Discrete Inverse Sobolev Inequalities with Applications to the Edge and Face Lemma for the Finite Element Method

Master’s Thesis
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Abstract

In this thesis we set out to prove certain inverse inequalities for piecewise linear Lagrangian finite element spaces. More specifically, we give a proof of the edge lemma and face lemma from [23]. First we discuss some preliminaries on Sobolev spaces and the finite element method. Then we move on to the main part of the thesis concerning newly discovered results and proofs which is the content of Chapters 4 and 5. In Chapter 4 well-known inverse inequalities are stated and proved together with some essential new ones for certain slices through centroids of tetrahedral elements. In chapter 5 we discuss some erroneous proofs of the edge and face lemma from certain research papers followed by our own proofs of these lemmas together with some key inverse inequalities. We conclude everything by giving a detailed exposition of results which use the edge and/or face lemma.
Acknowledgements

First and foremost I would like to thank my thesis advisor professor Dr. Ralf Hiptmair for proposing a project that was not only challenging but also very close to current research. I’m not going to lie, setting out with the sole task to prove certain results as my master’s thesis was a bit daunting at first. A thought which popped up in my mind countless times was a variant of ‘What if I’m not able to prove these results or what if I’m only able to give a partial solution?’ It took me a couple of weeks to gather my thoughts and figure out a way to attack the problems that I was given head on. Thankfully professor Hiptmair made time for coaching me which was very helpful because I had little prior experience on the topic of inverse inequalities. After putting in a lot of effort to catch up on the necessary concepts and definitions I was able to derive new results on a high technical level which required a lot of scientific creativity. I combined these results to derive a proof of the edge and face lemma which albeit quite long, is in essence surprisingly elegant.

Next I would like to thank my parents for giving me the opportunity to study abroad at ETH Zürich. Studying abroad at ETH has been a wonderful experience which I will cherish for the rest of my life. I would finally like to thank Dr. Jonas Bekaert for introducing and helping me work with the vector graphics editor Inkscape which was an essential part of the construction of the figures in this thesis. I would also like to thank him for the many chats we had during this period which helped me unwind a bit and keep my mind off my work.
# Contents

<table>
<thead>
<tr>
<th>Section</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>Preliminaries</td>
<td>3</td>
</tr>
<tr>
<td>2.1</td>
<td>The Euclidean Norm and Volume in $\mathbb{R}^n$</td>
<td>3</td>
</tr>
<tr>
<td>2.2</td>
<td>$L^p$-spaces</td>
<td>3</td>
</tr>
<tr>
<td>2.3</td>
<td>Lipschitz Domains</td>
<td>4</td>
</tr>
<tr>
<td>2.4</td>
<td>Sobolev Spaces</td>
<td>5</td>
</tr>
<tr>
<td>2.4.1</td>
<td>Positive Integer Order Sobolev Spaces</td>
<td>6</td>
</tr>
<tr>
<td>2.4.2</td>
<td>Fractional Order Sobolev Spaces</td>
<td>7</td>
</tr>
<tr>
<td>2.5</td>
<td>Sobolev Spaces on the Boundary</td>
<td>8</td>
</tr>
<tr>
<td>2.5.1</td>
<td>Negative Order Sobolev Spaces</td>
<td>9</td>
</tr>
<tr>
<td>2.5.2</td>
<td>Sobolev Spaces $H(\text{curl}; \Omega)$, $H_0(\text{curl}; \Omega)$ and $H^3(\text{curl}; \Omega)$</td>
<td>11</td>
</tr>
<tr>
<td>2.5.3</td>
<td>Sobolev Spaces $H(\text{div}; \Omega)$, $H_0(\text{div}; \Omega)$ and $H_0(\text{div}_0; \Omega)$</td>
<td>12</td>
</tr>
<tr>
<td>2.6</td>
<td>The Dirichlet Trace</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>The Finite Element Method</td>
<td>15</td>
</tr>
<tr>
<td>3.1</td>
<td>Variational Problems</td>
<td>15</td>
</tr>
<tr>
<td>3.2</td>
<td>Symmetric and Nonsymmetric Variational Problems</td>
<td>16</td>
</tr>
<tr>
<td>3.3</td>
<td>The Finite Element Method</td>
<td>17</td>
</tr>
<tr>
<td>3.4</td>
<td>Meshes and Regularity</td>
<td>17</td>
</tr>
<tr>
<td>3.5</td>
<td>Mesh Refinement</td>
<td>19</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Finite Elements and Local Shape Functions</td>
<td>20</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Affine equivalence</td>
<td>21</td>
</tr>
<tr>
<td>3.5.3</td>
<td>Global Shape Functions and piece-wise polynomial spaces</td>
<td>21</td>
</tr>
<tr>
<td>3.5.4</td>
<td>Ritz-Galerkin Approximation</td>
<td>23</td>
</tr>
<tr>
<td>4</td>
<td>Inverse Inequalities</td>
<td>25</td>
</tr>
<tr>
<td>4.1</td>
<td>Discrete variants</td>
<td>26</td>
</tr>
</tbody>
</table>
4.1.1 Scott-Zhang Quasi-Interpolant .................................. 26
4.1.2 The (Generalized) Discrete Harmonic Extension ........... 27
4.1.3 Discrete $L^2$-NORMS ............................................ 30
4.2 Well-known Inverse Inequalities .................................. 31
4.3 Geometric Considerations ......................................... 43
4.4 Inverse inequalities for centroid slices ......................... 45

5 The Edge and Face Lemma .............................................. 61
5.1 Erroneous Proofs .................................................... 61
5.1.1 Attempt 1 ......................................................... 61
5.1.2 Attempt 2 ......................................................... 62
5.1.3 Attempt 3 ......................................................... 64
5.2 Proof of the Edge Lemma ............................................ 66
5.3 Proof of the Face Lemma ............................................ 70
5.3.1 Implications ..................................................... 75

6 Applications in Recent Research ................................... 77
6.1 Direct References to the edge and face lemma ................. 77
6.2 Paper 1: A Substructuring Preconditioner with Vertex-Related
Interface Solvers for Elliptic-Type Equations in Three Dimensions 79
6.2.1 Goal of Paper 1 .................................................. 79
6.2.2 Results Based on the edge and face lemma ................. 79
6.3 Paper 2: Nonoverlapping Domain Decomposition Methods with
a Simple Coarse Space for Elliptic Problems ....................... 85
6.3.1 Goal of Paper 2 .................................................. 85
6.4 The (Preconditioned) Conjugate Gradient Method ............. 85
6.4.1 The Conjugate Gradient (CG) Method ......................... 85
6.4.2 The Preconditioned Conjugate Gradient (PCG) Method .... 87
6.4.3 Results Based on the edge and face lemma ................. 89
6.5 Paper 3: Substructuring Preconditioners with a Simple Coarse
Space for 2-D 3-T Radiation Diffusion Equations .................. 93
6.5.1 Goal of Paper 3 .................................................. 93
6.5.2 The SFVE Method for Radiation-Diffusion Equations .... 93
6.5.3 Results Based on the edge and face lemma ................. 95
6.6 Paper 4: A Mortar Edge Element Method with Nearly Optimal
Convergence for Three-Dimensional Maxwell’s Equations ....... 99
6.6.1 Goal of Paper 4 .................................................. 99
6.6.2 Maxwell’s Equations ............................................ 99
6.6.3 Weak Formulation of Maxwell’s Equations ................... 100
6.6.4 The Mortar Edge Element Method ............................ 101
6.6.5 A Generalized Edge Element Interpolation Operator ..... 104
6.6.6 Results Based on the edge and face lemma ................. 107
6.7 Paper 5: A Non-overlapping Domain Decomposition Method
for Maxwell’s Equations in Three Dimensions ...................... 110
<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.7.1 Goal of Paper 5</td>
<td>110</td>
</tr>
<tr>
<td>6.7.2 Results Based on the edge and face lemma</td>
<td>110</td>
</tr>
<tr>
<td>7 Conclusion</td>
<td>113</td>
</tr>
<tr>
<td>A List of Symbols</td>
<td>115</td>
</tr>
<tr>
<td>Bibliography</td>
<td>119</td>
</tr>
</tbody>
</table>
Chapter 1

Introduction

In this thesis we are interested in a theoretical aspect of the finite element method. The finite element method uses finite dimensional function spaces to construct approximations to solutions of partial differential equations. The fact that these spaces are finite dimensional allows us to use norm equivalence to obtain so-called inverse inequalities. The tricky part is therefore not proving that such inequalities exist but rather getting an explicit expression for these inequalities in terms of the mesh parameters.

The norms in which we are interested are Sobolev and $L^p$-norms because they are associated to the natural function spaces used for analyzing the finite element method. We will be mainly interested in two specific inverse inequalities, namely the edge and face lemma. The inequalities in these lemmas state a logarithmic bound in terms of the mesh parameters on a certain Sobolev norm of a restriction of a finite element function. The interesting fact to notice here is the logarithmic bound. Such a factor usually only appears in inverse inequalities for two-dimensional domains while the edge and face lemma hold for three-dimensional domains.

Our main objective is to give a complete proof of the edge and face lemma. Currently, there is no known complete proof of these results. However, they are used in current research to prove estimates on the condition number of certain preconditioners resulting from finite element discretization.

The proofs that we will give will be quite detailed to highlight the fact that there aren’t any ambiguities anywhere. The problem with the current proofs of the edge and face lemma is a spurious use of certain inverse inequalities. It would be counterproductive to claim that we have found a correct proof while omitting many of the details. This is precisely what lead to the errors in the known proofs.

In the final chapter we will be less concerned with detailed proofs because all results stated are from research papers and therefore not our own. They also
1. **Introduction**

do not contribute to the proofs of the edge and face lemma. This chapter is mostly an exposition of the results which use the these lemmas. It also highlights the importance of these results and that giving a proof actually bridges a gap between theory and experiment.
Chapter 2

Preliminaries

2.1 The Euclidean Norm and Volume in $\mathbb{R}^n$

In this section we introduce notation that we will use very frequently. We start by fixing the notation for Euclidean norms.

**Definition 2.1 (Euclidean norm)** The Euclidean norm of a vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is defined as

$$|\mathbf{x}| := \left( \sum_{k=1}^{n} x_k^2 \right)^{1/2}.$$ 

Next we fix the notation for the surface area and volume of a set.

**Definition 2.2 (Volume)** Let $\Omega \subseteq \mathbb{R}^n$ be a Borel-measurable set. We define the volume of $\Omega$ as

$$\text{vol}_n(\Omega) := \int_{\mathbb{R}^n} 1_{\Omega}(\mathbf{x}) \, d\mathbf{x} = \int_{\Omega} d\mathbf{x},$$

where $d\mathbf{x}$ is the Lebesgue measure on $\mathbb{R}^n$.

**Remark.** For $n = 2$ we will refer to $\text{vol}_2(\Omega)$ as the surface area of $\Omega$.

2.2 $L^p$-spaces

In this section we define the elementary $L^p$-spaces which we will do in full generality. However, we will only need the special case of the Lebesgue measure for all our results.

**Definition 2.3** Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and $p > 0$ and integer. The corresponding $L^p$-space is defined as

$$L^p(\Omega, \mathcal{F}, \mu) = \left\{ u : \Omega \to \mathbb{R} \mid u \text{ is } \mathcal{F}\text{-measurable, } \int_{\Omega} |u|^p \, d\mu < +\infty \right\},$$
where as usual we understand $f$ as the equivalence class of functions which are $\mu$-a.e. equal to $f$.

The norm
\[ \|u\|_{L^p(\Omega, \mathcal{F}, \mu)} := \left( \int_{\Omega} |u|^p \, d\mu \right)^{1/p}, \quad \forall u \in L^p(\Omega, \mathcal{F}, \mu), \]
turns $L^p(\Omega, \mathcal{F}, \mu)$ into a Banach space.

**Notation:** For $(\Omega, \mathcal{F}, \mu) = (\Omega, \mathcal{B}(\Omega), \lambda)$ where $\Omega \subseteq \mathbb{R}^n$ we write
\[ L^p(\Omega) := L^p(\Omega, \mathcal{B}(\Omega), \lambda). \]
Moreover, when writing an integral with respect to the Lebesgue measure, we will write $d\mathbf{x}$ instead of $d\lambda$ or $d\lambda(\mathbf{x})$.

For $p = \infty$ we have the following definition:
\[ L^\infty(\Omega, \mathcal{F}, \mu) := \{ u : \Omega \to \mathbb{R} \mid u \text{ is } \mathcal{F}\text{-measurable}, \|u\|_{L^\infty(\Omega, \mathcal{F}, \mu)} < +\infty \}, \]
where
\[ \|u\|_{L^\infty(\Omega, \mathcal{F}, \mu)} := \text{ess sup}_{x \in \Omega} |u(x)|. \]

### 2.3 Lipschitz Domains

We will generally work with domains that have polygonal boundaries which are examples of a more general type of domain, so-called Lipschitz domains. To define this, we need the concept of Lipschitz spaces.

**Definition 2.4 (Lipschitz Space)** Let $\Omega \subseteq \mathbb{R}^n$, we define the vector space $\text{Lip}(\Omega)$ as
\[ \text{Lip}(\Omega) := \{ f \in L^\infty(\Omega) \mid \|f\|_{\text{Lip}(\Omega)} < +\infty \}, \]
where
\[ \|f\|_{\text{Lip}(\Omega)} := \|f\|_{L^\infty(\Omega)} + \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|}. \]

**Remark:** Functions $f \in L^\infty(\Omega)$ which have finite Lipschitz seminorm
\[ |f|_{\text{Lip}(\Omega)} := \sup_{x \neq y \in \Omega} \frac{|f(x) - f(y)|}{|x - y|} \]
are called Lipschitz continuous.

**Definition 2.5 (Lipschitz Domain)** (Definition 2.2.7 in [19]) Let $\Omega \subseteq \mathbb{R}^n$ be a domain. Then $\Omega$ is a **Lipschitz domain** if there exists a finite cover $U$ of open subsets in $\mathbb{R}^n$ and bijective mappings $\{ \psi_U : \overline{B_2} \to \overline{U} \}_{U \in U}$ which have the following properties:
2.4. Sobolev Spaces

- \( \psi_U \in \text{Lip}(B_2, U), \psi_U^{-1} \in \text{Lip}(U, B_2) \),
- \( \psi_U(B_2^0) = U \cap \partial \Omega \)
- \( \psi_U(B_2^-) = U \cap \Omega \),
- \( \psi_U(B_2^+) = U \cap (\mathbb{R}^d \setminus \Omega) \),

where \( B_2^+ := \{ \xi \in B_r \mid \xi_n > 0 \} \), \( B_2^- := \{ \xi \in B_r \mid \xi_n < 0 \} \) and \( B_2^0 := \{ \xi \in B_r \mid \xi_n = 0 \} \).

**Remark.** \( \Omega \) is called a \( C^k \)-domain if the first property can be replaced by

\[ \psi_U \in C^k(B_2, U), \psi_U^{-1} \in C^k(U, B_2). \]

**Example 2.6 (Polyhedral Domain)** A domain \( \Omega \subseteq \mathbb{R}^n \) is called polyhedral if \( (n = 2) \) it is a polygon or if \( (n = 3) \) \( \partial \Omega \) is a finite union of polygons (called faces) such that for any face \( F \subseteq \partial \Omega \) and any edge \( e \subseteq \partial F \), there exists another face \( F' \subseteq \partial \Omega \) such that \( e \subseteq \partial F' \).

![Figure 2.1: Example of a polyhedral domain in \( \mathbb{R}^3 \)](image)

2.4 Sobolev Spaces

Sobolev spaces are function spaces with certain integrability and differentiability properties that allow for an easier analysis than smooth functions. It turns out that these spaces are indispensible in the analysis of the finite element method. In this section we will define various Sobolev spaces which we will need later on.
2. Preliminaries

2.4.1 Positive Integer Order Sobolev Spaces

Sobolev spaces contain functions which are differentiable in a more general sense, this more general definition let’s us talk about the derivative of functions in $L^p$-spaces for which point evaluations are not well-defined.

Definition 2.7 (Weak Derivative) (Definition 5.8 in [6]) Let $k \in \mathbb{N}$, $\alpha \in \mathbb{N}^\ell$ for some $\ell \leq k$ and $\Omega \subseteq \mathbb{R}^n$ an open set. Let $u \in L^2(\Omega)$, then $u$ has a weak $\alpha$-partial derivative if there exists a $g \in L^2(\Omega)$ such that

$$\int_{\Omega} u(x) \partial^\alpha \varphi(x) \, dx = (-1)^{\|\alpha\|_1} \int_{\Omega} g(x) \varphi(x) \, dx, \quad \forall \varphi \in C^\infty_c(\Omega).$$

Then $g$ is called the weak $\alpha$-partial derivative of $u$.

Notation: We denote the weak $\alpha$-partial derivative of $u$ by $\partial^\alpha u$.

This notation does not conflict with the notation for regular partial derivatives of differentiable functions. Indeed, if $u$ has a regular derivative $\partial^\alpha u$, then it is also the weak $\alpha$-partial derivative because the definition of the weak derivative is just the integration by parts formula applied $\|\alpha\|_1$-times.

This generalized definition of partial derivative is crucial for the construction of Sobolev spaces which are an essential tool in the analysis of the finite element method.

Definition 2.8 (Sobolev space of Positive Integer Order) Let $k, p \in \mathbb{N}$, then we define the Sobolev space $W^{k,p}(\Omega)$ as

$$W^{k,p}(\Omega) := \{ u \in L^p(\Omega) \mid \forall \|\alpha\|_1 \leq k : \partial^\alpha u \in L^p(\Omega) \}.$$

The associated norm on this space is defined as

$$\|u\|_{W^{k,p}(\Omega)} := \left( \sum_{\|\alpha\|_1 \leq k} \|\partial^\alpha u\|_{L^p(\Omega)}^p \right)^{1/p}, \quad p \in \mathbb{N},$$

$$\|u\|_{W^{k,\infty}(\Omega)} := \max_{\|\alpha\|_1 \leq k} \|\partial^\alpha u\|_{L^\infty(\Omega)}, \quad p = \infty.$$

Notation: For $p = 2$ we write $H^k(\Omega) := W^{k,2}(\Omega)$.

In order to prove properties about functions in Sobolev spaces the following famous theorem by Meyers and Serrin is frequently invoked.

Theorem 2.9 [18] Let $\Omega$ be an open set and $p < \infty$ an integer. Then $C^\infty(\Omega) \cap W^{k,p}(\Omega)$ is dense in $W^{k,p}(\Omega)$. 

6
More specifically, this means that for any \( u \in W^{k,p}(\Omega) \), there exists a sequence \((u_n)_n\) in \( C^{\infty}(\Omega) \cap W^{k,p}(\Omega) \) such that
\[
\|u_n - u\|_{W^{k,p}(\Omega)} \to 0, \quad \text{as } n \to +\infty.
\]
The last positive integer order Sobolev space that we define is a Sobolev space which will be useful when considering Dirichlet boundary value problems where the solution is supposed to vanish on the boundary.

**Definition 2.10** Let \( \Omega \subseteq \mathbb{R}^n \) be open and \( k, p \) be non-negative integers. We define the **Sobolev space** \( W_0^{k,p}(\Omega) \) as
\[
W_0^{k,p}(\Omega) := \overline{C^\infty_c(\Omega)}_{\| \cdot \|_{W^{k,p}(\Omega)}}.
\]

The intuitive way to think about this space is as a space of functions which ‘vanish’ on \( \partial \Omega \). These functions obviously don’t vanish in the pointwise sense since pointwise values of functions in \( L^p \)-spaces aren’t well defined.

### 2.4.2 Fractional Order Sobolev Spaces

Sobolev spaces of fractional order can be defined in three different ways. The first way is as the closure of a certain space with respect to an appropriate norm, the second way is using the Fourier transform and the third way is by using interpolation. All these definitions are equivalent under certain assumptions on the boundary of the domain. We will only use the first definition which uses the so-called Sobolev-Slobodeckij norms which are similar to Hölder norms but integrated. We start by introducing the Sobolev-Slobodeckij (semi)norms.

**Definition 2.11** (Sobolev-Slobodeckij (semi)norm) (Page 56 in [19]) Let \( \Omega \subseteq \mathbb{R}^n \) be an open set and \( s \in \mathbb{R}^+, \theta \in (0, 1), p \in \mathbb{N} \) such that \( s = \lfloor s \rfloor + \theta \).

Then we define the **Sobolev-Slobodeckij seminorm** \( |\cdot|_{W^{s,p}(\Omega)} \) as
\[
|u|_{W^{s,p}(\Omega)} := \left( \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{\theta p + n}} \, dx \, dy \right)^{1/p}, \quad \forall u \in C^\infty(\Omega)
\]
and the **Sobolev-Slobodeckij norm** as
\[
\|u\|_{W^{s,p}(\Omega)} := \left( \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha u\|_{L^p(\Omega)}^p + \sum_{|\alpha| \leq \lfloor s \rfloor} \int_{\Omega} \int_{\Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^p}{|x - y|^{\theta p + n}} \, dx \, dy \right)^{1/p},
\]
\[
= \left( \sum_{|\alpha| \leq \lfloor s \rfloor} \|\partial^\alpha u\|_{L^p(\Omega)}^p + \sum_{|\alpha| \leq \lfloor s \rfloor} |\partial^\alpha u|_{W^{s,p}(\Omega)}^p \right)^{1/p}, \quad \forall u \in C^\infty(\Omega)
\]
2. Preliminaries

Now we have the tools to define fractional order Sobolev spaces.

**Definition 2.12 (Sobolev Space $W^{s,p}$)** (Page 56 in [19]) Let $\Omega \subseteq \mathbb{R}^n$ and $s^+ \in \mathbb{R}$. Then we define the space $W^{s,p}(\Omega)$ as

$$W^{s,p}(\Omega) := \{ u \in C^\infty(\Omega) \mid \| u \|_{W^{s,p}(\Omega)} < +\infty \}.$$

We get a similar definition for the space $W^{s,p}_0(\Omega)$.

**Definition 2.13** Let $\Omega \subseteq \mathbb{R}^n$, $s > 0$ and $p$ a non-negative integer. We define the Sobolev space $W^{s,p}_0(\Omega)$ as

$$W^{s,p}_0(\Omega) := \{ u \in C^\infty_c(\Omega) \mid \| u \|_{W^{s,p}(\Omega)} < +\infty \}.$$

### 2.5 Sobolev Spaces on the Boundary

In the following we will need Sobolev spaces for functions defined on the boundary of a bounded domain $\Omega$. These spaces are somewhat harder to define than regular Sobolev spaces because we have to take into account the regularity of the boundary.

**Definition 2.14** (Definition 2.4.1 in [19]) Let $\Omega$ be a Lipschitz or $C^k$-domain for some $k \geq 1$ and $\ell \leq 1$ for Lipschitz domains and $\ell \leq k$ for $C^k$-domains. Fix a measurable subset $\Gamma \subseteq \Omega$. Let $\{\psi_U : \overline{B}_2 \to \overline{U}\}_{U \in \mathcal{U}}$ be the associated bijective mappings from Definition 2.5. For any function $u_U : U \to \mathbb{R}$ defined on a $U \in \mathcal{U}$ we define $\hat{u}_U$ as

$$\hat{u}_U := u_U \circ \psi_U : B_2^0 \to \mathbb{R}.$$

Then we define the **Sobolev space** $H^{\ell}(\Gamma)$ as

$$H^{\ell}(\Gamma) := \{ u : \Gamma \to \mathbb{R} \mid \forall U \in \mathcal{U} : \hat{u}_U \in H^{\ell}_0(B_2^0) \}.$$

Moreover, we equip this space with the norm

$$\| u \|^2_{H^{\ell}(\Gamma)} := \sum_{|\alpha| \leq \ell} \| \partial^\alpha u \|^2_{L^2(\Gamma)},$$

for $\ell \in \mathbb{N}$ and

$$\| u \|^2_{H^{\ell}(\Gamma)} := \sum_{|\alpha| \leq \lfloor \ell \rfloor} \| \partial^\alpha u \|^2_{L^2(\Gamma)} + \sum_{|\alpha| = \lfloor \ell \rfloor} \int_{\Gamma} \int_{\Gamma} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{n-1+2\theta}} ds(x) ds(y),$$

for $\ell \in \mathbb{R} \setminus \mathbb{N}$ where $\theta := \ell - \lfloor \ell \rfloor$ and

$$\partial^\alpha u(x) := \sum_{U \in \mathcal{U}} \partial^\alpha_{\xi}(\hat{u}_U)(\xi),$$
with \( x := \psi(U, \xi) \). Here \( \partial^\xi \) just means differentiation with respect to the variable \( \xi \). The norms above derived is derived from the inner products defined by

\[
\langle u, v \rangle_{H^\ell(\Gamma)} := \sum_{\|\alpha\|_1 \leq \ell} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Gamma)}^2
\]

for \( \ell \in \mathbb{N} \) and

\[
\langle u, v \rangle_{H^\ell(\Gamma)} := \sum_{\|\alpha\|_1 \leq \ell} (\partial^\alpha u, \partial^\alpha v)_{L^2(\Gamma)}^2 + \sum_{\|\alpha\|_1 \leq \ell} \int_{\Gamma} \left| \frac{\partial^\alpha u(x) - \partial^\alpha u(y)}{|x - y|^{n-1+2\theta}} \right| \, ds(x) ds(y),
\]

for \( \ell \in \mathbb{R} \setminus \mathbb{N} \).

The next Sobolev space is specifically needed because of the fact that extending a function \( u \in H^{1/2}(\Gamma) \) for some measurable \( \Gamma \subseteq \partial \Omega \) by 0 onto \( \partial \Omega \) does not imply that \( u \in H^{1/2}(\partial \Omega) \).

**Definition 2.15 (Lions-Magenes space)** (Section 4.1 in [23]) Let \( \Omega \subseteq \mathbb{R}^n \) be a bounded domain and \( \Gamma \) be a measurable subset of \( \partial \Omega \). Then we define the **Lions-Magenes space** as

\[
H^{1/2}_{00}(\Gamma) := \{ u \in L^2(\Gamma) \mid \tilde{u} \in H^{1/2}(\partial \Omega) \},
\]

where \( \tilde{u} \) is the extension by zero of \( u \in L^2(\Gamma) \) on \( \partial \Omega \setminus \Gamma \). The norm on this space is given by

\[
\| u \|_{H^{1/2}_{00}(\Gamma)} := \| u \|_{L^2(\partial \Omega)} + \| u \|_{H^{1/2}(\partial \Omega)},
\]

where

\[
| u \|_{H^{1/2}_{00}(\Gamma)}^2 := | u \|_{H^{1/2}(\Gamma)}^2 + \int_{\Gamma} \frac{u^2(x)}{d(x, \partial \Gamma)} \, ds(x).
\]

This turns \( H^{1/2}_{00}(\Gamma) \) into a Hilbert space.

**Remark.** Usually \( n = 3, \Omega \) is a polyhedral domain and \( \Gamma \) is one of its faces. Note that a Hilbert space requires an inner product while we only defined a norm on \( H^{1/2}_{00}(\Gamma) \). However, the inner product which induces this norm is defined in the obvious way since all norms are just of the \( L^2 \)-kind.

### 2.5.1 Negative Order Sobolev Spaces

In this section we define Sobolev spaces of negative order which will be denoted by \( W^{-k,p} \). These are constructed using dual spaces of Banach spaces. For this we first define the dual space.
Definition 2.16 (Definition 2.53 in [6]) Let $X$ be a Banach space over $K = \mathbb{R}$ or $\mathbb{C}$. The dual space of $X$ is defined as the space

$$X^* := \{ L \in \text{Hom}(X, K) \mid L \text{ continuous} \}$$

equipped with the norm

$$\|L\|_{\text{op}} := \sup_{\|x\| \leq 1} |L(x)|.$$

**Remark.** $(X^*, \|\cdot\|_{\text{op}})$ is again a Banach space.

The only thing we now need is to fix a Banach space which will be the predual of this space. For this we use the spaces $W_0^{k,p}$. This leads us to the following definition.

**Definition 2.17 (Negative Order Sobolev Space)** Let $\Omega \subseteq \mathbb{R}^n$ and $p \in \mathbb{N}$ and $k > 0$. Then we define the Sobolev space $W^{-k,p}(\Omega)$ as

$$W^{-k,p}(\Omega) := W_0^{k,p}(\Omega)^*.$$

**Remark.** If $\Gamma \subseteq \mathbb{R}^n$ is a closed $C^k$-surface, then it has no boundary. Hence

$$H_0^\ell(\Gamma) = H^\ell(\Gamma), \quad \forall \ell \leq k.$$

Therefore we have

$$H^{-\ell}(\Gamma) = H^\ell(\Gamma)^*.$$

Moreover, we have an $L^2$-dual pairing between $H^{-\ell}(\Gamma)$ and $H^\ell(\Gamma)$ given by

$$H^{-\ell}(\Gamma) \times H^\ell(\Gamma) \to K : (u, v) \mapsto \langle \tilde{u}, v \rangle_{H^\ell(\Gamma)},$$

where $\tilde{u}$ is the unique element of $H^\ell(\Gamma)$ which corresponds to $u$ under the Fréchet-Riesz representation theorem (Corollary 3.19 in [6]).

**Definition 2.18 (Dual Sobolev Norms)** Let $\Omega \subseteq \mathbb{R}^n$ be a bounded Lipschitz or $C^k$-domain and consider the Sobolev space $H^{-\ell}(\partial \Omega)$ for some $\ell \leq k$. We define a norm on this space by setting

$$\|u\|_{H^{-\ell}(\partial \Omega)} := \|u\|_{\text{op}},$$

as usual for the dual space of a Banach space. Because of the $L^2$-dual pairing between $H^\ell(\partial \Omega)$ and $H^{-\ell}(\partial \Omega)$ we can rewrite this as

$$\|u\|_{H^{-\ell}(\partial \Omega)} = \sup_{0 \neq v \in H^\ell(\partial \Omega)} \frac{|\langle \tilde{u}, v \rangle_{H^\ell(\partial \Omega)}|}{\|v\|_{H^\ell(\partial \Omega)}},$$

where $\tilde{u}$ is the unique element of $H^\ell(\Gamma)$ which corresponds to $u$ under the Fréchet-Riesz representation theorem (Corollary 3.19 in [6]).
2.5.2 Sobolev Spaces \( H(\text{curl}; \Omega) \), \( H_0(\text{curl}; \Omega) \) and \( H^\delta(\text{curl}; \Omega) \)

One application of the edge and face lemma is in the numerical simulation of Maxwell’s equation. For the analysis of these methods we need certain vectorial Sobolev spaces which we introduce in the next couple of sections. For the weak formulation of Maxwell’s equations we need Sobolev spaces involving the curl operator.

**Definition 2.19 (Sobolev Space \( H(\text{curl}; \Omega) \))** (A.5.3 in [21]) Let \( \Omega \subseteq \mathbb{R}^3 \) be a Lipschitz domain. The **Sobolev space** \( H(\text{curl}; \Omega) \) is defined as

\[
H(\text{curl}; \Omega) := \{ u \in (L^2(\Omega))^3 \mid \text{curl} \ u \in (L^2(\Omega))^3 \},
\]

equipped with the inner product

\[
\langle u, v \rangle_{H(\text{curl}; \Omega)} = \langle u, v \rangle_{(L^2(\Omega))^3} + \langle \text{curl} u, \text{curl} v \rangle_{(L^2(\Omega))^3}, \quad \forall u, v \in H(\text{curl}; \Omega).
\]

To define the Sobolev space \( H_0(\text{curl}; \Omega) \), we need a trace theorem which allows us to define the tangential component of a function \( u \in H(\text{curl}; \Omega) \).

**Lemma 2.20 (Tangential Trace)** (Lemma A.22 in [21]) Let \( \Omega \subseteq \mathbb{R}^3 \) be a Lipschitz domain with unit outward normal \( n \). Then the operator

\[
\gamma_t : (C^\infty(\Omega))^3 \rightarrow (C^\infty(\partial\Omega))^3 : u \mapsto u \times n,
\]

can be extended to a bounded operator

\[
\gamma_t : H(\text{curl}; \Omega) \rightarrow (H^{-1/2}(\partial\Omega))^3,
\]

called the **tangential trace**. Moreover, the following Green’s formula holds:

\[
\int_{\Omega} \text{curl} \ u(x) \cdot v(x) \, dx - \int_{\Omega} u(x) \cdot \text{curl} \ v(x) \, dx = \int_{\partial\Omega} (n \times u)(x) \cdot v(x) \, ds(x)
\]

Figure 2.2: Example of a smooth vector field \( u \) together with its tangential trace \( u \times n \) at a single point
2. Preliminaries

Definition 2.21 (Sobolev Space $H_0(\text{curl}; \Omega)$) (A.5.3 in [21]) Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain. The Sobolev space $H_0(\text{curl}; \Omega)$ is defined as

$$H_0(\text{curl}; \Omega) := \{ u \in H(\text{curl}; \Omega) \mid \gamma_t u = 0 \}.$$ 

Definition 2.22 (Sobolev Space $H^\delta(\text{curl}; \Omega)$) (Section 2 in [15]) Let $\Omega$ be a Lipschitz domain and $\delta > 0$ fixed. Then we defined the Sobolev space $H^\delta(\text{curl}; \Omega)$ as

$$H^\delta(\text{curl}; \Omega) := \{ v \in (H^\delta(\Omega))^3 \mid \text{curl } v \in (H^\delta(\Omega))^3 \},$$

equipped with the norm

$$\| v \|_{H^\delta(\text{curl}; \Omega)}^2 := \| v \|_{H^\delta(\Omega)}^2 + \| \text{curl } v \|_{H^\delta(\Omega)}^2.$$ 

So $H^\delta(\text{curl}; \Omega)$ is just a fractional order analogue of $H^\delta(\Omega)$ for the $H(\text{curl}; \Omega)$ Sobolev spaces.

2.5.3 Sobolev Spaces $H(\text{div}; \Omega)$, $H_0(\text{div}; \Omega)$ and $H_0(\text{div}_0; \Omega)$

In this section we define Sobolev spaces involving the divergence operator. The definitions are similar to those for the $H(\text{curl}, \Omega)$ spaces defined in the previous section but in this case for the divergence operator.

Definition 2.23 (Sobolev Space $H(\text{div}; \Omega)$) (A.5.1 in [21]) Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain. We define the Sobolev space $H(\text{div}; \Omega)$ as

$$H(\text{div}; \Omega) := \{ u \in (L^2(\Omega))^3 \mid \text{div } u \in L^2(\Omega) \},$$

equipped with the inner product

$$\langle u, v \rangle_{H(\text{div}; \Omega)} := \langle u, v \rangle_{(L^2(\Omega))^3} + \langle \text{div } u, \text{div } v \rangle_{L^2(\Omega)}, \quad \forall u, v \in H(\text{div}; \Omega).$$ 

Next we need a lemma similar to Lemma 2.20, but for the normal component of functions in $H(\text{div}; \Omega)$.

Lemma 2.24 (Normal Trace) (Lemma A.19 in [21]) Let $\Omega$ be a Lipschitz domain with unit outward normal $n$. Then the operator

$$\gamma_n : (C^\infty(\Omega))^3 \rightarrow C^\infty(\partial \Omega) : u \rightarrow u \cdot n,$$

can be extended to a bounded operator

$$\gamma_n : H(\text{div}; \Omega) \rightarrow H^{-1/2}(\partial \Omega),$$

called the normal trace.
2.6. The Dirichlet Trace

Consider the following Dirichlet boundary condition:

\[ u = f, \quad \text{on } \partial \Omega, \]

for some \( f : \partial \Omega \to \mathbb{R} \). If \( u \) is a strong solution to a PDE, then it is at least continuous and therefore the interpretation of the Dirichlet boundary condition is evident. On the boundary of \( \Omega \) we have pointwise equality

\[ u(x) = f(x), \quad \forall x \in \partial \Omega. \]

However, when considering weak solutions to a PDE, these will generally be elements of a certain Sobolev space. In general, pointwise evaluation of a function in an \( L^p \)-space is not well defined. Therefore we need to generalize what it means to evaluate a function or more generally to restrict a function to a subset of measure 0 (for example \( \partial \Omega \)). The way this will be done is by extending the operator which assigns to a continuous function \( u \) its restriction \( u|_{\partial \Omega} \) to the boundary of \( \Omega \). This generalization will be sufficient for our purposes since Dirichlet boundary conditions are only prescribed on the boundary of the domain of interest.
Theorem 2.26 (Dirichlet Trace for $H^1$) (Theorem 4.1 in [23]) Let $\Omega$ be a Lipschitz domain. Then there exists a surjective bounded linear operator

$$\gamma_0 : H^1(\Omega) \to H^{1/2}(\partial\Omega),$$

such that

$$\gamma_0 u = u|_{\partial\Omega}, \quad \forall u \in H^1(\Omega) \cap C^1(\bar{\Omega}),$$

and

$$|\gamma_0 u|_{H^{1/2}(\partial\Omega)} \leq C|v|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega)$$

for some constant $C > 0$. Moreover, $\gamma_0$ has a right-continuous bounded inverse

$$E : H^{1/2}(\Omega) \to H^1(\Omega)$$

such that

$$|Ev|_{H^1(\Omega)} \leq \tilde{C}|v|_{H^{1/2}(\partial\Omega)}, \quad \forall v \in H^{1/2}(\Omega),$$

for some constant $\tilde{C} > 0$. 
The Finite Element Method

The finite element method is a numerical method for approximating solutions to PDEs. The special approach that the finite element method takes is that it approximates solutions to the variational form of the PDE and not the strong form. Therefore we commence by deriving the variational formulation of a model PDE followed by the general definition of a variational problem.

3.1 Variational Problems

Let us start with an example to illustrate why we’re concerned with variational problems. Consider the following Dirichlet boundary value problem:

\[
\begin{align*}
-\Delta u &= f, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial \Omega,
\end{align*}
\]

for some bounded domain \( \Omega \subseteq \mathbb{R}^n \). If we want to find a weak solution \( u \) in some \( L^p \)-space, then pointwise values of \( u \) aren’t necessarily well-defined. However, we know that integrals of \( u \) can be defined. Let us therefore integrate the above PDE, but we must take care because \( \Delta u \) is not necessarily integrable. Therefore we first multiply the PDE by a test function \( \varphi \in C_0^\infty(\Omega) \) and then integrate. This gives

\[
\int_\Omega -\Delta u(x) \varphi(x) \, dx = \int_\Omega f(x) \varphi(x) \, dx.
\]

Next we apply integration by parts to the left-hand side:

\[
\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \, dx = \int_\Omega f(x) \varphi(x) \, dx.
\]

Since this holds for all \( \varphi \in C_0^\infty(\Omega) \) we get the following equation which we call the variational form of (3.1):

\[
\int_\Omega \nabla u(x) \cdot \nabla \varphi(x) \, dx = \int_\Omega f(x) \varphi(x) \, dx, \quad \forall \varphi \in H_0^1(\Omega),
\]
by density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$ with respect to $\|\cdot\|_{H^1(\Omega)}$. The reason why we take this norm is because (3.1) only requires $u$ to have (weak) derivatives up to order one. Notice that the boundary condition for the Dirichlet problem is incorporated in the Sobolev space $H_0^1(\Omega)$ of test functions. If we now set $V := H_0^1(\Omega)$ and define the bilinear form $a$ as

$$a : V \times V \to \mathbb{R} : (u, v) \mapsto \int_\Omega \nabla u(x) \cdot \nabla v(x) \, dx,$$

then we can reformulate (3.1) as

$$a(u, v) = \langle u, f \rangle_{L^2(\Omega)}, \quad \forall v \in V.$$

This last form is what we will abstractly deal with when talking about the finite element method.

### 3.2 Symmetric and Nonsymmetric Variational Problems

The finite element method approximates solutions to symmetric and nonsymmetric variational problems. In this section we abstractly define what these problems are.

**Definition 3.1 (Symmetric Variational Problem)** (Page 57 in [3]) Let $(H, \langle \cdot, \cdot \rangle_H)$ be a Hilbert space over $K = \mathbb{R}$ or $\mathbb{C}$, $V \subseteq H$ a closed subspace and $a : V \times V \to K$ a bounded, symmetric and coercive bilinear form. Then the associated symmetric variational problem is given by:

Given $F \in V^*$, find $u \in V$ such that

$$a(u, v) = \langle u, f \rangle_{L^2(\Omega)}, \quad \forall v \in V.$$

**Remark.** The nonsymmetric variational problem is similar to the symmetric one with one difference, $a$ is not necessarily symmetric. Moreover, the conditions that $a$ is coercive and bounded is not necessary to define a variational problem. However, we will always assume that these are fulfilled to guarantee the existence of solution for a symmetric variational problem and hence add them to the definition of a variational problem.

We can rewrite a variational problem in a more compact way by introducing an operator associated to the bilinear form $a$. The existence and unicity of this operator is guaranteed by the Fréchet-Riesz representation theorem (Corollary 3.19 in [6]).

**Definition 3.2** let $a$ a bilinear form as in Definition 3.1. The operator $A : V \to V$ associated to $a$ is defined as the unique bounded operator such that

$$\langle Au, v \rangle_H = a(u, v), \quad \forall u, v \in V.$$
Using the above definition, we can rewrite the variational problem from Definition 3.1 as:

Given \( f \in V \), find \( u \in V \) such that

\[ Au = f, \]

where \( f \in V \) is the unique element such that

\[ F(v) = \langle f, v \rangle_H, \quad \forall v \in V. \]

Again, the existence and unicity of \( f \) is guaranteed by the Fréchet-Riesz representation theorem (Corollary 3.19 in [6]).

### 3.3 The Finite Element Method

Now we can describe how the finite element method works. Given the variational problem: Given \( F \in V^* \), find \( u \in V \) such that

\[ a(u, v) = F(v), \quad \forall v \in V. \]

Under the assumption that this is the weak form of some PDE, the finite element method constructs an approximation of the solution \( u \) from some finite-dimensional subspace \( S \subseteq V \). The choice of approximand from this subspace will be an optimal approximation (to be made precise) to the solution of the weak form of the PDE. We start by explaining how the finite-dimensional space \( S \) is constructed. The essential ingredient is the mesh which is chosen on the domain where the PDE is defined.

### 3.4 Meshes and Regularity

The finite dimensional space \( S \) is constructed by first considering a mesh on the domain on which the PDE is defined. Let \( \Omega \subseteq \mathbb{R}^n \) be bounded any domain. We first define precisely what a mesh is.

**Definition 3.3 (Mesh)** (Definition 4.2.1 in [10]) A mesh (or triangulation) on a bounded domain \( \Omega \subseteq \mathbb{R}^n, n = 2, 3 \), is a finite family \( \mathcal{M} = \{K_i\}_{i=1}^M \), \( M \in \mathbb{N} \), of open non-degenerate (curvilinear) polygons (\( n = 2 \)) or polyhedra (\( n = 3 \)) such that

1. \( \overline{\Omega} = \bigcup_{i=1}^M \overline{K_i} \)
2. \( \forall i \neq j : K_i \cap K_j = \emptyset \),
3. \( \forall i \neq j : \overline{K_i} \cap \overline{K_j} \) is either the empty set, a vertex, an edge or a face of both \( K_i \) and \( K_j \).
We will denote by $N(M)$ the set nodes of the mesh $M$.

Our results will crucially depend on certain assumptions regarding such a mesh. The two assumptions which we will frequently use are shape regularity and quasi-uniformity. Before giving these definitions we will fix some notation.

**Definition 3.4 (Inscribed Ball)** Let $T \subseteq \mathbb{R}^n, n \in \mathbb{N}$ be an open set with compact closure and piecewise smooth boundary. We denote by $B_T$ the largest ball contained in $T$ such that $T$ is star-shaped with respect to $B_T$.

![Inscribed Ball](image)

Figure 3.1: Example of the inscribed ball (disk) as in Definition 3.4

Now we are set to give one of the most crucial definitions which is indispensable in the proofs that we will give.

**Definition 3.5 (Definition 4.4.13 in [3])** Fix a bounded domain $\Omega \subseteq \mathbb{R}^n, n = 2, 3$, and a family of subdivisions $\{T^h\}_{0 < h \leq 1}$ such that

$$h = \max_{T \in T^h} \frac{\text{diam } T}{\text{diam } \Omega},$$

i.e. $h$ is the mesh width scaled by $1/\text{diam } \Omega$. The family $\{T^h\}_{0 < h \leq 1}$ is quasi-uniform if there exists a $\rho > 0$ such that

$$\min_{T \in T^h} \text{diam } B_T \geq \rho h \text{ diam } \Omega,$$

for all $h \in (0, 1]$. The family $\{T^h\}_{0 < h \leq 1}$ is non-degenerate or shape regular if there exists a $\rho > 0$ such that for all $T \in T^h$ and $h \in (0, 1]$:

$$\text{diam } B_T \geq \rho \text{ diam } T.$$

**Remark.** Notice that the condition for shape regularity is equivalent to the condition

$$\inf_{T \in T^h, h \in (0, 1]} \frac{\text{diam } B_T}{\text{diam } T} \geq \rho.$$
Which means we can find for any element $T$ in the family of meshes $\{T^h\}_{0<h\leq1}$, a ball $B_T$ inscribed in $T$, such that $B_T$ is star-shaped with respect to $T$ such that the size of the ball $B_T$ relative to the size of $T$ is bounded by some uniform constant $\rho > 0$. We also remark that the interval $(0, 1]$ is not essential to the definition. Any set $H \subseteq (0, +\infty)$ can be chosen instead.

### 3.5 Mesh Refinement

In the last part of this section we describe what kind of mesh refinement we will consider. For our results, we will only consider meshes of triangles and tetrahedrons and therefore will only discuss mesh refinement for these cases. We will describe in detail how regular mesh refinement is done for two dimensional meshes of triangles which result in families of quasi-uniform meshes. A similar (but albeit more complicated) construction can be made for tetrahedral meshes. A graphical illustration of two consecutive refinements is given in Figure 3.2.

![Figure 3.2: Example of midpoint refinement of a triangle](image)

Consider the triangle on the left. This can be thought of as one triangle in a mesh of triangles covering a two dimensional domain. The regular mesh refinement that we will consider is midpoint refinement which goes as follows. The mesh is locally refined inside each triangle of the mesh, this means that the refinement of the mesh inside one triangle is independent of what happens for a different triangle. Therefore we illustrate this only with one triangle. The refinement is constructed by connecting the midpoints of each edge of the triangle. This results in the middle triangle in Figure 3.2 which now consists of four smaller triangles. The next refinement is done in exactly the same way by connecting the midpoints of the edges of each triangle with each other. The result of this is shown in the right triangle in Figure 3.2. When we talk about a family of quasi-uniform meshes, we will assume that they are the result of such a refinement.
3. The Finite Element Method

3.5.1 Finite Elements and Local Shape Functions

Now that we have a mesh on our domain, we can construct the finite-dimensional space of functions $S$ which we talked about before. This is done by choosing a basis for the functions defined on each element of the mesh separately. This results in the following definition.

Definition 3.6 (Finite Element) (Definition 3.3.1 in [3]) A finite element is a triple $(K, P_K, N_K)$ where

1. $K \subseteq \mathbb{R}^n$ is an open set with compact closure and piecewise smooth boundary (element domain),
2. $P_K$ is a finite-dimensional space of functions on $K$ (local shape functions),
3. $N_K \subseteq P_K^*$ is a basis for $P_K^*$ (nodal variables/degrees of freedom).

Let us illustrate this concept with an easy example.

Example 3.7 (Linear Lagrangian Finite Elements in 2D) (Example 3.2.1 in [3]) Let $K \subseteq \mathbb{R}^2$ be a triangle and set

$$P_K := \{ \mathbf{x} \in K \mapsto \mathbf{a}^T \mathbf{x} + b \mid \mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R} \}.$$ 

If we define $N_K$ as

$$N_K := \{ \text{ev}_1, \text{ev}_2, \text{ev}_3 \} \subseteq P_K^*,$$

which are the point evaluations at the three vertices of the triangle, then $(K, P_K, N)$ is a finite element. Indeed, $N_K$ is the dual basis associated to the barycentric coordinate functions $\{ \lambda_1, \lambda_2, \lambda_3 \}$ with respect to the nodes of the triangle. See Figure 3.3 for an illustration of these functions.

![Figure 3.3: The barycentric coordinate functions associated to the triangle $K$](image)

The main idea of the finite element method is to approximate the solution of a PDE on each element $K$ by a linear combination of functions in $P_K$. Before we discuss the global finite-dimensional space $S$ we discuss a special case where each element is related to one another.
3.5. Mesh Refinement

3.5.2 Affine equivalence

We will be interested in a special type of finite element. Namely, the ones which are all just transformations of one single element in the mesh. By this we mean that after fixing a single finite element in the given mesh, all the other finite elements are just transformations of this one element.

**Definition 3.8 (Affine Equivalence)** *(Definition 3.4.1 in [3])* Let \((K, P, N)\) be a finite element with \(K \subseteq \mathbb{R}^n\) and let \(F : \mathbb{R}^n \to \mathbb{R}^n\) be an invertible affine map. The finite element \((\hat{K}, \hat{P}, \hat{N})\) is **affine equivalent** to \((K, P, N)\) if

1. \(F(K) = \hat{K}\),
2. \(F^*\hat{P} = P\),
3. \(F_*N = \hat{N}\).

Here \(F^*\) is the **pullback** of \(F\), defined by

\[ F^*(\hat{f}) := \hat{f} \circ F, \quad \forall \hat{f} \in \hat{P}, \]

and \(F_*\) is the **pushforward** of \(F\), defined by

\[ (F_*(N))(\hat{f}) := N(F^*(\hat{f})), \quad \forall N \in N, \hat{f} \in \hat{P}. \]

3.5.3 Global Shape Functions and piece-wise polynomial spaces

In the previous section we introduced the notion of local shape functions. These are functions defined locally on an element domain. To obtain an approximation to a solution of a PDE we need a function defined on the entire domain we’re considering. These globally defined functions will be called (global) shape functions. There is no single definition of global shape functions since the properties of the global shape functions differ depending on the function space setting. Some examples of these properties are global continuity, continuity of the tangential or normal components etc.

To illustrate this concept we give an example. A commonly used choice of local shape functions are polynomials of a certain degree. If (piece-wise) polynomials are used as shape functions, then the name Lagrangian finite element method is used as in Example 3.7. It is this finite element method in which we are interested. First we need to introduce the concept of polynomial spaces.

**Definition 3.9 (Polynomial spaces)** *(Definition 4.2.2 in [10])* Let \(K \subseteq \mathbb{R}^n\) be a set (we don’t require any regularity). We define the set of **polynomials of degree** \(p \in \mathbb{N}\) on \(K\) as

\[ \mathcal{P}_p(K) := \left\{ q \in C(K) \left| \exists (a_\alpha)_{\|\alpha\|_1 \leq p} \subseteq \mathbb{R} : \forall x \in K : q(x) = \sum_{\|\alpha\|_1 \leq p} a_\alpha x^\alpha \right. \right\}. \]
So $\mathcal{P}_p(K)$ is the space of multivariate polynomials in $n$ variables on $K$ such that the (total) degree is at most $p$.

**Remark.** In the above definition, the $\alpha \in \mathbb{N}_0$. Therefore the sum in the definition is always finite and hence well-defined.

Now that we have defined polynomial spaces, we can define the relevant space of global shape functions which will be a piece-wise polynomial space. As we will see, this function space will locally be a polynomial space.

**Definition 3.10 (Lagrangian Shape Functions)** *(Definition 4.3.1 in [10])*

Let $\Omega \subseteq \mathbb{R}^n$ be a Lipschitz domain equipped with a simplicial mesh $\mathcal{M}$ and $p \in \mathbb{N}$. The space of $p$-th degree Lagrangian global shape functions is defined as

$$\mathcal{S}^0_p(\mathcal{M}) := \{ v \in C(\overline{\Omega}) \mid \forall K \in \mathcal{M} : v|_K \in \mathcal{P}_p(K) \}.$$

We denote by $\mathcal{S}^0_{p,0}(\mathcal{M})$ the subspace of functions in $\mathcal{S}^0_p(\mathcal{M})$ that vanish on $\partial \Omega$, i.e.

$$\mathcal{S}^0_{p,0}(\mathcal{M}) := \{ v \in \mathcal{S}^0_p(\mathcal{M}) \mid v|_{\partial \Omega} = 0 \}.$$

Last, for any $\Omega' \subseteq \Omega$, we define $\mathcal{S}^0_p(\mathcal{M}|_{\Omega'})$ as the $p$-th degree Lagrangian global shape functions with respect to the mesh on $\Omega'$ induced by $T^h$.

![Figure 3.4: Example of a Lagrangian global shape function $v$ which is a basis function for $\mathcal{S}^0_p(\mathcal{M})$ where $\mathcal{M}$ is the mesh on $\Omega$](image)

**Remark.** If the mesh is just a grid (a Cartesian product of 1-dimensional meshes), then the above definition of the Lagrangian shape functions simplifies. Indeed, we can replace $\mathcal{P}_p(K)$ by the space of tensor product polynomials $\mathcal{P}^\otimes_p(K)$ which is defined as

$$\mathcal{P}^\otimes_p(K) := \left\{ q \in \mathcal{P}_p(K) \mid \exists p_1, \ldots, p_n \in \mathcal{P}_p(K) : \forall x \in K : q(x) = \prod_{k=1}^n p_k(x_k) \right\}.$$
3.5. Mesh Refinement

3.5.4 Ritz-Galerkin Approximation

Now that we have defined the finite-dimensional space of approximands with respect to a given mesh we explain in what way an approximation is constructed to the solution of a PDE. The finite element method approximates the solution of a PDE on each element by the Ritz-Galerkin method. This method is surprisingly simple and elegant to describe. We state it in the following definition.

**Definition 3.11 (Ritz-Galerkin Approximation Problem)** (Section 0.2 in [3]) Let \( H \) be a Hilbert space over \( K (= \mathbb{R} \text{ or } \mathbb{C}) \), \( V \subseteq H \) a closed subspace and \( a : V \times V \to K \) a bounded and coercive bilinear form. Consider the symmetric variational problem: Given \( F \in V^* \), find \( u \in V \) such that

\[
a(u,v) = F(v), \quad \forall v \in V.
\]

Now fix a finite-dimensional subspace \( V^h \subseteq V \). Then the associated Ritz-Galerkin approximation problem is defined as: Find \( u^h \in V^h \) such that

\[
a(u^h,v^h) = F(v^h), \quad \forall v^h \in V^h.
\]

So the Ritz-Galerkin approximation problem associated to a symmetric variational problem is just that same variational problem with a smaller solution space, i.e. the restriction of \( V \) to some finite dimensional subspace.

**Remark 3.12** The Ritz-Galerkin approximation problem for a nonsymmetric variational problem is defined in exactly the same way as for the symmetric case.

We end this section by describing the philosophy of the finite element method in a bit more detail since we now have defined all the relevant concepts. Given a variational problem (resulting from a PDE), a mesh is chosen on the domain \( \Omega \). On each element domain, suitable local shape functions and nodal variables are chosen. Then the solution to the variational problem is approximated by computing the solution to the associated Ritz-Galerkin approximation problem which boils down to solving a system of linear equations (since it is a linear equation on a finite-dimensional vector space).
Chapter 4

Inverse Inequalities

In this chapter we are concerned with inverse inequalities (also called inverse estimates). These inequalities will be indispensable when we prove the edge and face lemma. Let us first clarify the term inverse inequality and motivate why such inequalities exist.

Consider for example the spaces $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, for some bounded domain $\Omega \subseteq \mathbb{R}^n$ and $p \in \mathbb{N}$. Then we have the obvious continuous embeddings

$$W^{1,p}(\Omega) \hookrightarrow W^{2,p}(\Omega) \hookrightarrow \ldots \hookrightarrow W^{k,p}(\Omega) \hookrightarrow W^{k+1,p}(\Omega) \hookrightarrow \ldots$$  \hspace{1cm} (4.1)

Moreover it is obvious that there are no embeddings

$$L^\infty(\Omega) \hookrightarrow W^{k,p}(\Omega), L^p(\Omega) \hookrightarrow W^{k,p}(\Omega).$$  \hspace{1cm} (4.2)

However, for finite element spaces certain converses of (4.1) are true. Even (4.2) turns out to be true for certain finite element spaces. This is not very surprising because of norm equivalence on finite dimensional normed spaces. Therefore, any two well-defined norms on a finite element space are related by two-sided inequalities. The nontrivial inequalities of this kind are inverse inequalities. So why are we interested in these inequalities if norm equivalence already tells us that these two-sided inequalities exist? The essential part here is that we want to state these inequalities with an explicit dependence on $h$, i.e. as an inequality of the form

$$\|u^h\|_{W^{k,p}(\Omega)} \leq Cb(h)\|u^h\|_{W^{\ell,q}(\Omega)},$$

for some constant $C > 0$ (independent of $u^h$ and $h$) and $k > \ell$ or $k = \ell$ and $p > q$, where $b : \mathbb{R}^+ \rightarrow \mathbb{R}$ is some function and $\ell < k$. Here $u^h$ is an arbitrary finite element function on some domain $\Omega \subseteq \mathbb{R}^n$ with respect to a mesh $T^h$ with mesh width $h > 0$. For the sake of argument assume that all norms are well defined.
These inequalities are essential for error analysis and for constructing preconditioners for the linear systems resulting from Galerkin discretization. In this section we will state and prove certain inverse inequalities which we will need in the proofs of the edge and face lemma.

Before continuing, we fix some notation that will be used in the proofs from now on. The symbols \( \lesssim, \gtrsim, \approx \) will be frequently used, but never in the statements of the results. The interpretation is as follows, let \( y_1, y_2 \) be two quantities of interest. We write \( y_1 \lesssim y_2 \) if there exists a constant \( C > 0 \) independent of the mesh parameters such that \( y_1 \lesssim y_2 \). The interpretation of \( \gtrsim \) is analogous. We say \( y_1 \approx y_2 \) if both \( y_1 \lesssim y_2 \) and \( y_1 \gtrsim y_2 \).

### 4.1 Discrete variants

This section is devoted to introducing some concepts which will be needed to help prove the inverse inequalities that will be stated in this section. This will mostly consist of generalizing known concepts like harmonic functions to the discrete setting of finite element spaces.

#### 4.1.1 Scott-Zhang Quasi-Interpolant

In contrast to the nodal value interpolant which just interpolates a continuous function on the nodes of the mesh, the average nodal value interpolant is used to interpolate possibly discontinuous functions.

**Definition 4.1 (Scott-Zhang Quasi-Interpolant)** (Section 4.2.1 in [23])

Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a mesh \( T^h \). Let \( T_k \) be an \( (n-1) \)-simplex from the mesh \( T^h \) with vertices \( \{z_k\}_{k=1}^{N_h} \) such that \( z_1 = z_k \) for any \( z_k \in N(T^h) = \{z_1, \ldots, z_{N_h}\} \). Let \( \theta_k \in \mathcal{P}_1(T_k) \) be the unique function such that

\[
\int_{T_k} \theta_k(x) \lambda_k(x) \, dx = \delta_{k1}, \quad k = 1, \ldots, n,
\]

where \( \lambda_k \) is the barycentric coordinate of \( T_k \) with respect to \( z_k \). The **Scott-Zhang quasi-interpolant** \( \Pi_h : H^1(\Omega) \rightarrow S^0_1(T^h) \) is defined as

\[
(\Pi_h v)(x) := \sum_{k=1}^{N_h} \varphi_k(x) \int_{T_k} \theta_k(\xi)v(\xi) \, d\xi, \quad \forall x \in \Omega,
\]

where \( (\varphi_k)_{k=1}^{N_h} \) is the nodal basis for \( S^0_1(T^h) \).

**Remark.** The choice of \( T_k \) is not unique in general, but if \( x_k \in \partial \Omega \), then we can choose \( T_k \) such that \( T_k \subseteq \partial \Omega \). For an illustration of this, we refer to Figure 4.1.
4.1. Discrete variants

\[ \partial \Omega \]

\[ T_k \]

\[ \tilde{T}_k \]

\[ z_1 = \tilde{z}_1 = x_k \]

\[ z_3 = \tilde{z}_3 \]

Figure 4.1: Example of two possibilities \( T_k \) and \( \tilde{T}_k \) of \((n-1)\)-simplices associated to the node \( x_k \). Since \( x_k \in \partial \Omega \), the choice of \((n-1)\)-simplex can indeed be chosen to in \( \partial \Omega \).

It turns out that the Scott-Zhang quasi-interpolant \( \Pi_h \) is not only a bounded operator, but also that the family \( \{ \Pi_h \} \) forms an equicontinuous family of operators. This is stated in the following lemma.

**Lemma 4.2** \([20]\) Let \( \Pi_h \) be the Scott-Zhang quasi-interpolant as in Definition 4.1. Then

\[ \| \Pi_h v \|_{H^1(\Omega)} \leq C_1 \| v \|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]

\[ |\Pi_h v|_{H^1(\Omega)} \leq C_2 |v|_{H^1(\Omega)}, \quad \forall v \in H^1(\Omega), \]

where the constants \( C_1 \) and \( C_2 \) are independent of \( h \) and depend only on the shape regularity of the mesh.

4.1.2 The (Generalized) Discrete Harmonic Extension

Consider a polyhedral domain \( \Omega \subseteq \mathbb{R}^n, n = 2, 3 \) equipped with a mesh \( \mathcal{T}^h \) with mesh width \( h > 0 \). Fix a function \( u \in C(\partial \Omega) \). The harmonic extension of \( u \) into \( \Omega \) is defined as the solution to the Dirichlet boundary value problem

\[
\begin{align*}
-\Delta \bar{u} &= 0, & \text{in } \Omega, \\
\bar{u} &= u, & \text{on } \partial \Omega.
\end{align*}
\]
4. **Inverse Inequalities**

We want to generalize this concept of harmonic extension to piece-wise linear and globally continuous functions. We start by looking at the variational form of this PDE as we did in the beginning of Chapter 3. The same computation gives us the following variational formulation

\[ \int_{\Omega} \nabla \tilde{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) \, d\mathbf{x} = 0, \quad \forall \tilde{u}, v \in H^1_0(\Omega). \]

Since we don’t want to look at the values on the boundary, we take \( H^1_0(\Omega) \) as the space of test functions. Imposing the correct boundary values gives us a generalization of harmonic extensions, but this isn’t enough. Finite element functions which vanish at the boundary don’t live in the whole space \( H^1_0(\Omega) \) since they form only a finite dimensional subspace of this Sobolev space. Therefore we should take \( S^0_{1,0}(T^h) \) as the space of test functions. This motivates the following definition.

**Definition 4.3 (Discrete Harmonic Extension)** (Section 4.2.2 in [23]) Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a quasi-uniform family of meshes \( \{T^h\} \). Fix a \( u \in S^0_1(T^h|\partial\Omega) \), the **discrete harmonic extension** of \( u \) into \( \Omega \) is defined as the solution \( u^h \in S^0_0(T^h) \) of the variational problem

\[ \int_{\Omega} \nabla u^h(\mathbf{x}) \cdot \nabla v^h(\mathbf{x}) \, d\mathbf{x} = 0, \quad \forall v^h \in S^0_{1,0}(T^h). \]

\[ u^h = u, \quad \text{on } \partial\Omega. \]

This definition is further generalized by considering the \( H^1 \)-inner product instead of the \( H^1 \)-pseudo-inner product.

**Definition 4.4 (Generalized Discrete Harmonic Extension)** (Section 4.2.2 in [23]) Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a quasi-uniform family of meshes \( \{T^h\} \). Fix a \( u \in S^0_0(T^h|\partial\Omega) \), we define the **generalized discrete harmonic extension** of \( u \) into \( \Omega \) as the solution \( u^h \in S^0_1(T^h) \) of the variational problem

\[ \langle u^h, v^h \rangle_{H^1(\Omega)} = 0, \quad \forall v^h \in S^0_{1,0}(T^h). \]

\[ u^h = u, \quad \text{on } \partial\Omega. \]

**Remark.** Any function \( u \in S^0_1(T^h) \) satisfying the definition of the (generalized) discrete harmonic extension is called a (generalized) discrete harmonic function.

The crucial property of these harmonic extensions is stated and proved in the following proposition. As we can intuitively expect, a discrete harmonic function can be controlled in a certain sense by its values on the boundary.
Proposition 4.5 (Section 4.2.2 in [23]) Let $\Omega \subseteq \mathbb{R}^n$ be a polyhedral domain equipped with a quasi-uniform family of meshes $\{T^h\}$ and let $u^h \in S^0_1(T^h)$ be a generalized discrete harmonic finite element function. Then

$$\|u^h\|_{H^1(\Omega)} = \inf_{v^h - u^h \in S^0_{1,0}(T^h)} \|v^h\|_{H^1(\Omega)} \leq C \|u^h\|_{H^{1/2}(\partial \Omega)} ,$$

for some constant $C > 0$ independent of $h$.

Proof Fix a generalized discrete harmonic function $u^h \in S^0_1(T^h)$ and let $v^h \in S^0_1(T^h)$ be such that $v^h - u^h \in S^0_{1,0}(T^h)$. Then

$$\langle u^h, v^h - u^h \rangle_{H^1(\Omega)} = 0.$$ 

Therefore,

$$\|u^h\|^2_{H^1(\Omega)} \leq \|u^h\|^2_{H^1(\Omega)} + \|v^h - u^h\|^2_{H^1(\Omega)}$$

$$= \|u^h\|^2_{H^1(\Omega)} + \langle u^h, v^h - u^h \rangle_{H^1(\Omega)} - \langle u^h, v^h - u^h \rangle_{H^1(\Omega)}$$

$$= \|u^h\|^2_{H^1(\Omega)} + \langle u^h, v^h - u^h \rangle_{H^1(\Omega)}$$

$$= \|v^h\|^2_{H^1(\Omega)} + \langle u^h - v^h, u^h \rangle_{H^1(\Omega)}$$

$$= \|v^h\|^2_{H^1(\Omega)} .$$

Hence

$$\|u^h\|_{H^1(\Omega)} \leq \inf_{v^h - u^h \in S^0_{1,0}(T^h)} \|v^h\|_{H^1(\Omega)} .$$

For equality, notice that $u^h - u^h = 0 \in S^0_{1,0}(T^h)$. To prove the last inequality we proceed as follows. Let $U$ be the solution to the Dirichlet boundary value problem

$$\begin{cases}
-\Delta U = 0, & \text{on } \Omega, \\
U = u^h, & \text{on } \partial \Omega.
\end{cases}$$

By Theorem 2.26,

$$\|U\|_{H^1(\Omega)} \lesssim \|u^h\|_{H^{1/2}(\partial \Omega)} .$$

Let $\Pi_h$ be the Scott-Zhang quasi-interpolant. Then for any $v^h \in S^0_1(T^h)$ such that $v^h - u^h \in S^0_{1,0}(T^h)$, it holds that

$$\|u^h\|_{H^1(\Omega)} \lesssim \|\Pi_h U\|_{H^1(\Omega)} \lesssim \|U\|_{H^1(\Omega)} \lesssim \|u^h\|_{H^{1/2}(\partial \Omega)} ,$$

by Lemma 4.2. □
4. Inverse Inequalities

4.1.3 Discrete $L^2$-Norms

Let $\Omega \subseteq \mathbb{R}^n$ be a polyhedral domain equipped with a quasi-uniform family of meshes $\{T^h\}$. As in the previous section we denote by $\mathcal{S}_0^1(T^h)$ the piecewise linear and continuous finite element functions on $T^h$. First we state a well-known result for discrete $L^2$-norms.

**Lemma 4.6 (Discrete $L^2$-norm)** (Section 4.2.3 in [23]) For any $v^h \in \mathcal{S}_0^1(T^h)$ it holds that

$$\|v^h\|_{L^2(\Omega)}^2 \sim h^n \sum_{n \in N(T^h)} v^h(n)^2,$$

where the implicit constants are independent of $h$ and only depend on the quasi-uniformity and shape regularity of the mesh.

In the case that we have a function $u \in \mathcal{S}_0^1(T^h)$ that is nonzero on only a face or a union of edges of $\partial \Omega$. Then we can get a more precise equivalence than the one in the above lemma.

**Definition 4.7 (Local $L^2$-inner product/norm)** (Section 4.2.3 in [23]) Let $K \subseteq \Omega$ be $\Omega$ itself, the entire boundary $\partial \Omega$, a face $F$ of $\partial \Omega$ or a union of edges of $\partial \Omega$. Then we define the local $L^2$-inner product with respect to $K$ as

$$\langle u^h, v^h \rangle_{h,K} := h^\alpha \sum_{n \in K \cap N(T^h)} u^h(n)v^h(n), \quad \forall u^h, v^h \in \mathcal{S}_0^1(T^h|_K),$$

where $\alpha := \dim K$. The corresponding norm is defined as

$$\|u^h\|_{h,K}^2 := h^\alpha \sum_{n \in K \cap N(T^h)} u^h(n)^2, \quad \forall u^h \in \mathcal{S}_0^1(T^h|_K).$$

It turns out that for functions in $\mathcal{S}_0^1(T^h|_K)$, this local norm is equivalent to the $L^2$-norm on $K$ as long as the mesh is quasi-uniform.

**Lemma 4.8 ((Local) norm equivalence)** (Section 4.2.3 in [23]) Let $K \subseteq \Omega$ be $\Omega$ itself, the entire boundary $\partial \Omega$, a face $F$ of $\partial \Omega$ or a union of edges of $\partial \Omega$. Then, for any $u^h \in \mathcal{S}_0^1(T^h|_K)$

$$\|u^h\|_{h,K} \approx \|u^h\|_{L^2(K)}.$$
4.2 Well-known Inverse Inequalities

Here we will state some well-known inverse inequalities. We first state and prove a standard local inverse inequality, followed by a compactness result and a global inverse inequality. We end the section by proving an $L^\infty - H^1$ inverse inequality which will be essential for our proof of the edge lemma. By local inverse inequality we mean that we only look at one fixed element and not at the whole mesh. This section is heavily inspired by [3] and most of the notation and proofs are a carbon copy of chapter 4 in [3]. However, the proofs of the inverse inequalities in this section are given in full detail without omitting any steps.

The way the first inverse inequality is proved is by using norm equivalence on finite dimensional normed spaces. However, this constant may depend on the mesh width. To remedy this, we first normalize everything such that the element has diameter 1 and then use norm equivalence. Hereafter we rescale everything back which gives an explicit dependence on the mesh width instead of an unknown one in the constant obtained from norm equivalence. We first introduce the notation that will be used for this rescaling.

**Definition 4.9 (Normalization of a bounded set) (Section 4.5 in [3])** Let $K \subseteq \mathbb{R}^n$ be a bounded set. We define $\hat{K} \subseteq \mathbb{R}^n$ to be the set

$$\hat{K} := \left\{ \frac{1}{\text{diam}(K)} x \mid x \in K \right\}.$$

**Definition 4.10 (Section 4.5 in [3])** Let $K \subseteq \mathbb{R}^n$ be a bounded domain and $v : K \to \mathbb{R}$ a function. We define the function $\hat{v} : \hat{K} \to \mathbb{R}$ as

$$\hat{v}(\hat{x}) := v((\text{diam}(K)) \hat{x}), \quad \forall \hat{x} \in \hat{K}.$$  

If $\mathcal{P}$ is a vector space of functions on $K$, then we define $\hat{\mathcal{P}}$ as

$$\hat{\mathcal{P}} := \{ \hat{v} \mid v \in \mathcal{P} \}.$$

Now we prove the first inverse inequality which, as we stated above, is a local inverse inequality.

**Lemma 4.11 (Lemma 4.5.3 in [3])** Let $K \subseteq \mathbb{R}^n$ be a bounded domain, $0 < \rho < 1$ be such that $0 < \rho h \leq \text{diam}(K) \leq h$, where $0 < h \leq 1$ and $\mathcal{P}$ a finite dimensional subspace of $W^{\ell,p}(K) \cap W^{m,q}(K)$, where $1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $0 \leq m \leq \ell$. Then there exists a constant $C = C(\mathcal{P}, K, \ell, p, q, \rho, n)$ such that for all $v \in \mathcal{P}$, we have

$$\|v\|_{W^{\ell,p}(K)} \leq C h^{m-\ell+n/p-n/q} \|v\|_{W^{m,q}(K)}.$$
4. Inverse Inequalities

**Proof** First we will prove the lemma for the case $m = 0$. Let $\mathcal{P}$ be a finite dimensional space as in the lemma. By norm equivalence on finite dimensional normed spaces we get
\[
\|\hat{v}\|_{W^\ell,p(\hat{K})} \lesssim \|\hat{v}\|_{L^q(\hat{K})}, \quad \forall v \in \mathcal{P}.
\]
Notice that the implicit constant is independent of $h$ because the domain now has diameter 1. Next we rescale everything back to get an explicit $h$-dependence. By the chain rule we have for any $0 \leq j \leq \ell$,
\[
\partial^\alpha \hat{v} = (\text{diam } K)^j \partial^\alpha v,
\]
for any $\alpha \in \mathbb{N}^j$ such that $|\alpha| = j$ and $v \in W^{j,p}(K)$. Using the change of variables from $\hat{K}$ to $K$ we obtain
\[
\int_K |\partial^\alpha \hat{v}(\hat{x})|^p d\hat{x} = (\text{diam } K)^j \int_K |\partial^\alpha v(x)|^p dx.
\]
This shows that $v \in W^{j,p}(K)$ if and only if $\hat{v} \in W^{j,p}(\hat{K})$ and that
\[
|v|_{W^{j,p}(K)} = (\text{diam } K)^j |v|_{W^{j,p}(\hat{K})}.
\]
Applying this to $\mathcal{P}$, we get
\[
|v|_{W^{j,p}(K)} (\text{diam } K)^j \lesssim \|v\|_{L^q(K)} (\text{diam } K)^{-n/q}, \quad \forall v \in \mathcal{P},
\]
for $0 \leq j \leq \ell$. By assumption $\rho h \leq \text{diam } K \leq h$, which gives
\[
|v|_{W^{j,p}(K)} \lesssim h^{j-l+n/p-n/q} \|v\|_{L^q(K)}, \quad \forall v \in \mathcal{P},
\]
for $0 \leq j \leq \ell$. Since $h \leq 1$,
\[
h^{-i} \leq h^{-j}, \quad \forall i = 0, \ldots, j - 1.
\]
Therefore,
\[
\|v\|_{W^{j,p}(K)} \lesssim h^{j-l+n/p-n/q} \|v\|_{L^q(K)}, \quad \forall v \in \mathcal{P}, \tag{4.3}
\]
for $0 \leq j \leq \ell$. If we take $j = \ell$, then we have proven the lemma for the case $m = 0$.

For the general case where $m \leq \ell$, we argue as follows. For $\ell - m \leq k \leq \ell$ and $|\alpha| = k$, we can write
\[
\partial^\alpha v = \partial^\beta \partial^\gamma v, \quad \forall v \in \mathcal{P},
\]
for $|\beta| = \ell - m$ and $|\gamma| = k + m - \ell$. Using this and (4.3) for $\partial^\gamma \mathcal{P}$, we obtain
\[
\|\partial^\alpha v\|_{L^p(K)} \leq \|\partial^\gamma v\|_{W^{\ell-m,p}(K)} \lesssim h^{-(\ell-m)+n/p-n/q} \|\partial^\gamma v\|_{L^q(K)}
\]
4.2. Well-known Inverse Inequalities

\[ |v|_{W^{k,p}(K)} \lesssim h^{-(\ell-m)+n/p-n/q}|v|_{W^{m,q}(K)}, \quad \forall v \in \mathcal{P}, \]

for all \( v \in \mathcal{P} \). Since \( |\alpha| = k \) was arbitrary, we get

\[ |v|_{W^{k,p}(K)} \lesssim h^{-(\ell-m)+n/p-n/q}|v|_{W^{m,q}(K)}, \quad \forall v \in \mathcal{P}, \]

for any \( k \) such that \( \ell - m \leq k \leq \ell \). In particular, this implies that

\[ |v|_{W^{k,p}(K)} \lesssim h^{-(\ell-m)+n/p-n/q}\|v\|_{W^{m,q}(K)}, \quad \forall v \in \mathcal{P}, \]

for any \( k \) such that \( \ell - m \leq k \leq \ell \), since the latter implies \( k + m - \ell \leq m \). Now, (4.3) for \( j = \ell - m \) is

\[ \|v\|_{W^{\ell-m,p}(K)} \lesssim h^{\ell-m+n/p-n/q}\|v\|_{W^{m,q}(K)}, \quad \forall v \in \mathcal{P}, \]

Using these last two inequalities, we obtain

\[ \|v\|_{W^{\ell,p}(K)}^p = \|v\|_{W^{\ell-m,p}(K)}^p + \sum_{k=\ell-m}^{\ell} \|v\|_{W^{k,p}(K)}^p \lesssim h^{p-(\ell-m)+n/p-n/q}\|v\|_{W^{m,q}(K)}, \]

for any \( v \in \mathcal{P} \) which proves the lemma. \( \square \)

Next we state two intermediate results which we will need to prove the global inverse inequality.

**Lemma 4.12** (Proof of Theorem 4.4.20 in [3]) Let \( \{T_h\} \) be a quasi-uniform family of meshes of a polyhedral domain \( \Omega \subseteq \mathbb{R}^n \) and assume that all finite elements are affine equivalent. Let \( (K, \mathcal{P}, \mathcal{N}) \) be a reference finite element. Then

\[ \mathcal{A} := \{ A_T \in GL_n(\mathbb{R}) \mid T \in \mathcal{T}_h, 0 < h \leq 1 \} \]

is contained in a compact subset of \( GL(\mathbb{R}^n) \), where \( A_T \) is the matrix of the affine transformation \( A_T \) which transforms \( K \) to \( \hat{T} \).

**Proof** For this fix an \( A_T \in \mathcal{A} \) and let \( (K, \mathcal{P}, \mathcal{N}) \) be a reference element. Since the mesh is quasi-uniform and a fortiori shape regular, there exists a ball \( B \subseteq \hat{T} \) such that \( \text{diam } B \geq \rho \text{ diam } \hat{T} = \rho > 0 \),

where \( \rho \) is independent of \( h \). It follows that

\[ \text{vol}_n(B) \leq \text{vol}_n(\hat{T}) = |\det A_T| \int_K dx \leq |\det A_T|\text{vol}_n(K). \]

On the other hand we have

\[ \text{vol}_n(B) \geq C_n \rho^n \]
for some constant $C_n$ only depending on $n$. Therefore,

$$0 < C_n \rho^n \leq |\det A_T| \text{vol}_n(K).$$

Rewriting this gives

$$|\det A_T| \geq \varepsilon(\rho, K, n) > 0.$$ 

This immediately implies that

$$A_T \in \{ B \in GL(\mathbb{R}^n) \mid |\det B| \geq \varepsilon \},$$

which is a closed a set. Since $K$ is bounded, there is a choice of coordinates such that

$$\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i \leq t_0, x_j > 0, \text{ for } j = 1, \ldots, n \} \subseteq K,$$

where $t_0$ depends only on $K$. Denote by $e^i$ the $i$-th standard unit vector. Then

$$A_T(te^i) = tA_Te^i + b_T \in \hat{T}, \quad \forall t \in [0, t_0],$$

where $b_T$ is the translation part of the affine map which maps $K$ to $\hat{T}$. Therefore,

$$\|A_Te^i\| \leq \text{diam } \hat{T}/t_0 = 1/t_0, \quad \forall i = 1, \ldots, n,$$

which immediately implies

$$|(A_T)_{ij}| \leq 1/t_0, \quad \forall i, j = 1, \ldots, n.$$ 

We used $\|\cdot\|$ here to denote the Euclidean norm on $\mathbb{R}^{n \times n}$. This shows that

$$A_T \in \{ B \in GL_n(\mathbb{R}) \mid |\det B| \geq \varepsilon, |B_{ij}| \leq 1/t_0, \forall i, j = 1, \ldots, n\}.$$ 

Denote this set by $B$. We show that $B$ is closed with respect to $\|\cdot\|$. The choice of norm doesn’t matter because of norm equivalence on finite dimensional normed vector spaces. Next note that $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is continuous with respect to $\|\cdot\|$. Let $(A_n)_n$ be a sequence in $B$ converging to some $A \in GL_n(\mathbb{R})$. By continuity of $\det$ and basic properties of inequalities, we have

$$\det A \geq \varepsilon, |A| \leq 1/t_0, \quad \forall i, j = 1, \ldots, n.$$ 

Therefore $A \in B$ which shows that $B$ is closed in $GL_n(\mathbb{R})$. By definition of $B$ it is bounded with respect to $\|\cdot\|$. Therefore $B$ is a compact subset of $GL_n(\mathbb{R})$. □

**Lemma 4.13 ($\ell^q(\mathbb{N}) \subseteq \ell^p(\mathbb{N})$)** Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of positive numbers and $1 \leq q \leq p$ be integers. Then

$$\|a\|_{\ell^p(\mathbb{N})} := \left( \sum_{n=1}^\infty a_n^p \right)^{1/p} \leq \left( \sum_{n=1}^\infty a_n^q \right)^{1/q} =: \|a\|_{\ell^q(\mathbb{N})}.$$
4.2. Well-known Inverse Inequalities

**Proof** We start by proving that for any $0 < \alpha < 1$,
\[
\left( \sum_{n=1}^{\infty} a_n^{\alpha} \right) \leq \sum_{n=1}^{\infty} a_n^{\alpha}.
\]
(4.4)

To show this, it is sufficient to show for any $m \in \mathbb{N}$,
\[
\left( \sum_{n=1}^{m} a_n^{\alpha} \right) \leq \sum_{n=1}^{m} a_n^{\alpha}.
\]

This follows by induction if
\[
(x + y)^{\alpha} \leq x^{\alpha} + y^{\alpha}, \quad \forall x, y \geq 0,
\]
which in turn follows if
\[
(1 + t)^{\alpha} \leq 1 + t^{\alpha}, \quad \forall t > 0.
\]
To show this last property, we calculate the derivative of
\[
\beta(t) := 1 + t^{\alpha} - (1 + t)^{\alpha-1}
\]
for $t > 0$. This gives
\[
\beta'(t) = \alpha(t^{\alpha-1} - (1 + t)^{-1}) \geq 0.
\]
Therefore, we have
\[
\beta(t) \geq \beta(0) = 0, \quad \forall t \geq 0,
\]
which proves (4.4). Using this, we get
\[
\left( \sum_{n=1}^{\infty} a_n^{p} \right)^{1/p} \leq \left( \sum_{n=1}^{\infty} a_n^{q} \right)^{q/(pq)} \leq \left( \sum_{n=1}^{\infty} a_n^{pq/p} \right)^{q} = \left( \sum_{n=1}^{\infty} a_n^{q} \right)^{q},
\]
because $q/p \leq 1$. \hfill \Box

We continue by proving a global version of 4.11. The crucial assumption here is that all finite elements are affine equivalent and that the family of meshes is quasi-uniform.

**Theorem 4.14** (Theorem 4.5.11 in [3]) Let $\{T^h\}$ be a quasi-uniform family of subdivisions of a polyhedral domain $\Omega \subseteq \mathbb{R}^n$ and assume that all elements are affine equivalent. Let $(K, \mathcal{P}, \mathcal{N})$ be a reference finite element such that
\[
\mathcal{P} \subseteq W^{k,p}(K) \cap W^{m,q}(K),
\]
where $1 \leq p \leq \infty, 1 \leq q \leq \infty$ and $0 \leq m \leq k$. For $T \in T^h$, let $(T, \mathcal{P}_T, \mathcal{N}_T)$ be the corresponding element affine-equivalent to $(K, \mathcal{P}, \mathcal{N})$ and
\[
V^h(\Omega) := \{ v \mid v \text{ is measurable and } v|_T \in \mathcal{P}_T, \forall T \in T^h \}.\]
4. INVERSE INEQUALITIES

Then there exists a constant $C = C(\ell, p, q, \rho, n)$ such that

$$
\left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{\ell,p}(T)}^p \right)^{1/p} \leq C h^{m-\ell+\min(0, n/p-n/q)} \left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{m,q}(T)}^q \right)^{1/q},
$$

for all $v \in V^h(\Omega)$. When $p = \infty$, the left-hand side should be interpreted as

$$
\max_{T \in \mathcal{T}^h} \|v\|_{W^{\ell,\infty}(T)}.
$$

The same convention is used for the right hand side.

**Proof** The quasi-uniformity of the mesh allows us to use Lemma 5.2 for all elements in $\mathcal{T}^h$ simultaneously. Therefore, we have

$$
\|v\|_{W^{\ell,p}(T)} \leq C(\hat{P}_T, \hat{T}, \ell, p, q, \rho) h^{m-\ell+n/p-n/q} \|v\|_{W^{m,q}(T)},
$$

for all $T \in \mathcal{T}^h$ and $v \in P_T$. Due to the fact that $\hat{T}$ continuously depends on $A_T$, we obtain

$$
C(\hat{P}_T, \hat{T}, \ell, p, q, \rho) \leq \zeta(A_T) C(\ell, p, q, \rho),
$$

where $\zeta$ is a positive function which depends continuously on $A_T \in GL(\mathbb{R}^n)$. From Lemma 4.12, we know that the set

$$
\{ A_T \in GL(\mathbb{R}^n) \mid T \in \mathcal{T}^h, 0 < h \leq 1 \}
$$

is contained in a compact subset of $GL(\mathbb{R}^n)$. Combining the above two inequalities with this result we obtain

$$
\|v\|_{W^{\ell,p}(T)} \leq C(\ell, p, q, \rho) h^{m-\ell+n/p-n/q} \|v\|_{W^{m,q}(T)},
$$

for all $T \in \mathcal{T}^h$ and $v \in P_T$. This proves the theorem for the case $p = \infty$. Therefore, assume that $p < \infty$. From now on, we will use $C$ as a generic constant which only depends on $\ell, p, q$ and $\rho$. By using the previous inequality and summing over all elements in $\mathcal{T}^h$, we get

$$
\left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{\ell,p}(T)}^p \right)^{1/p} \leq h^{m-\ell+n/p-n/q} \left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{m,q}(T)}^q \right)^{1/q},
$$

for all $v \in V^h(\Omega)$. Now, if $p \geq q$, then Lemma 4.13 implies that

$$
\left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{m,q}(T)}^p \right)^{1/p} \leq \left( \sum_{T \in \mathcal{T}^h} \|v\|_{W^{m,q}(T)}^q \right)^{1/q},
$$

for all $v \in V^h(\Omega)$. When $p = \infty$, the left-hand side should be interpreted as

$$
\max_{T \in \mathcal{T}^h} \|v\|_{W^{m,\infty}(T)}.
$$

The same convention is used for the right hand side.
for all \( v \in V^h(\Omega) \) which proves the theorem for this case. if \( p \leq q \), then the Hölder inequality implies

\[
\left( \sum_{T \in T^h} \| v \|^p_{W^{m,p}(T)} \right)^{1/p} \leq \left( \sum_{T \in T^h} 1 \right)^{1/p-1/q} \left( \sum_{T \in T^h} \| v \|^q_{W^{m,q}(T)} \right)^{1/q},
\]

for all \( v \in V^h(\Omega) \). By the shape regularity of the mesh we obtain

\[
\text{vol}_n(\Omega) = \sum_{T \in T^h} \text{vol}_n(T) \geq \sum_{T \in T^h} \text{vol}_n(B_T) \geq C \sum_{T \in T^h} (\text{diam } B_T)^n \geq C h^n \sum_{T \in T^h} 1,
\]

where the constant \( C \) only depends on \( n \) and the shape regularity of the mesh. Therefore, we obtain

\[
\sum_{T \in T^h} 1 \lesssim h^{-n}.
\]

Using this, we obtain

\[
\left( \sum_{T \in T^h} \| v \|^p_{W^{\ell,p}(T)} \right)^{1/p} \lesssim h^{m-\ell} \left( \sum_{T \in T^h} \| v \|^q_{W^{m,q}(T)} \right)^{1/q},
\]

for all \( v \in V^h(\Omega) \). which proves the theorem for this case. \( \square \)

**Remark.** In the case that we are working with \( W^{\ell,p} \cap W^{m,q} \)-conforming finite elements, we can replace the sum by global norms. The inequality in the above theorem then becomes

\[
\| v \|_{W^{\ell,p}(\Omega)} \lesssim h^{m-\ell+\min(0,n/p-n/q)} \| v \|_{W^{m,q}(\Omega)}.
\]

We end this section with a discrete Sobolev inequality which was the inspiration for the proof of an inverse inequality which is used in the proof of the edge lemma. The following lemma holds for higher order conforming Lagrangian finite elements, but we will not need this more general case. However, the proof of this more general result is exactly the same. First we define what a centroid of a measurable subset of \( \mathbb{R}^n \) is which will be an essential concept in the proof of this lemma.

**Definition 4.15 (Centroid)** Let \( K \subseteq \mathbb{R}^n \) be a measurable subset for some \( n \in \mathbb{N} \). The centroid \( c_K \) of \( K \) is defined as the point

\[
c_K := \frac{1}{\text{vol}_n(K)} \int_K x \, dx.
\]

**Remark.** If \( K \) is convex, then \( c_K \in \overline{K} \).
Lemma 4.16 (Discrete Sobolev Inequality) (Lemma 4.9.2 in [3]) Let \( \Omega \subseteq \mathbb{R}^2 \) be a polyhedral domain and \( \{ T^h \} \) a family of shape regular triangular meshes such that
\[
|\log h_T| \asymp |\log h| \quad \forall T \in T^h,
\]
where the implicit constants are independent of \( h \). Then
\[
\|v^h\|_{L^\infty(\Omega)} \leq C(1 + |\log h|)^{1/2}\|v^h\|_{H^1(\Omega)},
\]
for all \( v^h \in \mathcal{S}_0^1(T^h) \). The constant \( C > 0 \) is independent of \( h \).

Proof Since \( \Omega \) is polyhedral, it has the cone property, i.e. for any point \( x \in \Omega \) there is a cone \( K_x \) with vertex \( x \) congruent to the cone \( K \) defined (in polar coordinates) by
\[
K := \{(r,\theta) \mid 0 < r < d < +\infty, 0 < \theta < \omega < 2\pi\}.
\]

It is crucial here that \( d \) is independent of \( h \), this will become clear later on. For polar coordinates we use the convention \((r,\theta) \in [0, +\infty) \times (-\pi, \pi]\). Let \( T \) be a triangle of \( T^h \) and \( c \) the centroid of \( T \). We consider the two cases \( h_T \geq d/2, h_T < d/2 \) separately.

Case 1 (\( h_T \geq d/2 \)): For this case we have
\[
\|v^h\|_{L^\infty(T)} \leq \|v^h\|_{W^{1,\infty}(T)} \lesssim h_T^{-1}\|v^h\|_{H^1(T)},
\]
by Lemma 5.2. Therefore, by our assumption on \( h_T \), we get
\[
\|v^h\|_{L^\infty(T)} \lesssim d^{-1}\|v^h\|_{H^1(T)} \lesssim \|v^h\|_{H^1(\Omega)},
\]
Since \( 1 + |\log h| \to +\infty \) as \( h \to 0^+ \), we get that
\[
\|v^h\|_{L^\infty(T)} \lesssim (1 + |\log h|)^{1/2}\|v^h\|_{H^1(\Omega)}.
\]

Case 2 (\( h_T < d/2 \)): After translating everything by \( c \) and rotating (if needed), we may assume that \( c = 0 \) and that \( K_c = K \). The shape-regularity of \( T^h \) implies that there is an \( 0 < \eta < 1 \) independent of \( T \) such that the cone
\[
K_\eta := \{(r,\theta) \mid 0 < r < \eta h_T, 0 < \theta < \omega\}
\]
is contained in \( T \).

Now let \( v^h \in \mathcal{S}_0^1(T^h) \) and set \( \alpha := v^h(c) \). From the fundamental theorem of calculus, we get
\[
\alpha = v^h(r,\theta) - \int_0^r \frac{\partial v^h}{\partial \rho}(\rho,\theta) \, d\rho, \quad \text{for } \frac{d}{2} < r < d,
\]
4.2. Well-known Inverse Inequalities

Figure 4.2: Geometric construction in the proof of Lemma 4.16 for the case where \( h_T < d/2 \)

and hence

\[
\alpha^2 \leq 2v^h(r, \theta)^2 + 2 \left( \int_0^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho \right)^2, \quad \text{for } \frac{d}{2} < r < d,
\]

by using that \((a - b)^2 \leq 2a^2 + 2b^2\) for any \(a, b \in \mathbb{R}\). Next we estimate the term with the integral. Specifically, using the Cauchy-Schwarz inequality we find

\[
\int_0^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho = \int_0^{\eta h_T} \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho + \int_{\eta h_T}^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho
\]

\[
\leq \eta h_T |v|_{W^{1, \infty}(T)} + \left( \int_{\eta h_T}^r \left( \frac{\partial v}{\partial \rho}(\rho, \theta) \right)^2 \rho \, d\rho \right)^{1/2} \sqrt{\log \frac{d}{\eta h_T}}.
\]

Notice that we used the Cauchy-Schwarz inequality for the functions \( \rho \mapsto \sqrt{\rho} \frac{\partial v}{\partial \rho}(\rho, \theta) \) and \( \rho \mapsto 1/\sqrt{\rho} \). Combining the above two inequalities, we obtain

\[
\alpha^2 \int_0^\omega \int_{d/2}^d r \, dr \, d\theta \leq 2 \int_0^\omega \int_{d/2}^d v^h(r, \theta)^2 r \, dr \, d\theta + 4(\eta h_T)^2 |v^h|_{W^{1, \infty}(T)} \int_0^\omega \int_{d/2}^d r \, dr \, d\theta
\]

\[
+ \log(d/(\eta h_T)) \int_{d/2}^d \int_0^\omega \int_{\eta h_T}^r \partial_\rho v^h(\rho, \theta)^2 \rho \, d\rho \, d\theta \, dr. \quad (4.5)
\]

It is here that it is essential that \( d \) is independent of \( h \). Indeed, on the left-hand side we have the factor

\[
\int_0^\omega \int_{d/2}^d r \, dr \, d\theta =: \text{vol}_2(\tilde{K}).
\]
4. INVERSE INEQUALITIES

If this would somehow depend on \( h \), then we would not be able to just divide both sides of the inequality by \( \text{vol}_2(K) \) and absorb it in a constant independent of \( h \). Now,

\[
|v^h|_{W^{1,\infty}(T)} \lesssim h^{-1}|v^h|_{H^1(T)} \leq h^{-1}|v^h|_{H^1(\Omega)} \leq \|v^h\|_{H^1(\Omega)}.
\]

Here the implicit constant only depends on \( T \) and the shape regularity of the mesh. Indeed, we have

\[
|v^h|_{W^{1,\infty}(T)} = |a_T|, \quad |v^h|_{H^1(T)} = |a_T|\text{vol}_2(T) \sim |a_T|h,
\]

where \( a_T \in \mathbb{R}^2 \) is the linear part of \( v \) on \( T \). Moreover, by switching from polar to Cartesian coordinates we obtain

\[
\int_0^\omega \int_{d/2}^d v^h(r,\theta)^2 r \, dr \, d\theta \leq \int_{K_r} v^h(x)^2 \, dx \leq \|v^h\|_{L^2(K)}^2 \leq \|v^h\|_{H^1(\Omega)}^2.
\]

Last, by again switching from polar to Cartesian coordinates, we straightforwardly get

\[
\int_{d/2}^d \int_0^\omega \partial_\rho v^h(\rho,\theta)^2 \rho \, d\rho \, d\theta \, r \, dr \leq \int_{d/2}^d \int_{K_r} |\nabla v^h(x)|^2 \, dx \, r \, dr \lesssim \|v^h\|_{H^1(\Omega)}^2,
\]

since \( d \) is independent of \( h \). The implicit constant is just \((2\pi)^{-1}\) and hence independent of \( h \). Here we used the fact that for any domain \( X \subseteq \mathbb{R}^2 \) and any \( u \in H^1(X) \), the gradient of \( u \) in polar coordinates is given by

\[
\nabla u := (\partial_\rho u, 1/\rho \partial_\theta u)^T.
\]

The set \( K_r \) used in the above estimate is defined as

\[
K_r := \{ (\rho,\theta) \mid \eta h_T < \rho < r, 0 < \theta < \omega \}.
\]

Applying the above three estimates to (4.5) gives

\[
|v^h(x)| \lesssim (1 + |\log h_T|)^{1/2}\|v^h\|_{H^1(\Omega)}.
\]

Since \( v^h \) is linear on \( T \), we get by the fundamental theorem of calculus

\[
|v^h(x) - v^h(c)| \leq h_T|v^h|_{W^{1,\infty}(T)}
\]

for any \( x \in T \). Applying Lemma 4.11 to the right-hand side for \( \ell = 1, p = \infty, m = 1 \) and \( q = 2 \), we obtain

\[
|v^h(x) - v^h(c)| \lesssim |v^h|_{H^1(T)}.
\]

Therefore we get

\[
|v^h(x)| \leq |v^h(c)| + |v^h(x) - v^h(c)| \lesssim (1 + |\log h_T|)^{1/2}\|v^h\|_{H^1(\Omega)}.
\]
4.2. Well-known Inverse Inequalities

for any $x \in T$. Hence, we have proven for any $T \in \mathcal{T}^h$, that

$$\|v^h\|_{L^\infty(T)} \lesssim (1 + |\log h_T|)^{1/2} \|v^h\|_{H^1(\Omega)}.$$ Using that $|\log h_T| \sim |\log h|$ and taking the maximum over $T \in \mathcal{T}^h$ on the left-hand side gives the lemma. □

**Remark.** We don’t need the assumption $|\log h_T| = |\log h|$ if we assume that the mesh is quasi-uniform. Indeed, by definition of $h$ and $h_T$ we have $h_T \leq h$. By quasi-uniformity of the mesh there exists a $\rho > 0$ such that

$$\text{diam } B_T \geq \rho h, \quad \forall T \in \mathcal{T}^h.$$ Hence

$$h \lesssim h_T, \quad \forall T \in \mathcal{T}^h.$$ Thus we can conclude that

$$h \sim h_T, \quad \forall T \in \mathcal{T}^h,$$ which shows that the assumption on the diameters of the elements is satisfied if the mesh is quasi-uniform.

The next result is an analogue of Lemma 4.16 but for a domain in $\mathbb{R}^3$.

**Lemma 4.17** Let $\Omega \subseteq \mathbb{R}^3$ be a polyhedral domain and $\{\mathcal{T}^h\}$ a quasi-uniform family of tetrahedral meshes, where $h_T := \text{diam } T$ and $h := \max_{T \in \mathcal{T}^h} h_T$. Then

$$\|v^h\|_{L^\infty(\Omega)} \leq C h^{-1/2} \|v^h\|_{H^1(\Omega)},$$ for any $v^h \in S_0^1(\mathcal{T}^h)$. Here the constant $C > 0$ is independent of $h$.

**Proof** Fix a $v^h \in S_0^1(\mathcal{T}^h)$. The only thing different in this proof compared to the proof of Lemma 4.16, are the local inverse inequalities and the fact that we need to use spherical instead of polar coordinates. We will denote spherical coordinates by $(\rho, \theta, \psi)$. We have the well known relation

$$d\mathbf{x} = \rho^2 \sin \psi \, d\rho d\theta d\psi,$$ where $d\mathbf{x}$, $dr$, $d\theta$ and $d\psi$ are the Lebesgue measures on $\mathbb{R}^3$, $\mathbb{R}^+$, $[0, 2\pi]$ and $[0, \pi]$ respectively. Analogous to (4.5), but using the Cauchy-Schwarz inequality for $\rho$ instead of $\sqrt{\rho}$, gives

$$\left( \int_{\eta h_T}^{r} \partial_\rho v^h(\rho, \theta, \psi) \, d\rho \right)^2 \leq \left( \frac{1}{\eta h_T} - \frac{1}{r} \right) \int_{\eta h_T}^{r} (\partial_\rho v^h)^2 \rho^2 \, d\rho \lesssim h_T^{-2} \int_{\eta h_T}^{r} (\partial_\rho v^h)^2 \rho^2 \, d\rho.$$
Moreover, by Lemma 4.11 for $\ell = 1, p = \infty, m = 1$ and $q = 2$ we have

$$|v^h|_{W^{1,\infty}(T)} \lesssim h^{-3/2} |v^h|_{H^1(T)},$$

for any $v^h \in S_0^0(T^h)$ because the family $\{T^h\}$ is quasi-uniform. Therefore,

$$|v^h(x) - v^h(c)| \leq h_T |v^h|_{W^{1,\infty}(T)} \lesssim h^{-1/2} |v^h|_{H^1(T)}.$$

Again, we used the fact that the quasi-uniformity of the mesh to switch from $h_T$ to $h$. Putting everything together in exactly the same way as in the proof of Lemma 4.16 completes the proof.

Using Lemma 4.16 we can also derive an inverse inequality for finite element functions on the boundary $\partial \Omega$.

**Lemma 4.18 (Discrete Sobolev Inequality for $\partial \Omega$) (Lemma 4.5 in [23])**

Let $\Omega \subseteq \mathbb{R}^2$ be a polyhedral domain equipped with a quasi-uniform family of triangular meshes $\{T^h\}$. Then

$$\|v^h\|_{L^\infty(\partial \Omega)} \leq C(1 + |\log h|)^{1/2} \|v^h\|_{H^{1/2}(\partial \Omega)},$$

for all $v^h \in S_0^0(T^h|_{\partial \Omega})$. The constant $C > 0$ is independent of $h$.

**Proof** We follow the proof as in [23]. Let $\tilde{v}^h$ be the generalized discrete harmonic extension of $v^h$ in $\Omega$. By Lemma 4.16 we then get

$$\|v^h\|_{L^\infty(\partial \Omega)} \lesssim \|\tilde{v}^h\|_{L^\infty(\Omega)} \lesssim (1 + |\log h|)^{1/2} \|\tilde{v}^h\|_{H^1(\Omega)}.$$

By Proposition 4.5 we obtain

$$\|v^h\|_{L^\infty(\partial \Omega)} \lesssim (1 + |\log h|)^{1/2} \|v^h\|_{H^{1/2}(\partial \Omega)} \lesssim (1 + |\log h|)^{1/2} \|v^h\|_{H^{1/2}(\partial \Omega)},$$

which concludes the proof.

We conclude this section by stating an inverse inequality obtained by interpolation.

**Proposition 4.19 (Section 4.2.4 in [23])** Let $\Omega \subseteq \mathbb{R}^3$ be equipped with a quasi-uniform family of tetrahedral meshes $\{T^h\}$. Then for any $v^h \in S_1^0(T^h)$,

$$\|v^h\|_{H^{1/2}(\partial \Omega)} \leq C h^{-1/2} \|v^h\|_{L^2(\partial \Omega)},$$

where $C > 0$ is a constant independent of $h$. 

42
4.3 Geometric Considerations

In this section we prove some results purely about the geometry of the elements in a finite element mesh. These results will play a critical role in switching from two to three dimensions in inequalities.

We will call an \((n-1)\)-dimensional affine subspace of \(\mathbb{R}^n\) a (hyper)plane and will use \(d(\cdot, \cdot)\) to denote the point-plane distance. So, for any point \(p \in \mathbb{R}^n\) and plane \(\alpha \subseteq \mathbb{R}^n\),

\[
d(p, \alpha) := \min_{q \in \alpha} |p - q|.
\]

**Lemma 4.20 (Sphere around Centroid)** Let \(\Omega \subseteq \mathbb{R}^3\) be a polyhedral domain equipped with a shape regular family of tetrahedral meshes \(\{T^h\}\). Let \(P \in T^h\) be a tetrahedron and \(c_P\) its centroid. Then the largest ball \(B\) centered at \(c_P\) and contained in \(P\) satisfies

\[
diam B \sim h.
\]

*The implicit constants depend only on the shape regularity of the mesh and are independent of \(h\) and \(P\).*

**Proof** Fix a \(P \in T^h\) and let \(A, B, C, D\) be the vertices of \(P\). It is obvious that we can choose our coordinate system such that

\[
A = (0, 0, 0), B = (b_1, 0, 0), C = (c_1, c_2, 0), D = (d_1, d_2, d_3).
\]

Then we trivially have

\[
c_P = \frac{1}{4}(b_1 + c_1 + d_1, c_2 + d_2, d_3).
\]

Now let \(F_1\) be the triangle \(ABC\) or equivalently the bottom face of the tetrahedron \(P\) in our chosen coordinate system. Obviously

\[
d(c_P, F_1) = \frac{1}{4}d_3.
\]

Now notice that the largest ball \(B\) centered at \(c_P\) and contained in \(P\) satisfies

\[
diam B = 2 \min_{1 \leq i \leq 4} d(c_P, F_i),
\]

where \(\{F_i\}_{i=1}^4\) are the faces of \(P\). By symmetry it is therefore sufficient to show that

\[
d_3 \sim h.
\]

To see this, notice that by the shape regularity of \(\{T^h\}\),

\[
d_3 \leq h_P \leq h \lesssim \text{diam} B_P \leq d_3,
\]

where \(B_P\) is the largest ball inscribed in \(P\) such that \(P\) is star-shaped with respect to \(B_P\). \(\square\)
4. Inverse Inequalities

Figure 4.3: Illustration of Lemma 4.20, a tetrahedron $P$ and the largest sphere $B$ centered around its centroid $c_P$ with radius drawn in blue

**Lemma 4.21** Let $\{T^h\}$ be a quasi-uniform family of tetrahedral meshes on a polyhedral domain $\Omega \subseteq \mathbb{R}^3$. Fix a $P \in T^h$ and denote by $c_P$ its centroid. Let $T$ be the intersection of $P$ with any plane through $P$. Then it holds that

$$h \text{vol}_2(T) \leq C \text{vol}_3(P),$$

where $C > 0$ depends only on the quasi-uniformity of the mesh and is independent of $h$ and $P$.

**Proof** Fix a tetrahedron $P \in T^h$. First, notice that $T$ has to be a triangle or a quadrilateral if it is not a point. If it is a point, then the statement in the lemma is trivial. In any other case, the intersection of $P$ with a plane can only have a corner on the lines where two faces of $P$ intersect. Therefore, the maximum number of corners that $T$ can have is four because a plane intersecting a line cannot have more than one intersection point if the line does not lie in the plane. Therefore, $T$ has to be a triangle or a quadrilateral (if the intersection is nonempty). By quasi-uniformity we have

$$h \approx h_Q, \quad \forall Q \in T^h.$$

Now let $B_P$ be the largest ball inside $P$ such that $P$ is star-shaped with respect to $B_P$. Now,

$$h \text{vol}_2(T) \leq h h_P^2 \leq h^3.$$

Here we used the fact that the area of a general quadrilateral is bounded by the product of the lengths of its diagonals. Therefore, its area is bounded by $h_P^2$. For a triangle, this upper bound is trivial. Finally by shape regularity we have that

$$\text{vol}_3(P) \gtrsim (\text{diam}B_P)^3 \gtrsim h^3.$$

Putting the above two inequalities together gives

$$h \text{vol}_2(T) \lesssim \text{vol}_3(P).$$
4.4 Inverse inequalities for centroid slices

In this section we prove some inverse inequalities involving the centroids of elements of a tetrahedral mesh. The first one is a variant of Lemma 4.16. It shows that a part of the proof still holds locally without the assumption of shape regularity.

The general setting in which we will work for the results in this section is the following: Fix a $d > 0$ and let $\Omega := (0,d)^3$ be equipped with a quasi-uniform family of tetrahedral meshes $\{T^h\}$ and let $P \in T^h$ be any tetrahedron from the mesh. We will use $\pi_2 : \Omega \to \mathbb{R}$ to denote the orthogonal projection onto the $y$-axis. Define $h_P := \text{diam } P$, denote by $c_P$ the centroid of $P$ and define $\Delta^P$ as the square $\Delta^P := \Omega \cap \Delta_{c_P}$, where $\Delta_{c_P}$ is defined as

$$\Delta_{c_P} := \{(x,y,z) \in \Omega \mid y = \pi_2 c_P\}.$$

Next, let $T(P)$ be any equilateral triangle inside $B\cap\Delta^P$, where $B$ is the largest ball centered at $c_P$ that is contained in $P$ (see Figure 4.5).

It is immediately evident that $c_{T(P)} = c_P$. Define $\Gamma$ as the set defined by

$$\Gamma := (0,d)^2 \times \{0\} \cong (0,d)^2,$$

so $\Gamma$ is the face on the bottom of $\Omega$. We will denote by $T^h|\Gamma$ the induced mesh on $\Gamma$ by $T^h$. Last we define $T^h(\Gamma)$ as the set of tetrahedra in $T^h$ with one of its faces on $\Gamma$.

First we state a result comparing the diameters of some of the above defined domains.
Lemma 4.22 For the setting described above, it holds that
\[ h_{T(P)} = h, \quad \forall P \in \mathcal{T}^h, \]
where the implicit constants only depend on the quasi-uniformity of the mesh.

**Proof** Fix a tetrahedron \( P \in \mathcal{T}^h \). By Lemma 4.20, the largest ball \( B \) centered at \( c_P \) contained in \( P \) satisfies
\[ \text{diam } B = \sim h. \]
Next, notice that \( B \cap \Delta P \) is a disk with diameter \( \text{diam } B \) because \( B \) is centered at \( c_P \). Since this intersection is precisely the circumcircle of \( T(P) \), we have that
\[ h_{T(P)} = \sim \text{diam } (B \cap \Delta P) = \text{diam } B = \sim h. \]
This follows from the fact that the diameter of the circumcircle of \( T(P) \) is \( \sqrt{3} h_{T(P)} \).

Lemma 4.23 For \( d = 1 \) and any \( v^h \in \mathcal{S}_1^0(\mathcal{T}^h) \) we have
\[ |v^h(c_P)| \leq C(1 + |\log h_{T(P)}|)^{1/2} \|v^h\|_{H^1(\Delta c_P)}, \]
where the constant \( C > 0 \) is independent of \( h \) and \( P \).
4.4. Inverse inequalities for centroid slices

**Proof** The proof of this result is quite similar to the proof of Lemma 4.16. We immediately see that \( \Delta_{c_P} \) has the cone property. Therefore, any point \( x \in T(P) \subseteq \Delta_{c_P} \) is contained in a cone congruent to

\[
K := \{(r, \theta) \mid 0 < r < d, 0 < \theta < \omega\} \subseteq \Delta_{c_P},
\]

for some \( d > 0 \) independent of the mesh parameters. After translation we may assume that \( c_P = 0 \). Since \( c_P \in T(P) \) there is also such a cone around \( c_P = 0 \). Let \( 0 < \eta < 1 \) be such that

\[
K_\eta := \{(r, \theta) \mid 0 < r < \eta h_T(P), 0 < \theta < \omega\}
\]

is contained in \( T(P) \).

Notice that \( K_\eta \) is centered at \( c_P \) and that \( \eta \) is independent of \( T(P) \). Indeed, quasi-uniformity guarantees that if we set \( \eta := 1/(4\sqrt{3}) \), then \( B_{\eta h_T(P)}(c_P) \subseteq T(P) \) because the incircle of \( T(P) \) has a diameter equal to \( 1/\sqrt{3} \).

**Case 1** \( (h_T(P) \geq d/2) \): In this case, we apply Lemma 5.2 to \( T(P) \). This is warranted since \( v^h \) is linear on \( T(P) \) because it is linear on \( P \). Therefore we obtain the estimate

\[
\|v^h\|_{L^\infty(T(P))} \leq \|v^h\|_{W^{1,\infty}(T(P))} \lesssim h_T(P)^{\frac{1}{2}} \|v^h\|_{H^1(T(P))}.
\]

The reason why the implicit constant is independent of the mesh parameters is because \( h = h_T(P) \) by Lemma 4.23. Therefore, the \( \rho \) in the implicit constant which is used in Lemma 5.2 is independent of the mesh parameters since we
can take the same $\rho$ as in the definition of a quasi-uniform family of meshes (in this case the $\rho$ for $\{T^h\}$). Given our assumption on the diameter of $T(P)$, we obtain the estimate

$$\|v^h\|_{L^\infty(T(P))} \lesssim \|v^h\|_{H^1(T(P))}.$$  

Now notice that $1 + |\log h| \geq 1$ for all $h$. Since $d$ is independent of the mesh parameters, we have

$$\|v^h\|_{L^\infty(T(P))} \lesssim (1 + |\log h_{T(P)}|)^\frac{1}{2}\|v^h\|_{H^1(T(P))} \lesssim (1 + |\log h_{T(P)}|)^\frac{1}{2}\|v^h\|_{H^1(\Delta e_P)}.$$  

**Case 2** ($h_{T(P)} < d/2$): Now let $v^h \in S^0_1(T^h)$ and set $\alpha := v^h(c_P)$. From the fundamental theorem of calculus, we get

$$\alpha = v^h(r, \theta) - \int_0^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho, \quad \text{for } \frac{d}{2} < r < d,$$

and hence

$$\alpha^2 \leq 2v^h(r, \theta)^2 + 2\left(\int_0^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho\right)^2, \quad \text{for } \frac{d}{2} < r < d,$$

by using that $(a - b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$. Next we estimate the term with the integral. Specifically, using the Cauchy-Schwarz inequality we find

$$\int_0^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho = \int_{\eta h_{T(P)}}^\eta \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho + \int_{\eta h_{T(P)}}^r \frac{\partial v^h}{\partial \rho}(\rho, \theta) \, d\rho$$

$$\leq \eta h_{T(P)} |v^h|_{W^{1,\infty}(T(P))} + \left(\int_{\eta h_{T(P)}}^r \left(\frac{\partial v}{\partial \rho}(\rho, \theta)\right)^2 \rho \, d\rho\right)^{1/2} \sqrt{\log \frac{d}{\eta h_{T(P)}}}.$$

Notice that we used the Cauchy-Schwarz inequality for the functions $\rho \mapsto \sqrt{\rho \frac{\partial v}{\partial \rho}(\rho, \theta)}$ and $\rho \mapsto 1/\sqrt{\rho}$. Combining the above two inequalities, we obtain

$$v^h(c_P)^2 \int_0^\omega \int_{d/2}^d r \, dr \, d\theta \leq 2 \int_0^\omega \int_{d/2}^d v^h(r, \theta)^2 r \, dr \, d\theta + 4(\eta h_{T(P)})^2 |v^h|_{W^{1,\infty}(T(P))} \int_0^\omega \int_{d/2}^d r \, dr \, d\theta$$

$$\log(d/(\eta h_{T(P)})) \int_0^\omega \int_{d/2}^d \int_{\eta h_{T(P)}}^r \partial \rho v^h(\rho, \theta)^2 \rho \, d\rho \, d\theta \, dr.$$

Indeed, this part of the proof is just calculus and uses no assumptions on the mesh or the elements considered. Because the cone property is independent of the chosen mesh, we have that the area of the set

$$\tilde{K} := \{(r, \theta) \mid \frac{d}{2} < r < d, 0 < \theta < \omega\},$$
which is given by
\[ \int_0^\omega \int_0^{d/2} r \, dr \, d\theta, \]
is independent of the mesh parameters. Next we want to change the \( W^{1,\infty} \)-seminorm in the second term of (4.6) into an \( H^1 \)-seminorm. This is done as follows. As stated before, we know that \( v^h \) is linear on \( T(P) \). This implies that \( v^h \) is of the form
\[ v^h(x) = a^T x + b, \quad \forall x \in T(P), \]
for some \( a \in \mathbb{R}^n \) and \( b \in \mathbb{R}^n \). Therefore we get that \( |v^h|_{W^{1,\infty}(T(P))} = |a|, \quad |v^h|_{H^1(T(P))} = |a|\text{vol}_2(T(P))^{1/2}. \)

Since \( T(P) \) is an equilateral triangle, we immediately get that
\[ \text{vol}_2(T(P))^{1/2} \sim h_{T(P)}. \]
This in turn implies that
\[ |v^h|_{W^{1,\infty}(T(P))} \lesssim h^{-1}_{T(P)}|v^h|_{H^1(T(P))}. \]

Hence
\[ |v^h|_{W^{1,\infty}(T(P))} \lesssim h^{-1}_{T(P)}|v^h|_{H^1(T(P))} \lesssim h^{-1}_{T(P)}|v^h|_{H^1(\Delta_{\epsilon_p})} \lesssim h^{-1}_{T(P)}\|v^h\|_{H^1(\Delta_{\epsilon_p})}. \]
Here the implicit constant only depends on the shape regularity of the mesh. Moreover, by switching from polar to cartesian coordinates we obtain
\[ \int_0^\omega \int_0^{d/2} v^h(r, \theta)^2 r \, dr \, d\theta = \int_{\tilde{K}} v^h(x)^2 \, dx \leq \|v^h\|_{L^2(\tilde{K})}^2 \leq \|v^h\|_{H^1(\Delta_{\epsilon_p})}^2. \]

Last, by again switching from polar to cartesian coordinates, we straightforwardly get
\[ \int_{d/2}^d \int_0^\omega \partial_\rho v^h(\rho, \theta)^2 \rho d\rho d\theta r \, dr \leq \int_{d/2}^d \int_{\tilde{K}_r} |\nabla v^h(x)|^2 \, dx \, r \, dr \lesssim \|v^h\|_{H^1(\Delta_{\epsilon_p})}^2, \]
since \( d \) is independent of \( h \). The implicit constant is just \((2\pi)^{-1}\) and hence independent of \( h \). The set \( \tilde{K}_r \) used in the above estimate is defined as
\[ \tilde{K}_r := \{ (\rho, \theta) \mid \eta h_{T(P)} < \rho < r, 0 < \theta < \omega \}. \]

Applying the above three estimates to (4.5) gives
\[ |v^h(c_P)| \lesssim (1 + |\log h_{T(P)}|)^{1/2}\|v^h\|_{H^1(\Delta_{\epsilon_p})}. \]
Lemma 4.24 Let \( \{T^h\} \) be a quasi-uniform family of tetrahedral meshes on \((0, d)^3\). Define \( \Gamma := (0, d)^2 \times \{0\} \) and fix a \( v^h \in S_0^1(T^h|\Gamma) \). Then there exists a sequence of nodes \( \{n_1, \ldots, n_N\} \) on \( \Gamma \) and a partition \( \{\Lambda_1, \ldots, \Lambda_N\} \) of \( T^h(\Gamma) \) such that

- any nonconsecutive nodes (with respect to the index order) are different,
- \( n_k \in N_{\Gamma}(\Lambda_k) \), \( \forall k = 1, \ldots, N \),
- \( \pi_2(n_k) \geq \pi_2(n_{k+1}) \), \( \forall k \geq \ell \),

where \( N_{\Gamma}(\Lambda_k) \) denotes the set of nodes on \( \Gamma \) of the elements in \( \Lambda_k \). Moreover,

\[
\hat{d} \max_{0 \leq x \leq d} v^h(x, y, 0)^2 \, dy \leq 3h \sum_{k=1}^{N} v^h(n_k)^2.
\]

Proof Fix a \( v^h \in S_0^1(T^h|\Gamma) \). Now define

\[
\Lambda_k := \{ P \in T^h(\Gamma) \mid (k-1)h < \pi_2(c_P) \leq kh \},
\]

for \( k = 1, \ldots, N \) where \( N \) is the smallest number such that each \( P \in T^h(\Gamma) \) is contained in some \( \Lambda_k \). As usual we denoted by \( c_P \) the centroid of \( P \). First notice that \( \{\Lambda_k\}_{k=1}^N \) forms a partition of \( T^h(\Gamma) \). For any \( k \in \{1, \ldots, N\} \), we will denote by \( N_{\Gamma}(\Lambda_k) \) the nodes of \( \Lambda_k \) that lie on \( \Gamma \). Now notice that for all \( 0 \leq y \leq h \),

\[
\max_{0 \leq x \leq d} v^2(x, y, 0) \leq \max_{n \in N_{\Gamma}(\Lambda_1 \cup \Lambda_2)} v^2(n),
\]

because a linear function on a triangle attains its maximum at one of its vertices. For \((k-1)h < y \leq kh\),

\[
\max_{0 \leq x \leq d} v^h(x, y, 0)^2 \leq \max_{n \in N_{\Gamma}(\Lambda_k \cup \Lambda_{k+1})} v^h(n)^2, \quad k = 2, \ldots, N - 1.
\]

Last, for \((N-1)h < y \leq d\),

\[
\max_{0 \leq x \leq d} v^h(x, y, 0)^2 \leq \max_{n \in N_{\Gamma}(\Lambda_{N-1} \cup \Lambda_N)} v^h(n)^2.
\]

Now define \( n_k \) for \( k = 1, \ldots, N \) as the node \( n_k \) such that

\[
v^h(n_k)^2 = \max_{n \in N_{\Gamma}(\Lambda_k)} v^h(n)^2.
\]

Next, define \( I_k := ((k-1)h, kh] \) for \( k = 1, \ldots, N - 1 \) and \( I_N := ((N-1)h, d] \). Using this, we get

\[
\int_0^d \max_{0 \leq x \leq d} v^h(x, y, 0)^2 \, dy = \sum_{k=1}^{N} \int_0^d \max_{0 \leq x \leq d} v^h(x, y, 0)^2 1_{I_k}(y) \, dy
\]
4.4. Inverse inequalities for centroid slices

\[ \leq (v^h(n_1)^2 + v^h(n_2))^2 h + (v^h(n_{N-1})^2 + v^h(n_N)^2) h + \sum_{k=2}^{N-1} (v^h(n_{k-1})^2 + v^h(n_k)^2 + v^h(n_{k+1})^2) h \]

\[ \leq 3h \sum_{k=1}^{N} v^h(n_k)^2. \]

We now only need to check that the nodes \( \{n_k\}_{k=1}^{N} \) satisfy the properties in the statement of the Lemma. The first property follows immediately from the construction of the partition \( \{\Lambda_k\}_{k=1}^{N} \). The second property holds because \( n_k \) was chosen from the nodes in \( N_\Gamma(\Lambda_k) \) for any \( k = 1, \ldots, N \). The third property is again immediate from the construction of the partition \( \{\Lambda_k\}_{k=1}^{N} \).

Indeed, for any \( k > \ell \in \{1, \ldots, N\} \) we have

\[ \pi_2(n_k) \geq (k - 1)h \geq \ell h \geq \pi_2(n_\ell). \]

This completes the proof. \( \square \)

![Figure 4.7: Example of the partition \( \{\Lambda_k\}_k \) for \( N = 3 \). The blue lines denote the integer multiples of \( h \).](image)

**Remark.** Notice that \( N \) is independent of the finite element function \( v \in S^0_1(\mathcal{T}^h|\Gamma) \) chosen.

Next we state a similar result but for the gradient of functions in \( S^0_1(\mathcal{T}^h|\Gamma) \).

The proof is very similar to that of the previous lemma. However, there is a possible ambiguity here. Due to the fact that \( S^0_1(\mathcal{T}^h|\Gamma) \) contains piece-wise...
linear functions, evaluating the gradient of such functions at an arbitrary point in \( \Gamma \) is not well-defined. However, as we will see, the gradient is only evaluated at the centroids of certain elements and therefore can just be interpreted as the usual gradient of the restriction of the finite element function to that element (which is that of just a linear function). For the same reason we can’t just take the maximum of the gradient over a horizontal slice. Therefore we will need to be a bit more careful over which set we take the maximum. We first introduce this set in the next definition.

**Definition 4.25** Let \( \{ T^h \} \) be a quasi-uniform family of tetrahedral meshes on \( (0,d)^3 \) and fix a \( \overline{y} \in (0,d) \). Then we define the set \( I(T^h|\Gamma)_{\overline{y}} \) as the disjoint union of line segments defined by

\[
I(T^h|\Gamma)_{\overline{y}} := \bigsqcup_{T \in T^h|\Gamma} \hat{T} \cap \{ y = \overline{y} \}.
\]

![Figure 4.8: Example of the set \( I(T^h|\Gamma)_{\overline{y}} \). The red lines indicate the line segments in \( I(T^h|\Gamma)_{\overline{y}} \) and the blue dots denote the points where a possible ambiguity is for defining the gradient](image)

**Lemma 4.26** Let \( \{ T^h \} \) be a quasi-uniform family of tetrahedral meshes on \( (0,d)^3 \). Define \( \Gamma := (0,d)^2 \times \{ 0 \} \) and fix a \( v^h \in S_0^1(T^h|\Gamma) \). Then there exists a partition \( \{ \Lambda_1, \ldots, \Lambda_N \} \) of \( T^h \) and a set of distinct elements \( T^h_{\partial_1 v} (\Gamma) := \{ P^1, \ldots, P^N \} \) of \( T^h(\Gamma) \) such that \( P^k \in \Lambda_k \), for all \( k = 1, \ldots, N \).

Moreover,

\[
\int_0^d \max_{x \in I(T^h|\Gamma)_{\overline{y}}} |\partial_1 v^h(x, y, 0)|^2 \, dy \leq 3h \sum_{k=1}^N |\partial_1 v^h(c_{P^k})|^2.
\]
4.4. Inverse inequalities for centroid slices

**Proof** We start the proof in the same way as for the previous lemma. Fix a $v^h \in S_1^0(T^h|\Gamma)$. Now define

$$\Lambda_k := \{ P \in T^h(\Gamma) \mid (k-1)h < \pi_2(c_P) \leq kh \},$$

for $k = 1, \ldots, N$ where $N$ is the smallest number such that each $P \in T^h(\Gamma)$ is contained in some $\Lambda_k$. First notice that $\{\Lambda_k\}_{k=1}^N$ forms a partition of $T^h(\Gamma)$. Now notice that for all $0 \leq y \leq h$,

$$\max_{x \in I(T^h|\Gamma)_y} \partial_1 v^h(x, y, 0)^2 \leq \max_{P \in \Lambda_{1} \cup \Lambda_{2}} \partial_1 v^h(c_P)^2.$$

For $(k - 1)h < y \leq kh$,

$$\max_{x \in I(T^h|\Gamma)_y} |\partial_1 v(x, y, 0)|^2 \leq \max_{P \in \Lambda_{k-1} \cup \Lambda_{k+1}} |\partial_1 v(c_P)|^2, \quad k = 2, \ldots, N - 1.$$

Last, for $(N - 1)h < y \leq d$,

$$\max_{x \in I(T^h|\Gamma)_y} |\partial_1 v^h(x, y, 0)|^2 \leq \max_{P \in \Lambda_{N-1} \cup \Lambda_N} |\partial_1 v^h(c_P)|^2.$$

These inequalities trivially follow from the fact that $\partial_1 v^h$ is piece-wise constant on $\Gamma$. Now define $P^k$ for $k = 1, \ldots, N$ as the element in $\Lambda_k$ such that

$$|\partial_1 v^h(c_P)|^2 = \max_{P \in \Lambda_k} |\partial_1 v^h(c_P)|^2.$$

Notice that all these elements are distinct because $\{\Lambda_k\}_{k=1}^N$ forms a partition of $T^h(\Gamma)$. Next, define $I_k := ((k - 1)h, kh]$ for $k = 1, \ldots, N - 1$ and $I_N := ((N - 1)h, d]$. Using this, we get

$$\int_0^d \max_{x \in I(T^h|\Gamma)_y} |\partial_1 v^h(x, y, 0)|^2 \, dy = \sum_{k=1}^N \int_0^d \max_{x \in I(T^h|\Gamma)_y} |\partial_1 v^h(x, y, 0)|^2 I_k(y) \, dy$$

$$\leq (|\partial_1 v^h(c_{P_1})|^2 + |\partial_1 v^h(c_{P_2})|^2)h + (|\partial_1 v^h(c_{P_{N-1}})|^2 + |\partial_1 v^h(c_{P_N})|^2)h$$

$$+ \sum_{k=2}^{N-1} (|\partial_1 v^h(c_{P_{k-1}})|^2 + |\partial_1 v^h(c_{P_k})|^2 + |\partial_1 v^h(c_{P_{k+1}})|^2)h \leq 3h \sum_{k=1}^N |\partial_1 v^h(c_{P_k})|^2.$$  

**Definition 4.27** Let $\{T^h\}$ be a quasi-uniform family of meshes on $\Omega := (0, d)^3$, $\Gamma$ its bottom face and $v^h \in S_1^0(T^h)$ a fixed function. Then we define the set $T_v^h(\Gamma)$ as any fixed subset $\{P^1, \ldots, P^N\}$ of $T^h(\Gamma)$ such that

- $n_k$ is a node of $P^k$,
- $P^k \in \Lambda_k$.

Here $\{\Lambda_k\}_{k=1}^N$ and $\{n_k\}_{k=1}^N$ are as in Lemma 4.24.
Remark. It is very important to notice that the elements in $T^h(\Gamma)$ and $T^h_{\partial_1v}(\Gamma)$ depend on $v$. Therefore we have to be very careful when working with estimates concerning this set of elements. Hence we will always indicate that the constants in the estimates are independent of the choice of $v$ if necessary.

A crucial property of these sets of elements is that they are sufficiently spaced apart from each other. We summarize this in the following Lemma.

**Lemma 4.28** Fix a $d > 0$ and let $\Omega = (0,d)^3$ be equipped with a quasi-uniform mesh $\{T^h\}$ and let $v^h \in S^0(T^h)$ be fixed. Now consider two nonconsecutive elements (with respect to the index order) $P^k, P^\ell \in T^h(\Gamma)$ (or $T^h_{\partial_1v}(\Gamma)$) and the slices $\Delta_{c_{pk}}, \Delta_{c_{p\ell}}$ together with an arbitrary element $P \in T^h$. Then at most one of the following two statements holds:

- $P \cap \Delta_{c_{pk}} \neq \emptyset$,
- $P \cap \Delta_{c_{p\ell}} \neq \emptyset$.

**Proof** Fix two conconsecutive elements $P^k, P^\ell \in T^h(\Gamma)$ and consider the slices $\Delta_{c_{pk}}, \Delta_{c_{p\ell}}$. Without loss of generality we may assume $k \geq \ell + 2$ (because they are nonconsecutive). Assume that

$$P \cap \Delta_{c_{pk}} \neq \emptyset.$$

The vertical distance between $\Delta_{c_{pk}}$ and $\Delta_{c_{p\ell}}$ is at least $h$ because the elements $P^k$ and $P^\ell$ are nonconsecutive. By our assumption on $P$, the fact that $h_P \leq h$ and that of $P^\ell \in \Lambda_\ell$, we have

$$\pi_2(x) > (k - 2)h, \pi_2(c_{p\ell}) \leq \ell h,$$

for any $x \in P$. Hence

$$\pi_2(x) - \pi_2(c_{p\ell}) > (k - \ell - 2)h \geq 0.$$

Therefore, $P \cap \Delta_{c_{p\ell}} = \emptyset$. The proof that $P \cap \Delta_{c_{pk}} \neq \emptyset$ implies $P \cap \Delta_{c_{p\ell}} = \emptyset$ follows by symmetry. The proof for $T^h_{\partial_1v}(\Gamma)$ is exactly the same. □

Remark. The reason why we don’t care about the intersection of $\partial P$ with any of the slices is because the intersection can only be at most a point if $P$ and the slice considered are disjoint. This is a set of measure 0 and therefore not relevant for our discussion.
Figure 4.9: Illustration of Lemma 4.28, the small blue and red dots indicate the centroids of the other elements in $\Lambda_k$ and $\Lambda_\ell$. The pink dashed lines indicate the partition of $(0,d)$ into multiples of $h$.

**Proposition 4.29** For $d = 1$ and the setting described in the beginning of this section,

$$h \sum_{P \in T_h^h(\Gamma)} \|v^h\|_{H^1(\Delta e_p)}^2 \leq C \|v^h\|_{H^1(\Omega)}^2,$$

for any $v^h \in S^1_0(T^h)$, where $C$ depends only on the quasi-uniformity of the mesh.

**Proof** By explicit calculation we get

$$h \sum_{P \in T_h^h(\Gamma)} \|v^h\|_{H^1(\Delta e_p)}^2 = h \sum_{P \in T_h^h(\Gamma)} \left( \|v^h\|_{L^2(\Delta e_p)}^2 + \|
abla v^h\|_{L^2(\Delta e_p)}^2 \right)$$

$$\leq h \sum_{P \in T_h^h(\Gamma)} \|v^h\|_{L^\infty(\Delta e_p)} \text{vol}_2(\Delta e_p) + h \sum_{P \in T_h^h(\Gamma)} \|
abla v^h\|_{L^2(\Delta e_p)}^2. \quad (4.7)$$
Next we have
\[ h \sum_{P \in T_h(\Gamma)} \|v^h\|_{L^\infty(\Delta_{cP})}^2 \leq h \|v^h\|_{L^\infty(\Omega)}^2. \]

This follows from the construction of \( T^h(\Gamma) \). Indeed, we have that
\[ \Delta_{cP_k} \cap \Delta_{cP_{\ell}} = \emptyset, \quad \forall k \neq \ell, \tag{4.8} \]
because \( \pi_2(c_{P_k}) \) and \( \pi_2(c_{P_{\ell}}) \) are not contained in the same interval of length \( h \) by construction as can be seen in the proof of Lemma 4.24. An illustration of this can be seen in Figure 4.10

Figure 4.10: Illustration of claim (4.8), the dashed line indicates the horizontal cutoff between two multiples of \( h \)

Now, let’s look at the second summation in (4.7). Since \( v^h \) is piece-wise linear and continuous on \( \Omega \), it is also piece-wise linear on \( \Delta_{cP} \). Therefore we get
\[ \|\nabla v^h\|_{L^2(\Delta_{cP})}^2 = \int_{\Delta_{cP}} |\nabla v^h(x)|^2 \, dx = \sum_{T \in \mathcal{R}^h_{cP}} \int_T |\nabla v^h(x)|^2 \, dx = \sum_{T \in \mathcal{R}^h_{cP}} |a_T|^2 \text{vol}_2(T), \]
where \( \{\mathcal{R}^h_{cP}\} := \{T^h|_{\Delta_{cP}}\} \) is the family of meshes on \( \Delta_{cP} \) induced by the family \( T^h \). Here, for any \( T \in \mathcal{R}^h_{cP}, \) \( a_T \in \mathbb{R}^3 \) is such that
\[ v^h|_T(x) = (a_T)^T x + b_T, \quad \forall x \in T, \tag{4.9} \]
for some \( b_T \in \mathbb{R} \). Similarly, we have
\[ \|\nabla v^h\|_{L^2(\Omega)}^2 = \int_{\Omega} |\nabla v^h(x)|^2 \, dx = \sum_{P \in \mathcal{T}^h} \int_P |\nabla v^h(x)|^2 \, dx = \sum_{P \in \mathcal{T}^h} |a_P|^2 \text{vol}_3(P), \]
4.4. Inverse inequalities for centroid slices

where, for any $P \in T^h$, $a_P \in \mathbb{R}^3$ is such that

$$v^h(x) = (a_P)^T x + b_P, \quad \forall x \in P.$$ 

Therefore, to prove our claim, it is sufficient to show that

$$h \sum_{P \in T^h} \sum_{T \in R^h_P} |a_T|^2 \text{vol}_2(T) \lesssim \sum_{P \in T^h} |a_P|^2 \text{vol}_3(P).$$

For any $P \in T^h_v(\Gamma)$ and $T \in R^h_P$, we define $Q_T^P$ as the unique element in $T^h$ that satisfies

$$T = Q_T^P \cap \Delta_{c_P}.$$

We straightforwardly have that

$$h \sum_{P \in T^h} \sum_{T \in R^h_P} |a_T|^2 \text{vol}_2(T) \leq h \sum_{P \in T^h} \sum_{T \in R^h_P} |a_{Q_T^P}|^2 \text{vol}_2(T).$$

Now, Lemma 4.21 applied to $T$ and $Q_T^P$ and Lemma 4.28 show that

$$h \sum_{P \in T^h} \sum_{T \in R^h_P} |a_{Q_T^P}|^2 \text{vol}_2(T) \lesssim 2 \sum_{P \in T^h} |a_P|^2 \text{vol}_3(P) = |v^h|_{H^1(\Omega)}^2.$$ 

Putting everything together and using Lemma 4.17 for the mesh $T^h$, gives

$$h \sum_{P \in T^h_v(\Gamma)} \|v^h\|^2_{L^\infty(\Delta_{c_P})} \text{vol}_2(\Delta_{c_P}) + h \sum_{P \in T^h_v(\Gamma)} \|\nabla v^h\|^2_{L^2(\Delta_{c_P})} \lesssim h \|v^h\|^2_{L^\infty(\Omega)} + |v^h|_{H^1(\Omega)}^2$$

$$\lesssim \|v^h\|^2_{H^1(\Omega)} + \|v^h\|^2_{H^1(\Omega)} = \|v^h\|^2_{H^1(\Omega)}. \quad \square$$

This concludes the results needed to prove the edge lemma. Next we state and prove the results which we will need to prove the face lemma.

Lemma 4.30 Let $\Omega = (0,d)^3$ be equipped with a quasi-uniform family of meshes $\{T^h\}$. Then for any $v^h \in S^0_1(T^h)$ and $P \in T^h$,

$$|\partial_1 v^h(c_P)| \leq Ch^{-1} \|v^h\|_{L^\infty(T(P))},$$

where $C > 0$ is independent of $h$ and only depends on the quasi-uniformity of the mesh.

Proof Fix a $v^h \in S^0_1(T^h)$ and a $P \in T^h$. After translation we may assume $c_P = 0$. First notice that if $\partial_1 v^h(c_P) = 0$, then the inequality is trivially true because the right-hand side of the inequality is always positive. Therefore we
may assume that \( \partial_1 v(c_P) \neq 0 \). Since \( v \) is a linear function on \( P \) there exist \( a_{T(P)} \in \mathbb{R}^3, b \in \mathbb{R} \) such that

\[
v^h(x) = a_{T(P)}^T x + b_{T(P)}, \quad \forall x \in T(P).
\]

We will denote by \((a_1, a_2, a_3)^T\) the Cartesian coordinates of \( a_{T(P)} \) and by our assumption \( a_1 \neq 0 \). Now, let \( \alpha_P > 0 \) be such that

\[
|\alpha_P h \left( \frac{(a_1, 0, 0)^T}{\|a_1, 0, 0\|^T} \right)| = \frac{1}{4} \frac{h_{T(P)}}{\sqrt{3}}.
\]

The reason for the \( \sqrt{3} \) is because the incircle of \( T(P) \) has a diameter of \( h_{T(P)}/\sqrt{3} \). We will denote the quantity inside the outer Euclidean norm on the left hand-side of the above equation by \( \partial_1 v_{T(P)} \). See Figure 4.11 for an illustration of this construction. By Lemma 4.22 we then have that \( h_{T(P)} \sim h \) and hence

\[
|\partial_1 v_{T(P)}| \gtrsim h_{T(P)} \sim h.
\]

Therefore,

\[
\alpha_P = \frac{1}{h} \left| \alpha_P h \left( \frac{(a_1, 0, 0)^T}{\|a_1, 0, 0\|^T} \right) \right| \gtrsim 1.
\]

By the above properties of \( \alpha_P \), we then obtain

\[
\|v^h\|_{L^\infty(T(P))} \geq \|\partial_1 (v^h_{T(P)})\| = |\alpha_P h| a_1 | + b|.
\]

This uses the fact that

\[
\partial_1 v_{T(P)} \in T(P),
\]

which follows from (4.10). Indeed, (4.10) guarantees that \( \partial_1 v_{T(P)} \) lies strictly inside \( C^\circ_{in} \) which is the incircle of \( T(P) \). By definition, the incircle lies inside \( T(P) \) which itself implies that \( \partial_1 v_{T(P)} \) lies inside \( T(P) \). By the reverse triangle inequality we get

\[
\alpha_P h|a_1| \leq \|v^h\|_{L^\infty(T(P))} + |b| \leq 2\|v^h\|_{L^\infty(T(P))},
\]

because \( 0 \in T(P) \). Therefore we obtain

\[
|\partial_1 v(c_P)| = |a_1| \lesssim h^{-1}\|v\|_{L^\infty(T(P))}.
\]

Next we state a result very similar to Proposition 4.29 but for the elements in \( T^h_{\partial_1 v}(\Gamma) \).

**Proposition 4.31** For \( d = 1 \) and the setting described in the beginning of this section,

\[
\sum_{P \in T^h_{\partial_1 v}(\Gamma)} \|v^h\|^2_{H^1(\Delta_{c_P})} \leq C\|v^h\|^2_{H^1(\Omega)},
\]

for any \( v^h \in S^0_1(T^h) \), where \( C \) depends only on the quasi-uniformity of the mesh.
4.4. Inverse inequalities for centroid slices

\[
h^2 \left( \max_{x \in T^h(P)} (\partial_1 v^h)(x, y, 0) \right)^2 \leq C \left( \log \frac{d}{h} \right) \|v^h\|_{H^{1/2}(\partial\Omega)}^2,
\]

where \( C > 0 \) is a constant independent of \( h \) and \( d \).

**Proof**
The proof is identical to that of Proposition 4.29. The reason for this is that the proof essentially only uses Lemma 4.28 which holds for both \( T_v^h(\Gamma) \) and \( T_{\partial_1 v}^h(\Gamma) \).

**Proposition 4.32**
Let \( \Omega = (0, d)^3 \) be equipped with a quasi-uniform mesh \( \{T^h\} \). Then for any \( v^h \in S_0^1(T^h|_{\partial\Omega}) \),

\[
h^2 \int_0^d \max_{x \in T^h(P)} (\partial_1 v^h)(x, y, 0)^2 \, dy \leq C \left( \log \frac{d}{h} \right) \|v^h\|_{H^{1/2}(\partial\Omega)}^2,
\]

where \( C > 0 \) is a constant independent of \( h \) and \( d \).

**Proof**
It is sufficient to prove this for \( d = 1 \). The general case follows by a standard scaling argument. Fix a \( v^h \in S_0^1(T^h) \). First we use Lemma 4.26, which implies that

\[
h^2 \int_0^d \max_{x \in T^h(P)} (\partial_1 v^h)(x, y, 0)^2 \, dy \leq 3h^2 \sum_{P \in T_{\partial_1 v}^h(\Gamma)} |\partial_1 v^h(c_P)|^2.
\]

Next, notice that

\[|v^h|_{W^{1,\infty}(T(P))} = |a_{T(P)}|, \quad |v^h|_{H^1(T(P))} = |a_{T(P)}| \text{vol}(T(P)) \approx |a_{T(P)}| h_{T(P)},\]

where \( a_{T(P)} \in \mathbb{R}^3 \) and \( b_{T(P)} \in \mathbb{R} \) are such that

\[v^h(x) = a_{T(P)} T(P) x + b_{T(P)}, \quad \forall x \in T(P).\]

Figure 4.11: Illustration of condition (4.10)
This follows from the fact that $T(P)$ is an equilateral triangle. Therefore,

$$|v^h(x) - v^h(c_P)| \leq h_T(P)|v^h|_{W^1,\infty(T(P))} \lesssim |v^h|_{H^1(T(P))}, \quad \forall x \in T(P).$$

Using this together with Lemma 4.23 implies that

$$\|v^h\|_{L^\infty(T(P))} \lesssim (1 + |\ln h|)^{1/2}\|v^h\|_{H^1(\Delta c_P)}.$$

Together with Lemma 4.30, this shows that

$$h^3 \sum_{P \in T_{\delta^1}(\Gamma)} |\partial_1 v^h(c_P)|^2 \lesssim (1 + |\ln h|)h \sum_{P \in T_{\delta^1}(\Gamma)} \|v^h\|_{H^1(\Delta c_P)}^2.$$

Now by Proposition 4.31,

$$(1 + |\ln h|)h \sum_{P \in T_{\delta^1}(\Gamma)} \|v^h\|_{H^1(\Delta c_P)}^2 \lesssim (1 + |\ln h|)\|v^h\|_{H^1(\Omega)}^2$$

$$\lesssim (1 + |\ln h|)\|v^h\|_{H^{1/2}(\partial \Omega)}^2,$$

which concludes the proof. \qed
5.1 Erroneous Proofs

The main result in this thesis are the proofs of the edge and face lemma. The motivation was due to the fact that multiple wrong proofs have been given in the past. In this section we will discuss these wrong proofs given in three different papers and explain what parts are wrong which result in the fact that a correct proof for both lemmas hasn’t been given yet.

The main mistake which is made is that an inverse inequality is applied for a mesh which is not shape regular or quasi-uniform while the existence of inverse inequalities depends crucially on these regularities.

5.1.1 Attempt 1

We start off by discussing the proof given in the paper 'A Method of Domain Decomposition for Three-Dimensional Finite Element Elliptic Problems' by M. Dryja ([5]). In this paper an erroneous proof is given for a slight variant of the edge lemma. More specifically we are referring to Lemma 3 in [5]. Let us first state this result.

**Lemma 5.1 (Lemma 3 in [5])** Let $\Omega$ be a cube or tetrahedron with diameter $d$ and faces $F_j$ equipped with a family of shape regular meshes $\{T^h\}$. Let $f^h \in S^h_{1,0}(T^h)$ be a finite element function which equals $f^h_j$ in the interior of the face $F_j$ and zero everywhere else. Then there exists an extension $u^h \in S^h_{1}(T^h)$ of $f^h$ into $\Omega$ such that $u^h = f^h$ on $\partial \Omega$ and

$$|u^h|_{H^1(\Omega)}^2 \leq C \left( 1 + \log \frac{d}{h} \right)^2 \|f^h_j + \alpha\|_{H^{1/2}(F_j)}^2,$$

for any $\alpha \in \mathbb{R}$. Here $C$ is a constant independent of $h$. 

---

61
5. THE EDGE AND FACE LEMMA

Before we discuss the proof we state another lemma. The motivation for this is because the error in the proof of the above lemma is due to a spurious use of this lemma. 4.18.

Lemma 5.2 (Lemma 1 in [5]) Let $(0, H)$ for some $H > 0$ be equipped with a mesh of intervals $\mathcal{R}^h$ of length $h$. Then for any $v^h \in S^0_1(\mathcal{R}^h)$,

$$
\|v^h\|_{L^\infty(0,H)} \leq C \left(1 + \log \frac{H}{h}\right)^{1/2} \|v^h\|_{H^{1/2}(0,H)},
$$

where $C > 0$ is a constant independent of $h$.

Now that we have stated the results we can start our discussion of the erroneous proof. First of all, although it is not stated in Lemma 5.2, an assumption on the regularity of the mesh has to be made if the mesh is not equidistant. Quasi-uniformity would be sufficient. Shape regularity only is not enough because every one-dimensional mesh is shape regular. However, since we are only dealing with a one dimensional mesh we can weaken this condition to demanding that $h = h_{\text{min}}$ where $h_{\text{min}}$ is the smallest width of an interval.

Now we can discuss the proof of the first lemma. The error is a flawed use of the inequality in Lemma 5.2. This inequality is used (wrongly) to argue that

$$
\max_{0 \leq x_2 \leq 1} \left\| f^h(x_1, \cdot) \right\|_{L^2(0,1)}^2 \leq C \int_0^1 \left\| f^h(\cdot, x_2) + \alpha \right\|_{H_{1/2}(0,1)}^2 \, dx_2,
$$

for some constant $C > 0$ independent of $h$. However, care must be taken when applying Lemma 5.2. If $\{T^h\}$ is only shape regular, then we cannot assume that the mesh on $(0, 1)$ is equidistant for all $x_2 \in (0, 1)$. If any implicit assumption is made on the regularity, then there is no argument given as to why this assumption is reasonable. If this problem is ignored, then the above inequality does not simply follow from Lemma 5.2. In any case, demanding quasi-uniformity or $h = h_{\text{min}}$ for the one-dimensional meshes is too strong and not necessary as we will see in our proof of the edge lemma.

5.1.2 Attempt 2

We continue by discussing the paper 'The Construction of preconditioners for Elliptic Problems by Substructuring, IV' by James H. Bramel, Joseph E. Pascia and Alfred H. Schatz ([2]). In this paper, a (wrong) proof is given for a weaker version of Lemma 5.8 (see Chapter 5). We first state the lemma.

Lemma 5.3 (Lemma 4.2 in [2]) Let $\bar{\Omega} := (0,1)^3$ be the unit cube equipped with a family of quasi-uniform meshes $\{T^h\}$. Then for any $v^h \in S^0_1(T^h|_{\bar{\Omega}})$,

$$
\|v^h\|_{L^3(\mathcal{W})} \leq C(1 + \ln(h^{-1})) \|v^h\|_{H^{1/2}(\bar{\Omega})}^2,
$$

where $C > 0$ is a constant independent of $h$ and $\mathcal{W}$ is the wire-basket set associated to $\bar{\Omega}$.
In the proof of this Lemma, a very bold assumption is made about the mesh which is obtained by slicing $\hat{\Omega}$ with a plane. More precisely, it is assumed that
\[
\|v^h\|_{L^\infty(\hat{\Omega})} \leq C(1 + \ln(h^{-1}))\|v^h\|_{H^1(\hat{\Omega})}, \quad \forall v^h \in S_0^1(\mathcal{T}^h),
\] (5.1)
for any two-dimensional slice $\hat{\Omega}$ of $\hat{\Omega}$. The reason why this is assumption is very bold is because the mesh on $\hat{\Omega}$ doesn’t inherit any regularity properties from $\mathcal{T}^h$ in general. However, for the above inequality to hold, quasi-uniformity is needed. Notice that this inequality is actually just Lemma 4.16 and thus needs quasi-uniformity. Therefore this assumption is too strong. This would have been fine in case a remark was made about the fact that such an assumption is not too restrictive, but this isn’t the case. The problem in this case with applying an inverse inequality to a two-dimensional irregular mesh is not solved by just assuming that the mesh satisfies a certain technical assumption which is most likely very strong without remarking anything about why such an assumption would be reasonable. As we will see, the lemma stated actually can be proven without this very strong assumption but the proof is a lot more involved and doesn’t contain spurious uses of inverse inequalities for general two-dimensional slices through the three-dimensional mesh.

Next we discuss the (erroneous) proof of a variant of the face lemma. Before we do this we need to introduce a new operator.

**Definition 5.4** (Section 4 in [2]) Let $\hat{\Gamma}_i^f$ be any face of $\partial \hat{\Omega}$. Call $\ell_0$ The operator associated to the bilinear form defined by
\[
a_0(u,v) := \int_{\hat{\Gamma}_i^f} \nabla u^h \cdot \nabla v^h \, ds, \quad \forall u^h, v^h \in S^{0,e}_{1,0}(\mathcal{T}^h|_{\hat{\Gamma}_i^f}),
\]
as in Definition 3.2. Here $S^{0,e}_{1,0}(\mathcal{T}^h|_{\hat{\Gamma}_i^f})$ is the subset functions in $S^0_{1,0}(\mathcal{T}^h|_{\hat{\Gamma}_i^f})$ that vanish on the edges of $\hat{\Omega}$. Moreover, denote by $\ell_0^{1/2}$ the positive square root of this symmetric positive definite operator.

Now that we have defined the relevant concepts, we state the variant of the face lemma in [2].

**Lemma 5.5** (Lemma 4.3 in [2]) Let $v^h \in S^0_1(\mathcal{T}^h|_{\partial \hat{\Omega}})$ and let $v_f$ be either $v^h$ set equal to 0 on the edges of $\hat{\Omega}$ or equal to $u^h - \hat{u}^h$ where $\hat{u}^h$ coincides with $u^h$ on the edges of $\hat{\Omega}$ and is discrete harmonic on the faces of $\partial \hat{\Omega}$. Then
\[
\langle \ell_0^{1/2}v_f, v_f \rangle_{L^2(\hat{\Gamma}_i^f)} \leq C(1 + \ln(h^{-1}))^2\|v_f\|^2_{H^{1/2}(\partial \hat{\Omega})},
\]
for any face $\hat{\Gamma}_i^f$ of $\partial \hat{\Omega}$. Here $C > 0$ is a constant independent of $h$.

From equation (4.24) in [23], we know that
\[
\|v^h\|^2_{H^{1/2}(\hat{\Gamma}_i^f)} \approx \langle \ell_0^{1/2}v^h, v^h \rangle_{L^2(\hat{\Gamma}_i^f)}, \quad \forall v^h \in S^0_1(\mathcal{T}^h|_{\hat{\Gamma}_i^f})
\]
63
We also clearly have that \( v_f \in S_{1,0}^0(T^h|\hat{\Gamma}_f) \) since any choice of \( v_f \) vanishes on the edges of \( \hat{\Omega} \). Therefore we can rewrite the inequality in the previous lemma as

\[
\|v_f\|^2_{H^{1/2}(\hat{\Gamma}_f)} \leq C(1 + \ln(h^{-1}))^2 \|v_f\|^2_{H^{1/2}(\partial\hat{\Omega})}.
\]

Hence the lemma above is just the face lemma summed over all faces of \( \partial\hat{\Omega} \).

Next we discuss the proof. Again (5.1) is used to argue that a certain inverse inequality is true. The inequality that is claimed to hold is the following one:

\[
\int_0^1 \int_0^h v_f(x, y, 0)^2 \frac{dxdy}{x} \leq C(1 + \ln(h^{-1})) \int_0^1 \|\tilde{v}_f(x, y, \cdot)\|^2_{H^1(\Delta_y)} dy.
\]

It is stated in [2] that this follows from an inverse inequality for functions in \( S_{1,0}^0(T^h|\hat{\Gamma}) \) restricted to the plane \( y = c \) for some constant \( c \in (0,1) \) together with the (strong) assumption (5.1). There are two remarks to be made here. The first one concerns this inverse inequality. It is not true that it can just be applied to the two-dimensional slice \( y = c \). We will not go into detail about what this inverse inequality precisely is because we will deal with it in detail in our proof of the face lemma. The only remark that we make about it here is that it is only true locally for slices through the centroids of elements. This will be made rigorous later on. The second remark to be made concerns again assumption (5.1). For the same reasons as stated for the variant of the edge lemma, it is poorly motivated why such an assumption is reasonable. As stated before, if we look at the proof of Lemma 4.16, then quasi-uniformity plays an essential role in the derivation of this inequality. This regularity for the induced meshes on the two-dimensional slices is definitely not a simple consequence of the quasi-uniformity of the whole mesh.

### 5.1.3 Attempt 3

The third and last paper which we discuss concerning erroneous attempts at proving the edge and face lemma is the paper titled 'Some Nonoverlapping Domain Decomposition Methods' by Jinchao Xu and Jun Zou ([23]). We start our discussion with the proof of the edge lemma (lemma 4.9 in [23]). The proof of this lemma is in itself correct. However, it references another Lemma which contains an error in its proof. The lemma we are referring to is Lemma 4.8 in [23]. This lemma will be proved in detail when we give a correct proof of the edge lemma.

**Lemma 5.6** (Lemma 4.8 in [23]) Let \( \Omega \subseteq \mathbb{R}^3 \) be a polyhedral domain equipped with a family of quasi-uniform tetrahedral meshes \( \{T^h\} \). Then for any \( v^h \in S_{1,0}^0(T^h|\partial\Omega) \),

\[
\int_0^d \max_{0 \leq y \leq d} v^h(x, y, 0)^2 \, dy \leq C \left( \log \frac{d}{h} \right) \|v^h\|^2_{H^{1/2}(\partial\Omega)},
\]

64
where the constant $C > 0$ is independent of $h$ and $v^h$. Similar results also hold by interchanging the positions of $x$, $y$, $z$, and for a polyhedron.

The problem with the proof of this lemma is that it again uses an inverse inequality for an arbitrary two-dimensional slice through a three-dimensional mesh. More precisely, it is claimed that

$$ \int_0^d \max_{0 \leq x \leq d} v^h(x, y, 0)^2 \, dy \lesssim \left( \log \frac{d}{h} \right) \int_0^d \| \tilde{v}^h \|^2_{H^1(\Delta_y)} \, dy,$$

where $\tilde{v}^h$ is the generalized discrete harmonic extension of $v^h$ into $\Omega$ and $\Delta_\alpha := \Omega \cap \{ y = \alpha \}$ for any $\alpha \in (0, d)$. Again, the claim that

$$ \| \tilde{v}^h \|^2_{L^\infty(\Delta_y)} \lesssim \left( \log \frac{d}{h} \right) \| v^h \|^2_{H^1(\Delta_y)}$$

for any $y$ is false because the mesh on $\Delta_y$ is not quasi-uniform in general.

Next we turn to the face lemma (Lemma 4.10 in [23]). There are multiple errors in the proof of this lemma. Before turning to the proof, we state the result.

**Lemma 5.7 (Face Lemma)** (Lemma 4.10 in [23]) Let $F$ be a face ($n = 3$) or an edge of $\partial \Omega$ ($n = 2$). Then for any $v^h \in S_0^1(T_h|_{\partial \Omega})$,

$$ \| I^0_F v^h \|_{H^{1/2}(\partial \Omega)} \lesssim \left( \log \frac{d}{h} \right) \| v^h \|_{H^{1/2}(\partial \Omega)}.$$

The first error is presumably caused by misinterpreting the used notation. For the square $F := (0, d)^2$ and any point $x \in F$, it is not true that

$$ d(x, \partial F) = x,$$

as can easily verified. There are in fact four different possibilities depending on where $x$ is located in $F$. The second error is another spurious use of an inverse inequality. This time it is claimed that

$$ h^2 \int_0^d \max_{0 \leq x \leq x_1} | \partial_1 (I^0_F v^h)(x, y, 0) |^2 \, dy \lesssim \int_0^d \max_{0 \leq x \leq d} | v^h(x, y, 0) |^2 \, dy.$$

To obtain this inequality, the following (wrong) inequality is used:

$$ \| \partial_1 (I^0_F v^h)(\cdot, y, 0) \|_{L^\infty(0, d)} \lesssim h^{-1} \| v^h(\cdot, y, 0) \|_{L^\infty(0, d)}.$$

The above inequality only holds if the mesh on $(0, d)$ is quasi-uniform. This is definitely not the case for slices through general quasi-uniform meshes and general $y$. This problem will be dealt with in detail when we give our proof of the face lemma.
5. THE EDGE AND FACE LEMMA

5.2 Proof of the Edge Lemma

In this section we prove Lemmas 4.8 and 4.10 (the edge lemma) from [23]. Our approach will be similar to the attempt in [23], but instead of subdividing $\Gamma$ by horizontal slices for each $y \in (0, d)$, we will only consider slices through centroids of certain elements adjacent to the boundary. The proof has two parts. In the first part we derive an inequality that compares the values of a function $v^h \in S_1^0(T^h)$ on the boundary to the values on slices through the centroids of certain elements. In the second part we compare the values of $v^h$ on the slices through the centroids with the values of $v^h$ on the entire cube $\Omega$. Combining these two inequalities will then give the required inequality. As usual we will denote by $\Gamma$ the bottom face of $\Omega$.

Lemma 5.8 Let $\Omega = (0, d)^3$ be equipped with a quasi-uniform mesh $\{T^h\}$. Then for any $v^h \in S_1^0(T^h|_{\partial \Omega})$,

$$\int_0^d \max_{0 \leq x \leq d} v^h(x, y, 0)^2 \, dy \leq C \left( \log \frac{d}{h} \right) \|v^h\|^2_{H^{1/2}(\partial \Omega)},$$

where the constant $C > 0$ is independent of $h, v$ and $d$. Similar results also hold by interchanging the positions of $x, y, z$, and for a polyhedron.

Proof It is sufficient to prove this for the case $d = 1$. The result for general $d > 0$ follows by a standard scaling argument. Fix a $v^h \in S_1^0(T^h|_{\partial \Omega})$, we will denote by $\tilde{v}^h \in S_1^0(T^h)$ the generalized discrete harmonic extension of $v^h$ into $\Omega$. By Lemma 4.22 and Lemma 4.23, we have

$$|\tilde{v}^h(c_{P_k})| \lesssim (1 + |\ln h|)^{1/2} \|\tilde{v}^h\|_{H^1(\Delta_{c_{P_k}})}, \quad k = 1, \ldots, N. \quad (5.2)$$

Squaring this inequality, multiplying by $h$ and summing over $T^h_v(\Gamma)$ gives

$$h \sum_{P \in T^h_v(\Gamma)} \tilde{v}^h(c_P)^2 \lesssim h(1 + |\ln h|) \sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|^2_{H^1(\Delta_{c_P})}. \quad (5.3)$$
5.2. Proof of the Edge Lemma

|\Gamma| \Lambda_3
|\Lambda_2|
|\Lambda_1|

\[ h \sum_{k=1}^{N} v^h(n_k)^2 \lesssim h \sum_{k=1}^{N} (\hat{v}^h(n_k) - \hat{v}^h(c_{P_k}))^2 + h \sum_{k=1}^{N} \hat{v}^h(c_{P_k})^2, \quad (5.5) \]

because \( v^h = \hat{v}^h \) on \( \Gamma \). This follows from the inequality \((a+b)^2 \leq 2a^2 + 2b^2\) for any \( a, b \in \mathbb{R} \) and hence the implicit constants are independent of \( v^h \). Since we have a one-to-one correspondence between the elements in \( T^h(\Gamma) \) and the nodes above, we get

\[ h \sum_{k=1}^{N} v^h(n_k)^2 \lesssim h \sum_{P \in T^h(\Gamma)} (\hat{v}^h(n_P) - \hat{v}^h(c_P))^2 + h \sum_{P \in T^h(\Gamma)} \hat{v}^h(c_P)^2, \]

for any \( P \in \mathcal{T}^h(\Gamma) \) and any \( x \in P \cap \Gamma \). For the first inequality we simply bounded the derivative of \( \hat{v}^h \) by the \( W^{1,\infty} \)-seminorm times the maximum distance between two points in \( P \) which is obviously bounded by \( h \). The implicit constant in the last inequality above is independent of \( P \) and \( h \) because the mesh \( \{\mathcal{T}^h\} \) is quasi-uniform. Next we have the straightforward estimate

\[ |\hat{v}^h(x) - \hat{v}^h(c_P)| \leq h|\hat{v}^h|_{W^{1,\infty}(P)} \lesssim h^{-1/2}|\hat{v}^h|_{H^1(P)}, \quad (5.4) \]
5. THE EDGE AND FACE LEMMA

where $n_P$ is the unique node in $\{n_1, \ldots, n_N\}$ that corresponds to $P$ as in Definition 4.27. Now we use inequalities (5.3) and (5.4) to obtain

$$h \sum_{P \in T^h_v(\Gamma)} (\tilde{v}^h(n_P) - \tilde{v}^h(c_P))^2 + h \sum_{P \in T^h_v(\Gamma)} \tilde{v}^h(c_P)^2 \lesssim \sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|_{H^1(P)}^2,$$

$$+ (1 + |\ln h|)h \sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|_{H^1(\Delta_{cP})}^2. \tag{5.6}$$

Again, the implicit constants don’t depend on $v^h$ as was shown in the proof of Lemma 4.29. The first sum can be bounded by simply using linearity of the integral and the definition of the $H^1$-seminorm as follows

$$\sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|_{H^1(P)}^2 \leq \sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|_{H^1(\Omega)}^2.$$

The last inequality follows from the fact that any two different tetrahedra in $T^h_v(\Gamma)$ are disjoint. For the second summation in (5.6) we use Proposition 4.29. This gives

$$\sum_{P \in T^h_v(\Gamma)} \|\tilde{v}^h\|_{H^1(\Delta_{cP})}^2 \lesssim h^{-1} \|\tilde{v}^h\|_{H^1(\Omega)}^2.$$

Putting everything together we obtain

$$\int_0^d \max_{0 \leq x \leq d} v^h(x, y, 0)^2 \, dy \lesssim (1 + |\ln h|)\|\tilde{v}^h\|_{H^1(\Omega)}^2 + \|\tilde{v}^h\|_{H^1(\Omega)}^2$$

$$\lesssim (1 + |\ln h|)\|\tilde{v}^h\|_{H^1(\Omega)}^2 \lesssim (1 + |\ln h|)\|\tilde{v}^h\|_{H^{1/2}(\partial \Omega)}^2.$$

Notice that the implicit constants are independent of the function $v^h$ since all implicit constants in each step were independent of $v^h$. \qed

We need one more definition before we can state and prove the edge lemma. Given a subset $K \subseteq \Omega$ we will need a restriction operator $I^0_K$ which sets any finite element function defined on $\Omega$ equal to 0 outside the interior of $K$.

**Definition 5.9 (Restriction Operator)** (Section 4.3 in [23]) Let $\Omega \subseteq \mathbb{R}^n$ be a polyhedral domain equipped with a quasi-uniform mesh $\{T^h\}$ and $K \subseteq \Omega$ a set. Denote by $\{\varphi_k\}_k$ the nodal basis for $S^0_1(T^h)$. The restriction operator $I^0_K : S^0_1(T^h) \to S^0_1(T^h|_K)$ is defined as

$$(I^0_K u^h) := u^h|_{\hat{K}} = \sum_{x_k \in K \cap N(T^h)} u^h(x_k) \varphi_k.$$ 

Now that we have proved the preliminary results and defined all the notation that we need, we proceed by stating and proving the edge lemma.
5.2. Proof of the Edge Lemma

**Lemma 5.10 (Edge Lemma)** Let \( \Omega \subseteq \mathbb{R}^3 \) be a polyhedral domain equipped with a family of quasi-uniform tetrahedral meshes \( \{T^h\} \) and let \( e \) be any edge of \( \partial \Omega \). Then,

\[
\| I^0_e u^h \|_{H^{1/2}(\partial \Omega)} \leq C_1 \| u^h \|_{L^2(e)} \leq C_2 \left( \log \frac{d}{h} \right)^{1/2} \| u^h \|_{H^{1/2}(\partial \Omega)},
\]

for any \( u^h \in S_0^1(T^h|_{\partial \Omega}) \). The constants \( C_1, C_2 > 0 \) are independent of \( h \) and \( d \).

**Proof** First we may assume that the edge is on the \( z \)-axis by an appropriate choice of coordinates. Next, using Proposition 4.19 gives

\[
\| I^0_e u^h \|_{H^{1/2}(\partial \Omega)} \lesssim h^{-1/2} \| I^0_e u^h \|_{L^2(\partial \Omega)}.
\]

Hence by the definition of \( I^0_e \), Lemma 4.6 and Lemma 4.8, we obtain

\[
\| I^0_e u^h \|^2_{L^2(\partial \Omega)} \approx h^2 \sum_{n \in e \cap N(T^h)} \nu(n)^2 = h\| u^h \|^2_{h,e} \approx \| u^h \|^2_{L^2(e)},
\]

because \( \dim e = 1 \). Putting everything together we find

\[
\| I^0_e u^h \|_{H^{1/2}(\Omega)} \lesssim h^{-1/2} \| I^0_e u^h \|_{L^2(\partial \Omega)} \approx \| u^h \|_{h,e}.
\]

This proves the first inequality. For the second inequality we use Lemma 5.8 to obtain

\[
\| u^h \|^2_{L^2(e)} = \int_0^d u^h(0,0,z)^2 \, dz \leq \int_0^d \max_{0 \leq y \leq d} u^h(0,y,z)^2 \, dz \lesssim \left( \log \frac{d}{h} \right)^{1/2} \| u^h \|_{H^{1/2}(\Omega)}.\]

\[\square\]

**Figure 5.2:** Geometric setup for the edge lemma in the case of a cube
Remark. To bridge the gap between a cube and a general polyhedral domain, we can look at all results which are used to prove the edge lemma. These are Proposition 4.19 and Lemmas 4.6, 4.8 and 5.8. The proposition and the first two lemmas already hold for polyhedral domains. Therefore, we only need to check that Lemma 5.8 also holds for a polyhedral domain. This is indeed true, the only thing that needs to be changed is one step in Lemma 4.29. If \( \Omega \) is not a cube, then we don’t have that \( \text{vol}_2(\Delta_{c_P}) = 1 \). However, we can just substitute this by using the estimate \( \text{vol}_2(\Delta_{c_P}) \leq 1 \) which does hold for a general polyhedral domain with diameter 1.

### 5.3 Proof of the Face Lemma

Now that the preliminary results have been proven in the previous proposition and lemma, we turn to the proof of the Face lemma. This proof will be largely the same as in [23] except for two parts where the proof is incorrect. The preliminary results proved above will be the remedy.

**Lemma 5.11 (Face Lemma)** Let \( \Omega \subseteq \mathbb{R}^3 \) be a polyhedral domain equipped with a family of quasi-uniform tetrahedral meshes \( \{T^h\} \) and let \( F \) be a face of \( \partial \Omega \) (\( n = 3 \)) or an edge of \( \partial \Omega \) (\( n = 2 \)). Then for any \( u^h \in S_0^1(T^h|_{\partial \Omega}) \),

\[
\| I_F^0 u^h \|_{H^{1/2}(\partial \Omega)} \lesssim \left( \frac{\log d}{d} \right) \| u^h \|_{H^{1/2}(\partial \Omega)}.
\]

**Proof** From [23], we know that

\[
\| I_F^0 u^h \|_{H^{1/2}(\partial \Omega)}^2 \approx \| I_F^0 u^h \|_{H^{1/2}(F)}^2 = \| I_F^0 u^h \|_{H^{1/2}(F)}^2 + \int_F \frac{(I_F^0 u^h)^2(x)}{d(x, \partial F)} \, ds(x). \tag{5.7}
\]

Note that on \( F, I_F^0 u^h = u^h - I_{\partial F}^0 u^h \). Indeed, by definition of \( I_F^0 \) and \( I_{\partial F}^0 \)

\[
u^h(x) - (I_{\partial F}^0 u^h)(x) = \sum_{y \in F \cap N(T^h)} u^h(y) \varphi(x) - \sum_{y \in \partial F \cap N(T^h)} u(y) \varphi(x) = \sum_{y \in F \cap N(T^h)} u^h(y) \varphi(x) = (I_F^0 u^h)(x),
\]

for any \( x \in F \). By the triangle inequality and Lemma 5.10 it follows that

\[
| I_F^0 u^h |_{H^{1/2}(F)} \leq | u^h |_{H^{1/2}(F)} + | I_{\partial F}^0 u^h |_{H^{1/2}(F)} \lesssim \left( \frac{\log d}{d} \right) \| u^h \|_{H^{1/2}(\partial \Omega)}.
\]

70
To estimate the second term in (5.7) we assume without loss of generality that \( \Omega = (0, d)^n \). We consider the cases \( n = 2 \) and \( n = 3 \) separately.

**Case 1** (\( n = 2 \)): Without loss of generality, we can assume that \( F = \Pi \cap \{ y = 0 \} \). For convenience of notation we set \( w(x) := I_0^F u^h(x, 0) \). By definition of \( I_0^F \) we have \( w(x) = 0 \) for all \( x \in (0, d) \). We distinguish between the cases \( x_1 \leq d/2 \) and \( x_1 > d/2 \) where \( x_1 \) is the node on \( (0, d) \) closest to 0.

![Figure 5.3: Illustration of the cases \( x_1 \leq d/2, x_1 > d/2 \)](image.png)

**Case 1.1** (\( x_1 \leq d/2 \)): By linearity we trivially have

\[
\int_F \frac{w(x)^2}{d(x, \partial F)} \, ds(x) = \int_0^{d/2} \frac{w(x)^2}{x} \, dx + \int_{d/2}^d \frac{w(x)^2}{d-x} \, dx.
\]

After the change of variables \( x \mapsto d-x \), the second integral becomes

\[
\int_{d/2}^d \frac{w(x)^2}{d-x} \, dx = \int_0^{d} \frac{w(d-x)}{x} \, dx.
\]

Using the fundamental theorem of calculus for \( H^1 \)-functions and Lemma 4.14 for \( m = 0, \ell = 1, p = q = \infty \) gives,

\[
\int_0^{d/2} \frac{w(x)^2}{x} \, dx = \int_0^{x_1} \frac{w(x)^2}{x} \, dx + \int_{x_1}^{d/2} \frac{w(x)^2}{x} \, dx
\]

\[
\lesssim \|w\|_{L^\infty(0,d/2)}^2 \int_0^{x_1} x \, dx + \|w\|_{L^\infty(0,d/2)}^2 \int_{x_1}^{d/2} \frac{1}{x} \, dx
\]

\[
\lesssim \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d/2)}^2.
\]

In the last step we used the quasi-uniformity of the mesh which gives \( x_1 \approx h \).

Last, we do a similar sequence of estimates as we just did above,

\[
\int_0^{d/2} \frac{w(d-x)}{x} \, dx = \int_0^{x_1} \frac{w(d-x)}{x} \, dx + \int_{x_1}^{d/2} \frac{w(d-x)}{x} \, dx
\]

\[
\lesssim \|w\|_{L^\infty(d/2,d)}^2 \int_0^{x_1} x \, dx + \|w\|_{L^\infty(d/2,d)}^2 \int_{x_1}^{d/2} \frac{1}{x} \, dx
\]

\[
\lesssim \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(d/2,d)}^2.
\]
In the last step we again used Lemma 4.14 for $m = 0$, $\ell = 1$, $p = q = \infty$. This shows that
\[
\hat{F}_w(x) \leq \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d/2)}^2 + \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(d/2,d)}^2
\]
\[
= \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d)}^2
\]
\[
\leq \left( \log \frac{d}{h} \right) \|u\|_{L^\infty(\partial \Omega)}^2
\]
\[
\lesssim \left( \log \frac{d}{h} \right)^2 \|u\|_{H^{1/2}(\partial \Omega)}^2.
\]
In the last step we used Lemma 4.18.

**Case 1.2** ($x_1 > d/2$): By linearity of the integral we have
\[
\int_F \frac{w(x)^2}{d(x,\partial F)} \, ds(x) = \int_0^{d/2} \frac{w(x)^2}{x} \, dx + \int_{d/2}^d \frac{w(x)^2}{d-x} \, dx.
\]
For the second integral we obtain the following estimate by using that $w(d) = 0$,
\[
\int_{d/2}^d \frac{w(x)^2}{d-x} \, dx = \int_{d/2}^{x_1} \frac{w(x)^2}{d-x} \, dx + \int_{x_1}^d \frac{w(x)^2}{d-x} \, dx
\]
\[
\leq \|w\|_{L^\infty(d/2,d)}^2 \int_{d/2}^{x_1} \frac{1}{d-x} \, dx + \int_{x_1}^d \frac{1}{d-x} \left( - \int_x^d w'(t) \, dt \right)^2 \, dx
\]
\[
\leq \left( \log \frac{d}{2(d-x_1)} \right) \|w\|_{L^\infty(d/2,d)}^2 + \|w'\|_{L^\infty(d/2,d)} \int_{x_1}^d d-x \, dx
\]
\[
= \left( \log \frac{d}{2(d-x_1)} \right) \|w\|_{L^\infty(d/2,d)}^2 + \|w'\|_{L^\infty(d/2,d)} \int_0^{d-x_1} x \, dx
\]
\[
\lesssim \left( \log \frac{d}{h} \right)^2 \|w\|_{L^\infty(d/2,d)}^2 + h^2 \|w'\|_{L^\infty(d/2,d)}.
\]
In the last step we used the fact that $d - x_1 = h$. Indeed, since $h \geq x_1 \geq d/2$, we have
\[
d - x_1 \leq d/2 \leq h.
\]
For the other inequality, let $x_2$ be the second node on the $x$-axis. Then
\[
d - x_1 \geq x_2 - x_1 \geq h,
\]
because the mesh is quasi-uniform. For the first integral we perform the change of variables $x \mapsto d - x$ which gives
\[
\int_0^{d/2} \frac{w(x)^2}{x} \, dx = \int_{d/2}^d \frac{w^2(d-x)}{d-x} \, dx.
\]
As before, we split this last integral into two parts and use the same estimates as for the second integral,
\[
\int_{\frac{d}{2}}^{d} \frac{w^2(d-x)}{d-x} \, dx = \int_{\frac{d}{2}}^{x_1} \frac{w^2(d-x)}{d-x} \, dx + \int_{x_1}^{d} \frac{w^2(d-x)}{d-x} \, dx
\]
\[
\leq \left( \log \frac{d}{2(d-x_1)} \right) \|w\|_{L^\infty(0,d)}^2 + \int_{x_1}^{d} \frac{1}{d-x} \left( \int_{0}^{d-x} w'(t) \, dt \right)^2 \, dx
\]
\[
\leq \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d)}^2 + \|w'\|_{L^\infty(0,d)} \int_{0}^{d-x_1} x \, dx
\]
\[
\leq \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d)}^2 + h^2 \|w'\|_{L^\infty(0,d)}.
\]
Therefore,
\[
\int_{F} w(x)^2 \, d(x, \partial F) = \int_{0}^{d} \frac{w(x)^2}{x} \, dx + \int_{\frac{d}{2}}^{d} \frac{w(x)^2}{d-x} \, dx
\]
\[
\leq \left( \log \frac{d}{h} \right) \|w\|_{L^\infty(0,d)}^2 + h^2 \|w'\|_{L^\infty(0,d)}
\]
\[
\leq \left( \log \frac{d}{h} \right)^2 \|u_h\|_{H^{1/2}(\partial \Omega)}^2.
\]

The last step is again obtained by using Lemma 4.14 for \( m = 0, \ell = 1, p = q = \infty \) followed by Lemma 4.18. This completes the proof for \( n = 2 \).

**Case 2** (\( n = 3 \)): Without loss of generality we can assume that \( F = \Omega \cap \{ z = 0 \} \). Again we need to distinguish the two cases \( x_1 \leq d/2 \), \( x_1 > d/2 \), where \( x_1 \) is the node on \((0, d) \times \{0\} \) closest to 0. For each case we need to estimate again the second term in (5.7). This boils down to estimating nine double integrals because \( d(x, \partial F) \) has four different expressions \((x, y, d-x) \) or \((d-y) \) depending on where \( x \in F \) is precisely located. This is illustrated in Figure 5.4.

We will compute two of the integrals in detail and leave the rest as is, because all the estimates of the other cases are similar (as we have seen for the case \( n = 2 \)). For notational convenience we set \( v^h := I_0^h u^h \). We will explicitly derive the estimate for the case \( x_1 \leq d/2 \) and the integrals
\[
\int_{0}^{x_1} \int_{0}^{x} \frac{v^h(x,y,0)^2}{y} \, dy \, dx, \int_{0}^{x_1} \int_{d-x}^{d} \frac{v^h(x,y,0)^2}{y} \, dy \, dx.
\]
The domains of integration can also be seen in Figure 5.4. For the first integral we use the fact that \( v^h(0, y, 0) = 0 \) to obtain the sequence of estimates which is similar to the case \( n = 2 \),
\[
\int_{0}^{x_1} \int_{0}^{x} \frac{v^h(x,y,0)^2}{y} \, dy \, dx = \int_{0}^{x_1} \int_{0}^{x} \frac{1}{y} \left( \int_{0}^{y} \partial_2 v^h(x,t,0) \, dt \right)^2 \, dy \, dx
\]
5. The Edge and Face Lemma

Figure 5.4: Illustration of the two cases $x_1 \leq d/2, x_1 > d/2$ together with the partition (in red) of $F$ such that $d(x,F)$ equals $(x,y, d-x)$ or $(d-y)$ on each subdomain

\[
\leq \int_0^{x_1} \int_0^x y \max_{t \in I(T^h|_F)_x} |\partial_2 v^h(x,t,0)| \, dy \, dx \lesssim h^2 \int_0^{x_1} \max_{t \in I(T^h|_F)_x} |\partial_2 v^h(x,t,0)| \, dx.
\]

Next we apply Proposition 4.32 to obtain

\[
h^2 \int_0^{x_1} \max_{t \in I(T^h|_F)_x} |\partial_2 v^h(x,t,0)| \, dx \leq h^2 \int_0^d \max_{t \in I(T^h|_F)_x} |\partial_2 v^h(x,t,0)| \, dx
\]

\[
\lesssim \left( \log \frac{d}{h} \right) \|v^h\|_{H^{1/2}(\partial \Omega)}^2.
\]

For the second integral we get

\[
\int_0^{x_1} \int_{d-x}^d \frac{v^h(x,y,0)^2}{y} \, dy \, dx \leq \log \left( \frac{d}{d-x_1} \right) \int_0^{x_1} \max_{0 \leq y \leq d/2} |v^h(x,y,0)|^2 \, dx
\]

\[
\lesssim \left( \log \frac{d}{h} \right) \int_0^d \max_{0 \leq y \leq d} |v^h(x,y,0)|^2 \, dx.
\]

Here we again used that $d - x_1 \sim h$ as for the case $n = 2$. Hence, by Lemma 5.8, we find

\[
\int_0^{x_1} \int_{d-x}^d \frac{v^h(x,y,0)^2}{y} \, dy \, dx \lesssim \left( \log \frac{d}{h} \right) \int_0^d \max_{0 \leq y \leq d} |v^h(x,y,0)|^2 \, dx
\]

\[
\lesssim \left( \log \frac{d}{h} \right) \|v^h\|_{H^{1/2}(\partial \Omega)}^2.
\]

This completes the proof. □
### 5.3. Proof of the Face Lemma

**Remark.** The proof for the case where $\Omega$ is a polyhedral domain is very similar to what we explained for the edge lemma. Lemma 4.26 and 4.30 and Proposition 4.31 all hold trivially for polyhedral domains. The extension of Proposition 4.32 to polyhedral domains is the same as for Proposition 5.8. Hence the only thing that still needs to be checked is that all the estimates for the integrals still hold in the proof of the face lemma itself. For the case $n = 2$ this is trivial because $F$ is just an edge. For $n = 3$ the proof becomes even more complicated because $d(x, F)$ depends strongly on the geometry of $F$. However, since we can always partition $F$ into triangles, the proofs estimates be very similar to the case of that of the cube.

#### 5.3.1 Implications

This section serves as a short addendum. Here we state and prove the one result from [23] which relies on the edge and face lemma. The proof of this can be found in [23].

**Lemma 5.12** *(Lemma 4.11 in [23])* Let $F$ be a face ($n = 3$) or an edge ($n = 2$) of $\partial \Omega$. Then

$$
\|I_0^F 1\|_{H^{1/2}(\partial \Omega)}^2 \leq C d^{(n-1)/2} \left( \frac{\log d}{h} \right)^{1/2},
$$

where $C > 0$ is a constant independent of $h$ and $d$.

**Proof** Choosing $v \equiv 1$ in Lemma 5.11 gives

$$
\|I_0^F 1\|_{H^{1/2}(\partial \Omega)}^2 \lesssim \left( \frac{\log d}{h} \right) \left( \|1\|_{H^{1/2}(\partial \Omega)}^2 + \|1\|_{L^2(\partial \Omega)}^2 \right) = \left( \frac{\log d}{h} \right) \|1\|_{L^2(\partial \Omega)} \leq \left( \frac{\log d}{h} \right) d^{n-1}.
$$

**Remark.** In [23] the power of $d$ is $(n - 2)/2$, this is due to the fact that they use scaled Sobolev norms. Indeed, they use $d^{-1}\|v\|_{L^2(\partial \Omega)}^2$ instead of $\|v\|_{L^2(\partial \Omega)}^2$ in the definition of the scaled $H^{1/2}$-norm. This explains the extra $-1$ in the exponent of $d$. 

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75
In this chapter we present the results from research papers which rely on the edge and face lemma and therefore on the proofs that we gave in Chapter 5. The discussion will be structured as follows; for each research paper we will first give a brief description of what the main goal of that is paper followed by the results which rely on the edge and face lemma. In each section we will present a different paper and will introduce the relevant definitions and results. To make everything as clear as possible we will prec isely define what all the notation is that is used in each result. More specifically if any definition is more general than the definition we introduced in a previous chapter, then we will reintroduce the more general definition as is given in the paper. Moreover, to make all notation consistent with the notation we already introduced, we will modify the notation used in these papers. This will also have the added benefit that the notation throughout all the papers will be the same.

6.1 Direct References to the edge and face lemma

We start our discussion by listing the papers which only refer to the edge and face lemma directly but don’t use them to prove any new results. It is worthwhile to note beforehand that only the face lemma is referenced in these papers and not the edge lemma. Due to the fact that the results in these papers are not dependent on the edge and face lemma, we will only briefly list them. The more detailed discussions are reserved for the papers with results that actually use the edge and/or face lemma.

The first paper which makes a reference to the face lemma is the paper titled ’A Regularized Domain Decomposition Method with Lagrange Multiplier’ by Qiya Hu ([11]). In this paper, Lemma 5.5 is a direct reference to the face lemma in [23].

The second paper that references the face lemma is the paper ’Efficient Solvers
6. APPLICATIONS IN RECENT RESEARCH

for Saddle-Point Problems Arising from Domain Decompositions with Lagrange Multipliers’ by Qiya Hu, Zhongci Shi and Dehao Yu ([13]). In this paper, again a direct reference is made to the face lemma from [23] in the form of Lemma 4.4.

The third and last paper that I found that made a direct reference which was left unused is the paper titled 'Substructuring Preconditioners for Saddle-Point Problems Arising from Maxwell’s equations in Three Dimensions’ by Qiya Hu and Jun Zou ([17]). Again, a reference is made to the face lemma from [23] as one part of Lemma 4.10.

Now that we have briefly listed the direct references, we turn to a more detailed description of the results of the papers that actually use the edge and face lemma to prove new results. For each paper we will discuss the following:

- Goal of the paper,
- Algorithms (if relevant),
- Results based on the edge and/or face lemma.

Moreover, as we stated before, we will change some notation to make everything as consistent as possible with the notation we used elsewhere. However, this does not mean that any of the stated results are our own. In some places we may have elaborated more on a specific part or added a small number of additional steps to some computations, but all ideas for the proofs and results belong to the writers of these research articles. We will therefore always correctly refer to each paper when needed for the statements of results and the proofs.

Last we make a very important remark about notation. The main applications of the edge and face lemma are in domain decomposition methods. Here not only the mesh width $h$ is important, but also the diameter of the subdomains $d$. Therefore, the notations $\lesssim, \gtrsim$ and $\sim$ will not only denote an inequality with an implicit constant independent of $h$ but also independent of $d$. This should not be confused with the diameter of the entire domain, which is as usual not of interest and can be absorbed in these implicit constants.
6.2 Paper 1: A Substructuring Preconditioner with Vertex-Related Interface Solvers for Elliptic-Type Equations in Three Dimensions

In this section we discuss the research paper 'A Substructuring Preconditioner With Vertex-Related Interface Solvers for Elliptic-Type Equations in Three Dimensions' by Qiya Hu and Shaoliang Hu ([12]). We commence by giving a description of the main goal of the paper.

6.2.1 Goal of Paper 1

This research paper concerns the construction of a preconditioner for the systems of linear equations which are obtained from overlapping domain decomposition methods for three dimensional elliptic PDEs. The main focus is on PDEs with strongly discontinuous coefficients (an example will be given in Theorem 6.10). These discontinuities arise for example from modelling physical processes where different media are present. The important property of this new preconditioner is that its construction and efficiency are independent of the form in which the PDE is considered.

6.2.2 Results Based on the edge and face lemma

The results from this paper which rely on the edge and face lemma are Lemma 4.7 which uses our Lemma 5.11 in its proof and Theorem 4.2 which uses Lemma 4.7 from [12]. Before we state this result we introduce some basic concepts and explain a certain geometric construction which is needed.

Definition 6.1 (Course Partition) (Section 2.2 in [12]) Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2, 3$) be a polyhedral domain and let $T_d := \{\Omega_k\}_{k=1}^N$ be a family of tetrahedra (n = 3) (or hexahedra) or polygons (n = 2) in $\Omega$ of diameter $d \in (0, 1)$, for some $N \in \mathbb{N}_0$, such that the following holds:

- $\Omega = \bigcup_{k=1}^N \Omega_k$,
- $\Omega_i \cap \Omega_j = \emptyset$, $\forall i \neq j$,
- If $i \neq j$ and $\partial \Omega_i \cap \partial \Omega_j \neq \emptyset$, then $\partial \Omega_i \cap \partial \Omega_j$ is a common face, edge or vertex of $\Omega_i$ and $\Omega_j$.

Then $T_d$ is called a course partition of $\Omega$.

Next we define the notation for the interfaces between the subdomains in a given course partition.

Definition 6.2 (Interfaces of $T_d$) (Section 2.2 in [12]) Let $\Omega \subseteq \mathbb{R}^n$ ($n = 2, 3$) be a polyhedral domain equipped with a course partition...
6. Applications in Recent Research

\[ T_d = \{ \Omega_k \}_{k=1}^N \] for some \( d \in (0, 1) \) and \( N \in \mathbb{N} \). We define the various interfaces between the elements in \( T_d \) as follows:

- (Mutual Interface) If \( \partial \Omega_i \cap \partial \Omega_j \) is a common face of \( \Omega_i \) and \( \Omega_j \), we set
  \[ \Gamma_{ij} := \partial \Omega_i \cap \partial \Omega_j, \quad \forall i, j = 1, \ldots, N; \]

- (Full Interface) \( \Gamma := \bigcup_{i,j=1}^N \Gamma_{ij}, \)

- (Interfaces on Subdomains) \( \Gamma_k := \Gamma \cap \partial \Omega_k, \quad \forall k = 1, \ldots, N. \)

Last, we need a mesh on each subdomain \( \Omega_k \) of the course partition to be able to apply the finite element method for each of these subdomains. Some regularity and a certain relation between the different meshes is assumed. For this we make the following definition.

**Definition 6.3 (Mesh on \( \Omega \))** (Section 2.2 in [12]) Let \( \Omega \subseteq \mathbb{R}^n \) \( (n = 2, 3) \) be a polyhedral domain equipped with a course partition \( T_d = \{ \Omega_k \}_{k=1}^N \) for some \( d \in (0, 1) \) and \( N \in \mathbb{N} \). Assume each \( \Omega_k \in T_d \) is equipped with a shape regular tetrahedral (or hexahedral) mesh \( \mathcal{T}^h_k \) and that the meshes match on each interface between the subdomains such that the union of all the meshes constitutes a mesh on \( \Omega \) which is quasi-uniform. We denote this mesh by \( \mathcal{T}^h \) where \( h \) is defined as
  \[ h := \max_{T \in \mathcal{T}^h} \text{diam} T. \]

Now that we have a mesh on \( \Omega \) we can define the space of finite element functions with respect to this mesh.

**Definition 6.4 (Finite element functions)** (Section 2.2 in [12]) Let \( \mathcal{T}^h \) be the mesh from Definition 6.3 and let \( \mathcal{H}_\Omega \) be some Hilbert space of functions on \( \Omega \). For each \( K \in \mathcal{T}^h \) we denote by \( R(K) \) the span of the set of basis functions on the element \( K \). The space of finite element functions with respect to \( \{ R(K) \}_{K \in \mathcal{T}^h} \) is then defined as
  \[ V^h(\Omega) := \{ v^h \in \mathcal{H}_\Omega \mid \forall K \in \mathcal{T}^h : v^h|_K \in R(K) \}. \]

This concludes our introduction of basic concepts and notation relating to domain decomposition methods. Next, we discuss a geometric construction which is made in [12].

Using a course partition, an auxiliary subdomain \( \Omega^\text{half}_v \) is associated to each vertex \( v \in N(T_d) \). We consciously choose to not use a bold \( v \) to avoid confusion between the notation for vertices and the notation for vector-valued functions \( v \) which is frequently used in papers. Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a course partition \( T_d \) for some \( d \in (0, 1) \). Then for each \( v \in \)
6.2. Paper 1: A Substructuring Preconditioner with Vertex-Related Interface Solvers for Elliptic-Type Equations in Three Dimensions

Figure 6.1: Example of Definitions 6.1, 6.2 and 6.3

\[ N(T_d) \text{ we associate an open set } \Omega_{v}^{\text{half}} \text{ whose "center" is } v \text{ and size is about } d. \]
If \( v \in \partial \Omega \), then \( \Omega_v^{\text{half}} \) is chosen as the part which is inside \( \Omega \). Moreover, we assume the following:

- Each \( \Omega_v^{\text{half}} \) is the union of some elements in \( T^h \);
- \( \Gamma = \bigcup_{v \in N_d} (\Omega_v^{\text{half}} \cap \Gamma) =: \bigcup_{v \in N_d} \Gamma_v^{\text{half}}. \)

Now that we have defined all the preliminary notation and concepts, we can describe the statement of Lemma 4.7 in [12]. This is a kind of face lemma but for specific faces constructed from the sets \( \Omega_v^{\text{half}} \).

**Lemma 6.5 (Face Inequality)** (Lemma 4.7 in ) Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a course partition \( T_d \) for some \( d \in (0, 1) \). Let \( \Omega_k \in T_d \) be any subdomain and \( F_v^{\text{in}} \subseteq \partial \Omega_k \) the set defined as

\[ F_v^{\text{in}} = F \cap \Gamma_v^{\text{half}}, \]

where \( F \) is any face of \( \partial \Omega_k \) with \( v \) as a vertex. Then

\[ \| f_{F_v^{\text{in}}}^h \|_{H^{1/2}_0(F_v^{\text{in}})} \lesssim \log(d/h) \| v^h \|_{H^1(\Omega_k)}, \quad \forall v^h \in V^h(\Omega_k). \]

**Proof** We will discuss the part of the proof which references the edge and/or face lemmas only because giving a detailed description of each proof would take us too far.
Define $F^{\text{aux}}_v$ as the square $SVIJ$ in Figure 6.2. In the last step of the proof, they claim that

$$
\int_{F^{\text{aux}}_v} \frac{|v^h(x)|^2}{d(x, \partial F^{\text{in}}_v)} \, ds(x) \lesssim \log(d/h)^2 \|v^h\|_{H^1(\Omega_k)}^2.
$$

It is claimed that the proof of this inequality follows from Lemma 4.10 in [23] which is precisely the face lemma.

Next we turn to the second result which uses the previous Lemma and the edge and face lemma. This is Theorem 4.2 from [12]. For the setting of this theorem we refer partly to [12] for a more detailed description of the problem setting. Before we state this theorem we need to fix some notation and define the preconditioner proposed in this paper. We start off by defining some more finite element spaces

**Definition 6.6 (Section 3.1 in [12]):** Let $V^h(\Omega)$ be as in Definition 6.4 and let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a real, symmetric, continuous and coercive bilinear form. Define for any vertex $v \in N(T_d)$ the set $\Omega_v$ as

$$
\Omega_v := \bigcup_{k \in \Lambda_v} \Omega_k,
$$

where

$$
\Lambda_v := \{k \in \{1, \ldots, N\} \mid \Omega_k \text{ has } v \text{ as a vertex}\}.
$$
Using this we the following finite element spaces:

\[ V_d(\Omega) := \{ v_d \in H_\Omega \mid \forall \Omega_k \in T_d : v_d|_{\Omega_k} \in R(\Omega_k) \}, \]
\[ V_0^h(\Omega_k) := \{ v^h \in V^h(\Omega) \mid v^h|_{\partial\Omega_k} = 0 \}, \]
\[ V_0^h(T_{v}) := \{ v^h \in V^h(\Omega) \mid \text{supp} v^h \subseteq \Gamma^\text{half}_v \}, \]
\[ V_\perp^h(\Omega_k) := \{ v^h \in V^h(\Omega) \mid \forall v^h \in V_0^h(\Omega_k) : a(v^h, w^h) = 0 \}, \]
\[ V_\perp^h(\Omega_v) := \{ v^h \in V_\perp^h(\Omega) \mid \gamma^0_{\Gamma^\text{half}}(v^h) \in V_0^h(\Gamma^\text{half}) \}, \]

where \( \gamma^0_{\Gamma^\text{half}} \) is the Dirichlet trace operator mapping into the domain \( \Gamma^\text{half}_v \).

Next we define an operator associated to the bilinear form \( a \). It is similar to Definition 3.2, but instead of the whole space we restrict \( a \) to a subspace of functions in \( V_\perp^h(\Omega_v) \).

**Definition 6.7 (Section 3.2 in [12])** Fix a vertex \( v \in N(T_d) \). Let \( B_v : V^h_\perp(\Omega_v) \to V^h_\perp(\Omega_v) \) be the unique symmetric positive definite operator satisfying

\[ \langle B_v v, w \rangle_{L^2(\Omega_v)} = a(v^\text{half}, w^\text{half}), \quad \forall v, w \in V^h_\perp(\Omega_v), \]

where \( v^\text{half} \in V^h_\perp(\Omega_v) \) is defined as the function which equals \( v \) on \( \Gamma^\text{half}_v \).

Now that we have defined the relevant finite element functions spaces we give the definition of the preconditioner proposed in [12].

**Definition 6.8 (Proposed Preconditioner) (Section 3.2 in [12])** Let \( Q_d : V^h(\Omega) \to V_d(\Omega), Q_k : V^h(\Omega) \to V_0^h(\Omega_k) \) and \( Q_v : V^h(\Omega) \to V^h_\perp(\Omega_v) \) be the \( L^2 \)-orthogonal projectors onto the given codomains. The proposed preconditioner for the operator \( A \) associated to \( a \) as in Definition 3.2 is defined as

\[ B^{-1} := A_d^{-1}Q_d + \sum_{k=1}^{N} A_k^{-1}Q_k + \sum_{v \in N(T_d)} B_v^{-1}Q_v, \]

where \( A_d \) and \( A_k \) are the operators associated to the bilinear form \( a \) restricted to \( V_d(\Omega) \) and \( V_0^h(\Omega_k) \).

**Definition 6.9 (Notation for Theorem 4.1) (Section 4.2 in [12])** Let \( \tilde{V}^h(\Omega) \) be a subspace of \( V^h(\Omega) \) and let \( m_0 := \dim V^h(\Omega) - \dim \tilde{V}^h(\Omega) \). Then we denote by \( \lambda_{m_0+1}(B^{-1}A) \) the minimal eigenvalue of the restriction of \( B^{-1}A \) to \( \tilde{V}^h(\Omega) \) and define \( \kappa_{m_0+1}(B^{-1}A) \) as the reduced condition number of \( B^{-1}A \) associated to the subspace \( \tilde{V}^h(\Omega) \). More precisely,

\[ \kappa_{m_0+1}(B^{-1}A) := \frac{\lambda_{\text{max}}(B^{-1}A)}{\lambda_{m_0+1}(B^{-1}A)}. \]

Here \( A \) is the operator associated to the variational problem as in Definition 3.2 and \( B \) the (new) proposed preconditioner for \( A \) as in Definition 6.8.
6. Applications in Recent Research

We will now state Theorem 4.2. This result is a bound for the condition number of $B^{-1}A$ for so-called linear elasticity problems.

**Theorem 6.10 (Linear Elasticity Problem) (Theorem 4.2 in [12])**

Let $u \in (C^1(\Omega))^3$ and consider the linear elasticity problem

$$
\begin{align*}
-\sum_{j=1}^3 \partial_j \sigma_{ij}(u) &= f_i, \quad \text{in } \Omega, i = 1, 2, 3, \\
\mathbf{u} &= 0, \quad \text{on } \partial \Omega,
\end{align*}
$$

where $(f_1 f_2 f_3)^T$ is an internal volume force, e.g. gravity and

$$
\sigma_{ij}(u) := \lambda \delta_{ij} \text{div } \mathbf{u} + 2\mu \varepsilon_{ij}(u)
$$

for some positive functions $\lambda, \mu$ called the Lamé parameters. The functions $\varepsilon_{ij}(u)$ defined as

$$
\varepsilon_{ij}(u) := \frac{1}{2}(\partial_i u_i + \partial_j u_j),
$$

are the elements of the linearized stress tensor. For the rest of the setting we refer to [12]. For this problem we have

$$
\kappa_{m_0+1}(B^{-1}A) \lesssim \log(1/d) \log^2(d/h).
$$

When the coefficients $\mu(x), \lambda(x)$ have no large jump across the interface $\Gamma$, or there is no cross-point in the distribution of the jumps of the coefficients, the factor $\log(1/d)$ in the above inequality can be removed.

**Proof** The parts of the proof which use the face and edge lemma and Lemma 6.5 in [12] are the following: First Lemma 6.5 is used to obtain

$$
\|I_0^0 \tilde{v}_H^F\|_{H^{1/2}(\partial \Omega)} \lesssim \log^2(d/h)\|\tilde{v}_H^F\|_{H^1(\Omega_h)},
$$

where $\tilde{v}_H^F \in V_h^H(\Omega_v)$ is some function constructed from $\tilde{v}_h \in \tilde{V}^h(\Omega)$ in a previous part of the proof but is unimportant for our exposition. For more detail on these function spaces we refer again to [12]. The second part of the proof which uses the results we proved is the following sequence of estimates which uses the the edge lemma (Lemma 5.10 in Chapter 5):

$$
\|I_0^0 \tilde{v}_H^F\|_{H^{1/2}(\partial \Omega)} \lesssim \|I_0^0 \tilde{v}_h\|_{H^{1/2}(\partial \Omega_h)} \lesssim \log^{1/2}(d/h)\|\tilde{v}_h\|_{H^1(\Omega_h)}. \quad \square
$$
6.3 Paper 2: Nonoverlapping Domain Decomposition Methods with a Simple Coarse Space for Elliptic Problems

In this section we elaborate on the paper 'Nonoverlapping Domain Decomposition Methods with a Simple Coarse Space for Elliptic Problems' by Qiya Hu, Shi Shu and Junxian Wang ([14]). The structure of our exposition will be the same as in the previous section.

6.3.1 Goal of Paper 2

This paper presents a new substructuring preconditioner for solving three-dimensional elliptic PDEs with strongly discontinuous coefficients. This is a variant of the substructuring preconditioner in [2]. In this paper they are not able to derive a good bound on the condition number of the preconditioned system. However, the convergence rate of the PCG method using this preconditioner is nearly optimal.

6.4 The (Preconditioned) Conjugate Gradient Method

Many of the papers which we discuss construct preconditioners for certain systems of linear equations which are then solved using the preconditioned conjugate gradient method. It is for this reason why we will first describe this algorithm in this section. We choose to do this now because in [14] an example is given of a problem in which the PCG method is applied.

We start with the conjugate gradient method (CG method) followed by the example in [14] and the preconditioned conjugate gradient method. The reason for this is simple, the intuition behind the CG method is quite easy to explain and the PCG method is just a variant of this method so it makes sense to follow this order.

6.4.1 The Conjugate Gradient (CG) Method

This section is based on Lecture 38 of [22] but we will generalize everything to the general setting of operators on inner product spaces. The conjugate gradient method is an iterative method for solving certain systems of linear equations. More precisely, the CG method is an algorithm to solve linear equations of the form: Find $v \in V$ such that

$$Av = b,$$  \hspace{1cm} (6.1)

where $A : V \rightarrow V$ is a symmetric and positive definite operator, $b \in \text{Im}(A)$ a vector and $(V, \langle \cdot , \cdot \rangle_V)$ a finite-dimensional real inner product space. The
Applications in Recent Research

philosophy of the conjugate gradient method is very simple. First we define
the norm with respect to which we will measure the approximation error.

**Definition 6.11 (A-norm)** (Equation (38.2) in [22]) Let \( A : V \to V \) be a
symmetric positive definite operator on a finite-dimensional real inner product
space \((V, \langle \cdot, \cdot \rangle_V)\). The \textbf{A-norm} \( \| \cdot \|_A \) is defined as
\[
\|v\|_A^2 := \langle Av, v \rangle_V, \quad \forall v \in V.
\]

Next we define a sequence of subspaces which will contain the approximations
obtained from the CG method.

**Definition 6.12 (Krylov Subspaces)** (Equation 38.1 in [22]) Let \( A : V \to V \)
be a symmetric operator on a finite-dimensional vector space \( V \) and \( b \in V \).
The \textit{n-th Krylov subspace generated by} \( b, K_n \subseteq V \), is defined as
\[
K_n := \text{span}\{ b, Ab, \cdots, A^{n-1}b \}.
\]

Now that we have introduced the relevant notation and concepts, we explain
how the CG method works for each iteration. The idea is surprisingly simple
and is as follows: Given \( v^* \in V \), the exact solution to (6.1), then the CG
method constructs in step \( n \) the vector \( v_n \in K_n \) such that
\[
\|v^* - v_n\|_A = \min_{w \in K_n} \|v^* - w\|_A.
\]

In full detail this becomes:

**Algorithm 1:** The CG Method (Algorithm 38.1 in [22])

\[
\begin{align*}
v_0 &= 0, r_0 = b, p_0 = r_0; \\
\text{for } n \in \mathbb{N} \text{ do} \\
\alpha_n &:= \frac{\|r_{n-1}\|_V}{\|p_{n-1}\|_A}; \\
v_n &:= v_{n-1} + \alpha_n p_{n-1}; \\
r_n &:= r_{n-1} - \alpha_n Ap_{n-1}; \\
\beta_n &:= \frac{\|r_n\|_V}{\|r_{n-1}\|}; \\
p_n &:= r_n + \beta_n p_{n-1}
\end{align*}
\]

The rate of convergence of this algorithm strongly depends on the condition
number of \( A \). This is summarized in the following theorem.

**Theorem 6.13** (Theorem 38.5 in [22]) Let \( A : V \to V \) be a symmetric posi-
tive definite operator on a finite-dimensional real inner product space \((V, \langle \cdot, \cdot \rangle_V)\),
6.4. The (Preconditioned) Conjugate Gradient Method

K_0

K_1

v_0 = 0

v_1

v^*

\min_{w \in K_1} \|v^* - w\|_A

Figure 6.3: Examples of the approximations v_0, v_1 and the Krylov subspaces K_0, K_1

\|v^* - v_n\|_A \leq \frac{2}{\left(\sqrt{\kappa + 1} - 1\right)^n + \left(\sqrt{\kappa + 1} + 1\right)^n} \leq 2 \left(\frac{\sqrt{\kappa - 1}}{\sqrt{\kappa + 1}}\right)^n.

This indeed shows that the condition number of A is extremely important to guarantee a fast convergence rate. Heuristically we can see this as follows. If \( \kappa = 1 + \varepsilon \) for some \( \varepsilon \ll 1 \), then

\[ \frac{\sqrt{\kappa - 1}}{\sqrt{\kappa + 1}} = \frac{\varepsilon}{4} + O(\varepsilon^2), \quad \varepsilon \to 0^+. \]

By the previous theorem we then obtain

\[ \|v^* - v_n\|_A \leq 2 \left(\frac{\sqrt{\kappa - 1}}{\sqrt{\kappa + 1}}\right)^n \|v^*\|_A = 2^{1-2n} \varepsilon^n \|v^*\|_A + O(\varepsilon^{n+1}), \quad \varepsilon \to 0^+, \]

which shows that the guaranteed convergence speed is fast if \( \varepsilon \) is very small or equivalently, if \( \kappa \) is close to 1. It is of course possible that the convergence is faster in practice because we only have computed an upper bound for the convergence rate. To speed up this rate of convergence, the preconditioned conjugate gradient method is used which constructs a preconditioner for A and applies the CG method to the preconditioned linear system.

6.4.2 The Preconditioned Conjugate Gradient (PCG) Method

We base our discussion on a model elliptic boundary value problem which illustrates the need for the PCG method over the CG method. This example can be found in Section 2 of [14]. Let \( \Omega \subseteq \mathbb{R}^n \) be a polyhedral domain equipped with a shape regular tetrahedral (or hexahedral) mesh \( \{T^h\} \) such that h is the
maximum diameter of the elements in $T^h$. The space of global shape functions will be $S_{1,0}^0(T^h) \subseteq H_0^1(\Omega)$. Now consider the following model problem:

$$
\begin{aligned}
&-\text{div}(\omega \nabla u) = f, \quad \text{in } \Omega, \\
u = 0, &\quad \text{on } \partial \Omega,
\end{aligned}
$$

where $\omega \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. The standard finite element approximation problem is then given by: Find $u^h \in S_{1,0}^0(T^h)$ such that

$$
a(u^h, v^h) = \langle f, v^h \rangle_{L^2(\Omega)}, \quad \forall v^h \in S_{1,0}^0(T^h),
$$

where $f \in L^2(\Omega)$ and $a : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ is the bounded, symmetric and coercive bilinear form given by

$$
a(u, v) := \int_\Omega \omega(x) \nabla v(x) \cdot \nabla w(x) \, dx, \quad \forall v, w \in H_0^1(\Omega).
$$

Denote by $A : S_{1,0}^0(T^h) \to S_{1,0}^0(T^h)$ the operator associated to the restriction of $a$ to $S_{1,0}^0(T^h) \times S_{1,0}^0(T^h)$ as in Definition 3.1. Then we can as usual write the above variational problem in operator form: Find $u^h \in S_{1,0}^0(T^h)$ such that

$$
Au^h = f^h,
$$

where $f^h$ is the $L^2$-orthogonal projection of $f$ onto $S_{1,0}^0(T^h)$. In general dim $S_{1,0}^0(T^h)$ is very large and therefore direct factorization cannot be used. Hence we are left with iterative methods of which the CG method is one. For the condition number of $A$ it is known that

$$
\kappa(A) \lesssim \frac{1}{h^2} \max_{\Omega} \omega \min_{\Omega} \omega,
$$

which is very large for small $h$ and large jumps in $\omega$. The PCG method constructs a solution to the equivalent system

$$
BAu^h = Bf^h, \quad (6.2)
$$

by applying the CG method to this system of equations. Here $B : S_{1,0}^0(T^h) \to S_{1,0}^0(T^h)$ is a preconditioner for $A$. Similar to the CG method we get the following result for the convergence of the PCG method.

**Theorem 6.14** (Equation (3.2) in [14]) Let $u^h$ be the exact solution of (6.2) and let $u_n$ be the $n$-th iterate of the CG method with starting value $u_0$ for equation (6.2). Then

$$
\|u^h - u_n\|_A \leq 2 \left( \frac{\sqrt{\kappa(BA)} - 1}{\sqrt{\kappa(BA)} + 1} \right)^n \|u^h - u_0\|_A.
$$
6.4. The (Preconditioned) Conjugate Gradient Method

**Remark.** This result can easily be generalized to any preconditioned system of linear equations which needn’t be obtained from a finite element discretization of an elliptic PDE.

The same calculation can be made as we did at the end of the previous section from which we can conclude that to guarantee fast convergence, we want that \( \kappa(BA) \) is as close to 1 as possible. Hence finding a good preconditioner for \( A \) is essential for the efficiency of finite element and domain decomposition methods.

### 6.4.3 Results Based on the edge and face lemma

The result in [14] which uses the edge and/or face lemma is Theorem 4.1 which is an estimate for the reduced condition number of the proposed additive preconditioner. We note beforehand that the proposed preconditioner and Theorem 4.1 are very similar to what we discussed for the linear elasticity problem in [12]. The setting for this paper is also very similar to that of [12]. We will take the setting to be as in in Definitions 6.2, 6.3 and 6.4. For \( R(K) \) the linear functions on \( K \) are chosen such that the functions in \( V^h(\Omega) \) are globally continuous. To keep notation consistent, we will use \( S^0_{1,0}(T^h) \) instead of \( V^h(\Omega) \). Last, we assume that \( \omega \) is piece-wise constant and constant on each subdomain of the coarse partition \( T_d \). We start by defining the basket-sets corresponding to the coarse partition \( T_d \).

**Definition 6.15 (Basket-Sets)** (Section 4.1 in [14]) Let \( T_d \) be a course partition of a polyhedral domain \( \Omega \subseteq \mathbb{R}^3 \) consisting of tetrahedra (or hexahedra). The **basket-set** \( W_k \) of \( \Omega_k \in T_d \) is defined as

\[
W_k := \left( \bigcup_{e_k \in E(\Omega_k)} e_k \right) \cup \left( \bigcup_{v_k \in V(\Omega_k)} v_k \right),
\]

and we denote by \( \mathcal{W} \) the union of all these basket-sets, i.e.

\[
\mathcal{W} := \bigcup_{k=1}^{N} W_k.
\]

Here \( E(\Omega_k) \) denotes the set of edges of \( \Omega_k \) and \( V(\Omega_k) \) the set of vertices of \( \Omega_k \).

**Remark.** We chose on purpose a different notation to denote the vertices of \( \Omega_k \) to distinguish between the nodes of the mesh on \( \Omega_k \) and the vertices of the domain \( \Omega_k \) itself.

Next we need some more function spaces which are subspaces of \( S^0_{1,0}(T^h) \) before we can describe the space decomposition.
Figure 6.4: Example of two subdomains $\Omega_1$ and $\Omega_2$ together with the associates basket-sets $W_1$ and $W_2$

**Definition 6.16 (wire-basket subspace)** Let $\mathcal{W}$ be the union of all basket-sets as in Definition 6.15. The **wire-basket subspace** of $S_{1,0}^0(\mathcal{T}^h)$ is defined as

$$S_{1,0}^W(\mathcal{T}^h) := \{ v^h \in S_{1,0}^0(\mathcal{T}^h) \mid \forall v \in \mathcal{V}(\Omega) \setminus \mathcal{V}(\mathcal{W}) : v^h(v) = 0 \}.$$  

Next we define the finite element functions which are associated to the coarse partition $T_d$ as

$$S_{1,0}^0(T_d) := \{ v_d \in C(\Omega) \mid v_d|_{\partial\Omega} = 0, \forall \Omega_k \in T_d : v_d|_{\Omega_k} \in P_1(\Omega_k), \}.$$  

Last, for two neighbouring domains $\Omega_i$ and $\Omega_j$, define

$$\Omega_{ij} := \Omega_i \cup \Omega_j \cup \Gamma_{ij}$$

and set

$$S_{1,0}^0(\mathcal{T}^h)_{ij} := \{ v^h \in S_{1,0}^0(\mathcal{T}^h) \mid \text{supp } v^h \subseteq \Omega_{ij} \}.$$  

From the above definitions we evidently get the following decomposition of $S_{1,0}^0(\mathcal{T}^h)$:

$$S_{1,0}^0(\mathcal{T}^h) = S_{1,0}^0(T_d) + S_{1,0}^W(\mathcal{T}^h) + \sum_{\Gamma_{ij}} S_{1,0}^0(\mathcal{T}^h)_{ij}.$$  

Using this decomposition, we define some operators with certain spectral properties.
Definition 6.17 (Section 4.2 in [14]) Let $B_d : S^0_{1,0}(T_d) \to S^0_{1,0}(T_d)$ and for any interface $\Gamma_{ij}$, $B_{ij} : S^0_{1,0}(T^h)_{ij} \to S^0_{1,0}(T^h)_{ij}$ be symmetric and positive definite operators which are spectrally equivalent to the restrictions of $A$ to $V_d(\Omega)$ and $V^0_h(\Omega_{ij})$ respectively. Here $A$ is the operator associated to the bilinear form

$$a(u^h, v^h) = \int_{\Omega} \omega(x)\nabla u^h(x) \cdot \nabla v^h(x) \, dx, \quad \forall u^h, v^h \in S^0_{1,0}(T^h).$$

More specifically $B_d$ and $B_{ij}$ satisfy

$$\langle B_d v^h, v^h \rangle_{L^2(\Omega)} \approx \int_{\Omega} \omega(x)|\nabla v^h(x)|^2 \, dx, \quad \forall v^h \in S^0_{1,0}(T_d)$$

and

$$\langle B_{ij} v^h, v^h \rangle_{L^2(\Omega)} \approx \int_{\Omega_i} |\nabla v^h(x)|^2 \, dx + \int_{\Omega_j} |\nabla v^h(x)|^2 \, dx, \quad \forall v^h \in S^0_{1,0}(T^h)_{ij}.$$

Definition 6.18 (Jacobi-Smoother) The Jacobi-Smoother $B^{-1}_W : S^0_{1,0}(T^h) \to S^0_{1,0}(T^h)$ is the operator defined by

$$B^{-1}_W(v^h) := \sum_{p \in N(T^h) \cap W} \frac{\langle v^h, \varphi_p \rangle_{L^2(\Omega)} \varphi_p}{a(\varphi_p, \varphi_p)} \varphi_p, \quad \forall v^h \in S^0_{1,0}(T^h),$$

where $\varphi_p$ is the nodal basis function associated to $p$.

Using the above space decomposition and the above defined operators we define the proposed additive preconditioner.

Definition 6.19 (Additive Preconditioner) (Equation (4.4) in [14]) The additive preconditioner $B$ for $A$ is defined as

$$B := B^{-1}_d Q_d + B^{-1}_W Q_W + \sum_{\Gamma_{ij}} B^{-1}_{ij} Q_{ij},$$

where $Q_d, Q_W$ and $Q_{ij}$ are the $L^2$-orthogonal projectors onto the subspaces $S^0_{1,0}(T_d), S^0_{1,0}(T^h)$ and $S^0_{1,0}(T^h)_{ij}$ respectively.

Now that we have defined the additive preconditioner we finally turn to Theorem 4.1 in [14].

Theorem 6.20 (Theorem 4.1 in [14]) Let $B$ be the preconditioner as in Definition 6.19. Then

$$\lambda_{m_0+1}(BA) \gtrsim \frac{1}{\log(1/d) \log^2(d/h)} \quad \text{and} \quad \kappa_{m_0+1}(BA) \lesssim \log(1/d) \log^2(d/h).$$

When the coefficient $\omega$ has no large jump across the interface $\Gamma$, or there is no cross-point in the distribution of the jumps of the coefficient, the factor $\log(1/d)$ in the above inequalities can be removed.
6. Applications in Recent Research

**Proof** The face lemma is used to prove the estimate

\[
(\omega_i + \omega_j) |I_{\Gamma_{ij}}^0 \tilde{v}^H|_{H^{1/2}(\Gamma_{ij})}^2 \lesssim \log^2 (d/h) (\omega_i \| \tilde{v}^H \|_{H^{1/2}(\partial \Omega_i)}^2 + \omega_j \| \tilde{v}^H \|_{H^{1/2}(\partial \Omega_j)}^2),
\]

where \( \tilde{v}^H_{ij} \in \mathcal{S}_{1,0}^H(\mathcal{T}^h)_{ij} \) which is some subspace of \( \mathcal{S}_{1,0}^0(\mathcal{T}^h) \) which is unimportant for our discussion. \qed
6.5. Paper 3: Substructuring Preconditioners with a Simple Coarse Space for 2-D 3-T Radiation Diffusion Equations

In this section we elaborate on the paper ‘Substructuring Preconditioners with a Simple Coarse Space for 2-D 3-T Radiation Diffusion Equations’ by Xiaoqiang Yue, Shi Shu, Junxian Wang and Zhiyan Zhou ([24]). The structure of our exposition will be the same as in the previous section.

6.5.1 Goal of Paper 3

This paper presents two nonoverlapping domain decomposition preconditioners for the preserving-symmetry finite volume element (SFVE) scheme for two-dimensional three-temperature radiation diffusion equations with strongly discontinuous coefficients. An almost optimal estimate of the condition numbers is proved for the preconditioned systems. The theoretical results are tested numerically to show that they hold in practice.

6.5.2 The SFVE Method for Radiation-Diffusion Equations

This section is largely based on Section 2 in [24]. The results in [24] involve solving 2-D 3-T radiation diffusion equations of the form

\[
\begin{align*}
\partial_t E_R - \text{div} \left( \frac{c \lambda(E_R)}{k_R} \nabla E_R \right) &= c k_p (E_p - E_R), \\
\rho c_E \partial_t T_E - \text{div} \left( \kappa_E T_E^{2.5} \nabla T_E \right) &= -c k_p (E_p - E_r) + w_{EI} (T_I - T_E), \\
\rho c_I \partial_t T_I - \text{div} \left( \kappa_I T_I^{2.5} \nabla T_I \right) &= -w_{EI} (T_I - T_E),
\end{align*}
\]

where \(E_R\) is the spectral radiation energy density, \(c\) is the speed of light, \(\lambda(E_R)\) is a non-constant linear limiter, \(k_R\) and \(k_P\) are Rosseland and Planck mean absorption coefficients, \(E_p\) is the electron scattering energy density, \(\rho\) is the medium density updated in the hydrodynamic process, \(c_\alpha\) and \(T_\alpha (\alpha = E, I)\) are specific heats and temperatures of an electron and an ion, \(\kappa_E\) and \(\kappa_I\) are constants, and \(w_{EI}\) is the electron-ion energy exchange coefficient.

Applying the Eulerian and frozen-in coefficient methods for the above system gives

\[
\begin{align*}
-\text{div} (\omega R \nabla T_R) + P_{RR} T_R + P_{RE} T_E &= S_R, \\
-\text{div} (\omega E \nabla T_E) + P_{RE} T_R + P_{EE} T_E + P_{EI} T_I &= S_E \tag{6.3}, \\
-\text{div} (\omega I \nabla T_I) + P_{EI} T_E + P_{II} T_I &= S_I.
\end{align*}
\]

The homogeneous Neumann boundary conditions read

\[
(\omega_\alpha \nabla T_\alpha) \cdot n = 0, \quad \text{on } \partial\Omega, \alpha = R, E, I,
\]
where \( \mathbf{n} \) is the outer unit normal to \( \partial \Omega \). The constants in the above equations are defined as

\[
\begin{align*}
P_{RR} &= 4aT_R^3 + w_{ER} \Delta t, \quad P_{RE} = -w_{ER} \Delta t, \\
P_{EE} &= \rho c_E + w_{EI} \Delta t + w_{ER} \Delta t, \\
P_{EI} &= -w_{EI} \Delta t, \quad P_{II} = \rho c_I + w_{EI} \Delta t,
\end{align*}
\]

where \( T_R \) is the electron-photon energy exchange coefficient.

\[
\omega_{R} = \kappa_{R}T_R^{\beta}, \\
\omega_{E} = \kappa_{E}T_E^{5}, \\
\omega_{I} = \kappa_{I}T_I^{2.5},
\]

where \( T_R \) is the previous approximation of \( T_{\alpha} \), \( \Delta t \) is the current time step size and \( w_{ER} \) is the electron-photon energy exchange coefficient.

Next we assume that \( \Omega \subseteq \mathbb{R}^2 \) is polyhedral and equipped with a coarse partition of triangles \( \mathcal{T}_d := \{ \Omega_k \}_{k=1}^N \) of diameter \( d \). We use the same notation for the interfaces as in Definition 6.2. Moreover we assume that \( \omega_{\alpha} \), for \( \alpha = R, E, I \), is constant on each subdomain, i.e.

\[
\omega_{\alpha}(x) = \omega_{\alpha}^\ell > 0, \quad \forall x \in \Omega_\ell,
\]

for all \( \ell = 1, \ldots, N \) and for \( \alpha = R, E, I \). We also assume that there is a mesh on each subdomain as in Definition 6.3 but without the assumption of quasi-uniformity of the global mesh on \( \Omega \). Next we define the relevant global finite element function spaces

**Definition 6.21** (Section 2.1 in [24]) For the setting described above, we define the finite element spaces \( S_{1,3}^0(T^h) \) and \( S_{1,3}^0(T^h)_{ij} \) as

\[
S_{1,3}^0(T^h) := \{ u^h = (u^h_R, u^h_E, u^h_I)^T \in (C(\bar{\Omega}))^3 \mid \forall T^h \in T^h : u^h_\alpha \in P_1(T^h), \alpha = R, E, I \},
\]

\[
S_{1,3}^0(T^h)_{ij} := \{ u^h_{ij} = (u^h_{R,ij}, u^h_{E,ij}, u^h_{I,ij})^T \in (C(\bar{\Omega}))^3 \mid \forall T^d \in T_d : u^d_\alpha \in P_1(T_d), \alpha = R, E, I \},
\]

and

\[
S_{1,3}^0(T^h)_{ij} := \{ u^h_{ij} = (u^h_{R,ij}, u^h_{E,ij}, u^h_{I,ij})^T \in (C(\bar{\Omega}))^3 \mid \text{supp} u^h_{\alpha,ij} \subseteq \Omega_{ij}, \alpha = R, E, I \},
\]

where \( \Omega_{ij} := \Omega_i \cup \Omega_j \cup \Gamma_{ij} \), i.e. it is the subdomains \( \Omega_i \) and \( \Omega_j \) together with the associated interface \( \Gamma_{ij} \).

Now we can describe the variational formulation of (6.3). This is done by using Green’s formula for \( S_{1,3}^0(T^h) \), using \( u^h := (T^h_R, T^h_E, T^h_I) \in S_{1,3}^0(T^h) \) as an approximation for \( u := (T_R, T_E, T_I) \). This gives us

\[
a(u^h, v^h) = F(v^h), \quad \forall v^h = (v^h_R, v^h_E, v^h_I)^T \in S_{1,3}^0(T^h),
\]

where the associated bilinear form \( a \) and bounded linear operator \( F \) (as in Definition 3.1) are given by

\[
a(u^h, v^h) := \sum_{n \in N(T^h)} \left( \sum_{\alpha} \int_{b_n} (\omega_{\alpha}^n \nabla T^h_\alpha) \cdot \nabla u^h_\alpha + \langle J^h_R P_{\alpha} s^h T^h_\alpha, J^h_R s^h T^h_\alpha \rangle_{L^2(b_n)} \right) d\mathbf{x}
\]

(6.4)
6.5. Paper 3: Substructuring Preconditioners with a Simple Coarse Space for 2-D 3-T Radiation Diffusion Equations

\[ +\langle P_{E}T_{E}^{h},v_{E}^{h}\rangle_{L^{2}(b_{n})} + \langle P_{T}T_{E}^{h},v_{E}^{h}\rangle_{L^{2}(b_{n})} + \langle P_{E}T_{R}^{h},v_{R}^{h}\rangle_{L^{2}(b_{n})} + \langle P_{E}T_{E}^{h},v_{I}^{h}\rangle_{L^{2}(b_{n})} \]

and

\[ F(v^{h}) := \sum_{\alpha} \langle S_{\alpha},v_{\alpha}^{h}\rangle_{L^{2}(\Omega)} \]

for any \( u^{h}, v^{h} \in \mathcal{S}_{0,13}(\mathcal{T}^{h}) \). Here

\[ \omega_{\alpha}^{h}(x) = \omega_{\alpha}(c_{i}), \quad \forall x \in \Delta_{i}X_{\ell_{i}}X_{\ell_{i+1}}, i = 1, \ldots, 6, \]

\( b_{n} \) is the dual element associated to \( n \), \( \Delta_{i} := \Delta_{i}X_{\ell_{i}}X_{\ell_{i+1}} \) (triangle with vertices \( X_{\ell_{i}}X_{\ell_{i}} \) and \( X_{\ell_{i+1}} \)), \( D_{i} := b_{n} \cap \Delta_{i} \) and \( c_{i} \) is the centroid of \( \Delta_{i} \) for \( i \in \{1, \ldots, 6\} \). The dual element \( b_{n} \) is also called the control volume as is customary for finite volume schemes. We refer to Figure 6.5 for an illustration of this construction. The resulting linear system

\[ A^{h}u^{h} = f^{h}, \]

is called the SFVE linear system of (6.3).

\[ \begin{align*}
  X_{\ell_{1}} & \quad \Delta_{1}c_{1} \\
  M_{1} & \quad M_{2} \\
  D_{1} & \quad n
\end{align*} \]

Figure 6.5: Illustration of the dual element \( b_{n} \) (in grey) associated to the node \( n \)

6.5.3 Results Based on the edge and face lemma

Now we elaborate on the results in this paper which follow from the face lemma because the edge lemma is not used in this paper. The first result is Lemma 2.1 from [24] which is a direct reference to the face lemma in [23]. The second
and final result which uses the results we proved is Theorem 3.1 which is a bound on the condition number of the preconditioned system for the additive preconditioner. First, using the finite element function spaces defined in the previous section a space decomposition is obtained:

\[ S_{1,3}^0(T^h) = S_{1,3}^0(T_d) + \sum_{i,j=1}^N S_{1,3}^0(T_{hij}). \]

We associate to this decomposition the orthogonal projectors (with respect to the \( L^2 \)-inner product)

\[ Q_d : S_{1,3}^0(T^h) \to S_{1,3}^0(T_d), Q_{ij} : S_{1,3}^0(T^h) \to S_{1,3}^0(T_{hij}), \]

i.e.

\[ (Q_d u^h, v^h)_{(L^2(\Omega))^3} = (u^h, v^h)_{(L^2(\Omega))^3}, \quad \forall u^h \in S_{1,3}^0(T^h), v^h \in S_{1,3}^0(T_d), \]

\[ (Q_{ij} u^h, v^h)_{(L^2(\Omega))^3} = (u^h, v^h)_{(L^2(\Omega))^3}, \quad \forall u^h \in S_{1,3}^0(T^h), v^h \in S_{1,3}^0(T_{hij}). \]

Next we define weighted \( L^2 \)-spaces which are used in the proof of Theorem 3.1 in [24].

**Definition 6.22 (Vectorially Weighted \( L^2 \)– and \( H^1 \)-Spaces)** *(Section 2.2 in [24])* Let \( \Omega \subseteq \mathbb{R}^2 \) and let \( \omega : \Omega \to \mathbb{R}^3 \) be a piece-wise constant function (taking the role of \( \omega \) in the previous section). Then the \( \omega \)-weighted \( L^2 \)-inner product is defined as

\[ \langle u, v \rangle_{L_\omega^2(\Omega)} := \sum_{n \in N(T^h)} \sum_{\alpha} \int_{b_n} \omega_{\alpha}^{\lambda}(x) u_{\alpha}(x)v_{\alpha}(x) \, dx, \quad \forall u, v \in (L^2(\Omega))^3, \]

and the \( \omega \)-weighted \( H^1 \)-inner product as

\[ \langle u, v \rangle_{H_\omega^1(\Omega)} := \sum_{n \in N(T^h)} \sum_{\alpha} \int_{b_n} \omega_{\alpha}^{\lambda}(x) \nabla u_{\alpha}(x) \cdot \nabla v_{\alpha}(x) \, dx, \quad \forall u, v \in (H^1(\Omega))^3. \]

The \( \omega \)-weighted \( H^1 \)-norm is defined as

\[ \| v \|_{H_\omega^1(\Omega)}^2 := \| v_{H_\omega^1(\Omega)}^2 + \| v \|^2_{L_\omega^2(\Omega)}, \quad \forall u, v \in (H^1(\Omega))^3, \]

where \( | \cdot |_{H_\omega^1(\Omega)} \) is the semi-norm induced by \( \langle \cdot, \cdot \rangle_{H_\omega^1(\Omega)} \).

**Remark.** It is worthwhile to note that \( \langle \cdot, \cdot \rangle_{H_\omega^1(\Omega)} \) is not a scalar product because it is not definite and therefore only induces a semi-norm. Indeed, if \( \langle v, v \rangle_{H_\omega^1(\Omega)} = 0 \) for some \( v \in (H^1(\Omega))^3 \) then \( v \) has to be only locally constant which is obviously not the same as being 0.

We need one last function space which is the vectorial analogue of the Lions-Magenes space \( H_{00}^{1/2} \) defined in Chapter 1.
We only need to make one more definition before we state Theorem 3.1 and Theorem 3.1.

This operator is called the additive preconditioner for $A$ (Theorem 3.1 in [24]). Define the operator

\[ \forall u \in H^{1/2}(\partial \Omega), \]

\[ \langle A u, \psi \rangle := A u, \psi \rangle := (H^{1/2}(\partial \Omega))^3, \]

where $\tilde{u}$ is the extension by zero of $u$ into $\partial \Omega$. This set is equipped with the norm

\[ \|u\|^2_{H^{1/2}(\Gamma)} := \|u\|^2_{(L^2(\Gamma))^3} + \int_{\Gamma} \int_{\Gamma} \frac{|u(x) - u(y)|^2}{|x - y|^2} \, dx \, dy + \int_{\Gamma} d(x, \partial \Gamma) \, dx, \]

for any $u \in H^{1/2}(\Gamma)$.

We only need to make one more definition before we state Theorem 3.1 and discuss its proof. What we still need are the operators associated to the bilinear for $a$ in (6.4) for the various finite element subspaces we introduced.

**Definition 6.24** (Section 2.2 in [24]) Let $a$ be as in (6.4). Then we define the symmetric positive definite operators $A_h : S_{1,3}^0(\mathcal{T}^h) \to S_{1,3}^0(\mathcal{T}^h)$, $A_d : S_{1,3}^0(\mathcal{T}_d) \to S_{1,3}^0(\mathcal{T}_d)$ and $A_{ij} : S_{1,3}^0(\mathcal{T}^h)_{ij} \to S_{1,3}^0(\mathcal{T}^h)_{ij}$ as the unique operators satisfying

\[ \langle A^h u^h, v^h \rangle_{(L^2(\Omega))^3} = A(u^h, v^h), \quad \forall u^h, v^h \in S_{1,3}^0(\mathcal{T}^h), \]

\[ \langle A_d u_d, v_d \rangle_{(L^2(\Omega))^3} = A(u_d, v_d), \quad \forall u_d, v_d \in S_{1,3}^0(\mathcal{T}_d), \]

\[ \langle A_{ij} u_{ij}, v_{ij} \rangle_{(L^2(\Omega))^3} = A(u_{ij}, v_{ij}), \quad \forall u_{ij}, v_{ij} \in S_{1,3}^0(\mathcal{T}^h)_{ij}. \]

Now that we have introduced the relevant notation and background information we proceed with the statement of and the discussion of the proof of Theorem 3.1.

**Theorem 6.25** (Condition Number of The Additive Preconditioner) (Theorem 3.1 in [24]) Define the operator $B_h^n$ as

\[ B_h^n := A_h^{-1} Q_d + \sum_{ij} A_{ij}^{-1} Q_{ij}. \]

This operator is called the additive preconditioner for $A^h$ and satisfies the estimate

\[ \kappa(B_h^n A^h) \lesssim \left(1 + \log \frac{d}{h}\right)^3. \]

**Proof** The part of the proof which uses the face lemma is in the following estimate:

\[ \|\omega_{ij}^1 \tilde{v}_{ij}^h\|^2_{H^{1/2}(\Gamma_{ij})} + \|\omega_{ij}^1 \tilde{v}_{ij}^h\|^2_{H^{1/2}(\Gamma_{ij})} \lesssim \left(\log \frac{d}{h}\right)^2 (\|\omega_{ij}^1 \tilde{v}_{ij}^h\|^2_{H^{1/2}(\partial \Omega_{ij})}). \]
which is indeed just the face lemma but in vectorial form. Here $\mathbf{I}_{ij}^0 \tilde{\mathbf{v}}^h$ is the vectorial restriction operator which is just Definition 5.9 for each component of $\tilde{\mathbf{v}}^h$. The function $\tilde{\mathbf{v}}^h$ is some constructed function in $S_{1,3}^0(T^h)$ which is unimportant for our discussion of this part of the proof.

**Remark.** There is a typo in the proof of this theorem in [24]. In the second term on the left-hand side of the above equation it should be $\omega_j$ as we wrote instead of $\kappa_j$ as in [24]. Another remark is on the regularity of the mesh on $\Omega$. The face lemma is applied, but the mesh to which it is applied is only shape regular which is not enough for the face lemma to hold. It is possible that there is a reason why this doesn’t pose any issues but this is unclear.

In this section we elaborate on the paper 'A Mortar Edge Element Method with Nearly Optimal Convergence for Three-Dimensional Maxwell’s Equations' by Qiya Hu, Shi Shu and Jun Zou ([15]). The structure of our exposition will be similar to the sections before.

6.6.1 Goal of Paper 4

This paper is concerned with mortar edge element methods for solving three dimensional Maxwell’s equations. The system of PDEs that has to be solved is the following three dimensional system:

\[
\begin{align*}
\text{curl}(\alpha \text{curl } u) + \beta u &= f, & \text{in } \Omega, \\
u \times n &= 0, & \text{on } \partial \Omega,
\end{align*}
\]  

(6.5)

where \( \Omega \subseteq \mathbb{R}^3 \) is a (not necessarily convex) polyhedral domain, \( n \) is the outward unit normal to \( \partial \Omega \) and \( \alpha, \beta \in L^\infty(\Omega) \). To numerically compute the solutions of Maxwell’s equations the above system has to be solved multiple times.

In this paper a new type of Lagrange multiplier space is introduced and the mortar edge element method is shown to possess a nearly optimal convergence rate under certain mild assumptions on the meshes. Last, a generalized edge element interpolant is introduced which is used to prove the optimal convergence of the mortar edge element method. All these results are backed up by numerical experiments.

6.6.2 Maxwell’s Equations

In this section we motivate why (6.5) has to be solved when numerically solving Maxwell’s equations. First we state Maxwell’s equations.

Definition 6.26 (Maxwell’s Equations) (Introduction of [4]) Let \( \Omega \subseteq \mathbb{R}^3 \) be a domain and \( T > 0 \). Then Maxwell’s equations are the coupled system of PDEs given by

\[
\begin{align*}
\varepsilon \frac{\partial E}{\partial t} + \sigma E - \text{curl } H &= J, & \text{in } \Omega \times (0, T), \\
\mu \frac{\partial H}{\partial t} + \text{curl } E &= 0, & \text{in } \Omega \times (0, T),
\end{align*}
\]

where \( \varepsilon \in L^\infty_{+}(\Omega) \) is the dielectric constant and \( \sigma \in L^\infty_{+}(\Omega) \) the conductivity of the medium, while \( \mu \in L^\infty_{+}(\Omega) \) is the magnetic permeability of the material in \( \Omega \) and \( J \) is the applied current density.
Remark. Under the assumption that $\partial \Omega$ is a perfect conductor, the boundary condition
$$E \times n = 0, \quad \text{on } \partial \Omega \times (0, T)$$
is added.

The PDE in (6.5) is derived from Maxwell’s equations as follows. By taking the time derivative of the first equation in Definition 6.26 we obtain
$$\varepsilon \partial_t^2 E + \sigma \partial_t E - \text{curl} \partial_t H = \partial_t J, \quad \text{in } \Omega \times (0, T).$$

By solving the second equation in Definition 6.26 for $\partial_t H$ and plugging it into the previous equation we find
$$\varepsilon \partial_t^2 E + \sigma \partial_t E - \text{curl} \left( \frac{1}{\mu} \text{curl} E \right) = \partial_t J, \quad \text{in } \Omega \times (0, T).$$

This looks similar to (6.5) except for the time derivatives. Moreover, there is no time dependence in (6.5). This is because the time derivatives are discretized to obtain that equation. For example, after applying the Crank-Nicolson scheme, we find (as in [8]),
$$\frac{1}{4} \Delta t^2 \text{curl} \left( \frac{1}{\mu} \text{curl} E_n \right) + \left( \varepsilon + \frac{1}{2} \sigma \Delta t \right) E_n = F(E_{n-1}, E_{n-2}, J, \Delta t), \quad \text{in } \Omega,$$

where $\Delta t > 0$ is the length of the time step and $E_n$ is the unknown approximation of the electric field for the current time step. The function $F$ depends on the quantities between the brackets which are known quantities at the current time step. This equation is exactly in the form of (6.5) for $\alpha := \Delta t^2/(4\mu), \beta := \varepsilon + \sigma \Delta t/2$ and $f := F$, which motivates the analysis of these types of equations. So in the following sections $u$ should be thought of as an approximation for the electric field $E$ in a uniform medium where the boundary is a perfect conductor.

6.6.3 Weak Formulation of Maxwell’s Equations

The PDE in (6.5) is not yet formulated in an appropriate weak form to solve numerically. The derivation of the weak form is done in this section.

Lemma 6.27 (Equation (2.3) in [15]) The weak formulation of (6.5) is:
Find $u \in H_0(\text{curl}; \Omega)$

$$a(u, v) = \langle f, v \rangle_{(L^2(\Omega))^3}, \quad \forall u, v \in H_0(\text{curl}; \Omega),$$

where the bilinear form $a(\cdot, \cdot)$ is defined by

$$a(u, v) = \langle \alpha \text{curl } u, \text{curl } v \rangle_{(L^2(\Omega))^3} + \langle \beta u, v \rangle_{(L^2(\Omega))^3}, \quad \forall u, v \in H(\text{curl}; \Omega).$$
Proof We follow the steps from [15] but work out everything in full detail. Take any $v \in (C^\infty(\Omega))^3$ such that $v \times n = 0$ ($\gamma_t v = 0$), where $n$ is the outward unit normal to $\partial \Omega$ and $u \in (C^\infty(\Omega))^3$ such that $u \times n = 0$ and solves (6.5). Then

\[ \int_\Omega (\nabla \times (\alpha(x) \nabla u(x)) + \beta(x) \nabla u(x)) \cdot v(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx. \]

By Green’s formula (for smooth functions) we obtain

\[ \int_\Omega (\alpha(x) \nabla u(x) + \beta(x) u(x)) \cdot \nabla v(x) \, dx = \int_\Omega f(x) \cdot v(x) \, dx. \]

Hence by linearity,

\[ a(u, v) = \int_\Omega f(x) \cdot v(x) \, dx, \]

which proves the statement for $u, v \in \{ w \in (C^\infty(\Omega))^3 \mid \gamma_t w = 0 \}$.

By Lemma 2.20, this is a dense subspace of $H_0(\nabla; \Omega)$ with respect to the $H(\nabla; \Omega)$-norm. Therefore, by continuity of the $H(\nabla; \Omega)$-scalar product with respect to the $H(\nabla; \Omega)$-norm, we can conclude that the statement holds for $H_0(\nabla; \Omega)$.

6.6.4 The Mortar Edge Element Method

Here we will again assume we are in the setting as in Definitions 6.2 and 6.4. We commence by stating an assumption on the meshes of the subdomains. This will be a bit more general than the setting in Definition 6.3. For any $k \in \{1, \ldots, N\}$ we denote by $T_{h_k}$ the mesh on $\Omega_k$ where $h_k$ is the maximum diameter of the elements in the mesh. Moreover, we assume that $h := \min_k h_k$ (in contrast to $\max_k h_k$ in the usual case), i.e. $h$ is the fine mesh size. The extra assumption on these meshes is the following:

Assumption. Let $T_{h_i}$ and $T_{h_j}$ be the meshes on two neighbouring subdomains $\Omega_i$ and $\Omega_j$. We assume that these meshes match on the interface $\Gamma_{ij}$ or are strictly nested.

Next we describe the relevant local shape function spaces. These will be the so-called Nédélec edge element spaces. In contrast to the Lagrangian finite element spaces which are associated to the nodes of the mesh, these elements are associated to the edges of the mesh.

Definition 6.28 (Nédélec Edge Elements) (Section 2 in [15] and [9]) Let $\Omega \subseteq \mathbb{R}^3$ be a polyhedral domain equipped with a tetrahedral mesh $T^h$. The Nédélec edge element space is defined as

\[ \mathcal{N}^h \{ \Omega \} := \{ v \in H_0(\nabla; \Omega) \mid \forall K \in T^h : v|_K \in R(K) \}, \]
where \( R(K) \) is the space of linear functions on \( K \) of the form

\[
R(K) := \{ x \in K \mapsto a + b \times x \mid a, b \in \mathbb{R}^3 \}.
\]

The corresponding (local) degrees of freedom are defined as

\[
\lambda_e(v^h) := \int_e v^h \cdot t_e \, ds, \quad \forall v^h \in \mathcal{N} \mathcal{D}_1^h(\Omega),
\]

for any edge \( e \in \mathcal{E}(\mathcal{T}^h) \). Here \( t_e \) is the unit tangent vector along \( e \). The dual basis \( \{ L_e \}_{e \in \mathcal{E}(\mathcal{T}^h)} \) to \( \{ \lambda_e \}_{e \in \mathcal{E}(\mathcal{T}^h)} \) of \( \mathcal{N} \mathcal{D}_1^h(\Omega) \) is defined in the usual way as the unique functions such that

\[
\lambda_e'(L_e) = \delta_{ee'},
\]

where \( \delta_\cdot \) is the Kronecker delta. Explicitly we have the formula

\[
L_e := c^e(\lambda_1^e \nabla \lambda_2^e - \lambda_2^e \nabla \lambda_1^e), \quad \forall e \in \mathcal{E}(\mathcal{T}^h),
\]

where \( \lambda_1^e \) and \( \lambda_2^e \) are the barycentric coordinates of the endpoints of \( e \) with respect to \( e \) and \( c^e \) is chosen such that (6.6) holds.

**Remark.** [15] The tangential component of any \( v^h \in \mathcal{N} \mathcal{D}_1^h(\Omega) \) is continuous on all edges of every element in \( \mathcal{T}^h \).

**Example 6.29** The definition of the Nédelec edge elements may seem a bit abstract so we will compute the basis functions for the triangle with vertices \( p_1 := (-1,0,0), p_2 := (1,0,0), p_3 := (0,1,0) \) and edges \( e_1 := p_1p_2, e_2 := p_1p_3, e_3 := p_2p_3 \). First we need to compute the barycentric coordinates of the vertices of the triangle. A quick computation gives

\[
\begin{align*}
\lambda_1(x, y, 0) &= \frac{1}{2} - \frac{x}{2} - \frac{y}{2}, \\
\lambda_2(x, y, 0) &= y, \\
\lambda_3(x, y, 0) &= \frac{1}{2} + \frac{x}{2} - \frac{y}{2}.
\end{align*}
\]

We therefore obtain the following expressions for the Nédelec edge element basis functions for the given triangle:

\[
\begin{align*}
L_{e_1}(x, y, 0) &= c^{e_1}(\lambda_1 \nabla \lambda_2 - \lambda_2 \nabla \lambda_1)(x, y, 0) = \left( -\frac{y}{2}, \frac{x-1}{2} \right) = \frac{1}{2}(p_1 + (-y, x)), \\
L_{e_2}(x, y, 0) &= c^{e_2}(\lambda_1 \nabla \lambda_3 - \lambda_3 \nabla \lambda_1)(x, y, 0) = \left( -\frac{y+1}{2}, \frac{x}{2} \right) = \frac{1}{2}(p_2 + (-y, x)), \\
L_{e_3}(x, y, 0) &= c^{e_3}(\lambda_2 \nabla \lambda_3 - \lambda_3 \nabla \lambda_2)(x, y, 0) = \left( -\frac{y}{2}, \frac{x+1}{2} \right) = \frac{1}{2}(p_3 + (-y, x)).
\end{align*}
\]

In this case \( c^e = 1 \) for \( k = 1, 2, 3 \). Therefore we see that each basis function is a sort of ‘vortex’ vector field centered at a certain node and rescaled such that it satisfies the condition in (6.6).
Now we describe the mortar edge element method which is used in [15]. For this we need to define the mortar edge element space. Before we introduce this space we need some more function spaces. We will denote the space of finite element functions whose tangential components agree across the boundary of each interface by $\widetilde{\mathcal{N}D}_1^h(\Omega)$. More specifically we have

$$\widetilde{\mathcal{N}D}_1^h(\Omega) := \{ v \in (L^2(\Omega))^3 \mid \forall k = 1, \ldots, N : v_k \in \mathcal{N}D_1^{h_k}(\Omega_k)$$

and $\forall \Gamma_{ij} : v_j \cdot t_{ij} = v_j \cdot t_{ij}$, on $\partial \Gamma_{ij}$,

where $v_k := v|_{\Omega_k}$ for all $k = 1, \ldots, N$ and $t_{ij}$ is the tangential unit vector along $\partial \Gamma_{ij}$. Next we need to define so-called (local) multiplier spaces for each interface. For this we need tangential restrictions of the Nédélec edge element spaces on each subdomain.

**Definition 6.30 (Tangential Restriction Spaces)** (Section 2 in [15]) For any interface $\Gamma_{ij}$ for two neighbouring subdomains $\Omega_i$ and $\Omega_j$, we define the tangential restrictions of $\mathcal{N}D_1^{h_i}(\Omega_i)$ and $\mathcal{N}D_1^{h_j}(\Omega_j)$ as

$$\mathcal{N}D_1^{h_i}(\Omega_i) := \{(v^{h_i} \times n)|_{\Gamma_{ij}} \mid v^{h_i} \in \mathcal{N}D_1^{h_i}(\Omega_i)\},$$

and

$$\mathcal{N}D_1^{h_j}(\Omega_j) := \{(v^{h_j} \times n)|_{\Gamma_{ij}} \mid v^{h_j} \in \mathcal{N}D_1^{h_j}(\Omega_j)\},$$

respectively if the meshes $T^{h_i}$ and $T^{h_j}$ are strictly nested on $\Gamma_{ij}$. Here $n$ is the unit outward normal to $\Gamma_{ij}$. If the meshes match, then the above two spaces coincide.

We use these tangential restriction spaces to define the multiplier spaces.

**Definition 6.31 (Local Multiplier Space)** (Section 2 in [15]) For every two neighbouring subdomains $\Omega_i$, $\Omega_j$ and interface $\Gamma_{ij}$, we define the local multiplier space $W(\Gamma_{ij})$ as the two-dimensional Nédélec edge element space

$$W(\Gamma_{ij}) := \{(n \times v^{h_i} \times n)|_{\Gamma_{ij}} \mid v^{h_i} \in \mathcal{N}D_1^{h_i}(\Omega_i)\},$$

where $n$ is the unit outward normal to $\Gamma_{ij}$ if the mesh $T^{h_i}$ is strictly contained in $T^{h_i}$. The other case is similar. If the meshes match, then we define $W(\Gamma_{ij})$ as

$$W(\Gamma_{ij}) := \{\mu^{h_i} \in \mathcal{N}D_1^{h_i}(\Omega_i) \mid \mu^{h_i} \cdot \tau = 0 \text{ on } \partial \Gamma_{ij}\},$$

where $\tau$ is the unit normal to $\partial \Gamma_{ij}$ parallel to $\Gamma_{ij}$.

**Definition 6.32 (Mortar Edge Element Space)** (Section 3 in [15]) For the setting described above, the mortar edge element space on $\Omega$ is defined as

$$\mathcal{M}_1^h(\Omega) := \{v^h \in \widetilde{\mathcal{N}D}_1^h(\Omega) \mid \forall \Gamma_{ij} : \forall \mu^{h_i} \in W(\Gamma_{ij}) : \langle v^h \times n, \mu^{h_i} \rangle_{L^2(\Gamma_{ij})} = \langle v^h_j \times n, \mu^{h_i} \rangle_{L^2(\Gamma_{ij})}\}.$$
Remark. The space $\mathcal{M}_1^h(\Omega)$ is not a subspace of $H(\text{curl}; \Omega)$ ([15])!

Now that the mortar edge element space is defined we turn to the mortar edge element problem.

**Definition 6.33 (Mortar Edge Element Problem)** (Section 3 in [15]) For the setting described above, the mortar edge element problem associated to the variational problem in Lemma 6.27 is formulated as follows:

Find $u^h \in \mathcal{M}_1^h(\Omega)$ such that

$$\sum_{k=1}^{N} a_k(u^h_k, v^h_k) = \langle f, v^h \rangle_{L^2(\Omega)}, \quad \forall v^h \in \mathcal{M}_1^h(\Omega),$$

where the bilinear forms $a_k(\cdot, \cdot)$ (defined on $\mathcal{N}\mathcal{D}_1^h(\Omega_k)$) are defined by

$$a_k(u^h_k, v^h_k) = \langle \alpha \text{curl } u^h_k, \text{curl } v^h_k \rangle_{L^2(\Omega_k)} + \langle \beta u^h_k, v^h_k \rangle_{L^2(\Omega_k)},$$

for any $u^h_k, v^h_k \in \mathcal{N}\mathcal{D}_1^h(\Omega_k)$ and $k = 1, \ldots, N$. By definition of $\mathcal{N}\mathcal{D}_1^h(\Omega)$ these bilinear forms are well-defined.

**6.6.5 A Generalized Edge Element Interpolation Operator**

In this section we describe the construction of a new generalized edge element interpolation operator $r_h$. The functions we want to interpolate are the functions $v$ which lie in the subspace

$$H(\text{curl}; \Omega) \cap \left( \bigcap_{k=1}^{N} H^\delta_k(\text{curl}; \Omega_k) \right),$$

where each $\delta_k > 1/2$. The constructed interpolant $r_h v$ will have the property that $r_h v \in H(\text{curl}; \Omega) \cap \mathcal{M}_1^h(\Omega)$ such that $r_h v$ possesses some ideal approximation properties which we will describe in the next section since (some of) these will be based on the edge lemma. As in [15] we will write $v^h_k$ to denote $(r_h v)|_{\Omega_k}$. We will define $r_h v$ by defining it on each interface and then on all subdomains. We define $r_h v$ by first defining $v_{h_1}$, followed by $v_{h_2}$ and so on.

**Step 1.** We define $v_{h_1}$ as the standard edge element interpolation of $v_1$, i.e.

$$v_{h_1} := r_{h_1} v_1 \quad \text{in } \Omega_1.$$

Here $r_{h_1}$ is defined by

$$r_{h_1} v_1 := \sum_{e \in \mathcal{E}(T^{h_1})} \lambda_e(v_1) L_e.$$

**Step 2.** Next we define $v_{h_2}$ by using $v_{h_1}$. We consider two cases: the case where $\Gamma_1 \cap \Gamma_2 = \emptyset$ or where the meshes $T^{h_1}$ and $T^{h_2}$ match on $\Gamma_{12}$ and the
We denote these extended meshes by $\tilde{T}^{h_1}$ and $\tilde{T}^{h_2}$ are strictly nested or $\Gamma_1 \cap \Gamma_2 \neq \emptyset$.

**Case 1.** If $\Gamma_1 \cap \Gamma_2 = \emptyset$ or the meshes $\tilde{T}^{h_1}$ and $\tilde{T}^{h_2}$ match on $\Gamma_{12}$, then we define $v_{h_2}$ as

$$v_{h_2} := \sum_{e \in \mathcal{E}(\tilde{T}^{h_2})} \lambda_e(v_2) L_e, \quad \text{in } \Omega_2.$$ 

**Case 2.** For the case where $\tilde{T}^{h_1}$ and $\tilde{T}^{h_2}$ are strictly nested or $\Gamma_1 \cap \Gamma_2 \neq \emptyset$ the situation is more precarious. Again we have two cases: $\Gamma_1 \cap \Gamma_2 = \Gamma_{12}$ is a face or $\Gamma_1 \cap \Gamma_2 = e_{12}$ is an edge.

**Case 2.1.** For the case where $\Gamma_1 \cap \Gamma_2 = \Gamma_{12}$ is a face, we define

$$v_{h_2} := r_{h_2} v_2 + R_{T_{12},h_2}^{0}((r_{h_1} v_1 - r_{h_2} v_2)|_{\Gamma_{12}}),$$

where the discrete extension-type operator $R_{G,h_k}^{0} : \mathcal{N}^{h_k}(\Omega_k|G) \rightarrow \mathcal{N}^{h_k}(\Omega_k)$ for any closed subset $G \subseteq \Gamma_k$ is defined as

$$(R_{G,h_k}^{0} v^{h_k}|G)(x) := \sum_{e \in \mathcal{E}(T^{h_k}) \cap G} \lambda_e(v^{h_k}) L_e(x), \quad \forall x \in \bar{\Omega}_k$$

for any $v^{h_k}|G \in \mathcal{N}^{h_k}(\Omega_k|G)$. The space $\mathcal{N}^{h_k}(\Omega_k|G)$ is defined as the space of restrictions of functions in $\mathcal{N}^{h_k}(\Omega_k)$ to $G$.

**Remark.** Notice that any degree of freedom of $R_{G,h_k}^{0} v^{h_k}|G$ associated to an edge $e$ outside of $G$ vanishes. Indeed, let $\bar{e}$ be any such edge, then

$$\lambda_{\bar{e}}(R_{G,h_k}^{0} v^{h_k}|G) = \sum_{e \in \mathcal{E}(T^{h_k}) \cap G} \lambda_e(v^{h_k}) \lambda_{\bar{e}}(L_e) = \sum_{e \in \mathcal{E}(T^{h_k}) \cap G} \lambda_e(v^{h_k}) \delta_{\bar{e}} = 0.$$ 

**Case 2.2.** More care has to be taken for the case when $\Gamma_1 \cap \Gamma_2$ is an edge. Denote by $T_{12}$ and $T_{21}$ the restrictions of $\tilde{T}^{h_1}$ and $\tilde{T}^{h_2}$ on $\Gamma_{12}$ respectively. These meshes are assumed to be nested on $\Gamma_{12}$ hence we can extend these restrictions to (nested) meshes on $\Omega_1$ with mesh sizes $h_1$ and $h_2$ respectively. We denote these extended meshes by $\bar{T}_{12}$ and $\bar{T}_{21}$. Similarly we denote by $\bar{r}_{h_1}$ and $\bar{r}_{h_2}$ the associated standard edge element interpolation operators. Then from [7] we know that we have a Helmholtz decomposition (sum of a divergence-free and an irrotational vector field) for $\bar{r}_{h_1} v_1 - \bar{r}_{h_2} v_1$, i.e. there exist $w \in H(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega)$ and $p \in H^1(\Omega_1)$ such that

$$\bar{r}_{h_1} v_1 - \bar{r}_{h_2} v_1 = w + \nabla p.$$

Applying the interpolation operator $\bar{r}_{h_2}$ to both sides of this equation and by using the fact that the meshes are nested, we obtain

$$\bar{r}_{h_1} v_1 - \bar{r}_{h_2} v_1 = \tilde{w}_{h_2} + \tilde{p}_{h_2} + \nabla \tilde{h}_{h_2}.$$
where \( \hat{w}_{h_2} := \tilde{r}_{h_2} w \) and \( \tilde{p}_{h_2} \in H^1(\Omega_1) \) is now a piece-wise linear function associated to the mesh \( \mathcal{T}_{h_2} \) on \( \Omega_1 \). Then we define \( v_{h_2} \) as

\[
v_{h_2} := r_{h_2} v_2 + R_{e_{12},h_2}^0(\hat{w}_{h_2}|_{e_{12}}) + \nabla \tilde{p}_{h_2},
\]

where \( p_{h_2} \in H^1(\Omega_2) \) is the discrete harmonic extension of \( \pi_{e_{12},h_2}^0(\tilde{p}|_{\Gamma_{12}}) \) into \( \Omega_2 \) with respect to the mesh \( \mathcal{T}_{h_2} \). The definition of the operator \( \pi_{e_{12},h_2}^0 \) is given in the next section.

**Remark.** In [15] there is presumably a typo on the left-hand side of the previous equation. It should be \( v_{h_2} \) as we wrote and not \( v_{h_k} \) as in [15].

**Step 3.** Now that we have defined \( v_1 \) and \( v_2 \) we commence with the induction step. To this end assume that \( v_1, \ldots, v_{k-1} \) are defined. Again, we look at two different cases similar to what we did in Step 2. The first case is where \( \Gamma_r \cap \Gamma_k = \emptyset \) for \( r = 1, \ldots, k-1 \) or the meshes \( \mathcal{T}_{h_r} \) and \( \mathcal{T}_{h_k} \) match on \( \Gamma_{rk} \) for some \( 1 \leq r \leq k-1 \). The second case is where \( \Gamma_r \cap \Gamma_k \neq \emptyset \) for a subset of \( r \) in \( \{1, \ldots, k-1\} \) or where the meshes \( \mathcal{T}_{h_r} \) and \( \mathcal{T}_{h_k} \) are strictly nested.

**Case 1.** In this case we again just set \( v_{h_k} \) equal to the standard edge element interpolation of \( v_k \) as we did for \( v_2 \) in Step 2. Hence we have

\[
v_{h_k} := r_{h_k} v_k.
\]

**Case 2.** Let \( \Omega_{k_1}, \ldots, \Omega_{k_n} \) with \( k_1 < \ldots < k_n < k \) be the subdomains such that \( \Gamma_{k_\ell} \cap \Gamma_k \neq \emptyset \) for \( \ell = 1, \ldots, n \). Without loss of generality we may assume that

- \( \Gamma_{k_\ell} \cap \Gamma_k = \Gamma_{k_{\ell}k} \) is a face for \( \ell = 1, \ldots, n-1 \),
- \( \Gamma_{k_n} \cap \Gamma_k = e_{k_n,k} \) is an edge.

The face case is treated in the same way as Case 2.1. For the edge case we consider two cases: The first case is where there exists one subdomain in \( \{\Omega_{k_\ell}\}_{\ell<n} \) such that \( e_{k_n,k} \) is one of its edges. The second case is where \( e_{k_n,k} \) isn’t an edge of any subdomain in \( \{\Omega_{k_\ell}\}_{\ell<n} \).

**Case 2.1.** For the first case we define \( v_{h_k} \) directly as

\[
v_{h_k} := r_{h_k} v_k + \sum_{\ell=1}^{n-1} R_{\Gamma_{k_\ell}k,h_k}^0([r_{h_{k_\ell}} v_{k_\ell} - r_{h_k} v_k]|_{\Gamma_{k_\ell}k}).
\]

**Case 2.2.** For the second case we define (similar to Case 2.2) two interpolation operators \( \tilde{r}_{h_{k_n}} \) and \( \tilde{r}_{h_k} \) on \( \Omega_{k_n} \). As before, let \( \hat{w}_{h_{k_n}} \) and \( \tilde{p}_{h_{k_n}} \) be the two parts of the Helmholtz decomposition of \( \tilde{r}_{h_{k_n}} v_{k_n} - \tilde{r}_{h_k} v_k \). Then we define \( v_{h_k} \) as

\[
v_{h_k} := r_{h_k} v_k + \sum_{\ell=1}^{n-1} R_{\Gamma_{k_\ell}k,h_k}^0([r_{h_{k_\ell}} v_{k_\ell} - r_{h_k} v_k]|_{\Gamma_{k_\ell}k}) + R_{\Gamma_{k_{n}}k,h_k}^0(\hat{w}_{h_{k_n}}|_{e_{k_n,k}}) + \nabla \tilde{p}_{h_{k_n}},
\]

where \( p_{h_{k_n}} \) is the discrete harmonic extension into \( \Omega_k \) of \( \pi_{e_{k_n,k},h_k}(\tilde{p}_{h_{k_n}}|_{\Gamma_{k_n}}) \). Now that all the \( v_{h_i} \) are defined for all \( i = 1, \ldots, N \), we define the action of the generalized edge interpolation operator \( r_h v \) as just \( v_{h_i} \) on each subdomain \( \Omega_i \) for all \( i = 1, \ldots, N \).
6.6.6 Results Based on the edge and face lemma

Now the turn to the results in [15] which uses the edge and/or face lemma. First we discuss Lemma 4.6. Before we state this lemma we need to define an extension operator which we also used in the previous section. We define the extension operator \( \pi_{e,h_k} : S^0_{1,3}(\Gamma_r) \rightarrow S^0_{1,3}(\Gamma_k) \) for any common closed edge \( e \) of two subdomains \( \Omega_r \) and \( \Omega_k \) as

\[
(\pi_{e,h_k} \varphi^{hr}(x)) := \begin{cases} 
\varphi^{hr}(x), & x \in e \cap N(T^{h_k}), \\
\gamma_{\Gamma_r}(\varphi^{hr}), & x \in (\Gamma_k \setminus e) \cap N(T^{h_k}), 
\end{cases}
\]

for any \( \varphi^{hr} \in S^0_{1,3}(\Gamma_r) \) where \( \gamma_{\Gamma_r}(\varphi^{hr}) \) is the average value of \( \varphi^{hr} \) on \( \Gamma_r \). The function spaces \( S^0_{1,3}(\Gamma_k) \) for any \( k \in \{1, \ldots, N\} \) are defined as

\[ S^0_{1,3}(\Gamma_k) := \{ v^{hk} | _{\Gamma_k} | v^{hk} \in S^0_{1,3}(T^{h_k}) \}, \]

where \( S^0_{1,3}(T^{h_k}) \) is defined as in Definition 6.21. Now we state Lemma 4.6 and discuss its proof.

**Lemma 6.34** *(Lemma 4.6 in [15])* Let \( \Omega_r \) and \( \Omega_k \) be two subdomains sharing a common edge \( e \). Then for any \( \varphi^{hr} \in S^0_{1,3}(\Gamma_r) \), we have

\[
|\pi_{e,h_k} \varphi^{hr}|_{(H^{1/2}(\Gamma_k))^3} \lesssim (1 + \log(d/h_r))^{1/2} |\varphi^{hr}|_{(H^{1/2}(\Gamma_r))^3}.
\]

**Proof** The proof is just a concatenation of inverse inequalities of which the edge lemma is one. It uses the edge lemma directly to show the estimate

\[
\| \varphi^{hr} - \gamma_{\Gamma_r}(\varphi^{hr}) \|_{(L^2(\Gamma_k))^3} \lesssim (1 + \log(d/h_k))^{1/2} \| \varphi^{hr} - \gamma_{\Gamma_r}(\varphi^{hr}) \|_{(H^{1/2}(\Gamma_r))^3},
\]

which is indeed just the edge lemma applied to each component function. \( \square \)

Now we turn to the next result which is Theorem 4.1 in [15] which is an interpolation error estimate. This theorem uses the lemma above (Lemma 4.6 in [15]) and hence also the edge lemma.

**Theorem 6.35** *(Theorem 4.1 in [15])* Let \( r_h \) the generalized interpolation operator as was defined in the previous section. Then for any \( v \in H(\text{curl}; \Omega) \cap \left( \bigcap_{k=1}^N H^{\delta_k}(\text{curl}; \Omega_k) \right) \) with \( 1/2 < \delta_k \leq 1 \), we have

\[
\| r_h v - v \|_{H(\text{curl}; \Omega)} \lesssim (1 + \log(d/h))^{1/2} \left( \sum_{k=1}^N h_k^{2\delta_k} \| \text{curl} v \|_{H^{\delta_k}(\Omega_k)}^2 \right)^{1/2}.
\]

**Proof** Lemma 4.6 is used in the proof to derive the estimates

\[
\| R_{e_1 e_2, h_2} (\bar{w}_{h_2} |_{e_1}) \|_{H(\text{curl}; \Omega_2)}^2 \lesssim h_1^{2\delta_1} (1 + \log(d/h_2)) \| \text{curl} v_1 \|_{H^{\delta_1}(\Omega_1)}^2.
\]


6. Applications in Recent Research

and

\[ \| \nabla p_{h2} \|_{L^2(\Omega_2)}^2 \lesssim (1 + \log(d/h_1)) \| \tilde{p}_{h2} \|_{H^1(\Omega_1)}^2, \]

where \( \nabla p_{h2} \) is the irrotational vector field in the Helmholtz decomposition of \( \tilde{r}_{h1} v_1 - \tilde{r}_{h2} v_1 \) and \( p_{h2} \) is the discrete harmonic extension of \( \pi_e^{0, h_2} (\tilde{p}_{h2} \mid \Gamma_{12}) \) into \( \Omega_2 \).

**Remark.** In the [15] \( \prod_{k=1}^N H^\delta_k(\text{curl}; \Omega_k) \) is written instead of \( \bigcap_{k=1}^N H^\delta_k(\text{curl}; \Omega_k) \). This is presumably a typo.

The last result which indirectly uses the edge lemma is Theorem 5.1 from [15] through the use of Theorem 4.1 from [15]. This theorem is the nearly optimal error estimate that we mentioned when we discussed the goal of this paper.

**Theorem 6.36 (Nearly Optimal Error Estimate)** (Theorem 5.1 in [15])

Let \( u \) be the solution to the variational problem in Definition 6.27 and \( u^h \) the finite element solution to the mortar element system in Definition 6.33. Assume that

- \( u \in H(\text{curl}; \Omega) \) and \( f \in (H^5(\Omega))^3 \) with \( \delta > 1/2 \),
- In each subdomain \( \Omega_k \), we have \( u \mid_{\Omega_k} \in H^\delta_k(\text{curl}; \Omega_k) \), \( f \mid_{\Omega_k} \in (H^\delta_k(\Omega_k))^3 \) with \( \delta \in (1/2, 1] \), for all \( k = 1, \ldots, N \).

Then

\[ \| u - u^h \| \lesssim (1 + \log(d/h))^{1/2} \left( \sum_{k=1}^N h_k^{2\delta_k} \| \text{curl} u \|_{H^\delta_k(\Omega_k)}^2 \right)^{1/2}. \]

The norm on the left-hand side is defined by

\[ \| v \| := \sum_{k=1}^N \| v \|_{H(\text{curl}; \Omega_k)}, \]

for any \( v \in \bigcap_{k=1}^N H^\delta_k(\text{curl}; \Omega_k) \).

**Proof** In the first step of the proof, the estimate

\[ \| u - u^h \|_{H(\text{curl}; \Omega)} \leq \inf_{v_h \in \mathcal{M}_1^h(\Omega)} \| u - v^h \|_{H(\text{curl}; \Omega)} + (I)_2, \]

is proved, where \( (I)_2 \) is an unimportant term for our discussion. Therefore, we won’t elaborate on this term. To estimate the infimum Theorem 4.1 from the paper is used. This immediately implies

\[ \inf_{v_h \in \mathcal{M}_1^h(\Omega)} \| u - v^h \|_{H(\text{curl}; \Omega)} \leq \| r_h v - v \|_{H(\text{curl}; \Omega)} \]

\[ \lesssim (1 + \log(d/h))^{1/2} \left( \sum_{k=1}^N h_k^{2\delta_k} \| \text{curl} v \|_{H^\delta_k(\Omega_k)}^2 \right)^{1/2}. \]
which is (5.9) in the proof of this Theorem in [15]. The fact that $\mathbf{r}_h \mathbf{v} \in \mathcal{M}_1^h(\Omega)$ is the content of Lemma 4.2 from [15] and is unimportant for our discussion since it does not use the edge or face lemma. □
6. Applications in Recent Research


In this section we elaborate on the paper ‘A Non-overlapping Domain Decomposition Method for Maxwell’s Equations in Three Dimensions’ by Qiya Hu and Jun Zou ([16]). The structure of our exposition will be similar to the sections before.

6.7.1 Goal of Paper 5

In this research paper a new nonoverlapping domain decomposition method is proposed for solving Maxwell’s equations in three dimensions. This is done by using Nédélec edge element spaces. A proof is given for the claim that the condition number of the new preconditioner grows polylogarithmically with the ratio between $d$ (the subdomain diameter) and $h$ (the finite element mesh size) but also possibly depends on the jumps in the coefficients.

6.7.2 Results Based on the edge and face lemma

Statement (6.2) in Lemma 6.1 in [16] is a direct reference to the face lemma. There are two other results which use this lemma and therefore also the face lemma. We will first discuss Lemma 6.5 from [16] which is one of these results. For this we first need to fix some notation and define an extension operator. We will start by fixing notation. As usual we assume that we are in the setting as in Definitions 6.2 and 6.3.

Let $F$ be any open face of a subdomain $\Omega_k$. We will denote by $F_b$ the union of all closed triangles on $F$ induced by the mesh $T^h_k$ which have at least one edge on $\partial F$ and $F_0$ as the (open) set $F \setminus F_b$. Next we define some tangential restrictions of Nédélec edge element spaces which are slightly more general than the ones we defined before.

**Definition 6.37 (Tangential Restriction Spaces)** (Section 2 in [16]) Let $G \subseteq \Gamma$ where $\Gamma$ is the interface associated course partition $T_d$ on $\Omega$. We define the following **tangential restrictions** of the usual Nédélec edge element space $\mathcal{N}^h_1(\Omega)$,

$$\mathcal{N}^h_{1, ||}(\Omega | G) := \{(v \times n)|_G | v \in \mathcal{N}^h_1(\Omega)\},$$

$$\mathcal{N}^h_{1, ||, 0}(\Omega_i | F) := \{v \times n \in \mathcal{N}^h_{1, ||}(\Omega_i | F) | \forall e \subseteq \partial F \cap \mathcal{E}(T^h) : \lambda_e(v) = 0\}$$

for any open face $F$ of $\Omega_i$ and subdomain $\Omega_i \in T_d$. Here $\cdot \times n$ should be interpreted as in Lemma 2.20.

Using these tangential restriction spaces we define the following extension operator.

Definition 6.38 (Extension Operator) [16] Let $F$ be any open face of an interface $\Gamma_k$. Then we define the extension operator

$$
\Gamma^0_{F_0} : \mathcal{N}\mathcal{D}^h_{1,\|}(\Omega|\Gamma_i) \rightarrow E^0_F(\mathcal{N}\mathcal{D}^h_{1,\|,0}(\Omega_i | F)) \subseteq (L^2(F))^3
$$

by

$$(\Gamma^0_{F_0}[(v \times n)|\Gamma_i])(x) := \sum_{e \subseteq F_0} \lambda_e(v)(L_e \times n)(x), \quad \forall x \in \Gamma_i,
$$

where $E^0_F : \mathcal{N}\mathcal{D}^h_{1,\|,\|}(\Omega_i|\Gamma_i) \rightarrow (L^2(F))^3$ is the natural extension-by-zero operator.

To clarify, the sum over $e \subseteq F_0$ denotes the sum over all edges which lie in $F_0$.

Now that we have introduced the appropriate extension operator and tangential restriction spaces, we state and discuss the proof of Lemma 6.5 in [16].

Lemma 6.39 (Lemma 6.5 in [16]) For any $\Phi \in \mathcal{N}\mathcal{D}^h_{1,\|}(\Omega_i | \Gamma_i)$,

$$
\|\Gamma^0_{F_0} \Phi\|_{H^{-1/2}(\Gamma_i)} \leq C([1 + \log(d/h)]\|\Phi\|_{H^{-1/2}(\Gamma_i)} + h^{1/2}\|\Phi\|_{*,F_0}),
$$

where

$$
\|\Phi\|_{*,F_0} := \sum_{K \in F_0} \|\Phi\|_{L^2(\partial K)}^2.
$$

Proof In the proof of this lemma, the face lemma (together with some other results) is used to argue that

$$
\|\Phi\|_{H^{-1/2}(\Gamma_i)} \|\Gamma^0_{F_0} v^h\|_{H^{1/2}(\Gamma_i)} + C h^{1/2}\|\Phi\|_{L^2(\Gamma_i)}\|v^h\|_{L^2(\partial F)}
$$

$$
\leq C([1 + \log(d/h)]\|\Phi\|_{H^{-1/2}(\Gamma_i)}\|v^h\|_{H^{1/2}(\Gamma_i)}),
$$

where $v^h$ is some nodal finite element function in $(\mathcal{S}^0_{1,1}(T^h|\Gamma_i))^3$. \(\square\)

The second and last result which uses the face lemma is Lemma 4.7 in [16]. Before we state this result we need the concept of tangential divergence.

Definition 6.40 (Tangential Divergence) (Definition 2.1 in [1]) Let $\lambda \in (H^{-1/2}(\partial \Omega))^3$ be a vector field with

$$(\lambda \cdot n)|_{\partial \Omega} = 0,$$

where $n$ is the unit outward normal to $\partial \Omega$. The tangential divergence $\text{div}_T \lambda \in H^{-3/2}(\partial \Omega)$ of $\lambda$ is defined by

$$(\text{div}_T \lambda)(\psi) := -\langle \lambda, (\nabla \psi^*)|_{\partial \Omega}\rangle_{L^2(\partial \Omega)}, \quad \forall \psi \in H^{3/2}(\partial \Omega),$$

where $\psi^*$ is any extension of $\psi$ into $\Omega$.

Now we state Lemma 4.7 from [16].

111
Lemma 6.41 (Lemma 4.7 in [16]) Let \( w \in (H^1(\Omega_i))^3 \) and as before, let \( r_h \) be the usual edge element interpolant which maps into \( \mathcal{N}\mathcal{D}_h(\Omega_i) \). Let \( v^h \in \mathcal{N}\mathcal{D}_h(\Omega_i) \) be any function such that

\[
\text{curl } w = \text{curl } v^h,
\]

and set \( \Phi := (r_h w) \times n \) on \( \Gamma_i \). Then for any face \( F \subseteq \Gamma_i \), we have

\[
\| I^h_0 \|_{H^1/2(\Gamma_i)} \leq C [1 + \log(d/h)] \left( \| \Phi \|_{H^1(\Omega_i)} + \| w \|_{H^1(\Omega_i)} + \| \text{curl } v^h \|_{L^2(\Omega_i)} \right),
\]

where the norm \( \| \cdot \|_{H^1/2(\Gamma_i)} \) is defined as

\[
\| \Phi \|_{H^1/2(\Gamma_i)} := d^{-1}\| \Phi \|_{H^{-1/2}(\Gamma_i)} + \| \text{div} \Phi \|_{H^{-1/2}(\Gamma_i)}, \quad \forall \Phi \in (H^{-1/2}(\partial\Omega))^3.
\]

**Proof** It is claimed that this is a direct consequence of Lemma 6.5 in [16] together with some other results. The proof of this lemma uses the face lemma which results in the fact that the face lemma is used in the proof of this lemma. \( \square \)
Chapter 7

Conclusion

Inverse inequalities are crucial tools for the analysis of the finite element and domain decomposition methods. They are used for error analysis and estimation of the condition numbers of preconditioners for the linear systems of equations resulting from finite element discretization to just name a few examples. In this thesis we focused on two such inequalities, namely the edge and face lemma.

Although the notation of inverse inequalities looks quite elegant and compact, the proofs of these inequalities are not. This is easily seen in our proof of the edge and face lemma with requires multiple pages of delicate estimates. At every step of the way it has to be carefully checked that the implicit constants are independent of the mesh parameters and the chosen finite element function.

In this thesis we set out to first prove the edge lemma. After completing the proof of the edge lemma, we noticed similar issues with the proof of the face lemma. To resolve these issues, we used similar tools as in the proof of the edge lemma to correct the proof of this result. The result of this is that there are now complete proofs of the edge and face lemma available. This is essential for further and existing research. Indeed, as we have extensively shown in Chapter 6, there are already results which use these lemmas. Moreover, the publication dates of these papers span at least 15 years with the most recent one being published as recent as in 2019. This shows that the edge and face lemma have been and still are tools which are actively used to derive new results.

The way in which we proved the edge and face lemma was strongly inspired by the incorrect proofs from [23]. Here the issue was a spurious use of various inverse inequalities for meshes lacking the required regularity. The way we fixed these proofs was by discretizing the integrals involved into finite sums and using certain new inverse inequalities that we proved for centroids of tetrahedra to get a sort of equivalence between these integrals and sums. The
centroids played a crucial role due to the fact that they have the nice property of being centers of inscribed balls with sufficiently large radii. This allowed us to bypass the problem of having to use inverse inequalities for irregular meshes. We were able to choose these finite sums to be sums over a certain number of well-chosen tetrahedra such that we could use these new inverse inequalities. The precise choice of these tetrahedra turned out to be crucial to be able to complete the proof.

We have to remark that there is a downside to our proofs. The proofs we have given are very long compared to the existing erroneous proofs. We don’t claim that we have given the most elegant or shortest proofs possible for these results. The problem of finding a more elegant proof of these results is left to further research. Even though our proofs are quite technical, the main ideas can be easily used for other results whose proofs have similar issues as the edge and face lemma did.
Appendix A

List of Symbols

Basic Concepts

| · | Euclidean norm |
| a | Vector |
| A | Matrix |
| d(·,·) | Point-plane distance |
| (ρ,θ) | Polar coordinates |
| (ρ,θ,ψ) | Spherical coordinates |
| c_K | Centroid of a measurable subset \( K \subseteq \mathbb{R}^n \) |
| vol_n | \( n \)-dimensional volume measure |
| Γ | Face of the cube \( (0,d)^3 \) on the \( xy \)-plane |
| κ | Spectral condition number of an operator |
| κ_{m_0+1} | Reduced spectral condition number of an operator |
| K_n | \( n \)-th Krylov subspace |
| \( \hat{K} \) | Normalization of a bounded set \( K \subseteq \mathbb{R}^n \) |
| P | Tetrahedron |

Function Spaces

\( L^p(\Omega) \) Banach space of functions with finite \( p \)-th moment with respect to the Lebesgue measure
\( L^2_\omega(\Omega) \) vectorially \( \omega \)-weighted \( L^2 \)-space
\( \mathcal{H} \) Hilbert space
\( W^{k,p}(\Omega) \) Sobolev space of order \( k \in \mathbb{R} \) for \( p \in \mathbb{N} \cup \{\infty\} \) on \( \Omega \)
\( H^k(\Omega) \) Sobolev space of order \( k \in \mathbb{R} \) for \( p = 2 \) on \( \Omega \)
\( H^k(\partial \Omega) \) Sobolev space of order \( k \in \mathbb{R} \) for \( p = 2 \) on \( \partial \Omega \)
\( H^1_\omega(\Omega) \) vectorially \( \omega \)-weighted \( H^1 \)-space
### A. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>( H^{1/2}_{00}(Γ) )</td>
<td>Lions-Magenes space on ( Γ \subseteq ∂Ω )</td>
</tr>
<tr>
<td>( \mathbf{H}^{1/2}_{00}(Γ) )</td>
<td>Vectorial Lions-Magenes space on ( Γ \subseteq ∂Ω )</td>
</tr>
<tr>
<td>( H(\text{curl}; Ω) )</td>
<td>Square integrable vector fields with square integrable curl</td>
</tr>
<tr>
<td>( H_0(\text{curl}; Ω) )</td>
<td>Subspace of vector fields in ( H(\text{curl}; Ω) ) with vanishing tangential component</td>
</tr>
<tr>
<td>( H^{δ}(\text{curl}; Ω) )</td>
<td>Subspace of ((H^{δ}(Ω))^3) with curl in ((H^{δ}(Ω))^3)</td>
</tr>
<tr>
<td>( H(\text{div}; Ω) )</td>
<td>Square integrable vector fields with square integrable divergence</td>
</tr>
<tr>
<td>( H_0(\text{div}; Ω) )</td>
<td>Subspace of ( H(\text{div}; Ω) ) with vanishing normal component</td>
</tr>
<tr>
<td>( S^0_p(\mathcal{M}) )</td>
<td>( H^1 )-conforming degree ( p ) global Lagrangian shape functions with respect to the mesh ( \mathcal{M} ) on ( Ω )</td>
</tr>
<tr>
<td>( S^0_p(\mathcal{M}</td>
<td>_{Ω'}) )</td>
</tr>
<tr>
<td>( \mathcal{P}_p(K) )</td>
<td>Space of multivariate polynomials of degree ( p ) on ( K )</td>
</tr>
<tr>
<td>( \mathcal{P}^0_p(K) )</td>
<td>Space of multivariate tensor product polynomials of degree ( p ) on ( K )</td>
</tr>
<tr>
<td>( \mathcal{ND}^1_h(Ω) )</td>
<td>Degree 1 Nédélec edge element space on ( Ω )</td>
</tr>
<tr>
<td>( \mathcal{M}^h_1(Ω) )</td>
<td>Mortar edge element space</td>
</tr>
</tbody>
</table>

### Norms

- \( \| \cdot \|_{L^p(Ω)} \): \( L^p \)-norm for the Borel-measurable set \( Ω \)
- \( |·|_{W^{k,p}} \): Sobolev seminorm of order \( k ∈ \mathbb{R} \) for \( p ∈ \mathbb{N} \cup \{∞\} \)
- \( \|·\|_{W^{k,p}} \): Sobolev norm of order \( k ∈ \mathbb{R} \) for \( p ∈ \mathbb{N} \cup \{∞\} \)
- \( \|·\|_{h,K} \): Local discrete \( L^2 \)-norm

### Operators

- \( γ_0 \): Dirichlet trace
- \( γ_t \): Tangential trace
- \( γ_n \): Normal trace
- \( π_2 \): Projection onto the \( y \)-coordinate
- \( \Pi_h \): Scott-Zhang quasi-interpolant
- \( I^0_h \): Restriction operator restricting finite element functions in \( S^0_h(\mathcal{M}) \) to the interior of \( K \)
- \( \text{curl} \): Curl operator
- \( \text{div} \): Divergence operator
- \( \text{div}_τ \): Tangential divergence operator
- \( \nabla \): Gradient operator
\[ \lambda_e \quad \text{Nédélec local degree of freedom associated to the edge } e \]
\[ B_W^{-1} \quad \text{Jacobi smoother} \]

**Meshes**

\[ \mathcal{M} \quad \text{Mesh on a bounded domain } \Omega \]
\[ n \quad \text{Node of a mesh} \]
\[ \{T^h\} \quad \text{Nested family of meshes with mesh width } h \]
\[ N(T^h) \quad \text{Nodes in to the mesh } T^h \]
\[ \mathcal{E}(T^H) \quad \text{Edges in the mesh } T^h \]
\[ B_T \quad \text{Largest ball inside } T \subseteq \mathbb{R}^n \text{ such that } T \text{ is star-shaped with respect to } B_T \]
\[ (K, P_K, N_K) \quad \text{Finite element on } K \subseteq \mathbb{R}^n \]
\[ h_P \quad \text{Diameter of a compact set } P \subseteq \mathbb{R}^n \]
\[ \Delta_{c_P} \quad \text{Slice through a domain } \Omega \text{ parallel to the } x\text{-axis containing } c_P \]
\[ \Delta_P \quad \text{Closure of the intersection of } P \text{ with the slice } \Delta_{c_P} \]
\[ T(P) \quad \text{Arbitrary equilateral triangle inside } B \cap \Delta_P \text{ where } B \text{ is the largest ball centered at } c_P \text{ contained in } P \]
\[ \mathcal{T}_d \quad \text{Course partition of a polyhedral domain} \]
\[ \Gamma \quad \text{Total interface with respect to a course partition } \mathcal{T}_d \]
\[ \mathcal{V}(\mathcal{T}_d) \quad \text{Nodes of the course partition } \mathcal{T}_d \]
\[ \Gamma_{ij} \quad \text{Interface between neighbouring subdomains } \Omega_i \text{ and } \Omega_j \]
\[ W_k \quad \text{Basket-set of the subdomain } \Omega_k \]
\[ \mathcal{W} \quad \text{Union of all basket sets associated to a course partition } \mathcal{T}_d \]
Bibliography


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