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Author(s):
Boroumand, Parham; Guillén i Fàbregas, Albert

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Error Exponents of Mismatched Likelihood Ratio Testing

Parham Boroumand
University of Cambridge
pb702@cam.ac.uk

Albert Guillén i Fàbregas
ICREA & Universitat Pompeu Fabra
University of Cambridge
guillen@ieee.org

***Abstract*—We study the problem of mismatched likelihood ratio test. We analyze the type-I and II error exponents when the actual distributions generating the observation are different from the distributions used in the test. We derive the worst-case error exponents when the actual distributions generating the data are within a relative entropy ball of the test distributions. In addition, we study the sensitivity of the test for small relative entropy balls.**

I. Introduction and Prelimenaries

Consider the binary hypothesis testing problem [1] where an observation $\boldsymbol{x} = (x_1, \dots, x_n)$ is generated from two possible distributions P_1^n and P_2^n defined on the probability simplex $\mathcal{P}(\mathcal{X}^n)$. We assume that P_1^n and P_2^n are product distributions, i.e., $P_1^n(\boldsymbol{x}) = \prod_{i=1}^n P_1(x_i)$, and similarly for P_2^n. For simplicity, we assume that both $P_1(x) > 0$ and $P_2(x) > 0$ for each $x \in \mathcal{X}$.

Let $\phi : \mathcal{X}^n \to \{1, 2\}$ be a hypothesis test that decides which distribution generated the observation $\boldsymbol{x}$. We consider deterministic tests ϕ that decide in favor of P_1^n if $\boldsymbol{x} \in \mathcal{A}_1$, where $\mathcal{A}_1 \subset \mathcal{X}^n$ is the decision region for the first hypothesis. We define $\mathcal{A}_2 = \mathcal{X}^n \setminus \mathcal{A}_1$ to be the decision region for the second hypothesis. The test performance is measured by the two possible pairwise error probabilities. The type-I and type-II error probabilities are defined as

$$\epsilon_1(\phi) = \sum_{\boldsymbol{x} \in \mathcal{A}_2} P_1^n(\boldsymbol{x}), \quad \epsilon_2(\phi) = \sum_{\boldsymbol{x} \in \mathcal{A}_1} P_2^n(\boldsymbol{x}). \tag{1}$$

A hypothesis test is said to be optimal whenever it achieves the optimal error probability tradeoff given by

$$\alpha_\beta = \min_{\phi : \epsilon_2(\phi) \leq \beta} \epsilon_1(\phi). \tag{2}$$

The likelihood ratio test defined as

$$\phi_\gamma(\boldsymbol{x}) = \mathbb{1}\left\{ \frac{P_2^n(\boldsymbol{x})}{P_1^n(\boldsymbol{x})} \geq e^{n\gamma} \right\} + 1. \tag{3}$$

was shown in [2] to attain the optimal tradeoff (2) for every γ. The type of a sequence $\boldsymbol{x} = (x_1, \dots, x_n)$ is $\hat{T}_{\boldsymbol{x}}(a) = \frac{N(a|\boldsymbol{x})}{n}$, where $N(a|\boldsymbol{x})$ is the number of occurrences of the symbol $a \in \mathcal{X}$ in the string. The likelihood ratio test can also be expressed as a function of the type of the observation $\hat{T}_{\boldsymbol{x}}$ as [3]

$$\phi_\gamma(\hat{T}_{\boldsymbol{x}}) = \mathbb{1}\big\{ D(\hat{T}_{\boldsymbol{x}} \| P_1) - D(\hat{T}_{\boldsymbol{x}} \| P_2) \geq \gamma \big\} + 1. \tag{4}$$

where $D(P\|Q) = \sum_{\mathcal{X}} P(x) \log \frac{P(x)}{Q(x)}$ is the relative entropy between distributions P and Q.

In this paper, we are interested in the asymptotic exponential decay of the pairwise error probabilities. Therefore, it is sufficient to consider deterministic tests. The optimal error exponent tradeoff (E_1, E_2) is defined as

$$E_2(E_1) \triangleq \sup \big\{ E_2 \in \mathbb{R}_+ : \exists \phi, \exists n_0 \in \mathbb{Z}_+ \text{ s.t. } \forall n > n_0 \\ \epsilon_1(\phi) \leq e^{-nE_1} \quad \text{and} \quad \epsilon_2(\phi) \leq e^{-nE_2} \big\}. \tag{5}$$

By using the Sanov's Theorem [3], [4], the optimal error exponent tradeoff (E_1, E_2), attained by the likelihood ratio test, can be shown to be [5], [6]

$$E_1(\phi_\gamma) = \min_{Q \in \mathcal{Q}_1(\gamma)} D(Q \| P_1), \tag{6}$$

$$E_2(\phi_\gamma) = \min_{Q \in \mathcal{Q}_2(\gamma)} D(Q \| P_2), \tag{7}$$

where

$$\mathcal{Q}_1(\gamma) = \big\{ Q \in \mathcal{P}(\mathcal{X}) : D(Q\|P_1) - D(Q\|P_2) \geq \gamma \big\}, \tag{8}$$

$$\mathcal{Q}_2(\gamma) = \big\{ Q \in \mathcal{P}(\mathcal{X}) : D(Q\|P_1) - D(Q\|P_2) \leq \gamma \big\}. \tag{9}$$

The minimizing distribution in (6), (7) is the tilted distribution

$$Q_\lambda(x) = \frac{P_1^{1-\lambda}(x) P_2^\lambda(x)}{\sum_{a \in \mathcal{X}} P_1^{1-\lambda}(a) P_2^\lambda(a)}, \quad 0 \leq \lambda \leq 1 \tag{10}$$

whenever γ satisfies $-D(P_1\|P_2) \leq \gamma \leq D(P_2\|P_1)$. In this case, λ is the solution of

$$D(Q_\lambda \| P_1) - D(Q_\lambda \| P_2) = \gamma. \tag{11}$$

Instead, if $\gamma < -D(P_1\|P_2)$, the optimal distribution in (6) is $Q_\lambda(x) = P_1(x)$ and $E_1(\phi_\gamma) = 0$, and if $\gamma > D(P_2\|P_1)$, the optimal distribution in (7) is $Q_\lambda(x) = P_2(x)$ and $E_2(\phi_\gamma) = 0$.

Equivalently, the dual expressions of (6) and (7) can be derived by substituting the minimizing distribution (10) into the Lagrangian yielding [4], [5]

$$E_1(\phi_\gamma) = \max_{\lambda \geq 0} \lambda\gamma - \log \Big(\sum_{x \in \mathcal{X}} P_1^{1-\lambda}(x) P_2^\lambda(x) \Big), \tag{12}$$

$$E_2(\phi_\gamma) = \max_{\lambda \geq 0} -\lambda\gamma - \log \Big(\sum_{x \in \mathcal{X}} P_1^\lambda(x) P_2^{1-\lambda}(x) \Big). \tag{13}$$

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The Stein regime is defined as the highest error exponent under one hypothesis when the error probability under the other hypothesis is at most some fixed $\epsilon \in (0, \frac{1}{2})$ [3]

$$E_2^{(\epsilon)} \triangleq \sup\big\{E_2 \in \mathbb{R}_+ : \exists \phi, \exists n_0 \in \mathbb{Z}_+ \text{ s.t. } \forall n > n_0 \\ \epsilon_1(\phi) \le \epsilon \quad \text{and} \quad \epsilon_2(\phi) \le e^{-nE_2}\big\}. \quad (14)$$

The optimal $E_2^{(\epsilon)}$, given by [3]

$$E_2^{(\epsilon)} = D(P_1\|P_2), \quad (15)$$

can be achieved by setting the threshold in (4) to be $\gamma = -D(P_1\|P_2) + \frac{C_2}{\sqrt{n}}$, where C_2 is a constant that depends on distributions P_1, P_2 and ϵ.

In this work, we revisit the above results in the case where the distributions used by the likelihood ratio test are not known precisely, and instead, fixed distributions $\hat{P}_1$ and $\hat{P}_2$ are used for testing. In particular, we find the error exponent tradeoff for fixed $\hat{P}_1$ and $\hat{P}_2$ and we study the worst-case tradeoff when the true distributions generating the observation are within a certain distance of the test distributions. The literature in robust hypothesis testing is vast (see e.g., [7]–[9] and references therein). Robust hypothesis testing consists of designing tests that are robust to the inaccuracy of the distributions generating the observation. Instead, we study the error exponent tradeoff performance of the likelihood ratio test for fixed test distributions. The proofs of our results can be found in [10].

II. Mismatched Likelihood Ratio Testing

Let $\hat{P}_1(x)$ and $\hat{P}_2(x)$ be the test distributions used in the likelihood ratio test with threshold $\hat{\gamma}$ given by

$$\hat{\phi}_{\hat{\gamma}}(\hat{T}_{\boldsymbol{x}}) = \mathbb{1}\big\{D(\hat{T}_{\boldsymbol{x}}\|\hat{P}_1) - D(\hat{T}_{\boldsymbol{x}}\|\hat{P}_2) \ge \hat{\gamma}\big\} + 1. \quad (16)$$

For simplicity, we assume that both $\hat{P}_1(x) > 0$ and $\hat{P}_2(x) > 0$ for each $x \in \mathcal{X}$. We are interested in the achievable error exponent of the mismatched likelihood ratio test, i.e.,

$$\hat{E}_2(\hat{E}_1) \triangleq \sup\big\{\hat{E}_2 \in \mathbb{R}_+ : \exists \hat{\gamma}, \exists n_0 \in \mathbb{Z}_+ \text{ s.t. } \forall n > n_0 \\ \epsilon_1(\hat{\phi}_{\hat{\gamma}}) \le e^{-n\hat{E}_1} \text{ and } \epsilon_2(\hat{\phi}_{\hat{\gamma}}) \le e^{-n\hat{E}_2}\big\}. \quad (17)$$

Theorem 1. *For fixed $\hat{P}_1, \hat{P}_2 \in \mathcal{P}(X)$ the optimal error exponent tradeoff in* (17) *is given by*

$$\hat{E}_1(\hat{\phi}_{\hat{\gamma}}) = \min_{Q \in \hat{\mathcal{Q}}_1(\hat{\gamma})} D(Q\|P_1) \quad (18)$$

$$\hat{E}_2(\hat{\phi}_{\hat{\gamma}}) = \min_{Q \in \hat{\mathcal{Q}}_2(\hat{\gamma})} D(Q\|P_2) \quad (19)$$

where

$$\hat{\mathcal{Q}}_1(\hat{\gamma}) = \big\{Q \in \mathcal{P}(\mathcal{X}) : D(Q\|\hat{P}_1) - D(Q\|\hat{P}_2) \ge \hat{\gamma}\big\}, \quad (20)$$

$$\hat{\mathcal{Q}}_2(\hat{\gamma}) = \big\{Q \in \mathcal{P}(\mathcal{X}) : D(Q\|\hat{P}_1) - D(Q\|\hat{P}_2) \le \hat{\gamma}\big\}. \quad (21)$$

The minimizing distributions in (18) *and* (19) *are*

$$\hat{Q}_{\lambda_1}(x) = \frac{P_1(x)\hat{P}_1^{-\lambda_1}(x)\hat{P}_2^{\lambda_1}(x)}{\sum_{a\in\mathcal{X}} P_1(a)\hat{P}_1^{-\lambda_1}(a)\hat{P}_2^{\lambda_1}(a)}, \quad \lambda_1 \ge 0, \quad (22)$$

$$\hat{Q}_{\lambda_2}(x) = \frac{P_2(x)\hat{P}_2^{-\lambda_2}(x)\hat{P}_1^{\lambda_2}(x)}{\sum_{a\in\mathcal{X}} P_2(a)\hat{P}_2^{-\lambda_2}(a)\hat{P}_1^{\lambda_2}(a)}, \quad \lambda_2 \ge 0 \quad (23)$$

respectively, where λ_1 is chosen so that

$$D(\hat{Q}_{\lambda_1}\|\hat{P}_1) - D(\hat{Q}_{\lambda_1}\|\hat{P}_2) = \hat{\gamma}, \quad (24)$$

whenever $D(P_1\|\hat{P}_1) - D(P_1\|\hat{P}_2) \le \hat{\gamma}$, and otherwise, $\hat{Q}_{\lambda_1}(x) = P_1(x)$ and $\hat{E}_1(\hat{\phi}_{\hat{\gamma}}) = 0$. Similarly, $\lambda_2 \ge 0$ is chosen so that

$$D(\hat{Q}_{\lambda_2}\|\hat{P}_1) - D(\hat{Q}_{\lambda_2}\|\hat{P}_2) = \hat{\gamma}, \quad (25)$$

whenever $D(P_2\|\hat{P}_1) - D(P_2\|\hat{P}_2) \ge \hat{\gamma}$, and otherwise, $\hat{Q}_{\lambda_2}(x) = P_2(x)$ and $\hat{E}_2(\hat{\phi}_{\hat{\gamma}}) = 0$. Furthermore, the dual expressions for the type-I and type-II error exponents are

$$\hat{E}_1(\hat{\phi}_{\hat{\gamma}}) = \max_{\lambda\ge 0} \lambda\hat{\gamma} - \log\Big(\sum_{x\in\mathcal{X}} P_1(x)\hat{P}_1^{-\lambda}(x)P_2^{\lambda}(x)\Big), \quad (26)$$

$$\hat{E}_2(\hat{\phi}_{\hat{\gamma}}) = \max_{\lambda\ge 0} -\lambda\hat{\gamma} - \log\Big(\sum_{x\in\mathcal{X}} P_1^{\lambda}(x)P_2(x)\hat{P}_2^{-\lambda}(x)\Big). \quad (27)$$

Remark 1: For mismatched likelihood ratio testing, the optimizing distributions $\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}$ can be different, since the decision regions only depend on the mismatched distributions. However, if $\hat{P}_1, \hat{P}_2$ are tilted with respect to P_1 and P_2, then both $\hat{Q}_{\lambda_1}, \hat{Q}_{\lambda_2}$ are also tilted respect to P_1 and P_2. This implies the result in [11], where for any set of mismatched distributions $\hat{P}_1, \hat{P}_2$ that are tilted with respect to generating distributions, the mismatched likelihood ratio test achieves the optimal error exponent tradeoff in (5).

Theorem 2. *In the Stein regime, the mismatched likelihood ratio test achieves*

$$\hat{E}_2^{(\epsilon)} = \min_{Q \in \hat{\mathcal{Q}}_2(\hat{\gamma})} D(Q\|P_2), \quad (28)$$

with threshold

$$\hat{\gamma} = D(P_1\|\hat{P}_1) - D(P_1\|\hat{P}_2) + \frac{\hat{C}_2}{\sqrt{n}}, \quad (29)$$

and $\hat{C}_2$ is a constant that depends on distributions $P_1, \hat{P}_1, \hat{P}_2$, and ϵ.

Remark 2: Note that since P_1 satisfies the constraint in (28) then $\hat{E}_2^{(\epsilon)} \le E_2^{(\epsilon)}$. In fact, if $\hat{P}_1, \hat{P}_2$ are tilted respect to P_1, P_2 then this inequality is met with equality. Moreover, it is easy to find a set of data and test distributions where $\hat{E}_2^{(\epsilon)} < E_2^{(\epsilon)}$.

III. Mismatched Likelihood Ratio Testing with Uncertainty

In this section, we analyze the worst-case error exponents tradeoff when the actual distributions P_1, P_2 are close to the mismatched test distributions $\hat{P}_1$ and $\hat{P}_2$. More specifically,

$$P_1 \in \mathcal{B}(\hat{P}_1, R_1), \quad P_2 \in \mathcal{B}(\hat{P}_2, R_2) \quad (30)$$

where the D-ball

$$\mathcal{B}(Q, R) = \big\{P \in \mathcal{P}(\mathcal{X}) : D(Q\|P) \le R\big\} \quad (31)$$

is a ball centered at distribution Q containing all distributions whose relative entropy is smaller or equal than radius R. This model was used in robust hypothesis testing in [12]. Figure 1 depicts the mismatched probability distributions and the mismatched likelihood ratio test as a hyperplane dividing the probability space into the two decision regions.

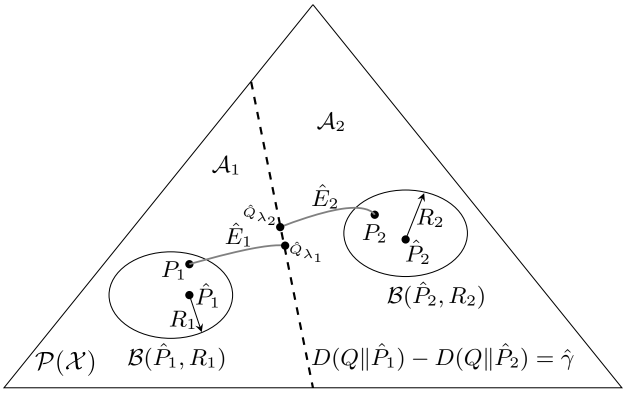

Fig. 1. Mismatched likelihood ratio test over distributions in D-balls.

We study the worst-case error-exponent performance of mismatched likelihood ratio testing when the distributions generating the observation fulfill (30). In particular, we are interested in the least favorable distributions P_1^L, P_2^L in $\mathcal{B}(\hat{P}_1, R_1), \mathcal{B}(\hat{P}_2, R_2)$, i.e., the distributions achieving the lowest error exponents $\hat{E}_1^L(R_1), \hat{E}_2^L(R_2)$.

Theorem 3. *For every $R_1, R_2 \geq 0$ let the least favorable exponents $\hat{E}_1^L(R_1), \hat{E}_2^L(R_2)$ defined as*

$$\hat{E}_1^L(R_1) = \min_{P_1 \in \mathcal{B}(\hat{P}_1, R_1)} \min_{Q \in \hat{\mathcal{Q}}_1(\hat{\gamma})} D(Q\|P_1), \tag{32}$$

$$\hat{E}_2^L(R_2) = \min_{P_2 \in \mathcal{B}(\hat{P}_2, R_2)} \min_{Q \in \hat{\mathcal{Q}}_2(\hat{\gamma})} D(Q\|P_2), \tag{33}$$

where $\hat{\mathcal{Q}}_1(\hat{\gamma}), \hat{\mathcal{Q}}_2(\hat{\gamma})$ are defined in (20)*,* (21)*. Then, for any distribution pair $P_1 \in \mathcal{B}(\hat{P}_1, R_1), P_2 \in \mathcal{B}(\hat{P}_2, R_2)$, the corresponding error exponent pair $(\hat{E}_1, \hat{E}_2)$ satisfies*

$$\hat{E}_1^L(R_1) \leq \hat{E}_1(\hat{\phi}_{\hat{\gamma}}), \quad \hat{E}_2^L(R_2) \leq \hat{E}_2(\hat{\phi}_{\hat{\gamma}}). \tag{34}$$

Furthermore, the optimization problem in (32) *is convex with optimizing distributions*

$$Q_{\lambda_1}^L(x) = \frac{P_1^L(x)\hat{P}_1^{-\lambda_1}(x)\hat{P}_2^{\lambda_1}(x)}{\sum_{a\in\mathcal{X}} P_1^L(a)\hat{P}_1^{-\lambda_1}(a)\hat{P}_2^{\lambda_1}(a)}, \tag{35}$$

$$P_1^L(x) = \beta_1 Q_{\lambda_1}^L(x) + (1-\beta_1)\hat{P}_1(x), \tag{36}$$

where $\lambda_1 \geq 0, 0 \leq \beta_1 \leq 1$ are chosen such that

$$D(Q_{\lambda_1}^L\|\hat{P}_1) - D(Q_{\lambda_1}^L\|\hat{P}_2) = \hat{\gamma}, \tag{37}$$

$$D(\hat{P}_1\|P_1^L) = R_1, \tag{38}$$

when

$$\max_{P_1 \in \mathcal{B}(\hat{P}_1, R_1)} D(P_1\|\hat{P}_1) - D(P_1\|\hat{P}_2) \leq \hat{\gamma}. \tag{39}$$

Otherwise, we can find a least favorable distribution $P_1^L \in \mathcal{B}(\hat{P}_1, R_1)$ such that $\hat{E}_1(\hat{\phi}_{\hat{\gamma}})$ for this distribution is $\hat{E}_1(\hat{\phi}_{\hat{\gamma}}) = 0$. Similarly, the optimization (33) *is convex with optimizing distributions*

$$Q_{\lambda_2}^L(x) = \frac{P_2^L(x)\hat{P}_2^{-\lambda_2}(x)\hat{P}_1^{\lambda_2}(x)}{\sum_{a\in\mathcal{X}} P_2^L(a)\hat{P}_2^{-\lambda_2}(a)\hat{P}_1^{\lambda_2}(a)}, \tag{40}$$

$$P_2^L(x) = \beta_2 Q_{\lambda_2}^L(x) + (1-\beta_2)\hat{P}_2(x), \tag{41}$$

where $\lambda_2 \geq 0, 0 \leq \beta_2 \leq 1$ are chosen such that

$$D(Q_{\lambda_2}^L\|\hat{P}_2) - DQ_{\lambda_2}^L\|\hat{P}_1) = \hat{\gamma}, \tag{42}$$

$$D(\hat{P}_2\|P_2^L) = R_2, \tag{43}$$

whenever,

$$\min_{P_2 \in \mathcal{B}(\hat{P}_2, R_2)} D(P_2\|\hat{P}_1) - D(P_2\|\hat{P}_2) \geq \hat{\gamma}. \tag{44}$$

Otherwise, we can find a distribution $P_2^L \in \mathcal{B}(\hat{P}_2, R_2)$ such that $\hat{E}_2(\hat{\phi}_{\hat{\gamma}})$ for this distribution is $\hat{E}_2(\hat{\phi}_{\hat{\gamma}}) = 0$.

The worst-case achievable error exponents of mismatched likelihood ratio testing for data distributions in a D-ball are essentially the minimum relative entropy between two sets of probability distributions. Specifically, the minimum relative entropy $\mathcal{B}(\hat{P}_1, R_1)$ and $\hat{\mathcal{Q}}_2(\hat{\gamma})$ gives $\hat{E}_1^L(R_1)$, and similarly for $\hat{E}_2^L(R_2)$.

IV. Mismatched Likelihood Ratio Testing Sensitivity

In this section, we study how the worst-case error exponents $(\hat{E}_1^L, \hat{E}_2^L)$ behave when the D-ball radii R_1, R_2 are small. In particular, we derive a Taylor series expansion of the worst-case error exponent. This approximation can also be interpreted as the worst-case sensitivity of the test, i.e., how does the test perform when actual distributions are very close to the mismatched distributions.

Theorem 4. *For every $R_i \geq 0$, $\hat{P}_i \in \mathcal{P}(\mathcal{X})$ for $i = 1, 2$, and*

$$-D(\hat{P}_1\|\hat{P}_2) \leq \hat{\gamma} \leq D(\hat{P}_2\|\hat{P}_1), \tag{45}$$

we have

$$\hat{E}_i^L(R_i) = E_i(\hat{\phi}_{\hat{\gamma}}) - S_i(\hat{P}_1, \hat{P}_2, \hat{\gamma})\sqrt{R_i} + o\big(\sqrt{R_i}\big), \tag{46}$$

where

$$S_i^2(\hat{P}_1, \hat{P}_2, \hat{\gamma}) = 2Var_{\hat{P}_i}\left(\frac{\hat{Q}_\lambda(X)}{\hat{P}_i(X)}\right) \tag{47}$$

and $\hat{Q}_\lambda(X)$ is the minimizing distribution in (10) *for test $\hat{\phi}_{\hat{\gamma}}$.*

Lemma 5. *For every $\hat{P}_1, \hat{P}_2 \in \mathcal{P}(\mathcal{X})$, and $\hat{\gamma}$ satisfying* (45)

$$\frac{\partial}{\partial\hat{\gamma}} S_1(\hat{P}_1, \hat{P}_2, \hat{\gamma}) \geq 0, \quad \frac{\partial}{\partial\hat{\gamma}} S_2(\hat{P}_1, \hat{P}_2, \hat{\gamma}) \leq 0. \tag{48}$$

This lemma shows that $S_1(\hat{P}_1, \hat{P}_2, \hat{\gamma})$ is a non-decreasing function of $\hat{\gamma}$, i.e., as $\hat{\gamma}$ increases from $-D(\hat{P}_1\|\hat{P}_2)$ to $D(\hat{P}_2\|\hat{P}_1)$, the worst-case exponent $\hat{E}_1^L(R_1)$ becomes more sensitive to mismatch with likelihood ratio testing. Conversely, $S_2(\hat{P}_1, \hat{P}_2, \hat{\gamma})$ is a non-increasing function of $\hat{\gamma}$, i.e., as $\hat{\gamma}$

increases from $-D(\hat{P}_1\|\hat{P}_2)$ to $D(\hat{P}_2\|\hat{P}_1)$, the worst-case exponent $\hat{E}_2^L(R_2)$ becomes less sensitive (more robust) to mismatch with likelihood ratio testing. Moreover, when $\lambda = \frac{1}{2}$, we have

$$\hat{Q}_{\frac{1}{2}}(x) = \frac{\sqrt{\hat{P}_1(x)\hat{P}_2(x)}}{\sum_{a\in\mathcal{X}}\sqrt{\hat{P}_1(a)\hat{P}_2(a)}}, \tag{49}$$

and then $S_1(\hat{P}_1, \hat{P}_2, \hat{\gamma}) = S_2(\hat{P}_1, \hat{P}_2, \hat{\gamma})$. In addition, $\hat{Q}_{\frac{1}{2}}$ minimizes $E_1(\hat{\phi}_{\hat{\gamma}}) + E_2(\hat{\phi}_{\hat{\gamma}})$ yielding [13]

$$E_1(\hat{\phi}_{\hat{\gamma}}) + E_2(\hat{\phi}_{\hat{\gamma}}) = \min_{Q\in\mathcal{P}(\mathcal{X})} D(Q\|\hat{P}_1) + D(Q\|\hat{P}_2) \tag{50}$$

$$= 2B(\hat{P}_1, \hat{P}_2) \tag{51}$$

where $B(\hat{P}_1, \hat{P}_2)$ is the Bhattacharyya distance between the mismatched distributions $\hat{P}_1$ and $\hat{P}_2$. This suggests that having equal sensitivity (or robustness) for both hypotheses minimizes the sum of the exponents.

Example 1. When $\gamma = 0$ the likelihood ratio test becomes the maximum-likelihood test, which is known to achieve the lowest average probability of error in the Bayes setting for equal priors. For fixed priors π_1, π_2, the error probability in the Bayes setting is $\bar{\epsilon} = \pi_1\epsilon_1 + \pi_2\epsilon_2$, resulting in the following error exponent [3]

$$\bar{E} = \lim_{n\to\infty} \frac{1}{n} \log \bar{\epsilon} = \min\{E_1, E_2\}. \tag{52}$$

Consider $\hat{P}_1 = \text{Bern}(0.1)$, $\hat{P}_2 = \text{Bern}(0.8)$. Also, assume $R_1 = R_2 = R$. Figure 2 shows the worst-case error exponent in the Bayes setting given by $\min\{\hat{E}_1^L, \hat{E}_2^L\}$ by solving (32) and (33) as well as $\min\{\tilde{E}_1^L, \tilde{E}_2^L\}$ using the approximation in (46). We can see that the approximation is good for small R. Moreover, it can be seen that error exponents are very sensitive to mismatch for small R, i.e., the slope of the worst-case exponent goes to infinity as R approaches to zero.

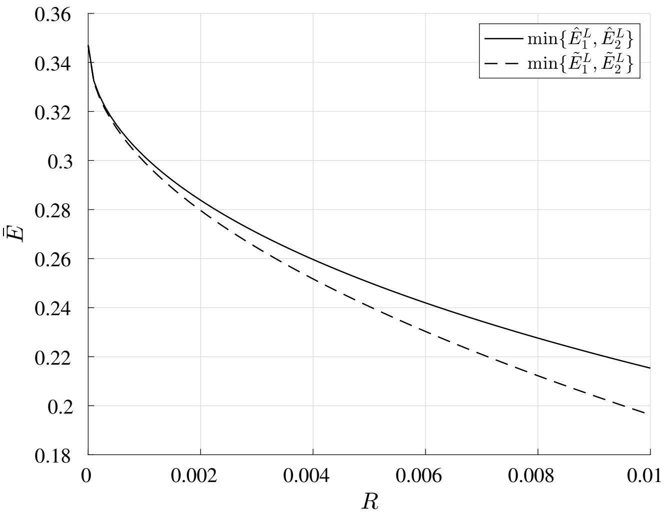

Fig. 2. Worst-case achievable Bayes error exponent.

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