in B$. Clearly, $z_x = f(x)$ for all $x \in S$. It is not hard to check that the map $\overline{f}\colon X \to E$ given by $x \mapsto z_x$ is a K-Lipschitz extension of $f\colon S \to E$. This completes the proof. □

3.6.2 – In this paragraph, we collect some facts about Lipschitz free spaces. Throughout, let (X, d_X) denote a bounded non-empty metric space. We set

$$\operatorname{Lip}(X) := \big\{f\colon X \to \mathbb{R} : f \text{ is Lipschitz}\big\}.$$

The map $\operatorname{Lip}(\cdot)\colon \operatorname{Lip}(X) \to \mathbb{R}$ given by

$$f \mapsto \operatorname{Lip}(f) := \inf\{L \in [0, +\infty) : \; f \text{ is L-Lipschitz }\}$$

is a semi-norm on X. Moreover, $\operatorname{Lip}(f + c) = \operatorname{Lip}(f)$ for all $f \in \operatorname{Lip}(X)$ and $c \in \mathbb{R}$. Let $L(X)$ denote the quotient vector space obtained from $\operatorname{Lip}(X)$ by the equivalence relation $f \sim g$ if and only if the function $f - g$ is constant. We equip $L(X)$ with the quotient norm

$$[f] \mapsto \inf_{c\in\mathbb{R}} \operatorname{Lip}(f + c).$$

The space $L(X)$ is a dual space, cf. [Kai78, p. 326]. This motivates the following definition.

Definition 3.15 (Lipschitz free space). *Let (X, d) denote a bounded metric space. Let $(E, \|\cdot\|)$ be a Banach space. We say that E is a Lipschitz free space over X if the dual space of E is isometric to $L(X)$.*

By a result of Weaver, it follows that if $(E, \|\cdot\|)$ and $(E', \|\cdot\|)$ are Lipschitz free spaces over X, then E and E' are isometric, cf. [Wea99, Theorem 3.26]. Thus, the Lipschitz free space over X is unique up to isometry.

3.6.3 – In this paragraph we retain the notation from Section 2.1. Our goal is to show that the space of signed measures on a bounded non-empty metric space (X, d_X) can be equipped with a norm such that it is a Lipschitz free space over X. We set

$$M^0(X) := \big\{\mu\colon \mathcal{B}_X \to \mathbb{R} : \mu \text{ is a signed finite Radon measure with } \mu(X) = 0\big\}.$$

It is not hard to check that $M^0(X)$ is a vector space and that for all $\mu \in M^0(X)$:

$$\int_X d_X(x, x_0)\ |\mu|\,(dx) < +\infty$$

for $x_0 \in X$, as X is bounded. The map $\|\cdot\|_{KR}\colon M^0(X) \to \mathbb{R}$ given by

$$\mu \mapsto \sup\left\{\int_X f\, d\mu : f \in \mathrm{Lip}_1(X)\right\}.$$

defines a norm on $M^0(X)$, cf. [Edw11] for historical remarks. The following theorem characterizes the dual space of $(M^0(X), \|\cdot\|_{KR})$.

Theorem 3.16. *Let (X, d) denote a bounded metric space. Then $(M^0(X), \|\cdot\|_{KR})$ is a Lipschitz free space over X.*

A proof of Theorem 3.16 can be found in [Edw11, Theorem 7.3]. From now on, we set $\mathcal{F}(X) := M^0(X)$. Now, it is readily verified that every Lipschitz map $f\colon X \to Y$ induces a linear map $\phi\colon \mathcal{F}(X) \to \mathcal{F}(Y)$ with $\|\phi\| = \mathrm{Lip}(f)$ such that $\phi(\delta_{x_1} - \delta_{x_2}) = \delta_{f(x_1)} - \delta_{f(x_2)}$ for all $x_1, x_2 \in X$.

3.6.4 – Using the result from the previous paragraph, we obtain the subsequent proposition that relates linear and non-linear Lipschitz extension moduli.

Proposition 3.17. *Let (X, d_X) be a metric space and let $S \subset X$ be a finite subset. Then*

$$\mathsf{e}_{\mathrm{fin}}(S; X) = \sup\big\{\,\Pi(\mathcal{F}(S), \mathcal{F}(X')) : X' \subset X \textit{ finite and } S \subset X'\big\}.$$

Proof. Due to Lemma 3.14, we may suppose that X is finite. Let $S \subset X$ be a non-empty subset and let $x_0 \in S$ be a point and let $\mathrm{ev}\colon S \to L(S)^*$ denote the map given by

$$s \mapsto \begin{cases} \mathrm{ev}(s)\colon L(S) \to \mathbb{R} \\ [\ell] \mapsto \ell(s) - \ell(x_0). \end{cases} \tag{3.39}$$

Further, let D denote the infimum over those $D' \geq 1$ such that there exists a Lipschitz map $\overline{\mathrm{ev}}\colon X \to L(S)^*$ that extends ev and satisfies $\mathrm{Lip}(\overline{\mathrm{ev}}) \leq D' \,\mathrm{Lip}(\mathrm{ev})$. Clearly, $D \leq \mathsf{e}_{\mathrm{fin}}(S; X)$. On the other hand, every map $f\colon S \to E$ induces a map linear map $L\colon \mathcal{F}(S) \to \mathcal{F}(E)$ such that $\|L\| = \mathrm{Lip}(f)$ and $\beta_E \circ L(\delta_s - \delta_{x_0}) = f(s)$ for all $s \in S$; consequently, since $\mathcal{F}(S) \cong L(S)^*$, we infer

$$\mathsf{e}_{\mathrm{fin}}(S; X) = D.$$

Next, we show that $D = \Pi(\mathcal{F}(S), \mathcal{F}(X))$. Let $\iota\colon X \to \mathcal{F}(X)$ denote the isometric embedding given by $x \mapsto \delta_x - \delta_{x_0}$; using this isometric embedding, it is readily verified that $D \leq \Pi(\mathcal{F}(S), \mathcal{F}(X))$. Now, let $\overline{\mathrm{ev}}\colon X \to L(S)^*$ be a Lipschitz extension of ev. The linear map $\phi\colon L(S) \to L(X)$ given by

$$[\ell] \mapsto \begin{cases} X \to \mathbb{R} \\ x \mapsto \overline{\mathrm{ev}}(x)([\ell]) \end{cases}$$

satisfies $\phi([\ell])(s) - \phi([\ell])(x_0) = \ell(s) - \ell(x_0)$ for all $[\ell] \in L(S)$ and $s \in S$. Moreover, a short calculation reveals that $\|\phi\| = \mathrm{Lip}(\overline{\mathrm{ev}})$. Let $\phi^*\colon \mathcal{F}(X) \to \mathcal{F}(S)$ denote the adjoint of ϕ. By construction, ϕ^* is a linear projection of $\mathcal{F}(X)$ onto $\mathcal{F}(S)$. Since $\|\phi^*\| = \|\phi\| = \mathrm{Lip}(\overline{\mathrm{ev}})$, we conclude $\Pi(\mathcal{F}(S), \mathcal{F}(X)) \leq D$. This completes the proof. $\square$

Proof of Theorem 1.11. The formula for æ(S) is a direct consequece of Proposition 3.17. The estimate (1.8) follows readily from the first part and the classical fact that every finite-dimensional Banach space admits a linear isometric embedding into $\ell_\infty(\mathbb{N})$. $\square$

3.6.5 – In this paragraph we construct a three-point metric space (S, d_S) such that æ$(S) \geq \frac{4}{3}$.

Example 3.18. Let $S := \{1, 2, 3\}$ equipped with the discrete metric $d_S(i,j) = 2(1 - \delta_{ij})$. We let $X := \{0\} \cup S$ denote the metric space endowed with the metric

$$d_X(0,0) := 0, \; d_X(0,i) := 1 \;\text{ and }\; d_X(i,j) := d_S(i,j)$$

for all $i = 1, 2, 3$. X is a metric tree with leaves S and branch point 0. Let $h\colon S \to \ell_1^3$ denote the map given by

$$1 \mapsto (0, 0, 0),\ 2 \mapsto (1, 0, 1),\ 3 \mapsto (0, 1, 1).$$

Clearly, h is 1-Lipschitz. Let $E \subset \ell_1^3$ denote the linear span of $h(S)$. We define the map $f\colon S \to E$ via $f(i) := h(i)$ for all $i = 1, 2, 3$. The unit ball of E is equal to the closed convex hull of $\pm\frac{1}{2}f(2), \pm\frac{1}{2}f(3), \pm\frac{1}{2}(f(3) - f(2))$; thus, E is linearly isometric to $\mathbb{R}^2$ equipped with the hexagonal norm (via the linear map determined by $\frac{1}{2}f(2) \mapsto (1, 0)$ and $\frac{1}{2}f(3) \mapsto \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$). It follows that

$$\inf \left\{ r \geq 0 : \bigcap_{i=1}^{3} B_r(f(i)) \neq \varnothing \right\} = \frac{4}{3}.$$

Hence, for every Lipschitz extension $\overline{f}\colon X \to E$ of f it holds that

$$\operatorname{Lip}(\overline{f}) \geq \frac{4}{3},$$

as desired.

3.7 A formula for Π_n

3.7.1 – Let $d \geq 1$ be an integer and set

$$\mathcal{A}_d := \left\{ \mathbb{1}_d + S : S \text{ is a Seidel adjacency matrix of a simple graph of order } d \right\}.$$

Moreover, we use $\mathcal{D}_d$ to denote the set of all diagonal $d \times d$-matrices that have trace equal to one and whose diagonal entries are non-negative.

For $A \in \mathcal{A}_d$ and $D \in \mathcal{D}_d$ we write $\lambda_1(\sqrt{D}A\sqrt{D}) \geq \ldots \geq \lambda_d(\sqrt{D}A\sqrt{D})$ for the eigenvalues of the symmetric matrix $\sqrt{D}A\sqrt{D}$ (counted with multiplicity). The subsequent result, due to Chalmers and Lewicki, characterizes the values $\Pi(n, d)$ in terms of maximal sums of eigenvalues of matrices of the form $\sqrt{D}A\sqrt{D}$.

Theorem 3.19 (Theorem 2.3 in [CL10]). *Let $1 \leq n \leq d$ be integers. The value $\Pi(n, d)$ is attained and equals*

$$\max \left\{ \sum_{k=1}^{n} \lambda_k \left(\sqrt{D}A\sqrt{D} \right) : A \in \mathcal{A}_d \text{ and } D \in \mathcal{D}_d \right\}.$$

3.7.2 – Let $i \geq 1$ be an integer and consider the map

$$\mathrm{bl}_i \colon \bigcup_{d \geq i} \mathcal{A}_d \to \bigcup_{d \geq i+1} \mathcal{A}_d, \quad A \mapsto \mathrm{bl}_i(A) := \begin{bmatrix} A & a_i^t \\ a_i & 1, \end{bmatrix}$$

where a_i denotes the i-th row of A. By construction, the i-th row of $\mathrm{bl}_i(A)$ and the last row of $\mathrm{bl}_i(A)$ coincide. We say that the matrix $\mathrm{bl}_i(A)$ is a *blow-up* of A (with respect to the i-th row).

If $A \in \mathcal{A}_d$ is a matrix and $D \in \mathcal{D}_d$ is positive-definite, then all eigenvalues of AD are real, for AD is equivalent to the symmetric matrix $\sqrt{D}A\sqrt{D}$. With a similar argument, one can show that even if D is positive-semidefinite, then all eigenvalues of AD are real. We use the notation

$$\lambda(AD) := (\lambda_1(AD), \ldots, \lambda_d(AD)),$$

where $\lambda_1(AD) \geq \ldots \geq \lambda_d(AD)$ are the eigenvalues of AD (counted with multiplicity). The lemma below is the key step in the proof of Theorem 1.14.

Lemma 3.20. *Let $A' \in \mathcal{A}_{d-1}$ be a matrix, let $A := \mathrm{bl}_i(A')$ for some integer $1 \leq i \leq d-1$ and let $D := \mathrm{Diag}(d_1, \ldots, d_d) \in \mathcal{D}_d$ be an invertible matrix. We set $D' := \mathrm{Diag}(d_1, \ldots, d_{i-1}, d_i + d_d, d_{i+1}, \ldots, d_{d-1})$. Then $D' \in \mathcal{D}_{d-1}$ is invertible, $\lambda(AD)$ has a zero entry and*

$$\lambda(A'D') \text{ is obtained from } \lambda(AD) \text{ by deleting a zero entry .}$$

Proof. For each integer $1 \leq k \leq d$ let s_k denote the k-th row of A. By assumption,

$$s_d = s_i.$$

Let λ be an eigenvalue of $A'D'$ and let $x' := (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$ be a corresponding eigenvector. We define $x := (x_1, \ldots, x_{d-1}, x_i)$. For all $1 \leq k < d$ we compute

$$\begin{aligned} \langle Ds_k, x \rangle_{\mathbb{R}^d} &= s_{ki} d_i x_i + s_{kd} d_d x_i + \sum_{\ell \neq d,i}^{d} s_{k\ell} d_\ell x_\ell \\ &= s_{ki} d_i x_i + s_{ki} d_d x_i + \sum_{\ell \neq d,i}^{d} s_{k\ell} d_\ell x_\ell = \langle D' s'_k, x' \rangle_{\mathbb{R}^{d-1}}. \end{aligned} \tag{3.40}$$

Thus, for all $1 \leq k < d$ we have

$$\langle Ds_k, x \rangle_{\mathbb{R}^d} = \langle D' s'_k, x' \rangle_{\mathbb{R}^{d-1}} = \lambda x_k.$$

Furthermore,

$$\langle Ds_d, x\rangle_{\mathbb{R}^d} = \langle Ds_i, x\rangle_{\mathbb{R}^d} = \lambda x_i;$$

as a result, the vector x is an eigenvector of AD with corresponding eigenvalue λ.

Next, we show that AD and $A'D'$ have the same rank. There exists a principal submatrix T of A such that T is invertible and $\mathrm{rk}(A) = \mathrm{rk}(T)$. This is well-known, cf. for example [Tho68, Theorem 5]. Clearly, T cannot be obtained from A by keeping the i-th and d-th column simultaneously; thus, T is also a principal submatrix of A'. Therefore,

$$\mathrm{rk}(A') \leq \mathrm{rk}(A) = \mathrm{rk}(T) \leq \mathrm{rk}(A')$$

and thereby $\mathrm{rk}(A) = \mathrm{rk}(A')$. Now, via Sylvester's law of interia

$$\mathrm{rk}(AD) = \mathrm{rk}(\sqrt{D}A\sqrt{D}) = \mathrm{rk}(A) = \mathrm{rk}(A') = \mathrm{rk}(A'D'),$$

as claimed. To summarize, AD and $A'D'$ have the same rank and if λ is an eigenvalue of $A'D'$, then λ is an eigenvalue of AD. This completes the proof. $\square$

3.7.3 – Now, we have everything at hand to verify Theorem 1.14.

Proof of Theorem 1.14. We set

$$\Phi_n := \sup_{d\geq 1} \max\left\{\frac{1}{d}\sum_{k=1}^{n} \lambda_k(A) : A \in \mathcal{A}_d\right\}.$$

First, we show for all $d \geq n$ that

$$\Pi(n, d) \leq \Phi_n.$$

We abbreviate

$$\pi_n(AD) := \sum_{k=1}^{n} \lambda_k(AD).$$

Due to Theorem 3.19, there exist matrices $A \in \mathcal{A}_d$ and $D \in \mathcal{D}_d$ such that

$$\Pi(n, d) = \pi_n(AD).$$

Choose a sequence $D_k \in \mathcal{D}_d$ of invertible matrices with rational entries satisfying

$$\Pi(n, d) \leq \pi_n(AD_k) + \frac{1}{2^k}. \tag{3.41}$$

This is possible since $\pi_n(AD) = \pi_n(\sqrt{D}A\sqrt{D})$ and because the map $\pi_n(\cdot)$ is continuous on the set of symmetric matrices, cf. [OW92, p. 44]. Fix $k \geq 1$. By finding a common denominator, we may write

$$D_k = \frac{1}{m}\operatorname{Diag}(n_1, \ldots, n_d),$$

where $n_i \geq 1$ for all $1 \leq i \leq d$ and $m = n_1 + \cdots + n_d$. We set

$$A_k := \mathrm{bl}_d^{(n_d-1)}(\cdots(\mathrm{bl}_1^{(n_1-1)}(A))\cdots),$$

where we use the convention $\mathrm{bl}_i^0(A) = A$. Note that $A_k \in \mathcal{A}_m$. By applying Lemma 3.20 repeatedly, we get that $\lambda(AD_k)$ is obtained from $\lambda\left(A_k\frac{1}{m}\mathbb{1}_m\right)$ by deleting exactly $(m-d)$ zero entries. As a result,

$$\pi_n(AD_k) \leq \frac{\pi_n(A_k)}{m} \leq \Phi_n. \tag{3.42}$$

Thus, by combining (3.42) with (3.41), we obtain

$$\Pi(n,d) \leq \Phi_n.$$

It is well-known that

$$\Pi_n = \lim_{d\to+\infty} \Pi(n,d).$$

Hence,

$$\Pi_n \leq \Phi_n.$$

The inequality $\Phi_n \leq \Pi_n$ is a direct consequence of Theorem 3.19. Putting everything together, we conclude

$$\Pi_n = \Phi_n.$$

We are left to show that it suffices to consider K_{n+2}-free two-graphs. To this end, fix an integer $d > n$ and let $A \in \mathcal{A}_d$ be a matrix such that

$$\pi_n(A) = \max\left\{\pi_n(A') : A' \in \mathcal{A}_d\right\}.$$

As the symmetric matrix A is orthogonally diagonalizable, there are orthonormal vectors $u_1, \ldots, u_n \in \mathbb{R}^d$ such that

$$\pi_n(A) = \operatorname{tr}(AUU^t),$$

where U is the matrix that has the vectors u_i as columns. Let r_k for $1 \leq k \leq d$ be the rows of the matrix U. We use $e_1, \ldots, e_d \in \mathbb{R}^d$ to denote the standard basis. Fix $1 \leq i,j \leq d$ and let $\epsilon \in \mathbb{R}$ be a real number. We set

$$A(i,j;\epsilon) := \begin{cases} \epsilon \operatorname{sgn}\left(\langle r_i, r_j\rangle_{\mathbb{R}^n}\right) e_i e_j^t & \text{if } \langle r_i, r_j\rangle \neq 0 \\ \epsilon\, e_i e_j^t & \text{otherwise} \end{cases}$$

and

$$\widehat{A}_\epsilon := A + \frac{1}{2}\big(A(i,j;\epsilon) + A(j,i;\epsilon)\big).$$

Clearly, $\widehat{A}_\epsilon$ is symmetric. Hereafter, we show that $\langle r_i, r_j\rangle \neq 0$. To this end, suppose that $\langle r_i, r_j\rangle = 0$.

We set $\epsilon_\star := -4\,\mathrm{sgn}(a_{ij})$, and we observe that $\widehat{A}_{\epsilon_\star} \in \mathcal{A}_d$. Further, we abbreviate $\widehat{A} := \widehat{A}_{\epsilon_\star}$. It holds that

$$\pi_n(A) = \mathrm{tr}\left(AUU^t\right) = \mathrm{tr}\left(\widehat{A}UU^t\right) - \epsilon_\star\,\mathrm{sgn}\left(\langle r_i, r_j\rangle\right)\langle r_i, r_j\rangle. \tag{3.43}$$

Via von Neumann's trace inequality, cf. [Mir75], we obtain

$$\mathrm{tr}\left(\widehat{A}UU^t\right) \leq \pi_n(\widehat{A}) \leq \pi_n(A);$$

thus,

$$\mathrm{tr}\left(\widehat{A}UU^t\right) = \pi_n(\widehat{A}) = \pi_n(A).$$

The equality case of von Neumann's trace inequality occurs. Therefore, the diagonalizable matrices UU^t and $\widehat{A}$ are simultaneously orthogonally diagonalizable and thereby commute. This implies that UU^t and $\frac{1}{2}\big(A(i,j;\epsilon_\star) + A(j,i;\epsilon_\star)\big)$ commute; as a result, we get that

$$\begin{aligned}
\langle r_i, r_i\rangle &= \langle r_j, r_j\rangle, \\
\langle r_i, r_k\rangle &= 0, \qquad \text{for all } k \neq i \text{ with } k \in \{1, \ldots d\}, \\
\langle r_j, r_k\rangle &= 0, \qquad \text{for all } k \neq j \text{ with } k \in \{1, \ldots d\}.
\end{aligned}$$

By applying the same argument to $\langle r_i, r_k\rangle = 0$ for every $k \neq i, k \in \{1, \ldots, d\}$, we may conclude that the vectors $r_1, \ldots, r_d \in \mathbb{R}^n$ are orthogonal and none of them is equal to the zero vector. However, this is only possible if $n = d$. Therefore, we have shown for $d > n$ that $\langle r_i, r_j\rangle \neq 0$ for all integers $1 \leq i, j \leq d$.

We claim that

$$a_{ij} = \mathrm{sgn}\left(\langle r_i, r_j\rangle_{\mathbb{R}^n}\right) \tag{3.44}$$

for all $1 \leq i, j \leq d$. Because $\langle r_i, r_j\rangle \neq 0$, this is a direct consequence of the maximality of $\pi_n(A)$ and equality (3.43). Hence, we have shown that A and UU^t have the same sign pattern, which allows us to invoke [CW13, Lemma 2.1]. From this result we see that A does not have a principal $(n+2) \times (n+2)$-submatrix which has only -1 as off-diagonal elements. Such a matrix is the Seidel adjacency matrix of the complete graph on $n+2$ vertices. For that reason, we have shown that

$$\frac{1}{d}\pi_n(A) = \max\left\{\frac{n}{d} + \frac{1}{d}\sum_{k=1}^{n}\lambda_k(T) : T \text{ is a } K_{n+2}\text{-free two-graph of order } d\right\}.$$

This completes the proof. $\square$

We conclude this section with the proof of Corollary 1.16.

Proof of Corollary 1.16. Let $J_2 \in \mathcal{A}_2$ denote the all-ones matrix. For every $A \in \mathcal{A}_d$, the matrix $A \otimes J_2$ is contained in $\mathcal{A}_{2d}$, where $\otimes$ denotes the Kronecker product of matrices. Moreover, since the eigenvalues of $A \otimes J_2$ are precisely all possible products of an eigenvalue of A (counted with multiplicity) and an eigenvalue of J_2 (counted with multiplicity), it is readily verified that

$$\frac{\pi_n(A)}{d} = \frac{\pi_n(A \otimes J_2)}{2d}.$$

Let $(\epsilon_\ell)_{\ell \geq 1}$ be a sequence of positive real numbers that converges to zero. Due to Theorem 1.14 and the above, there exists a strictly increasing sequence $(d_\ell)_{\ell \geq 1}$ of integers and matrices $A_\ell \in \mathcal{A}_{d_\ell}$ such that

$$\Pi_n \leq \frac{\pi_n(A_\ell)}{d_\ell} + \epsilon_\ell.$$

We have

$$\pi_n(A_\ell) = d_\ell - \sum_{k=n+1}^{d_\ell} \lambda_k(A_\ell) = d_\ell + \sum_{k=1}^{d_\ell - n} \lambda_k(-A_\ell);$$

thus,

$$\pi_n(A_\ell) = d_\ell + \sum_{k=1}^{d_\ell - n} \lambda_k(\overline{A_\ell}) - (d_\ell - n)2,$$

where $\overline{A_\ell} = 2\mathbb{1}_{d_\ell} - A_\ell$. Consequently,

$$\Pi_n \leq \frac{2n}{d_\ell} - 1 + \frac{\pi_{d_\ell - n}(\overline{A_\ell})}{d_\ell} + \epsilon_\ell.$$

Since $\overline{A_\ell} \in \mathcal{A}_{d_\ell}$, we obtain

$$\Pi_n \leq \frac{2n}{d_\ell} - 1 + \Pi(d_\ell - n, d_\ell) + \epsilon_\ell.$$

Proposition 2 in [FS17] tells us that

$$\Pi(d - n, d) \leq \Pi_n + 1$$

for all $d \geq 1$. Thus,

$$\Pi_n \leq \frac{2n}{d_\ell} - 1 + \Pi(d_\ell - n, d) + \epsilon_\ell \leq \Pi_n + \frac{2n}{d_\ell} + \epsilon_\ell;$$

for that reason, the desired result follows. $\square$

3.8 Polyhedral maximizers of $\Pi_n(\cdot)$

3.8.1 – Let $(E, \|\cdot\|)$ be a Banach space and let $V \subset E$ and $F \subset E^*$ denote linear subspaces. We set

$$V^0 := \left\{ \ell \in E^* \,:\, \ell(v) = 0 \text{ for all } v \in V \right\} \subset E$$

and

$$F_0 := \left\{ x \in E \,:\, f(x) = 0 \text{ for all } f \in F \right\} \subset E^*.$$

Suppose that $U \subset E$ is a linear subspace such that $E = V \oplus U$. The map

$$P_V^U : E \to V, \quad v + u \mapsto v$$

is a linear projection onto V. In the subsequent lemma we gather classical results from functional analysis.

Lemma 3.21. *Let $(E, \|\cdot\|)$ be a Banach space.*

1. *If there exist closed linear subspaces $V, U \subset E$ such that V is finite-dimensional and $E = V \oplus U$, then $E^* = V^0 \oplus U^0$, $\dim(U^0) = \dim(V)$,*

$$(V^0)_0 = V \text{ and } (U^0)_0 = U.$$

2. *If there exist closed linear subspaces $F, G \subset E^*$ such that F is finite-dimensional and $E^* = F \oplus G$, then $E = F_0 \oplus G_0$, $\dim(G_0) = \dim(F)$,*

$$(F_0)^0 = F \text{ and } (G_0)^0 = G.$$

3. *If there exist closed linear subspaces $V, U \subset E$ such that V is finite-dimensional and $E = V \oplus U$, then*

$$\|P_V^U\| = \|P_{U^0}^{V^0}\|.$$

Proof. The first two items follow from elementary properties of the annihilator of a linear subspace. The third item is a straightforward computation. □

It is worth to point out that [CL14, Theorem 3.2] may be established via the first and the third item of Lemma 3.21.

3.8.2 – The following theorem translates the calculation of relative projection constants to second preduals (if such a space exists).

Theorem 3.22. *Let $(E, \|\cdot\|)$ be a Banach space and let $F \subset E$ denote a finite-dimensional linear subspace. If $(X, \|\cdot\|)$ is a Banach space such that $E = X^{**}$, then there exist a linear subspace $V \subset X$ with $\dim(V) = \dim(F)$ and*

$$\Pi(F, E) = \Pi(V, X).$$

Proof. It is not hard to check that

$$\Pi(F, E) := \inf \left\{ \|P_F^G\| : E = F \oplus G, G \subset E \text{ closed linear subspace } \right\}.$$

We set $V := (F_0)_0$. On the one hand, using the second and third item of Lemma 3.21, we obtain

$$\Pi(V, X) \leq \Pi(F, E);$$

on the other hand, using the first and third item of Lemma 3.21, we infer

$$\Pi(F, E) \leq \Pi(V, X).$$

This completes the proof. □

We conclude this section with the proof of Theorem 1.17.

Proof of Theorem 1.17. Let $F \subset \ell_\infty$ be an n-dimensional linear subspace with

$$\Pi_n = \Pi(F, \ell_\infty).$$

Via Theorem 3.22, there exists an n-dimensional linear subspace $V \subset c_0$ such that

$$\Pi(F, \ell_\infty) = \Pi(V, c_0).$$

As $\Pi(V, c_0) \leq \Pi(V)$, we get

$$\Pi_n = \Pi(V).$$

This completes the proof, since due to a result of Klee, cf. [Kle60, Proposition 4.7], every finite-dimensional subspace of c_0 is polyhedral. □

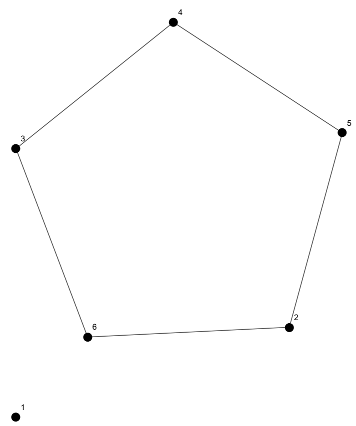

Figure 3.1: The graph that has $A_6 - \mathbb{1}_6$ as Seidel adjacency matrix.

3.9 Computation of Π_2

3.9.1 – Let $n \geq 1$ be an integer, let $\mathcal{R}_{2n+1} \subset \mathbb{R}^2$ be a regular $(2n+1)$-gon centred at the origin and let $V(\mathcal{R}_{2n+1})$ denote the vertices of $\mathcal{R}_{2n+1}$. Further, we let $\mathcal{T}_{2n+1}$ denote the two-graph that has $V(\mathcal{R}_{2n+1})$ as vertex set and $\{v_1, v_2, v_3\} \subset V(\mathcal{R}_{2n+1})$ is an edge if and only if the origin is contained in the closed convex hull of v_1, v_2, v_3. It is readily verified that $\delta(R_{2n+1} - \mathbb{1}_{2n+1}) = \mathcal{T}_{2n+1}$ for

$$R_{2n+1} := \begin{pmatrix} 1 & j^t & -j^t \\ j & J_n & J_n - 2L_n \\ -j & J_n - 2L_n^t & J_n \end{pmatrix},$$

where $j \in \mathbb{R}^n$ is the all-ones vector, J_n is the all-ones $n \times n$-matrix and L_n is the $n \times n$-matrix given by

$$(L_n)_{ij} := \begin{cases} -1 & i > j \\ 0 & \text{otherwise} . \end{cases}$$

Note that L_n has only -1's below the diagonal and only 0's above the first sub-diagonal. We set

$$A_6 := \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & 1 & 1 \end{pmatrix}.$$

One can check that $A_6 - \mathbb{1}_6$ is the Seidel adjacency matrix of the graph depicted in Figure 3.1. We abbreviate

$$\Omega := \big\{ S : S \text{ is a principal submatrix of } A_6 - \mathbb{1}_6 \big\} \cup \big\{ R_{2n+1} - \mathbb{1}_{2n+1} : n \geq 1 \big\}.$$

In [FF84], Frankl and Füredi showed that each non-empty K_4-free two-graph belongs to the set

$$\delta(\Omega) \cup \big\{\delta(B) : \text{ B is a blow-up of a matrix in } \Omega\big\}.$$

3.9.2 – Given a matrix $A \in \mathcal{A}_d$, we denote by $\mathrm{Stab}(A)$ the set

$$\big\{Q \in O_d(\mathbb{Z}) : A = QAQ^t\big\}.$$

We use $O_d(\mathbb{Z})$ to denote the group of orthogonal $d \times d$-matrices with integer entries. Every $Q \in \mathrm{Stab}(A)$ has a unique decomposition $Q = PD$, where P is a permutation matrix and D is a diagonal matrix consisting only of 1's and -1's. We write $P_\tau := P$ if the permutation matrix P is associated to the permutation τ, that is, $P_{ij} = (e_{\tau(i)})_j$. The group $\mathrm{Stab}(A)$ acts on $\{1, \dots, d\}$ via

$$(P_\tau D, k) \mapsto \tau(k).$$

Two Seidel adjacency matrices S_f and S_g are called *switching equivalent* if $\delta(f) = \delta(g)$. This gives rise to an equivalence relation, equivalence classes are called switching classes. The lemma below tells us that the orbit decomposition of the action $\mathrm{Stab}(A) \curvearrowright \{1, \dots, d\}$ may be obtained by determining the switching class of every principal $(d-1)-$dimensional submatrix of A.

Lemma 3.23. *Let $A \in \mathcal{A}_d$ be a matrix, let $1 \leq i, j \leq d$ be two integers and for $k = i, j$ let T_k denote the submatrix of $A - \mathbb{1}_d$ obtained by deleting the k-th column and the k-th row of $A - \mathbb{1}_d$.*

Then, the matrices T_i and T_j are switching equivalent if and only if the integers i and j lie in the same orbit under the action $\mathrm{Stab}(A) \curvearrowright \{1, \dots, d\}$.

Proof. This is a straightforward consequence of the definitions. $\square$

Let M be a diagonalizable $d \times d$-matrix over the real numbers. We set

$$\pi_n(M) := \sum_{k=1}^{n} \lambda_k(M)$$

for each integer $1 \leq n \leq d$. The following lemma simplifies the calculation of the maximum value of the function $D \mapsto \pi_n(AD)$ if the action $\mathrm{Stab}(A) \curvearrowright \{1, \dots, d\}$ is transitive.

Lemma 3.24. *Let $A \in \mathcal{A}_d$ be a matrix and let $1 < n \le d$ be an integer. If $\Lambda \in \mathcal{D}_d$ is a invertible matrix such that*

$$\pi_n(A\Lambda) = \max_{D \in \mathcal{D}_d} \pi_n(AD),$$

then

$$Q^2 \Lambda (Q^2)^t = \Lambda$$

for all $Q \in \mathrm{Stab}(A)$. In particular, if d is odd and the action $\mathrm{Stab}(A) \curvearrowright \{1, \ldots, d\}$ is transitive, then $\Lambda = \frac{1}{d}\mathbb{1}_d$.

Proof. For each $Q \in \mathrm{Stab}(A)$ we have

$$Q^t \sqrt{\Lambda} A \sqrt{\Lambda} Q = Q^t \sqrt{\Lambda} Q A Q^t \sqrt{\Lambda} Q = \sqrt{\Lambda_{Q^t}} A \sqrt{\Lambda_Q},$$

where $\Lambda_Q := Q \Lambda Q^t$. Consequently,

$$\pi_n(\sqrt{\Lambda} A \sqrt{\Lambda}) = \pi_n\left(\sqrt{\Lambda_{Q^t}} A \sqrt{\Lambda_Q}\right) = \pi_n\left(A \sqrt{\Lambda_Q}\sqrt{\Lambda_{Q^t}}\right).$$

Thus, using that Λ is a maximizer, we get

$$1 \le \mathrm{tr}\left(\sqrt{\Lambda_Q}\sqrt{\Lambda_{Q^t}}\right).$$

Via the Cauchy-Schwarz inequality, we deduce

$$\mathrm{tr}\left(\sqrt{\Lambda_Q}\sqrt{\Lambda_{Q^t}}\right) \le 1;$$

as a result, there exists a real number $\alpha \ge 0$ such that

$$\Lambda_Q = \alpha \Lambda_{Q^t}.$$

Since $\mathrm{tr}(\Lambda_Q) = \mathrm{tr}(\Lambda_{Q^t}) = 1$, we get $\alpha = 1$ and thus

$$\Lambda_Q = \Lambda_{Q^t},$$

which is equivalent to

$$\Lambda_{Q^2} = \Lambda.$$

Now, suppose that d is odd and assume that the action $\mathrm{Stab}(A) \curvearrowright \{1, \ldots, d\}$ is transitive. We claim that $\Lambda = \frac{1}{d}\mathbb{1}_d$. The statement follows via elementary group theory. Indeed, let H denote the subgroup of $\mathrm{Stab}(A)$ generated by the squares. By basic algebra, H is normal and the action of $\mathrm{Stab}(A)/H$ on the orbits of $H \curvearrowright \{1, \ldots, d\}$ is transitive. Since $|\mathrm{Stab}(A)/H| = 2^k$ for some integer $k \ge 0$, the action $H \curvearrowright \{1, \ldots, d\}$ has either one orbit or an even number of orbits. Because d is odd and the orbits of $H \curvearrowright \{1, \ldots, d\}$ all have the same cardinality, we may conclude that $H \curvearrowright \{1, \ldots, d\}$ is transitive. This completes the proof. $\square$

3.9.3 – In the following we retain the notation from the first paragraph of this section. By the use of Theorem 1.14, Lemma 3.20 and the classification of all K_4-free two-graphs, we obtain

$$\Pi_2 = \max_{(A-\mathbb{1})\in\Omega} \max_{D\in\mathcal{D}_d} \pi_2(AD).$$

Clearly, all induced sub-graphs of $\mathcal{T}_{2n+1}$ that are obtained by deleting one vertex are isomorphic (as two-graphs) to each other. Thus, via Lemma 3.23 and Lemma 3.24, we get that

$$\max_{D\in\mathcal{D}_d} \pi_2\left(R_{2n+1}D\right) = \pi_2\left(\tfrac{1}{2n+1}R_{2n+1}\right).$$

Moreover, if B is a principal submatrix of A_6, then it is not hard to see that

$$\max_{D\in\mathcal{D}_d} \pi_2(BD) \leq \max\left\{\pi_2\left(\tfrac{1}{5}R_5\right), \pi_2\left(\tfrac{1}{3}R_3\right)\right\};$$

thereby,

$$\Pi_2 = \max_{n\geq 1} \pi_2\left(\tfrac{1}{2n+1}R_{2n+1}\right).$$

Thus, we are left to consider the eigenvalues of the matrices R_{2n+1} for $n \geq 1$. Due to the following lemma it suffices to calculate the eigenvalues of R_3.

Lemma 3.25. *Let $n' \geq n \geq 1$ be integers. It holds*

$$\pi_2\left(\tfrac{1}{2n+1}R_{2n+1}\right) \geq \pi_2\left(\tfrac{1}{2n'+1}R_{2n'+1}\right). \tag{3.45}$$

Proof. We abbreviate $N := 2n+1$. Let R'_N denote the $2n \times 2n$-matrix that is obtained from R_N by deleting the second row and second column. Clearly, R'_N is a blow-up of R_{N-2}; thus, via Lemma 3.20, we obtain

$$\max_{D\in\mathcal{D}_{2n}} \pi_2(R'_N D) \leq \pi_2(R_{N-2}).$$

If for all integers $k \geq 1$

$$\pi_2\left(R_{2k+1}\right) = 2\lambda_1\left(R_{2k+1}\right), \tag{3.46}$$

then

$$\pi_2\left(R_N\tfrac{1}{N}\right) = 2\lambda_1\left(R'_N\tfrac{1}{N-1}\right)\tfrac{N-1}{N} \leq 2\lambda_1\left(R_{N-2}\tfrac{1}{N-2}\right)\tfrac{N-1}{N}$$

and thus (3.45) follows. We are left to show that (3.46) holds.

Suppose that $\lambda_1(R_N)$ has multiplicity one. Below, we show that this leads to a contradiction.

Let $x \in \mathbb{R}^N$ be an eigenvector of R_N associated to the eigenvalue $\lambda_1(R_N)$. As we assume that $\lambda_1(R_N)$ has multiplicity one, we get $Qx = x$ or $Qx = -x$ for each $Q \in$

$\mathrm{Stab}(R_N)$. We know that the action $\mathrm{Stab}(R_N) \curvearrowright \{1, \ldots, N\}$ is transitive; thus all entries of x differ only by a sign. Without loss of generality we may suppose the entries of x consist only of 1's and -1's. For each integer $1 \leq i \leq N$ let A_i denote the matrix that is obtained from R_N by replacing the i-th column with x. Cramers rule tells us that

$$x_i \det(R_N) = \det(A_i)$$

for all $1 \leq i \leq N$. It is easy to see (via the definition of R_N) that for all $n < i < N$: if x_{i-n+1} and x_{i-n} have the same sign, then $\det(A_i) = 0$. But this is impossible; for that reason, for all $n < i < N$ we have $x_{i-n} = -x_{i-n+1}$. Similarly,

$$x_{i+n} = x_{i+n-1}$$

for all $2 < i \leq n+1$ and $x_1 = -x_{n+1}$, $x_2 = -x_N$. Thus, if we suppose that $x_1 = 1$, then

$$x = (1, \underbrace{-1, 1, -1, 1, \ldots 1, -1}_{n \text{ times}}, \underbrace{1, -1, 1, -1, \ldots, -1, 1}_{n \text{ times}}) \quad \text{if } n \text{ is odd}$$

and

$$x = (1, \underbrace{1, -1, 1, -1, \ldots, 1, -1}_{n \text{ times}}, \underbrace{1, -1, 1, -1, \ldots, 1, -1}_{n \text{ times}}) \quad \text{if } n \text{ is even.}$$

Therefore, if $j \in \mathbb{R}^N$ denotes the all-ones vector we obtain

$$\langle x, j \rangle = 1,$$

and consequently it holds that

$$\lambda_1(R_N) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even.} \end{cases}$$

This is a contradiction, since $\mathrm{tr}(R_N) = N$ and we assume that $\lambda_1(R_N)$ has multiplicity one. Hence, we have shown that the eigenvalue $\lambda_1(R_N)$ has multiplicity greater than or equal to two. As a result, (3.46) holds, which was left to show. This completes the proof. $\square$

Employing Lemma 3.25, we get

$$\Pi_2 = \pi_2\left(\tfrac{1}{3}R_3\right) = \frac{1}{3}\big(2\lambda_1(R_3)\big) = \frac{1}{3}\big(3 - \lambda_3(R_3)\big) = \frac{4}{3},$$

as conjectured by Grünbaum.

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