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Cardinal Characteristics of the Continuum Based on Asymptotic Density

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German Abstract

Eine Kardinalzahlcharakteristik des Kontinuums ist eine Kardinalzahl echt grösser als $\aleph_0$, die Kardinalität der natürlichen Zahlen, und kleinder als $2^{\aleph_0}$, die Kardinalität des Kontinuums (das heisst der reellen Zahlen). Im Allgemeinen sind wir an solchen Kardinalzahlcharakteristiken interessiert, die konsistent strikt kleiner als $2^{\aleph_0}$ sein können.

Asymptotische Dichte ist ein Konzept aus der Zahlentheorie, das verwendet wird, um die Grösse gewisser Teilmengen der natürlichen Zahlen zu beschreiben. Deswegen wird sie im Englischen zum Teil auch als "natural density" bezeichnet.

Zunächst führen wir verschiedene Kardinalzahlcharakteristiken im Zusammenhang mit der independence number $\mathfrak{i}$, der reaping number $\mathfrak{r}$ und der splitting number $\mathfrak{s}$ ein und verwenden dabei asymptotische Dichte, um verschiedene Durchschnittseigenschaften von unendlichen Mengen zu charakterisieren.

Eine dieser Charakteristiken, $\mathfrak{s}_{1/2}$, taucht auf, wenn wir die Relation "splitting" zu "im Limes halbieren" verstärken. Wir zeigen verschiedene untere und obere Schranken sowie Konsistenzresultate, zum Beispiel die Konsistenz von $\mathfrak{s} < \mathfrak{s}_{1/2}$ und $\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N})$, sowie verschiedene Resultate über mögliche Werte von $\mathfrak{i}_{1/2}$. Abschliessend diskutieren wir kurz Kardinalzahlcharakteristiken im Zusammenhang mit der homogeneity number $\mathfrak{hom}$ und der partition number $\mathfrak{par}$, indem wir das Konzept der asymptotische Dichte auf Färbungen von Mengen ausdehnen.

Diese Arbeit basiert auf [BHK+18]. Es werden sowohl bekannte Resultate klarer präsentiert als auch neue Erkenntnisse vorgestellt.

Cardinal Characteristics of the Continuum Based on Asymptotic Density

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Abstract

A cardinal characteristic of the continuum is a cardinal number strictly greater than $\aleph_0$, the cardinality of the natural numbers, and less than $2^{\aleph_0}$, the cardinality of the continuum (i. e. the real numbers). In general, we are only interested in those cardinal characteristics of the continuum that consistently are strictly less than $2^{\aleph_0}$.

Asymptotic density is a concept from number theory used to describe how large certain subsets of the natural numbers are. Therefore, it is also referred to as natural density.

We start by introducing several cardinal characteristics related to the independence number $\mathfrak{i}$, the reaping number $\mathfrak{r}$ and the splitting number $\mathfrak{s}$ by using the notion of asymptotic density to characterise different properties of intersections of infinite sets.

One of the new characteristics, $\mathfrak{s}_{1/2}$, arises when refining the notion of splitting to what we call bisection in the limit. We prove several bounds and consistency results, e. g. the consistency of $\mathfrak{s} < \mathfrak{s}_{1/2}$ and $\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N})$, as well as several results about possible values of $\mathfrak{i}_{1/2}$. Later, we briefly discuss cardinal characteristics related to the homogeneity number $\mathfrak{hom}$ and the partition number $\mathfrak{par}$ by extending the concept of asymptotic density to colourings of sets.

This thesis is an extension of [BHK+18], both presenting several of the old results more clearly and introducing new findings.

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Contents

Chapter 1

Introduction

This research forms part of the study of cardinal characteristics of the continuum. A cardinal characteristic of the continuum is a cardinal number strictly greater than $\aleph_0$, the cardinality of the natural numbers denoted by ω, and less than $2^{\aleph_0}$, the cardinality of the continuum (i. e. the real numbers). For a general overview of cardinal characteristics, see [Bla10], [Hal17, chapter 9] and [Vau90] as well as [BJ95].

Consider the following well-known cardinal characteristics:

- $\mathfrak{s} := \min\{|\mathcal{S}| \mid \mathcal{S} \subseteq [\omega]^\omega \text{ and } \forall\, X \in [\omega]^\omega\ \exists\, S \in \mathcal{S}\colon |X \cap S| = |X \setminus S| = \aleph_0\}$ (the splitting number),
- $\mathfrak{r} := \min\{|\mathcal{R}| \mid \mathcal{R} \subseteq [\omega]^\omega \text{ and } \nexists\, X \in [\omega]^\omega\ \forall\, R \in \mathcal{R}\colon |R \cap X| = |R \setminus X| = \aleph_0\}$ (the reaping number), and
- $\mathfrak{i} := \min\{|\mathcal{I}| \mid \mathcal{I} \subseteq [\omega]^\omega, \forall\, \mathcal{A} \cup \mathcal{B} \in \operatorname{fin}(\mathcal{I})\colon \left|\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)\right| = \aleph_0 \text{ and } \mathcal{I} \text{ is maximal}\}$ (the independence number),

We defined specialised variants of these (all of them related in some way to asymptotic density, in particular asymptotic density $1/2$) and obtained a number of bounds and consistency results for them.

We use the standard notation. In addition to $\mathfrak{s}$, $\mathfrak{r}$ and $\mathfrak{i}$ mentioned above, we will refer to a few other well-known cardinal characteristics.

Given an ideal $\mathcal{I}$ on some base set X, we can define four cardinal characteristics:

- the additivity number $\operatorname{add}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\}$,
- the covering number $\operatorname{cov}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} = X\}$,
- the uniformity number $\operatorname{non}(\mathcal{I}) := \min\{|Y| \mid Y \subseteq X \text{ and } Y \notin \mathcal{I}\}$, and
- the cofinality $\operatorname{cof}(\mathcal{I}) := \min\{|\mathcal{A}| \mid \mathcal{A} \subseteq \mathcal{I} \text{ and } \forall\, B \in \mathcal{I}\ \exists\, A \in \mathcal{A}\colon B \subseteq A\}$.

1.1 Asymptotic Density

Recall the following concepts from number theory.

Definition 1.1.1. For $X \in [\omega]^\omega$ and $0 < n < \omega$, define the *initial density* (of X up to n) as

$$d_n(X) := \frac{|X \cap n|}{n},$$

and the *lower* and *upper density* of X as

$$\underline{d}(X) := \liminf_{n\to\infty}(d_n(X)) \quad \text{and} \quad \bar{d}(X) := \limsup_{n\to\infty}(d_n(X)),$$

respectively. In case of convergence of $d_n(X)$, call

$$d(X) := \lim_{n\to\infty}(d_n(X))$$

the *asymptotic density* or just the *density* of X.

We define the following relations on $[\omega]^\omega \times [\omega]^\omega$.

Definition 1.1.2. Let $S, X \in [\omega]^\omega$, $\rho \in (0,1)$. We say:

- S *splits* X, written as $S \mid X$, *iff* both $S \cap X$ and $X \setminus S$ are infinite sets.
- S *bisects* X *in the limit* (or just S *bisects* X), written as $S \mid_{1/2} X$, *iff*

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

- S *ρ-splits* X *in the limit* (or just S *ρ-splits* X), written as $S \mid_\rho X$, *iff*

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \rho.$$

We prove some simple facts about densities of sets.

Lemma 1.1.3. *Given $X, Y \in [\omega]^\omega$ such that both X and $(X \cap Y)$ have a density, we have*

$$d(X \setminus Y) = d(X) - d(X \cap Y).$$

If additionally $Y \subseteq X$, this simplifies to $d(X \setminus Y) = d(X) - d(Y)$. For $X = \omega$, in particular, we get $d(\omega \setminus Y) = 1 - d(Y)$.

Proof. We see

$$\frac{|(X \setminus Y) \cap n|}{n} = \frac{|X \cap n|}{n} - \frac{|(X \cap Y) \cap n|}{n},$$

and taking the limit for n to ∞ on both sides of the equation yields the claimed identity. □

Lemma 1.1.4 (Intermediate Value Theorem for Density $1/2$). *Given a set $X \in [\omega]^\omega$ such that there are $m, n \in \omega$ with $m < n$ and $d_m(X) < 1/2 < d_n(X)$, there is a $k \in (m, n)$ such that $d_k(X) = 1/2$.*

Proof. Let $k \in [m, n]$ be maximal such that $d_k(X) \leq 1/2$ and let $\ell := k + 1$. By our assumptions, $k < n$ and $\ell \leq n$. Assume towards a contradiction that $d_k(X) < 1/2$. This means we have

$$d_k(X) \leq \frac{\lceil k/2 \rceil - 1}{k},$$

but then

$$d_\ell(X) \leq \frac{\lceil k/2 \rceil}{k+1} \leq \frac{1}{2},$$

contradicting the maximality of k. Hence $d_k(X) = 1/2$. □

Corollary 1.1.5. *Given sets $Y, Z \in [\omega]^\omega$ such that there are $m < n$ in ω with*

$$\frac{d_m(Y \cap Z)}{d_m(Z)} < \frac{1}{2} < \frac{d_n(Y \cap Z)}{d_n(Z)},$$

there is a k in (m, n) such that

$$\frac{d_k(Y \cap Z)}{d_k(Z)} = \frac{1}{2},$$

Proof. The proof is analogous to the proof of Lemma 1.1.4.

Alternatively, if χ_Y is the characteristic function of Y and f_Z is the enumeration of Z, let X be the set with characteristic function $\chi_Y \circ f_Z$,. Then, for all $n \in \omega$,

$$d_n(X) = \frac{d_{f_Z(n)}(Y \cap Z)}{d_{f_Z(n)}(Z)},$$

and applying Lemma 1.1.4 to X also proves the claim. □

Lemma 1.1.6. *Given $X, Y, Z \in [\omega]^\omega$ and $\rho, \tau \in (0, 1)$, the following implication holds:*

$$X \mid_\rho (Y \cap Z) \quad \text{and} \quad Y \mid_\tau Z \quad \Longrightarrow \quad (X \cap Y) \mid_{\rho\tau} Z\,.$$

Proof. We can calculate

$$\frac{|(X \cap Y) \cap Z \cap n|}{|Z \cap n|} = \frac{|X \cap (Y \cap Z) \cap n|}{|(Y \cap Z) \cap n|} \cdot \frac{|Y \cap Z \cap n|}{|Z \cap n|}\,.$$

Thus, assuming $X \mid_\rho (Y \cap Z)$ and $Y \mid_\tau Z$, and taking the limit on both sides yields

$$\lim_{n \to \infty} \frac{|(X \cap Y) \cap Z \cap n|}{|Z \cap n|} = \rho \cdot \tau\,,$$

which proves $(X \cap Y) \mid_{\rho\tau} Z$. □

1.2 Tukey Connections

The following is an adapted and extended version of the corresponding definition found in [KM18].

For notational simplicity, we describe the cardinal characteristics we are interested in, whenever possible, through relational systems as below.

Definition 1.2.1. A *relational system* is a triplet $\mathbf{R} := \langle X, Y, \sqsubset \rangle$ where $\sqsubset$ is a relation contained in $X \times Y$. The *cardinal characteristics associated with* $\mathbf{R}$ are

- $\mathfrak{b}(\mathbf{R}) := \min\{|B| \mid B \subseteq X \text{ and } \nexists\, y \in Y\ \forall x \in B \colon x \sqsubset y\}$,
- $\mathfrak{d}(\mathbf{R}) := \min\{|D| \mid D \subseteq Y \text{ and } \forall x \in X\ \exists y \in D \colon x \sqsubset y\}$.

We call a family $B \subseteq X$ with $\nexists\, y \in Y\ \forall x \in B \colon x \sqsubset y$ an $\mathbf{R}$-*unbounding* family and a family $D \subseteq Y$ with $\forall x \in X\ \exists y \in D \colon x \sqsubset y$ an $\mathbf{R}$-*dominating* family.

The *dual of* $\mathbf{R}$ is the relational system $\mathbf{R}^\perp := \langle Y, X, \not\sqsupset \rangle$.

Let $\mathbf{R}' := \langle X', Y', \sqsubset' \rangle$ be another relational system.

- A pair (F, G) is a *Tukey connection from* $\mathbf{R}$ *to* $\mathbf{R}'$ *iff* $F \colon X \to X'$, $G \colon Y' \to Y$ and for any $x \in X$ and $y' \in Y'$, $F(x) \sqsubset' y'$ implies $x \sqsubset G(y')$.
- When there exists a Tukey connection from $\mathbf{R}$ to $\mathbf{R}'$, we say that $\mathbf{R}$ *is Tukey-below* $\mathbf{R}'$, which is denoted by $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$.
- We say that $\mathbf{R}$ and $\mathbf{R}'$ are *Tukey-equivalent*, denoted by $\mathbf{R} \cong_{\mathrm{T}} \mathbf{R}'$, *iff* $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ and $\mathbf{R}' \preceq_{\mathrm{T}} \mathbf{R}$.

Note that $\mathbf{R} \preceq_{\mathrm{T}} \mathbf{R}'$ implies that $\mathfrak{d}(\mathbf{R}) \leq \mathfrak{d}(\mathbf{R}')$ and $\mathfrak{b}(\mathbf{R}') \leq \mathfrak{b}(\mathbf{R})$. Also, $\mathfrak{b}(\mathbf{R}^\perp) = \mathfrak{d}(\mathbf{R})$ and $\mathfrak{d}(\mathbf{R}^\perp) = \mathfrak{b}(\mathbf{R})$. In this section, we will use such Tukey connections to prove inequalities between cardinal invariants.

Finally, we will refer to two more cardinal characteristics:

- $\mathfrak{b} := \min\{|B| \mid B \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega\ \exists f \in B \colon f \not\leq^* g\}$ (the unbounding number) and
- $\mathfrak{d} := \min\{|D| \mid D \subseteq \omega^\omega \text{ and } \forall g \in \omega^\omega\ \exists f \in D \colon g \leq^* f\}$ (the dominating number).

Example 1.2.2. We give two examples of relational systems.

(1) Let $\mathbf{Dm} := \langle \omega^\omega, \omega^\omega, \leq^* \rangle$. Then $\mathfrak{b} := \mathfrak{b}(\mathbf{Dm})$ (the unbounding number) and $\mathfrak{d} := \mathfrak{d}(\mathbf{Dm})$ (the dominating number).

(2) Let $\mathcal{I}$ be an ideal in $\mathscr{P}(X)$, i. e. a family of subsets of a set X that satisfies
  (i) $\mathrm{fin}(X) \subseteq \mathcal{I}$,
  (ii) $X \notin \mathcal{I}$, and
  (iii) whenever $Y \in \mathcal{I}$ and $X \subseteq Y$, then $X \in \mathcal{I}$.
  Consider the relational systems $\mathbf{Cf}(\mathcal{I}) := \langle \mathcal{I}, \mathcal{I}, \subseteq \rangle$ and $\mathbf{Cv}(\mathcal{I}) := \langle X, \mathcal{I}, \in \rangle$. Note that

$$\mathfrak{b}(\mathbf{Cf}(\mathcal{I})) = \mathrm{add}(\mathcal{I}), \qquad \mathfrak{d}(\mathbf{Cf}(\mathcal{I})) = \mathrm{cof}(\mathcal{I}),$$
$$\mathfrak{b}(\mathbf{Cv}(\mathcal{I})) = \mathrm{non}(\mathcal{I}), \qquad \mathfrak{d}(\mathbf{Cv}(\mathcal{I})) = \mathrm{cov}(\mathcal{I}),$$

which are the *cardinal invariants associated with* $\mathcal{I}$. Now, both $\mathbf{Cv}(\mathcal{I}) \preceq_{\mathrm{T}} \mathbf{Cf}(\mathcal{I})$ and $\mathbf{Cv}(\mathcal{I})^{\perp} \preceq_{\mathrm{T}} \mathbf{Cf}(\mathcal{I})$, so the well-known fact that $\mathrm{add}(\mathcal{I})$ is below both $\mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I})$ which in turn are below $\mathrm{cof}(\mathcal{I})$ is easily proved through the relational systems.
If $\mathcal{J}$ is another ideal in $\mathscr{P}(X)$ and $\mathcal{I} \subseteq \mathcal{J}$, then $\mathbf{Cv}(\mathcal{J}) \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{I})$, so $\mathrm{cov}(\mathcal{J}) \leq \mathrm{cov}(\mathcal{I})$ and $\mathrm{non}(\mathcal{I}) \leq \mathrm{non}(\mathcal{J})$.

In particular, we will refer to these cardinal characteristics for

- the ideal $\mathcal{N} := \{A \subseteq 2^\omega \mid \lambda(A) = 0\}$ of *Lebesgue null sets* and
- the ideal $\mathcal{M} := \{A \subseteq 2^\omega \mid A = \bigcup_{n<\omega} A_n \text{ and } \forall\, n < \omega\colon A_n \text{ nowhere dense}\}$ of *meagre sets*.

We will use the following concept in a few of the proofs:

Definition 1.2.3. A *chopped real* is a pair (x, Π) where $x \in 2^\omega$ and Π is an interval partition of ω. We say a real $y \in 2^\omega$ *matches* (x, Π) if $y{\restriction}_I = x{\restriction}_I$ for infinitely many $I \in \Pi$.

We note that the set $\mathrm{Match}(x, \Pi)$ of all reals matching (x, Π) is a comeagre set (see [Bla10, Theorem 5.2]).

We remark that we will not rigidly distinguish between a real r in 2^ω and the set $R := r^{-1}(1) \in \mathscr{P}(\omega)$, or conversely, between a subset of ω and its characteristic function.

Chapter 2

Characteristics Related to $\mathfrak{r}$ and $\mathfrak{s}$

We introduce characteristics related to $\mathfrak{r}$ and $\mathfrak{s}$. The characteristics related to $\mathfrak{s}$ were defined and studied in [BHK+18]. Some of the results about the dual characteristics related to $\mathfrak{r}$ were independently shown by Barnabás Farkas in correspondence with the authors.

In section 2.1 we unified these results using Tukey connections and were able to prove new ones. Further characteristics from [BHK+18] and their duals will be discussed in section 4.1.

In section 2.2 we introduce a model witnessing the consistency of $\mathfrak{s}_{1/2} > \mathrm{cof}(\mathcal{M})$.

Finally, section 2.3 remains as seen in [BHK+18].

2.1 Definitions and Bounds

Based on the relations in Definition 1.1.2 (or, more precisely, on their negations) we can define the following relational systems:

Definition 2.1.1.

- Let $\mathbf{Rp} := \langle [\omega]^\omega, [\omega]^\omega, \not\vert \rangle$. Then

$$\mathfrak{b}(\mathbf{Rp}) = \mathfrak{s}, \quad \mathfrak{d}(\mathbf{Rp}) = \mathfrak{r}.$$

- Let $\mathbf{Rp}_{1/2} := \langle [\omega]^\omega, [\omega]^\omega, \not\vert_{1/2} \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}_{1/2}) =: \mathfrak{s}_{1/2}, \quad \mathfrak{d}(\mathbf{Rp}_{1/2}) =: \mathfrak{r}_{1/2}.$$

  An $\mathbf{Rp}_{1/2}$-unbounding family is called *bisecting* or *1/2-splitting* whereas an $\mathbf{Rp}_{1/2}$-dominating family is called *1/2-reaping*.

- Let $\mathbf{Rp}_\rho := \langle [\omega]^\omega, [\omega]^\omega, \not\vert_\rho \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}_\rho) =: \mathfrak{s}_\rho, \quad \mathfrak{d}(\mathbf{Rp}_\rho) =: \mathfrak{r}_\rho.$$

  An $\mathbf{Rp}_\rho$-unbounding family is called *ρ-splitting* and an $\mathbf{Rp}_\rho$-dominating family is called *ρ-reaping*.

We discuss a few elementary Tukey connections.

Lemma 2.1.2. *For every $\rho \in (0,1)$, we have $\mathbf{Rp}_\rho \preceq_{\mathrm{T}} \mathbf{Rp}$.*
By Definition 1.2.1, this implies

$$\mathfrak{s}_\rho \geq \mathfrak{s} \qquad \text{and} \qquad \mathfrak{r}_\rho \leq \mathfrak{r}\,.$$

Proof. If both $F\colon [\omega]^\omega \to [\omega]^\omega$ and $G\colon [\omega]^\omega \to [\omega]^\omega$ are the identity map, then the Tukey connection $(F,G)\colon \mathbf{Rp}_\rho \to \mathbf{Rp}$ has the desired properties.

To see this, we have to prove the implication $S \mid_\rho X \Rightarrow S \mid X$ for all $S, X \in [\omega]^\omega$.

Assume $S \mid_\rho X$ and note that X is an infinite set. So, for $|S \cap X \cap n|/|X \cap n|$ to tend to $\rho > 0$, the sequence of cardinalities $|S \cap X \cap n|$ has to tend to infinity, i. e. $S \cap X$ is an infinite set. Moreover, since we have $\rho < 1$, the sequence $|(X \smallsetminus S) \cap n|$ has to tend to infinity, as well. Hence, also $X \smallsetminus S$ is an infinite set. In other words, S splits X. □

Lemma 2.1.3. *For every $\rho \in (0,1)$, we have $\mathbf{Rp}_\rho \cong_{\mathrm{T}} \mathbf{Rp}_{1-\rho}$ and hence, by Definition 1.2.1:*

$$\mathfrak{s}_\rho = \mathfrak{s}_{1-\rho} \qquad \text{and} \qquad \mathfrak{r}_\rho = \mathfrak{r}_{1-\rho}\,.$$

Proof. Let $F\colon [\omega]^\omega \to [\omega]^\omega, S \mapsto \omega \smallsetminus S$ and let $G\colon [\omega]^\omega \to [\omega]^\omega$ be the identity map. Then, $(F,G)\colon \mathbf{Rp}_\rho \to \mathbf{Rp}_{1-\rho}$ is a Tukey connection, as

$$F(S) \nmid_{1-\rho} X \quad \Rightarrow \quad S \nmid_\rho G(X)\,,$$

which we will prove by contraposition. Assume $S \mid_\rho X = G(X)$, then

$$(\omega \smallsetminus S) \cap X = X \smallsetminus (S \cap X)$$

and we get

$$\lim_{n\to\infty} \frac{d_n((\omega \smallsetminus S) \cap X)}{d_n(X)} = \lim_{n\to\infty} \frac{d_n(X) - d_n(S \cap X)}{d_n(X)} = 1 - \rho,$$

hence $F(S) = (\omega \smallsetminus S) \mid_{1-\rho} X$. This shows $\mathbf{Rp}_\rho \preceq_{\mathrm{T}} \mathbf{Rp}_{1-\rho}$ and, by symmetry, $\mathbf{Rp}_\rho \cong_{\mathrm{T}} \mathbf{Rp}_{1-\rho}$. □

We are now ready to prove the following result.

Proposition 2.1.4. *For every $\rho, \tau \in (0,1)$, we have $\mathfrak{r}_\rho = \mathfrak{r}_\tau$.*

Proof. By Lemma 2.1.3 it is enough to show for $\rho \in (1/2, 1)$ that $\mathfrak{r}_\rho = \mathfrak{r}_{1/2}$.

To show $\mathfrak{r}_\rho \geq \mathfrak{r}_{1/2}$, let $\mathcal{R} \subseteq [\omega]^\omega$ such that $|\mathcal{R}| < \mathfrak{r}_{1/2}$. Our goal is to construct $S_\rho \in [\omega]^\omega$ such that $S_\rho \mid_\rho \mathcal{R}$.

For $n \in \omega$, let $\mathcal{R}_n \subseteq [\omega]^\omega$ and $S_{n+1} \in [\omega]^\omega$ with $\mathcal{R}_0 := \mathcal{R} \cup \{\omega\}$ and

$$\mathcal{R}_{n+1} = \{X \cap S_{n+1} \mid X \in \mathcal{R}_n\}\,,$$

where S_{n+1} will be such that $S_{n+1} \mid_{1/2} \mathcal{R}_n$, which exists because $|\mathcal{R}_n| = |\mathcal{R}| < \mathfrak{r}_{1/2}$. (To get a unique S_{n+1} in each step, initially fix a well-order on $[\omega]^\omega$ and always take the smallest S_{n+1} satisfying the condition.) For $n \in \omega$, define

$$I_n := \bigcap_{k=1}^{n} S_k \quad \text{and} \quad D_{n+1} := I_n \setminus I_{n+1}\,,$$

so $I_0 = \omega$.

Claim 1. *For $n \geq 1$, we have $d(I_n) = d(D_n) = 1/2^n$ as well as both $I_n \mid_{1/2^n} \mathcal{R}$ and $D_n \mid_{1/2^n} \mathcal{R}$.*

We prove this by induction. For $n = 1$ we have $d(I_1) = d(S_1) = 1/2$ and

$$d(D_1) = d(\omega \setminus S_1) = 1 - d(S_1) = 1 - 1/2 = 1/2\,.$$

Since $I_1 = S_1 \mid_{1/2} \mathcal{R}$, also $D_1 = \omega \setminus S_1 \mid_{1/2} \mathcal{R}$.

Now, for some $k \in \omega$, our induction hypothesis is that $d(I_k) = d(D_k) = 1/2^k$ as well as both $I_k \mid_{1/2^k} \mathcal{R}$ and $I_k \mid_{1/2^k} \mathcal{R}$. We make use of the fact that $I_k \in \mathcal{R}_k$ and thus $S_{k+1} \mid_{1/2} I_k$, hence

$$d(I_{k+1}) = d(S_{k+1} \cap I_k) = 1/2 \cdot d(I_k) = 1/2 \cdot 1/2^k = 1/2^{k+1}$$

and

$$d(D_{k+1}) = d(I_k \setminus I_{k+1}) = d(I_k) - d(I_{k+1}) = 1/2^k - 1/2^{k+1} = 1/2^{k+1}\,,$$

by Lemma 1.1.3.

Moreover, for all $X \in \mathcal{R}$ we have $X \cap I_k \in \mathcal{R}_k$ and thus $S_{k+1} \mid_{1/2} (X \cap I_k)$, which (together with $I_k \mid_{1/2^k} X$) implies the following, using Lemma 1.1.6:

$$I_{k+1} = (S_{k+1} \cap I_k) \mid_{1/2^{k+1}} X \quad \text{and} \quad D_{k+1} = (I_k \setminus S_{k+1}) \mid_{1/2^{k+1}} X\,.$$

This proves the claim. ■

Let $P \subseteq \omega$ be such that $\sum_{n \in P} 1/2^n = \rho$ and define $S_\rho := \bigcup_{n \in P} D_n$. As the D_n are mutually disjoint and satisfy Claim 1, we get $d(S_\rho) = \rho$ and $S_\rho \mid_\rho \mathcal{R}$.

The proof of the converse inequality $\mathfrak{r}_\rho \leq \mathfrak{r}_{1/2}$ is analogous, with the roles of $1/2$ and ρ exchanged.

Claim 2. *For $n \geq 1$, we have $d(I_n) = \rho^n$, $d(D_n) = \rho^n - \rho^{n+1} = \rho^n(1-\rho)$ as well as both $I_n \mid_{\rho^n} \mathcal{R}$ and $D_n \mid_{\rho^n(1-\rho)} \mathcal{R}$.*

Let $H \subseteq \omega$ be such that $\sum_{n\in H} \rho^n(1-\rho) = 1/2$. Indeed, this is possible, since it is equivalent to finding $H \subseteq \omega$ such that $\sum_{n\in H} \rho^n = \frac{1}{2(1-\rho)}$, i.e. finding a representation of $\frac{1}{2(1-\rho)}$ in the non-integer base $\rho^{-1} < 2$.

Define $S_{1/2} := \bigcup_{n\in H} D_n$. As the D_n are mutually disjoint and satisfy Claim 2, we get $d(S_{1/2}) = 1/2$ and $S_{1/2} \mid_{1/2} \mathcal{R}$. □

Note that we did not make use of Tukey connections in the above proof. Therefore, the dual equality of ρ-splitting numbers for different parameters ρ does not follow. In fact, we only showed that a family too small to be $1/2$-reaping is not ρ-reaping and vice versa – which does not yield a method to turn a $1/2$-reaping into a ρ-reaping family or vice versa.

For the dual characteristics we have a weaker result.

Lemma 2.1.5. *For every ρ, $\tau \in (0,1)$, we have $\max\{\mathfrak{s}_\rho, \mathfrak{s}_\tau\} \geq \mathfrak{s}_{\rho\tau}$; in particular (for $\tau = \rho$), we get $\mathfrak{s}_\rho \geq \mathfrak{s}_{\rho^2}$.*

Proof. Given a ρ-splitting family $\mathcal{S}_\rho$ and a τ-splitting family $\mathcal{S}_\tau$, we know that for every $X \in [\omega]^\omega$, there are $S_\tau \in \mathcal{S}_\tau$ such that $S_\tau \mid_\tau X$ and $S_\rho \in \mathcal{S}_\rho$ such that $S_\rho \mid_\rho (S_\tau \cap X)$. Thus, using Lemma 1.1.6 or directly calculating

$$\lim_{n\to\infty} \frac{d_n(S_\rho \cap S_\tau \cap X)}{d_n(X)} = \lim_{n\to\infty} \frac{d_n(S_\rho \cap S_\tau \cap X)}{d_n(S_\tau \cap X)} \frac{d_n(S_\tau \cap X)}{d_n(X)} = \rho\,\tau\,,$$

we get that $\{S_\rho \cap S_\tau \mid S_\rho \in \mathcal{S}_\rho,\, S_\tau \in \mathcal{S}_\tau\}$ is a $\rho\tau$-splitting family of cardinality $\max\{|\mathcal{S}_\rho|, |\mathcal{S}_\tau|\}$, hence $\max\{\mathfrak{s}_\rho, \mathfrak{s}_\tau\}$ is an upper bound of $\mathfrak{s}_{\rho\tau}$. □

We will examine the characteristic $\mathfrak{s}_{1/2}$ in more detail.

Theorem 2.1.6. *The bounds shown in Figure 2.1 hold for $\mathfrak{s}_{1/2}$.*

Proof. We consider the individual inequalities.

$\boldsymbol{\mathfrak{s}_{1/2} \geq \mathrm{cov}(\mathcal{M})}$: Let $G\colon [\omega]^\omega \to [\omega]^\omega$ be the identity map and define $F\colon [\omega]^\omega \to \mathcal{M}$ by

$$F(S) := \{X \in [\omega]^\omega \mid S \mid_{1/2} X\}\,.$$

Then, $(F,G)\colon \mathbf{Rp}_{1/2} \to \mathbf{Cv}(\mathcal{M})^\perp$ is a Tukey connection, as

$$F(S) \not\ni X \quad \Rightarrow \quad S \nmid_{1/2} G(X)\,.$$

(We even have equivalence.)

It remains to show that F is well-defined, specifically that, for all $S \in [\omega]^\omega$, the set $F(S)$ is meagre.

Define a chopped real (x,Π) as follows: Let $x := S$ and let $f_S\colon \omega \to S$ be the ascending enumeration of S. Let Π consist of the intervals $[0, f_S(1))$ and, for all n in ω, $[f_S(3^n), f_S(3^{n+1}))$.

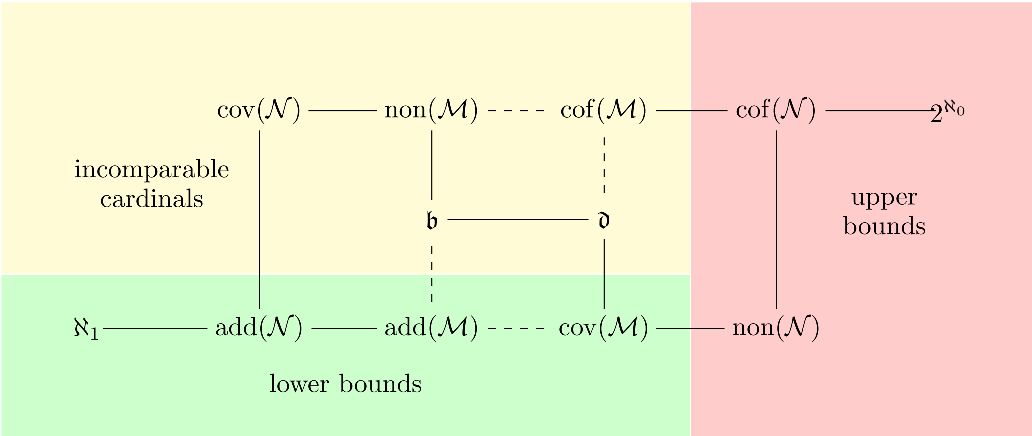

Figure 2.1: The ZFC-provable bounds of $\mathfrak{s}_{1/2}$ as well as incomparable cardinals, i. e. cardinals both consistently less and consistently greater than $\mathfrak{s}_{1/2}$, in Cichoń's diagram.
(Original code generously provided in https://arxiv.org/abs/1901.06055 by William Brian and Alan Dow.)

The sets matching this chopped real form a comeagre set which consists of reals not bisected by S. Indeed, let $Y \in \mathrm{Match}(x, \Pi)$; for infinitely many n in ω and for $k := f_S(3^n)$, we have

$$\frac{d_k(Y \cap S)}{d_k(S)} \geq \frac{2}{3}\,.$$

Hence the family $F(S)$ of those reals that *are* bisected by S is a meagre set (as its complement is a superset of a comeagre set), and $\{F(S) \mid S \in \mathcal{S}\}$ is a 2^ω-covering consisting of meagre sets.

Hence we have $\mathbf{Rp}_{1/2} \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{M})^\perp$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}_{1/2} \geq \mathrm{cov}(\mathcal{M}) \qquad \text{and} \qquad \mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M})\,.$$

$\boldsymbol{\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})}$: We define $F\colon [\omega]^\omega \to [\omega]^\omega$ as the identity map and $G\colon [\omega]^\omega \to \mathcal{N}$ as

$$G(X) := \{S \in [\omega]^\omega \mid S \not|_{1/2} X\}\,.$$

Then, $(F, G)\colon \mathbf{Cv}(\mathcal{N}) \to \mathbf{Rp}_{1/2}$ is a Tukey connection, as

$$F(S) \not|_{1/2} X \quad \Rightarrow \quad S \in G(X)\,.$$

(We even have equivalence.)

It remains to show that G is well-defined, specifically that for all $X \in [\omega]^\omega$ the set $G(X)$ is a null set.

Enumerating $X =: \{x_0, x_1, x_2, \ldots\}$, we define functions $f_{X,n}$ and f_X as follows:

$$f_{X,n}\colon [\omega]^\omega \to \{0,1\}, \quad Y \mapsto \begin{cases} 0 & x_n \notin Y \\ 1 & x_n \in Y \end{cases}$$

$$f_X\colon [\omega]^\omega \to [0,1], \quad Y \mapsto \begin{cases} \lim_{k\to\infty} \frac{\sum_{n=1}^k f_{X,n}(Y)}{k} & \text{if the limit exists} \\ 0 & \text{otherwise} \end{cases}$$

It is clear that for all $n \in \omega$ we have $\lambda(f_{X,n}^{-1}(\{1\})) = 1/2$. Hence, the $f_{X,n}$ are identically distributed random variables on the probability space $([\omega]^\omega, \lambda)$ with λ the Lebesgue measure. Moreover, they are independent and have finite variance. By the law of large numbers it follows that f_X is almost surely equal to $1/2$, in other words $\lambda(f_X^{-1}(\{1/2\})) = 1$. This means that with

$$\mathcal{S}_X := \{S \in [\omega]^\omega \mid f_X(S) = 1/2\} = \{S \in [\omega]^\omega \mid S \mid_{1/2} X\},$$

we have that $\lambda(\mathcal{S}_X) = 1$ and hence $G(X) = [\omega]^\omega \setminus \mathcal{S}_X \in \mathcal{N}$.

Hence we have $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}_{1/2}$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N}) \qquad \text{and} \qquad \mathfrak{r}_{1/2} \geq \mathrm{cov}(\mathcal{N})\,.$$

Con($\mathfrak{s}_{1/2} >$ non($\mathcal{M}$)) and Con($\mathfrak{r}_{1/2} <$ cov($\mathcal{M}$)): This is implied by the consistency of $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ as witnessed by the Cohen model.

Con($\mathfrak{s}_{1/2} <$ non($\mathcal{M}$)) and Con($\mathfrak{s}_{1/2} < \mathfrak{b}$): Two suitable models are mentioned in [BJ95, Model 7.6.7]. The second one is obtained adding $\aleph_2$ random reals to a model of $\mathfrak{b} = \aleph_2 = 2^{\aleph_0}$. Since random forcing is ω^ω-bounding and adds a nonmeasurable set of size $\aleph_1$, we get $\aleph_1 = \mathfrak{s}_{1/2} = \mathrm{non}(\mathcal{N}) < \mathrm{cov}(\mathcal{N}) = \mathfrak{b} = 2^{\aleph_0}$.

Con($\mathfrak{s}_{1/2} >$ cof($\mathcal{M}$)): See section 2.2.

Con($\mathfrak{s}_{1/2} <$ non($\mathcal{N}$)): See Lemma 2.3.2, Lemma 2.3.4 and Theorem 2.3.5 in section 2.3. □

Corollary 2.1.7. *It follows that* $\mathrm{Con}(\mathfrak{s} = \mathfrak{r}_{1/2} < \mathfrak{s}_{1/2} = \mathfrak{r})$.

Proof. In the Cohen model we have $\aleph_1 = \mathfrak{s} = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathfrak{r} = 2^{\aleph_0}$. With the boundaries seen in Theorem 2.1.6 above, this yields the claim. □

2.2 A Model for $\mathfrak{s}_{1/2} > \operatorname{cof}(\mathcal{M})$

(This is joint work with Lukas Daniel Klausner.)

Recall the standard definition of infinitely equal forcing (denoted by $\mathbb{EE}$ in literature and by $\mathbb{IE}$ here) as per [BJ95, Definition 7.4.11]:

Definition 2.2.1. For a partial function p from ω to $2^{<\omega}$ we say $p \in \mathbb{IE}$ *iff*

- $p(n) \in 2^n \ \forall\, n \in \operatorname{dom}(p)$, and
- $|\omega \setminus \operatorname{dom}(p)| = \aleph_0$.

A condition q is stronger than p *iff* $p \subseteq q$.

Forcing with $\mathbb{IE}$ is ω^ω-bounding and preserves nonmeagre sets, and as both of these properties are preserved under countable support iterations, both $\mathfrak{d}$ and $\operatorname{non}(\mathcal{M})$ remain small when forcing with a CS iteration of $\mathbb{IE}$, and consequently, so does $\operatorname{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \operatorname{non}(\mathcal{M})\}$.

We will instead work with $p(n) \in 2^{2^n} \ \forall\, n \in \operatorname{dom}(p)$ to make sure each new interval is about as large as the entire initial domain of the generic real so far. Define a variant of infinitely equal forcing, *exponentially growing infinitely equal forcing*:

Definition 2.2.2. For a partial function p from ω to $2^{<\omega}$ we say $p \in \mathbb{XIE}$ *iff*

- $p(n) \in 2^{2^n} \ \forall\, n \in \operatorname{dom}(p)$, and
- $|\omega \setminus \operatorname{dom}(p)| = \aleph_0$.

A condition q is stronger than p *iff* $p \subseteq q$.

The generic real is equivalent to a total function $g \in 2^\omega$. We partition ω into finite intervals $I_n := [2^n - 1, 2^{n+1} - 1)$ corresponding to the size of the domain of the n-th part of conditions in $\mathbb{XIE}$.

Lemma 2.2.3. *Given a real x in the ground model, a condition p and an integer k, we can find an $\ell \geq k$ and a stronger condition q forcing that*

$$\left| \frac{|\dot{g} \cap x \cap \ell|}{|x \cap \ell|} - \frac{1}{2} \right| > \frac{1}{12}.$$

Thus, assuming CH *in the ground model,* $\mathbb{XIE}$ *forces that* $\mathfrak{s}_{1/2} \geq \aleph_2$.

Proof. We denote $\frac{|\dot{g} \cap x \cap \ell|}{|x \cap \ell|}$ by $X(\dot{g}, x, \ell)$

Given x, p and k, there is a minimal $n \in \omega \setminus \operatorname{dom}(p)$ with $\min I_n \geq k$. Consider $x \cap I_n$: This contains either more than 2^{n-1} 1s (case 1) or at least 2^{n-1} 0s (case 2). Strengthen p to q by extending its domain to include n and setting $q(n)$ to be $x \cap I_n$ (case 1) or $I_n \setminus x$ (case 2). Now consider the following (in the extension):

1. In case 1, if $\rho := X(\dot{g}, x, 2^n - 1)$, then $X(\dot{g}, x, 2^{n+1} - 1)$ is at least

$$\frac{|\dot{g} \cap x \cap (2^n - 1)| + 2^{n-1}}{|x \cap (2^n - 1)| + 2^{n-1}} = \frac{\rho\,|x \cap (2^n - 1)| + 2^{n-1}}{|x \cap (2^n - 1)| + 2^{n-1}} \overset{(*)}{\geq} \frac{\rho\, 2^n + 2^{n-1}}{2^n + 2^{n-1}} = \frac{2\rho + 1}{3}\,,$$

where $(*)$ is due to the fact that $\frac{\rho\, a + 2^{n-1}}{a + 2^{n-1}}$ is monotonously decreasing in a and the fact that $|x \cap (2^n - 1)| < 2^n$.
Now, if $\rho \geq 1/2$, then

$$\frac{2\rho + 1}{3} \geq \frac{2}{3}\,,$$

so $X(\dot{g}, x, 2^{n+1} - 1)$ is greater than $1/2$ by at least $1/6$.
If $\rho < 1/2$, then

$$\frac{2\rho + 1}{3} = \rho + \frac{1 - \rho}{3} \geq \rho + \frac{1}{6}\,,$$

so $X(\dot{g}, x, 2^{n+1} - 1)$ is greater than $X(\dot{g}, x, 2^n - 1)$ by at least $1/6$ and thus one of them must differ from $1/2$ by at least $1/12$.

2. Case 2 is analogous: If $\rho := X(\dot{g}, x, 2^n - 1)$, then $X(\dot{g}, x, 2^{n+1} - 1)$ is at most

$$\frac{|\dot{g} \cap x \cap (2^n - 1)|}{|x \cap (2^n - 1)| + 2^{n-1}} = \frac{\rho\,|x \cap (2^n - 1)|}{|x \cap (2^n - 1)| + 2^{n-1}} \overset{(*)}{\leq} \frac{\rho\, 2^n}{2^n + 2^{n-1}} = \frac{2\rho}{3}\,,$$

where $(*)$ is due to the fact that $\frac{\rho\, a}{a + 2^{n-1}}$ is monotonously increasing in a and the fact that $|x \cap (2^n - 1)| < 2^n$.
Now, if $\rho \leq 1/2$, then

$$\frac{2\rho}{3} \leq \frac{1}{3}\,,$$

so $X(\dot{g}, x, 2^{n+1} - 1)$ is less than $1/2$ by at least $1/6$.
If $\rho > 1/2$, then

$$\frac{2\rho}{3} = \rho - \frac{\rho}{3} \leq \rho - \frac{1}{6}\,,$$

so $X(\dot{g}, x, 2^{n+1} - 1)$ is less than $X(\dot{g}, x, 2^n - 1)$ by at least $1/6$ and thus one of them must differ from $1/2$ by at least $1/12$.

□

Lemma 2.2.4. *Forcing with* $\mathbb{XIE}$ *is* ω^ω*-bounding and preserves nonmeagre sets. Thus, assuming* CH *in the ground model,* $\mathbb{XIE}$ *forces that*

$$\operatorname{cof}(\mathcal{M}) = \max\{\mathfrak{d}, \operatorname{non}(\mathcal{M})\} = \aleph_1\,.$$

The following proof is based on [BJ95, Lemma 7.4.14].

Proof. By [BJ95, Lemma 6.3.21], it is enough to show that every meagre set in $V^{\mathbb{XIIE}}$ is covered by a Borel meagre set coded in V. This is also preserved under countable support iteration, cf. [BJ95, Theorem 6.3.22]. Suppose that $\{\dot{F}_n \mid n \in \omega\}$ is a sequence of $\mathbb{XIIE}$-names and $p \in \mathbb{XIIE}$ is such that

$$p \Vdash_{\mathbb{XIIE}} \bigcup_{n\in\omega} \dot{F}_n \in \mathcal{M}.$$

We will find $q \leq p$ and a function $f\colon \omega \longrightarrow 2^{<\omega}$ such that

$$q \Vdash_{\mathbb{XIIE}} \left(\bigcap_{n\in\omega} \bigcup_{m>n} \bigcup_{s\in 2^{2^m}} [s^\frown f(m)] \right) \cap \bigcup_{n\in\omega} \dot{F}_n = \varnothing,$$

which will finish the proof since $\bigcup_{m>n} \bigcup_{s\in 2^{2^m}} [s^\frown f(m)]$ is a dense open set for every $n \in \omega$.

Define a sequence $\langle p_n \mid n \in \omega\rangle$ such that $p_0 = p$, and for all $n \in \omega$:

1. $p_{n+1} \leq_n p_n$, which means $p_{n+1} \leq p_n$ and the first n elements of $\omega \setminus \mathrm{dom}(p_n)$ are also in $\omega \setminus \mathrm{dom}(p_{n+1})$, and
2. $p_{n+1} \Vdash_{\mathbb{XIIE}} \forall j \leq n\colon \forall s \in 2^{2^n}\colon\ [s^\frown f(n)] \cap \dot{F}_j = \varnothing$.

Suppose that p_n and $f(n)$ are already defined. Let $A_n = \{k_n^1, \ldots, k_n^n\}$ be the first n elements of $\omega \setminus \mathrm{dom}(p_n)$. Let $\{r_n^j \mid j < k\}$ be the list of all functions $r\colon A_n \longrightarrow 2^{<\omega}$ such that $r(k_n^i) \in 2^{2^{k_n^i}}$ for $i \leq n$, i. e. functions in $\mathbb{XIIE}$ with domain A_n. Successively extend $p_n = p_n^0 \geq p_n^1 \geq \cdots \geq p_n^k = p_{n+1}$ and $\varnothing \subseteq s_n^1 \subseteq \cdots \subseteq s_n^k = f(n)$ such that for all $j \leq k$ we have

$$r_n^j \cup p_n^j \Vdash_{\mathbb{XIIE}} \forall i \leq j\colon \forall s \in 2^{2^n}\colon\ \left[s^\frown s_n^i\right] \cap \dot{F}_i = \varnothing.$$

Let q be the fusion condition of the sequence $\langle p_n \mid n \in \omega\rangle$ for all n. It is straightforward to check that q has the required properties. □

Theorem 2.2.5. $\mathrm{Con}(\mathfrak{s}_{1/2} > \mathrm{cof}(\mathcal{M}))$.

Proof. Assume CH in the ground model. Then the statement follows by combining Lemma 2.2.3 and Lemma 2.2.4. □

2.3 Separating $\mathfrak{s}_{1/2}$ and $\mathrm{non}(\mathcal{N})$

To prove $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$, we will use a typical creature forcing construction to increase $\mathrm{non}(\mathcal{N})$ and show that the forcing poset does not increase $\mathfrak{s}_{1/2}$.

We will not go into too much detail regarding creature forcing; see [RS99] for the most general and most detailed explanation. The specific forcing poset we use here also appears in [FGKS17] and [GK18].

Definition 2.3.1. We define a forcing poset $\mathbb{P}$ as follows: A condition $p \in \mathbb{P}$ is a sequence of *creatures* $p(k)$ such that each $p(k)$ is a non-empty subset of

$$\mathrm{POSS}_k := \left\{ F \subseteq 2^{I_k} \;\middle|\; \frac{|F|}{|2^{I_k}|} \geq 1 - \frac{1}{2^{a_k}} \right\}$$

for some sufficiently large consecutive intervals $I_k \subseteq \omega$ and strictly increasing $a_k < \omega$ (for our construction, let I_k be an interval of length 2^{2^k} and let $a_k := k$) and such that, letting the *norm* $\|\cdot\|$ of a creature C be defined by $\|C\| := \log_2 |C|$, p fulfils $\limsup_{k\to\infty} \|p(k)\| = \infty$. The order is $q \leq p$ *iff* $q(k) \subseteq p(k)$ for all $k < \omega$ (i. e. stronger conditions consist of smaller subsets of POSS_k). Note that $\mathbb{P} \neq \varnothing$ since $\limsup_{k\to\infty} \|\mathrm{POSS}_k\| = \infty$.

Given a condition p such as above, the finite initial segments in $p{\restriction}_{k+1}$ (for $k < \omega$) are sometimes referred to as *possibilities* and denoted by $\mathrm{poss}(p, {\leq}k) := \prod_{\ell \leq k}[p(\ell)]^1 = \{\langle\{z(\ell)\} \mid \ell \leq k\rangle \mid \forall \ell \leq k\colon z(\ell) \in p(\ell)\}$. We may also use the notation $\mathrm{poss}(p, {<}k) := \mathrm{poss}(p, {\leq}k-1)$. When $\eta \in \mathrm{poss}(p, {\leq}k)$, we write $p \wedge \eta$ to denote $\eta^\frown p{\restriction}_{[k+1,\omega)}$.[1]

Define the forcing poset $\mathbb{Q}$ as the countable support product $\mathbb{Q} := \prod_{\alpha<\omega_2} \mathbb{Q}_\alpha$, where each $\mathbb{Q}_\alpha = \mathbb{P}$. We will work with the dense subset of *modest* conditions of $\mathbb{Q}$, i. e. conditions $p \in \mathbb{Q}$ such that for each $k < \omega$, there is at most one index α_k such that $|p(\alpha_k, k)| > 1$. We call such creatures $p(\alpha_k, k)$ *non-trivial.* (An easy bookkeeping argument shows that the modest conditions do indeed form a dense subset of $\mathbb{Q}$.) Modest conditions p have the advantage that for each $k < \omega$, $\mathrm{poss}(p, {<}k)$ is finite and even bounded by $\mathrm{maxposs}({<}k) := \prod_{j<k} |\mathrm{POSS}_k|$, which makes iterating over all possibilities below a certain level possible.

By the usual Δ-system argument, CH implies that $\mathbb{Q}$ is $\aleph_2$-cc. (For details, see [FGKS17, Lemma 3.3.1] or [GK18, Lemma 4.18].) By the usual creature forcing arguments, it is clear that $\mathbb{Q}$ satisfies the finite version of Baumgartner's axiom A and hence is proper and ω^ω-bounding, that $\mathbb{Q}$ continuously reads all reals and that $\mathbb{Q}$ preserves all cardinals and cofinalities. (For details, see [FGKS17, section 5] or [GK18, sections 6–7].) In particular, given any condition $p \in \mathbb{Q}$ and any name $\dot{r}$

[1] The usual creature forcing notation defines the set of possibilities more abstractly as $\mathrm{poss}(p, {\leq}k) := \prod_{\ell\leq k} p(\ell)$ and defines $p \wedge \eta$ as a condition with an extended *trunk* (a concept which we did not deem necessary to introduce in our paper). Since working with possibilities η as sequences of singletons suffices for our proofs and is conceptually easier, we instead opted for this simpler definition.

for a real, we can find $q \le p$ such that each $\eta \in \mathrm{poss}(q, {<}k)$ already decides $\dot r{\restriction}_{\min(I_k)}$ (which we refer to as "*q reads $\dot r$ rapidly*"). We will reproduce an abbreviated version of the proof of $V^{\mathbb{Q}} \vDash \mathrm{non}(\mathcal{N}) \ge \aleph_2$ here:

Lemma 2.3.2. *Assuming* CH *in the ground model,* $\mathbb{Q}$ *forces that* $\mathrm{non}(\mathcal{N}) \ge \aleph_2$.

Proof. First, note that for $\alpha < \omega_2$, the generic object $\dot R_\alpha$ is a sequence of $\dot R_\alpha(k) \subseteq 2^{I_k}$ of relative size at least $1 - 1/2^{a_k}$. Since $\langle a_k \mid k < \omega\rangle$ is strictly increasing, it is clear that

$$\prod_{k<\omega}\left(1 - \frac{1}{2^{a_k}}\right) > 0$$

and hence the set

$$\{r \in 2^\omega \mid \forall\, k < \omega \colon r{\restriction}_{I_k} \in \dot R_\alpha(k)\}$$

is positive and

$$\dot N_\alpha := \{r \in 2^\omega \mid \exists^\infty\, k < \omega \colon r{\restriction}_{I_k} \notin \dot R_\alpha(k)\}$$

is a name for a null set.

Now, given a name $\dot r \in 2^\omega$ for a real and a $p \in \mathbb{Q}$ which reads $\dot r$ rapidly, we can pick an $\alpha < \omega_2$ not in the support of p and add it to the support to get a (without loss of generality) modest condition p'; then p' still reads $\dot r$ rapidly not using the index α. Since we only require the lim sup of the norms to go to infinity, one can then show that $p' \Vdash \dot r \in \dot N_\alpha$. From this fact and $\aleph_2$-cc, it follows that for any $\kappa < \omega_2$, any sequence of names of reals $\langle \dot r_i \mid i < \kappa\rangle$ is contained in a null set of $V^{\mathbb{Q}}$.[2] □

We will now prove that the ground model reals are a bisecting family in $V^{\mathbb{Q}}$. To show this, we will use the following combinatorial lemma.

Lemma 2.3.3. *If $R, S \subseteq \omega$ are disjoint finite sets of sizes r and s, respectively, $s = c \cdot r$ for some $c > 1$, and $A \subseteq R$, $B \subseteq S$ such that*

$$\frac{|B|}{|S|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right)$$

for some $\varepsilon > 0$, then

$$\frac{|A \cup B|}{|R \cup S|} \in \left(\frac{1}{2} - \varepsilon - \frac{1}{c}, \frac{1}{2} + \varepsilon + \frac{1}{c}\right).$$

[2] The actual argument for $p \Vdash \dot r \in \dot N_\alpha$ involves a slightly more complicated norm than we defined above; however, since the parameters of the creature forcing poset $\mathbb{P}$ are immaterial for the more complicated proof in Lemma 2.3.4 below, we opted to omit the details for this paper. Details can be found in [GK18, section 11].

Proof. Since

$$\frac{1}{1+1/c} \geq 1 - \frac{1}{c},$$

we have the lower bound

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &> \frac{s \cdot (1/2 - \varepsilon)}{r+s} = \frac{s \cdot (1/2 - \varepsilon)}{s \cdot 1/c + s} = \frac{1/2 - \varepsilon}{1 + 1/c} \\ &\geq \left(\frac{1}{2} - \varepsilon\right)\left(1 - \frac{1}{c}\right) \geq \frac{1}{2} - \varepsilon - \frac{1}{c}.\end{aligned}$$

For the upper bound, we get

$$\begin{aligned}\frac{|A \cup B|}{|R \cup S|} &< \frac{r + s \cdot (1/2 + \varepsilon)}{r+s} = \frac{s \cdot 1/c + s \cdot (1/2 + \varepsilon)}{s \cdot 1/c + s} \\ &= \frac{1/2 + \varepsilon + 1/c}{1 + 1/c} \leq \frac{1}{2} + \varepsilon + \frac{1}{c}.\end{aligned}$$

□

Lemma 2.3.4. *$2^\omega \cap V$ is a bisecting family in $V^{\mathbb{Q}}$.*

Proof. We will show the following: Given a modest condition $p \in \mathbb{Q}$ and a name $\dot{Y}$ for a real, we can find $q \leq p$ and a ground model real X such that $q \Vdash X \,|_{1/2}\, \dot{Y}$.

In order to do this, we will construct $p^* \leq p$ as well as $m_0 := 0 < m_1 < m_2 < \dots$ and choose $\langle P_i \mid i < \omega \rangle$ with $P_0 := 1/2$, $P_i > 0$ for all $i < \omega$ and $\lim_{i \to \infty} P_i = 0$ such that the following statements hold:

(i) The condition p^* is not only modest, but even fulfils that for each interval $J_i := [m_i, m_{i+1})$, there is exactly one $k_i \in J_i$ such that $|p^*(\alpha_{k_i}, k_i)| > 1$, i. e. such that the creature $C_i := p^*(\alpha_{k_i}, k_i)$ is non-trivial.

(ii) Due to continuous reading, we can find for each $\eta \in \mathrm{poss}(p^*, {<}k_i)$ and each $S \in C_i$ finite sets $Y_{\eta,S} \subseteq m_{i+1}$ and $Z_{\eta,S} \subseteq J_i$ such that

$$p^* \wedge (\eta^\frown\{S\}) \quad \Vdash \quad \dot{Y}{\restriction}_{m_{i+1}} = Y_{\eta,S} \quad \text{and} \quad \dot{Y}{\restriction}_{J_i} = Z_{\eta,S}.$$

(iii) Note that due to property (i), $N_{i+1} := |\,\mathrm{poss}(p^*, {<}m_{i+1})| = |\,\mathrm{poss}(p^*, {\leq}k_i)|$ only depends on the i-th creature $C_i = p^*(\alpha_{k_i}, k_i)$, since from $k_i + 1$ to m_{i+1}, there are only singletons in p^*. Hence we can choose m_{i+1} such that $m_{i+1} \gg N_{i+1}$.

(iv) For all $0 < i < \omega$, we have $N_i \geq i^6$. Additionally, let $N_1 = |C_0| \geq 100$. (This is possible without loss of generality since we can just "skip" creatures which do not have sufficiently many elements to fulfil these bounds.)

(v) Letting the name $\dot{M}_i$ denote the number of elements in $\dot{Y}{\restriction}_{[m_i, m_{i+1})}$, we can ensure that p^* forces for all $i < \omega$ that $\dot{M}_i \geq \max\{2\,i\,m_i, N_{i+1}\}$.

(vi) Letting $E_i := \lceil N_i \cdot P_i \rceil$, letting $e_i(\eta, S)$ be the E_i-th element of $Z_{\eta,S}$ and letting $e_i := \max_{\eta,S} e_i(\eta, S)$, we can finally choose m_{i+1} large enough such that $m_i + e_i < m_{i+1}$.

We now make a probabilistic argument using the following formulation of Chernoff's bound (see [AS16, Theorem A.1.1]): Given mutually independent random variables $\langle x_i \mid 1 \leq i \leq k\rangle$ with $\Pr[x_i = 0] = \Pr[x_i = 1] = {}^1\!/\!_2$ for all $1 \leq i \leq k$ and letting $S_k := \sum_{1\leq i\leq k} x_i$, it follows that for any $a > 0$,

$$\Pr\left[S_k - \frac{k}{2} > a\right] < \exp\left(-\frac{a^2}{2k}\right).$$

We use this bound as follows: Fix $n < \omega$. Let X be some randomly chosen subset of J_n and denote the probability space by Ω. Fix $\eta \in \mathrm{poss}(p^*, {<}k_n)$, $S \in C_n$ and $m \in J_n$ with $m \geq m_n + e_n(\eta, S)$. We consider the probability that this randomly chosen X does *not* bisect $Z_{\eta,S} \cap m$ with error at most $\frac{1}{2n}$; denote this event by $\mathrm{FAIL}(X, \eta, S, m)$.

Let $k \geq E_n$ denote the number of elements in $Z_{\eta,S} \cap m$. Then the choice of X (or, more precisely, the choice of the initial part of X relevant for this argument) amounts to tossing k fair coins x_j with values in $\{0, 1\}$, summing up the results and dividing by k, and considering the gap between the result and ${}^1\!/\!_2$. By Chernoff's bound above we have

$$\Pr[\mathrm{FAIL}(X,\eta,S,m)] = \Pr\left[\sum_{1\leq i\leq k} \frac{x_i}{k} - \frac{1}{2} > \frac{1}{2n}\right] = \Pr\left[\sum_{1\leq i\leq k} x_i - \frac{k}{2} > \frac{k}{2n}\right]$$
$$< \exp\left(-\frac{({}^k\!/\!_{2n})^2}{2k}\right) = \exp\left(-\frac{k}{8n^2}\right).$$

Hence the probability of failing for at least one $m \in J_n$ (with $Z_{\eta,S} \cap m \geq E_n$) is bounded as follows (note that we only have to sum over the elements of $Z_{\eta,S} \cap m$):

$$\Pr[\mathrm{FAIL}(X,\eta,S)] := \Pr[\exists\, m \geq m_n + e_n(\eta,S)\colon\ \mathrm{FAIL}(X,\eta,S,m)]$$
$$< \sum_{k\geq E_n} \exp\left(-\frac{k}{8n^2}\right) = \frac{\exp(-{}^{E_n}\!/\!_{8n^2})}{1-\exp(-{}^1\!/\!_{8n^2})}$$

Using the fact that $\frac{1}{1-\exp(-x)} \leq \frac{2}{x}$ for $x \in (0,1)$, we get

$$\Pr[\mathrm{FAIL}(X,\eta,S)] < 16n^2 \cdot \exp\left(-\frac{E_n}{2n^2}\right) = 16n^2 \cdot \exp\left(-\frac{\lceil N_n \cdot P_n\rceil}{2n^2}\right).$$

For the final step of our probabilistic estimate, we want to bound the probability of failing for at least one η, and we get

$$\Pr[\mathrm{FAIL}(X,S)] := \Pr[\exists\,\eta\colon\ \mathrm{FAIL}(X,\eta,S)] \leq N_n \cdot 16n^2 \cdot \exp(-{}^{\lceil N_n\cdot P_n\rceil}\!/\!_{2n^2}) =: \delta_n.$$

It is easy to see that $\delta_n < {}^1\!/\!_2$ holds for e.g. $P_n := \max\{{}^1\!/\!_2, {}^1\!/\!_n\}$ and $N_n \geq \min\{n^6, 100\}$, which holds by property (iv).

Now we make the following observation: If we count the number of pairs $\{\langle X, S\rangle \mid X \in \Omega, S \in C_n\}$ with $\mathrm{FAIL}(X,S)$, this total number of failures is bounded from above by $\delta_n \cdot |C_n| \cdot |\Omega|$. If we now assume that for each $X \in \Omega$, the number of

$S \in C_n$ with $\mathrm{FAIL}(X, S)$ is at least F, then the total number of failures is bounded from below by $F \cdot |\Omega|$ – but this shows that $F \leq \delta_n \cdot |C_n| < {}^{|C_n|}/_2$.

Summing up the entire probabilistic argument, this means that we can find some $X =: X_n \subseteq J_n$ and some $D_n \subseteq C_n$ with $|D_n| > {}^{|C_n|}/_2$ (and hence $\|D_n\| > \|C_n\| - 1$) such that for each $\eta \in \mathrm{poss}(p^*, {<}k_n)$, each $S \in D_n$ and each $m \geq m_n + e_n(\eta, S)$, we have that

$$\frac{|X_n \cap Z_{\eta,S} \cap m|}{|Z_{\eta,S} \cap m|} \in \left(\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right).$$

Now we perform the usual fusion construction, starting with $q_0 := p^*$, shrinking the creature C_n to D_n in the n-th step (and keeping everything below that from q_{n-1}), and constructing a fusion condition $q := \bigcap_{n<\omega} q_n$ as well as sets $X_n \subseteq J_n$. It is clear that the q constructed this way is a valid condition. We now claim that the set $X := \bigcup_{n<\omega} X_n$ is as required; in particular, we claim that for each $\varepsilon > 0$, there is an m_ε such that for all $m \geq m_\varepsilon$, we have

$$q \Vdash \frac{|X \cap \dot{Y} \cap m|}{|\dot{Y} \cap m|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

We prove this inductively and will show that the error at any point $m < \omega$ is bounded by an expression that goes to 0 as n goes to infinity. Let $X_{<n} := \bigcup_{i<n} X_i$ for each $n < \omega$. For our induction hypothesis, assume that we already know that at m_n, the bisection error of $X_{<n}$ with each possible $Y_{\eta,S}\restriction_{m_n}$ is at most ${}^1/_{(n-1)}$. For each $m \in [m_n + 1, m_{n+1}]$, we now have to consider the bisection error of $X_{<n+1}$ at m with each such $Y_{\eta,S}$.

- For $m \in [m_n + 1, m_n + e_n(\eta, S))$, note that $Y_{\eta,S}\restriction_{m_n}$ has at least N_n elements by property (v), while $Y_{\eta,S}\restriction_{[m_n,m]}$ has at most $E_n = N_n \cdot P_n$ elements by property (vi). Thus we can apply Lemma 2.3.3 with $R := Y_{\eta,S}\restriction_{[m_n,m]}$, $S := Y_{\eta,S}\restriction_{m_n}$, $\varepsilon := {}^1/_{(n-1)}$ and some $c > {}^1/_{P_n}$ to get

$$\begin{aligned} \frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} &\in \left(\frac{1}{2} - \frac{1}{n-1} - \frac{1}{c}, \frac{1}{2} + \frac{1}{n-1} + \frac{1}{c}\right) \\ &\subseteq \left(\frac{1}{2} - \frac{1}{n-1} - P_n, \frac{1}{2} + \frac{1}{n-1} + P_n\right) \\ &\subseteq \left(\frac{1}{2} - \frac{2}{n-1}, \frac{1}{2} + \frac{2}{n-1}\right). \end{aligned}$$

- For $m \in [m_n + e_n(\eta, S), m_{n+1}]$, it is clear that

$$\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m|}{|Y_{\eta,S} \cap m|} \in \left(\frac{1}{2} - \frac{1}{n-1}, \frac{1}{2} + \frac{1}{n-1}\right),$$

  since the error on $Y_{\eta,S}\restriction_{m_n}$ is at most ${}^1/_{(n-1)}$ and the error on $Y_{\eta,S}\restriction_{[m_n,m]}$ is at most ${}^1/_n$.

- For $m = m_{n+1}$, however, we have to show even more to ensure that our induction hypothesis remains true for the next step. Note that $Y_{\eta,S}{\restriction}_{m_n}$ has at most m_n elements, while $Y_{\eta,S}{\restriction}_{[m_n,m_{n+1}]}$ has at least $2\,n\,m_n$ elements by property (v). Thus we can apply Lemma 2.3.3 once more with $R := Y_{\eta,S}{\restriction}_{m_n}$, $S := Y_{\eta,S}{\restriction}_{[m_n,m_{n+1}]}$, $\varepsilon := 1/2n$ and some $c \geq 2n$ to get

$$\begin{aligned}\frac{|X_{<n+1} \cap Y_{\eta,S} \cap m_{n+1}|}{|Y_{\eta,S} \cap m_{n+1}|} &\in \left(\frac{1}{2} - \frac{1}{2n} - \frac{1}{c}, \frac{1}{2} + \frac{1}{2n} + \frac{1}{c}\right) \\ &\subseteq \left(\frac{1}{2} - \frac{1}{n}, \frac{1}{2} + \frac{1}{n}\right),\end{aligned}$$

  which is precisely the induction hypothesis for $n+1$.

Given any $\varepsilon > 0$, pick some n_ε such that $\frac{2}{n_\varepsilon - 1} < \varepsilon$ and let $m_\varepsilon := m_{n_\varepsilon}$. Then for all $m \geq m_\varepsilon$, by the bounds above

$$q \Vdash \frac{|X \cap \dot{Y} \cap m|}{|\dot{Y} \cap m|} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right),$$

finishing the proof. □

Theorem 2.3.5. $\mathrm{Con}(\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N}))$.

Proof. Assume CH in the ground model; then the statement follows by combining Lemma 2.3.2 and Lemma 2.3.4. □

Chapter 3

Characteristics Related to $\mathfrak{i}$

The results in this chapter were obtained in [BHK+18]. While section 3.2 and section 3.3 remain unchanged, in section 3.1, the proof of Theorem 3.1.9 was streamlined using Tukey connections and some lemmas and definitions were added or expanded.

3.1 Definitions and Bounds

We define a second set of properties more closely related to $\mathfrak{i}$, although characteristics related to $\mathfrak{r}$ and $\mathfrak{s}$ do reappear.

Definition 3.1.1. A set $X \in [\omega]^\omega$ is *moderate iff* $\underline{d}(X) > 0$ as well as $\bar{d}(X) < 1$.[1]

Definition 3.1.2. A family $\mathcal{I}_* \subseteq [\omega]^\omega$ is *statistically independent* or $*$*-independent iff* for any set $X \in \mathcal{I}_*$ we have that X is moderate and for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$, the following holds:

$$\lim_{n\to\infty} \left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)} \right) = 1\,. \tag{3.1}$$

In the case of convergence of $d_n\big(\bigcap_{E\in\mathcal{E}} E\big)$ for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$, this simplifies to asking for $0 < d(X) < 1$ to hold for all $X \in \mathcal{I}_*$ and

$$d\left(\bigcap_{E\in\mathcal{E}} E\right) = \prod_{E\in\mathcal{E}} d(E) \tag{3.2}$$

to hold for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_*$.

We denote the least cardinality of a maximal $*$-independent family by $\mathfrak{i}_*$, the *statistical independence number* or $*$*-independence number*.

[1] It is not clear whether weakening this to $\bar{d}(X) > 0$ as well as $\underline{d}(X) < 1$ would still yield all the desired results.

Lemma 3.1.3. *Let $\mathcal{I}_* \subseteq [\omega]^\omega$ be $*$-independent and $\mathcal{U} \subseteq \mathcal{I}_*$. Then*

$$\mathcal{J} = (\mathcal{I}_* \setminus \mathcal{U}) \cup \{\omega \setminus E \mid E \in \mathcal{U}\}$$

is $*$-independent, too.

Proof. Let $\mathcal{E} \subseteq \mathcal{I}_\rho$ be a finite subfamily, $E_0 \in \mathcal{E}$ and $\mathcal{E}' := \mathcal{E} \setminus \{E_0\}$. We will show that $\mathcal{E}'' := \mathcal{E}' \cup \{\omega \setminus E_0\}$ still satisfies condition 3.2. Indeed,

$$\bigcap_{E\in\mathcal{E}''} E = (\omega \setminus E_0) \cap \bigcap_{E\in\mathcal{E}'} E = \bigcap_{E\in\mathcal{E}'} E \setminus \bigcap_{E\in\mathcal{E}} E\,.$$

Now, since $\bigcap_{E\in\mathcal{E}} E$ is a subset of $\bigcap_{E\in\mathcal{E}'} E$, we get

$$\begin{aligned} d\left(\bigcap_{E\in\mathcal{E}''} E\right) &= d\left(\bigcap_{E\in\mathcal{E}'} E\right) - d\left(\bigcap_{E\in\mathcal{E}} E\right) = \prod_{E\in\mathcal{E}'} d(E) - \prod_{E\in\mathcal{E}} d(E) \\ &= (1-d(E_0)) \prod_{E\in\mathcal{E}'} d(E) = d(\omega\setminus E_0) \prod_{E\in\mathcal{E}'} d(E) = \prod_{E\in\mathcal{E}''} d(E)\,, \end{aligned}$$

which proves the claim. In fact, by the same argument, every subfamily of $\mathcal{E}''$ still satisfies condition 3.2. Hence, $\mathcal{E}''$ is $*$-independent, too.

Iterating the above argument, taking an arbitrary finite subfamily $\mathcal{E} \subseteq \mathcal{I}_\rho$ and replacing an arbitrary subset of its members by their complements, the resulting family is still $*$-independent.

Now, since $*$-independence only depends on finite subfamilies, we can conclude that $\mathcal{J}$ is $*$-independent. $\square$

Recall that a family $\mathcal{I}$ of subsets of ω is called *independent* if for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}$, the set

$$\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega \setminus B)$$

is infinite. Generalising this notion leads to the following definitions (which are more obviously related to the classical $\mathfrak{i}$):

Definition 3.1.4. Let $\rho \in (0,1)$. A family $\mathcal{I}_\rho \subseteq [\omega]^\omega$ is *ρ-independent iff* for any disjoint finite subfamilies $\mathcal{A}, \mathcal{B} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A \cap \bigcap_{B\in\mathcal{B}} (\omega\setminus B)\right) = \rho^{|\mathcal{A}|} \cdot (1-\rho)^{|\mathcal{B}|},$$

which simplifies to $= 1/2^{|\mathcal{A}|+|\mathcal{B}|}$ in the case of $\rho = 1/2$. By an argument similar to the one in Lemma 3.1.3, the above definition is equivalent to demanding that for any finite $\mathcal{A} \subseteq \mathcal{I}_\rho$, the following holds:

$$d\left(\bigcap_{A\in\mathcal{A}} A\right) = \rho^{|\mathcal{A}|}$$

We denote the least cardinality of a maximal ρ-independent family by $\mathfrak{i}_\rho$, the *ρ-independence number.*

Lemma 3.1.5. *For $\rho \in (0,1)$ a family $\mathcal{I}_\rho \subseteq [\omega]^\omega$ is ρ-independent if and only if $\mathcal{I}_\rho \subseteq [\omega]^\omega$ is $*$-independent and every E in $\mathcal{I}_\rho$ has density ρ.*

Proof. "$\Rightarrow$" Let $\mathcal{I}_\rho \subseteq [\omega]^\omega$ be ρ-independent. Clearly, all its members have density ρ. Moreover, given any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_\rho$, we can show condition 3.1:

$$\lim_{n\to\infty}\left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)}\right) = \frac{\lim_{n\to\infty}(d_n\left(\bigcap_{E\in\mathcal{E}} E\right))}{\lim_{n\to\infty}\left(\prod_{E\in\mathcal{E}} d_n(E)\right)} = \frac{d\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d(E)} = \frac{\rho^{|\mathcal{E}|}}{\rho^{|\mathcal{E}|}} = 1$$

We could have also showed condition 3.2.

"$\Leftarrow$" Let $\mathcal{I}_\rho \subseteq [\omega]^\omega$ be $*$-independent and let every E in $\mathcal{I}_\rho$ have density ρ. Now, given any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_\rho$, we can use condition 3.2 to show

$$d\left(\bigcap_{E\in\mathcal{E}} E\right) = \prod_{E\in\mathcal{E}} d(E) = \prod_{E\in\mathcal{E}} \rho = \rho^{|\mathcal{E}|},$$

which proves the claim.

□

Lemma 3.1.6. *For every $\rho \in (0,1)$, we have $\mathfrak{i}_\rho = \mathfrak{i}_{1-\rho}$.*

Proof. Let $\mathcal{I}_\rho \subseteq [\omega]^\omega$ be ρ-independent and define $\mathcal{I}_{1-\rho} := \{\omega \setminus E \mid E \in \mathcal{I}_\rho\}$. By Lemma 3.1.3 and Lemma 3.1.5, $\mathcal{I}_{1-\rho}$ is $(1-\rho)$-independent.

If $\mathcal{I}_\rho$ is maximal, so is $\mathcal{I}_{1-\rho}$. Indeed, if $\mathcal{I}_{1-\rho}$ was not maximal, we would find a set $X \in [\omega]^\omega \setminus \mathcal{I}_{1-\rho}$ such that $\mathcal{I}_{1-\rho} \cup \{X\}$ is still $(1-\rho)$-independent. By the same reasoning as above, this would imply that $\mathcal{I}_\rho \cup \{\omega \setminus X\}$ is ρ-independent, contradicting the assumption.

Hence, each maximal ρ-independent family of minimal cardinality yields a maximal $(1-\rho)$-independent family of the same cardinality, proving $\mathfrak{i}_\rho \geq \mathfrak{i}_{1-\rho}$ and, by symmetry, $\mathfrak{i}_\rho = \mathfrak{i}_{1-\rho}$. □

Motivated by the equivalence seen in Lemma 3.1.5 and by the equality seen in Lemma 3.1.6, we extend the definition of ρ-independence to $\rho \in \{0,1\}$, even though sets with density 0 or 1 are not moderate. We operate as follows:

Definition 3.1.7. A family $\mathcal{I}_0 \subseteq [\omega]^\omega$ is 0-*independent iff* all its members have density 0 and for any finite subfamily $\mathcal{E} \subseteq \mathcal{I}_0$, the following holds:

$$\lim_{n\to\infty}\left(\frac{d_n\left(\bigcap_{E\in\mathcal{E}} E\right)}{\prod_{E\in\mathcal{E}} d_n(E)}\right) = 1\,.$$

A family $\mathcal{I}_1 \subseteq [\omega]^\omega$ is 1-*independent iff* its pointwise complement $\{\omega \setminus E \mid E \in \mathcal{I}_1\}$ is 0-independent.

For $\rho \in \{0,1\}$, we denote the least cardinality of a maximal ρ-independent family by $\mathfrak{i}_\rho$. By definition, $\mathfrak{i}_0 = \mathfrak{i}_1$.

For $\rho \in (0,1)$ we will later see a proof of $\mathfrak{r}_\rho \leq \mathfrak{i}_\rho$ based on a proof of $\mathfrak{r} \leq \mathfrak{i}$. Given the above connection between ρ-independence and $*$-independence, the natural question is: Can we define $\mathfrak{r}_*$ analogously? Consider the following definitions:

Definition 3.1.8. Let $S, X \in [\omega]^\omega$.

- *S statistically splits X* or *S $*$-splits X*, written as $S \mid_* X$, *iff* S is moderate and

$$\lim_{n\to\infty} \left(\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} \right) = 1\,.$$

- Let $\mathbf{Rp}_* := \langle [\omega]^\omega, [\omega]^\omega, \nmid_* \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}_*) =: \mathfrak{s}_*, \quad \mathfrak{d}(\mathbf{Rp}_*) =: \mathfrak{r}_*.$$

  An $\mathbf{Rp}_*$-unbounding family is called *statistically splitting* or *$*$-splitting* and an $\mathbf{Rp}_*$-dominating family is called *statistically reaping* or *$*$-reaping*.

Theorem 3.1.9. *The relations shown in Figure 3.1 hold.*

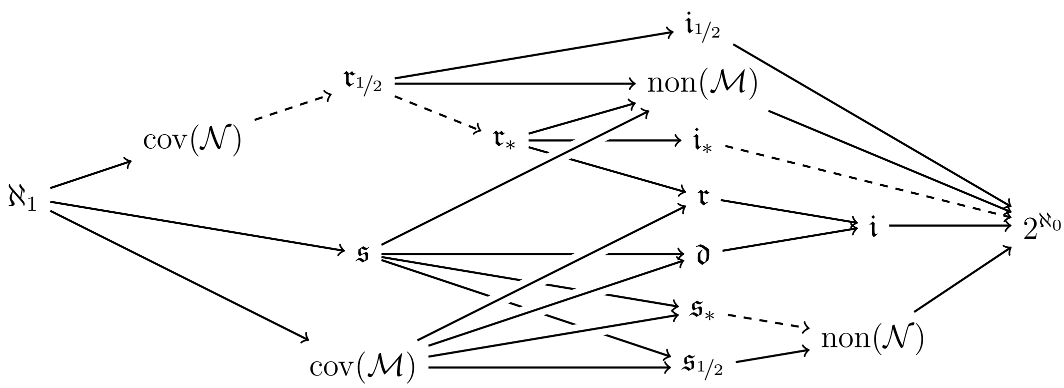

Figure 3.1: The ZFC-provable and/or consistent inequalities between $\mathfrak{i}_{1/2}$, $\mathfrak{i}_*$, $\mathfrak{r}_{1/2}$, $\mathfrak{r}_*$, $\mathfrak{s}_{1/2}$, $\mathfrak{s}_*$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

Proof. $\boldsymbol{\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})}$ and $\boldsymbol{\mathfrak{r}_{1/2} \geq \mathrm{cov}(\mathcal{N})}$ has been proved in Theorem 2.1.6.

$\boldsymbol{\mathfrak{s}_* \leq \mathrm{non}(\mathcal{N})}$ and $\boldsymbol{\mathfrak{r}_* \geq \mathrm{cov}(\mathcal{N})}$: The proof is mostly analogous to proving $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}_{1/2}$ as in Theorem 2.1.6.

Define $F\colon [\omega]_* \to [\omega]^\omega$ as the inclusion map and $G\colon [\omega]^\omega \to \mathcal{N}$ as

$$G(X) := \{S \in [\omega]^\omega \mid S \nmid_* X\}\,.$$

Then, $(F, G)\colon \mathbf{Cv}(\mathcal{N}) \to \mathbf{Rp}_*$ is a Tukey connection, since

$$F(S) \nmid_* X \quad \Rightarrow \quad S \in G(X)\,.$$

(We even have equivalence.)

It remains to show that G is well-defined, specifically that for all $X \in [\omega]^\omega$ the set $G(X)$ is a null set.

Let $X \in [\omega]^\omega$. As seen in Theorem 2.1.6 above, letting

$$\mathcal{S}_X := \{S \in [\omega]^\omega \mid S \mid_{1/2} X\},$$

we have that $\lambda(\mathcal{S}_X) = 1$. This is true in particular for $X = \omega$ and

$$\mathcal{S}_\omega = \{S \in [\omega]^\omega \mid S \mid_{1/2} \omega\} = \{S \in [\omega]^\omega \mid d(S) = 1/2\}.$$

Thus, $\lambda(\mathcal{S}_X \cap \mathcal{S}_\omega) = 1$. Now, since $S \mid_{1/2} X$ and $d(S) = 1/2$ implies $S \mid_* X$, we have $G(X) \subseteq [\omega]^\omega \setminus (\mathcal{S}_X \cup \mathcal{S}_\omega) \in \mathcal{N}$.

Hence, we have $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}_*$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}_* \leq \mathrm{non}(\mathcal{N}) \qquad \text{and} \qquad \mathfrak{r}_* \geq \mathrm{cov}(\mathcal{N})\,.$$

$\mathfrak{r}_{1/2} \leq \mathfrak{r}_*$: Let $\mathcal{R}_*$ be a $*$-reaping family and let $\mathcal{R}_{1/2} := \mathcal{R}_* \cup \{\omega\}$; clearly, $|\mathcal{R}_{1/2}| = |\mathcal{R}_*|$. Now, any S which bisects all $R \in \mathcal{R}_{1/2}$ also $*$-splits all $R \in \mathcal{R}_*$ – this follows from the fact that $S \mid_{1/2} \omega$ implies $d(S) = 1/2$. Hence, for any $R \in \mathcal{R}_*$, we now have

$$\frac{d_n(S \cap R)}{d_n(S) \cdot d_n(R)} = \frac{d_n(S \cap R)}{d_n(R)} \cdot \frac{1}{d_n(S)} \xrightarrow{n \to \infty} 1\,.$$

Indeed, $S \mid_{1/2} R$ implies that the first factor converges to $1/2$, while $d(S) = 1/2$ implies that the second factor converges to 2.

$\mathfrak{s}_{1/2} \geq \mathbf{cov}(\mathcal{M})$ and $\mathfrak{r}_{1/2} \leq \mathbf{non}(\mathcal{M})$ has been seen in Theorem 2.1.6.

$\mathfrak{s}_* \geq \mathbf{cov}(\mathcal{M})$ and $\mathfrak{r}_* \leq \mathbf{non}(\mathcal{M})$:

This is analogous to the proof of $\mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M})$, since the set of all reals $*$-split by a fixed moderate real S is a meagre set, as well. To see this, define a chopped real based on S with the interval partition having the partition boundaries at the $n!$-th elements of S. The sets matching this chopped real form a comeagre set which consists of reals X not $*$-split by S: As the matching intervals grow longer and longer, they "pull" $\frac{d_n(S \cap X)}{d_n(X)}$ above $1 - 1/n$, which implies that $\frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)}$ cannot converge to 1 as $d_n(S)$ does not converge to 1 by the moderacy of S.

$\mathfrak{s}_* \geq \mathfrak{s}$ and $\mathfrak{r}_* \leq \mathfrak{r}$: In fact, we will show $\mathbf{Rp}_* \preceq_{\mathrm{T}} \mathbf{Rp}$. If $F\colon [\omega]_* \to [\omega]^\omega$ is the inclusion map and $G\colon [\omega]^\omega \to [\omega]^\omega$ is the identity map, then the Tukey connection $(F, G)\colon \mathbf{Rp}_\rho \to \mathbf{Rp}$ has the desired properties.

To see this, we will prove by contradiction that $S \mid_* X$ implies $S \mid X$ for all $S \in [\omega]_*$, $X \in [\omega]^\omega$.

Suppose not, that is, suppose that for some $S \mid_* X$, either (a) $S \cap X$ is finite or (b) $X \setminus S$ is finite.

In case (a), we use the fact that S is moderate to see that there is $\varepsilon > 0$ such that for infinitely many n, we have $d_n(S) > \varepsilon$. Since $S \cap X$ is finite, we see that $|S \cap X \cap n|$ is bounded by some $k^* \in \omega$. Letting $k_n := |X \cap n|$, this yields

$$\exists^\infty n \in \omega\colon \frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} \leq \frac{k^*/n}{\varepsilon \cdot k_n/n} = \frac{k^*}{\varepsilon \cdot k_n} \xrightarrow{n\to\infty} 0\,,$$

which contradicts $S \mid_* X$.

Similarly, in case (b) we use the moderacy of S to see that there is $\delta > 0$ such that for infinitely many n, we have $d_n(S) < 1 - \delta$. Since $X \setminus S$ is finite, we see that $|S \cap X \cap n|$ is bounded from below by $k_n - k^*$ for some k^*. (This bound simply states that after finitely many exceptions, S contains all elements of X.) Taken together, we have

$$\begin{aligned}\exists^\infty n \in \omega\colon \frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} &\geq \frac{(k_n - k^*)/n}{(1-\delta) \cdot k_n/n} \\ &= \frac{1}{1-\delta} - \frac{k^*}{(1-\delta) \cdot k_n} \xrightarrow{n\to\infty} \frac{1}{1-\delta} = 1 + \varepsilon\end{aligned}$$

for some suitable $\varepsilon > 0$, which again contradicts $S \mid_* X$.

$\mathfrak{r}_{1/2} \leq \mathfrak{i}_{1/2}$ **and** $\mathfrak{r}_* \leq \mathfrak{i}_*$: For the first claim, let $\mathcal{I}_{1/2}$ be a maximal $1/2$-independent family. Define

$$\mathcal{R}_{1/2} := \left\{ \bigcap_{A \in \mathcal{A}} A \cap \bigcap_{B \in \mathcal{B}} (\omega \setminus B) \;\middle|\; \mathcal{A}, \mathcal{B} \subseteq \mathcal{I}_{1/2},\ \mathcal{A} \cap \mathcal{B} = \varnothing \right\}.$$

Then $\mathcal{R}_{1/2}$ is a $1/2$-reaping family, since the existence of an $S \in [\omega]^\omega$ bisecting each $R \in \mathcal{R}_{1/2}$ would contradict the maximality of $\mathcal{I}_{1/2}$.

The proof of the second claim is analogous: Take all finite tuples of sets in the witness $\mathcal{I}_*$ of the value of $\mathfrak{i}_*$ and collect their Boolean combinations in a family $\mathcal{R}_*$. This family must then be $*$-reaping, because a set S $*$-splitting each $R \in \mathcal{R}_*$ would violate the maximality of $\mathcal{I}_*$, and thus $\mathcal{R}_*$ witnesses $\mathfrak{r}_* \leq \mathfrak{i}_*$.

$\mathfrak{i}_\rho \leq 2^{\aleph_0}$ **and** $\mathfrak{i}_* \leq 2^{\aleph_0}$: For $\mathfrak{i}_\rho$, consider the collection $\mathbb{I}_\rho$ of all ρ-independent families. Now, $\mathbb{I}_\rho$ has finite character, i. e. for each $\mathcal{I} \subseteq 2^{\aleph_0}$, $\mathcal{I}$ belongs to $\mathbb{I}_\rho$ if and only if every finite subset of $\mathcal{I}$ belongs to $\mathbb{I}_\rho$. Hence we can apply Tukey's Lemma and see that $\mathbb{I}_\rho$ has a maximal element with respect to inclusion. Therefore, $\mathfrak{i}_\rho$ is well defined and hence $\mathfrak{i}_\rho \leq 2^{\aleph_0}$.

The proof for $\mathfrak{i}_*$ is analogous.

Con($\mathfrak{r}_* < \mathfrak{r}$**)**: This follows from Con(non($\mathcal{M}$) < cov($\mathcal{M}$)).

Con($\mathfrak{r}_{1/2}$ < non($\mathcal{M}$)) and Con($\mathfrak{r}_*$ < non($\mathcal{M}$)): This follows from Con($\mathfrak{r} <$ non($\mathcal{M}$)), see [BJ95, Model 7.5.9].

Con($\mathfrak{s} < \mathfrak{s}_*$): Just like Con($\mathfrak{r}_* < \mathfrak{r}$), this follows from Con(non($\mathcal{M}$) < cov($\mathcal{M}$)).

Con(cov($\mathcal{M}$) < $\mathfrak{s} \leq \mathfrak{s}_*$): Follows as in the proof of Con(cov($\mathcal{M}$) < $\mathfrak{s} \leq \mathfrak{s}_{1/2}$).

Con($\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2}$) and Con($\mathfrak{r}_* < \mathfrak{i}_*$): See Lemma 3.2.1 and Corollary 3.2.2 below.

Con($\mathfrak{i}_{1/2} < 2^{\aleph_0}$): This follows from Lemma 3.2.4 below. □

3.2 Bounds for $\mathfrak{i}_{1/2}$

Lemma 3.2.1. Con($\mathfrak{r}_{1/2} < \mathfrak{i}_{1/2}$).

Proof. We will prove the following: Assume CH in the ground model and let $\lambda > \mu > \aleph_1$ be regular cardinals with $\lambda = \lambda^{\aleph_0}$. Then there is a forcing extension satisfying $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mathfrak{r}_{1/2} = \mu$ and $\mathfrak{c} = \mathfrak{i}_{1/2} = \lambda$.

We prove this by using the forcing $\mathbb{P}{\restriction}_{(L,\mathcal{I})}$ and the model from [Bre02, Proposition 4.7]; this is essentially the Jörg Brendle's original template model (see [Bre02, Theorem 3.3]) with localisation forcing instead of Hechler forcing. It is shown in [Bre02] that this model satisfies $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = \mu$; since we know that $\mathrm{add}(\mathcal{N}) \leq \mathrm{cov}(\mathcal{N}) \leq \mathfrak{r}_{1/2} \leq \mathrm{non}(\mathcal{M}) \leq \mathrm{cof}(\mathcal{N})$, we also have $\mathfrak{r}_{1/2} = \mu$.

To show that $\mathfrak{i}_{1/2} = \lambda$ holds in this model, we use the isomorphism-of-names argument from [Bre02, Theorem 3.3]. Although the original proof of Theorem 3.3 uses Hechler forcing, it was already remarked in [Bre02] that this is irrelevant to the isomorphism-of-names argument as long as we use the same template. We will not reproduce the full extent of the argument here, but instead only point out the few differences.[2]

Let $\dot{\mathcal{A}} = \{\dot{A}^\alpha \mid \alpha < \kappa\}$ be a name for a $1/2$-independent family of size $\kappa < \lambda$; we have to show that $\dot{\mathcal{A}}$ is not maximal in $V^{\mathbb{P}{\restriction}_{(L,\mathcal{I})}}$. By $\mathfrak{r}_{1/2} \leq \mathfrak{i}_{1/2}$, we may assume $\mu \leq \kappa$; for technical reasons, we actually want to assume that $\omega_2 \cdot 2 \leq \kappa$. We now obtain the B^α as in the proof of Theorem 3.3 and use them to construct B^κ and the name $\dot{A}^\kappa$ in the same way. The pruning arguments and other details of the construction depend neither on the specific forcing poset nor on the particular properties of the names $\dot{A}^\alpha$, but only on the structure of the template, so every step of the proof works exactly as in [Bre02].

The only part we need to replace is the final paragraph ([Bre02, p. 23]). We instead observe that for any finite $F \subseteq \kappa$, we can find $\alpha < \omega_1$

- such that $B^\kappa \cup \bigcup_{\beta\in F} B^\beta$ and $B^\alpha \cup \bigcup_{\beta\in F} B^\beta$ are order isomorphic via the mapping fixing nodes of $\bigcup_{\beta\in F} B^\beta$ and moving B^κ to B^α, and
- such that the template restricted to $B^\kappa \cup \bigcup_{\beta\in F} B^\beta$ is basically the same as the template restricted to $B^\alpha \cup \bigcup_{\beta\in F} B^\beta$.[3]

Hence the posets $\mathbb{P}{\restriction}_{B^\kappa\cup\bigcup_{\beta\in F} B^\beta}$ and $\mathbb{P}{\restriction}_{B^\alpha\cup\bigcup_{\beta\in F} B^\beta}$ are isomorphic (and both are subforcings of the forcing poset $\mathbb{P}{\restriction}_{(L,\mathcal{I})}$). Since we know that $\mathbb{P}{\restriction}_{B^\alpha\cup\bigcup_{\beta\in F} B^\beta}$ forces that $\{\dot{A}^\alpha\} \cup \{\dot{A}^\beta \mid \beta \in F\}$ is a $1/2$-independent family, $\mathbb{P}{\restriction}_{B^\kappa\cup\bigcup_{\beta\in F} B^\beta}$ forces that $\{\dot{A}^\kappa\} \cup \{\dot{A}^\beta \mid \beta \in F\}$ is a $1/2$-independent family. Since $F \subseteq \kappa$ was arbitrary, this shows that $\{\dot{A}^\alpha \mid \alpha \leq \kappa\}$ is forced to be a $1/2$-independent family in $V^{\mathbb{P}{\restriction}_{(L,\mathcal{I})}}$, which shows that $\dot{\mathcal{A}}$ is not maximal in $V^{\mathbb{P}{\restriction}_{(L,\mathcal{I})}}$. □

[2] For a general approach to and explanation of template forcing, see [Bre05].

[3] Using the terms of [Bre02], this means α is such that $\mathcal{I}{\restriction}_{B^\kappa\cup\bigcup_{\beta\in F} B^\beta}$ is an innocuous extension of the image of $\mathcal{I}{\restriction}_{B^\alpha\cup\bigcup_{\beta\in F} B^\beta}$.

We remark that the construction in [Bre03] can be modified analogously to show that $\mathfrak{i}_{1/2}$ can have countable cofinality; see the subsequent section.

Corollary 3.2.2. Con($\mathfrak{r}_* < \mathfrak{i}_*$).

Proof. Replacing the names for 1/2-independent families $\dot{\mathcal{A}}$ with names for $*$-independent families, the same proof as in Lemma 3.2.1 shows the analogous result. □

For the final proof of this section, we will require another combinatorial lemma.

Lemma 3.2.3. *If $R, S \subseteq \omega$, $0 < r < 1$, $\varepsilon > 0$ and $m < n$ are such that*

$$\frac{|R \cap m|}{m} \in (r - \varepsilon, r + \varepsilon)$$

and for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|S \cap \ell|}{\ell} \in (r - \varepsilon, r + \varepsilon),$$

then for all ℓ with $m \leq \ell \leq n$, we have

$$\frac{|(R \cap m) \cup (S \cap [m, \ell))|}{\ell} \in (r - 3\varepsilon, r + 3\varepsilon).$$

Proof. Suppose this were false for some $\ell^* \geq m$; then without loss of generality,

$$\frac{|(R \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} \geq r + 3\varepsilon.$$

Since

$$\frac{|R \cap m|}{m} < r + \varepsilon,$$

we get

$$\frac{|S \cap [m, \ell^*)|}{\ell^*} \geq r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon).$$

But then

$$\frac{|S \cap m|}{m} > r - \varepsilon$$

implies

$$\frac{|S \cap \ell^*|}{\ell^*} = \frac{|(S \cap m) \cup (S \cap [m, \ell^*))|}{\ell^*} > \frac{m}{\ell^*}(r - \varepsilon) + r + 3\varepsilon - \frac{m}{\ell^*}(r + \varepsilon)$$
$$= r + 3\varepsilon - \frac{2m}{\ell^*} \cdot \varepsilon \geq r + \varepsilon,$$

which is a contradiction. □

Lemma 3.2.4. $\mathrm{Con}(\mathfrak{i}_{1/2} < \mathfrak{i})$.

Proof. The proof is analogous to the classical proof of $\mathrm{Con}(\aleph_1 = \mathfrak{a} < 2^{\aleph_0})$ (see e. g. [Hal17, Proposition 18.5]).

Assume CH in the ground model and let $\lambda \geq \aleph_2$. We force with the λ-Cohen forcing poset $\mathbb{C}_\lambda$; letting G be a $\mathbb{C}_\lambda$-generic filter, it is clear that $V[G] \vDash \mathfrak{i} = 2^{\aleph_0} = \lambda$. We will now show $V[G] \vDash \mathfrak{i}_{1/2} = \aleph_1$ by constructing a maximal $1/2$-independent family $\mathcal{A}$ in the ground model such that $\mathcal{A}$ remains maximal $1/2$-independent in $V[G]$. By the usual arguments, it suffices to consider what happens to a countably infinite $1/2$-independent family when forcing with just $\mathbb{C} := \langle 2^{<\omega}, \subseteq\rangle$.

Let $\mathcal{A}_0 := \{A_n \subseteq [\omega]^{\aleph_0} \mid n < \omega\}$ be such a family. Fix (in the ground model) an enumeration $\{(p_\alpha, \dot{X}_\alpha) \mid \omega \leq \alpha < \omega_1\}$ of all pairs $(p, \dot{X})$ such that $p \in \mathbb{C}$ and $\dot{X}$ is a nice name for a subset of ω.[4] In particular, this means that for any $\langle\check{n}, p_1\rangle, \langle\check{n}, p_2\rangle \in \dot{X}$, either $p_1 = p_2$ or $p_1 \perp p_2$. Note that since $V \vDash$ CH, there are just $\aleph_1$ many nice names for subsets of ω in V.

We now construct $\mathcal{A}$ from $\mathcal{A}_0$ iteratively as follows: Let $\omega \leq \alpha < \omega_1$ and assume we have already defined sets $A_\beta \subseteq \omega$ for all $\beta < \alpha$. Below, we will construct $A_\alpha \subseteq \omega$ such that the following two properties hold:

(i) The family $\{A_\beta \mid \beta \leq \alpha\}$ is $1/2$-independent.

(ii) If $p_\alpha \Vdash |\dot{X}_\alpha| = \aleph_0 \wedge$ "$\{A_\beta \mid \beta < \alpha\} \cup \{\dot{X}_\alpha\}$ is $1/2$-independent", then for all $m < \omega$, the set $D^\alpha_m := \{q \in \mathbb{C} \mid \exists n \geq m \colon q \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})\}$ is dense below p_α.

We first show that the $\mathcal{A} := \{A_\beta \mid \beta \leq \omega_1\}$ constructed this way is a maximal $1/2$-independent family in $V^{\mathbb{C}}$. Clearly, $\mathcal{A}$ is $1/2$-independent, so only maximality could fail. Suppose it were not maximal; then there is a condition p and a nice name $\dot{X}$ for a subset of ω such that $p \Vdash$ "$\mathcal{A} \cup \{\dot{X}\}$ is $1/2$-independent". Let α be such that $(p, \dot{X}) = (p_\alpha, \dot{X}_\alpha)$ and let $\varepsilon > 0$ be sufficiently small (e. g. $\varepsilon < 1/16$). We can then find $q \leq p_\alpha$ and $m < \omega$ such that

$$q \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap \ell|}{\ell} \in \left(\frac{1}{4} - \varepsilon, \frac{1}{4} + \varepsilon\right) \text{ for all } \ell \geq 2^m \tag{$*_1$}$$

(because p_α forces that $\{A_\alpha, \dot{X}_\alpha\}$ is $1/2$-independent) and

$$\frac{|A_\alpha \cap [2^n, 2^{n+1})|}{2^n} > \frac{1}{2} - \varepsilon \text{ for all } n \geq m.$$

Now by the density of D^α_m below p_α, we can find $r \leq q$ and some $n \geq m$ such that $r \Vdash A_\alpha \cap [2^n, 2^{n+1}) = \dot{X}_\alpha \cap [2^n, 2^{n+1})$. But this implies that

$$\begin{aligned} r \Vdash \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^{n+1}|}{2^{n+1}} &= \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap 2^n|}{2^n} + \frac{1}{2} \cdot \frac{|A_\alpha \cap \dot{X}_\alpha \cap [2^n, 2^{n+1})|}{2^n} \\ &> \frac{1/4 - \varepsilon}{2} + \frac{1/2 - \varepsilon}{2} = \frac{3}{8} - \varepsilon > \frac{1}{4} + \varepsilon, \end{aligned}$$

[4] The reason the index set of the enumeration is $[\omega, \omega_1)$ instead of $[0, \omega_1)$ is just to make the notation more convenient.

which contradicts Eq. ($*_1$).

We finally have to show that we can find such an A_α satisfying (i) and (ii) for any $\omega \le \alpha < \omega_1$. We only have to consider those α such that $\dot{X}_\alpha$ satisfies the assumption in property (ii), since finding an A_α with property (i) is straightforward. Enumerate $\{A_\beta \mid \beta < \alpha\}$ as $\{B_n \mid n < \omega\}$. For $n < \omega$ and any partial function $f\colon n \to \{-1, 1\}$, we let

$$B^f := \bigcap_{i \in \mathrm{dom}(f)} B_i^{f(i)},$$

where $B_i^1 := B$ and $B_i^{-1} := \omega \setminus B$. We further pick some strictly decreasing sequence of real numbers $\langle \delta_n \mid n < \omega \rangle$ with $\delta_0 := 3$ and $\lim_{n\to\infty} \delta_n = 0$ and let $\langle q_n \mid n < \omega \rangle$ be some sequence enumerating all conditions below p_α infinitely often. We will now construct, by induction on $n < \omega$, conditions $r_n \le q_n' \le q_n$, a strictly increasing sequence of natural numbers $\langle k_n \mid n < \omega \rangle$ and initial segments $Z_n = A_\alpha \cap 2^{k_n}$ of A_α such that for all $n < \omega$ and all partial functions $f\colon n \to \{-1, 1\}$, the following four statements will hold (with $F := |\mathrm{dom}(f)| + 1$)

(R1) $\dfrac{|B^f \cap Z_n \cap 2^{k_n}|}{2^{k_n}}, \dfrac{|(B^f \setminus Z_n) \cap 2^{k_n}|}{2^{k_n}} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$,

(R2) $q_n' \Vdash \dfrac{|B^f \cap \dot{X}_\alpha \cap \ell|}{\ell}, \dfrac{|(B^f \setminus \dot{X}_\alpha) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \dfrac{\delta_n}{3}, \dfrac{1}{2^F} + \dfrac{\delta_n}{3}\right)$
for all ℓ with $2^{k_n} \le \ell \le 2^{k_{n+1}}$,

(R3) $\dfrac{|B^f \cap Z_{n+1} \cap \ell|}{\ell}, \dfrac{|(B^f \setminus Z_{n+1}) \cap \ell|}{\ell} \in \left(\dfrac{1}{2^F} - \delta_n, \dfrac{1}{2^F} + \delta_n\right)$
for all ℓ with $2^{k_n} \le \ell \le 2^{k_{n+1}}$, and

(R4) $r_n \Vdash Z_{n+1} \cap [2^{k_n}, 2^{k_{n+1}}) = \dot{X}_\alpha \cap [2^{k_n}, 2^{k_{n+1}})$.

It is clear that (R1)–(R4) taken together for all $n < \omega$ imply that $A_\alpha := \bigcup_{n<\omega} Z_n$ is as required by (i) and (ii).

For $n = 0$, let $k_0 := 0$, $q_0' := q_0$ and $Z_0 := \varnothing$; then (R1) and (R2) hold vacuously by our choice of δ_0, and there is nothing to show yet for (R3) and (R4).

Now assume that we have obtained k_n, $q_n' \le q_n$ and Z_n such that (R1) and (R2) hold for n; we will construct $r_n \le q_n'$, k_{n+1}, $q_{n+1}' \le q_{n+1}$ and Z_{n+1} such that (R3) and (R4) hold for n and such that (R1) and (R2) hold for $n+1$. We first find $q_{n+1}' \le q_{n+1}$ and $k_n' \ge k_n$ such that for all partial functions $f\colon n+1 \to \{-1, 1\}$, we have that (with $F := |\mathrm{dom}(f)| + 1$)

$$q_{n+1}' \Vdash \frac{|B^f \cap \dot{X}_\alpha \cap \ell|}{\ell}, \frac{|(B^f \setminus \dot{X}_\alpha) \cap \ell|}{\ell} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{3}, \frac{1}{2^F} + \frac{\delta_{n+1}}{3}\right)$$

for all $\ell \ge 2^{k_n}$ (hence satisfying (R2) for $n+1$); this is possible since the assumption in property (ii) is true. Next we find $r_n \le q_n'$ and a sufficiently large $k_{n+1} \ge k_n'$ such that for all partial functions $f\colon n+1 \to \{-1, 1\}$, we have that (still with

$F := |\operatorname{dom}(f)| + 1)$

$$r_n \Vdash \frac{|B^f \cap \dot{X}_\alpha \cap 2^{k_{n+1}}|}{2^{k_{n+1}}}, \frac{|(B^f \setminus \dot{X}_\alpha) \cap 2^{k_{n+1}}|}{2^{k_{n+1}}} \in \left(\frac{1}{2^F} - \frac{\delta_{n+1}}{6}, \frac{1}{2^F} + \frac{\delta_{n+1}}{6}\right) \qquad (*_2)$$

and that r_n decides $\dot{X}_\alpha \cap 2^{k_{n+1}}$; in particular, let $X_n \subseteq [2^{k_n}, 2^{k_{n+1}})$ be such that $r_n \Vdash \dot{X}_\alpha \cap [2^{k_n}, 2^{k_{n+1}}) = X_n$. All this is also possible since the assumption in property (ii) is true. Let $Z_{n+1} := Z_n \cup X_n$.

Now, (R4) holds for n by definition of Z_{n+1}. Apply Lemma 3.2.3 to $R := Z_n$, $S := \dot{X}_\alpha[r_n]$, $r := 1/2^F$, $\varepsilon := \delta_n$, $m := 2^{k_n}$ and $n := 2^{k_{n+1}}$ to see that (R3) for n follows from (R1) and (R2) for n and our choice of Z_{n+1}. Finally, (R1) for $n+1$ follows from Eq. ($*_2$), (R4) for n and the choice of a sufficiently large k_{n+1} (e.g. using the argument from Lemma 2.3.3).

By the usual arguments, our construction implies that $\mathcal{A}$ remains maximal $1/2$-independent in $V^{\mathbb{C}_\lambda}$. □

3.3 Adding $^1/_2$-independent Families and Forcing $\mathrm{cf}(\mathfrak{i}_{1/2}) = \omega$

We describe a forcing for adding a maximal $^1/_2$-independent family generically with a product-style forcing (like Hechler's forcing for adding a mad family [Hec72]). This gives an alternative proof of the consistency of $\mathfrak{i}_{1/2} < \mathfrak{c}$, while also showing that there can be (consistently) simultaneously maximal $^1/_2$-independent families of many different sizes and that $\mathrm{cf}(\mathfrak{i}_{1/2}) = \omega$ is consistent. We note in this context that the consistency of $\mathrm{cf}(\mathfrak{i}) = \omega$ is a well-known open problem.

Definition 3.3.1. Fix an uncountable cardinal κ. We define the forcing $\mathbb{P} = \mathbb{P}_\kappa$ as follows. Conditions are of the form $p = (F^p, n^p, \bar{a}^p, \varepsilon^p)$ such that

(C1) $F^p \subseteq \kappa$ is finite,

(C2) $n^p \in \omega$,

(C3) $\bar{a}^p = \langle a^p_\alpha \subseteq n^p \mid \alpha \in F^p \rangle$,

(C4) $\varepsilon^p \colon 2^{\leq F^p} \longrightarrow \mathbb{Q}^+$ (where $2^{\leq F^p}$ denotes the partial functions from F^p to 2) is such that $\varepsilon^p(f) \leq \varepsilon^p(g)$ whenever $f \subseteq g$,

(C5) for all $f \in 2^{\leq F^p}$, we have

$$\left| \frac{|\bigcap_{f(\alpha)=1} a^p_\alpha \cap \bigcap_{f(\alpha)=0} (n^p \setminus a^p_\alpha)|}{n^p} - \frac{1}{2^{|\mathrm{dom}(f)|}} \right| < \frac{\varepsilon^p(f)}{8},$$

and

(C6) we have

$$\frac{2^{2|F^p|}}{n^p} < \frac{\varepsilon^p}{8},$$

where $\varepsilon^p := \varepsilon^p(\varnothing) = \min\{\varepsilon^p(f) \mid f \in 2^{\leq F^p}\}$

The order is given by $q \leq p$ if

(D1) $F^p \subseteq F^q$,

(D2) $n^p \leq n^q$,

(D3) $a^p_\alpha = a^q_\alpha \cap n^p$ for all $\alpha \in F^p$,

(D4) $\varepsilon^p(f) \geq \varepsilon^q(f)$ for all $f \in 2^{\leq F^p}$, and

(D5) for all i with $n^p \leq i \leq n^q$ and all $f \in 2^{\leq F^p}$, we have

$$\left| \frac{|\bigcap_{f(\alpha)=1} (i \cap a^q_\alpha) \cap \bigcap_{f(\alpha)=0} (i \setminus a^q_\alpha)|}{i} - \frac{1}{2^{|\mathrm{dom}(f)|}} \right| < \varepsilon^p(f).$$

We first need to check we can extend conditions arbitrarily.

Definition 3.3.2. Given a condition p and $E \subseteq \kappa$, we define the restriction $p' = p\restriction_E$ of p to E by

(i) $F^{p'} = F^p \cap E$,

(ii) $n^{p'} = n^p$,

(iii) $a^{p'}_\alpha = a^p_\alpha$ for $\alpha \in F^{p'}$, and

(iv) $\varepsilon^{p'} = \varepsilon^p\restriction_{2^{\leq F^{p'}}}$.

It is easy to see that $p' \in \mathbb{P}$ and that $p \leq p'$. Also, for $f \in 2^{\leq F^p}$, let

$$b^p_f := \bigcap_{f(\alpha)=1} a^p_\alpha \cap \bigcap_{f(\alpha)=0} (n^p \smallsetminus a^p_\alpha).$$

Lemma 3.3.3 (extendibility lemma)**.** *Let $p \in \mathbb{P}$, $E \subseteq \kappa$, $p' = p\restriction_E$, $m \in \omega$, and $\varepsilon\colon 2^{\leq F^p} \longrightarrow \mathbb{Q}^+$ with $\varepsilon(f) \leq \varepsilon(g)$ whenever $f \subseteq g$ and $\varepsilon(f) \leq \varepsilon^p(f)$ for all $f \in 2^{\leq F^p}$. Assume $q' \leq p'$ is such that $F^{q'} \subseteq E$. Then there is a condition $q \in \mathbb{P}$ with $q \leq p$, $q \leq q'$, $F^q = F^p \cup F^{q'}$, $n^q \geq m$, and*

- $\varepsilon^q(f) = \min\{\varepsilon(f), \varepsilon^{q'}(f)\}$ *for all* $f \in 2^{\leq F^{p'}}$,
- $\varepsilon^q(f) = \varepsilon(f)$ *for all* $f \in 2^{\leq F^p} \smallsetminus 2^{\leq F^{p'}}$,
- $\varepsilon^q(f) = \varepsilon^{q'}(f)$ *for all* $f \in 2^{\leq F^{q'}} \smallsetminus 2^{\leq F^{p'}}$, *and*
- $\varepsilon^q(f) = 16$ *for all other* $f \in 2^{\leq F^q}$.

Proof. Let $F := F^q := F^{q'} \cup F^p$. Define $\varepsilon^q\colon 2^{\leq F} \longrightarrow \mathbb{Q}^+$ as stipulated in the statement of the lemma. Finally, let $n := n^q \geq \max\{m, n^{q'}\}$ be so large that

- $n - n^{q'}$ is divisible by $2^{|F|}$,
- $\frac{n^{q'}}{n} < \frac{\varepsilon^q}{8}$, and
- $\frac{2^{2|F|}}{n} < \frac{\varepsilon^q}{8}$.

Note that the last item immediately guarantees (C6). We produce the required extension in two steps. The main point is to prove (D5) for $q \leq p$ and $q \leq q'$ and condition (C5) for $q \in \mathbb{P}$.

In the first step we extend to $n^{q'}$. This step is only necessary if $E \neq \varnothing$ and $n^{q'} > n^p$. Let $\{\alpha_\ell \mid \ell \in |F^p \smallsetminus E|\}$ enumerate $F^p \smallsetminus E$. For each $f \in 2^{F^{p'}}$, let $c_f := b^{q'}_f \smallsetminus b^p_f = b^{q'}_f \smallsetminus n^p$. Note that the c_f are pairwise disjoint, that their union is the interval $[n^p, n^{q'})$ and that in case $F^{p'} = \varnothing$, we have $c_\varnothing = [n^p, n^{q'})$.

Let $\{c_f(j) \mid j \in m_f\}$ be the increasing enumeration of c_f. For each $\ell \in |F^p \smallsetminus E|$ and each $f \in 2^{F^{p'}}$, define

$$a^q_{\alpha_\ell} \cap c_f := \left\{ c_f(j) \;\middle|\; j \in m_f \cap \bigcup_k [2^{\ell+1}k, 2^{\ell+1}k + 2^\ell) \right\}. \qquad (*_3)$$

Thus $a^q_{\alpha_\ell} \cap [n^p, n^{q'})$ is the disjoint union of the sets $a^q_{\alpha_\ell} \cap c_f$. We need to see that (D5) is satisfied for all i with $n^p \le i \le n^{q'}$ and all $g \in 2^{\le F^p}$. Hence we fix such i and g. We may assume that $\mathrm{dom}(g) \not\subseteq E$ (otherwise, (D5) holds by $q' \le p'$). We will only show that

$$\frac{|i \cap b^q_g|}{i} < \frac{1}{2^{|\mathrm{dom}(g)|}} + \varepsilon^p(g);$$

the second inequality is analogous.

Let $f = g\restriction_E = g\restriction_{F^{p'}} \in 2^{\le F^{p'}}$, hence $f \subsetneq g$. By (C5) for p and f, we know that

$$\left|n^p \cap b^{q'}_f\right| = \left|b^p_f\right| > n^p \cdot \left(\frac{1}{2^{|\mathrm{dom}(f)|}} - \frac{\varepsilon^p(f)}{8}\right),$$

and by (D5) for $q' \le q$ and f,

$$\left|i \cap b^{q'}_f\right| < i \cdot \left(\frac{1}{2^{|\mathrm{dom}(f)|}} + \varepsilon^p(f)\right);$$

thus

$$\left|[n^p, i) \cap b^{q'}_f\right| < \frac{i - n^p}{2^{|\mathrm{dom}(f)|}} + \frac{9i \cdot \varepsilon^p(f)}{8}.$$

For $f' \in 2^{F^{p'}}$ with $f \subseteq f'$ we have, by Eq. ($*_3$),

$$\left|[n^p, i) \cap b^q_{f' \cup g}\right| = \left|i \cap c_{f'} \cap b^q_{g\restriction_{F^p \setminus E}}\right| \le \frac{1}{2^{|\mathrm{dom}(g) \setminus E|}} \cdot |i \cap c_{f'}| + 2^{|F^p \setminus E|}.$$

Since $[n^p, i) \cap b^q_g$ is the disjoint union of the $[n^p, i) \cap b^q_{f' \cup g}$ and $[n^p, i) \cap b^{q'}_f$ is the disjoint union of the $i \cap c_{f'}$, we see that

$$\begin{aligned}
\left|[n^p, i) \cap b^q_g\right| &= \sum_{f \subseteq f' \in 2^{F^{p'}}} \left|[n^p, i) \cap b^q_{f' \cup g}\right| \\
&\le \frac{1}{2^{|\mathrm{dom}(g) \setminus E|}} \cdot \sum_{f \subseteq f' \in 2^{F^{p'}}} |i \cap c_{f'}| + 2^{|F^{p'} \setminus \mathrm{dom}(f)|} \cdot 2^{|F^p \setminus E|} \\
&\le \frac{1}{2^{|\mathrm{dom}(g) \setminus E|}} \cdot \left|[n^p, i) \cap b^{q'}_f\right| + 2^{|F^p|} \\
&< \frac{i - n^p}{2^{|\mathrm{dom}(g)|}} + \frac{9i \cdot \varepsilon^p(f)}{8 \cdot 2^{|\mathrm{dom}(g) \setminus E|}} + 2^{|F^p|}
\end{aligned}$$

and thus, by (C5) for p and g and (C6) for p, and using that g strictly extends f,

$$\begin{aligned}
\frac{|i \cap b^q_g|}{i} &= \frac{|n^p \cap b^q_g|}{i} + \frac{|[n^p, i) \cap b^q_g|}{i} \\
&< \frac{1}{2^{|\mathrm{dom}(g)|}} + \frac{\varepsilon^p(g)}{8} + \frac{9 \cdot \varepsilon^p(f)}{16} + \frac{\varepsilon^p}{8} < \frac{1}{2^{|\mathrm{dom}(g)|}} + \frac{7 \cdot \varepsilon^p(g)}{8},
\end{aligned} \tag{$*_4$}$$

as required.

We now extend from $n^{q'}$ to $n = n^q$. Let $\{\alpha_\ell \mid \ell \in |F^{p'}|\}$ enumerate $F^{p'}$. Next let $\tilde{\ell} = \min\{|F^p \smallsetminus F^{p'}|, |F^{q'} \smallsetminus F^{p'}|\}$. Let $\{\alpha_{2\ell+|F^{p'}|} \mid \ell < \tilde{\ell}\}$ enumerate the next $\tilde{\ell}$ many elements of $F^p \smallsetminus F^{p'} = F^p \smallsetminus E$, and let $\{\alpha_{2\ell+1+|F^{p'}|} \mid \ell < \tilde{\ell}\}$ enumerate the next $\tilde{\ell}$ many elements of $F^{q'} \smallsetminus F^{p'}$. Finally let $\{\alpha_\ell \mid |F^{p'}| + 2\tilde{\ell} \leq \ell < |F|\}$ enumerate the remaining elements of F. Define

$$a^q_{\alpha_\ell} \cap [n^{q'}, n) = \bigcup_k [n^{q'} + 2^{\ell+1}k, n^{q'} + 2^{\ell+1}k + 2^\ell) \qquad (*_5)$$

for $\ell < |F|$. First, we need to show (D5) for all i with $n^{q'} \leq i < n$ and all $g \in 2^{\leq F^p} \cup 2^{\leq F^{q'}}$. Fix such i and g. Without loss of generality, we may assume $g \in 2^{\leq F^p}$. (For $g \in 2^{\leq F^{q'}}$ the proof is the same.) Again, we only show the inequality

$$\frac{|i \cap b^q_g|}{i} < \frac{1}{2^{|\mathrm{dom}(g)|}} + \varepsilon^p(g).$$

By Eq. ($*_5$) and the choice of the sequence of the α_ℓ, we have

$$\left|[n^{q'}, i) \cap b^q_g\right| \leq \frac{i - n^{q'}}{2^{|\mathrm{dom}(g)|}} + 2^{2|F^p|}.$$

Thus, by Eq. ($*_4$) for $n^{q'}$, we have

$$\begin{aligned} \frac{|i \cap b^q_g|}{i} &= \frac{|n^{q'} \cap b^q_g|}{i} + \frac{|[n^{q'}, i) \cap b^q_g|}{i} \\ &< \frac{1}{2^{|\mathrm{dom}(g)|}} + \frac{7 \cdot \varepsilon^p(g)}{8} + \frac{2^{2|F^p|}}{i} < \frac{1}{2^{|\mathrm{dom}(g)|}} + \varepsilon^p(g), \end{aligned}$$

as required.

Finally, we need to show condition (C5) for q and $g \in 2^{\leq F}$. Since $n - n^{q'}$ is divisible by $2^{|F|}$, it is easy to see that

$$\left|[n^{q'}, n) \cap b^p_g\right| = \frac{n - n^{q'}}{2^{|\mathrm{dom}(g)|}}.$$

Thus

$$\frac{1}{2^{|\mathrm{dom}(g)|}} \cdot \frac{n - n^{q'}}{n} \leq \frac{|b^p_g|}{n} \leq \frac{1}{2^{|\mathrm{dom}(g)|}} \cdot \frac{n - n^{q'}}{n} + \frac{n^{q'}}{n},$$

and the required inequality follows from $\frac{n^{q'}}{n} < \frac{\varepsilon^q}{8}$. □

Corollary 3.3.4. *Let $p \in \mathbb{P}$ and $m \in \omega$. Then there is a condition $q \in \mathbb{P}$ with $q \leq p$ and $n^q \geq m$. Furthermore, we may require $F^q = F^p$ and $\varepsilon^q = \varepsilon^p$.*

Proof. Apply Lemma 3.3.3 with $E = \varnothing$ (so $p' = q'$ is the trivial condition) and $\varepsilon = \varepsilon^p$. □

Corollary 3.3.5. *Let $p \in \mathbb{P}$ and $\alpha \in \kappa$. Then there is a condition $q \in \mathbb{P}$ with $q \leq p$ and $\alpha \in F^q$.*

Proof. We may assume $\alpha \notin F^p$. Apply Lemma 3.3.3 with $E = \{\alpha\}$ (so p' is the trivial condition) and arbitrary q' with $F^{q'} = E = \{\alpha\}$. □

Corollary 3.3.6. *Let $p \in \mathbb{P}$ and $\varepsilon\colon 2^{\leq F^p} \longrightarrow \mathbb{Q}^+$ with $\varepsilon(f) \leq \varepsilon(g)$ whenever $f \subseteq g$. Then there is a condition $q \in \mathbb{P}$ with $q \leq p$ such that $\varepsilon^q(f) \leq \varepsilon(f)$ for all $f \in 2^{\leq F^p}$.*

Proof. Apply Lemma 3.3.3 with $E = \varnothing$ (so $p' = q'$ is the trivial condition). □

Lemma 3.3.7 (compatibility lemma)**.** *Assume $p, q \in \mathbb{P}$ are such that $n^p = n^q$, $a^p_\alpha = a^q_\alpha$ for all $\alpha \in F^p \cap F^q$, and $\varepsilon^p{\restriction}_{2^{\leq(F^p \cap F^p)}} = \varepsilon^q{\restriction}_{2^{\leq(F^p \cap F^p)}}$. Then p and q are compatible.*

Proof. Apply Lemma 3.3.3 with $p = p$, $E = F^q$, $m = n^p$, and $\varepsilon = \varepsilon^p$. Note that $q' = q$ satisfies the necessary assumptions. □

Corollary 3.3.8 (ccc)**.** *$\mathbb{P}$ is ccc and thus preserves cardinals.*

Proof. This follows from a Δ-system argument together with Lemma 3.3.7. □

Definition 3.3.9. For $X \subseteq \kappa$, let $\mathbb{P}_X$ be the collection of conditions $p \in \mathbb{P}_\kappa$ with $F^p \subseteq X$.

Corollary 3.3.10 (complete embeddability)**.** *For any $X \subseteq \kappa$, $\mathbb{P}_X$ completely embeds into $\mathbb{P}_\kappa$.*

Proof. By Lemma 3.3.3, $p' = p{\restriction}_X \in \mathbb{P}_X$ is a reduction of $p \in \mathbb{P}_\kappa$. □

Let G be $\mathbb{P}$-generic over V. For $\alpha < \kappa$, let $A_\alpha := \bigcup\{a^p_\alpha \mid p \in G\}$. By the corollaries of Lemma 3.3.3 (Corollary 3.3.4, Corollary 3.3.5 and Corollary 3.3.6), we immediately see:

Corollary 3.3.11. *$\{A_\alpha \mid \alpha < \kappa\}$ is a $1/2$-independent family.*

Next, combining the basic idea of Hechler's classical work [Hec72] with the combinatorics of Lemma 3.3.3, we have:

Lemma 3.3.12 (maximality)**.** *$\{A_\alpha \mid \alpha < \kappa\}$ is a maximal $1/2$-independent family.*

Proof. Let $\dot{B}$ be a $\mathbb{P}$-name for an infinite and coinfinite subset of ω. For each $i \in \omega$, let M_i be a maximal antichain of conditions deciding $i \in \dot{B}$. By Corollary 3.3.8, each M_i is at most countable. Thus we can find a countable $X \subseteq \kappa$ such that $F^p \subseteq X$ for all $p \in \bigcup_i M_i$. Let $\beta \in \kappa \setminus X$. Clearly, it suffices to show:

Claim. *Assume $p_0 \in \mathbb{P}$ forces that $\dot{B}$ is $1/2$-independent from all $\dot{A}_\alpha$ for $\alpha \in X$. Then p_0 forces that for all k, there is an $\ell > k$ such that*

$$\frac{|\ell \cap \dot{B} \cap \dot{A}_\beta|}{\ell} > \frac{3}{8}.$$

(Note that, analogously, we can show that p_0 forces that for all k there is an $\ell > k$ such that

$$\frac{|\ell \cap \dot{B} \cap \dot{A}_\beta|}{\ell} < \frac{1}{8},$$

and in fact, it is not difficult to see that an elaboration of the argument shows that p_0 forces $\underline{d}(\dot{B} \cap \dot{A}_\beta) = 0$ and $\bar{d}(\dot{B} \cap \dot{A}_\beta) = 1/2$.)

Fix $p \leq p_0$ and k. We need to find $\ell > k$ and $r \leq p$ forcing the required statement. We may assume $n^p \geq k$ and $\beta \in F^p$. We may also assume that for f_0 with $\mathrm{dom}(f_0) = \{\beta\}$ and $f_0(\beta) = 1$, $\varepsilon^p(f_0) < 1/2$.

Let $p' = p\restriction_X$. For $f \in 2^{\leq F^{p'} \cup \{\beta\}}$ with $\beta \in \mathrm{dom}(f)$, let $\dot{C}_f$ denote the name

$$\bigcap_{f(\alpha)=1} \dot{A}_\alpha \cap \bigcap_{f(\alpha)=0} (\omega \setminus \dot{A}_\alpha) \cap \dot{B}^{f(\beta)}$$

where $\dot{B}^1 = \dot{B}$ and $\dot{B}^0 = \omega \setminus \dot{B}$. By assumption on $\dot{B}$, we may find $q' \leq p'$ with $F^{q'} \subseteq X$ and $k' \geq n^p$ such that

$$q' \Vdash \forall i \geq k' \; \forall f \in 2^{\leq F^{p'} \cup \{\beta\}} \colon \left| \frac{|i \cap \dot{C}_f|}{i} - \frac{1}{2^{|\mathrm{dom}(f)|}} \right| < \frac{\varepsilon^p(f)}{16}.$$

We may assume $n^{q'} \geq k'$.

Now apply Lemma 3.3.3 with p, $E = X$, $m = k'$, $\varepsilon = \varepsilon^p$ and q' to obtain q such that $q \leq p$, $q \leq q'$, $F^q = F^{q'} \cup F^p$, $\varepsilon^q(f) = \varepsilon^p(f)$ for all $f \in 2^{\leq F^p} \setminus 2^{\leq F^{p'}}$, and $\varepsilon^q(f) = 16$ for all f whose domain is not contained in either F^p or $F^{q'}$. Let $q'' = q\restriction_{X \cup \{\beta\}}$. We may assume $q' = q\restriction_X = q''\restriction_X$.

Let $\ell \geq 8n^q$. We may find $r' \leq q'$ with $F^{r'} \subseteq X$ such that r' decides $\dot{B} \cap \ell$. By Corollary 3.3.4, we may also assume

$$\frac{2^{2(|F^{r'}|+1)}}{n^{r'}} < \frac{\varepsilon^{r'}}{8}. \tag{$*_6$}$$

Next, let $s \leq r'$ with $F^s \subseteq X$ such that s decides $\dot{B} \cap n^{r'}$. We now define a condition r'' with $r'' \leq r'$ and $r'' \leq q''$ as follows:

- $F^{r''} = F^{r'} \cup \{\beta\} = F^{r'} \cup F^{q''}$,

- $n^{r''} = n^{r'}$,
- $a^{r''}_\alpha = a^{r'}_\alpha$ for $\alpha \in F^{r'}$, $a^{r''}_\beta \cap n^q = a^q_\beta$, and, for $n^q \leq i < n^{r'}$, $i \in a^{r''}_\beta$ iff $s \Vdash i \in \dot{B}$, and
- $\varepsilon^{r''}\restriction_{2^{\leq F^{r'}}} = \varepsilon^{r'}$, $\varepsilon^{r''}(f) = \varepsilon^{q''}(f)$ for $f \in 2^{\leq F^{q''}}$ with $\beta \in \mathrm{dom}(f)$, and $\varepsilon^{r''}(f) = 16$ for all remaining f.

We need to check that r'' is indeed a condition and $r'' \leq q''$. ($r'' \leq r'$ then follows trivially.)

Fix i with $n^q \leq i \leq n^{r'}$. Also let $f \in 2^{\leq F^{p'} \cup \{\beta\}}$ with $\beta \in \mathrm{dom}(f)$. (There is nothing to show for other f, because they either belong to $2^{\leq F^{r'}}$ or they satisfy $\varepsilon^{r''}(f) = 16$.)

We will show only

$$\frac{|i \cap b^{r''}_f|}{i} < \frac{1}{2^{|\mathrm{dom}(f)|}} + \varepsilon^p(f),$$

since the other inequality is analogous. By assumption on q' and s, we know

$$s \Vdash \left| n^q \cap \dot{C}_f \right| > n^q \cdot \left(\frac{1}{2^{|\mathrm{dom}(f)|}} - \frac{\varepsilon^p(f)}{16} \right)$$

and

$$s \Vdash \left| i \cap \dot{C}_f \right| < i \cdot \left(\frac{1}{2^{|\mathrm{dom}(f)|}} + \frac{\varepsilon^p(f)}{16} \right).$$

Therefore

$$s \Vdash \left| [n^q, i) \cap \dot{C}_f \right| < \frac{i - n^q}{2^{|\mathrm{dom}(f)|}} + \frac{n^q \cdot \varepsilon^p(f)}{16} + \frac{i \cdot \varepsilon^p(f)}{16}.$$

By the definition of $a^{r''}_\beta$, we now see that

$$\left| [n^q, i) \cap b^{r''}_f \right| < \frac{i - n^q}{2^{|\mathrm{dom}(f)|}} + \frac{n^q \cdot \varepsilon^p(f)}{16} + \frac{i \cdot \varepsilon^p(f)}{16}.$$

On the other hand, by (C5) for q and f,

$$\left| n^q \cap b^{r''}_f \right| = \left| b^q_f \right| < n^q \cdot \left(\frac{1}{2^{|\mathrm{dom}(f)|}} + \frac{\varepsilon^p(f)}{8} \right).$$

Hence

$$\frac{\left| i \cap b^{r''}_f \right|}{i} < \frac{1}{2^{|\mathrm{dom}(f)|}} + \frac{n^q}{i} \cdot \frac{3 \cdot \varepsilon^p(f)}{16} + \frac{e^p(f)}{16} < \frac{1}{2^{|\mathrm{dom}(f)|}} + \varepsilon^p(f),$$

as required for (D5). Furthermore, using $n^{r'} \geq 8n^q$, the previous formula with $i = n^{r'}$ gives

$$\frac{\left| n^{r'} \cap b^{r''}_f \right|}{n^{r'}} < \frac{1}{2^{|\mathrm{dom}(f)|}} + \frac{e^p(f)}{8}$$

as required for (C5). On the other hand, since $|F^{r''}| = |F^{r'}| + 1$, condition (C6) is an immediate consequence of Eq. ($*_6$).

Finally, apply Lemma 3.3.3 with $p = q$, $E = X \cup \{\beta\}$, $p' = q''$, $m = \ell$, $\varepsilon = \varepsilon^q$ and $q' = r''$ to obtain r with $r \leq q$, $r \leq r''$. In particular, we have $r \leq p$, and since $r \leq r''$, r forces that $[n^q, \ell) \cap \dot{B} = [n^q, \ell) \cap \dot{A}_\beta$. Now note that

$$r' \Vdash \left|n^q \cap \dot{B}\right| < n^q \cdot \left(\frac{1}{2} + \frac{\varepsilon^p(f_0)}{16}\right)$$

and

$$r' \Vdash \left|\ell \cap \dot{B}\right| > \ell \cdot \left(\frac{1}{2} - \frac{\varepsilon^p(f_0)}{16}\right).$$

Therefore

$$r \Vdash \left|[n^q, \ell) \cap \dot{B}\right| = \left|[n^q, \ell) \cap \dot{B} \cap \dot{A}_\beta\right| > \frac{\ell - n^q}{2} - \frac{\ell \cdot \varepsilon^p(f_0)}{8}$$

and hence, using $\ell \geq 8n^q$ and $\varepsilon^p(f_0) < 1/2$,

$$r \Vdash \frac{\left|\ell \cap \dot{B} \cap \dot{A}_\beta\right|}{\ell} > \frac{\ell - n^q}{2\ell} - \frac{\varepsilon^p(f_0)}{8} > \frac{7}{16} - \frac{1}{16} = \frac{3}{8}$$

as required. □

Thus we obtain:

Theorem 3.3.13. *Let κ be an uncountable cardinal. There is a generic extension with a maximal $1/2$-independent family of size κ.*

Using a finite support product of forcings $\mathbb{P}_\kappa$ together with an argument due to Blass ([Bla93, Theorem 9]), we see:

Theorem 3.3.14. *Let V be a model of* ZFC *and* GCH*. In V, let C be a closed set of uncountable cardinals with $\aleph_1 \in C$, $\kappa \in C$ for $\aleph_1 \leq \kappa \leq |C|$ and $\lambda^+ \in C$ for $\lambda \in C$ with $\mathrm{cf}(\lambda) = \omega$.*

Then there is a ccc poset $\mathbb{Q}$ forcing $\mathfrak{c} = \max(C)$ and, in the generic extension, there is a maximal $1/2$-independent family of size κ if and only if $\kappa \in C$.

For a similar argument, cf. [BSZ00, Theorem 3.2].

Embedding the partial order $\mathbb{P}_\lambda$ (for λ of countable cofinality) into the template framework as in [Bre03], we see:

Theorem 3.3.15. *Assume* CH *and let λ be a singular cardinal of countable cofinality. Then there is a forcing extension satisfying $\mathfrak{i}_{1/2} = \lambda$. In particular, $\mathfrak{i}_{1/2} = \aleph_\omega$ is consistent.*

Proof. Assume $\mathrm{cf}(\lambda) = \omega$. Instead of Hechler's poset for adding a mad family of size λ, embed $\mathbb{P}_\lambda$ into the template framework of [Bre03]. (The argument works the same way as the modification of [Bre02] in the proof of Lemma 3.2.1.) □

For a similar argument, cf. [FT15].

Note that since $\mathrm{cov}(\mathcal{N})$ is a lower bound of $\mathfrak{i}_{1/2}$, it is clear (and much easier to prove) that $\mathfrak{i}_{1/2}$ can be a singular cardinal of uncountable cofinality (in the appropriate random model).

Chapter 4

Further Results

4.1 More Characteristics Related to $\mathfrak{r}$ and $\mathfrak{s}$

We introduce further characteristics related to $\mathfrak{r}$ and $\mathfrak{s}$. The characteristics related to $\mathfrak{s}$ were defined and studied in [BHK+18]. Some of the results about the dual characteristics related to $\mathfrak{r}$ were independently shown by Barnabás Farkas in correspondence with the authors. In this section, we unify these results using Tukey connections.

We define relations on $[\omega]^\omega \times [\omega]^\omega$ and their associated cardinal characteristics.

Definition 4.1.1. Let $S, X \in [\omega]^\omega$. We say:

- For $0 < \varepsilon < 1/2$, S *ε-almost bisects* X, written as $S \mid_{1/2\pm\varepsilon} X$, *iff* for all but finitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

- S *weakly bisects* X, written as $S \mid^w_{1/2} X$, *iff* for any $\varepsilon > 0$, for infinitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} \in \left(\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right).$$

- S *bisects* X *infinitely often*, written as $S \mid^\infty_{1/2} X$, *iff* for infinitely many $n < \omega$ we have

$$\frac{|S \cap X \cap n|}{|X \cap n|} = \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{2}.$$

Based on these relations (or, more precisely, on their negations) we can define the following relational systems:

Definition 4.1.2. Let

- $\mathbf{Rp}_{1/2\pm\varepsilon} := \langle [\omega]^\omega, [\omega]^\omega, \not\propto_{1/2\pm\varepsilon} \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}_{1/2\pm\varepsilon}) =: \mathfrak{s}_{1/2\pm\varepsilon}, \quad \mathfrak{d}(\mathbf{Rp}_{1/2\pm\varepsilon}) =: \mathfrak{r}_{1/2\pm\varepsilon}.$$

  An $\mathbf{Rp}_{1/2\pm\varepsilon}$-unbounding family is called *ε-almost bisecting* and an $\mathbf{Rp}_{1/2\pm\varepsilon}$-dominating family is called *ε-almost $1/2$-reaping.*

- $\mathbf{Rp}^w_{1/2} := \langle [\omega]^\omega, [\omega]^\omega, \not\propto^w_{1/2} \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}^w_{1/2}) =: \mathfrak{s}^w_{1/2}, \quad \mathfrak{d}(\mathbf{Rp}^w_{1/2}) =: \mathfrak{r}^w_{1/2}.$$

  An $\mathbf{Rp}^w_{1/2}$-unbounding family is called *weakly bisecting* and an $\mathbf{Rp}^w_{1/2}$-dominating family is called *weakly $1/2$-reaping.*

- $\mathbf{Rp}^\infty_{1/2} := \langle [\omega]^\omega, [\omega]^\omega, \not\propto^\infty_{1/2} \rangle$ and let

$$\mathfrak{b}(\mathbf{Rp}^\infty_{1/2}) =: \mathfrak{s}^\infty_{1/2}, \quad \mathfrak{d}(\mathbf{Rp}^\infty_{1/2}) =: \mathfrak{r}^\infty_{1/2}.$$

  An $\mathbf{Rp}^\infty_{1/2}$-unbounding family is called *infinitely often bisecting* and an $\mathbf{Rp}^\infty_{1/2}$-dominating family is called *infinitely often $1/2$-reaping.*

Theorem 4.1.3. *The relations shown in Figure 4.1 hold. We also give proofs for the dual relations, except for* $\mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2})$.

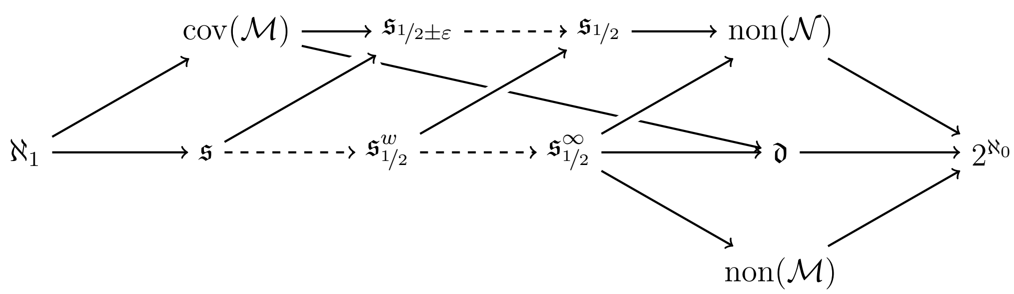

Figure 4.1: The ZFC-provable and/or consistent inequalities between $\mathfrak{s}_{1/2}$, $\mathfrak{s}_{1/2\pm\varepsilon}$, $\mathfrak{s}^w_{1/2}$, $\mathfrak{s}^\infty_{1/2}$ and other well-known cardinal characteristics, where $\longrightarrow$ means "$\leq$, consistently $<$" and $\dashrightarrow$ means "$\leq$, possibly $=$".

Proof. Recall that it is known that $\mathfrak{s} \leq \mathrm{non}(\mathcal{M})$ and $\mathfrak{s} \leq \mathrm{non}(\mathcal{N})$ (see e. g. [Bla10, Theorem 5.19]) as well as $\mathfrak{s} \leq \mathfrak{d}$ (see e. g. [Hal17, Theorem 9.4] or [Bla10, Theorem 8.13]). All mentioned cardinal characteristics and their duals are strictly larger than $\aleph_1$ in a model of "ZFC + MA + $2^{\aleph_0} > \aleph_1$" and they are consistently strictly smaller than $2^{\aleph_0}$, e. g. in the Sacks model (see [BJ95, Model 7.6.2]).

The following inequalities

$$
\begin{aligned}
\mathbf{Rp}_{1/2\pm\varepsilon} &\preceq_{\mathrm{T}} \mathbf{Rp}, & \mathfrak{s}_{1/2\pm\varepsilon} &\geq \mathfrak{s}, & \mathfrak{r}_{1/2\pm\varepsilon} &\leq \mathfrak{r},\\
\mathbf{Rp}^{w}_{1/2} &\preceq_{\mathrm{T}} \mathbf{Rp}, & \mathfrak{s}^{w}_{1/2} &\geq \mathfrak{s}, & \mathfrak{r}^{w}_{1/2} &\leq \mathfrak{r},\\
\mathbf{Rp}_{1/2} &\preceq_{\mathrm{T}} \mathbf{Rp}_{1/2\pm\varepsilon}, & \mathfrak{s}_{1/2} &\geq \mathfrak{s}_{1/2\pm\varepsilon}, & \mathfrak{r}_{1/2} &\leq \mathfrak{r}_{1/2\pm\varepsilon},\\
\mathbf{Rp}_{1/2} &\preceq_{\mathrm{T}} \mathbf{Rp}^{w}_{1/2}, & \mathfrak{s}_{1/2} &\geq \mathfrak{s}^{w}_{1/2}, & \mathfrak{r}_{1/2} &\leq \mathfrak{r}^{w}_{1/2},\\
\mathbf{Rp}^{\infty}_{1/2} &\preceq_{\mathrm{T}} \mathbf{Rp}^{w}_{1/2}, & \mathfrak{s}^{\infty}_{1/2} &\geq \mathfrak{s}^{w}_{1/2}, & \mathfrak{r}^{\infty}_{1/2} &\leq \mathfrak{r}^{w}_{1/2}
\end{aligned}
$$

can all be proved using the identity Tukey connection, i. e. the Tukey connection (F, G) where both F and G are the identity map on $[\omega]^\omega$. To give one example, proving for all S and X in $[\omega]^\omega$ that $S \mid_{1/2} X \Rightarrow S \mid^{w}_{1/2} X$, yields $\mathbf{Rp}_{1/2} \preceq_{\mathrm{T}} \mathbf{Rp}^{w}_{1/2}$. Therefore, in the five cases mentioned above, we will not explicitly repeat the fact that we are using Tukey connections.

$\boldsymbol{\mathfrak{s} \leq \mathfrak{s}^{w}_{1/2} \leq \mathfrak{s}^{\infty}_{1/2}}$: Clearly, an infinitely often bisecting real is a weakly bisecting real; being equal to $1/2$ infinitely often implies entering an arbitrary ε-neighbourhood of $1/2$ infinitely often.
Moreover, a weakly bisecting real is a splitting real. Indeed, if a real X does not split another real Y, the relative initial density of X in Y, that is

$$\frac{d_n(X \cap Y)}{d_n(Y)},$$

cannot be close to $1/2$ for infinitely many n.

$\boldsymbol{\mathfrak{s} \leq \mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{s}_{1/2}}$: The first claim follows since an ε-almost bisecting real is a splitting real by the fact that finite sets have density 0 and cofinite sets have density 1, and hence if X does not split Y, the relative initial densities of X and $\omega \setminus X$ in Y tend to 0 and 1, respectively (or vice versa).
The second claim follows since a bisecting real is an ε-almost bisecting real by definition.

$\boldsymbol{\mathfrak{s}_{1/2\pm\varepsilon} \geq \mathrm{cov}(\mathcal{M})}$: Let $G\colon [\omega]^\omega \to [\omega]^\omega$ be the identity map and let $F\colon [\omega]^\omega \to \mathcal{M}$ be

$$F(S) := \{X \in [\omega]^\omega \mid S \mid_{1/2\pm\varepsilon} X\}\,.$$

Then, $(F, G)\colon \mathbf{Rp}_{1/2\pm\varepsilon} \to \mathbf{Cv}(\mathcal{M})^\perp$ is a Tukey connection, as

$$F(S) \not\ni X \quad \Rightarrow \quad S \nmid_{1/2\pm\varepsilon} G(X)\,.$$

(We even have equivalence.)

It remains to show that F is well-defined, specifically that for all $S \in [\omega]^\omega$ the set $F(S)$ is meagre.

We inductively define a chopped real (S, Π) based on S as follows: Let the first interval of the partition Π be $I_0 = [0, \min(S)]$. For any $n \in \omega$, given $m_n := \max(I_n)$, chose m_{n+1} minimal such that $I_{n+1} := [m_n + 1, m_{n+1}]$ contains $n\, m_n + 1$ elements of S.

Now, any real X matching this chopped real is not ε-almost bisected by S. Indeed, whenever such an X is equal to S on one of the intervals I_n, we have

$$\frac{d_{m_n}(S \cap X)}{d_{m_n}(X)} \geq \frac{n-1}{n}\,.$$

As $\varepsilon < {}^1\!/_2$, for n large enough we get $1 - {}^1\!/_n > {}^1\!/_2 + \varepsilon$, and since such an X is equal to S on I_n for infinitely many $n \in \omega$, S does not ε-almost bisect X in the limit.

Now, the family $M := \mathrm{Match}(S, \langle I_n \rangle_{n\in\omega})$ is a comeagre set, and thus $F(S)$, being a subset of $[\omega]^\omega \setminus M$, is meagre.

Hence we have $\mathbf{Rp}_{1/2\pm\varepsilon} \preceq_{\mathrm{T}} \mathbf{Cv}(\mathcal{M})^\perp$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}_{1/2\pm\varepsilon} \geq \mathrm{cov}(\mathcal{M}) \qquad \text{and} \qquad \mathfrak{r}_{1/2\pm\varepsilon} \leq \mathrm{non}(\mathcal{M})\,.$$

$\boldsymbol{\mathfrak{s}^{w}_{1/2} \leq \mathfrak{s}_{1/2}}$: A bisecting real is a weakly splitting real. Indeed, for the relative density to converge to ${}^1\!/_2$, it has to eventually be arbitrarily close to ${}^1\!/_2$, and hence also within an arbitrary ε-neighbourhood of ${}^1\!/_2$ infinitely often.

$\boldsymbol{\mathfrak{s}^{\infty}_{1/2} \leq \mathrm{non}(\mathcal{M})}$: We define $F\colon [\omega]^\omega \to [\omega]^\omega$ as the identity map and $G\colon [\omega]^\omega \to \mathcal{M}$ as

$$G(X) := \{S \in [\omega]^\omega \mid S \not\mid^{\infty}_{1/2} X\}\,.$$

Then, $(F, G)\colon \mathbf{Cv}(\mathcal{M}) \to \mathbf{Rp}^{\infty}_{1/2}$ is a Tukey connection, as we get

$$F(S) \not\mid^{\infty}_{1/2} X \quad \Rightarrow \quad S \in G(X)\,.$$

(We even have equivalence.)

It remains to show that G is well-defined, specifically that for all $X \in [\omega]^\omega$ the set $G(X)$ is a meagre set.

Define a chopped real (R, Π) as follows: Let $f_X\colon \omega \to X$ be the ascending enumeration of $X \in [\omega]^\omega$. For all $n \in \omega$ define intervals $J_n := [f_X(3^n), f_X(3^{n+1}))$, $I_{n+1} := J_{2n} \cup J_{2n+1}$ as well as $I_0 := [0, f_X(1))$, and let $\Pi := \langle I_n \rangle_{n\in\omega}$. Define $R \subsetneq X$ such that for each $n \in \omega$,

$$R \cap J_{2n} = \varnothing\,, \qquad\qquad R \cap J_{2n+1} = X \cap J_{2n+1}\,.$$

Suppose the real R_0 matches (R, Π) and is equal to R on I_{n+1}. Let $k := \max(J_{2n})$ and $\ell := \max(J_{2n+1}) = \max(I_{n+1})$. Then we have:

$$\frac{d_k(R_0 \cap X)}{d_k(X)} \leq \frac{1}{3}\,, \qquad\qquad \frac{d_\ell(R_0 \cap X)}{d_\ell(X)} \geq \frac{2}{3}\,.$$

By Corollary 1.1.5, this implies that R_0 bisects X infinitely often.

Now, the family $M := \mathrm{Match}(R, \langle I_n \rangle_{n\in\omega})$ is a comeagre set, and thus $G(S)$, being a subset of $[\omega]^\omega \smallsetminus M$, is meagre.

Hence we have $\mathbf{Cv}(\mathcal{M}) \preceq_{\mathrm{T}} \mathbf{Rp}^\infty_{1/2}$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}^\infty_{1/2} \leq \mathrm{non}(\mathcal{M}) \qquad \text{and} \qquad \mathfrak{r}^\infty_{1/2} \geq \mathrm{cov}(\mathcal{M}) \,.$$

$\mathfrak{s}^\infty_{1/2} \leq \mathfrak{d}$: We will define a Tukey connection $(F, G)\colon \mathbf{Dm}^\perp \to \mathbf{Rp}^\infty_{1/2}$. For $X \in [\omega]^\omega$, let f_X be its enumeration. Define $G\colon [\omega]^\omega \to \omega^\omega, X \mapsto f_X$.

To define $F\colon \omega^\omega \to [\omega]^\omega$, start with $g \in \omega^\omega$. Without loss of generality assume g is strictly increasing and satisfies $g(0) > 0$. Define $\bar{g}\colon \omega \to \omega$ by $\bar{g}(n) := g^{(n+1)}(0)$ for every $n < \omega$. Then, for any $X \in [\omega]^\omega$ with $f_X \leq^* g$ and for sufficiently large n,

$$\bar{g}(n) \leq f_X(\bar{g}(n)) < g(\bar{g}(n)) = \bar{g}(n+1).$$

Hence (for sufficiently large n) every interval $[\bar{g}(n), \bar{g}(n+1))$ contains at least one element of X and at most $\bar{g}(n+1) - \bar{g}(n)$ many. Now iteratively define a function $\Gamma_g\colon \omega \to \omega$ by $\Gamma_g(0) := 0$, $\Gamma_g(1) := \bar{g}(0) = g(0)$ and for $n \in \omega$:

$$\Sigma_n := \sum_{k=0}^{n} \Gamma_g(k)\,, \qquad\qquad \Gamma_g(n+1) := \bar{g}(\Sigma_n)$$

and consider the interval partition with partition boundaries $\langle \Gamma_g(n) \mid n < \omega \rangle$. For sufficiently large n, every interval

$$\begin{aligned} I_n :=& \Big[\Gamma_g(n), \Gamma_g(n+1)\Big) = \Big[\bar{g}\big(\sum_{k=0}^{n-1}(\Gamma_g(k))\big), \bar{g}\big(\sum_{k=0}^{n}(\Gamma_g(k))\big)\Big) \\ =& \Big[\bar{g}(\Sigma_{n-1}), \bar{g}(\Sigma_{n-1}+1)\Big) \cup \ldots \cup \Big[\bar{g}(\Sigma_{n-1} + \Gamma_g(n) - 1), \bar{g}(\Sigma_{n-1} + \Gamma_g(n))\Big) \end{aligned}$$

contains at least $\Gamma_g(n)$ many elements of X and at most $\Gamma_g(n+1) - \Gamma_g(n)$ many of them.

Finally, let $Y_g := \bigcup_{k\in\omega} I_{2k} = \bigcup_{k\in\omega} [\Gamma_g(2k), \Gamma_g(2k+1))$.

Then, for any $X \in [\omega]^\omega$ with $f_X \leq^* g$, Y_g bisects X infinitely often: Indeed, the number of elements of X which are in any interval I_n is at least as large as the lower boundary of I_n. Moreover, Y_g is defined to alternate between consecutive intervals. This means the relative initial density

$$\frac{d_\ell(Y_g \cap X)}{d_\ell(X)}$$

is infinitely often above $1/2$ after all but finitely many even intervals I_{2k}, infinitely often below $1/2$ after all but finitely many odd intervals I_{2k+1}. Thus, by Corollary 1.1.5, Y_g bisects X infinitely often.

Define $F\colon \omega^\omega \to [\omega]^\omega$ as $F(g) := Y_g$. We have just proved that for all $g \in \omega^\omega$ and $X \in [\omega]^\omega$ we have

$$g \geq^* G(X) \Rightarrow F(g) \mid^\infty_{1/2} X\,.$$

This proves that $(F, G)\colon \mathbf{Dm}^\perp \to \mathbf{Rp}^\infty_{1/2}$ is indeed a Tukey connection. Hence we have $\mathbf{Dm}^\perp \preceq_{\mathrm{T}} \mathbf{Rp}^\infty_{1/2}$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}^\infty_{1/2} \leq \mathfrak{d} \qquad \text{and} \qquad \mathfrak{r}^\infty_{1/2} \geq \mathfrak{b}\,.$$

$\boldsymbol{\mathfrak{s}^\infty_{1/2} \leq \mathrm{non}(\mathcal{N})}$: We are using a Tukey connection analogous to the one used to prove $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}_{1/2}$ in Theorem 2.1.6. Define $F\colon [\omega]^\omega \to [\omega]^\omega$ as the identity map and $G\colon [\omega]^\omega \to \mathcal{N}$ as

$$G(X) := \{S \in [\omega]^\omega \mid S \nmid^\infty_{1/2} X\}\,.$$

Then, $(F, G)\colon \mathbf{Cv}(\mathcal{N}) \to \mathbf{Rp}^\infty_{1/2}$ is a Tukey connection, as

$$F(S) \nmid^\infty_{1/2} X \quad \Rightarrow \quad S \in G(X)\,.$$

(We even have equivalence.)

It remains to show that G is well-defined, specifically that for all $X \in [\omega]^\omega$ the set $G(X)$ is a null set.

Consider $X, S \in [\omega]^\omega$ and let f_X be the enumerating function of X. Define the function $g_S(n) := |X \cap S \cap f_X(n)| - n/2$. Consider the probability space $([\omega]^\omega, \lambda)$ with λ the Lebesgue measure. Then $g_S(n)$ defines a balanced random walk with step size $1/2$, since

$$g_S(n+1) - g_S(n) = \begin{cases} +1/2 & f_X(n) \in S, \\ -1/2 & f_X(n) \notin S. \end{cases}$$

From probability theory we know that for λ-almost all $S \in [\omega]^\omega$, $g_S(n)$ will be 0 for infinitely many $n \in \omega$. Equivalently, λ-almost surely,

$$\frac{g_S(n)}{n} + \frac{1}{2} = \frac{|X \cap S \cap f_X(n)|}{n}$$

will be $1/2$ infinitely often.

In other words, for any $X \in [\omega]^\omega$, the set of all S not bisecting X infinitely often is a null set.

Hence we have $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}^\infty_{1/2}$ and, by Definition 1.2.1, this implies

$$\mathfrak{s}^\infty_{1/2} \leq \mathrm{non}(\mathcal{N}) \qquad \text{and} \qquad \mathfrak{r}^\infty_{1/2} \geq \mathrm{cov}(\mathcal{N})\,.$$

$\mathfrak{s}_{1/2} \leq \mathrm{non}(\mathcal{N})$: This has been proved in Theorem 2.1.6.

Con($\mathrm{non}(\mathcal{M}) < \mathfrak{s}_{1/2\pm\varepsilon}$) and hence **Con($\mathfrak{s}^{\infty}_{1/2} < \mathfrak{s}_{1/2\pm\varepsilon}$):** This is implied by the consistency of $\mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M})$ as witnessed by the Cohen model, see [BJ95, Model 7.5.8].

Con($\mathfrak{s}^{\infty}_{1/2} < \mathrm{non}(\mathcal{M})$), Con($\mathfrak{s}^{\infty}_{1/2} < \mathfrak{d}$) and **Con($\mathfrak{s}^{\infty}_{1/2} < \mathrm{non}(\mathcal{N})$):** In the Cohen model, we have $\aleph_1 = \mathfrak{s} = \mathfrak{s}^{\infty}_{1/2} = \mathrm{non}(\mathcal{M}) < \mathrm{non}(\mathcal{N}) = \mathfrak{d}$, see [BJ95, Model 7.5.8]; and in the random model, we have $\aleph_1 = \mathfrak{s}^{\infty}_{1/2} = \mathfrak{d} < \mathrm{non}(\mathcal{M})$, see [BJ95, Model 7.6.8].

Con($\mathrm{cov}(\mathcal{M}) < \mathfrak{s} \leq \mathfrak{s}_{1/2}$): In the Mathias model, we have $\mathrm{cov}(\mathcal{M}) < \mathfrak{s} = 2^{\aleph_0}$, see [Hal17, Theorem 26.14].

Con($\mathfrak{s}_{1/2} < \mathrm{non}(\mathcal{N})$): See Theorem 2.3.5 above. □

4.2 Cardinals Related to $\mathfrak{hom}$ and $\mathfrak{par}$

Definition 4.2.1. Let

$$P_{2,2} := \{f\colon [\omega]^2 \to 2\}$$

be the set of 2-colourings (or 2-partitions) of pairs of natural numbers.

The following relations on $P_{2,2} \times [\omega]^\omega$ are analogous to the ones in Definition 1.1.2 and are therefore denoted by the same symbols.

Definition 4.2.2. Let $X \in [\omega]^\omega$, $f \in P_{2,2}$ and $\rho \in (0,1)$. We say:

- f *splits* X, written as $f \mid X$, *iff* f is not almost monochromatic on X, i. e. if both $f^{-1}(0) \cap [X]^2$ and $f^{-1}(1) \cap [X]^2$ are infinite sets.
- f *bisects* X *in the limit* (or just f *bisects* X), written as $f \mid_{1/2} X$, *iff*

$$\lim_{n\to\infty} \frac{|f^{-1}(1) \cap [X \cap n]^2|}{|[X \cap n]^2|} = \frac{1}{2}\,.$$

- f ρ*-splits* X *in the limit* (or just f ρ*-splits* X), written as $f \mid_\rho X$, *iff*

$$\lim_{n\to\infty} \frac{|f^{-1}(1) \cap [X \cap n]^2|}{|[X \cap n]^2|} = \rho\,.$$

Recall the following well-known cardinal characteristics:

- $\mathfrak{par} := \min\{|\mathcal{P}| \mid \mathcal{P} \subseteq P_{2,2} \text{ and } \forall H \in [\omega]^\omega\ \exists p \in \mathcal{P}\colon p \mid H\}$ (the partition number),
- $\mathfrak{hom} := \min\{|\mathcal{H}| \mid \mathcal{H} \subseteq [\omega]^\omega \text{ and } \nexists p \in P_{2,2}\ \forall H \in \mathcal{H}\colon p \mid H\}$ (the homogeneity number).

Definition 4.2.3. We define the following relational systems:

- Let $\mathbf{Hm} := \langle P_{2,2}, [\omega]^\omega, \nmid \rangle$. Then

$$\mathfrak{b}(\mathbf{Hm}) = \mathfrak{par}, \quad \mathfrak{d}(\mathbf{Hm}) = \mathfrak{hom}.$$

- Let $\mathbf{Hm}_{1/2} := \langle P_{2,2}, [\omega]^\omega, \nmid_{1/2} \rangle$ and let

$$\mathfrak{b}(\mathbf{Hm}_{1/2}) =: \mathfrak{par}_{1/2}, \quad \mathfrak{d}(\mathbf{Hm}_{1/2}) =: \mathfrak{hom}_{1/2}.$$

- Let $\mathbf{Hm}_\rho := \langle P_{2,2}, [\omega]^\omega, \nmid_\rho \rangle$ and let

$$\mathfrak{b}(\mathbf{Hm}_\rho) =: \mathfrak{par}_\rho, \quad \mathfrak{d}(\mathbf{Hm}_\rho) =: \mathfrak{hom}_\rho.$$

Lemma 4.2.4. *For every $\rho \in (0,1)$, we have $\mathbf{Rp}_\rho \preceq_{\mathrm{T}} \mathbf{Hm}_\rho$. By Definition 1.2.1, this implies*

$$\mathfrak{s}_\rho \geq \mathfrak{par}_\rho \qquad \textit{and} \qquad \mathfrak{r}_\rho \leq \mathfrak{hom}_\rho\,.$$

Proof. Let $G\colon [\omega]^\omega \to [\omega]^\omega$ be the identity map and define $F\colon [\omega]^\omega \to P_{2,2}$ as $F(S) := f_S$, where

$$\begin{aligned} f_S\colon [\omega]^2 &\longrightarrow 2 \\ a &\longmapsto \begin{cases} 1 & \min(a) \in S, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

To see that $(F, G)\colon \mathbf{Rp}_\rho \to \mathbf{Hm}_\rho$ is the desired Tukey connection, we have to prove the implication

$$F(S) \nmid_\rho X \quad \Rightarrow \quad S \nmid_\rho G(X)\,.$$

We instead prove the contraposition, i. e. $S \mid_\rho X \Rightarrow f_S \mid_\rho X$ for all $S, X \in [\omega]^\omega$. Assume $S \mid_\rho X$, let $X = \{x_n \in X \mid n \in \omega\}$ be the (canonical) ascending enumeration of X and set $X_n := \{x_i \mid i \leq n\}$. Note that $\langle X_n \rangle$ is a subsequence of $\langle X \cap n \rangle$, which itself is monotone, hence instead of

$$\lim_{n\to\infty} \frac{|f_S^{-1}(1) \cap [X \cap n]^2|}{|[X \cap n]^2|}$$

we can work with

$$\lim_{n\to\infty} \frac{|f_S^{-1}(1) \cap [X_n]^2|}{|[X_n]^2|} = \lim_{n\to\infty} \frac{2}{n(n+1)} \cdot |f_S^{-1}(1) \cap [X_n]^2|\,. \tag{4.1}$$

The equality $|[X_n]^2| = \frac{n(n+1)}{2}$ follows from $|X_n| = n + 1$. For any $n, i \in \omega$, the number of pairs in $[X_n]^2$ with minimal element x_i is exactly $n-i$, as there are $n-i$ elements greater than x_i in X_n. This yields the following equality:

$$|f_S^{-1}(1) \cap [X_n]^2| = \sum_{i=0}^{n} \left(\chi_S(x_i) \cdot (n-i)\right) = \sum_{i=0}^{n-1} |S \cap X_i|\,. \tag{4.2}$$

Since $S \mid_\rho X$, for any $\varepsilon > 0$, there is $N_\varepsilon \in \omega$ such that for all $n \geq N_\varepsilon$ we have

$$\rho - \varepsilon < \frac{|S \cap X_n|}{n+1} < \rho + \varepsilon\,. \tag{4.3}$$

Using Eq. (4.1) and Eq. (4.2), we can proceed as follows (for $n > N_\varepsilon$):

$$\begin{aligned} \frac{|f_S^{-1}(1) \cap [X_n]^2|}{|[X_n]^2|} &= \frac{2}{n(n+1)} \sum_{i=0}^{n-1} |S \cap X_i| \\ &= \frac{2}{n(n+1)} \left(\sum_{i=N_\varepsilon}^{n-1} \left((i+1) \cdot \frac{|S \cap X_i|}{i+1} \right) + \sum_{i=0}^{N_\varepsilon - 1} |S \cap X_i| \right) \end{aligned} \tag{4.4}$$

Combining Eq. (4.3) and Eq. (4.4) with generous applications of the well-known formula $\sum_{i=0}^{n} i = \frac{n^2+n}{2}$, we arrive at the following bounds for $n > N_\varepsilon$:

$$0 \leq \frac{2}{n^2+n} \sum_{i=0}^{N_\varepsilon - 1} |S \cap X_i| \leq \frac{N_\varepsilon^2 + N_\varepsilon}{n^2+n}$$

and

$$\frac{(n^2+n)-(N_\varepsilon^2+N_\varepsilon)}{n^2+n}\cdot(\rho-\varepsilon)<\frac{2}{n^2+n}\sum_{i=N_\varepsilon}^{n-1}\left((i+1)\cdot\frac{|S\cap X_i|}{i+1}\right)<(\rho+\varepsilon)\,.$$

Summing these two bounds, we have

$$\frac{(n^2+n)-(N_\varepsilon^2+N_\varepsilon)}{n^2+n}\cdot(\rho-\varepsilon)<\frac{|f_S^{-1}(1)\cap[X_n]^2|}{|[X_n]^2|}<(\rho+\varepsilon)+\frac{N_\varepsilon^2+N_\varepsilon}{n^2+n}\,,$$

and hence, for n large enough, we have

$$\rho-2\varepsilon<\frac{|f_S^{-1}(1)\cap[X_n]^2|}{|[X_n]^2|}<\rho+2\varepsilon\,,$$

which proves the claim. □

Lemma 4.2.5. *For every $\rho\in(0,1)$, we have $\mathbf{Hm}_\rho\preceq_{\mathrm{T}}\mathbf{Hm}$. By Definition 1.2.1, this implies*

$$\mathfrak{par}_\rho\geq\mathfrak{par}\qquad\textit{and}\qquad\mathfrak{hom}_\rho\leq\mathfrak{hom}\,.$$

Proof. If both $F\colon[\omega]^\omega\to[\omega]^\omega$ and $G\colon[\omega]^\omega\to[\omega]^\omega$ are the identity map, then the Tukey connection $(F,G)\colon\mathbf{Hm}_\rho\to\mathbf{Hm}$ has the desired properties.

To see this, we have to prove the implication

$$f\mid_\rho X\Rightarrow f\mid X$$

for all $f\in P_{2,2}$ and $X\in[\omega]^\omega$. Assume $f\mid_\rho X$ and note that X is an infinite set. For $|f^{-1}(1)\cap[X\cap n]^2|/|[X\cap n]^2|$ to tend to $\rho>0$, the sequence of cardinalities $|f^{-1}(1)\cap[X\cap n]^2|$ has to tend to infinity, i.e. $f^{-1}(1)\cap[X]^2$ has to be an infinite set. Moreover, since we have $\rho<1$, the sequence $|f^{-1}(0)\cap[X\cap n]^2|$ has to tend to infinity, as well. Hence, also $f^{-1}(0)\cap[X]^2$ is an infinite set. In other words, f splits X. □

Lemma 4.2.6. *We have $\mathbf{Hm}_{1/2}\preceq_{\mathrm{T}}\mathbf{Cv}(\mathcal{M})^\perp$. By Definition 1.2.1, this implies*

$$\mathfrak{par}_{1/2}\geq\mathrm{cov}(\mathcal{M})\qquad\textit{and}\qquad\mathfrak{hom}_{1/2}\leq\mathrm{non}(\mathcal{M})\,.$$

Proof. Let $G\colon[\omega]^\omega\to[\omega]^\omega$ be the identity map and define $F\colon P_{2,2}\to\mathcal{M}$ by

$$F(f):=\{X\in[\omega]^\omega\mid f\mid_{1/2}X\}\,.$$

Then, $(F,G)\colon\mathbf{Hm}_{1/2}\to\mathbf{Cv}(\mathcal{M})^\perp$ is a Tukey connection, as

$$F(f)\not\ni X\quad\Rightarrow\quad f\nmid_{1/2}G(X)\,.$$

(We even have equivalence.)

It remains to show that F is well-defined, specifically that for all $f \in P_{2,2}$ the set $F(f)$ is meagre. We inductively define a chopped real based on f as follows: by Ramsey's Theorem, we know there is a homogeneous set $H \in [\omega]^\omega$ for f, i. e. a set H such that $f|_{[H]^2}$ is constant. Let the first interval of the partition be $I_0 = [0, \min(H)]$. Now, for any $n \in \omega$, given $m_n := \max(I_n)$, chose m_{n+1} minimal such that $I_{n+1} := [m_n + 1, m_{n+1}]$ contains $m_n + 1$ elements of H.

Any real X matching this chopped real is not bisected by f. Indeed, whenever such an X is equal to H on one of the intervals I_n, X is at least $1/12$ away from being bisected by f, either immediately before or immediately after I_n (analogous to the argument in Lemma 2.2.3). Since such an X is equal to H on I_n for infinitely many $n \in \omega$, f does not bisect X in the limit.

Now, the family $M := \mathrm{Match}(H, \langle I_n \rangle_{n\in\omega})$ is a comeagre set, and thus $F(f)$, being a subset of $[\omega]^\omega \setminus M$, is meagre. □

4.3 On an Alternate Definition of $\mathfrak{s}_\rho$

In Theorem 3.1.9 we proved, among other results, that $\mathfrak{r}_{1/2} \leq \mathfrak{r}_*$, but were unable to show that $\mathfrak{s}_{1/2} \geq \mathfrak{s}_*$. This motivated our attempts at finding an alternate and potentially more natural definition of ρ-splitting and its related cardinal characteristics. In this section, we introduce this alternate definition and discuss its benefits and drawbacks.

Definition 4.3.1. For $\rho \in (0,1)$ let

$$[\omega]_\rho := \{X \subseteq \omega \mid d(X) = \rho\} = \{X \subseteq \omega \mid X \mid_\rho \omega\} ,$$

i. e. the family of subsets of ω with density ρ.

Definition 4.3.2. We define the following relational system:

- Let $\mathbf{Rp}_\rho^+ := \langle [\omega]_\rho, [\omega]^\omega, \nmid_\rho \rangle$ and

$$\mathfrak{b}(\mathbf{Rp}_\rho^+) =: \mathfrak{s}_\rho^+, \quad \mathfrak{d}(\mathbf{Rp}_\rho^+) =: \mathfrak{r}_\rho^+.$$

Lemma 4.3.3. *For every $\rho \in (0,1)$, we have $\mathbf{Rp}_\rho^+ \preceq_{\mathrm{T}} \mathbf{Rp}_\rho$, which, by Definition 1.2.1, implies*

$$\mathfrak{s}_\rho^+ \geq \mathfrak{s}_\rho \qquad \text{and} \qquad \mathfrak{r}_\rho^+ \leq \mathfrak{r}_\rho .$$

In fact, $\mathfrak{r}_\rho^+ = \mathfrak{r}_\rho$.

Proof. If $F\colon [\omega]_\rho \to [\omega]^\omega$ is the inclusion map and $G\colon [\omega]^\omega \to [\omega]^\omega$ is the identity map, then the Tukey connection $(F, G)\colon \mathbf{Rp}_\rho \to \mathbf{Rp}$ has the desired properties.

To see the equality $\mathfrak{r}_\rho^+ = \mathfrak{r}_\rho$, let $\mathcal{R} \subseteq [\omega]^\omega$ such that $|\mathcal{R}| < \mathfrak{r}_\rho$. Let $\mathcal{R}_0 := \mathcal{R} \cup \{\omega\}$. Since $|\mathcal{R}_0| < \mathfrak{r}_\rho$, as well, there is $S \in [\omega]^\omega$ such that

$$\forall X \in \mathcal{R}_0\colon \quad S \mid_\rho X .$$

Moreover, as $\omega \in \mathcal{R}_0$ (and thus $S \mid_\rho \omega$), we have $S \in [\omega]_\rho$. Hence, $|\mathcal{R}| = |\mathcal{R}_0| < \mathfrak{r}_\rho^+$, and, since $\mathcal{R}$ was arbitrary, $\mathfrak{r}_\rho^+ \geq \mathfrak{r}_\rho$. ☐

With the new definition we can now prove the following

Lemma 4.3.4. *For every $\rho \in (0,1)$, we have $\mathbf{Rp}_\rho^+ \preceq_{\mathrm{T}} \mathbf{Rp}_*$, which, by Definition 1.2.1, implies*

$$\mathfrak{s}_\rho^+ \geq \mathfrak{s}_* \qquad \text{and} \qquad \mathfrak{r}_\rho^+ \leq \mathfrak{r}_* .$$

Proof. The inclusion map $F\colon [\omega]_\rho \to [\omega]_*$ together with the identity map $G\colon [\omega]^\omega \to [\omega]^\omega$ yield the desired Tukey connection $(F, G)\colon \mathbf{Rp}_\rho \to \mathbf{Rp}_*$.

To see this, we have to prove the implication $S \mid_\rho X \Rightarrow S \mid_* X$ for all $S \in [\omega]_\rho$ and $X \in [\omega]^\omega$. Assume $S \mid_\rho X$ and note that

$$\lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(S) \cdot d_n(X)} = \frac{1}{d(S)} \cdot \lim_{n\to\infty} \frac{d_n(S \cap X)}{d_n(X)} = \frac{1}{\rho} \cdot \rho = 1\,,$$

which implies $S \mid_* X$. □

We need to slightly modify the corresponding Tukey connection in Theorem 2.1.6 to get the analogous result for $\mathbf{Rp}^+_{1/2}$:

Lemma 4.3.5. *We have* $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}^+_{1/2}$*, which, by Definition 1.2.1, implies*

$$\mathrm{non}(\mathcal{N}) \geq \mathfrak{s}^+_{1/2} \qquad \textit{and} \qquad \mathrm{cov}(\mathcal{N}) \leq \mathfrak{r}^+_{1/2}\,.$$

Proof. Take any $F\colon [\omega]^\omega \to [\omega]_{1/2}$ which is the identity map on $[\omega]_{1/2}$. Moreover, define $G\colon [\omega]^\omega \to \mathcal{N}$ as

$$G(X) := \{S \in [\omega]^\omega \mid S \nmid_{1/2} X \text{ or } d(S) \neq 1/2\}$$

and we get

$$F(S) \nmid_{1/2} X \quad \Rightarrow \quad S \in G(X)\,.$$

It remains to show that $G(X)$ is a null set for every $X \in [\omega]^\omega$. Indeed, we have seen in Theorem 2.1.6 that for each X, the family $\{S \in [\omega]^\omega \mid S \nmid_{1/2} X\}$ is a null set, in particular, so is the family $\{S \in [\omega]^\omega \mid S \nmid_{1/2} \omega\}$. However, $G(X)$ is just the union of these two null sets, thus a null set itself. □

Corollary 4.3.6. *We have* $\mathbf{Cv}(\mathcal{N}) \preceq_{\mathrm{T}} \mathbf{Rp}_*$*, which, by Definition 1.2.1, implies*

$$\mathrm{non}(\mathcal{N}) \geq \mathfrak{s}_* \qquad \textit{and} \qquad \mathrm{cov}(\mathcal{N}) \leq \mathfrak{r}_*\,.$$

Proof. This is achieved by the composing the Tukey connections in Lemma 4.3.4 and Lemma 4.3.5. □

Unfortunately, our proof of Lemma 2.1.5 does not work anymore for this alternate definition: Given a ρ-splitting family $\mathcal{S}_\rho \subseteq [\omega]_\rho$ and a τ-splitting family $\mathcal{S}_\tau \subseteq [\omega]_\tau$, for every $X \in [\omega]^\omega$ we want to find $S \in [\omega]_{\rho\tau}$ such that $S \mid_{\rho\tau} X$.

If we take $S_\tau \in \mathcal{S}_\tau$ such that $S_\tau \mid_\tau X$ and $S_\rho \in \mathcal{S}_\rho$ such that $S_\rho \mid_\rho (S_\tau \cap X)$, indeed $S_\rho \cap S_\tau \mid_{\rho\tau} X$ as seen in Lemma 2.1.5. However, it is not clear that $S_\rho \cap S_\tau \in [\omega]_{\rho\tau}$.

If we take $S_\tau \in \mathcal{S}_\tau$ and $S_\rho \in \mathcal{S}_\rho$ such that $S_\rho \mid_\rho S_\tau$, indeed $S_\rho \cap S_\tau \in [\omega]_{\rho\tau}$ as seen in Lemma 1.1.6. However, it is not clear that $S_\rho \cap S_\tau \mid_{\rho\tau} X$.

Chapter 5

Open Questions

While we have shown that several of our newly defined cardinal characteristics are, in fact, new, there are still a number of open questions.

Question A. *We summarise the open questions related to Figure 3.1:*

(Q1) Is it consistent that $\mathfrak{i}_* < 2^{\aleph_0}$*?*

(Q2) Which relations between $\mathfrak{i}_{1/2}$*,* $\mathfrak{i}_*$ *and* $\mathfrak{i}$ *are true or consistent?*

(Q3) Are there any smaller upper bounds for $\mathfrak{i}_{1/2}$ *and* $\mathfrak{i}_*$*?*

(Q4) Which relations between $\mathfrak{s}_{1/2}$ *and* $\mathfrak{s}_*$ *are true or consistent?*

(Q5) Which of the following statements are true?

$$\begin{aligned} \mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2}) \quad &\textit{or} \quad \mathrm{cov}(\mathcal{N}) = \mathfrak{r}_{1/2}\,, \\ \mathrm{Con}(\mathfrak{r}_{1/2} < \mathfrak{r}_*) \quad &\textit{or} \quad \mathfrak{r}_{1/2} = \mathfrak{r}_*\,, \\ \mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N})) \quad &\textit{or} \quad \mathfrak{s}_* = \mathrm{non}(\mathcal{N})\,. \end{aligned}$$

We suspect that (Q1) might be provable (via $\mathrm{Con}(\mathfrak{i}_* < \mathfrak{i})$) using the same idea as in Lemma 3.2.4.

If the probabilistic argument from Lemma 2.3.4 can be reproduced for $\mathfrak{s}_*$, a similar approach as in section 2.3 might also work to answer the third part of (Q5) and prove $\mathrm{Con}(\mathfrak{s}_* < \mathrm{non}(\mathcal{N}))$.

Finally, since it is not too difficult to ensure that a creature forcing poset keeps $\mathrm{cov}(\mathcal{N})$ small (compare [FGKS17, Lemma 5.4.2] or [GK18, Lemma 7.7]), a clever creature forcing construction might be able to answer the first part of (Q5) and prove $\mathrm{Con}(\mathrm{cov}(\mathcal{N}) < \mathfrak{r}_{1/2})$.

Question B. *We summarise the open questions related to Figure 4.1:*

(Q6) Does $\mathrm{Con}(\mathfrak{d} < \mathfrak{s}_{1/2\pm\varepsilon})$ *hold or is* $\mathfrak{s}_{1/2\pm\varepsilon} \leq \mathfrak{d}$*?*

(Q7) Which of the following statements are true?

$$\begin{aligned} \mathrm{Con}(\mathfrak{s} < \mathfrak{s}^w_{1/2}) \quad &\textit{or} \quad \mathfrak{s} = \mathfrak{s}^w_{1/2}\,, \\ \mathrm{Con}(\mathfrak{s}^w_{1/2} < \mathfrak{s}^\infty_{1/2}) \quad &\textit{or} \quad \mathfrak{s}^w_{1/2} = \mathfrak{s}^\infty_{1/2}\,, \\ \mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}_{1/2}) \quad &\textit{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} = \mathfrak{s}_{1/2}\,. \end{aligned}$$

(Q8) *Given* $\varepsilon > \varepsilon'$ *and an* ε*-almost bisecting family, can one (finitarily) modify it to get an* ε'*-almost bisecting family of equal size? (If yes, then* $\mathfrak{s}_{1/2\pm\varepsilon}$ *is independent of* ε*. If not, then* $\inf_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *and* $\sup_{\varepsilon\in(0,1/2)} \mathfrak{s}_{1/2\pm\varepsilon}$ *might be interesting characteristics, as well.)*

(Q9) *Can characteristics in the upper row of the diagram consistently be smaller than ones in the lower row? Specifically, which of the following statements are true?*

$$\mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^{w}_{1/2}) \quad \textit{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^{w}_{1/2}\,,$$
$$\mathrm{Con}(\mathfrak{s}_{1/2\pm\varepsilon} < \mathfrak{s}^{\infty}_{1/2}) \quad \textit{or} \quad \mathfrak{s}_{1/2\pm\varepsilon} \geq \mathfrak{s}^{\infty}_{1/2}\,,$$
$$\mathrm{Con}(\mathfrak{s}_{1/2} < \mathfrak{s}^{\infty}_{1/2}) \quad \textit{or} \quad \mathfrak{s}_{1/2} \geq \mathfrak{s}^{\infty}_{1/2}\,.$$

Bibliography

[AS16] Noga Alon and Joel Harold Spencer, *The Probabilistic Method*, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Hoboken, NJ, ^{4}2016, DOI: 10.1002/9780470277331.

[BHK+18] Jörg Brendle, Lorenz J. Halbeisen, Lukas Daniel Klausner, Marc Lischka, and Saharon Shelah, *Halfway New Cardinal Characteristics*, arXiv: 1808.02442 [math.LO].

[BJ95] Tomek Bartoszyński and Haim Judah, *Set Theory: On the Structure of the Real Line*, A K Peters, Wellesley, MA, 1995, DOI: 10.1112/S0024609396222374.

[Bla93] Andreas Blass, *Simple Cardinal Characteristics of the Continuum*, Set Theory of the Reals (Haim Judah, ed.), Israel Mathematics Conference Proceedings, vol. 6, Gelbart Research Institute for Mathematical Sciences, Bar-Ilan University, Ramat-Gan, 1993, pp. 63–90, arXiv: 9405202 [math.LO].

[Bla10] ______, *Combinatorial Cardinal Characteristics of the Continuum*, Handbook of Set Theory (Matthew Foreman and Akihiro Kanamori, eds.), Springer, Dordrecht, 2010, pp. 395–489, DOI: 10.1007/978-1-4020-5764-9_7, also available at http://www.math.lsa.umich.edu/~ablass/hbk.pdf.

[Bre02] Jörg Brendle, *Mad Families and Iteration Theory*, Logic and Algebra, Contemp. Math., vol. 302, Amer. Math. Soc., Providence, RI, 2002, pp. 1–31, DOI: 10.1090/CONM/302/05083.

[Bre03] ______, *The Almost-Disjointness Number May Have Countable Cofinality*, Trans. Amer. Math. Soc. **355** (2003), no. 7, 2633–2649, DOI: 10.1090/S0002-9947-03-03271-9.

[Bre05] ______, *Templates and Iterations: Luminy 2002 lecture notes*, RIMS Kōkyūroku **1423 Forcing Method and Large Cardinal Axioms** (2005), 1–12, HDL: 2433/47211.

[BSZ00] Jörg Brendle, Otmar Spinas, and Yi Zhang, *Uniformity of the Meager Ideal and Maximal Cofinitary Groups*, J. Algebra **232** (2000), no. 1, 209–225, DOI: 10.1006/JABR.2000.8396, also available at https://core.ac.uk/download/pdf/82709895.pdf.

[FGKS17] Arthur Fischer, Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Creature Forcing and Five Cardinal Characteristics in Cichoń's Diagram*, Arch. Math. Logic **56** (2017), no. 7–8, 1045–1103, DOI: 10.1007/S00153-017-0553-8, arXiv: 1402.0367 [math.LO].

[FT15] Vera Fischer and Asger Törnquist, *Template Iterations and Maximal Cofinitary Groups*, Fund. Math. **230** (2015), 205–236, DOI: 10.4064/FM230-3-1, arXiv: 1310.3245 [math.LO].

[GK18] Martin Goldstern and Lukas Daniel Klausner, *Cichoń's Diagram and Localisation Cardinals*, arXiv: 1808.01921 [math.LO].

[Hal17] Lorenz J. Halbeisen, *Combinatorial Set Theory: With a Gentle Introduction to Forcing*, Springer Monographs in Mathematics, Springer, London, ^{2}2017, DOI: 10.1007/978-3-319-60231-8.

[Hec72] Stephen H. Hechler, *Short Complete Sequences in $\beta\mathbb{N}\setminus\mathbb{N}$ and Small Maximal Almost-Disjoint Families*, Gen. Topology Appl. **2** (1972), no. 3, 139–149, DOI: 10.1016/0016-660X(72)90001-3.

[KM18] Lukas Daniel Klausner and Diego Alejandro Mejía, *Many Different Uniformity Numbers of Yorioka Ideals*, arXiv: 1805.11005 [math.LO].

[RS99] Andrzej Rosłanowski and Saharon Shelah, *Norms on Possibilities I: Forcing with Trees and Creatures*, Mem. Amer. Math. Soc. **141** (1999), no. 671, xii+167, DOI: 10.1090/MEMO/0671, arXiv: MATH/9807172 [math.LO].

[Vau90] Jerry E. Vaughan, *Small Uncountable Cardinals and Topology*, Open Problems in Topology (Jan van Mill and George M. Reed, eds.), North-Holland, Amsterdam, 1990, pp. 195–218, available at https://pdfs.semanticscholar.org/9065/61a1e45b49a2ab816e088ffd33279c05a3ba.pdf.