Doctoral Thesis

Smooth Poly-Hypar Surface Structures
A new approach to design freeform surfaces by combining hyperbolic paraboloids

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Publication Date:
2019-12

Permanent Link:
https://doi.org/10.3929/ethz-b-000404744

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Smooth Poly-Hypar Surface Structures
A new approach to design freeform surfaces by combining hyperbolic paraboloids

A thesis submitted to attain the degree of

DOCTOR OF SCIENCES of ETH ZURICH
(Dr. sc. ETH Zurich)

presented by
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Prof. Dr. Toni Kotnik (Aalto University)

2019
Acknowledgement

First of all, I would like to thank my advisor, Prof. Dr. Joseph Schwartz, for guiding my research and for leading me to the world of structural arts. I deeply value his support of my research. I am particularly grateful to him for giving me the freedom to develop my research in the inspiring environment of the Chair of Structural Design at ETH Zurich. I also want to thank him for his encouragement to cooperate with other institutes and companies.

I would like to express my gratitude to my co-advisor, Prof. Dr. Toni Kotnik, for his guidance from the very beginning of my research, and for providing me with some key information regarding my research topic. I am also grateful to him for extending the theoretical background of my research in the field of architecture.

In addition, I would also like to thank my colleagues, Pierluigi D’Acunto, Enrique Monzo Lluis, Juan José Castellón, Mario Rinke, Jaray Bergianti Ursula, and Alessandro Tellini from Raplab. Pierluigi D’Acunto and Enrique Monzo Lluis gave me patient consultations during my learning of graphic statics, and helped me to adapt quickly to the teachings and research of our chair. Together with Alessandro Tellini and Juan José Castellón, Pierluigi D’Acunto also gave me suggestions regarding the design and implementations of some case studies in my research. I strongly appreciate their support during the workshop with Southeast University in China, in fabricating a case study project called the Hypar Pavilion. Together with them, I had a chance to explore the materialization and fabrication of smooth poly-hypar surfaces. Besides support in my field of research, Mario Rinke and Jaray Bergianti Ursula also provided help in my life in a foreign country, encouraging me to integrate and become acquainted with the cultural milieu in Switzerland.

Moreover, I am grateful for the chance provided by Prof. Dr. Zhang Hong in Southeast University to build the first prototype of a smooth poly-hypar surface.

Finally, I would like to thank my family: my parents, for their constant support, especially after my son was born; my husband, for his help in some of the digital implementations of my research; and last but not least, my son, for all the joy that he brings to me.
Abstract

This dissertation introduces smooth poly-hypar surfaces, a new category of freeform surfaces, which integrates architectural forms with structural efficiency and ease of construction. As a combination of hyperbolic paraboloids (hypars), a smooth poly-hypar surface is ruled locally, while globally appearing to be continuous freeform. It achieves structural stiffness through the double curved shape, ensures bending-free behavior by smoothly connecting adjacent hypars, meanwhile indicates a relatively easy fabrication method due to being locally ruled.

Based on the special geometrical and structural properties of smooth poly-hypar surfaces, the research presented in this dissertation aims to solve a conflict between the freedom of forms and the technical constraints in freeform surface design. In investigating this, it explicates the structural principles lying behind smooth poly-hypar surfaces, presents both intuitive visualizations and precise evaluations of static behaviors, and explores an operative method to design double curved freeform surfaces, which enables structural considerations to be involved from the initial design stages. Moreover, this research also shows the potential to approximate other types of freeform surfaces with smooth poly-hypar surfaces, thereby simplifying complex geometries, optimizing structural efficiencies, and reducing construction difficulties.

As an extension to existing research on hypars, this dissertation makes use of graphic statics to evaluate the behavior of hypars by finding a balancing point between complex mathematical calculations and oversimplified visualizations. It concludes that the behavior of an individual hypar can be considered as a combination of a wall and a shell, with reactions only along rulings and edges. Based on this, a new solution to resist the forces at the edges of a hypar arises, which replaces rigid edge beams, and thereby resulting smooth connections between adjacent hypars in poly-hypar surfaces. There are two primary preconditions to ensure the bending-free behavior and global equilibrium of smooth poly-hypar surfaces: the coplanarity principle and fully supported load paths. By respecting the coplanarity principle, hypars can be joined in such a way that all intersecting rulings and edges are coplanar. Thereby the interactions between adjacent hypars are always in plane without activating bending moments. The concept of load paths is also introduced to check the global equilibrium of smooth poly-hypar surfaces. Once all load paths are supported, a smooth poly-hypar surface turns from an abstract geometry into an efficient surface structure.

By following the coplanarity principle and ensuring fully supported load paths, a general method to design smooth poly-hypar surfaces is presented in this dissertation. All smooth poly-hypar surfaces can be dissolved into two basic prototypes. By free arrangement of these two prototypes, following an additive sequence, various smooth poly-hypar surfaces can be generated. Through the parametrizations of these two prototypes, a smooth poly-hypar surface becomes a parametric typology, which can continuously diverge into a collection of surfaces, adopting different design contexts.

To testify the design method and explore architectural potentials, smooth poly-hypar surfaces were applied in various teaching and design tasks. In several case studies, low-tech construction methods were developed, both in light-weight grid shells as well as concrete shells. The results
clearly show the advantages of smooth poly-hypar surfaces in construction. As a modular system, a smooth poly-hypar surface can be prefabricated as individual hypar modules, then assembled on site. Benefiting from their structural stiffness and the property of being ruled surfaces, neither grid shells nor concrete shells require scaffolding or formwork during construction.

As such, smooth poly-hypar surfaces can be seen as the mediators between architectural smoothness, structural efficiency and construction expediency.
Kurzfassung

Diese Dissertation präsentiert glatte Poly-hypar-Oberflächen, eine neue Kategorie von Freiformflächen, die identisch mit Strukturen und Konstruktionen sind. Als Kombination von hyperbolischen Paraboloiden (Hypars) wird die glatte polyhypare Oberfläche lokal zu einer Regelfläche, während sie global als Freiform erscheint. Es archiviert die strukturelle Steifigkeit durch die doppelt geschwungene Form, sorgt für ein freibiegendes Verhalten durch sanftes Verbinden benachbarter Hypars und zeigt gleichzeitig eine relativ einfache Herstellungsmethode durch die Eigenschaft der lokalen Beherrschbarkeit.


In Anlehnung an das Koplanaritätsprinzip und die Lastpfade wird in dieser Forschung eine allgemeine Methode zur Gestaltung einer glatten Poly-hypar-Oberfläche vorgestellt. Alle glatten Poly-hypar-Oberflächen können in zwei grundlegende Prototypen aufgelöst werden.


Letztlich werden glatte Poly-hypar-Oberflächen zu Vermittlern für integrierte architektonische Oberflächenglättung, strukturelle Effizienz und eine Vereinfachung der Bauweise.
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Nomenclature

Table 1 shows in this dissertation the main structural terms represented in colored graphic.

Table 1

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<th>Tension</th>
<th>Load</th>
<th>Reaction</th>
<th>Load component</th>
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The indexes of nodes, parabolas and rulings, the names of forces mentioned in Chapter 4 are listed as below:

Table 2

<table>
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<tr>
<th>Node((m, k))</th>
<th>The node where rulings (h_m) and ruling (l_k) intersect</th>
</tr>
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<tr>
<td>(P_{l(m,k/l,j)})</td>
<td>A parabola passing through any (Node(m, k)) and (Node(l, j))</td>
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<tr>
<td>(\vec{f}_{r(m,k)})</td>
<td>Axis component of point load (\vec{g}) applied to (Node(m, k)), parallel with axis (r)</td>
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<td>(\vec{f}_{h(m,k)})</td>
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<tr>
<td>(\vec{f}_{r, Ar(m,k)})</td>
<td>a part of the axis component (\vec{f}_{r(m,k)}), applied to the parabolic arch passing through (Node(m, k))</td>
</tr>
<tr>
<td>(\vec{f}_{r, C(l,m,k)})</td>
<td>the other part of the axis component (\vec{f}_{r(m,k)}), applied to the parabolic cable passing through (Node(m, k))</td>
</tr>
<tr>
<td>(\vec{R}_{h(m)})</td>
<td>The accumulated reactions along ruling (h_m)</td>
</tr>
<tr>
<td>(\vec{R}_{i(m)})</td>
<td>The accumulated reactions along ruling (i_m)</td>
</tr>
<tr>
<td>(\vec{R}<em>{h(0), h(2n)}), (\vec{R}</em>{i(0), i(2n)})</td>
<td>The maximal reactions along edges (h_0, h_{2n}, i_0, i_{2n}).</td>
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Part I

Introduction
1 Introduction

1.1 Introduction

This dissertation develops a new approach towards design and fabrication of double curved freeform surface structures in architecture, based on smooth combinations of hyperbolic paraboloids (smooth poly-hypar surfaces).

1.1.1 The continuity of new architectural forms in digital times

Since the beginning of the 1990s, architectural designs containing freeform surfaces have started appearing more and more frequently (Figure 1-1a). Such tendency towards form smoothness was derived from the theory of continuity: a stream merged in architectural theory to overcome the fragmentation of deconstructionism, while pursued coherent integration of heterogeneity (Lynn, 1993). In the early stage of such discourse, it mainly stressed the continuous variation process of forms, and the principles lying behind (Carpo, 2014). Particularly, such a generative process is an analogy to evolution involving the influence of the environmental context. As organic shapes evolve in a gradient environment, the result of a continuous and multiplicitous process in architectural design should likewise turn to be smooth and curvilinear, without distinct contours or edges (Lynn, 1999).

With an application of computer technology in architectural design, the differential calculus lying behind the continuity theory was reemphasized, and implemented in computers as mathematical functions. It enabled a parametric manipulation of forms as well as a visualization of smooth curvilinear shapes. In this way, the theory of continuity interacted with the simultaneous revolution in computer technology, aroused a discourse of new architectural forms in digital times (Carpo, 2014).

Figure 1-1: (a) Explorations of new architectural forms: Walt Disney Concert Hall by Frank Gehry, 1988 (Soko & Mafi, 2018). Innsbruck Railway Station by Zaha Hadid, 2004 (Innsbruckinfo, 2016). (b) Sydney opera house is considered to be a historical starting point of New Structuralism in digital times (Rice, 1998)
1.1.2 The rereading of architectural continuity with structural performance

However, in most cases, smooth freeform surfaces were shallowly treated as a symbol of new digital forms, while the coherent generation process, and the multiple influencing factors were ignored (Carpo, 2014). Once an architectural form was treated as a solely geometrical shape, a series of technical problems in performance and construction were raised.

In this context, an early discourse of formal and form generative continuity, moved beyond its original scope and extended into a more scientific, computational, and technological direction. One branch of this technical movement—Performative Design—focused on the influence of technical performance in the complex processes of design synthesis (Oxman & Oxman, 2014). It reemphasized the concept of evolution, and explicated technical performances as environmental factors, which affect the generation of architectural forms. As an influencing factor of a continuous evolution, technical performance should be taken into account throughout the entire design process, instead of just an optimization stage at the end (Oxman & Oxman, 2014). Such an idea of interdisciplinary integration, in terms of structures, materials, and construction, turned into a theory called the New Structuralism (Figure 1-1b), which proposed a highly iterative and interactive design process involving multiple technical feedbacks (Bollinger, et al., 2010). Searching for such design synthesis solution, the architectural theorists of New Structuralism referred to corresponding explorations in the engineering field by structural artists (Billington, 1985), or engineers-architects (Pogacnik, 2012) in the middle of 20th century. Their explorations to intuitively integrate the structural consideration into the entire form generating process, were praised in the New Structuralism as "a significant concept that changes the traditional meaning of structural performance" (Oxman, 2008).

1.1.3 Structural & architectural integration in works of structural artists

Even nowadays, many explorations by structural artists, referring to the interdisciplinary design synthesis proposed in the New Structuralism, are still rather inspiring. Along Heinz Isler, the work of Sergio Musmeci, Frei Otto, or Felix Candela nicely illustrates not only the generative continuity and form smoothness proposed in the early Theory of Continuity, but also shows the interdisciplinary interactions emphasized by the New Structuralism. This appears particularly in the design of curved surface structures.

The main focus of structural artists, from Thomas Telford to Robert Maillart, from Heinz Isler to Felix Candela, was aimed at the reconciliation of engineering and architectural thinking in the design process. Ever since the nineteenth century, when the engineer profession was formally separated from the one of the architect, the discourse regarding the integration of these two disciplines has never stopped (Saint, 2007). It revolves around two different perspectives concerning form and its development: on the one hand, structural form and the search for efficiency, on the other hand, architectural form and space. In order to bring these two aspects together, structural artists normally started the design intuitively, according to various experiments or experiences, and only later validated the design through calculations.
1.1 Introduction

(Faber, 1963) (Billington, 1985). The intuitive design stage enabled an interactive loop between the architectural design thinking and structural consideration, simultaneously avoiding complex mathematical calculations, which may impede the creativity in design (Billington, 1965).

Form-finding is an approach relying on experiments, which are necessary in understanding the relation between forms and forces. It was applied frequently in the design of curved surface structures by structural artists like Heinz Isler, Sergio Musmeci, or Frei Otto. They developed various form-finding methods based on physical equilibrium models, like hanging fabric (Figure 1-2a) or stretched membrane (Figure 2-5). The concept of these experiments is rather simple: upon a dominant loading, a flexible surface deforms until its internal tensile stresses are in equilibrium with external loads (Bechthold, 2008). Through the variation of boundary conditions, the resulting equilibrium surface can be adapted to a specific architectural requirement. The dominant loading applied to the physical model in order to cause the deformation, creates characteristic structural behavior in the generated shapes. The resulting forms, due to the effect of the distributed loads and the continuity of internal forces, appear as smooth freeform surfaces. From this perspective, form smoothness and generative continuity of form are spontaneously achieved through the form-finding. These form-finding methods are generally carried out considering one dominant loading situation. Consequently, the resulting forms are bending-free only for specific load configurations. Moreover, without considering the construction method during the form generation process, the discovered freeform geometries typically require complex molds and scaffoldings to be constructed (Figure 1-2), (Chilton, 2000).

Figure 1-2: (a) Isler’s hanging membrane model generated by self-setting polyester resin. (b) Drawings of the formwork profiles in the Grötzingen by Isler. (Chilton, 2000).
1.1.4 Geometries as mediators between architectural and structural design

Alternatively, another approach related to a special geometry—hyperbolic paraboloids (hypars)—was introduced into architectural design by structure artists, like Antonio Gaudi, Eduardo Torroja, Pier Luigi Nervi, Felix Candela, among others. Contrary to form-finding, in this approach the structures are not the result of a self-forming process, but they are defined based on the geometric properties of the hypars. As a double curved and double ruled geometry, the surface of a hypar can be defined through two sets of straight rulings, or two sets of parabolas (Figure 2-6a). The descriptive geometrical properties of a hypar enable a simple way to visualize its behavior as a set of two parabolas: one in tension, and one in compression (Figure 1-3), (Parme, 1956), (Garlock & Billington, 2008). Benefiting from the simple behavior of a hypar, designers can manipulate the geometrical variations, and also intuitively understand their structural behavior (Faber, 1963). In this way, the iterative integration of architectural concepts and engineering thinking becomes possible during the design process. Additionally, the double curved and double ruled properties enable hypars to support various loading conditions by activating only membrane forces (Figure 1-3), (Almond, 1933), (Tester, 1947), (Candela, 1955), (Parme, 1956), (Scordelis, et al., 1969), besides indicating the construction in a relatively simple way: with straight components or formwork (Figure 1-5). Furthermore, from an architectural standpoint, through simple manipulations of four vertices of a hypar, it can be transformed into a wide range of forms to satisfy various architectural requests (Figure 1-4), especially when they are combined and used creatively. The flexibility and versatility of architectural forms designed with hypars is reflected very well in the built projects of Felix Candela, through their different spatial qualities and functions (Figure 1-6), (Faber, 1963), (Richardson, 1989), (Nordenson, 2008).

Figure 1-3: Compared with flat surfaces or funicular shapes, hypars can still withstand membrane stresses under undistributed loads.
1.1 Introduction

In these precedents, hypars worked as geometrical mediators, showing an impressive potential in integrating different aspects like architectural space, structural efficiency, and construction availability in the design. However, there still remain some unsolved problems (Schnobrich, 1988a), both structural and geometrical. The current research on hypars did not cover the whole range of hypar variations, but focused on a special category (Candela, 1955),(Figure 2-6a). Moreover, the analysis of hypars, either in the membrane theory or the bending theory, (Almond, 1933), (Tester, 1947), (Candela, 1955), (Scordelis, et al., 1969) was so complex that
their application was only limited to a validation stage (Faber, 1963). Furthermore, regular rigid beams were used as a solution to resist forces perpendicular to the edges of hypars.

According to the current research, a hypar generally maintains pure membrane forces when two of its edges are fully supported. The rigid edge beams (Figure 1-6) were thought to be able to resist forces in any direction (Candela, 1960). However, such a solution was later proved inadequate in preventing horizontal displacement (Billington, 1965), and even causing additional loading rather than reinforcing the shell (Schnobrich, 1972). These shortcomings of the rigid edge beams cause the behavior of the surface shell to deviate from the expectations in the membrane theory. In his works, Candela avoided the horizontal displacement of the edge beams, by connecting the hypars symmetrically through folds (folded poly-hypar surfaces) (Figure 2-20 to Figure 2-22). In this case, the horizontal interactions between adjacent hypars were cancelled out, leaving only vertical loads taken by the rigid beams (Candela, 1960). Impeding by the complex calculation of internal stresses in a hypar, the understanding of interactions between adjacent hypars, turned very difficult to go beyond the beam solution at that time. At that time, hindered by the complex calculations of internal hypar stresses, the understanding of adjacent hypar interactions made it very difficult to go beyond the simple beam solution. Such a combination of curved folds and rigid beams not only led to problems in structural efficiency, but also interrupted the visual continuity and smoothness of the geometries. Actually, the geometrical discontinuity and the uneven distribution of forces in the folded poly-hypar surfaces may turn out to be two sides of the same problem. The study of poly-hypar surfaces, could be extended to encompass various levels of smoothness, which, through the interactions between adjacent hypars, may lead to a more design-oriented solution.

![Figure 1-7: (a)The steel sculpture of Angel Duarte, a smooth combination of six hypars, 1965-1967. (b) The steel sculpture of Angel Duarte in Lausanne, 1977. (c) Hyperbolic nets in the mathematic research of Huhnen Emanuel (Emanuel & Thilo, 2014)](image)

In the works of Spanish artist Angel Duarte (Figure 1-7b) (Duarte, 1992), and some recent research on architectural geometry (Craizer, et al., 2009), (Käferböck & Pottmann, 2013), (Emanuel & Thilo, 2014), instead of combing hypars in a discrete way, they were merged into smooth poly-hypar surfaces, sometimes referred as hyperbolic nets (Käferböck & Pottmann, 2013),
1.2 Statement of goals

According to the discussions above, there is a necessity to develop an operative method in order to steer the use of smooth poly-hypar surfaces in freeform design, simultaneously guaranteeing structural efficiency. Here are several goals aimed to meet in this thesis:

- A simple approach to intuitively visualize and precisely calculate the structural behavior of an individual hypar
- Geometrical principles to connect hypars smoothly and ensure the global equilibrium of poly-hypar surface, without causing a bending moment
- A method to calculate internal forces and reactions of smooth poly-hypar surfaces
- Design method to control and manipulate geometrical variations of smooth poly-hypar surfaces

1.3 Outline of chapters

The scope of this research is to introduce a novel approach to the design of freeform surface structures as smooth poly-hypar surfaces. Instead of joining hypars through folded edges, as was typically the case in the existing precedents, the approach proposed in this research combines different hypars smoothly, thus guaranteeing overall bending-free behavior of the structure.

This dissertation is divided into five parts.

Part I (Chapter 1 to 2) includes introductions and literature reviews. Chapter 1 explicates the motivation and goals of the research, and end up with current state of the art. Chapter 2 reviews the relevant literatures and discusses the origin and development of
freeform surfaces, both from architectural and structural standpoints. Previous design methods of curved surface structures are reviewed and assessed, followed by a critical evaluation of design approach towards surface structures based on hypars.

Part II and III present the smooth poly-hypar surface structures – the new approach developed in this research.

Part II (Chapter 3 to 5) focuses on the structural analysis of smooth poly-hypar surfaces. Chapter 3 states the fundamentals, assumptions, as well as outlines the main steps of the analysis. Chapter 4 describes the structural behavior of an individual hypar, presenting both intuitive visualization of its performance with graphic statics and precise calculation based on linear vector algebra. Chapter 5 extends the analysis to smooth poly-hypar surfaces, examining the interactions between adjacent hypars as well as introducing the coplanarity principle to ensure in-plane transmission of internal forces. It presents the load path as an intuitive way to structurally evaluate generated geometries, and formulates a precise calculation of internal forces as a linear algebraic problem.

Part III (Chapter 6) elaborates the design method. It first discusses how to create and manipulate smooth poly-hypar surfaces—starting with two basic prototypes, following a description of an additive form-making sequence based on these two prototypes—and ending with the digital implementation of design method. Later it also shows the results of using smooth poly-hypar surfaces for an approximation of existing freeform surface, indicating the potential in simplifying complex surface structures.

Part IV (Chapter 7) presents the applications of smooth poly-hypar surfaces. It first presents two case studies in architectural teaching to assess to the reliability of the design method. In order to validate this method and demonstrate its use in freeform design and fabrication, it introduces two built pavilions employing low-tech construction techniques and different building materials.

Part V (Chapter 8) lists general conclusions and outlines future work in this field.
2 Literature review

This chapter reviews the relevant literature of freeform surface structures in the following fields: architectural theories of continuity; interdisciplinary integration in design of curved surface structures, especially in shells and buildings made with hypars; and approximations of freeform surfaces with hypars in architectural geometry.

2.1 Architectural theories of continuity

The discourse on continuity in architecture appeared in the early 1990s, in response to the fragmentation of deconstructivism. Peter Eisenman was the first to introduce the notion of the fold as the generator of continuity (Carpo, 2014). Eisenman’s interest in the fold and its significance in the field of architecture derived from his reading of Gilles Deleuze’s book “The Fold: Leibniz and the Baroque” (Deleuze, 1988). After the first explorations by Eisenman, the generative continuity was proved to attract enough discourse not only to complete, but also to extend its content. This became especially evident in Greg Lyn’s interpretation, relating it with motion, time, and environmental influences (Lynn, 1999). The theory of continuity was later implemented in newly developed computer technologies, in such a way that the formal generative continuity is represented computationally through differential calculus. This change has also resulted in smoothness of forms. The development of digital technologies has not only encouraged architectural forms to progress from folds to smooth curvilinearity (Carpo, 2014), but also guided the early exploration of continuity into more scientific directions. One of these directions focuses on the synthesis of architectural performance, i.e. structural efficiency, in one of which focusing on how to integrate architecture performance like structural efficiency into the continuous design synthesis (Oxman & Oxman, 2014).

2.1.1 The generative continuity of form

Gilles Deleuze: The Fold, Leibniz and the Baroque

The fold, as a metaphor for continuity, originated from Deleuze’s interpretation of the works of the philosopher and mathematician Gottfried Wilhelm Leibniz. The fold was regarded by Deleuze as a certain variation instead of an actual physical entity (Deleuze, 1988). He mentioned that Baroque principles differentiate folds according to two different levels: the pleats of matter and the folds in the soul. He explained the former pleats using two distinct variations in matter: the inorganic fold, and the organic fold. The formation of the inorganic fold, with the fluidity of matter and the elasticity of bodies in mind, enables the matter to be divided into ‘cohering parts that form folds, such that they are not separated into parts but are rather divided to infinity in smaller and smaller folds that always retain a certain cohesion’ (Deleuze, 1988). In this case, the basic unit of matter is not a point but rather coherent folds: matter consists of an infinity of ever smaller, recursively folded folds, or differences. In contrast to the inorganic fold, Deleuze explained organic folds as forces exerted on matter, brought about mechanically by
the influence and actions of the surroundings. Note that the surroundings can be external or internal to the matter in question. As the mechanism of matter, it organizes the variation of matter, indicating “not only the infinite coherent folds, but also progressivity in the gain and loss of movement … the matter-fold becomes a matter-time” (Deleuze, 1988). Under the action of exterior surroundings and influences of the derived internal forces, the variation of matter does not go from small to large through aggregations, but rather from the general to the special. In this case, inorganic mass is developed into matter with life and organisms, and the organic folding turns into evolution.

Moving from the variation of matter, in the part of *Folds in the Soul*, Deleuze described his understanding of the universal world. The world was conceived by Deleuze as a developing temporal continuum (Lynn, 1999). The primitive element of such a world is not a point but the ‘*monad*’, which is a position in space that can only be calculated continuously as a vector flow (Benjamin, 1993). Deleuze’s conceptions of the world and the monad directly reflected Leibniz’s differential calculus description of a continuous curve: instead of a static curve, it indicates the way the curve varies. The mathematical notion of Leibniz’s inflection point is especially connected with variations of curves. In Deleuze’s interpretation, the inflecting point becomes a certain property of the curve, a variable of the curve, or variation itself (Lynn, 1999). Derived from such mathematical studies of variation is the notion of ‘*objectile*’: a new conception, which, instead of representing a static state of an object, expresses all its possible infinite variations (Carpo, 2014). It actually reflects the continuous multiplicity.

**Peter Eisenman’s reading of the Folds**

As Carpo described, the Deleuzian fold was granted a second life when Peter Eisenman began to elaborate an architectural version of it (Carpo, 2014). In the early stage of Eisenman’s reading of the Deleuze fold, he borrowed the ideal of ‘*objectile*’ to describe a new notion of form that can change, morph and move (Eisenman, 1992). While later in his essay *folding in time*, he replaced it with the contiguous conception of the ‘*object event*’ (Eisenman, 1993). His readings suggested that the event is the extension of a frame in time, implying the continuous variation of matter. Time in this case is not a linear narrative time in the mechanical paradigm, but a folded time in the electronic paradigm; the time of present also includes past and future. This understanding of time in the digital age can be used to identify a special property of forms: singularity, indicating the differences residing in time. In this case, Eisenman read the fold as a certain extension in time, which requests architectural forms capable of continuous variations: forms that move in time (Carpo, 2014).

**Greg Lyn: Animate Form**

After Eisenman introduced the Deleuze’s fold into architectural theory, this discourse of continuity produced a boom of publication, including the work of Greg Lyn. Ever since, the reading of Deleuze’s fold within the field of architectural theory evolved from literal interpretation to more architecturally inclined interpretations (Carpo, 2014). The main question Lyn discussed in his book *Animate Form* focused on how to reintroduce time and motion into
architectural design. As opposed to indexical responses to time in cubist or futurist approaches, Lyn pointed out that time and motion is inseparable for architecture (Lynn, 1999). Time and forces enable the context for design to become “an active space with a temporal dynamic”, and “direct the generation of shape; to be stored as information in the shape” (Lynn, 1999). In this case, the experience of architecture is modelled as a “participant immersed with dynamical flows, and received contextual specificity that cannot be achieved in an abstract space of fixed coordinates” (Lynn, 1999). Lynn's idea of such abstract active space originated from the monadological space of Leibniz (Tuinen & McDonnell, 2010), which describes the minimal element of space as vectors implying continuous motion and time.

![Figure 2-1: (a) A curve composed of segments of multiple radius elements (b) A similar curve described using splines, in which radii are replaced by control vertices which imply motions and flows (Lynn, 1999).](image)

Based on the differential calculus of Leibniz, this active space with time and forces can be simulated using a computer. According to Lyn, there are three fundamental properties of organization in the computer: topology, time and parameters (Lynn, 1999). The typological entities are defined with calculus which leads to infinite solutions representing multiple forms. Instead of points and centers, topology is characterized by flexible surfaces composed of splines, which are based on vectors or flows (Figure 2-1). The vector-based topology is able to incorporate time and motion into its shapes as inflections or continuous curvature.

Curvilinearity is a more sophisticated and complex form of organization than linear in two ways: it integrates multiple rather than single entities, and it is capable of expressing vectorial attributes, and therefore time and motion. Curvature in a temporal environment is the method by which the interaction of multiple forces can be structured, analyzed and expressed.” (Lynn, 1999)

As Lynn pointed out, using a computer, many forces in a temporal environment are represented as parameters to direct the formation of the curvilinear shapes. Lynn presented more examples about curvilinearity resulting from multiple parameters, as in the evolutionary theory or the idea of fitness landscape (Derrida & Luca, 1991). These theories, which defined the development of forms in a gradient environment with multiple factors, can be informative to architectural design. Lyn explained some examples in detail, like the deformable grid of the animal morphology by D’Arcy Thompson (D’Arcy & John, 1971), or the geological
development of landscapes. Based on these discourses, the animation for Lynn implied the evolution of forms within time, and the underlying forces shaping them. The focus in the architectural design shifted from a static figure or shape to a continuous generative process directed by multiple influencing factors in the environment (Lynn, 1999). As a reflection of gradient influence, the formed shapes also turn out to be curvilinear and smooth.

2.1.2 Shift of continuity theories to technological directions

Influenced by the development of computer technology, the early discourse of continuity moved into a more scientific, technological direction, in the beginning of new Millennial. One such direction, called new structuralism, is, related to the technical and structural performance of architecture.

**Performative Design**

The idea of performance-based design or performative design emerged together with digital design as a result of the rapid development of computer technology in the 1990s. The term *performative* represent a synthesis of two of the essential characteristics of digital design: the “transformation of a geometrical model” and the “analytical evaluation of environmental performances” (Oxman, 2008). It has the potential of an integration of technical evaluative processes with digital ‘form generation’ or ‘form modification’ models (Oxman, 2008). Generally, the main conception of performative design is about the analysis and understanding of how the environmental performance may inform complex processes of design synthesis. However, the concept of performances is contingent upon multiple parameters in an environment of a building and presents a complex problem. The integration and balance of these multiple factors implies a need for interdisciplinary integrations (Oxman & Oxman, 2010). How to format and apply these multiple technical factors in a design is the key question of performative design. It argues that instead of appending technical performance as optimizations at the end of the design, it should be integrated into the formation and generation process (Weinstock, 2006) (Weinstock & Stathopoulos, 2006). Thus, performative design is a highly iterative and interactive process, in which the technical data provides feedback for form generation and refinement in architectural design (Bollinger, et al., 2008) (Bollinger, et al., 2010). In order for performance data to be useful in the design process and manufacturing, an exact control of geometry and its topological variation is also necessary (Whitehead, 2003). In this case, geometry becomes a medium to connect performance feedback with the generation of architectural forms.

**New Structuralism**

New structuralism is a branch of performance-based design, focusing mainly on the interactions between architectural ideas and structural performance in a dynamic generative process. The motive of such an idea is to break down the sequential development of architectural forms (‘form, structure and material’) (Oxman & Oxman, 2010), while allowing multiple factors, such as structural performance, material efficiency and fabrication possibility,
2.2 Interdisciplinary integration in designs of curved surface structures

Influence the generation of architectural forms. Based on this idea, Oxman used the term ‘design engineering’ in New Structuralism as a medium for interdisciplinary integration. It has the ability to accommodate the structural, material and fabrication considerations early in the design process. Once such a consideration is added to the dynamic generative design process of architectural forms, it can help develop a perfect model of design collaboration and interdisciplinary interactions.

Oxman explained the meaning of design engineering using some historical examples, such as the structuring and materialization of the Sydney Opera House (Figure 2-2). As it is the result of interdisciplinary cooperation between Jorn Utzon, Ove Arup and Jack Zunz, it is considered as a historical starting point for a new structuralism in digital times (Rice, 1998), as the structural stability and fabrication possibilities effectively depend on the overall form of the roof. Oxman also mentioned the inspiring works of Heinz Isler, as a representative of structural artists and engineer architects in the non-digital time, who nicely integrated material, structural efficiency and architectural functions from the design decision phrase. Coming largely from an engineering perspective, the structural artists from the middle of 20th century, already explored similar ideas compared to those proposed by new structuralism architects. Namely the reconciliation of engineering and architectural thinking. The structural artists contribute inspiring references to bridge separations between different disciplines.

2.2 Interdisciplinary integration in designs of curved surface structures

Form generative continuity, formal smoothness raised in the early theory of continuity, and the interdisciplinary integration proposed by New Structuralism, were all reflected in the work of structural artists in the middle of the 20th century, especially in curved surfaces structures. As such, this section will mainly discuss the approaches of structural artists in the design of smooth curved surface structures. This discussion will start by briefly reviewing the early theories aiming at the reconciliation of engineering and architectural thinking, like the research of architectural theorist Viollet-le-Duc, the practices of structural artists like Thomas Telford in the 19th century or Robert Maillart in the beginning of 20th century. Next, this section will focus
on the architectural and structural integration in the design of curved surface structures in the second half of the 20th century; in particular the research and practices based on hyperbolic paraboloids, as they are directly related to the research presented in this thesis.

2.2.1 Design philosophies of structural artists

The ideas and practices aiming for reintegration of formal aesthetics and structural efficiency emerged both in architectural theories and practices of structural artists in the 19th century. In the research of gothic architecture by French architectural theorist Violett-le-Duc, he discussed the connection between architectural forms and natural forces. He deduced from historical research, that architecture must express its interaction with nature, in such a way that it demonstrates how its forms resist the effect of gravity (Frampton, 1995). This idea corresponded to the design principles of certain contemporary structural artists such as Thomas Telford, Gustave Eiffel or later Robert Maillart, who requested a dialogue between forms and forces in designs (Kotnik & Schwartz, 2011).

British engineer Thomas Telford, as a representative of other structural artists in the Age of Iron (Billington, 1985), was seeking efficiency in material, economic value in construction and desirable appearance of the final form as his main ideas for design. Although many of his works were related to the design of bridges (Figure 2-3), they demonstrated clearly that structural forms are not only responsible for load bearing but are also expressions of the designers’ aesthetic values (Telford, 1838).

![Figure 2-3: (a) The Craigellachie Bridge designed by Thomas Telford in 1814, (Wikipedia, 2019). (b) The Cement Hall for the National Exhibition in Zurich 1939 designed by Robert Maillart (Billington, 1985).](image)

In the Age of Concrete (Billington, 1985), Swiss structural artist Robert Maillart also had similar design principles: the aesthetic ideas precede and interact with the technical ones (Billington, 1985). His design principles and practices between 1900 and 1940 influenced a revolution in structural art. Billington described him as “the first twentieth-century designer to break completely with the masonry past and put concrete into forms technically appropriate to its
properties and yet visually surprising’ (Billington, 1985). With a thorough understanding of material properties and the structural functions of forms, Maillart set the criteria for structural art as: minimal materials, minimum cost, and maximum aesthetic expression. Guided with these ideas, he invented new forms for several types of buildings in which the structure and the form are united as one (Billington, 1997). The thin concrete shell (Figure 2-3b) is one of his inventions which marked a starting point of fruitful explorations of concrete surface shells in the 20th century. Such exploration in thin shell structures were continued by a group of structural artists or engineer-architects, like Sergio Musmeci, Heinz Isler, Pierlugi Nervi, Frei Otto and Felix Candela among others, in the second half of the 20th century.

2.2.2 Surface structures designed with form-finding methods

The use of form-finding in the design can be traced back to the simple hanging model of Heinrich Hübsch in the 19th century (Gerhardt, 2002). In working with physical models, the central idea was of using hanging chains to generate these forms. Throughout the 20th century, this idea was developed further by many structural artists, such as Sergio Musmeci, Heinz Isler and Frei Otto, into various form-finding methods aimed at designing surface structures. All the Physical models they developed inherently integrated the influence of force flows in the generation of forms. Furthermore, by manipulating the boundary conditions, the generated forms could be adopted to different architectural functions. However, the structural efficiency of these generated forms, as well as the buildability of the resulting freeform geometry remain problematic when multiple loads are applied.

Hanging membrane

In the 19th century, hanging membrane models were developed directly from hanging chain models and they were widely used in the design of surface shells during the second half of the 20th century. A hanging membrane model can be developed by simply saturating a fabric with resin or plaster and lifting up several points of the surface as supports (Bechthold, 2008). Once the fabric is dried, it is placed upside-down, so as to accurately model a surface shell.

Figure 2-4: (a) The reversed form-finding model for the tennis shell. (b) It is measured with a special instrument developed by Isler. (c) Curved profiles of formworks for the tennis hall shells at Düdingen. (Chilton, 2000)
The tennis hall shells at Düdingen designed by Swiss engineer Heinz Isler in 2000 were developed with such a method (Figure 2-4a). However, as mentioned previously, the freeform shape of this outdoor shell led to difficulties in its fabrication. This difficulty was exacerbated since, at the time of the hall’s planning and construction, 3D scanning equipment had yet to be invented. As such, Heinz Isler developed a measuring instrument (Figure 2-4b) to measure the principle curves of the models’ surfaces. He then scaled them up to generate the profiles of the curved formwork (Figure 2-4c), (Chilton, 2000).

**Soap film / Stretched fabric**

Soap film models and stretched fabric possess similar properties. Geometrically they are minimal surfaces, and therefore structurally uniformly stressed. Since soap films are very fragile compared to fabrics which can be pre-stressed to a higher degree, usually the soap films served as quick experiments for real tensile membrane systems (Escher, 2016). Early works of German engineer-architect Frei Otto and his team at the University of Stuttgart designed several tensile membrane surfaces drawing inspiration from soap films, like the dance pavilion designed for the Bundesgartenschau in the 1950s (Vrachliotis, et al., 2016). Besides membrane structures, soap film and stretched fabric models can also be applied to the design of concrete shells: using the same geometrical surfaces, the tensile stresses in a soap film or stretched fabric model turn into compressive stresses once the direction of external forces are reversed.

![Figure 2-5: (a) The soap film model and fabric model of the Basento bridge designed by Sergio Musmeci. (b) Basento bridge under construction. (Musmeci, 1977)](image)

This property of soap films and stretched fabrics was used in the design of Basento bridge by Italian engineer Sergio Musmeci. One segment of the bridge was at first inspired by a soap film model, and then developed further with a stretched fabric model (Figure 2-5a). Once the stretching forces applied to the fabric were translated into the vertical loads on the deck and the ground’s reactions, the model’s internal forces were changed into compression to represent the forces present in the bridge. However, the fabrication of this doubly curved bridge encountered problems typical for surfaces generated through form-finding methods: It was so
2.3 Structural analysis and architectural practices of hypar shells

As an alternative to form-finding, some structural engineers and architects of the 20th century explored the spatial and structural potentials of hyperbolic paraboloids (hypars) for the design of doubly curved shells. In contrast to form-finding, in this approach, the structures are not based on the deformations of physical models, but rather on the geometrical variations of hypars (Garlock & Billington, 2008). This approach leads to the generation of doubly curved surface structures that are able to support variable loads, while keeping to membrane forces (Almond, 1936), (Candela, 1955), (Parme, 1958), (Scordelis, et al., 1970). Furthermore, being double-ruled surfaces, hypars can be fabricated in a relatively simple way, while simultaneously ensuring structural integrity. As shown by the remarkable work of Eduardo Torroja, Felix Candela, Pier Luigi Nervi and Le Corbusier, among others, hypars can be effectively used for the design of complex surface structures with differing architectural functions.

2.3.1 Structural analysis methods of hypar shells

As hypar shells became popular throughout the world in the period between 1950 and 1980, many papers were written analyzing hypars and their behavior. There were generally two approaches: the membrane theory and the bending theory. Both approaches have many remaining unsolved questions, especially in regards to boundary conditions of a hypar shell. The membrane theory concludes that the behavior of a hypar is sufficiently explained through only in-plane membrane forces. As such, it is adequate for the calculation of stresses in the surface, but it does not provide enough information necessary for the design of supporting members, such as rigid beams. The bending theory on the other hand, is based on numerical approaches like the finite element method. However, since it normally requires unrealistic support conditions, it raises little practical interests (Schnobrich, 1988a).

Analysis with the membrane theory

Ferdinand Almond

The earliest publication of structural analyses in hypar shells was made by Ferdinand Almond in the 1930s (Almond, 1933). At that age, the structural analysis of shells was mainly reliant on the membrane theory. In this project, Almond developed a simple equilibrium equation to show that the vertical forces are uniformly distributed throughout the surface, and carried entirely by shear forces (Billington, 1965). For the boundary condition, he proposed to stiffen the edges of the hypar surface with beams. Later he published a mathematical treatise on hyperbolic paraboloids, which concluded, unlike domes or barrels, that the straight ruling hypar seems not to need the correction of bending theory. Thus, made the following statement
regarding the analysis of hypars: “the very simple calculations can be done in a very intuitive geometrical forms, which makes their checking easier.” He also mentioned that “constructing hypars in concrete or metal, is made easily by their mode of generation, which only uses straight lines.” (Almond, 1936)

Felix Candela
Hypar shells attracted more attentions when Felix Candela introduced this family of geometries into his structural art (Billington, 1965). To support his practices in hypar shell designs, Candela published several papers analyzing the structural of hypars in the 1950s, (Candela, 1955), (Candela, 1960), (Candela, 1963). These papers explained in detail an analytical approach based on differential calculus which evaluates the internal stresses of a hypar; and briefly discussed boundary conditions. The analysis of internal stresses presented in these papers started with hypars having a parallelogram shape in plan, and later extended to a more general case: hypars with an arbitrary quadrilateral projection in the plane. His key idea for calculating stress in doubly curved surfaces was originally presented by Austrian engineer Adolf Pucher (Pucher, 1934). This method projects the membrane stress on the horizontal plane, and solves these projected stresses first. Then, in turn, the real stresses in the original surface may be found. In this calculation of internal stresses, Candela defined three axis within a hypar: $x$, $y$ and $z$. $x$ and $y$ are two straight rulings passing through the center $O$ with an intersecting angle $w$; and $z$ is the vertical axis perpendicular to the $xy$ plane and parallel to the axes of the principle parabolas in the hypar (Candela, 1955) (Figure 2-6).

![Image](a) ![Image](b)

Figure 2-6: Drawings by Candela describing the geometry of a hypar surface and its projected surface (a) The three axes of a hypar. (b) A differential of the hypar surface and its projection on $xy$ plane. (Candela, 1955)

In the case of hypars with a parallelogram projection in plan (perpendicular to the axis of the hypar parallel with gravity), Candela took a differential of the hypar surface as the subject to study, and projected it on the $xy$ plane. By considering the external loads in the three directions, $x, y, z$ as forces per unit of projected area, and the projected stresses as forces per horizontal unit length in two directions ($x, y$), he constructed equations of equilibrium on the projected plane.
2.3 Structural analysis and architectural practices of hypar shells

and thus found out the projected stresses. Based on the geometrical relationship between the projected plane and the original surface differential, Candela could calculate the real internal stresses present in a hypar (Candela, 1955), (Candela, 1960). He took this further by studying the case of hypars with arbitrary quadrilateral projections in the plane. These are simply the same hypars rotated in space. In these cases, the axis \( z \) of the hypar is no longer parallel with gravity. Candela set up trigonometric equations to represent the external force components of the three rotated axes \( x_1, y_1, z_1 \) of a rotated hypar as components of the global \( x, y, z \) directions. He then substituted these equations into the previous formulas developed for the internal stresses of ‘regular’ hypars with \( x, y, z \) axes, thereby having developed a formula to determine the internal stresses of a hypar with arbitrary positions in space.

![Figure 2-7](image1.png)

Figure 2-7: Drawings by Candela to describe conditions of equilibrium of projected stresses. (Candela, 1955)

According to his calculations, Candela concluded for the first case that: the first special case represents the most well-known types of hypar shells with only shear forces along their edges. The second case on the other hand is a more general case, in which shear forces along edges are fixed, while oblique stresses in a surface are statically indeterminate. Thus to freeze these stresses, the designer should refer to the boundary conditions (Candela, 1960). This property gives designers some freedom to choose the position of supports.

![Figure 2-8](image2.png)

Figure 2-8: Drawings by Candela to describe different boundary conditions. (a) Symmetrically combined hypars. Simplification of support conditions may be obtained by considering symmetry. (b) Zero stress curved edge hypars. (Candela, 1960)
Based on these conclusions, Candela also studied two different boundary conditions: the straight edge hypars, and hypars with arbitrarily curved boundaries. In a straight edge hypar, it is possible to leave two edges free of oblique stresses; thus the other two edges become determinate (Candela, 1960). The oblique stresses applied to these edges must be absorbed by a continuous support which is able to resist forces in any direction. To solve these edge problems, Candela proposed to combine several modules of the same hypar together, which aids in cancelling the horizontal components of the oblique stresses. He then solved the vertical components of the oblique stresses and the shear forces with stiffened beams (Candela, 1960). In the case of curved edge hypars, and according to his formulas of internal stresses, the normal and tangential stresses on any sections not parallel to the straight rulings are variable and can have any arbitrary values, including zero. Thus, it is possible for the curved edges to be free of stresses when all the rulings which intersect such curved edges are fully supported to resist loads in any directions (Candela, 1960). This allowed Candela to create many elegant hypar shells with curved edges (Figure 2-20 to Figure 2-22).

Candela continued to thoroughly analyze the structural behavior of hypar shells in his research. Since his calculations were closely related to his experience in practices, they were also limited to them. The two cases he introduced in his papers were limited by a single geometrical condition. Whether a hypar had a parallelogram shape, or an arbitrary quadrilateral projection in plan, the z axis of a hypar is always perpendicular to the plan (the xy plain) defined by the two middle rulings of the hypar. However, there is still a more general case which remains unsolved. His solution to the oblique stress applied to the edges, was still limited with a rigid beam. This was proved later in the bending theory, and it was what caused the behavior of the shell to deviate from his expectations (Billington, 1965).

**Candela’s Successors**

After Candela introduced hypars into the design of thin shell construction, they became popular throughout the world between the 1950s and the 1980s (Schnobrich, 1988a). Many papers have been written concerning the analysis and behavior of hypars in the membrane theory. Examples of this are the work of Scordelis, Ramirez and Ngo (Scordelis, et al., 1969), (Scordelis, et al., 1970). In their research, the surface definitions were similar to those presented in Candela’s works, and the internal stresses were also calculated following Candela’s differential equation approach, though in more detail. Particularly in the case of hypars with an arbitrary quadrilateral shape in plane, the transformation between an orthogonal coordinate system and the other system defined by the three axes of the hypar was explained more in detail. In their approach, vectors were introduced to represent four edges and rulings in a hypar. By calculating the desired data for the transformation between two coordinate systems as sums or products of edge vectors and unit axes vectors, the external loads in orthogonal coordinate system was transformed into the coordinate system defined by the axes of a hypar (Scordelis, et al., 1970). Ramirez, Scordelis and Ngo also further developed the analysis of edge members. To evaluate the internal forces and bending moments in edge members, they specified the applied boundary stress as the sum of a vector along the edge member and another vector normal to the edge member. These two vectors were finally written as formulas of unit ruling vectors and edge
vectors to represent the stresses and bending moments at the edges (Scordelis, et al., 1970). With these calculations, they also proposed a rigid space frame as the solution for the boundary stresses. Another improvement included in their research is that a general computer program was written to perform the analysis described in the previous calculations. This program made it possible to compute the membrane stresses in the shell on a given mesh of points, as well as the boundary stresses on the edge members.

Besides Candela’s structural analysis itself, his ideas about the role of mathematical analysis in shell designs also influenced the way his successors presented the results of their own analysis. Following the idea of Candela that mathematical formulations may get in the way of design imagination (Billington, 1965), American engineer Alfred Parm began to develop a simple way to visualize hypar behaviors. In his publications, he first simplified the behavior of a hypar shell as two systems of parabolas, one in compression and one in tension. In hypars with straight edges, these parabolas carry loads to these edges and resolve them into edge shears, resulting in axially loaded edges (Parme, 1956), (Parme, 1958). This simplified explanation is helpful to understand the behavior in the design, but it cannot work for all cases. As Billington described in his book, troubles arose with an increase in the variety of possible hypar forms. Interestingly, Candela emphasized this as a virtue of hypar shells (Billington, 1965).

Analysis with the bending theory

As opposed to the membrane theory analysis, which focused mainly on the hypar shell itself, the structural analysis in bending theory treats a hypar as a shell-beam system (Schnobrich, 1988b). In other words, the application of bending theory in the analysis of hypar shells was mainly due to the rigid edge beams. The bending theory argued for several deficiencies of the membrane analysis, such as the inability to account for the dead weight of beams and the failure
to achieve compatibility between the shell and the stiffened beams, which can result in a significant separation between the calculated membrane results and reality.

The discussions of general bending theories applied to hypar shells and stiffened edges mainly existed in the publications of Apeland (Apeland, 1962), (Apeland & Popov, 1961); Billington (Billington, 1965); Schnobrich (Schnobrich, 1971), (Schnobrich, 1972); Mueller (Mueller, 1977), Ramaswamy (Ramaswamy, 1968), among others. In these publications, the equations for a bending theory of shallow shells were specialized to the case of the hypar shell and were solved with approximate numerical solutions, like the finite element method. The dead weight of edge beams was included in the analysis, and described a more precise bending solution with real support conditions (Schnobrich, 1988b). Schnobrich’s calculations and tests showed that the edge beams actually load the shell rather than the other way around. It behaves like a beam with three supports, where the middle support is the center of the gable crown of a hypar (Schnobrich, 1972). In this case, he suggested the designers reduce the size of beams. However, Billington pointed out another problem arose when the size or reinforcement of a beam is not enough: “as results shown by more complete solution using numerical methods, the straight edge beams do not prevent horizontal displacement” (Billington, 1965). Without enough horizontal reactions, the inverted parabolas in a hypar cannot develop tension, thus the compression parabolas have to carry most of the load to the supports. In the end, this leads to double the internal stresses predicted by the membrane theory (Billington, 1965).

The bending analysis just pointed out that the analysis of the membrane theory did not sufficiently consider the influence of stiffened beams on the behavior of a hypar shell. Therefore, the advantage of the bending analysis over membrane theory was to provide precise bending solutions of the edge beams. However, it did not propose any solution to the boundary conditions other than the solution to the stiffened beams mentioned above. On the other hand, the complex mathematic calculations of the bending theory can impede imagination in design, as the analysis results are typically only obtained during consultation or optimization at the end of the design process, and therefore struggle to influence the design in an early enough stage.

2.3.2 Architectural practices with hypars

Catalan vaults in the shape of hypars by Antoni Gaudi

Hypar geometry was first translated from shapes that appeared in descriptive geometry manuals in the 19th century into architectural designs by Spanish architect Antoni Gaudi (Huerta, 2006). Hypars nicely reflected Gaudi’s design ideal: “form did not follow structure and construction, it was identical with them” (Collins R., 1971). Gaudi was fascinated with such saddle shapes because they are thinner and stiffer than traditional Gothic vaults, and the double ruled and double curved properties enable them to be constructed with Catalan tile vaulting, which can carry both tension and compression. The porch of the Güell Colony designed by Gaudi in the early 1900s is the first hypar vault in architectural history (González & B., 2003). Gaudi’s use of hypar was described by González as an example of the anti-funicular method or the application of graphic statics. The way Gaudi applied hypars in the design of the portico roofs
suggests that he understood well the behavior of hypar vaults. For example, he placed these hypars in such a way that they look concave from underneath at first glance (Figure 2-10 a,b). These curved vaults seem to contradict the logic of funicular vaults, which can only take compression and are convex from underneath. Gaudi successfully made use of such illusive properties to achieve a sensation of instability (Huerta, 2006).

In the decoration and design of other elements, Gaudi also tried to emphasize the character of a hypar as a ruled surface. These hypar roofs were decorated with ceramic tiling which accentuate their geometrical feature of straight rulings (Figure 2-10b) (González & B., 2003). Moreover, every hypar vault was edged with mechanical arches formed by three straight sides (Figure 2-10a). The straight arch shape recalled the geometrical origin of a hypar. These arches were finally resting on two great brackets, which transmitted edge forces to the ground. The building process of a hypar vault also matches the ruled form and its cladding. The construction of a hypar vault used a process similar to that of Roman vaults. Later, he also used hypar geometry in the roofs of two houses in Park Güell, as well as the vault in the Sagrada Familia. The design and construction of Gaudi’s hypar vaults became a powerful inspiration to other structural artists; especially his Spanish followers Eduardo Torroja and Felix Candela who later implemented hypar geometry as elegant thin reinforced concrete shells.

Concrete hyperboloid shells by Eduardo Torroja

The first concrete structure designed by Eduardo Torroja is a cantilevered hyperboloid shell (Figure 2-11) from the middle of the 1930s. Geometrically, one sheet hyperboloids are also a hyperbolic paraboloids. This cantilevered shell covers the stands of a race track in Madrid called Zarzuela Hippodrome. In his autobiographical book, Torroja described the way to reach the final shape of such shell is neither purely rational, nor purely imaginative, “but rather both together. The imagination alone could not have reached such a decision unaided by reason, nor could a process of deduction, advancing by successive cycles of refinement, have been so logical and determinate as to lead inevitably to it.” (Torroja, 1958)
Chapter 2. Literature review

Such a design process drove Torroja to finally choose a hyperbolic doubly curved surface, similar to Gaudi’s. The whole shell is combined from several modules of the same hyperboloid; the intersecting edges of these modules are supported by the frame underneath (Figure 2-11). The shell cantilevers around 13 meters from the main support to cover the stands below, and goes 7 meters back to the other direction. There is a tie at around 1.5 meters back to prevent the shell falling over. The thickness of the shell increases from the free edge to the supports, with a minimum of 5 centimeters and a maximum of 14 centimeters. As a result, the shell appears extremely light, but structurally very strong. The builder of this shell was provided a chance to test a full-scale model of one section, in order to check the placement of reinforcement bars and the structural rigidity (Figure 2-12a). In the end, the shell carried three times its design load (Antuna, 2003), (Figure 2-12b). The shell was constructed with straight and modular wooden formworks, so that they could be reused for other sections (Juan, et al., 2015). In this project, Torroja made use of the rigidity of hyperbolic forms to achieve his aesthetic preference: lightness and smoothness. Furthermore, he also gained constructive convenience due to the ruled geometry of the shell.

![Figure 2-11: (a) The geometrical shape of one section of Zarzuela Hippodrome is part of a hyperboloid. (b) The lightness of Zarzuela Hippodrome. (photos by WikiArquitectura)](image1)

![Figure 2-12: (a) The construction test of one section. (b) The load test of one section. (photos by WikiArquitectura)](image2)
Concrete hypar shells and buildings by Felix Candela

The real impetus to apply hypar geometries in design came from Felix Candela. The Spanish architect carried a tradition of smooth surfaces, and visually drove thinness to new limits (Billington, 1985). The structural rigidity and geometrical simplicity of hypars enabled him to create new forms based on his experience in construction. Candela clearly saw the problem that mathematical formulations could impede the freedom of the imagination in design, so he used calculations only as a validation of the original design (Faber, 1963). The other virtue of hypars emphasized by Candela, was the variety of possible forms. The geometrical shapes of all his surface structures, which adapted to different architectural functions, were actually derived from several prototypes.

Umbrella shell

The umbrella shell is the one of the most well-known shell prototypes of Candela. It was originally made of four hypars with all straight edges on the same plane. This idea was derived from a sketch in an article by Almond (Almond, 1936) (Figure 2-13a), however Candela illustrated the possibility of more elegant solutions. Compared to the large edge beams and stubby columns in Almond’s sketch, Candela’s umbrellas had more graceful shapes, which appeared as the design result of an artist instead of a builder or engineer (Garlock & Billington, 2008), (Figure 2-13b).

To understand the behavior of such shapes, Candela did several structural tests in the 1950s. With the experience gained from these tests, together with his calculations based on the membrane theory, Candela represented the behavior of half an umbrella as a cantilever (Figure 2-14), in which all straight edges are in tension and the valleys are in compression (Candela, 1954). Based on this simplified model, he developed a simple formula to describe the proportion between the rise and the area of the umbrella (Garay, 1994).

Later, he designed and constructed a series of umbrella shells as roofs for factories, warehouses or markets. By placing several umbrellas side by side, Candela managed to cover the large areas
(Bowman, 1957). In some cases, individual umbrellas were tilted or lifted up to different heights, or pierced with glass bricks (Garlock & Billington, 2008), to allow light to enter the space, like in Rio’s warehouse (Figure 2-15a) or the High Life Textile Factory in Mexico City (Figure 2-15b). Such rectangular umbrella shells were the most economical standard shell structures developed by Candela. Their constructions were also fast, convenient and saved on labor. In terms of construction, each umbrella had four sets of formwork (the scaffolding and form boards), one for each piece of hypar. The formwork could be reused once the concrete of one hypar hardened enough, by lowering and moving it to the next hypar. According to Candela, such umbrellas had the advantage of straight formwork, and were thus much cheaper than other traditional vaults (Cassinello & Candela, 2010).

![Figure 2-14](image1.png)

Figure 2-14: (a) Cantilever method for analyzing an umbrella shell (Candela, 1954) (b) Sketch of scaffolding and form board used for one hypar in an umbrella (Faber, 1963).

![Figure 2-15](image2.png)

Figure 2-15: (a) Rio’s Warehouse (b) High Life Textile Factory (Garlock & Billington, 2008)

Starting from these standard umbrellas, Candela also explored more variations of such umbrella shapes. The entrance of Lederle Laboratories is an example of this (Figure 2-16a). By stretching two hypars in the umbrella, one end of the intersecting line between two hypars was moved further from the support, and an asymmetrical umbrella was created. Such asymmetrical
umbrellas can also be found in other projects of his. In the roof of the El Leon confectionery factory, he took the longer cantilevering part of an asymmetrical umbrella and mirrored it along the shorter horizontal edge. The resulting geometry was a roof spanning the entire width, creating a large open space uninterrupted by columns (Figure 2-16b).

La Virgen Milagrosa (derived from the umbrella shell)

Another project related to the umbrella shape is the Church La Virgen Milagrosa in Mexico City. The original idea was to create a traditional Gothic church (Faber, 1963). However, Candela reinterpreted the defining idea of Gothic by using hypars to minimize material over a large span. He mentioned he had tried to construct a church of traditional character with this material; one in which both the structure as well as the internal expression function, and are exclusively dependent on form (Candela, 1956). Following this idea, he maintained the traditional forms of a church: a traditional linear nave ending with an apse, in his design of Milagrosa, but played with the shape of the bays (Figure 2-17). One bay of the nave was designed as a combination of four hypars. Candela mentioned that he got the inspiration for the bays from a French engineering text in which a structure formed by hypars showed a Gothic ‘ascending tendency’ (Garlock & Billington, 2008). However, he designed each bay of Milagrosa by deforming his tilted asymmetrical umbrellas (Figure 2-17). By placing two deformed umbrellas against each other, one bay of the church was created and the whole nave was comprised of four of these bays (Figure 2-17), (Candela, 1956). Candela also designed the other part of the church following a similar geometrical language. Like the apse, it was designed with another four hypars. The columns, were shaped by intersecting several hypars to better match the form of the roof and to resist bending at the base (Figure 2-18c). The column foundations were designed as a small umbrella too.

Similar to the design process of other projects, Candela designed the church at Milagrosa initially through his rich experience obtained from practice, after which he validated his design.
using membrane theory calculations. Though most of his analysis confirmed the validity of the original design, an unexpected upward thrust was revealed at the roof’s apex, threatening to tear the umbrella apart (Faber, 1963). In this case, the ridges of the roof were thickened in order to counterbalance the upward thrust. According to the Candela’s calculations, the forces in the surface were negligible, and therefore the roof was made with reinforced concrete only 4 centimeters thick. Furthermore, the edges of the hypars were stiffened with beams at the points where internal forces were accumulated.

The construction process started with the foundations, excavating the forms of these inverted umbrellas out of the ground. Next, the formwork of stiffened edge beams along the perimeter of the nave was constructed (Figure 2-18a). Once the triangular perimeters were built, the workers then proceeded to construct the formwork for the bays (Figure 2-18b). Following the construction of the formwork, steel reinforcement bars were placed on top and concrete was placed by hand without any outer formworks (Garlock & Billington, 2008).

Figure 2-17: The design of the church of Milagrosa still kept the plan of a traditional gothic church while using hypars to make the nave and apse. Each bay comprising the nave of the church of Milagrosa by Candela was deformed from his asymmetric umbrella shell; a combinations of four hypars, (Faber, 1963).

Figure 2-18: (a) The construction of the church of Milagrosa: the formwork of stiffened edge beams along the perimeter of the nave were constructed first (b) The form board for the roof shell (c) The interior space shaped with hypar surfaces, (Garlock & Billington, 2008).
2.3 Structural analysis and architectural practices of hypar shells

For the design of the church of Milagrosa, Candela paid careful attention to the interior, as he considered the aesthetics of the exterior of a church primarily to be for inviting people in, and that of the interior as the key expressive feature of the building (Figure 2-18c). About the relationship between the interior space, the external appearance, and the structural efficiency, he mentioned that “it is about attaining an expressive interior space, a surrounding sculpture that one admires from the inside. But this sculpture cannot be capricious and arbitrary, since one has to response to the external laws of structural equilibrium” (Candela, 1956). This statement nicely interpreted his understanding of integration between structure, form and space. Different from other shell structures designed for industrial buildings, the use of hypar geometry in the church of Milagrosa did not only satisfy the efficiency of a shell structure, but also reflected the architectural potential to create interesting interior spaces with hypar geometry.

Curved edge hypar shells

Besides umbrella shells made with straight edges, there is also another prototype that appeared frequently in Candela’s works: hypar shells with curved edges. Candela started construction on his first hypar shell for the cosmic rays laboratory as two curved edge hypars in Mexico City in 1951 (Faber, 1963). He changed the original design of a barrel shell into double curved hypars (Seguí, 2004), reasoning that the extreme thinness of the shell would require the stiffness found in a hypar (Candela, 1954). In this project, each edge of the curved hypars was fully supported with arches and frames. Similarly, in the design of Chapel Lomas de Cuernavaca in 1958, which is a shape cut from a straight edge hypar, the two curved edges were both supported with stiffened beams. In these two cases, Candela reinforced the curved edges with stiffened beams to resist bending after he had analyzed the behavior of the shells.

![Image](image1.png)

Figure 2-19: (a) The components of the Cosmic Rays Laboratory (Garlock & Billington, 2008). (b) The geometry of the Lomas de Cuernavaca Chapel was cut from a straight edge hypar (Faber, 1963). (c) Two curved edges of Chapel Lomas de Cuernavaca were stiffen with beams, (Faber, 1963).
Chapter 2. Literature review

Figure 2-20: (a) The plan of the Los Manantiales restaurant in Xochimilco. (b) The finished Los Manantiales restaurant. (c) Stiff beams at the intersecting line of two hypars. (Faber, 1963)

Figure 2-21: (a) The plan of the sales office in Guadalajara. (b) The built sales office in Guadalajara. (Faber, 1963)

Figure 2-22: (a) The plan of one of the roof units of the Bacardi Rum Factory. (b) The built Bacardi Rum Factory. (Garlock & Billington, 2008)
Later in some of his projects, he successfully got rid of these beams by combining several curved edge hypars together. This design strategy is based on his calculations according to membrane theory; ‘the curved edges are free of stresses when all the rulings which intersect such curved edges, are fully supported to resist loads in any direction’ (Candela, 1960). In the designs of the restaurant Los Manantiales in Xochimilco (Figure 2-20), the sales office in Guadalajara or the Bacardi Rum Factory in Cuautitlan (Figure 2-21, Figure 2-22), all the rulings intersecting the curved edges of the hypars were supported by stiffened beams (Figure 2-20c). Benefiting from such a combination, the resultant forces in the curved edges all turned out to be zero, allowing them to appear elegantly thin.

**Philips Pavilion by Le Corbusier**

Besides that of structural artist, hypar geometry also appeared in the works of some architects like the Philips Pavilion in the 1958 Expo designed by architect Le Corbusier and musician Iannis Xenakis. Instead of using a single hypar or a repeated series of the same hypar, Le Corbusier combined several different hypars to shape his desired internal space. In this project, Le Corbusier tried to “connect the evolution of his mathematical thought on harmonic series and modular coordination with the idea of three-dimensional continuity” (Capanna, 2014).

The original idea of the design was to develop the interior space as an instrument which envelopes sound, light and space (Clarke, 2012). To design such an instrument as a container, Philips Pavilion was first presented as a stomach-shaped volume defined by conoids and hypars. Due to the easier static calculations of hypars, all the conoids were replaced by them in the end (Figure 2-23a). As Xenakis described, they chose hypars from a large family of double curved surfaces to inform the building geometry, due to their stiffness and relatively low costs and ease of construction.
During construction, benefiting from the straight rulings of a hypar, the large surface was divided into smaller portions and cut according to the network constituting of the intersection of these rulings (Corbusier & Capanna, 2000). The finished Philips Pavilion turned into “an orchestral work in which light, loudspeakers, film projections on curved surfaces, spectators shadows and their expression of wonder, objects hanging from the ceiling, and the containing space itself were all virtual instruments” (Capanna, 2014). The architecture in this case, through the double curved continuous surfaces, played the role of orchestral instrument. (Figure 2-23b)

2.4 Smooth freeform surfaces made with hypars

For most hypar shells or buildings designed by structural artists or architects, the hypars were either used as a single surface or joined with rigid beams. The resulting geometries appeared as “curved folds”. However, there were more explorations by artists and researchers in the field of architectural geometry, which showed a broader potential for the application of hypar geometry in the creation of innovative smooth surfaces.

2.4.1 Smooth surface sculptures made with hypars

Applying hypar geometries to architecture was taken into a different direction by Spanish artist Angel Duarte. Instead of creating surfaces similar to the curved folds present in the works of structural artists in the middle of 20th century, he combined hypars into smooth freeform surfaces. Duarte generally used two approaches in his designs with hypars: the smooth combination of the same hypar modules, or a combination of differing hypar modules (Durate, et al., 2004). In most of his larger scale sculptures, these innovative poly-hypar surfaces were made with concrete or steel bars.

Figure 2-24: Concrete sculptures Duarte designed with Isler between 1974 and 1979.

For several years, Duarte worked with Isler, and received advices from him regarding the construction of several large sculptures. Between 1974 and 1979, a total of four sculptures were made (Burkhalter, 2015) (Figure 2-24, Figure 2-25a). There are rare documents about the
2.4 Smooth freeform surfaces made with hypars

cooperation between Duarte and Isler, but it is clear that the engineer Isler was responsible for stability, materials and manufacturing, while Duarte was mainly driven by aesthetic motivations (Bösiger, 2011). Some of his sculptures were made with steel bars and each hypar was only made with bars following one group of rulings (Figure 2-25b,c), which is against the basic structural logic of hypar geometry. However, in the design of the relatively small scale sculptures without an architectural function, the aesthetic request mattered more than structural efficiency. From a design point view, Duarte’s sculptures showed a striking contrast between their straight edges and smooth surfaces. These sculptures provided an inspiration for the creation of new geometrical forms in-between simple ruled geometry and complex freeform surfaces.

![Figure 2-25: (a) Concrete sculpture at Saint-Guérin’s central school designed by Duarte between 1976 -1977. (b)(c)Scultures made with steel bars; designed by Durate in 1977.](image)

2.4.2 Approximating freeform surfaces with hypars

In the emerging field of architectural geometry, which provides architects with sophisticated geometrical knowledge to solve diverse problems (Emanuel & Thilo, 2014), more and more research is showing the inspiring potential of hypars in creating innovative freeform surfaces. One of the key problems to be solved within this field, is the fabrication of the freeform skins (Flöry & Pottmann, 2010). From the early solutions, through discrete freeform surfaces as planar faces (Glymph, et al., 2002) (Bobenko & Suris, 2009), to the approach approximating freeform surfaces using developable surfaces (Liu, et al., 2006) (Pottmann, et al., 2008) (Flöry, et al., 2012), these discretizations of freeform surfaces all shared the same challenge of visual smoothness. Even with individually smooth panels, the arising surface still exhibited kinks between adjacent panels (Käferböck & Pottmann, 2013). Thus, there was some research into replacing ruled strips with hypars, bilinear patches, to approximate double curved freeform surfaces.
The explorations to approximate freeform surfaces with hypars mainly existed in the research of Carizer (Craizer, et al., 2009), Käferböck (Käferböck & Pottmann, 2013) and Emanuel (Emanuel & Thilo, 2014). The main idea of these researches was to find a way to glue adjacent hypars smoothly. The smoothness in this case is $G^1$ smoothness, satisfying tangent continuity (Emanuel & Thilo, 2014). In these studies, they started from a quadrilateral net with planar vertex stars, which is classified as an $A$–net (Sauer, 1937) (Wallner, 2001). It is the only quad mesh which can be extended by bilinear patches to overall continuously differentiable surfaces (Käferböck & Pottmann, 2013). To ensure two adjacent hypars join smoothly along the adjacent edge, all the edges intersecting at one vertex should be coplanar, and the edges that join two neighbouring hypars are parallel to a plane (Emanuel & Thilo, 2014). Through the discretization of surfaces of constant negative Gaussian curvature as special $A$-nets, which are also called K-surfaces (Sauer, 1950) (Bobenko & Pinkall, 1996), freeform surfaces can be successfully discretized into piecewise smooth surfaces, called hyperbolic nets. The geometrical studies in this field aimed to solve the difficulties in the construction of freeform surfaces while maximising the visual smoothness of the original surfaces, despite that the structural potential of these double curved geometries was rarely studied.
2.5 Summary

This chapter presented a critical overview of the key references and precedents relevant to freeform surface structures, particularly in regards to the approach based on hyperbolic paraboloids.

Regarding the design of freeform surfaces, in the architectural field, the theoretic discourse of such new forms was complete and explicit, while its extension into the technical direction was still to be exemplified through implemented methods and applications. The practices and research of curved surface structures from an engineering point of view can be informative to the technical extension of the continuity theory. However, most of them, like form-finding, still contained many problems in terms of structural efficiency and construction economy. Adhering to the smoothness sought in conceptual architecture, and the synthesis with multiple technical factors such as material, structure and construction, surface structures designed using hypars emerged as a feasible solution: their double ruled and double curved properties provide convenience in construction, while simultaneously ensuring structural integrity. Though at the time, exploration in hypar surface structures was limited to folded poly-hypar surfaces, the folding lines in between adjacent hypars create defects in visual smoothness and internal stress distributions. Besides the folded case, more geometrical variations as smooth poly-hypar surfaces were explored. Regardless, it is not enough to transmit the pure geometrical concepts into architecture and construction without understanding the related structural and constructive feasibilities.

As such, there is a need to merge the relevant streams in architecture, engineering and geometry into one operative design method based on smooth poly-hypar surfaces, which motivates an iterative and interactive process to create constructively economic, structurally efficient smooth surfaces structures in architecture.
Part II

Structural Analysis
3 Overview and fundamentals

3.1 Overview of main steps

Part II is presented in two steps: an individual hyperbolic paraboloid (hypar), and the smooth poly-hypar surface (the smooth combination of hypars). In both cases, the structural analysis is developed using graphic statics, an equilibrium-based approach that relies on the theory of plasticity. The most outstanding property of graphic statics is its ability to qualitatively visualize the distribution of force flows, while quantitatively calculating forces in equilibrium. In this way, it helps designers to understand the relation between forms and forces, while also providing a solution for the precise force calculations. Benefiting from such advantages, graphic statics, unlike other complex analytical methods normally limited to validations or optimizations at the end of the design, can be involved in the design from the early decision making phase.

On the level of an individual hypar, the structural analysis in this research is closely related to the studies of hypars using the membrane theory (Almond, 1936), (Candela, 1955), (Parme, 1958), (Scordelis, et al., 1970). Unlike complex analytical approaches impeding the design freedom (Almond, 1933) (Candela, 1955), (Scordelis, et al., 1970), or over-simplified representations which are unable to cover all the geometrical variations (Parme, 1958), the approach based on graphic statics in this research finds a balancing point between complex calculations and simplified visualizations. Based on 3D graphic statics, the behavior of an individual hypar is represented as a strut and tie model with varying discreteness. The minimal discrete model introduced at the beginning is the simplest solution to visualize the behavior of a hypar, while the infinite discrete model presented later provides a more precise evaluation of internal forces and reactions based on vector algebraic calculations.

Though the global equilibrium of smooth poly-hypar surfaces is a new field of study with little precedent in the engineering field, there are several studies focusing only on the geometry in the mathematic field nonetheless (Craizer, et al., 2009), (Käferböck & Pottmann, 2013), (Emanuel & Thilo, 2014). Based on the results on the level of an individual hypar, this research proposes a new solution to the problem of interactions between adjacent hypars in poly-hypar surfaces. Instead of using rigid beams as the solution to bending moments at edges as the precedent for folded poly-hypar surfaces, this research sets up a geometrical constraint, called the coplanarity principle (Chapter 5.2.1), which ensures that the interactions between adjacent hypars are always coplanar. Thus, it enables force interactions to be transmitted from one hypar to another without any bending moments. Such a geometrical constraint was originally derived from the equilibrium requests, which ultimately ensures $G^1$ smoothness (Chapter 5.2.2) of the global geometry, leading to a new geometrical category: smooth poly-hypar surfaces (also referred to as hyperbolic nets in mathematics). Based on the coplanarity principle, the concept of load paths (Chapter 5.3) is introduced to intuitively check the global equilibrium of smooth poly-
hypar surfaces, helping designers to complete the design. Benefiting from the coplanarity of interactions between adjacent hypars, the complex 3D equilibrium problem in a smooth polyhypar surface can be solved with 2D graphic statics, and formulating force calculations becomes a linear vector algebraic problem.

3.2 Fundamentals

3.2.1 Theory of plasticity

The theory of plasticity argues that the elastic deformation of a particular material may be neglected if it is much smaller than the plastic deformation (Muttoni, et al., 1997). Based on this reasoning, the deformation and properties of the material can be ignored when performing the structural analysis, thus focusing only on the static equilibrium of forces: the fundamental of graphic statics. As a result, it is possible to represent the behaviors of structures with pin-joined strut and tie models, leaving only axial forces in the structural members. In this research, the structural analysis is limited to the calculation of forces in an equilibrium model. Consequently, the dimension of the structure can be completed with the lower bound theorem. According to the lower bound theorem, a structural system is not in danger if the axial stresses applied in all the members violate neither the yield condition of the material nor the limit of buckling. Based on the magnitude of forces in the strut and tie models, and the yield stresses of the material, it is possible to dimension the members in a structural system.

3.2.2 Graphic statics

Graphic statics is a synthetic vector based structural design and analysis method, which can be described as a set of geometric algorithms based on vector calculus and descriptive geometrical diagrams (Cremona, 1890). Two important and related diagrams in graphic statics are form, and force diagrams. Form diagrams represent the geometry of a structure, the location of applied loads and supports, and the distribution of force flows. Force diagrams on the other hand, suggest the direction and magnitude of forces inside a structure (Figure 3-1), (Allen & Michel, 2010). At little expense of time, graphic statics can provide visual information about the relation between forms and forces, allowing it to make the interaction between architectural and structural thinking explicit from the early stages of a design process.

Current research on graphic statics is mainly focused on 2D problems, with some newly developed studies extending graphic statics into the third dimension (Akbarzadeh, et al., 2015), (D’Acunto, et al., 2016), (Konstantatou & McRobie, 2017), such as the polyhedron and vector based approaches (D’Acunto, et al., 2016). The 3D graphic statics applied in the analysis of an individual hypar in this research uses the vector-based approach.
3.2 Fundamentals

Figure 3-1: internal forces and reactions of two supports beams loaded with a point load and calculated using 2D graphic statics. (a) form diagram (b) force diagram.

3.2.3 Assumptions

This research is based on the following assumptions:

• In the study of an individual hypar, to simplify the calculations, the centroid of a hypar is assumed to be at the intersecting point of two middle rulings (Figure 4-1). This simplification doesn’t qualitatively affect the distribution of internal forces and reactions.

• The self-weight of one hypar is discrete as a group of point loads. Such discretion is achieved by splitting the whole surface into small patches with two families of straight rulings (Figure 4-1). As the amount of rulings tends to infinity, the difference between the weight of each patch tends to zero. In this case, it is assumed the weight of every small patch is the same; the discrete point loads applied to a strut and tie model of a hypar are the same everywhere.

• The continuous principle parabolas (section 4.1) in a hypar are discretized as tangents to the original parabolas (Figure 4-3). This discreteness leads to the splitting of one node in a continuous hypar into two sub-nodes along a line parallel to the axis of a hypar (Figure 4-3). As the discreteness tends to infinity, the two nodes merge as one. In this way, the two sub-nodes are considered as one in the force diagrams.

• The internal forces in a smooth poly-hypar surface are structurally indeterminate even when all the load paths are only supported at one end. In the calculation of internal forces, it is assumed that the forces are transmitted to the supports through the shortest path.

• The structural analysis in this research is independent from the structure’s material. It assumes that the plastic deformations of the material are substantially greater than the elastic deformations.
4 Structural analysis of a single hypar

This chapter presents the analysis of an individual hypar with graphic statics, starting with the classic case: hypars with axis $r$ parallel to gravity, and extending to a more general case: hypars with axis $r$ nonparallel with gravity. The minimal discrete strut and tile model of a hypar is the simpler visualization of its structural behavior, while an infinitely discrete strut and tile mode of a general case illustrates calculations of internal forces and reactions in detail.

4.1 Geometrical definition of hyperbolic paraboloids

The structural and aesthetic characteristics of hyperbolic paraboloids (hypars) are closely related to their special geometrical properties as double ruled and double curved surfaces. To better explain the structural analysis based on graphic statics in the following sections, the surface definition of a hypar is given at first.

By assuming four non-coplanar points $A$, $B$, $C$, $D$ in space and connecting them in turn, four segments of lines $AB$, $BC$, $DC$ and $AD$ are defined. These are the four edges of the hypar $ABCD$ (Figure 4-1a). By connecting the midpoints of the diagonals $AC$ and $BD$, a segment $r$ can be defined as the axis of hypar $ABCD$ (Figure 4-1b). If $AB$ and $CD$ are divided into $2n$ numbers of segments, by subsequently connecting the dividing points, a family of rulings $h$ can

Figure 4-1: A hypar is a double curved and double ruled surface. (a) Two sets of rulings $h$ and $i$. (b) Two sets of principle parabolas. The axis $r$ of a hypar is parallel with axes of principle parabolas and can be represented as a vector $r$. 
be defined. Similarly, another family of rulings $i$ (Figure 4-1a) can be constructed using the same method for $AD$ and $CB$. Rulings $h$ and $i$ define the surface of a hypar, while rulings $h_n$ and $i_n$ are the middle rulings, and $h_0, h_2, i_0, i_2$ are the four edges (Figure 4-1a). By intersecting the hypar with planes $P_{AOC}$ or $P_{BOD}$, the resulting curves at the intersection are two families of principle parabolas in a hypar (Figure 4-1b). The axes of principle parabolas are always parallel to the axis $r$ of a hypar.

Figure 4-2: If rulings and edges of a hypar are represented as vectors, one ruling vector can be represented as a linear combination of two intersecting edge vectors and a ruling vector in the same family.

If all rulings and axis $r$ are represented as vectors, two rulings in the same family and axis $r$ are always in a linear constraint, which can be proven as shown below:

It is trivial that in Figure 4-1b, vector $\vec{r}$ can be written as the sum of vectors $\overrightarrow{AC}, \overrightarrow{DA}, \overrightarrow{BD}$

$$\vec{r} = \frac{1}{2} \overrightarrow{AC} + \overrightarrow{DA} + \frac{1}{2} \overrightarrow{BD} \quad (4-1)$$

And vectors $\overrightarrow{AC}, \overrightarrow{BD}$ can be written as

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} \quad (4-2)$$

$$\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} \quad (4-3)$$

Substituting (4-2),(4-3) for $\overrightarrow{AC}, \overrightarrow{BD}$ in (4-1), $\vec{r}$ can be written as below:
\[ \vec{r} = \frac{\vec{BC}}{2} - \frac{\vec{AD}}{2} \]  

(4-4)

Rewriting \( \vec{BC} \) and \( \vec{AD} \) in (4-4) as \( \vec{h}_{2n} \) and \( \vec{h}_0 \), gives

\[ \vec{r} = \frac{\vec{h}_{2n}}{2} - \frac{\vec{h}_0}{2} \]  

(4-5)

Following a similar process as (4-1) to (4-5), \( \vec{r} \) can also be written as

\[ \vec{r} = \frac{\vec{i}_0}{2} - \frac{\vec{i}_{2n}}{2} \]  

(4-6)

In Figure 4-2, \( \vec{h}_k \) can be represented as the sum of \( \vec{h}_m, \vec{i}_{2n}, \) and \( \vec{i}_0 \)

\[ \vec{h}_k = \frac{m - k}{2n} \vec{i}_{2n} + \vec{h}_m - \frac{m - k}{2n} \vec{i}_0 \]  

(4-7)

Rearranging (4-7) gives:

\[ \vec{h}_k - \vec{h}_m = \frac{k - m}{2n} (\vec{i}_0 - \vec{i}_{2n}) \]  

(4-8)

Substituting (4-7) for \( \vec{i}_0, \vec{i}_{2n} \) in (4-8), gives \( \vec{r} \) as a linear combination of \( \vec{h}_m, \vec{h}_k \)

\[ \frac{n}{k - m} (\vec{h}_k - \vec{h}_m) = \vec{r} \]  

(4-9)

Similarly, \( \vec{r} \) can also be represented as a linear combination of \( \vec{i}_k, \vec{i}_m \)

\[ \frac{n}{m - k} (\vec{i}_k - \vec{i}_m) = \vec{r} \]  

(4-10)

According to the linear algebra shown above (Strang, 2009), the formulas (4-9) and (4-10) show that axis \( r \) is always parallel with the plane defined by any two rulings in the same family.

### 4.2 Discrete strut and tie model of an individual hypar

This subchapter explains a number of minimum discrete models of an individual hypar, providing simple force distribution visualizations on its surface.

#### 4.2.1 Discrete strut and tie model of a classic hypar

This subsection introduces the classical case: a hypar with axis \( r \) parallel to gravity. In Figure 4-3a, a hypar is split into four patches by straight rulings, and the self-weight of a hypar is
4.2 Discrete strut and tie model of an individual hypar

discretized as four point loads and applied to the center of each patch separately. The continuous parabolas passing through these nodes are discretized as polylines, corresponding to the tangents of these parabolas.

This geometrical discreteness makes sure that two polylines and the edges intersecting at the same node are coplanar with the edge. Moreover, it also causes each node on the surface to be divided into two sub-nodes, which lie on the same action line of the point loads (Figure 4-3b). As such, each point load is divided in two and applied to the sub-nodes, thus, upward curving polylines and downward curving polylines each take half of the loads (Figure 4-4). This way, the internal forces in the polylines and reactions can be calculated with force diagrams (Figure 4-5).

Figure 4-3: (a) The self-weight applied to a hypar are discretized as four point loads. (b) The parabolas are discretized as polylines. Each point load is divided into two parts and applied to sub-nodes along action lines parallel to gravity.

Figure 4-4: The redistribution of loads at downward curving polylines (a) and upward curving polylines (b).
Chapter 4. Structural analysis of a single hypar

Figure 4-5: (a) Form diagram of a classical hypar. (b) The calculation of internal forces and reactions using a force diagram.

The result of the model in Figure 4-5 suggests that no matter how the other geometrical parameters vary, the axis $r$ is parallel to the load, hypars always keep internal forces as membrane forces, requiring reactions only along the four edges (Figure 4-5b).

4.2.2 Discrete strut and tie model of a general hypar

Based on the similar discreteness of distributed loads and principle parabolas in section 4.2.1, a strut and tie model can be set up for a general case: a hypar with axis $r$ nonparallel to gravity. In this case, each point load applied to each node has to be divided into three components: one axis component $f_r$ parallel to the axis, and another two ruling components $f_h$ and $f_i$, parallel to the two rulings intersecting this node (Figure 4-6).
4.2 Discrete strut and tie model of an individual hypar

Assuming that only ruling components $f_R^r$ and $f_H$ are applied to the hypar, they can be balanced with reactions along the rulings (Figure 4-7a). This is the first subsystem of the global structural model. The second subsystem is only loaded with axis components $f_R^r$ parallel to axis $r$ (Figure 4-7b), similar to the case studied in chapter 4.2.1. However, axis components $f_R^r$ applied in the second subsystem are not equally distributed. For example, components $f_{r-1}$ and $f_{r-4}$ have the same magnitude, while $f_{r-2}$ and $f_{r-3}$ have different magnitudes. These axis components have to be redistributed between the upward curving polylines (cables) and the downward curving polylines (arches) intersecting at the same node, until each polyline is loaded with distributed loads. However, the redistribution of axis components $f_R^r$ applied to the cables and arches has multiple solutions. Below, the solution in Figure 4-7 aims to ensure minimums difference in forces between the cables and arches, while keeping every parabola uniformly loaded.

![Figure 4-7](image)

Figure 4-7: (a) Subsystem I loaded with ruling components and its reactions. (b) Subsystem II loaded with axis components.

![Figure 4-8](image)

Figure 4-8: Redistribution of axis component and calculations of internal forces in subsystem I.
Figure 4-9: (a) the strut and tie model of subsystem II, (b) the calculation of its internal forces, and reactions with force diagrams.

Figure 4-10: (a) overlapping the first and the second subsystems gives this strut and tie model, (b) the force diagram of a general hypar.

Figure 4-8b shows that, after the redistribution of axis components $\mathbf{f}_r$ in the cables and arches, the internal forces can be calculated using a force diagram. For each node on an edge, the tangents of parabolas passing through the node are always coplanar with the edges. Thus in a general case, to achieve equilibrium of nodes on the edges, reactions are necessary along both edges and rulings (Figure 4-9b). Ultimately, by overlapping the first and the second subsystem together, a complete strut and tie model of a general hypar and its force diagrams can be developed (Figure 4-10b).
In some special cases, like the application in Chapter 7.2, two families of principle parabolas in a hypar can only take tension and all the principle parabolas turn into pre-stressed cables. Applied to the same distributed loads, hypars with all-tension parabolas get much higher internal forces than hypars with compression-tension parabolas. To illustrate, though the hypar in Figure 4-11 is loaded with distributed loads of half the magnitude as the loads present in Figure 4-10a, the internal forces in the hypar shown in Figure 4-11 are still higher than those in the hypar with compression-tension parabolas shown in Figure 4-10b.

Figure 4-11: A hypar with all principle parabolas in tension. (a) form diagram, (b) force diagram.

**Conclusion:**

If one family of principle parabolas in a hypar takes tension and another family takes compression, a hypar under self-weight and fully supported with two edges, generally exhibits behavior that lies between the behavior of a shell and that of a wall. The classical hypar studied in Chapter 4.2.1, is a special case in which rulings components are zero, thus, it works only as a shell. In another extreme case, when axis $r$ is perpendicular to the distributed loads, the axis components turn to zero and the hypar gets much higher internal forces along its rulings than along its parabolas. In this case, the load bearing behavior of a wall largely governs the behavior of the hypar (Cao & Schwartz, 2015).

Furthermore, if two families of parabolas in a hypar can only take tension, an individual hypar turns out to be a pre-stressed grid shell with rulings as struts and parabolas as cables.
4.3 Infinitely discrete strut and tie model of an individual hypar

This subchapter presents an infinitely discrete model of a hypar (in which the principle parabolas can take both compression and tension). The study of the infinitely discrete model includes three main steps. For the first step, point loads $\vec{g}$ are decomposed into three components: ruling components $\vec{f}_h$, $\vec{f}_i$ and axis components $\vec{f}_r$. For the second step, the model is divided into two subsystems. In subsystem I, the straight rulings $b$ and $i$ are loaded with ruling components $\vec{f}_h$ and $\vec{f}_i$, while in subsystem II, axis components $\vec{f}_r$ are redistributed and loaded according to two families of parabolas, ensuring that each parabola always takes distributed loads. Finally, subsystems I and II are overlapped together.

4.3.1 Decomposition of distributed loads

Starting with the first step, in Figure 4-12, a hypar is divided into $n^2$ numbers of patches ($n \in \mathbb{Z}^+$) and a point load $\vec{g}$ is applied to the center of each patch. Each point load $\vec{g}$ can be divided into three components along rulings $b$, $i$, and axis $r$:

$$\vec{g} = \vec{f}_h + \vec{f}_i + \vec{f}_r$$

(4-11)

If rulings $b$, $i$ and axis $r$ are represented as vectors $\vec{h}$, $\vec{i}$, $\vec{r}$, (4-11) can be rewritten as below. ($x$, $y$, $z$ are vector scalars)

$$\vec{g} = x\vec{i} + y\vec{h} + z\vec{r}$$

(4-12)

Figure 4-12: Infinite discreteness of a hypar. In this drawing, it compares axis components applied to two nodes which are on the same cable.
4.3 Infinitely discrete strut and tie model of an individual hypar

To ensure all the parabolas always take distributed loads, it is necessary to compare axis components $\vec{f}_r$ applied to every node of the same parabola. Node $(n,n)$ where two middle rulings pass through, and another node $(m,m)$ in the parabolic cable $P_l(0,0/2n,2n)$ are explained as an example at first in the following section (Figure 4-12). ($m=2N-1, N \in \mathbb{Z}^+$)

Point load $\vec{g}$ at node $(n,n)$ and node $(m,m)$ can be divided as below. ($x_{n,n}, y_{n,n}, z_{n,n}$ are vector scalars named after the nodes)

$$\vec{g} = x_{n,n}\vec{h}_n + y_{n,n}\vec{t}_n + z_{n,n}\vec{r} \quad (4-13)$$

$$\vec{g} = x_{m,m}\vec{h}_m + y_{m,m}\vec{t}_m + z_{m,m}\vec{r} \quad (4-14)$$

Form (4-9), following a linear constraint in $\vec{h}_m, \vec{h}_n, \vec{r}$

$$\vec{h}_m = \vec{h}_n + \frac{(m-n)}{n}\vec{r} \quad (4-15)$$

And using (4-10) $\vec{t}_m, \vec{t}_n, \vec{r}$ have a linear relation as below:

$$\vec{t}_m = \vec{t}_n - \frac{(m-n)}{n}\vec{r} \quad (4-16)$$

Substituting (4-15) and (4-16) for $\vec{h}_m$ and $\vec{t}_m$ in (4-14), gives

$$\vec{g} = x_{m,m}\vec{h}_n + y_{m,m}\vec{t}_n + \left[y_{m,m} + \frac{m-n}{n}(x_{m,m} - z_{m,m})\right]\vec{r} \quad (4-17)$$

Comparing (4-13) and (4-17), gives

$$y_{m,m} = y_{n,n} \quad (4-18)$$

$$x_{m,m} = x_{n,n} \quad (4-19)$$

$$z_{m,m} = \frac{m-n}{n}(y_{m,m} - x_{m,m}) + z_{n,n} \quad (4-20)$$

By replacing $x_{m,m}, y_{m,m}$ in (4-20) with (4-18) and (4-19), (4-20) can be rewritten as follows:

$$z_{m,m} = \frac{m-n}{n}(y_{n,n} - x_{n,n}) + z_{n,n} \quad (4-21)$$

After the study of nodes on the same parabola, nodes on different parabolas but the same ruling will be studied below, such as node $(n,k)$ and node $(m,k)$ in Figure 4-13. The point load $\vec{g}$
Chapter 4 . Structural analysis of a single hypar

applied to node\( (n,k) \) can be written as the sum of ruling components \( \vec{f}_{h(m,k)} \), \( \vec{f}_{i(m,k)} \) and axis components \( \vec{f}_{r(m,k)} \). \( (k = 2N-1, N \in \mathbb{Z}^+) \)

Figure 4-13: Comparing axis components applied to two nodes which are on the same ruling, but on different cables.

Representing \( \vec{g} \) with sum of vectors \( \vec{h}_n, \vec{t}_k, \vec{r} \):

\[
\vec{g} = x_{n,k} \vec{h}_n + y_{n,k} \vec{t}_k + z_{n,k} \vec{r} 
\] (4.22)

From (4.10) following a linear constraint in \( \vec{t}_k \):

\[
\vec{t}_k = \vec{t}_n + \frac{n - k}{n} \vec{r} 
\] (4.23)

Substituting (4.23) for \( \vec{t}_k \) in (4.22), gives

\[
g = x_{n,k} \vec{h}_n + y_{n,k} \vec{t}_n + \left[ z_{n,k} + y_{n,k} \frac{n - k}{n} \right] \vec{r} 
\] (4.24)

Comparing (4.13) with (4.24), gives the following results:

\[
x_{n,k} = x_{n,n} 
\] (4.25)

\[
y_{n,k} = y_{n,n} 
\] (4.26)
4.3 Infinitely discrete strut and tie model of an individual hypar

\[ z_{n,k} = z_{n,n} - \frac{n-k}{n} y_{n,n} \]  \hspace{1cm} (4-27)

Following a similar process as (4-22) to (4-27), for any node \((m,k)\) in Figure 4-13, it is possible to prove:

\[ z_{m,k} = z_{m,m} - \frac{m-k}{n} y_{n,n} \]  \hspace{1cm} (4-28)

Substituting (4-21) for \(Z_{m,m}\) in (4-28), gives

\[ z_{m,k} = z_{n,n} + \frac{k-n}{n} y_{n,n} - \frac{m-n}{n} x_{n,n} \]  \hspace{1cm} (4-29)

By representing \(x_{n,n}, y_{n,n}, z_{n,n}\) as \(x, y, z\), the ruling components and the axis components applied to node \((m,k)\) of a hypar \((m, k=2N-1, N \in \mathbb{Z})\), can be written as follows:

\[ g = \hat{f}_{h(m,k)} + \hat{f}_{l(m,k)} + \hat{f}_{r(m,k)} \]  \hspace{1cm} (4-30)

\[ \hat{f}_{h(m,k)} = x h_m \]  \hspace{1cm} (4-31)

\[ \hat{f}_{l(m,k)} = y l_k \]  \hspace{1cm} (4-32)

\[ \hat{f}_{r(m,k)} = \left[ \frac{m+k}{n} (x+y) + x + z - y \right] \hat{f} \]  \hspace{1cm} (4-33)

**Conclusion:**

Based on the results from formulas (4-30) to (4-33), the following can be concluded: all the nodes on ruling \(b_a\) of a hypar are applied with the same ruling components parallel to ruling \(b_c\). Similarly, all the nodes on the same ruling \(i_a\) are applied with the same ruling components parallel to ruling \(i_c\). Finally, the axis components applied to all the nodes of a hypar are changing linearly along the parabolic cables and the rulings. \((m, k=2N-1, N \in \mathbb{Z}^*)\)

4.3.2 Redistribution of the axis component

As mentioned in section 4.2.2, two parabolas passing through one node of a hypar each take a part of axis component \(\hat{f_R}\). This means that axis component \(\hat{f_R}\) at one node is divided into two parts: \(\hat{f}_{R,CI}\) and \(\hat{f}_{R,AR}\). \(\hat{f}_{R,CI}\) represents the part of the axis component applied to the sub-nodes of parabolic cable, and \(\hat{f}_{R,AR}\) represents the other part of the axis component applied to the sub-nodes of the parabolic arch. This subchapter will show how the axis components are redistributed at each node and how the values of \(\hat{f}_{R,CI}\) and \(\hat{f}_{R,AR}\) are calculated at each node.
According to the formula (4-33), the value of axis component $\vec{f}_{m,k}$ at node $(m,k)$ is a function of its indexes $(m,k)$. In relation to the geometry of a hypar in Figure 4-13, for all the nodes on the same arch, the sums of their two indexes, $m + k$, are always the same. Similarly, for all the nodes on the same cable, differences between their two indexes: $m - k$, are also always the same. In this case, a part of axis component $\vec{f}_{Ar}(m,k)$ applied to node $(m,k)$ on the arch can be written as the formula below, including $m + k$ and the known scalars $x, y, z$. ($v_x, v_y, v_z$ and $w_x, w_y, w_z$ are three unknown parameters of scalars $x, y, z$)

$$\vec{f}_{Ar}(m,k) = \left[ (m + k)(v_x x + v_y y + v_z z) + w_x x + w_y y + w_z z \right] \vec{r} \quad (4-34)$$

Similarly, the other part of axis components applied to node $(m,k)$ on the cable can be written as below, including $m + k$ and known scalars $x, y, z$. ($v'_x, v'_y, v'_z$ and $w'_x, w'_y, w'_z$ are three unknown parameters of scalars $x, y, z$)

$$\vec{f}_{Cl}(m,k) = \left[ (m - k)(v'_x x + v'_y y + v'_z z) + w'_x x + w'_y y + w'_z z \right] \vec{r} \quad (4-35)$$

The sum of (4-34) and (4-35) equals axis component $\vec{f}_{r(m,k)}$. Comparing their sum with (4-33) yields:

$$v_x = v'_x = - \frac{1}{2n}$$
$$v_y = -v'_y = \frac{1}{2n}$$
$$v_z = v'_z = 0$$
$$w_x + w'_x = w_z + w'_z = 1$$
$$w_y + w'_y = -1 \quad (4-36)$$

According to (4-36), the part of the axis components applied to the arch (4-34), and the other part of the axis components applied to the cable (4-35) can be expressed as follows:

$$\vec{f}_{Ar}(m,k) = \left[ (m + k)\frac{y - x}{2n} + w_x x + w_y y + w_z z \right] \vec{r} \quad (4-37)$$

$$\vec{f}_{Cl}(m,k) = \left[ (k - m)\frac{x + y}{2n} + w'_x x + w'_y y + w'_z z \right] \vec{r} \quad (4-38)$$

By varying $w_x, w_y, w_z$ and $w'_x, w'_y, w'_z$, (4-37) and (4-38) can have multiple solutions. This implies that the distribution of axis components between the cables and arches is indeterminate.

Since the value of $(m + k)\frac{y - x}{2n}$ in (4-37) and the value of $(k - m)\frac{x + y}{2n}$ in (4-38) are fixed to minimize the difference between (4-37) and (4-38), the value of $w_x x + w_y y + w_z z$ in (4-37)
should be equal to the value of \( w'x + w'y + w'z \) in (4-38). In this case, by considering (4-36), the value of \( w_x, w_y, w_z \) and \( w'_x, w'_y, w'_z \) can be found as below:

\[
w_x = w'_x = \frac{1}{2} \quad w_z = w'_z = \frac{1}{2} \quad w_y = w'_y = -\frac{1}{2}
\]  

(4-39)

Based on the values above, the best distribution of axis components on cables and arches are as below. This ensures every cable and arch are always loaded with evenly distributed point loads, and the difference between the force in the cables and arches is minimized.

\[
\mathbf{f}_r.Ar(m,k) = \left[ (m + k) \frac{y - x}{2n} + \frac{x - y + z}{2} \right] \mathbf{i}
\]  

(4-40)

\[
\mathbf{f}_r.Cl(m,k) = \left[ (k - m) \frac{x + y}{2n} + \frac{x - y + z}{2} \right] \mathbf{i}
\]  

(4-41)

Formulas (4-40) and (4-41) are two general formulas to calculate the axis components applied to the cable and the arch passing through node \((m,k)\). They describe how every cable or arch is always evenly loaded, and how the difference between the force in the cables and arches is minimized.

### 4.3.3 Calculations of internal forces and reactions

Below, the internal forces and reactions in a hypar under self-weight are calculated. To do this, the whole hypar is divided into two subsystems. Subsystem I is loaded with ruling components \( \mathbf{f}_h \) and \( \mathbf{f}_l \), while Subsystem II is loaded with axis components \( \mathbf{f}_r \).

![Figure 4-14: In subsystem I, every node is only loaded with ruling components \( \mathbf{f}_h \) and \( \mathbf{f}_l \) parallel to rulings \( h \) and \( i \).](image)
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Subsystem I - reactions along rulings

Subsystem I is only loaded with ruling components \( \vec{f}_h \) and \( \vec{f}_i \). According to (4-31) and (4-32), all the nodes on the same ruling are applied with the same ruling components, and there are \( n \) nodes on the same ruling. As such, the sum of all reactions along ruling \( h_{2N-1} \) (\( N \in \mathbb{Z}^+ \)) can be written as:

\[
R_{\vec{h}}(2N-1) = -x Nh_{2N-1}
\] (4-42)

And the sum of all reactions along ruling \( i_{2N-1} \) (\( N \in \mathbb{Z}^+ \)) can be written as below:

\[
R_{\vec{i}}(2N-1) = -yn_i_{2N-1}
\] (4-43)

Subsystem II - internal forces along parabola

In subsystem II, every parabola is loaded with axis component \( f_r \), and supported by edges and rulings.

To calculate maximal internal forces along parabolas, parabolic cable \( Pl_{(2N, 2n / 0, 2n-2N)} \) and parabolic arch \( Pl_{(2N, 2n / 2n, 2N)} \) are studied as examples (Figure 4-15):

In the case of parabolic cable \( Pl_{(2N, 2n / 0, 2n-2N)} \), according to (4-40), all the loads applied to it can be written as:

\[
\vec{F}_{\vec{r},c}(2N, 2n / 0, 2n-2N) = \left[ \frac{(n-N)(x+y)}{2n} + \frac{x-y+z}{8} \right] N \vec{r}
\] (4-44)

Based on (4-9) and (4-10), vectors \( \vec{h}_{2N}, \vec{h}_0, \vec{i}_{2n-2N}, \vec{i}_{2n} \) and \( \vec{r} \) are linearly constrained as:

\[
\vec{h}_{2N} = \vec{h}_0 - \vec{i}_{2n} + \vec{i}_{2n-2N} = \frac{4N}{n} \vec{r}
\] (4-45)

Comparing (4-44) and (4-45), and by multiplying (4-45) with \( \frac{(n-N)(x+y)}{4} + \frac{n(x-y+z)}{8} \) on both sides, reactions along rulings \( h_{2N} \) and \( i_{2n} \) can be written as:

\[
R^1_{\vec{h}}(2N) = - \left[ \frac{(n-N)(x+y)}{4} + \frac{n(x-y+z)}{8} \right] \vec{h}_{2N}
\] (4-46)

\[
R^1_{\vec{i}}(2n) = \left[ \frac{(n-N)(x+y)}{4} + \frac{n(x-y+z)}{8} \right] \vec{i}_{2n}
\] (4-47)

The maximum internal force along parabolic cable \( Pl_{(2N, 2n / 0, 2n-2N)} \) is at node \((2N, 2n)\), and is balanced by the reactions along rulings \( h_{2N} \) and \( i_{2n} \). Therefore, it can be written as the negative of the sum of (4-46) and (4-47):
4.3 Infinitely discrete strut and tie model of an individual hypar

\[
\vec{F}_{r.CI(2N,2n / 0,2n-2N)} = \left[ \frac{(n - N)(x + y)}{4} + \frac{n(x - y + z)}{8} \right] (\vec{h}_{2N} - \vec{i}_{2n}) \quad (4-48)
\]

Similar to (4-44), the maximum amount of reactions along parabolic arch Pl(2N,2n / 2n,2N) in (4-48) can be found through:

\[
\vec{F}_{r.Ar(2N,2n / 2n,2N)} = \left[ \frac{(N + n)(y - x)}{4} + \frac{n(x - y + z)}{8} \right] (\vec{h}_{2n} + \vec{i}_{2n}) \quad (4-49)
\]

Figure 4-15: Subsystem II is only loaded with axis components. Parabolic arches and cables in a hypar are supported by reactions along rulings and edges. Cable Pl(2N,2n / 0,2n-2N) and arch Pl(2N,2n / 2n,2N) are studied as examples of maximum internal forces along parabolas. Rulings \( h_{2n} \) are studied as examples of reactions along rulings, and edges \( i_{2n} \) are studied as examples of reactions along edges.

**Subsystem II - reactions along rulings**

In subsystem II, every parabola is loaded with axis components \( \vec{f}_r \) and is supported by edges and rulings. To study the internal forces applied to the rulings and edges in subsystem II in general, the internal forces applied to ruling \( h_{2N} \) \((N \in \mathbb{Z}^+)\) are explained as an example below.

In the case of a hypar in Figure 4-15, a ruling \( h_{2N} \) is connected with four parabolas: cables \( Pl(2N,2n / 0,2n-2N) \), \( Pl(2n,2n-2N / 2N,0) \), and arches \( Pl(0,2N / 2N,0) \), \( Pl(2N,2n / 2n,2N) \). All the loads applied to each parabola are finally transmitted to the rulings. Below, the load applied to these four parabolas, as well as the reactions along rulings \( h_{2N} \) are calculated separately.

Similar to the calculation from (4-44) to (4-47), in the case of parabolic cable \( Pl(2n,2n-2N / 2N,0) \), one of its reactions along a rulings \( h_{2N} \) can be represented as:
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\[ \vec{R}_{h(2N)}^2 = -\left[ \frac{n(x - y + z)}{8} - \frac{N(y + x)}{8} \right] \vec{h}_{2n} \]  
(4-50)

In the case of parabolic arch \( P_{L(2N, / 2N,0)} \), one of its reactions along a ruling \( b_{2N} \) can be represented as:

\[ \vec{R}_{h(2N)}^3 = \left[ \frac{n(x - y + z)}{8} + \frac{N(y - x)}{8} \right] \vec{h}_{2N} \]  
(4-51)

In the case of parabolic arch \( P_{L(2N,2N / 2N,2N)} \), one of its reactions along a ruling \( b_{2N} \) can be represented as:

\[ \vec{R}_{h(2N)}^4 = -\left[ \frac{n(x - y + z)}{4} + \frac{(N + n)(y - x)}{4} \right] \vec{h}_{2N} \]  
(4-52)

Based on the above calculations, the sum of all the reactions along rulings \( b_{2N} \) \( (N \in Z^+) \) can be expressed as the sum of (4-46), (4-50), (4-51) and (4-52):

\[ \vec{R}_{h(2N)}^N = -\frac{n}{2} x\vec{h}_{2n} \]  
(4-53)

Following a similar process to the calculation of reactions along ruling \( b_{2N} \) from (4-50) to (4-53), the forces along a ruling \( i_{2N} \) \( (N \in Z^+) \) can be written as:

\[ \vec{R}_{i(2N)}^N = -\frac{n}{2} y\vec{i}_{2n} \]  
(4-54)

Subsystem II - reactions along edges

In the example below, edge \( i_{2n} \) is taken as an example to calculate reactions along edges.

To study the reactions accumulated along edge \( i_{2n} \), a node \( (2N, 2n) \) on this edge \( i_{2n} \) is checked first.

For any parabolic cable \( P_{L(2N,2N / 2n,2N)} \) connected with edge \( i_{2n} \), the reactions applied to node \( (2N,2n) \) on this edge can be written as below, by following a similar calculation as seen in (4-44) to (4-47), \( (1 \leq N \leq n) \)

\[ \vec{R}_{i(2n)}^N = \left[ \frac{n(x - y + z)}{4} - \frac{(n - N)(y + x)}{4} \right] \vec{i}_{2n} \]  
(4-55)

There are \( n \) numbers of cables connected with edge \( i_{2n} \) (Figure 4-15), and according to (4-55), the sum of all reactions along this edge \( i_{2n} \) caused by cables can be written as follows:

\[ \sum_{N=1}^{n} \vec{R}_{i(2n)}^N = \left[ \frac{n^2(2x + z)}{8} - \frac{n(x + y)}{8} \right] \vec{i}_{2n} \]  
(4-56)
4.3 Infinitely discrete strut and tie model of an individual hypar

Similarly, for any parabolic arch $P_{2N,2n}$, one of its reactions applied to any node $(2N,2n)$ on edge $i_{2n}$ can be written as follows (assuming $1 \leq N \leq n$):

$$\vec{R}^N_{i(2n)} = \left[ \frac{n(x - y + z)}{4} - \frac{(N + n)(y - x)}{4} \right] \vec{i}_{2n} \quad (4-57)$$

There are also $n$ numbers of arches connected to edge $i_{2n}$ (Figure 4-15). As a result, the sum of all reactions along edge $i_{2n}$ caused by arches can be written as follows:

$$\sum_{N=0}^{n-1} \vec{R}^N_{i(2n)} = \left[ \frac{n^2(2y + z - 2x)}{8} - \frac{n(y - x)}{8} \right] \vec{i}_{2n} \quad (4-58)$$

By adding (4-56) and (4-58), the total reaction along edge $i_{2n}$ is given as:

$$\vec{R}_{i(2n)} = \left[ \frac{n^2(z + y)}{4} - \frac{n}{4} y \right] \vec{i}_{2n} \quad (4-59)$$

With similar calculations as seen in (4-58) and (4-59), the reactions accumulated on edges $i_0$, $h_0$ and $h_{2n}$ can be written as follows:

$$\vec{R}_{i(0)} = \left[ \frac{n^2(y - z)}{4} - \frac{n}{4} y \right] \vec{i}_0 \quad (4-60)$$

$$\vec{R}_{h(0)} = \left[ \frac{n^2(z + x)}{4} - \frac{n}{4} x \right] \vec{h}_0 \quad (4-61)$$

$$\vec{R}_{h(2n)} = \left[ \frac{n^2(x - z)}{4} - \frac{n}{4} x \right] \vec{h}_{2n} \quad (4-62)$$

**Conclusion:**

If the self-weight $\vec{G}$ of a hypar is discrete, as $n^2$ numbers of point loads (hypars include $2n$ of rulings $b$ and $2n$ of rulings $i$), the value of a point load $\vec{g}$ is written as below:

$$\vec{g} = \vec{G} \frac{1}{n^2} \quad (4-63)$$

Point load $\vec{g}$ can be further decomposed into three components along middle rulings $h_0$, $i_0$ and axis $r$. 
The maximum reactions on the four edges of a hypar can be written as follows:

\[ R_{(2n)} = \frac{n^2}{4} (z + y) - \frac{n}{4} y \hat{i}_{2n} \]

\[ R_{(0)} = \frac{n^2}{4} (y - z) - \frac{n}{4} y \hat{i}_0 \]

\[ R_{h(0)} = \frac{n^2}{4} (z + x) - \frac{n}{4} x \hat{h}_0 \]

\[ R_{h(2n)} = \frac{n^2}{4} (x - z) - \frac{n}{4} x \hat{h}_{2n} \] (4-64)

Similarly, the maximum reactions along rulings \( b \) can be written as follows (assuming \( N \in \mathbb{Z}^+ \)):

\[ \vec{R}_{h(2N)} = -\frac{n}{2} x \hat{h}_{2N} \]

\[ \vec{R}_{h(2N+1)} = -xn \hat{h}_{2N-1} \] (4-65)

Lastly, the maximum reactions along rulings \( i \) can be written as follows:

\[ \vec{R}_{i(2N)} = -\frac{n}{2} y \hat{h}_{2N} \]

\[ \vec{R}_{i(2N+1)} = -ny \hat{i}_{2N-1} \] (4-66)

In this general case, to achieve the equilibrium of a hypar loaded with distributed forces, reactions along all four edges and along all the rulings are required.

**Special case:**

When \( n = 1 \), structurally all the self-weight of a hypar is implied to be combined into one resultant, and the final reactions are also resultants:

Point load \( \vec{G} \) can be decomposed into three components along middle rulings \( h_n, i_i \) and axis \( r \):

\[ \vec{G} = Xh_n + Yi_i + Zr \]
This way, the maximum reactions accumulated on the edges can be written as follows (assuming \( N \in \mathbb{Z}^+ \)):

\[
\vec{R}_{h\{2n\}} = \frac{Z}{4} \vec{i}_{2n} \quad \vec{R}_{h\{0\}} = -\frac{Z}{4} \vec{i}_0
\]

\[
\vec{R}_{h\{0\}} = \frac{Z}{4} \vec{h}_0 \quad \vec{R}_{h\{2n\}} = -\frac{Z}{4} \vec{h}_{2n}
\]

Similarly, the maximum reactions along rulings \( h \) can be written as follows:

\[
\vec{R}_h = -X \vec{h}_n
\]

And maximum reactions along rulings \( i \) can be written as:

\[
\vec{R}_i = -Y \vec{i}_n
\]

In this case, the reactions necessary for equilibrium of a hypar loaded with distributed loads are simplified into shear forces along four edges and forces along the two middle rulings.

### 4.4 Distributed horizontal loads

In this section, a hypar loaded with self-weight and distributed horizontal forces will be studied. Similar to a hypar under self-weight in section 4.3, the distributed horizontal loads can also be decomposed into ruling components along rulings \( h \) and \( i \), and axis components along axis \( r \). By applying ruling components and axis components separately to the same two subsystems as in chapter 4.3, the internal forces along the parabolas and reactions along the rulings and edges can be calculated following a similar process what was used in section 4.3.

Besides horizontal loads, a hypar under distributed loads in other directions always keeps its internal forces in plane and reactions along its rulings and edges. When the resultant of all the externally distributed loads is parallel to the axis \( r \) of a hypar, the hypar turns out to be a pure shell which only has membrane forces along its parabolas, and reactions along its four edges. On the other hand, when the resultant is perpendicular to axis \( r \), the hypar behaves more like a wall. The formulas in section 4.3 for reactions and internal forces of a hypar under vertically distributed loads can generally be applied to calculate the behavior of a hypar under uniformly distributed loads in different directions.

### 4.5 Undistributed loads

In this section, a hypar loaded with undistributed loads is studied. The self-weight of a hypar is discretized as a group of point loads \( \vec{g} \), and an extra point load \( \vec{N} \) is applied to one node of the hypar (Figure 4-16). In terms of the simplification, similar to section 4.3.1, point loads \( \vec{g} \) are decomposed into ruling components \( \vec{f}_h, \vec{f}_i \) and axis components \( \vec{f}_r \). Axis components \( \vec{f}_r \) are redistributed between cables and arches to ensure that every parabola takes distributed loads
like in section 4.3.2. Load \( \vec{N} \) is also decomposed into three components along the rulings and axis (Figure 4-16): the ruling components are balanced by reactions along the rulings, and axis components are taken by the arches and cables. Therefore, the cable and the arch passing through the node where point load \( \vec{N} \) is applied are loaded with undistributed loads. This cable and this arch can be studied separately without affecting the global equilibrium of the whole hypar (Figure 4-17 and Figure 4-18).

In Figure 4-17 and Figure 4-18, the parabolic cable and the parabolic arch passing through the node where point load \( \vec{N} \) is applied are loaded with distributed loads and the axis component of point load \( \vec{N} \). To achieve equilibrium in the unevenly loaded parabola, reactions along the rulings which intersect the parabola, as well as reactions at the edges, are necessary. According to the force diagram in Figure 4-17b, the reactions along the rulings are balanced by the axis components of point load \( \vec{N} \).

Benefits of the geometry of a hypar include that the tangents of a parabola in a hypar are always coplanar with two rulings passing through the tangent point; and that the undistributed loads applied to any node of a hypar can always be balanced by two additional reactions along its rulings. The above ensure a hypar can always keep its internal forces in plane, and its reactions along its rulings and edges.

Figure 4-16: The self-weight and point load \( \vec{N} \) are decomposed into ruling components and axis components
4.5 Undistributed loads

Figure 4-17: A cable loaded with distributed loads and a part of the axis components of load $\vec{N}$. To keep the cable in equilibrium, it needs reactions not only at two ends but it also needs reactions of some of the nodes along the cable. (a) Form diagram (b) force diagram

Figure 4-18: An arch loaded with distributed loads and a part of the axis component of load $\vec{N}$. To keep the cable in equilibrium, it needs reactions not only at two ends but also at some nodes along the arch. (a) Form diagram (b) force diagram
4.6 Digital implementation

The results presented in chapter 4.3 are implemented as a grasshopper component in Rhino 6. This component can simply visualize the distribution of force flows in a hypar under vertically distributed loads by calculating the internal forces, thereby helping designers to judge the necessary supporting boundaries.

Based on formulas (4-64) to (4-66), there are three inputs for the grasshopper component: the geometry of the hypar, the parameter $n$ representing the discreteness, and the parameter $G$ representing the magnitude of vertically distributed loads. The geometry of the hypar is input as a surface in the grasshopper component. According to the value of parameter $n$, the input hypar surface is discretized as a strut and tie model with the corresponding density of rulings and parabolas. The component then translate geometrical rulings and edges into vectors in grasshopper. Consequently, the vertically distributed load applied to the hypar is the product of the surface area and parameter $G$. With these three inputs and by following the formulas (4-64) to (4-66), the function can proceed as a simple vector operation which can calculate the internal forces along the rulings and parabolas, as well as the necessary reactions. Based on the calculation results, the strut and tie model is visualized in varying thicknesses in Rhino, according to the magnitude of their axial forces. Finally, the reactions are presented as separated vectors.

4.7 Conclusion

The principle parabolas and rulings in a hypar can both be structurally activated due to its double curved and double ruled geometry, thus allowing it to exhibit the structural behavior to the likeness between that of a shell and a wall, or a membrane and a wall. Benefiting from their special geometrical properties, hypars are able to resist different loading cases without activating any bending moments; the required reactions are always along its four edges and along two families of rulings. In some very special cases (when axis $r$ of the hypar is parallel to the external loads), the required reactions occur only along its four edges.
5 Global equilibrium of smooth poly-hypar surfaces

This chapter discusses the global equilibrium of smooth poly-hypar surfaces made from several hypar modules. It first discusses the necessary support conditions of a hypar in general by proposing a new approach to replace rigid edge beams in poly-hypar surfaces. In this approach, one hypar is fully supported by another adjacent hypar instead of by rigid beams. Then, a geometrical constraint called the coplanarity principle is introduced to ensure that the interactions between adjacent hypars are unable to cause bending moments. Following the coplanarity principle, the resulting poly-hypar surfaces turn to satisfy the second order of surface smoothness in mathematics, which in this research is defined as the smooth poly-hypar surface. To guide designers intuitively in designing smooth poly-hypar surfaces while ensuring global equilibrium, the conception of load paths is also introduced to check the support conditions of a smooth poly-hypar surface. At the end of this chapter, a precise calculation based on vector algebra is presented to evaluate the internal forces and reactions in a smooth poly-hypar surface loaded with self-weight.

5.1 Structural behavior of a hypar module in poly-hypar surfaces

As discussed in chapter 4, generally, a hypar with self-weight needs reactions along its rulings and edges to achieve equilibrium. Furthermore, in the case of an individual hypar, the minimum amount of required supported boundary is two straight edges. In chapter 4, reactions were assumed to be applied directly to every node on the rulings and edges (Figure 5-1a), while in the case of a hypar supported by two edges, these reactions are transferred into internal forces (Figure 5-1b). These internal forces along rulings and edges are cancelled by reactions applied to two supported edges (Figure 5-1b).

Figure 5-1: (a) a hypar with reactions at every node. (b) a hypar supported by two edges, reactions are transferred into internal forces along the ruling and edges.
Based on the results from (4-64) to (4-66), in the hypar supported by two edges, the resultants of the internal force along the rulings and edges can be written as follows:

The maximum internal forces accumulated along the edges:

\[
\vec{F}_{i(2n)} = \left[ -\frac{n^2}{4}(z + y) + \frac{n}{4}y \right] \vec{i}_{2n}
\]

\[
\vec{F}_{i(0)} = \left[ -\frac{n^2}{4}(y - z) + \frac{n}{4}y \right] \vec{i}_0
\]

\[
\vec{F}_{h(0)} = \left[ -\frac{n^2}{4}(z + x) + \frac{n}{4}x \right] \vec{h}_0
\]

\[
\vec{F}_{h(2n)} = \left[ -\frac{n^2}{4}(z - x) + \frac{n}{4}x \right] \vec{h}_{2n}
\]

The maximum internal forces accumulated along rulings \( b \) (assuming \( N \in \mathbb{Z}^+ \)):

\[
\vec{F}_{h(2N)} = \frac{n}{2} \vec{h}_{2N}
\]

\[
\vec{F}_{h(2N+1)} =nx\vec{h}_{2N+1}
\]

The maximum internal forces accumulated along rulings \( i \) (assuming \( N \in \mathbb{Z}^+ \)):

\[
\vec{F}_{i(2N)} = \frac{n}{2} \vec{i}_{2N}
\]

\[
\vec{F}_{i(2N+1)} =ny\vec{i}_{2N+1}
\]

To provide the necessary reactions along the rulings and edges, one optional solution, unlike stiffened beams in the precedents of folded poly-hypar structures, is to ensure another hypar becomes the full support of the adjacent hypar. Once one hypar is connected with another, the reactions which are necessary for the equilibrium of the first hypar become actions applied to the adjacent hypar. In other words, these actions applied to the other hypar equal the internal forces along the rulings and edges in the first hypar.

5.2 Definition of smooth poly-hypar surfaces

5.2.1 Coplanarity principle

The coplanarity principle is a geometrical constraint derived from the equilibrium request. It ensures that the interactions between adjacent hypars are unable to cause bending moments. Since all the necessary reactions of a hypar are always along its rulings and edges, the only way to avoid bending moments at the shared edges between adjacent hypars, is to geometrically
5.2 Definition of smooth poly-hypar surfaces

ensure all the rulings and edges intersecting at each node are always coplanar. To achieve such coplanarity, the geometrical constraint between two adjacent hypars is studied below:

If the four edges of a hypar $H_1$ are given, the rulings and edges of $H_1$ can be represented as vectors (Figure 5-2). To ensure that edges of $H_1$ are coplanar with the intersecting edges of the adjacent hypar $H_2$, edge $b_{i'}$ of $H_2$ can be represented as a vector $\vec{h}_0^i$ which is in a linear constrain (5-1) with vectors $\vec{h}_0^3$ and $\vec{r}_{2n}^i$ (vector $\vec{r}_{2n}^i$ represents the shared edge between $H_1$ and $H_2$). Similarly, edge $b_{k'}$ of hypar $H_2$ can also be represented as a vector $\vec{h}_{2n}^j$, which is in a linear constraint with $\vec{h}_{2n}^1$ and $\vec{r}_{2n}^i$ (5-2).

\[ \vec{h}_0^i = a\vec{h}_0^1 + b\vec{r}_{2n}^i \]  \hspace{1cm} (5-1)

\[ \vec{h}_{2n}^j = c\vec{r}_{2n}^j + d\vec{r}_{2n}^i \]  \hspace{1cm} (5-2)

Figure 5-2: Two adjacent hypars satisfying the coplanarity principle, which ensures that all the intersecting rulings and edges in adjacent hypars are always coplanar.

According to (4-9), a vector $\vec{h}_m^i$ which represents one ruling $b_{i'}$ of hypar $H_i$, can be written as a linear function of $\vec{h}_{2n}^1$ and axis vector $\vec{r}$, or as a linear function of $\vec{h}_0^1$ and $\vec{r}$ as follows:

\[ \vec{h}_m^i - \vec{h}_{2n}^1 = \frac{m - 2n}{n} \vec{r} \]  \hspace{1cm} (5-3)
By cancelling \( \vec{r} \) in (5-3) and (5-4), \( \vec{h}_{m} \) can be written below as a linear function of \( \vec{h}_{2n} \) and \( \vec{h}_{0} \):

\[
\vec{h}_{m} = \frac{m}{2n} \vec{h}_{2n} + \frac{2n - m}{2n} \vec{h}_{0}
\] (5-5)

Similarly, vector \( \vec{h}_{m}^{2} \) which represents one ruling \( b_{m}^{2} \) of hypar \( H_{2} \) can be written as:

\[
\vec{h}_{m}^{2} = \frac{m}{2n} \vec{h}_{2n}^{2} + \frac{2n - m}{2n} \vec{h}_{0}^{2}
\] (5-6)

Substituting (5-1) and (5-2) for \( \vec{h}_{2n}^{1} \) and \( \vec{h}_{0}^{1} \) in (5-6), allows the vector \( \vec{h}_{m}^{2} \) to be written as a linear function of vectors \( \vec{h}_{2n}^{1} \), \( \vec{h}_{0}^{1} \) and \( \vec{r}_{2n}^{1} \):

\[
\vec{h}_{m}^{2} = \frac{m}{2n} a \vec{h}_{2n}^{1} + \frac{2n - m}{2n} c \vec{h}_{0}^{1} + (\frac{m}{2n} b + \frac{2n - m}{2n} d) \vec{r}_{2n}^{1}
\] (5-7)

Rewriting (5-5) as below:

\[
\vec{h}_{2n}^{1} = \frac{2n}{m} \vec{h}_{m}^{1} - \frac{2n - m}{m} \vec{h}_{0}^{1}
\] (5-8)

Replacing \( \vec{h}_{2n}^{1} \) in (5-7) with (5-8), gives:

\[
\vec{h}_{m}^{2} = a \vec{h}_{m}^{1} + (\frac{m}{2n} b + \frac{2n - m}{2n} d) \vec{r}_{2n}^{1} + (c - d) \frac{2n - m}{2n} \vec{h}_{0}^{1}
\] (5-9)

According to the coplanarity of rulings \( b_{m}^{2}, b_{m}^{1} \) and the shared edge \( i_{2n}^{1} \) between hypars \( H_{1} \) and \( H_{2} \), vector \( \vec{h}_{m}^{2} \) must be written as a linear function of vectors \( \vec{h}_{m}^{1} \) and \( \vec{r}_{2n}^{1} \). Therefore, the vector \( (c - d) \frac{2n - m}{2n} \vec{h}_{0}^{1} \) in (5-9) must be the zero vector for all \( m \). This means \( c = d \), and formulas (5-1) and (5-2) can be rewritten as below:

\[
\vec{h}_{0}^{2} = a \vec{h}_{0}^{1} + b \vec{r}_{2n}^{1}
\]

\[
\vec{h}_{2n}^{2} = a \vec{h}_{2n}^{1} + d \vec{r}_{2n}^{1}
\] (5-10)

**Conclusion**

Formulas (5-10) are the mathematical representations of the coplanarity principle. The geometrical principle ensures that in a pair of edge-joined hypars, all the edges and rulings intersecting the same node are always coplanar. It enables interactions between adjacent hypars.
never to be outside the surface, causing bending moments. It is the one of the two constraints required to construct smooth poly-hypar surfaces with multiple hypar modules

5.2.2 Smooth poly-hypar surfaces

The poly-hypar surface is a general description for geometries made from multiple hypar modules. Most approaches in the precedents can be considered to be folded poly-hypar surfaces (like the works of Candela), because of the way the hypar modules are joined by discontinuous seams. Smooth poly-hypar surfaces, as opposed to folded poly-hypar surfaces, are made with smoothly joined hypar modules with continuous seams.

Figure 5-3: Smooth poly-hypar surfaces combined from six hypar modules. (a) Locally ruled, all the intersecting rulings and edges are coplanar. (b) Globally freeform, the principle curves on a smooth poly-hypar surface are freeform curves.

The smoothness of poly-hypar surfaces results from the need to archive equilibrium, which ensures that every pair of adjacent hypar modules satisfies the coplanarity principle. According to the coplanarity principle, the most important geometrical property of smooth poly-hypar surfaces is that all the straight lines on the surface intersecting at one node should always be coplanar. In regards to this special property, smooth poly-hypar surfaces actually belong to a geometrical category sometimes referred to as hyperbolic nets (Emanuel & Thilo, 2014), which generally satisfy $G^1$ smoothness: continuity and tangency. In some special cases, smooth poly-hypar surfaces can reach the level of $G^2$ smoothness, where the curvature at the joined point is equal on both sides; and $G^3$ smoothness, where the curvature is continuous at the joined point (Mortenson, 2006), (Munford, 2014). Thus, the term smooth poly-hypar surface can have two levels of meaning: on the one hand, the local, individual module is a ruled hypar, while on the other hand, globally the constructed surface has a smooth, freeform appearance.
In relation to the global equilibrium, several hypar modules sharing the same load path are supported one by another until the internal forces are transmitted to supports at the boundaries. Thus, whether a smooth poly-hypar surface can turn into an efficient surface structure mainly depends on its supported boundaries. Besides the coplanarity principle, this is the other constraint required to construct the desired surface structures with hypar modules.

5.3 Load paths

To judge the necessary supporting boundaries of a smooth poly-hypar surface, this subchapter presents load paths as the intuitive reference.

In a smooth poly-hypar surface, by considering each hypar module as a sub-system in equilibrium, the reactions at the border of one hypar can be understood to be transferred as actions into the adjacent hypars. As such, the model used to study the global equilibrium of a smooth poly-hypar surface is only loaded with actions along rulings and edges (Figure 5-5a), since the internal forces along parabolas do not affect the global equilibrium (Cao, et al., 2017).

Based on the simplification above, following the coplanarity principle, rulings and edges intersecting at one node are always coplanar. This implies that the internal forces can always be transmitted along rulings and edges, thus generating specific paths to transfer the internal forces to the supports. The concept of load paths is used in this research to locate the necessary supports. A load path actually is a group of middle rulings connected end to end (Figure 5-5b). The boundaries intersecting the load paths are the optional locations of supports. Of the two intersecting boundaries with the same load path, it is only necessary for one to be supported. Thus, there are multiple choices for the locations of supports in a smooth poly-hypar surface (Figure 5-6), (Figure 5-7). Once all the load paths are supported, the smooth poly-hypar surfaces can achieve global equilibrium without causing bending moments. In this way, the designer can combine hypars freely as long as the coplanarity principle is fulfilled and each load path is directly supported.

Figure 5-4: Generally, a smooth poly-hypar surface satisfy $G^1$ smoothness (a), in some special cases, it can also satisfy $G^2$ (b) and $G^3$ (c) smoothness.
5.3 Load paths

Figure 5-5: (a) In the study of the global equilibrium, a smooth poly-hypar surface is only loaded with resultant forces along rulings and edges. The local behavior of each hypar (internal forces along parabolas) is ignored. (b) Load paths (green polylines) represent how forces are transmitted to the boundaries. To achieve global equilibrium, all the load paths should be supported.

Figure 5-6: (c) Supports variation I. Adding additional hypar modules at one end of the load path, bringing forces down to the ground and to the wall. (d) Force distribution along rulings and edges in variation I. The thickness of the blue (compression) and the red (tension) lines represent the density of the forces. The unsupported edges of the combined shells are all freestanding and without bending moments.
Figure 5-7: (a) Supports variation II. The smooth poly-hypar surface is supported by two sidewalls and the ground. (b) The force distribution along rulings and edges in variation II. The smooth poly-hypar surface globally behaves similar to a cantilever.

5.4 Calculations of internal forces and reactions

When several hypar modules are combined into a smooth poly-hypar surface, the internal forces along the principle parabolas in each hypar module still stay the same, while actions applied to the adjacent hypars are accumulated through load paths. Below, the internal forces accumulated along rulings and edges will be calculated by only loading the smooth poly-hypar surface with actions along its rulings and edges.

To achieve this, there are two steps: the first step focuses on the internal forces accumulated along rulings, while the second step, mainly solves the internal forces accumulated along edges. In the first step, actions are transmitted through load paths, from rulings of one hypar module to the rulings of another adjacent module, resulting in deviating forces along the shared edges between adjacent hypar modules (Figure 5-8). In the second step, these deviating forces are added to the actions along the edges of the hypar modules, and then transmitted into the supports similar to how the actions along rulings in the first step were transmitted.

5.4.1 Internal forces transmitted along rulings

Since actions applied along rulings of each hypar module are always accumulated in the same load path, in a smooth poly-hypar surface, it is possible to insulate a group of hypar modules along the same load path, thereby being able to study the accumulated internal forces along rulings separately without affecting the global equilibrium.
A group of hypars along one load path of a smooth poly-hypar surface is studied in detail as an example for the calculation of internal forces in poly-hypar surfaces in general (Figure 5-8). In particular, the internal forces along a ruling $h_m$ of this group of hypars is evaluated.

Figure 5-8: A load path through $q$ pieces of hypars. Actions along edges and rulings $h_m$ are applied.

In Figure 5-8, a group of hypars along the same load path are loaded with actions along rulings $h_m$. Hypars along the same load path are numbered from the starting point of the load path to the support. Following the load path $LP_m$ in compliance with the coplanarity principle, the action $\vec{F}_h(m)$ along the ruling $h_m$ of hypar $H_0$ can be decomposed into two components as follows: $s_1 \vec{F}_h(m)$ along ruling $h_m$ of hypar $H_1$ is the first component, and $t_1 \vec{F}_i(2n)$ along edge $i_{2n}$ of hypar $H_i$ is the second. ($\vec{F}_i(2n)$ is the action along edge $i_{2n}$ of hypar $H_i$, $s_1$ and $t_1$ are scalars)

$$F_{h(m)}^{0} = s_1 F_{h(m)}^{1} + t_1 F_{i(2n)}^{1} \quad (5-11)$$

Similarly, the action $\vec{F}_h(m)$ along ruling $h_m$ of $H_1$ can be decomposed into a force $s_2 \vec{F}_h(m)$ along $h_m$ of $H_2$ and a deviation force $t_2 \vec{F}_i(2n)$ along the shared edge $i_{2n}$ of $H_i$ and $H_2$. ($\vec{F}_i(2n)$ is the action along edge $i_{2n}$ of the hypar $H_2$)

$$F_{h(m)}^{1} = s_2 F_{h(m)}^{2} + t_2 F_{i(2n)}^{2} \quad (5-12)$$

A general formula to describe the decomposition of the action along ruling $h_m$ of hypar $H_{q-1}$ is as follows: ($\vec{F}_i(2n)$ is the action along edge $i_{2n}$ of the hypar $H_q$)

$$F_{h(m)}^{q(q-1)} = s_q F_{h(m)}^{q} + t_q F_{i(2n)}^{q} \quad (5-13)$$

According to (5-11) and (5-12), the internal forces accumulated along ruling $h_m$ of $H_i$ can be described as including two parts: the action $F_{h(m)}^{1}$ applied directly at $H_i$, and a component $s_1 F_{h(n)}^{1}$ deviating from action $F_{h(n)}^{0}$ applied to $H_i$.

$$\vec{A}_{h(m)}^{1} = (1 + s_1) \vec{F}_{h(m)}^{1}$$
Similarly, the internal forces accumulated along rulings \(b_e\) of \(H_2\) and \(H_3\) can be described as follows:

\[
\tilde{A}_{h(m)}^2 = (1 + s_1 + s_1s_2)\tilde{F}_{h(m)}^2
\]

\[
\tilde{A}_{h(m)}^3 = (1 + s_3 + s_2s_1 + s_3s_2s_1)\tilde{F}_{h(m)}^3
\]

In a general case, the internal forces accumulated along ruling \(b_m\) of hypar \(H_q\) and lying on the same load path can be written as follows:

\[
\tilde{A}_{h(m)}^q = (1 + s_q + s_q^2s_{q-1} + \cdots + s_q^2s_{q-1}s_{q-2} \cdots s_1)\tilde{F}_{h(m)}^q
\]

Which can be shortened to:

\[
\tilde{A}_{h(m)}^q = \left( 1 + \sum_{j=1}^{q} \prod_{i=j}^{q} s_i \right) \tilde{F}_{h(m)}^q
\]  \( (5-14) \)

Figure 5-9: Actions are accumulated along rulings \(b_m\) in a group of hypars along the same load path, and are deviated along the edges intersecting the load path.

5.4.2 Internal forces transmitted along edges

**Forces deviated along edges**

According to (5-14), internal forces accumulated along ruling \(b_m\) of hypar \(H_q\) can be written as follows:

\[
\tilde{A}_{h(m)}^{q-1} = \left( 1 + \sum_{j=1}^{q-1} \prod_{i=j}^{q-1} s_i \right) \tilde{F}_{h(m)}^{q-1}
\]  \( (5-15) \)
Based on formula (5-13), the accumulated internal force $\vec{A}^q_{h(m)}$ in (5-15) can also be decomposed into two components; one along ruling $b_m$ and the other along edge $i_{2n}$ of hypar $H_q$. Substituting (5-13) for $F^q_{h(m)}$ in (5-15), gives the deviation along edge $i_{2n}$ of $H_q$ in (5-16):

$$\vec{D}^q_{i(2n),m} = t_q \left( 1 + \sum_{j=1}^{q-1} \prod_{l=j}^{q-1} s_l \right) \vec{F}^q_{i(2n)}$$

(5-16)

Following (5-15), it is possible to calculate the forces accumulated along the other rulings $b$ that lie on the same load path in this group of hypars. Furthermore, according to (5-16), the edge forces deviating from forces accumulated along other rulings $b$ can be evaluated. Summing up all the deviations from different rulings along the same edge, results in the accumulation of the deviations on each edge intersecting with the load path.

$$\vec{D}^q_{i(2n)} = \sum_{m=1}^{2n-1} \vec{D}^q_{i(2n),m}$$

(5-17)

By adding $\vec{D}^q_{i(2n)}$ in (5-17) with the action force $\vec{F}^q_{i(2n)}$ parallel to edge $i_{2n}$ of hypar $H_q$, all the forces $\vec{A}^q_{i(2n)}$ applied to edge $i_{2n}$ of hypar $H_q$ can be written as follows:

$$\vec{A}^q_{i(2n)} = \vec{D}^q_{i(2n)} + \vec{F}^q_{i(2n)}$$

(5-18)

**Internal forces along edges**

Until now in the calculations, all the forces in a smooth poly-hypar surface can be distinguished into two categories: the internal forces along the rulings of hypar modules (5-14) which are already in equilibrium with external reactions, and the internal forces applied along the edges of hypar modules (5-18) which have not been transmitted to the supports. The latter can be applied separately to a subsystem as an edge-network loaded with edge forces (5-18).

With given supports, such a subsystem is statically indeterminate as there are multiple paths through which loads can be transmitted into the supports. In order to simplify, forces along the edges can be accumulated through a similar path as the forces on other rulings in the same group of hypars (Figure 5-10). By treating edge forces similar to actions along rulings, formula (5-14) can be used to calculate the accumulation of edge forces parallel to the load path. Similarly, formula (5-18) can be used to calculate deviations along other edges intersecting with the load path. In turn, these deviations along other edges can be used as the input of formula (5-14). They are transmitted into supports through other load paths and can cause secondary deviations along the edges. By using formulas (5-14) and (5-18) repeatedly, a loop can be started, which transmits and deviates edge forces until all the deviated forces along edges are directly
Chapter 5. Global equilibrium of smooth poly-hypar surfaces

connected with supports. Such a loop is illustrated in more detail in a diagram in Figure 5-11 of chapter 5.4.3.

Figure 5-10: An edge-network with given supports and loaded with edge forces is statically indeterminate. To simplify, edge forces are assumed to be transmitted along a load path similar to forces on other rulings in the same group of hypars.

5.4.3 Future implementation

The calculations of internal forces in a smooth poly-hypar surface shown in 5.4.3 can be implemented in the future as a Grasshopper function with python scripts in Rhino 6. By inputting the smooth poly-hypar surface as a combination of several individual hypar modules, the function can calculate the actions applied along rulings and edges on each hypar separately. Using the input geometry, the function defines all the load paths, and groups the hypar modules into hypar chains. Next, the function inputs the actions along the rulings of each hypar chain into formulas (5-14) and (5-16), thereby calculating the accumulated forces along the rulings and the deviated forces along the edges. Afterwards, the function sums up these deviated forces and actions along edges and repeats a loop based on formulas (5-14) and (5-16). This loop stops when all the deviated forces are supported. Finally, the accumulated forces along the rulings and edges are given as the output (Figure 5-11).

5.4.4 Other loading cases

In sections 4.4 and 4.5, it is mentioned that in a single hypar under horizontal, or undistributed loading conditions, the internal forces behave as membrane forces and reactions are parallel to rulings and edges. Similarly, when a smooth poly-hypar surface is loaded with horizontal or undistributed loads, the interactions between hypar modules are still limited to rulings and edges. Furthermore, since only forces along rulings and edges affect the global equilibrium of a smooth poly-hypar surface, the different loading cases can only quantitatively change the magnitude of internal forces, and cannot change the global equilibrium of the whole poly-hypar surface nor the position of the supports. Thus, when applied with different loads, smooth poly-hypar surfaces are still in equilibrium without activating any bending moments.
5.4 Calculations of internal forces and reactions

Figure 5-11: Flow chart showing a progression of steps required to calculate forces.
5.5 Summary

This chapter explained the global equilibrium of smooth poly-hypar surfaces. To achieve equilibrium, there are two constraints to follow: the coplanarity principle and the load paths. Once each pair of adjacent hypar modules in a poly-hypar surface satisfies the co-planarity principle, the geometry turns into a smooth poly-hypar surface, which enables force transmissions to always be in plane. To further ensure structural efficiency, all the load paths in a smooth poly-hypar surface should be supported. In this way, an abstract geometrical surface can evolve into a rigid and efficient structural form.
Part III
Design Method
6 Design method of smooth poly-hypar surfaces

This chapter explains a method to design smooth poly-hypar surfaces. In compliance with the coplanarity principle, there are two basic prototypes in all smooth poly-hypar surfaces: the negative, and the positive prototype. Based on these two prototypes, an additive sequence is proposed to guide the design process of smooth poly-hypar surfaces. Following the load paths, potential solutions to support conditions are also explained. Besides the design of smooth poly-hypar surfaces, the methodology presented in this chapter can also be extended to approximate and modify existing freeform surfaces.

6.1 Two basic prototypes

According to the coplanarity principle, in a smooth poly-hypar surface, all the intersecting edges of two adjacent hypars which form the shared plane, are always coplanar (Figure 6-1). Resulting from their double curved property, adjacent hypars are always connected in such a way that one lies above and one below the shared plane (Figure 6-1). This leads to the two simplest prototypes of smooth poly-hypar surfaces: the four-hypar surface and the six-hypar surface. In the first case, two hypars are below the shared plane and the other two above (Figure 6-1a). In the latter case, three hypars are below the shared plane and the other three above (Figure 6-1b).

In a four-hypar surface, the two hypars lying along the diagonals are on the same side of the shared plane and their diagonal profiles are joined positively (positive case), (Figure 6-2a). In the case of a six-hypar surface on the other hand, every set of hypars lying along the diagonals are on different sides of the shared plane and their diagonal profiles are joined negatively (negative case), (Figure 6-2b).

To better control the global geometry in the modelling of smooth poly-hypar surfaces, hypar modules can be joined through vertices instead of edges at first, since a diagonal profile can clearly outline the global shape of a smooth poly-hypar surface (Figure 6-9). As a double curved surface, the profile of the principle curves in one direction of a smooth poly-hypar surface is
dependent on the profile of these curves in the other direction. To start the design of the assembly of a smooth poly-hypar surface, designers can first outline the profile of the principle curves in only one direction by joining the hypar modules by vertices, and later complete the whole surface through an additive operation.

![Figure 6-2: Vertex-joined hypars](image)

Figure 6-2: Vertex-joined hypars: (a) Positive case: the two hypars are on the same side of the shared plane (b) Negative case: the two hypars are on different sides of the shared plane.

Based on the two basic prototypes: by freely adding the four-hypar surface (positive case) and the six-hypar surface (negative case), along the direction of diagonal, designers can extend the diagonal profile of a smooth poly-hypar surface to numerous variations.

In both cases, once the diagonal profile of two hypar modules is fixed, more hypar modules should be added between the two vertex-joined hypar modules to complete the shape. Below, in chapters 6.1.1 and 6.1.2, in compliance with the coplanarity principle, the geometrical constraint to join hypar modules with vertices both in the four-hypar surface and six-hypar surface is explained in detail.

### 6.1.1 Positive prototype: four-hypar surface

In the positive case, two hypar modules are joined with a shared vertex and lie on the same side of the shared plane. The diagonal profiles of two modules are positively joined. To complete the surface, one hypar is filled at each side of the two vertex-joined hypars (Figure 6-3).

If the edges of the hypar $H_1$ are given and represented as vectors, according to the coplanarity principle, the edges of unknown hypar $H_2$ can be represented as products of vectors in hypar $H_1$ and vector scalars ($a, b, c, d, e, p, q$ are vector scalars and $a, d > 0$):

\[
\overrightarrow{CE} = a \overrightarrow{DC} + b \overrightarrow{BC}
\]  \hspace{1cm} (6-1)

\[
\overrightarrow{CG} = c \overrightarrow{DC} + d \overrightarrow{BC}
\]  \hspace{1cm} (6-2)

\[
\overrightarrow{EF} = p \overrightarrow{BC} + q \overrightarrow{DC} + e \overrightarrow{AB}
\]  \hspace{1cm} (6-3)
6.1 Two basic prototypes

Figure 6-3: Four-hypar surface. Hypars $H_1$ and $H_2$ are joined with a vertex, and lie on the same side of the shared plane; the diagonal profiles of the two modules are positively connected. Another two hypar modules are filled in between $H_1$ and $H_2$.

The hypar $H_1$ and unknown hypar $H_3$ satisfy the coplanarity principle, as can be seen in (6-1) and (5-10). This leads to: ($a$ is an unknown vector scalar)

$$BH = a \overrightarrow{AB} + u \overrightarrow{BC}$$  \hspace{1cm} (6-4)

Rearranging (6-1) to write $\overrightarrow{DC}$ as the difference between $\overrightarrow{CE}$ and $\overrightarrow{BC}$, and replacing $\overrightarrow{DC}$ in (6-2) with the rearranged (6-1), gives an equation of $\overrightarrow{CG}$ as a sum of $\overrightarrow{BC}$ and $\overrightarrow{CE}$.

$$\overrightarrow{CG} = \left(d - \frac{cb}{a}\right)\overrightarrow{BC} + \frac{c}{a} \overrightarrow{CE}$$  \hspace{1cm} (6-5)

Because hypars $H_3$ and $H_2$ satisfy the coplanarity principle, according to (6-5), $\overrightarrow{EF}$ can be written as follows ($v$ is an unknown vector scalar):

$$\overrightarrow{EF} = \left(d - \frac{cb}{a}\right)\overrightarrow{HE} + v \overrightarrow{CE}$$  \hspace{1cm} (6-6)

According to the geometry in Figure 6-3, the difference between $\overrightarrow{BC}$ and $\overrightarrow{BH}$ equals the difference between $\overrightarrow{HE}$ and $\overrightarrow{CE}$. Substituting (6-4) for $\overrightarrow{BH}$ gives:

$$BC - BH = -a \overrightarrow{AB} + (1 - u) \overrightarrow{BC}$$  \hspace{1cm} (6-7)

Substituting (6-1) for $\overrightarrow{CE}$, and the rearranged (6-6) for $\overrightarrow{HE}$, the difference between $\overrightarrow{CE}$ and $\overrightarrow{HE}$ can be written as follows:

$$\overrightarrow{HE} - \overrightarrow{CE} = \frac{ae}{ad - bc} \overrightarrow{AB} \left(\frac{ap + abv}{ad - bc} - d\right)BC + \frac{aq + a^2v}{ad - bc} \overrightarrow{DC}$$  \hspace{1cm} (6-8)

Since the difference between $\overrightarrow{BC}$, $\overrightarrow{BH}$, and the difference between $\overrightarrow{HE}$, $\overrightarrow{CE}$ are the same, comparing (6-7) and (6-8) gives ($cb - ia < 0$):
Chapter 6 . Design method of smooth poly-hypar surfaces

\[ u = 1 - \frac{ap - bq}{ad - cb} \]
\[ v = i - \frac{cb + q}{a} \]
\[ e = b - ad \]

Conclusion:

In the case of the positive prototype, the two vertex-joined hypars are on the same side of the shared plane. In this case, if hypar \( H_1 \) is given, the four unknown edges of hypar \( H_2 \) can be represented as the following formulas (\( a, c > 0, bc - ad < 0 \)):

\[ \overrightarrow{CE} = a \overrightarrow{DC} + b \overrightarrow{BC} \]
\[ \overrightarrow{CG} = c \overrightarrow{DC} + d \overrightarrow{BC} \]
\[ \overrightarrow{EF} = p \overrightarrow{BC} + q \overrightarrow{DC} + (cb - ad) \overrightarrow{AB} \] \hspace{1cm} (6-9)

Two unknown edges of hypar \( H_1 \) which fills part of the space between hypars \( H_1 \) and \( H_2 \) can be represented as the following formulas:

\[ \overrightarrow{BH} = a \overrightarrow{AB} + (1 - \frac{ap - bq}{ad - bc}) \overrightarrow{BC} \]
\[ \overrightarrow{HE} = -a \overrightarrow{AB} + \frac{ap + ad - bc + q}{ad - bc} \overrightarrow{BC} + a \overrightarrow{DC} \] \hspace{1cm} (6-10)

Following the formulas (6-9) and (6-10), the edges of the fourth hypar in a four-hypar surface can also be represented with known vectors in hypar \( H_1 \). Furthermore, the parametric representation in (6-9) and (6-10) ensures that each pair of edge-connected hypars in a four-hypar surface satisfies the coplanarity principle.

6.1.2 Negative prototype: six-hypar surface

In the second case, when two hypar modules are on opposite sides of the shared plane, the diagonal profiles of these two modules are negatively connected. Two additional hypars are added between these two hypars at each side, making four added hypars, and a total of six.

Each pair of edge-connected hypars should satisfy the coplanarity principle. Similar to the first case in section 6.1.1, the edges of the given hypar \( H_1 \) can be represented as vectors. Furthermore, the edges of the unknown hypar \( H_2 \) can be represented as products of vectors in hypar \( H_1 \) and vector scalars as follows (\( a, b, c, i, p, q, r \) are vector scalars, \( n, j > 0 \)):

\[ \overrightarrow{CE} = a \overrightarrow{DC} + b \overrightarrow{BC} \] \hspace{1cm} (6-11)
6.1 Two basic prototypes

\[ \overrightarrow{CG} = c \overrightarrow{DC} + d \overrightarrow{BC} \quad (6-12) \]

\[ \overrightarrow{EF} = p \overrightarrow{BC} + q \overrightarrow{DC} + e \overrightarrow{AB} \quad (6-13) \]

Figure 6-4: A six-hypar surface. Hypars \( H_1 \) and \( H_2 \) are at different sides of the shared plane, the diagonal profiles of the two modules are negatively joined and another four hypars are added in between Hypars \( H_1 \) and \( H_2 \).

In the negative case, \( H_1 \) and \( H_2 \) satisfy the coplanarity principle, so vectors \( \overrightarrow{CK} \) and \( \overrightarrow{BH} \) can be written as the formulas below (\( a_1, b, b_2 \) are arbitrary parameters, \( a_1 > 0 \)):

\[ \overrightarrow{DC} = a_1 \overrightarrow{CR} + b_1 \overrightarrow{BC} \quad (6-14) \]

\[ \overrightarrow{AB} = a_1 \overrightarrow{BH} + b_2 \overrightarrow{BC} \quad (6-15) \]

Substituting (6-14) for \( \overrightarrow{DC} \) in (6-11), gives:

\[ \overrightarrow{CE} = (ab_1 + b) \overrightarrow{BC} + a a_1 \overrightarrow{CR} \quad (6-16) \]

Since hypars \( H_1 \) and \( H_2 \) satisfy the coplanarity principle, the following can be derived from (6-16) and (5-10): (\( v \) is an unknown vector scalar)

\[ \overrightarrow{KI} = (ab_1 + b) \overrightarrow{HK} + u \overrightarrow{CR} \quad (6-17) \]

Rearranging (6-11), (6-12) and (6-14), and canceling \( \overrightarrow{DC}, \overrightarrow{BC} \), gives a formula for \( \overrightarrow{CR}, \overrightarrow{GC} \) and \( \overrightarrow{EC} \):

\[ \overrightarrow{CR} = - \frac{b+ab_1}{a_1(bc-ab)} \overrightarrow{GC} - \frac{c(b+ab_1)}{a a_1(bc-ad)} \overrightarrow{EC} \quad (6-18) \]

Since hypars \( H_2 \) and \( H_3 \) satisfy the coplanarity principle, the following can be derived from (6-18): (\( v \) is an unknown vector scalar)

\[ \overrightarrow{EI} = - \frac{b+ab_1}{a_1(bc-ad)} \overrightarrow{FE} - v \overrightarrow{EC} \quad (6-19) \]
According to the geometry shown in Figure 4-1, $\overrightarrow{CI}$ can be written as the sum of $\overrightarrow{CE}$ and $\overrightarrow{EI}$. By substituting (6-11) for $\overrightarrow{CE}$, as well as (6-13) and (6-19) for $\overrightarrow{EI}$, vector $\overrightarrow{CE}$ can be written as the sum of $\overrightarrow{BC}$ and $\overrightarrow{DC}$, thus $\overrightarrow{CI}$ can be written as the sum of $\overrightarrow{BC}$, $\overrightarrow{DC}$ and $\overrightarrow{AB}$ as follows:

$$\overrightarrow{CI} = \overrightarrow{CE} + \overrightarrow{EI}$$

$$\overrightarrow{CI} = \left[\frac{p(b+ab_1)}{a_2(bc-ad)} + vb + b\right] \overrightarrow{BC} + \left[\frac{q(b+ab_1)}{a_1(bc-ad)} + av + a\right] \overrightarrow{DC} + \frac{e(b+ab_1)}{a_1(bc-ad)} \overrightarrow{AB}$$

$\overrightarrow{CI}$ can be also written as the sum of $\overrightarrow{CK}$ and $\overrightarrow{KI}$. By substituting (6-11), (6-14) and (6-18) for $\overrightarrow{CK}$, $\overrightarrow{CI}$ can be written as a sum of $\overrightarrow{BC}$ and $\overrightarrow{DC}$. Substituting $\overrightarrow{KI}$ below with (6-15), (6-17) and (6-18), $\overrightarrow{KI}$ can be written as the sum of $\overrightarrow{BC}$, $\overrightarrow{DC}$, $\overrightarrow{AB}$:

$$\overrightarrow{CI} = \overrightarrow{CK} + \overrightarrow{KI}$$

$$\overrightarrow{CI} = \left[\frac{ab_1+b}{a_1}(a_4 + b_2 - b_1) - \frac{(u+1)b_1}{a_1}\right] \overrightarrow{KI} + \frac{ab_1+b+u+1}{a_1} \overrightarrow{DC} - \frac{(b+ab_1)}{a_1} \overrightarrow{AB}$$

Comparing (6-20) and (6-21), gives:

$$e = b - ad$$

$$v = \frac{ab_1+b}{aa_1} \left(\frac{b_2+a_1}{b_1+1} + \frac{q}{ad-bc}\right) - 1$$

**Conclusion:**

In the positive prototype, the given hypar $H_1$ and the unknown hypar $H_2$ are at different sides of the shared plane and the unknown hypar $H_2$ can be represented with edge vectors of the given hypar $H_1$ as follows ($a, e > 0$, $ad-bc > 0$):

$$\overrightarrow{CE} = a \overrightarrow{DC} + b \overrightarrow{BC}$$

$$\overrightarrow{CG} = c \overrightarrow{DC} + i \overrightarrow{BC}$$

$$\overrightarrow{EF} = p \overrightarrow{BC} + q \overrightarrow{DC} + (ad - bc) \overrightarrow{AB}$$

The unknown point $K$ shared by the hypars $H_1$ and $H_6$ always lies on the shared plane between these hypars. Designers can decide the location of point $K$ in this plane by changing the arbitrary parameters $a_1$ and $b_1$ ($a_1 > 0$). The unknown point $H$ of hypar $H_1$ is partially dependent on point $K$ ($a_1$, $b_1$ are parameters to control the location of point $H$, $a_1 > 0$). The unknown point $I$ of hypar $H_1$ is depending on points $K$ and $H$. Once points $K$, $H$, $I$ are founded, hypars $H_1$ and $H_6$ are fixed. The locations of points $K$, $H$, $I$ can be represented by the vectors below:
6.2 Design of smooth poly-hypar surfaces

\[
BH = \frac{1}{a_1} \frac{AB}{AB} - b_2 \frac{BC}{BC}
\]

\[
CK = -\frac{b + ab_1}{a_1(bc - ad)} \frac{GC}{GC} - \frac{c(b + ab_1)}{aa_1(bc - ad)} \frac{EC}{EC}
\]

\[
EI = \frac{b + ab_1}{a_1(ad - bc)} \frac{FE}{FE} - \left[ \frac{ab_1 + b}{aa_1} \left( \frac{b_2 + a_1}{b_1 + 1} + \frac{q}{ad - bc} \right) - 1 \right] \frac{EC}{EC}
\]  

(6-23)

Similar to the positive case, in the negative case, by starting with a given hypar \( H_1 \), a set of variations of hypar \( H_2 \) can be developed. The difference is that the added hypars \( H_1 \) and \( H_2 \) in the negative case have unlimited options by varying the parameters \( a_1, a_2, b_2 \). Such variations provide a certain amount of freedom in the design of the final forms.

6.2 Design of smooth poly-hypar surfaces

As mentioned in chapter 6.1, designers can construct various surfaces based on the two previously mentioned prototypes. Following a descriptive additive sequence, various smooth poly-hypar surfaces can be assembled from these two prototypes. All the steps in these additive sequences can be expressed as three operations which are implemented in grasshopper as a design tool.

6.2.1 Additive design sequence

To design a smooth poly-hypar surface, designers can follow an aforementioned additive sequence, which can be summarized as two linear steps and one loop (Figure 6-8). The first step in such a sequence is to design the profile of a smooth poly-hypar surface in one direction by joining hypar modules through vertices only. Through the free arrangement of the positive and negative cases along a diagonal direction, the skeleton of a global surface can be constructed. The second step is to complete the surface by adding more hypar modules between the two vertex-joined hypar modules, connecting the edges created by the shared vertex. This is a repeated step until all the hypar modules in a poly-hypar surface are joined by edges. Until this step, a complete surface is created based on the skeleton developed in the first step. To extend the surface, more hypar modules can be attached to the existing surface through edges or vertices, which leads to a set of hypar modules along boundaries only connected through vertices. In this case, more hypar modules can be filled in between the two vertex-joined hypar modules. Such an extending and filling process can be looped iteratively to enable the surface to grow according to design requests.
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Figure 6-5: An additive sequence to construct a smooth poly-hypar surface. (a) Constructing a skeleton of a smooth poly-hypar surface by joining hypar modules through vertices. (b) Completing the surface by filling more hypar modules between vertex-joined hypar modules until all hypar modules along the boundaries are connected through edges. (c) An iterative loop to extend the surface by joining hypar modules through edges and completing the surface until all hypar modules along the boundaries are connected through edges.

For this additive sequence, a distinction should be made in the way to complete the surface between the positive and negative case, in compliance with the two basic prototypes explained in chapter 6.1. If the set of two vertex-joined hypar modules belongs to the positive case, it requires one hypar module to be added at each side of the two vertex-joined hypar modules. If
the set belongs to the negative case, it needs two hypar modules on each side of the two vertex-joined hypar modules. In the following step of constructing the skeleton of a smooth poly-hypar surface, the positive and negative cases can be easily distinguished according to the locations of the two vertex-joined hypars in reference to the shared plane. However, in the extend-complete loop, designers need to judge whether the set of hypars belongs to the positive or the negative case by examining the number of hypars connected at one vertex: if three hypar modules are connected at one vertex, one hypar module is missing to complete a four-hypar surface, indicating it is the positive case. If four hypar modules are connected at one vertex, it means two hypar modules should be added to complete a six-hypar surface, which indicates it belongs to the negative case. The distinction between the positive and negative cases is an important geometrical consideration to enable the surface to grow through an additive sequence.

Figure 6-6: Completing the boundary of a smooth poly-hypar surface can be concluded according to the two basic prototypes: (a) Positive case: three hypar modules intersecting at one vertex, missing one more hypar module. (b) Negative case: four hypar modules intersecting at one vertex, missing another two modules.

6.2.2 Digital implementations and surface variations

Following the additive sequence discussed in chapter 6.2.1, it is possible to construct a smooth poly-hypar surface, or more precisely, a typology of smooth poly-hypar surfaces. As is shown in Figure 6-6, any whole poly-hypar surface can be dissolved into two basic prototypes: the four-hypar surface (positive prototype) and the six-hypar surface (negative prototype). These two prototypes can be parametrized with vector scalars according to the calculations in chapter 6.1. As a combination of these two prototypes, the constructed smooth poly-hypar surfaces can also be parametrized, thereby representing a geometrical typology. By changing the vector scalars in each prototype, different variations can be developed from this typology of smooth poly-hypar surfaces.
According to the two basic prototypes and the additive sequence, there are generally three geometrical operations needed to construct a typology of smooth poly-hypar surfaces, join hypar modules through vertices, fill between vertex-joined hypar modules and join hypar modules through edges. The first two operations are derived from the two basic prototypes in chapter 6.1, and the last one is related to the coplanarity principle in chapter 5.2.1. These three operations are implemented in Grasshopper as geometry generating tools in Rhino 6.0.

Figure 6-7: Variations of the two basic prototypes through the manipulation of parameters in the vertex-joining and filling tools.
6.2 Design of smooth poly-hypar surfaces

**Edge-joining tool**

With one given hypar, the edge-joining tool can generate another hypar connected to the first through a shared edge. Such a tool is in fact a digital representation of the coplanarity principle. According to the formula (5-10), there are three independent parameters $a$, $b$, $c$ which control the shape of generated hypars. Parameter $a$ controls the scale of generated hypars; and parameters $b$ and $c$ manipulate the two unknown vertices of the generated hypars. In the digital implementation, besides these three parameters for form variation, there is another index to identify the shared edge between the original hypar and the generated hypar.

**Figure 6-8**: Variations of the edge-joined hypar modules through the manipulation of parameters in the edge-joining tool.

**Vertex-joining tools**

Vertex-joining tools are developed to join hypar modules through vertices. They are based on the two prototypes mentioned in chapter 6.1 where two vertex joined hypar modules are either placed on the same or different side of the shared plane. Thus, there is a different vertex-joining tool for both the positive and the negative case. By using the positive vertex-joining tool (Figure 6-7a) with a given hypar, another hypar is generated according to formula (6-9) in section 6.1.1. The original and generated hypar both share one vertex. Similarly, the negative vertex-tool is developed based on formulas (6-22) in section 6.1.2. In the digital implementation of both cases, there is one index to identify the shared vertex between the two hypars and another three pairs of parameters: $a$ and $b$, $c$ and $d$, and $p$ and $q$ from formulas (6-9),(6-22), which vary the shape of the generated hypars. Parameters $a$ and $b$ represent the length of the edges connected to the shared vertex; $c$ and $d$ represent the angles between the two edges connecting by the shared vertex; and $p$, $q$ represent the curvature of the generated hypar (Figure 6-7b).

**Filling tools**

Filling tools are utilized to fill additional hypars between the two vertex-joined hypars. They are also related to the two basic prototypes, positive and negative, from chapter 6.1. In the case of the positive filling tool (Figure 6-7a), according to formulas (6-10) in section 6.1.1, there is only one solution for the hypars added between two vertex-connected hypars. For the negative filling tool (Figure 6-7b), following the formula (6-23) in section 6.1.2, there are two hypars filled between each side of the vertex-joined hypars, corresponding to two sets of parameters needed to control the shape of the filled hypars. In each set, three parameters $a_1$, $b_1$, $b_2$ are
included in formula (6-23). This formula defines the two unknown vertices of one filled hypar; the other filled hypar has only one solution, depending on the first filled hypar.

In all the aforementioned grasshopper tools, the inputs are always the individual hypar surfaces, and the outputs are also a group of hypar surfaces in which two adjacent hypars always satisfy the coplanarity principle. With these grasshopper tools, designers can explore different typological variations by following the additive sequence (Figure 6-9).

Figure 6-9: A collection of smooth poly-hypar surfaces designed by manipulating diagonal curves. By joining hypars through vertices positively and negatively, designers can freely define different diagonal curves and thus shape versatile smooth poly-hypar surfaces.
6.2.3 Support variations

As mentioned in section 5.3, to turn a smooth poly-hypar surface into an efficient structure, all the load paths should be supported, or forces will be applied normal to the boundary intersecting the load paths, which can cause local bending moments (Figure 6-10a). This is a common problem in structural engineering work. The general solution to this problem is to increase the thickness of the structure, for which there are mainly two approaches: increasing the thickness through the twisting of the geometry, or by increasing the thickness of the material itself. These two approaches can both be applied to solve the forces normal to the boundary of a smooth polyhypar surface.

In a smooth poly-hypar surface, to complete the load paths and avoid bending moments at the edges, one solution is to follow the coplanarity principle and add more hypars smoothly. In this case, all load paths are directly connected to supports and all internal forces are still kept as membrane forces (Figure 6-10b). This solution increases the structural thickness by extending and twisting the geometry.

To minimize supports, another solution addressing the bending moments at the edges is to increase the material thickness. By following the same language as the ruled geometry, the material thickness can be enveloped by several twisted surfaces which behave like a stiffening frame, resisting forces normal to the boundary.

Designers can choose different solutions to boundary supports according to other design constraints.

![Figure 6-10: (a) Load paths in a poly-hypar surface are not directly supported (b) A solution completing load paths by adding more hypar modules following the coplanarity principle. (c) A solution increasing material thickness; forces normal to the boundary are taken by the thicker material.](image-url)
6.3 Smooth poly-hypar surfaces as approximations

Besides the creation of innovative surface structures, smooth poly-hypar surfaces can also be used to approximate existing double curved surfaces. Following a similar additive sequence, the approximations can start from the skeleton of the surface, approximating the diagonal profile, and then progressively add more hypars to complete it. After the geometrical approximation, the global equilibrium of the constructed surface can be intuitively checked through load paths. This way the defects of the original surface can be improved by modifying the boundary conditions.

6.3.1 Approximations of double curved surfaces

Minimal surfaces are a category of double curved surfaces which is very similar to the hyperbolic paraboloid (Piker, 2007). It is a surface that locally minimizes its area (Wikipedia, 2019). Stretched fabric and soap film are physical examples of minimal surfaces. Geometrically, the primary difference between minimal surfaces and hypars can be considered as the difference between their principle curves: the principle curves of hypars are parabolas, while the principle curves of minimal surfaces are catenaries. However, from a visual point of view, this difference is so small that hypars are frequently misunderstood as minimal surfaces (Piker, 2007). This confusion between minimal surfaces and hypars reflects their geometrical similarity and indicates the potential to approximate minimal surfaces with hypars. As mentioned in chapter 2.2.2, minimal surfaces were frequently used in form-finding through the use of physical models since the middle of the 20th century in the works of Musmeci and Frei Otto, among others. In chapter 6.3.2, Musmeci’s Basento bridge (Figure 6-12) is taken as an example to show how smooth poly-hypar surfaces can approximate double curved freeform surfaces.

Although form-finding result in geometries similar to smooth poly-hypar surface, the principles to initiate the design and involve construction and structures are different. This way, benefiting from the double ruled property and the structural stiffness of hypars, the construction of the approximated minimal surface can be simplified; and its structural behavior tested and improved.

Besides minimal surfaces, other double curved freeform surfaces can also be approximated with smooth poly-hypar surfaces. As discussed in chapter 2.4.2, currently approximations of double curved surfaces are made with planar surfaces or ruled stripes, which cannot lead to smooth approximations. When using smooth poly-hypar surfaces though, the resulting approximations are always smooth. Nonetheless, the difference between the original surfaces and the approximation still exists. If this difference doesn’t affect the architectural expression of the original geometry, the approximated geometry generated following the principle of smooth poly-hypar surfaces certainly leads to more advantages in structural efficiency and construction convenience.
6.3 Smooth poly-hypar surfaces as approximations

(a) Constructing the skeleton by joining hypar modules through vertices.

(b) Adding more hypar modules to complete the surface and approximate the boundary of a unit of the Basento bridge.

(c) Checking the load paths and identifying the locations of supports.

(d) Adding hypar modules to support all load paths, and modifying the support conditions and boundaries of the constructed smooth poly-hypar surface.

Figure 6-11: Approximating a unit of the Basento bridge with a smooth poly-hypar surface.
6.3.2 Case study

As mentioned in chapter 2.2.2, the Basento bridge was designed using a stretch fabric model. The resulting geometry was a double curved minimal surface, very close to a smooth poly-hypar surfaces. In the process to approximate original geometry with smooth poly-hypar, the design of a unite of the bridge is started by joining multiple hypar modules with vertices using the vertex-joining tools, thereby outlining the skeleton of the original surface (Figure 6-11a). During this stage, the principle curves of the original surface are the primary guide in the approximation of the skeleton. More hypars are added step by step using the filling tools according the boundaries of the original surface (Figure 6-11b). Finally, the load paths are checked after the geometrical approximation is finished (Figure 6-11c). The locations of the supports are decided according to the load paths in the constructed geometry. Unlike the stretched fabric model of the bridge, for the poly-hypar approximation there are two necessary locations for supports: one at the bottom and another at the top between two bridge units (Figure 6-11d). This difference results from the distribution of internal forces. The tension-only fabric model of the Basento bridge is reversed into a compression-only shell, while a similar geometry constructed as a smooth poly-hypar surface can take both compression and tension. Therefore, the approximation leads to different support conditions.

![Figure 6-12: The stretched fabric model for a unit of Musmeci's Basento bridge. (b) The profile of half of a unit.](image)

6.4 Summary

The methodology to design smooth poly-hypar surfaces presented in this chapter can be summarized into an additive sequence based on two basic prototypes: the four-hypar surface and the six-hypar surface. Implemented with digital tools, this methodology is helpful to design a variety of smooth poly-hypar surfaces. Moreover, the methodology to construct smooth poly-hypar surfaces can also be utilized to approximate other freeform surfaces, such as minimal surfaces, thereby optimizing their geometries and structural behaviors.
Part III
Design Method
7 Applications

This chapter presents some applications of smooth polyhypar surfaces. It first shows the use of such innovative surfaces in architectural teaching, later it explains its applications in shell designs, and shows how they are implemented using different materials and be fabricated using low-tech construction methods.

7.1 Architectural teaching

This section introduces two examples of polyhypar surfaces designed by architectural students. It suggests that this innovative approach is of value to interest architects in their education of structures, and shows the potential to apply it widely as a double curved shell design method.

7.1.1 Innovative stair design

This project was designed as a part of an elective course (wahlfacharbeit) for master students during the Autumn semester in 2017 at ETH Zurich: the design of a staircase for the Chair of Architecture and Building Process (CABP). The office of CABP is located close to the open space of the new ITA building (Institute of Technology in Architecture, ETH Zurich). To more easily access the lower floor of the open space (Figure 7-1a), CABP intended to build an extra stair connecting the corridor in front of their office on the mezzanine floor with the open space below.

![Figure 7-1: (a) The site for the new stair design: the open space in front of the Chair of Architecture and Building Process. (b) The initial design concept for the stair: an extension of the open space, working both as a platform and a cover for social activities.](image)

Two master students (Ruizhe Liang, Weilan Jiang) formed a design group. Their initial idea was to design a staircase as a spatial extension of the open space, not only to connect between different levels, but also as an open platform for social activities (Figure 7-1b). In their descriptions, they designed the staircase as a double-sided nest which envelopes the spaces above and below the staircase. The semi-open space can emotionally provide visitors a
motivation for staying, sitting or chatting there. The double curved property of smooth poly-hypar surfaces matches their initial design concept well, thus they chose it as the general geometrical and structural guide for their design.

They started the design with an unguided, free combination of hypars, which lead to a disordered appearance for resulting geometry (Figure 7-2a,b). Later, they introduced diagonal symmetry into their design, shaping the profile of the surface through the principle diagonal curve, and regulated the added hypar modules by adhering to a rectangular grid in plan (Figure 7-2c). By exploring different diagonal curves and matching the geometry with the site, they finally found their solution which preserved their initial design idea to use one surface to shape a social space above and below the surface.

In terms of structural efficiencies, they followed the load paths to support the surface. In their final design, they ensured every load path was connected to the two floors (Figure 7-3b). As such, the design ensured the stair is in global equilibrium.

In their work, they also considered the materialization of the stair. They referred to several projects fabricated using wood or bamboo, which create semi-transparent surfaces with linear
elements. Following this idea, the stair is also implemented as a wooden grid-shell with a wooden mesh covering (Figure 7-3c).

![Figure 7-3c: Final model 1:20. (a) top view (b) view for the space below the stair. (Photos by Ruizhe Liang, Weilan Jiang)](image)

### 7.1.2 Canopy in Zurich Central

This canopy is a design project in the master elective course ‘architektur und tragwerk’ (architecture and structure). In this course, students chose a site in the center of Zurich, and designed an urban scale structure to activate the urban space. The group of students (Rickli Laura, De Aguiar Parreira Luiza, Lengerer Sina and Strub Larissa) chose the site close to the public transport hub ‘Central’ in the city center of Zurich. According to their descriptions, the intersection of various tram lines, streets and pedestrian paths at Central made this cross into a confused and chaotic knot (Figure 7-5a). Therefore, they decided to design a large scale walk-on canopy to cover the bus and tram stops, creating a new path from the main station to Polyterasse.

![Figure 7-5: (a) The site in Zurich Central with a variety of overlapping transportation paths. (b) A section of the surface found through form-finding. The original idea is to create a walk-on canopy to connect Polyterasse with the main station. (Photos by Rickli Laura, De Aguiar Parreira Luiza, Lengerer Sina and Strub Larissa)](image)
This group started by using the form-finding method to shape the canopy (Figure 7-5 b). However, the supports of the resulting single-curved surface occupied a large area of the site. For this reason, they had to search for another approach to design the surface structure. Compared to single curved surfaces, double curved surfaces are normally stiffer and require less supports. Thus, the group intended to turn their original design into a double curved surface and a smooth polyhypar surface became their preferred solution.

In their design of the walk-on canopy using a smooth polyhypar surface, the team still designed one section of the surface using form-finding (Figure 7-5b): the section through Polyterasse, Central and Hauptbahnhof, which reflects their original concept of unifying the chaotic transportation knot. In the first step, they tried to approximate the original section with a group of vertex-joined hypar modules, thereby assembling the skeleton of the surface. Once the skeleton was defined, more hypar modules were added to complete the geometry. Benefiting from the stiffness of hypars, the resulting surface became a three-support shell.

In the material implementation stage, they turned the continuous surface into a grid-shell which allows for more light to penetrate to the space underneath (Figure 7-6a). The straight rulings in each hypar module were materialized as steel components and stiffened with polycarbonate plates as diagonals. The smooth polyhypar shell is a canopy for the space below and also lights up during the night. An extra path was placed on top of - and was supported by the shell (Figure 7-6b). It directly connects the pedestrian path to Polyterasse to Hauptbahnhof.

### 7.2 Application in shell designs

This subchapter presents two shells designed using smooth polyhypar surfaces, as well as their implementation using different materials. Benefiting from the double ruled property of each
hypar module, all the smooth polyhypar shells were able to be built using low tech construction methods.

7.2.1 Lightweight aluminium grid shell: the Hypar Pavilion

Background information and design conception

The design and fabrication of the Hypar Pavilion was embedded in a construction workshop between ETH Zurich and Southeast University, Nanjing, China. The main content of this workshop was to design a prototype of a smooth polyhypar surface, thereby exploring potential low-tech construction methods of such a surface structure. Fourteen students were involved in the fabrication workshop.

From an architectural point of views, the design of freeform surfaces relates to the concept of fluid space and the dissolution of discreet building elements into a structural and spatial continuum. The architectural concept developed for the Hypar Pavilion takes full advantage of such fluid properties in order to design a porous structure, in which the conventional walls and ceilings are merged into a continuous surface that in turn generates multiple relationships between interior and exterior spaces. From a structural standpoint, the design of the Hypar Pavilion provides a topological system that integrates structural efficiency and spatial qualities. In terms of its geometry, the pavilion was generated through the combination of forty hypar modules. The scale of the pavilion was conceived so as to enhance the circulation of people through it, all the while allowing visual interaction with the surrounding context (Figure 7-8).

![Figure 7-7: Two different types of subsystem within the Hypar Pavilion: (a) hyperbolic vault, (b) cantilever. The whole pavilion is combined from two hyperbolic vaults and one cantilever.](image)

The smooth polyhypar surface of the Hypar Pavilion incorporates two basic element: (a) the hyperbolic void, and (b) the cantilever (Figure 7-7). Each hyperbolic void is formed by nine hypar modules; two of which, one below and one above, are further combined into a cluster. This cluster constitutes half the Hypar Pavilion (Figure 7-8a). The other half of the pavilion, formed by ten hypars, was developed as an independent cluster and acts similar to a cantilever (Figure 7-8b). Finally, an additional vaulted surface was produced by joining the two previously mentioned clusters, thereby enhancing the way people walk through it (Figure 7-8c). The last step in the design process was to check the load paths. To ensure all load paths are connected to the ground, another twelve hypars were added, this addition on the one hand ensures that there are exclusively membrane forces in the shell, while on the other hand it completes the
smooth surface, produces more spatial differentiation and adds more potential uses (Figure 7-8b).

![Figure 7-8: (a) The surface of the Hypar Pavilion was combined from two clusters of hypar modules. (a) one cluster was combined from two hyperbolic vaults. (b) Another cluster acts as a cantilever. (c) All the load paths are supported by the ground.](image)

**Materialization**

The Hypar Pavilion locates on the campus of Southeast University. To reduce the impact of a temporary structure on the campus, light-weight aluminium rods and cables were chosen as the
fabrication material instead of heavy reinforced concrete, as this minimizes the weight of the structure. Additionally, the softness of the material juxtaposes the stiffness of the geometry itself. (Figure 7-9).

Synthetically, taking into account the construction difficulties and material costs, each hypar in the pavilion is discretized as ten rulings in both directions, as well as ten upward, and ten downward parabolas. The straight rulings have been materialized as aluminium rods, and the parabolas as stainless steel pre-stressed cables. As a result, the smooth polyhypar surface of the pavilion has been materialized in the form of a grid-shell (Figure 7-9).

![Figure 7-9: The Hypar Pavilion has been designed as a grid shell to reduce the weight of the structure](image)

**Structural conception**

Based on the discretization and materialization above, the internal forces within the Hypar Pavilion have been analyzed and controlled using graphic statics. Just below, a typical hypar of the pavilion is analyzed in order to describe the structural behavior of each unit when regarded as a sub-system in equilibrium. Thus, the global equilibrium of the whole pavilion is generally described.

Each of the hypar modules constituting the Hypar Pavilion can be regarded as a pre-stressed grid shell with rulings as struts and parabolas as cables. However, unlike the tension cables and compression arches in a continuous shell, (as described in section 4.2), all the parabolas in a grid shell work as pre-stressed cables. In Figure 7-10, the load distributions between hypars represented as (a) continuous shell with self-weight, (b) a grid shell with only pre-stressing forces and (c) a grid shell with both pre-stressing forces and self-weight are compared.

Figure 7-11 shows the case of a hypar loaded only with pre-stressing forces $p$ and $-p$. The magnitude of the internal forces due to the pre-stressing can be evaluated using the vector-based 3D force diagram in Figure 7-11b. If a self-stressed hypar is also loaded with self-weight, which is the case corresponding to the grid shell made with aluminium rods and steel cables present in the Hypar Pavilion, pre-stressing forces vary among the different cables, depending on the self-weight and the geometry of the hypar.
Chapter 7. Applications

Figure 7-10: Load distributions at one node. (a) one node in a continuous shell. A parabolic arch and cable each take half of the loads. (b) one node in a self-stressed hypar without loads, two parabolas are pre-stressed and both in tension; the sum of the two pre-stressing forces $p$ and $-p$ equals zero. (c) one node in a pre-stressed hypar with self-weight. Downward curving cables are pre-stressed with $p$; upward curving cables are both pre-stressed and loaded with shelf-weight.

Figure 7-11: A self-stressed grid hypar (a) form diagram (b) force diagram to calculate internal forces and reactions
7.2 Application in shell designs

Figure 7-12: (a) Each point load $g$ applied to a hypar module is decomposed into three components along its rulings and edges. (b) Subsystem I is only loaded with ruling components, which are balanced by reactions along rulings.

Figure 7-13: Subsystem II has only axis components applied to the hypar module. (a) form diagram (b) force diagram.
Similar to the strut and tie model in section 4.2, for a hypar module loaded with pre-stressing forces and self-weight, the continuous self-weight can be discretized as 25 point loads. Each point load is divided into three components: \( f_h \) and \( f_i \) along the rulings, and \( f_r \) parallel to the axis \( r \) of the hypar (Figure 7-12). Applying ruling components \( f_{h(m,k)} \), \( f_{i(m,k)} \), and axis components \( f_{r(m,k)} \) separately to the hypar, the whole hypar can be divided into two subsystems: subsystem I loaded with axis components \( f_{r(m,k)} \), and subsystem II loaded only with ruling components \( f_{h(m,k)} \) and \( f_{i(m,k)} \).

After overlapping subsystems I and II, the following conclusion can be drawn: to achieve equilibrium under distributed loads, one hypar acts as a grid shell, meaning that the reactions are always parallel to its rulings and edges.

As mentioned in chapter 5.4, considering each hypar as a subsystem each in equilibrium, the reactions at the border of one hypar in a polyhypar surface are transferred as actions to the adjacent hypar. As such, in the global equilibrium of the Hypar Pavilion, the forces along its rulings are always transmitted along one load path until they reach the relevant supports (Figure 7-14).

![Figure 7-14: A part of the Hypar Pavilion. The internal forces are always transmitted along rulings and edges.](image)

**Reconstruction of the geometry**

Taking into account the calculated internal forces, 8-mm-radius aluminium rods with a hollow section and 1-mm-thick stainless-steel cables were chosen as the construction materials. Due to the radius of the aluminium rods, the shared edges between two adjacent hypars in the digital model were built as two parallel rods during construction. To take into account the material thickness in the final assembly, every hypar was scaled uniformly to a smaller size. In order to
fulfill the coplanarity principle, the scaling was obtained by offsetting the vertex of each hypar in the shared plane defined by the intersecting edges of the adjacent hypars.

**Details designs**

The detail design for the Hypar Pavilion was comprised of two main steps: the first step consisted of the fabrication of a single hypar, and the second step of the combinations of two adjacent hypars.

In the case of a generic hypar, the rulings always intersect the edges at different angles, which means the joints between rulings and edges should be able to rotate 360 degrees. To be able to adapt to the different angles between rulings and edges, all joints were standardized as a combination of three hinges: one hinge rotates around the axis of the edge, while the another hinge rotates in a plane defined by an edge and a ruling, and the last hinge rotates around the axis of a ruling (Figure 7-15a)

![Figure 7-15](image)

Figure 7-15: (a) The connections to join ruling rods and edge rods: R-shaped connection and a rotating head, including two hinges and one roller. This connection can be adopted to different angles between rulings and edges. (b) Because of the R-shaped connection, two sets of rulings are shifted one above and one below, without interrupting each other.

![Figure 7-16](image)

Figure 7-16: (a) R-shaped connection. (b) Rotating head. (c) A ruling rod and edge rod are connected through an R-shaped connection and a rotating head.

The R-shaped connection (Figure 7-16a) is a hinge which can rotate around the axis of an edge, and can be manually manufactured according to three steps: cutting, punching and folding. A
small tool has been designed to fold a straight aluminium strip into an R-shaped connection in the final folding step. Benefiting from this R-shape, two sets of ruling rods can now be shifted up and down (Figure 7-15b) without obstructing each other during assembly. The other two hinges are both included in one connection: the rotating head (Figure 7-16b). This head is pressed from a short, hollow 8mm radius aluminium rod, which makes use of the soft and extensible material properties of aluminium. The pressed part is punched and joined to the R-shape connection with a pin, which thereby becomes a hinge rotating in a plane defined by an edge and a ruling. The last hinge is created by inserting a short 6mm radius aluminium rod into the 8mm radius rod.

![Figure 7-17: (a) Details on how to connect cables with aluminium rods. (b) Round plates joining several hypars at their vertices.](image)

The detail for fixing cables is also standardized (Figure 7-16c). Rings are fixed above and below the intersection points of two rulings with plastic cable ties and stainless steel cables passing through these rings are tensioned manually. Basically, all connections in the fabrication of a single hypar are standardized. The only variable is the length of the edge and ruling rods.

The other nonstandard connection is at the shared edges where adjacent hypar modules touch. Due to the coplanarity principle, intersecting edges of all adjacent hypar modules are always in a same plane. Such a geometrical constraint lead to an advantage in the design of the detail. The joining of several rods at the same node in this case is solved by a planar connection: two layers of round plates with holes punched in different places depending on the hypars (Figure 7-16c). The two plates are connected to the rods by pins as well.

To sum up, all the details designed for the Hypar Pavilion are solved using mechanical joints.

**Fabrication of individual hypars and the assembly of the hypar pavilion**

The fabrication and assembly involved a team of sixteen students. All connections and aluminium rods were prepared in advance.

**Prefabrication and numbering:** The prefabrication included standard details and non-standard rulings and edge rods. The standard connections were serially produced, while the non-standard rods with different lengths were labelled with their individual indexes. The full index of an edge node included three parts: the place of the hypar in the construction sequence, the location of
the edge in the hypar and the index of the two hypars which are joined by this shared edge. Similarly, the full index of a ruling node also included information such as the number of the parent hypar, the identification of the ruling’s group, and the ruling’s index within that group. All prefabricated components were organized into groups according to the hypar they belonged to.

**Fabrication of individual hypars:** Following a fabrication manual for each hypar module, students collected all the prefabricated components, marked the locations of the rulings on the four edges, and connected all four with screws. To fix four edges in position, students simply tied two timber sticks as diagonals (Figure 7-18); then rulings were fixed on the marked edges with R-shaped connections and screws. Cable rings were fixed with cable ties at the points of intersection between two ruling rods. The timber diagonals were kept in position until the fabrication of each hypar was completed. Cables were not placed yet in this step in order to count on certain flexibility for each hypar, thereby allowing the required tolerance for the final assembly.

![Figure 7-18: The manual fabrication process of one of the hypar modules of the Hypar Pavilion.](image)

**Assembly of the pavilion:** Exclusively students were involved in the assembly of the Hypar Pavilion. As described before, the whole pavilion can be divided into two self-supporting clusters, each of which is assembled separately before being joined together at the end (Figure 7-19). Adjacent hypars were joined at their shared vertex with two round plates, and their parallel and touching edges were tied together with plastic cable ties. After fixing the lower hypars at the bottom of the cluster to the ground with forks, they were already self-supporting due to the rigidity of their double curved geometry, even during the fabrication process. Furthermore, apart from some ladders to place the hypar modules located in the higher parts of the clusters, no scaffolding was required, throughout the whole process. Finally, after all the
hypar modules had been correctly placed in their respective positions, the cables were incorporated and pre-stressed.

Figure 7-19: The Hypar Pavilion’s process of assembly in progress.
7.2 Application in shell designs

Figure 7.20: The completed Hypar Pavilion

Figure 7.21: Top view of the Hypar Pavilion
Chapter 7 . Applications

Result

The whole pavilion was approximately 10 meters long, 6 meters wide, and the highest point measured about 3.4 meters. The size of each hypar varied from 1.2m x 1.2m to 1.5 m x 1.5 m. Benefiting from the use of light-weight materials, the weight of each hypar was less than 1.5kg, and the weight of the entire structure was around 60kg. The Hypar Pavilion was exhibited for one month on the campus of Southeast University in Nanjing (Figure 7-20 to Figure 7-22). Afterwards, it was disassembled and subsequently moved 700 kilometers away to the city of Dezhou, to be reassembled for a second exhibition, proving the success of this temporary and reusable fabrication concept.
7.2 Application in shell designs

7.2.2 Ferrocement shell: Cantilever

Project background

As described in chapter 4.2, a smooth polyhypar surface, can also be implemented as a pre-stressed grid shell as well as a continuous concrete shell, in which the principle parabolas take both tension and compression. Benefiting from the double ruled property of hypars, smooth polyhypar surfaces can be built using straight formwork similar to the method used to build the concrete shells of Candela. However, once this particular geometry is integrated with the technique called ferrocement, it is possible to develop a new fabrication method for concrete shells which does not require formwork. This was tested on the fabrication of a small piece of cantilevered concrete shell in the fabrication workshop of Southeast University, Nanjing, in June 2018.

Fabrication and design concept

Ferrocement is a technique where concrete is applied to a metal mesh layer made of woven expanded-metal or metal-fibers and closely spaced thin steel rebar (Wikipedia, 2018). This technique was widely applied in the fabrication of concrete shell designs by the Italian engineer Pier Luigi Nervi. In the case of a smooth polyhypar surface, by materializing a hypar’s straight rulings as rebars, it is possible to create a rebar mesh with enough density for the casting of concrete. However, in terms of structural stiffness, the density of rebar achieved using this method is not as high as usual for concrete casting. For the test of this cantilevered shell, the solution was to make a rebar skeleton first, before subsequently covering it with wire meshes. In this case, a high enough mesh density was achieved for concrete casting, at the cost of increasing the amount of rebar required in terms of structural stiffness.

The fabrication of the shell using ferrocement includes roughly four steps: First, weld the rebar to shape the skeleton of each hypar, then assemble all hypars into a continuous cantilevered surface. The third step is to cover the rebar skeleton with a thin wire mesh. The last step is to cast concrete on the mesh. The whole construction process should be free of formworks and scaffoldings.

\[\text{(a) The top view of the cantilevered shell (b) An elevation (c) An axonometric view}\]
Due to weight limitations, this shell is designed as a combination of six hypar modules, each with edges of around one meter. The height of the whole shell is less than 2 meters; the thickness of the shell is 3 centimeters and the total weight is around 600kg.

**Construction process**

The fabrication of the shell started with the hypar modules made of rebars. Similar to the fabrication of the aluminium grid hypars with four precut edges, the shape of each hypar can be defined by fixing the lengths of its two diagonals. The four vertices of each hypar were welded once the two diagonals were fixed. The hypars’ rulings were also welded to the edges. To stiffen the rebar mesh, some rulings were welded together at their intersection. These rebar modules could be prefabricated in the factory before being transported to the construction site by stacking them (Figure 7-24).

![Figure 7-24: (a) Six rebar modules. (b) Rebar modules can be stacked during transportation. (b) The wire meshes were cut into strips to match the double curved surfaces](image)

The next step was to assemble the rebar modules into a continuous skeleton of the shell. Benefiting from their double curved geometry, after tying three hypars together with metal wires, the assembled surface was already a stable structure (Figure 7-25a). In this case, the assembly process of the skeleton did not need any scaffolding. The whole assembly process started from the bottom and finished with the last hypar module on top. After having assembled six of the hypars into the skeleton, some points were welded along the shared edges between hypars, to stiffen the whole structure (Figure 7-25a). After completing the assembly of the rebar modules, the rebar skeleton was covered with two wire mesh layers (Figure 7-25b), one above, the other below. As undevelopable surfaces, hypars cannot be unfolded into a flat surface (Pottmann, 2007). As such, the only solution to cover a double curved surface with flat mesh was to approximate the surface with mesh strips (Figure 7-24c). As part of the fabrication process, the wire meshes were cut into long strips to match the surface along the direction of the parabolas and fixed to the rebar by wires.

The last step was to cast concrete. In this case, fast setting concrete was used - a mixture of sand, cement, water and some fibers to prevent the concrete from falling from the meshes. The casting of the concrete also started from the bottom to ensure that the whole structure maintained its stability during the whole casting process, without the need of additional supports (Figure 7-25c).
7.2 Application in shell designs

Figure 7-25: (a) The assembly of rebar modules. (b) Covering the rebar skeleton with a thin metal mesh. (c) Casting concrete on the mesh.
Further improvement

According the experience gained from this test, there is one main drawback in the current fabrication method: matching the flat mesh strips to a double curved surface. This was the most labor intensive stage of the whole fabrication process. To improve this process in the future, one solution could be to replace the wire meshes with closely spaced thin rebars, thereby finding a balance between the amount of rebar required for structural stiffness and the fabrication request. The other solution could be to replace the wire meshes with flexible membranes. Moreover, the concrete mixture can be improved by increasing the amount and the size of the fibers, in order to further prevent the concrete from falling from the rebar meshes.

To sum up, for future fabrication, concrete can be applied directly to a rebar mesh. In this case, the dimensions of, and the space between the rebars, as well as the concrete mixture should be studied and calculated more precisely.
Part V

Conclusions
8 Conclusions

8.1 Overview

The emergence of freeform surfaces was derived from the continuity theory in the 1990s, which mainly focuses on three levels of continuity in architectural design: generative continuity, form smoothness, and interdisciplinary integration (Einsenman, 1993), (Lynn, 1999), (Carpo, 2014). However, when deviating from the original concept of continuity, the main problem in freeform surface design becomes the separation between architectural forms and their technical performance, such as structural efficiency or ease of construction. Once the new geometrical category proposed in this dissertation—the smooth polyhypar surface—is applied in the design of freeform architecture, three levels of continuousness proposed by the continuity theory can be achieved:

- Generative continuity is realized using parametric representations of smooth polyhypar surfaces. The variables of the surface are defined according to the coplanarity principle, enabling the parametrization of different surface typologies.

- Form smoothness is achieved through a fluent combination of hypar modules. With the coplanarity principle as the geometrical constraint, the generated surfaces generally satisfy the second order of smoothness (chapter 5.2).

- Interdisciplinary integration is obtained as a result of the special geometrical properties of smooth polyhypar surfaces. According to the coplanarity principle and following its load paths, the generated surface enables the unity of structural and geometrical forms. As a locally ruled surface and a modular system, the geometry of smooth polyhypar surfaces is suited for various possibilities in low-tech prefabrication.

Based on the advantages offered by the three levels of continuity described above, the smooth polyhypar surface is set to become one of the new architectural forms of the digital age, as proposed by the continuity theory. Within the research field of freeform architecture and surface structures, the smooth polyhypar surfaces presented in this dissertation have the following primary characteristics:

- Structure: as an extension of current research on hypar structures, this dissertation provides both intuitive visualizations and precise calculations of the static behavior; as well as a new solution to the global equilibrium of polyhypar surfaces, by smoothly joining hypars, and simultaneously avoiding bending moments.
• Architecture: a new geometrical category—smooth polyhypar surfaces—has been defined, which is based on a development of an innovative design method for new freeform surfaces, as well as on an approximation of the already existing ones.

• Construction: the economic low-tech construction methods for surface structures, free of scaffolding or formwork

These characteristics are explained in more detail in chapter 8.2. Chapter 8.3 discusses the limitations of the presented smooth polyhypar surfaces, and suggests a range of directions for future development. At the end a final conclusion is given.

8.2 Contributions

This dissertation contributes to three closely related fields: surface structures, freeform architecture, and freeform construction. It extends the research on hyperbolic paraboloid shells in the historical context by solving past problems; and defines a new geometrical category derived from hypar geometry. Based on the new geometry—smooth polyhypar surfaces—the study develops an effective design method for freeform surfaces, which ensures structural strength as well as reasonable construction possibilities. It also proposes a new direction in approximating existing freeform surfaces using smooth polyhypar surfaces.

Extension of previous research on hyperbolic paraboloids

The two remaining problems from the previous research on hyperbolic paraboloids are the intuitive representation of force flows (Billington, 1965) and support conditions (Billington, 1965), (Schnobrich, 1971). This research provides possible solutions for the above-mentioned problems through the unique properties of smooth polyhypar surfaces.

As already mentioned in the chapter 2.3.1, in the past, most structural analysis generally shared a similar problem: it is too complex to be involved in the decisive stages of design (Billington, 1965). Since the calculation of internal forces only acted as a validation for the finalized design, it was difficult for it to be used to comprehend the generation of forms. Although this problem has already been addressed in some research (Parme, 1956), (Parme, 1958), the over-simplified visualization of structural behavior used in this research was unable to represent all the hypar types (Billington, 1965). The solution depicted in this research is based on graphic statics—it finds a balancing point between complex calculations and intuitive representation. By discretizing a hypar as a pin-joined strut and tie model, the distribution of internal forces and reactions can be intuitively visualized with the help of a form diagram in a minimal discrete hypar, thereby providing designers with real time feedback between the hypar's form and its forces. The precise calculation can also be completed using an infinitely discrete strut and tie model by using equilibrium functions at each node of the model.

The support conditions of hyperbolic paraboloids also remained unsolved in both the membrane theory as well as the bending theory approaches. The solution proposed in the membrane theory—to stiffen beams at the edges—proved to be counter-productive in the
bending theory, as it provided insufficient horizontal reactions and added extra load to the structure (Billington, 1965). These problems were partially solved in practice according to the engineers’ experiences. For example, in the work of Candela, hypars were joined symmetrically through folds to cancel out the horizontal reactions. Inspired by these early explorations in the field of folded polyhypar surfaces, this research presents a new solution that reinforces the conditions of each hypar module within the context of a polyhypar surface: each hypar module supports its adjacent modules without activating any bending moments. Two conditions are to be met in order to ensure bending-free behavior; the coplanarity principle and fully supported load paths. The coplanarity principle ensures that all the intersecting rulings and edges are coplanar, which consequently leads to geometrically smooth polyhypar surfaces, whilst structurally ensuring that forces along the rulings and edges are always transmitted in plane. Load paths in the second condition intuitively marks the location of supports, guaranteeing global equilibrium for the whole surface. In this case, smooth polyhypar surfaces avoid the use of rigid beams, which normally cause the behavior of hypars to deviate from how they are predicted to behave based on the analysis in the membrane theory. When compared to the cases with rigid beams, the solution presented in this research has a far more design-oriented approach, solving bending moments through the twisted geometrical forms themselves instead of through material thickness.

A new method for freeform surface design

From an architectural perspective, the new sub-category of freeform surfaces presented in this research—smooth polyhypar surfaces—points towards the possibility of an innovative design method for structurally efficient freeform surfaces. Just as freeform surfaces are controlled using splines, smooth polyhypar surfaces can be controlled and manipulated through the use of diagonal profiles. Following an additive sequence, the diagonal profile of a smooth polyhypar surface can be constructed first as a skeleton; then by adding more hypar modules, its geometrical shape can be completed or even extended. Skeletons with different diagonal profiles can be developed into various surface typologies. In compliance with the coplanarity principle, each surface typology can be parametrized to represent a group of variations, which is the ‘objectile’ in the continuity theory. Having satisfy the coplanarity principle, a surface typology becomes a dynamic form, which underlies the continuous morphing process. In this way, smooth polyhypar surfaces achieve the generative continuity of form, which is considered to be one of the fundamental properties of new architectural design in the digital time.

As a result of this new methodology being based on smooth polyhypar surfaces, the role of shell structures in architectural design can be brought to a whole new level. In addition to large span structures like roofs or bridges, smooth polyhypar surfaces can also be used to design structures with more diverse spatial qualities, such as museums, theaters, and so on. Like in the hypar pavilion presented in the chapter 7.2.1, traditional architectural components like walls and ceilings all merge into one continuous plane enveloping an architectural space with smooth polyhypar surfaces, separating the interior from the exterior, while still keeping the fluidity between different spaces.
The versatility of smooth polyhypar surfaces in architectural design is also reflected in their ability to approximate other freeform surfaces. When following a process similar to the design of smooth polyhypar surfaces, double curved freeform surfaces like the Basento Bridge by Musmeci or the National Taichung Theater by Toyo Ito, can be approximated with smooth polyhypar surfaces. Such approximations can provide a relatively simple geometrical description of complex freeform surfaces by breaking the large-scale surface into ruled modules, thus leading to a relatively easy construction method. Furthermore, the behavior of the original surface structure can also be analyzed using the logic applied to smooth polyhypar surfaces, which turns out to simply be a planar equilibrium problem. As such, with some intuitive understanding of internal forces, the original freeform surface can be developed into a more efficient structural form.

**Economic prefabrication methods for surface structures**

As a cluster of hypars, smooth polyhypar surfaces enable a new low-tech construction method for surface structures free of scaffolding or formwork. They allow the separate prefabrication of each hypar module using only straight components, all the while preserving the smoothness of the surface after assembly.

A smooth polyhypar surface can be manufactured of different materials, as has been shown in chapter 7. Both the lightweight grid shells as well as the concrete shells, can be manually fabricated in a cost-effective way. In the case of a grid shell as a modular system, a smooth polyhypar surface can be divided into multiple hypar segments, which, due to being ruled surfaces, can be easily prefabricated using only straight elements. Furthermore, since all double curved hypars are of a similar form, the modules can conveniently be stacked for transportation to the construction site. During the construction process, benefiting from the stability of smooth polyhypar surfaces, the assembled structure stays stable without the need for scaffolding. The fabrication material of a grid shell can also be recycled for the construction of other smooth polyhypar surfaces. In the case of a ferrocement shell, the processes of prefabrication, transportation, and assembly of rebar meshes are similar to those of a grid shell. Even during the casting of the concrete, no formwork or scaffolding is required.

This low-tech, modular prefabrication method for smooth polyhypar surfaces can also be applied to optimize the fabrication of other double curved freeform surfaces, thereby reducing their construction complexity and costs.

### 8.3 Limitation and future work

While the section above explains the advantages of the smooth polyhypar surfaces from architectural, structural, and building perspectives, there are still some limitations. The following text explains in detail the shortcomings of the smooth polyhypar surfaces and offers some potential solutions as a future extension to the current research.
8.3 Limitation and future work

Improving the design method

In terms of a design method, the way of generating and manipulating variations of smooth polyhypar surfaces still needs to be improved. Currently, variations of smooth polyhypar surfaces are controlled through variables in each hypar module. With the increasing number of hypar modules and corresponding variables, it becomes progressively difficult to precisely define the desired geometrical variations and their related variables, thus causing problems in controlling the global geometry. To overcome such limitations, one possible solution is to introduce parabolic networks to represent the edges of the hypar modules in a smooth polyhypar surface and setting a few numbers of variables to directly manipulate their variations.

Another limitation of the design methodology is the current approach towards the bracing of smooth polyhypar surfaces. A hypar module in a smooth polyhypar surface is always supported by an adjacent module, while the last module in a load path should be fully braced by two edge elements. The current solution is to add more hypar modules until the edge of the last module intersecting the load path is placed directly on the ground. This solution normally leads to a large modification of the original smooth polyhypar surface, thereby becoming a disturbing factor from an architectural design point of view. In future works, alternative approaches or techniques can be explored. One option could be to utilize the symmetry of a smooth polyhypar surface, as was done by Musmeci on his Basento bridge. In case several hypar modules are smoothly joined into a horizontally mirrored pair of identical clusters, the coplanarity principle will be satisfied; and the reactions at the boundary of each cluster will have equal magnitude, but opposing direction, thereby cancelling each other out. Yet another option is to introduce prestressing forces to balance out the internal forces along the edges. Further exploration in this direction could present solutions to free smooth polyhypar surfaces from the constraints of load paths, thereby providing more possibilities to explore different forms of surface structure.

Digital implementation of form-forces interactions

As explained in part III, the distributions of forces and geometrical variations in graphic statics are interactive through both form diagrams and force diagrams. The digital implementation for the form-forces interactions is available on the level of an individual hypar, but it has not yet been implemented as a digital tool on the level of smooth polyhypar surfaces. Nevertheless, there is an intuitive way to check the global equilibrium for a smooth polyhypar surface using load paths. That being said, a precise calculation of internal forces and reactions of smooth polyhypar surfaces in chapter 5.4 has not been implemented as an operative tool.

In order to visualize the interactions between the geometrical variations and the magnitude of forces in a smooth polyhypar surface, a digital tool can be developed according to formulas (5-14), (5-16), and the flow chart in Figure 5-11. With such a tool, designers could create a smooth polyhypar surface as a cluster of hypar modules. Following the logic of load paths, all modules would be organized into several hypar-chains, where internal forces would be repeatedly calculated until they were all balanced out with the reaction forces. The output forces
would be displayed in line thicknesses through a network of rulings and edges, which could change according to the geometrical variation.

Further improvements to the construction methods

Chapter 7.2 already presented the advantages of construction methods developed based on the locally ruled geometry of smooth polyhypar surfaces. However, the built examples of these construction methods are yet limited to a small scale. Moreover, from the perspectives of efficiency and cost-effectiveness, these construction methods—either for grid shells or for concrete shells—still present some drawbacks which can be improved.

In the case of grid shells, the concept of reuse can be introduced to enhance the construction performance. Since all the components are mechanically joined, it is possible to disassemble the grid shell into straight components which can be reused for the construction of other smooth polyhypar grid shells. The method of organizing the recycled components and designing a new grid shell based on them relates to the studies of truss design with reused components (Brüttinga, et al., 2018).

The construction of ferrocement shells also presents some disadvantages. As has already been discussed in chapter 7.2.2, although the ruled geometry of smooth polyhypar surfaces is convenient when constructing the rebar skeleton, the process of covering the skeleton with wire meshes is very time consuming. One option for future development could be to replace the wire meshes with a rebar mesh. It will also be necessary to research various methods of spraying concrete on the rebar meshes in order to achieve a smooth final appearance. For future studies, a bigger scale test using ferrocement will be conducted.

8.4 Conclusion

In this research, the smooth polyhypar surface—a modular system based on hypars—has been developed to achieve the smoothness called for in the design of freeform architecture, while simultaneously including technical considerations. Benefiting from its special structural stiffness and geometrical simplicity, this new geometrical category works as a mediator to integrate architectural smoothness, structural efficiency, and ease of construction from the initial design stage.

In digital times, this innovative approach can be utilized to search for new freeform surfaces, where architectural and structural forms can come together Furthermore, smooth polyhypar surfaces enable a new direction in approximating complex freeform surfaces with ruled geometries. Finally, in terms of construction, using locally ruled polyhypar surfaces provides simple and economic methods to fabricate complex double curved surfaces.
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