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Rational Points and Transcendental Points

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Contents

Abstract

In the second chapter we revisit the Ritt theory by the study of topological fundamental groups. Our main result of this chapter gives a presentation of $(\mathrm{End}(\mathbb{E}), \circ)$ which is the monoid of finite endomorphisms of the unit disk.

The third chapter, as a technical part of our thesis, is devoted to elliptic rational functions. Those functions were originally constructed by Zolotarev in terms of, essentially, descents of isogeny between elliptic curves. Our reconstruction of them is by means of monodromy representation, and this new viewpoint is essential for arithmetic or geometric applications. We shall verify that if the curve $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$, where f and g are elements of $\mathrm{End}(\mathbb{E})$, admits certain special arithmetic or geometric properties then f and g admit special representations in $(\mathrm{End}(\mathbb{E}), \circ)$. In addition elliptic rational functions contribute to most nontrivial cases of these representations.

The purpose of the fourth chapter is to prove that if two finite endomorphisms of the unit disk have orbits with infinitely many intersections then they have a common iteration. This implies as a corollary a dynamical analogue of Mordell-Lang and André-Oort type for finite endomorphisms of polydisks.

The last chapter is devoted to a generalization of Schneider's theorem in transcendence. The main result here is that under suitable conditions the image of $f\colon C \to X_{\mathbb{C}}$, which is a holomorphic map from an affine algebraic curve to a projective algebraic variety X defined over a number field, assumes finitely many times rational points.

Chapter 2 and Section 3.1 are joint with Tuen-Wai Ng and the last Chapter is joint with Gisbert Wüstholz.

Zusammenfassung

Im zweiten Kapitel geben wir zunächst eine Darstellung der Ritt'schen Theorie mittels Fundamentalgruppen. Im Hauptresultat des Kapitels wird dann $(\mathrm{End}(\mathbb{E}), \circ)$ als Monoid von endlichen Endomorphismen des Einheitskreises repräsentiert.

Das dritte Kapitel, welches den technischen Teil der Dissertation darstellt, ist elliptischen rationalen Funktionen gewidmet. In der formalen Korrespondenz zwischen elliptischen und trigonometrischen Funktionen fungieren die letztgenannten als Analoga der Tschebyshew'schen Polynome.

Ziel und Zweck des vierten Kapitels is der Beweis, dass zwei endliche Endomorphismen des Einheitskreises, welche Orbite mit unendlichem Durchschnitt besitzen, ein gemeinsames Interiertes aufweisen. Dies impliziert ein dynamisches Analogon für endliche Endomorphismen des Einheitskreises von Mordell-Weil and André-Oort.

Das letzte Kapitel ist einer Verallgemeinerung des Schneider'schen Transzendenzsatzes gewidmet. Das Hauptresultat hier ist, dass das Bild eine holomorphen Funktion $f\colon C \to X_{\mathbb{C}}$ von einer affinen algebraischen C in eine projektive algebraische Varietät X über einem Zahlkörper K unter gewissen Bedingungen nur endlich viele K-rationale Punkte als Wert annimmt.

Kapitel 2 und Abschnitt 3.1 wurden zusammen mit Tuen-Wai Ng ausgearbeitet. Das letzte Kapitel geht aus einem gemeinsammen Paper mit Gisbert Wüstholz hervor.

Introduction

The interaction of number theory and geometry is one of the most beautiful parts of mathematics. As examples we may take the Mordell-Lang conjecture which was the central problem of Diophantine geometry and the André-Oort conjecture which has partly its roots in transcendence. Despite their different roots in Diophantine geometry or in transcendence, these conjectures enjoy the same expectation: if a geometric object admits special arithmetic properties then it carries a special underlying geometric structure.

Considering topology which is the naivest geometry one might quickly come up with the well-known theorem of Belyi, that a compact Riemann surface is defined over $\overline{\mathbb{Q}}$ if and only if it is birational to a finite étale covering of $\mathbb{P}^1\backslash\{0,1,\infty\}$, which has the strong implication that arithmetic objects in dimension 1 are simply topological in nature. Moreover, celebrated theorems of Siegel on integral points and of Faltings on rational points also demonstrate the dominant role of topological structure on Diophantine equations.

The fundamental group is one of the simplest invariants in topology. Surprisingly it is also one of the most useful concepts in detecting many involved properties. Taking a special case of Mostow's rigidity as an example, two compact hyperbolic manifold with the same dimension greater than two and with isomorphic fundamental groups must be isometric. Subsequent development of fundamental groups in geometry is greatly influenced by Mostow's rigidity, and in parallel a vast program in number theory is sketched by Grothendieck in his letter to Faltings. As one of the main ideas of Grothendieck's letter, "dann ist ein Homomorphismus völlig bestimmt, wenn man die entsprechend Abblildung der entstprechenden äusseren Fundamentalgruppen kennt." Because a variety X over $\mathbb{K}$ can be regarded as a fibration morphism $X \to^{\pi} \operatorname{Spec}\mathbb{K}$ and a $\mathbb{K}$-rational point on X is regarded as a section morphism $\operatorname{Spec}\mathbb{K} \to^{s} X$, hopefully fundamental groups applied to these morphisms will in particular determine much of the geometry and arithmetic of X over $\mathbb{K}$.

If f is a finite endomorphism of an algebraic (or analytic) space X and if we are interested in its iterations

$$\cdots \to^{f} X \to^{f} X \to^{f} X \to^{f} \cdots$$

then from the viewpoint of birational geometry the dynamical property should be encoded in

$$\cdots \leftarrow_{f^\sharp} K \leftarrow_{f^\sharp} K \leftarrow_{f^\sharp} K \leftarrow_{f^\sharp} \cdots$$

which is a tower of finite extensions of the function field K of X. In terms of fundamental groups hopefully the tower of group homomorphisms

$$\cdots \rightarrow^{f_*} \pi_1 \rightarrow^{f_*} \pi_1 \rightarrow^{f_*} \pi_1 \rightarrow^{f_*} \cdots$$

imposes restrictions on the dynamics of f. This thesis might provide an example in supporting such a point of view.

In a fundamental case the André-Oort conjecture expects that: if a subcurve of $\Gamma\backslash\mathbb{H}^n$ (where Γ is a congruence subgroup) contains infinitely many "special points" then it is a "special curve". We shall work on the uniformization space $\mathbb{H}^n$ instead of its quotients and prove the following dynamical analogue of the conjectues of Mordell-Lang and André-Oort type: if an orbit of a finite endomorphism of $\mathbb{H}^n$ has infinitely many intersections with a complex geodesic V of $\mathbb{H}^n$ then there exists a positive integer n such that $f^n(V)=V$.

Similar to the case of subcurves of a product of modular curves in which the André-Oort conjecture reduces easily to the case of $\mathbb{C}^2$, the above result reduces easily to the main theorem of the first part of this thesis: if two non-linear finite endomorphisms of the unit disk have orbits with infinitely many intersections then they have a common iteration. The reduction is fulfilled by applying a classification theorem of Remmert-Stein and an argument on heights which involves the old technique of specialization.

Our main theorem is a hyperbolic analogue of the Ghioca-Tucker-Zieve theorem which has the same conclusion but with the unit disk replaced by the complex plane. Although have the same topology, the Poincaré unit disk differs greatly from the Gaussian complex plane in dynamics, for which we refer to the classical result of Denjoy-Wolff. Our major observation is that some arguments of the proof of Ghioca-Tucker-Zieve are simply topological in nature which makes it possible to adopt their strategy to the context of unit disk. Briefly the proof of our main theorem goes as follows.

- On the one hand the decomposition of a finite endomorphism of the unit disk is very rigid.

- On the other hand infinitely many intersections leads to special decompositions of given endomorphisms and of their iterations.

Finally the rigidity and the speciality of decompositions as obtained above are in many cases incompatible which enable us to continue with the proof by a lengthy analysis of the endomorphism monoid of the unit disk.

Concerning the rigidity part the starting point is that the decomposition property is determined by the monodromy action of the fundamental group and therefore is topological in nature. This enables us to transform Ritt's original theory from the Gaussian complex plane to the Poincaré unit disk. In the complex plane case the rigidity is encoded in close cycles around infinity, and in the unit disk case the rigidity is encoded in cycles around the unit circle. Full details is carried out in Chapter 2.

Regarding that special decompositions part the starting point is a decomposition principle saying that if the curve $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ defined by rational functions has special arithmetic properties then hopefully f and g can be expected to admit special decompositions. For polynomials this principle is supported by the work of Bilu-Tichy and of Avanzi-Zannier. In our disc case we have to verify this principle for curves defined by finite Blaschke products. This can be done because the arithmetic and the decomposition property are both of topological nature and because the Poincaré unit disk is the same as the Gaussian complex plane as topological space. Our main task is to determine when does the curve $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ defined by finite Blaschke products f and g has potentially dense rational points. This is achieved by the work in Section 1.6, Section 2.3, Section 3.3 and in Section 3.6. We summarize here several major points of the proof, and for simplicity we assume that $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ is irreducible and is of potentially dense rational points.

- Faltings' theorem: this reduces the arithmetic assumption to a topological one $\chi(\mathbb{P}^1 \times_{f,g} \mathbb{P}^1) \geq 0$.
- Schwarz reflection principle: this gives a symmetry or a real structure on $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$. Indeed $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ is a double of $\mathbb{E} \times_{f,g} \mathbb{E}$.
- Additivity of Euler characteristic: this gives $\chi(\mathbb{E} \times_{f,g} \mathbb{E}) \geq 0$.
- Deformation: $\mathbb{E} \times_{f,g} \mathbb{E}$ is not algebraic. Using an Änderung argument à la Hurwitz(or Riemann), we may modify it and obtain an algebraic curve $\mathbb{C} \times_{\overline{f},\overline{g}} \mathbb{C}$ which is defined by polynomials f, g with $\chi(\mathbb{C} \times_{\overline{f},\overline{g}} \mathbb{C}) \geq 0$.

- Siegel's theorem: our topological condition on $\mathbb{C}\times_{\overline{f},\overline{g}}\mathbb{C}$ is equivalent to that the algebraic curve $\mathbb{C}\times_{\overline{f},\overline{g}}\mathbb{C}$ admits potentially dense integral points.
- Bilu-Tichy criterion: used to obtain monodromy properties of $\overline{f}$ and of $\overline{g}$, and therefore those of f and of g.
- Riemann's existence theorem: f and g can be recovered from their underlying monodromy representations.

For the last step we have to handle concrete analytic functions, and this is not of topological nature any more. In the complex plane case Ghioca-Tucker-Zieve have to deal with Chebyshev polynomials which are related to trigonometric functions. In the unit disk case we have to play with elliptic rational functions which arise from elliptic functions. These functions, which are constructed from descents of cyclic isogenies by Kummer maps or from their special monodromy representations, are analogues of Chebyshev polynomials in the formal dictionary between elliptic functions and trigonometric functions and contribute to most of the nontrivial factors of special decompositions as discussed above. Technical preparations on these functions are contained in Chapter 3.

The second part of this thesis is devoted to transcendence. Faltings' theorem tells us that an algebraic curve other than rational or elliptic ones has only finitely may rational points. In transcendence it is expected that under suitable conditions any rational point on a transcendental curve has special geometric reasons.

Typically an infinite covering carries many transcendental properties, and the projection map from the uniformization space to an algebraic variety is of particular interest in transcendence. As it was remarked by Kollár in his book "one hopes that there are many interesting connections between the meromorphic function theory of a variety and the holomorphic function theory of its universal cover". One general principle in transcendence is that: an algebraic object of an algebraic variety lifts to a transcendental object of its universal covering space. Therefore a large part of transcendence theory should be closely related to the theory of uniformization. Indeed Schneider's theorem, which is a criterion about functions of the complex plane and applies to periods of the torus and of elliptic curves, serves as an outstanding example in illustrating such a point of view. In the theory of uniformization the complex Euclidean spaces are higher generalizations of the complex plane. For transcendence on the complex Euclidean spaces we have theorems of Bombieri and of Wüstholz.

Wüstholz's integral theorem tells us: a period on curves defined over $\overline{\mathbb{Q}}$ is either 0 or transcendental. His proof is based on the embedding of curves into their Jacobians which reduces the above result to his 1983 analytic subgroup theorem. If one is interested in the contribution of the action of the full topological fundamental groups on periods then it is natural to try to reprove Wüstholz's integral theorem by regarding periods as values of special functions on the uniformization space. As a consequence of the action of fundamental groups these special functions are hopefully to be of finite order. To sum up looking for a new proof of integral theorem amounts to establishing a suitable criterion of Schneider's type on the unit disk.

The main result here is that under suitable conditions the image of $f\colon C \to X_{\mathbb{C}}$, which is a holomorphic map from an affine algebraic curve to an projective algebraic variety X defined over a number field, assumes finitely many times rational points. This truly generalizes Schneider's criterion, and the proof is along that of Schneider's original work. One of our motivation is to understand an analogy between the first main theorem of Nevanlinna theory and Schneider's criterion.

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Backgrounds and General Remarks

1.1 Diophantine geometry on curves

The fundamental goal of Diophantine geometry is to describe arithmetical properties in terms of underlying geometric structures. The central object of the arithmetical part is the rational point, and the geometry part may refer to rather different contexts including algebraic geometry, complex geometry and topology. The theory on curves derived from Diophanti's book and culminated in Faltings' 1983 theorem, although up to date several longstanding problems such as the effective Mordell conjecture and the Birch and Swinnerton-Dyer conjecture remain out of reach. The plenty of rational points on rational curves is closely related to the existence of parametrization, see [39] and [69].

Theorem 1.1.1 (Diophanti, Hilbert-Hurwitz). *If X is a rational curve over $\mathbb{K}$ then either $X(\mathbb{K})$ is empty or X is $\mathbb{K}$-birational to $\mathbb{A}^1$.*

There are less rational points on genus one curves, see [96] and [133].

Theorem 1.1.2 (Mordell, Weil). *If E is an elliptic curve over a field $\mathbb{K}$ of finite type over $\mathbb{Q}$ then $E(\mathbb{K})$ is a finitely generated abelian group.*

The finiteness of rational points on curves of genus greater than one was proved in [42].

Theorem 1.1.3 (Faltings). *If X is a curve of general type over a field $\mathbb{K}$ of* *finite type over* $\mathbb{Q}$ *then* $X(\mathbb{K})$ *is finite.*

From the point of view of algebraic geometry arithmetics is reflected by whether the underling variety is Fano, of intermediate type or of general type. In the higher dimension the asymptotic behavior of the set of rational points of Fano varieties is predicted by Manin, and the pseudo mordellicity of varieties of general type is conjectured by Bombieri and Lang [82].

Conjecture 1.1.4 (Bombieri-Lang Conjecture)**.** *If* X *is a variety of general type over a field* $\mathbb{K}$ *of finite type over* $\mathbb{Q}$ *then* $X(\mathbb{K})$ *is not Zariski dense.*

This conjecture has only been verified by Faltings when X is a subvariety of an abelian variety in [43] which already covers the original Mordell conjecture.

With a slightly different flavor one could try to predict the arithmetical behavior by complex geometric properties. The existence of rational curves gives lots of rational points, and as a weaker condition the existence of non-constant holomorphic $\mathbb{C}$ curves also suggests the infiniteness of rational points. In the most interesting case the non-existence of non-constant holomorphic $\mathbb{C}$ curves, known as Brody hyperbolicity according to [28], hopefully implies the mordellicity of a projective variety. This is contained in another conjecture of Lang [82]. Thanks to the work of Bloch [17] and of Faltings [43] this has again been verified for subvarieties of abelian varieties.

The arithmetic of curves is dominated by the topology of the curves, and this topological viewpoint will prevail in this thesis. To restate Faltings's theorem in terms of topology we write $\chi(X)$ for the Euler characteristic of the smooth model of an irreducible curve X. Rational points of an algebraic variety X are called *potentially dense* if $X(\mathbb{K})$ is Zariski dense for sufficiently large fields k of finite type over $\mathbb{Q}$. Our definition is motivated by Bogomolov and Tschinkel, but for the purpose of application here we are interested in fields of finite type rather than number fields originally considered in [26].

Theorem 1.1.5 (Faltings)**.** *Rational points of an irreducible projective curve* X *are potentially dense if and only if* $\chi(X) \geq 0$.

Rational points can be regarded as integral points with respect to the empty divisor at infinity. Siegel's 1929 theorem on integral points with respect to positive divisors is the first major result on diophantine equations that depended only on the genus. We can also introduce the notion of potential density with respect to integral points, and then Siegel's theorem reads as follows.

Theorem 1.1.6 (Siegel [122]). *Integral points of an irreducible affine curve X are potentially dense if and only if $\chi(X) \geq 0$.*

Let $\mathbb{L}\subset\mathbb{C}$ and $\mathbb{K}\subset\mathbb{C}$ be fields of finite type over $\mathbb{Q}$ and $X \subset \mathbb{P}^n$ a projective algebraic variety over $\mathbb{L}$. We write $X(\mathbb{K})$ for the set $X(\mathbb{C})\cap\mathbb{P}^n(\mathbb{K})$, and the notation $X(\mathbb{K})$ causes no confusion even if X is not over $\mathbb{K}$. Theorem 1.1.3 still holds without the assumption that X is defined over $\mathbb{K}$, for which we give a short argument.

Proof. The curve X is always defined over a field $\mathbb{F}$ which is of finite type over $\mathbb{K}$ and Theorem 1.1.3 implies that $X(\mathbb{F})$ is finite. This together with the trivial fact that $X(\mathbb{K})$ is contained in $X(\mathbb{F})$ gives the finiteness of $X(\mathbb{K})$. □

One may also obtain an alternative argument by using the fact that if X contains infinitely many $\mathbb{K}$-rational points then it is actually defined over $\mathbb{K}$.

1.2 Abelian curves

Besides the Euler characteristic, the fundamental group is another topological invariant which reflects Diophantine properties. This was proposed by Grothendieck in [64] and led to the theory of anablian geometry.

As a fundamental example we take a curve X of type (g, ν) with g its genus and ν the number of points points at infinity with respect to a smooth model. The property of the curve having a dense set of integral points or not is independent of the model (see for instance [130, Proposition 12.4]) but depends on the base ring. We collect theorems of Siegel and of Faltings as follows

Theorem 1.2.1 (Siegel, Faltings). *Integral points of an algebraic curve X of type (g, ν) are potentially dense if and only if $\chi(X) = 2 - 2g - \nu \geq 0$. This is exactly the case the topological fundamental group of a smooth model of $X(\mathbb{C})$ is abelian.*

On the one hand a rational point is regarded as a section morphism and on the other hand a morphism is supposed to be encoded in fundamental groups. Therefore there is the hope that the study of fundamental groups and their rigidity will lead to new proofs of finiteness theorems in Diophantine geometry.

There are only four types of curves carrying abelian fundamental groups, namely ones of

$$(0,0),(0,1),(0,2) \text{ and } (1,0),$$

which will be called *abelian curves.*

Let f, g be polynomials and consider the curve $\mathbb{C}\times_{f,g}\mathbb{C}$. In [58] the authors introduce Siegel factors which are irreducible components of $\mathbb{C}\times_{f,g}\mathbb{C}$ of type $(0,1)$ or $(0,2)$. In our work we will call curves of type $(0,1)$ or $(0,2)$ *Siegel factors* and curves of type $(0,0)$ or $(1,0)$ *Faltings factors.* For finite Blaschke products f and g we will construct polynomials f_*, g_* for which there is a one-to-one correspondence between Siegel factors of $\mathbb{C}\times_{f_*,g_*}\mathbb{C}$ and Faltings factors of $\mathbb{P}^1\times_{f,g}\mathbb{P}^1$. In other words there exists the following duality.

$$(0,1)+(0,1) \longleftrightarrow (0,0)$$
$$(0,2)+(0,2) \longleftrightarrow (1,0)$$

We shall use this simple fact as one major argument in this thesis.

Fundamental groups played an important role in our work, and here they are used to detect decomposition properties of morphisms. We shall used the fact that if a finite map carries a closed cycle along which the topological monodromy action is transitive then its decomposition is very rigid. This rigidity property, which spring from the the monodromy action of a special part of topological fundamental group, is basically the starting point of Ritt's theory.

1.3 Finite maps

In this section we shall discuss finite maps between Riemann surfaces which, from the point of view of value distribution theory, should be regarded as generalizations of polynomials fitting into the fundamental theorem of algebra.

This set of maps is one of the most elementary category which demonstrates the following principle: fundamental groups detect morphisms.

In analytic geometry or algebraic geometry a *finite map* refers to an analytic or algebraic map which is proper and quasi-finite. In the simplest setting a holomorphic map between Riemann surfaces is finite if and only if it is non-constant and proper. This notion was first introduced by Radó who proved in [102] that: a holomorphic map $f\colon\mathfrak{M}\to\mathfrak{N}$ between Riemann surfaces is finite if and only if there exists an integer n such that $f(z)=c$ has n solutions for any point c of $\mathfrak{N}$, and we refer to [50, p.27] for a modern treatment. We shall define the number n given above to be the degree of f and denote it by $\deg f$. One may deduce readily from [27, p.99] that if $h\colon\mathfrak{M}\to\mathfrak{T}$ and $g\colon\mathfrak{T}\to\mathfrak{N}$ are holomorphic maps between Riemann surfaces then $g\circ h$ is finite if and only if both g and h are finite.

If f is a finite map from $\mathfrak{M}$ to $\mathfrak{N}$ then it is called *linear* if and only if $\deg f=1$. Let f be a nonlinear finite map and we call f *prime* if there do not exist nonlinear finite maps $g\colon\mathfrak{T}\to\mathfrak{N}$ and $h\colon\mathfrak{M}\to\mathfrak{T}$ for which $f{=}g{\circ}h$. Otherwise it is called *composite* or *factorized*. We shall call a factorization of f *proper* if all its factors are nonlinear and a maximal proper factorization is called a *prime factorization*. The *length* of f with respect to a prime factorization is defined to be the number of its factors.

From the point of view of birational geometry the finite map f is fully encoded in analytic function fields of $\mathfrak{N}$ and of $\mathfrak{M}$. We use $\mathbb{C}(\mathfrak{N})$ to denote the analytic function field of $\mathfrak{N}$ and then the Riemann surfaces $\mathfrak{N}$ can be uniquely recovered from $\mathbb{C}(\mathfrak{N})$ (see for instance [3]). Furthermore according to [119] finite maps f from some other Riemann surface $\mathfrak{M}$ to $\mathfrak{N}$ are in one-to-one correspondence with finite extensions of fields $\mathbb{C}(\mathfrak{N})\subset\mathcal{K}$ given by $f\mapsto f^{\sharp}\colon\mathbb{C}(\mathfrak{N})\to\mathbb{C}(\mathfrak{M})$. In terms of fundamental groups finite maps can be characterized as follows.

Theorem 1.3.1 ([119])**.** *Let Σ be a discrete subset of $\mathfrak{N}$ and $q\notin\Sigma$ a point in $\mathfrak{N}$. There is a one-to-one correspondence between finite maps $f\colon(\mathfrak{M},p)\to(\mathfrak{N},q)$ of degree n with $\mathfrak{d}_f$ contained in Σ and subgroups H of $\pi_1(\mathfrak{N}\backslash\Sigma,q)$ of index n given by $f\mapsto H=\pi_1(\mathfrak{M}\backslash f^{-1}(\Sigma),p)$.*

There are no finite maps between $\mathbb{C}$ and $\mathbb{E}$. This is a consequence of Liuville's theorem and the following

Lemma 1.3.2. *If f is a finite map from $\mathbb{E}$ to $\mathfrak{N}$ then $\mathfrak{N}$ is biholomorphic to $\mathbb{E}$.*

Proof. Let $\overline{\mathfrak{N}}$ be the universal covering of $\mathfrak{N}$. By [27, p.99] we deduce that the finiteness of f implies the finiteness of the lifting map $\overline{f}\colon \mathbb{E} \to \overline{\mathfrak{N}}$ and of the projection map $\pi\colon \overline{\mathfrak{N}} \to \mathfrak{N}$ which leads to $\pi_1(\mathfrak{N}) = 1$. Firstly $\mathfrak{N}$ cannot be the Riemann's sphere because there is no proper map from a non-compact space to a compact space. Now we claim that $\mathfrak{N}$ cannot be the complex plane and therefore it has to be biholomorphic to the unit disk as claimed. Otherwise we may assume that f is a finite map from $\mathbb{E}$ to $\mathbb{C}$ and then a bounded holomorphic function of $\mathbb{E}$ descends to a bounded holomorphic function of $\mathbb{C}$ by taking the symmetric product. This will imply that there is a non-constant bounded holomorphic function on $\mathbb{C}$, which is impossible. $\square$

We call two proper factorizations

$$\mathfrak{M} \xrightarrow{\phi_1} \mathfrak{T}_1 \xrightarrow{\phi_2} \mathfrak{T}_2 \to \cdots \to \mathfrak{T}_{r-1} \xrightarrow{\phi_r} \mathfrak{N}$$

and

$$\mathfrak{M} \xrightarrow{\psi_1} \mathfrak{R}_1 \xrightarrow{\psi_2} \mathfrak{R}_2 \to \cdots \to \mathfrak{R}_{s-1} \xrightarrow{\psi_s} \mathfrak{N}$$

equivalent if $r = s$ and there exist biholomorphic maps ε_i such that the diagram

$$\begin{tikzcd}
\mathfrak{M} \arrow[r, "\phi_1"] \arrow[d, "\mathrm{id}"] & \mathfrak{T}_1 \arrow[r, "\phi_2"] \arrow[d, "\varepsilon_1"] & \mathfrak{T}_2 \arrow[r, "\phi_3"] \arrow[d, "\varepsilon_2"] & \cdots\cdots \arrow[r, "\phi_{r-1}"] & \mathfrak{T}_{r-1} \arrow[r, "\phi_r"] \arrow[d, "\varepsilon_{r-1}"] & \mathfrak{N} \arrow[d, "\mathrm{id}"] \\
\mathfrak{M} \arrow[r, "\psi_1"] & \mathfrak{R}_1 \arrow[r, "\psi_2"] & \mathfrak{R}_2 \arrow[r, "\psi_3"] & \cdots\cdots \arrow[r, "\psi_{r-1}"] & \mathfrak{R}_{r-1} \arrow[r, "\psi_r"] & \mathfrak{N}
\end{tikzcd}$$

commutes.

Corollary 1.3.3. *Let Σ be a discrete subset of $\mathfrak{N}$, p a point in $\mathfrak{M}$, $q \notin \Sigma$ a point in $\mathfrak{N}$ and f a finite map from $(\mathfrak{M}, p)$ to $(\mathfrak{N}, q)$ with $\mathfrak{d}_f$ contained in Σ. There is a one-to-one correspondence between proper factorizations of f and proper chains of groups between $\pi_1(\mathfrak{M}\backslash f^{-1}(\Sigma), p)$ and $\pi_1(\mathfrak{N}\backslash\Sigma, q)$.*

Let f be a finite map of degree n from $\mathfrak{M}$ to $\mathfrak{N}$ and $q \notin \mathfrak{d}_f$ a point in $\mathfrak{N}$. If we write $f^{-1}(q) = \sum_{i=1}^{n}(p_i)$ then for all α in $\pi_1(\mathfrak{N}\backslash\mathfrak{d}_f, q)$ and for all $1 \le i \le n$ there is a uniquely determined point $(p_i)^\alpha$ supported in $f^{-1}(q)$ and a path β from p_i to $(p_i)^\alpha$, unique up to homotopy, such that $f_*\beta = \alpha$. There is a uniquely defined $\rho(\alpha)$ in S_n such that $(p_i)^\alpha = p_{i\rho(\alpha)}$ for all $1 \leq i \leq n$ and we call the group homomorphism $\rho\colon \pi_1(\mathfrak{N}\backslash\mathfrak{d}_f, q) \to S_n$ the *monodromy* and the image of ρ the *monodromy group* of f. The monodromy group of f is transitive because $\mathfrak{M}$ is connected. We shall need the following useful remark which complements Theorem 1.3.1:

$$\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{d}_f), p) = \{\alpha \in \pi_1(\mathfrak{N}\backslash\mathfrak{d}_f, q) : p^\alpha = p\}. \tag{1.1}$$

Here we write p^α instead of $(p)^\alpha$.

If f is the Chebyshev polynomial T_n then

$$\mathfrak{d}_f = \begin{cases} \{-1, 1\} & \text{for } \ n \geq 3; \\ \{-1\} & \text{for } \ n = 2; \\ \emptyset & \text{for } \ n = 1. \end{cases}$$

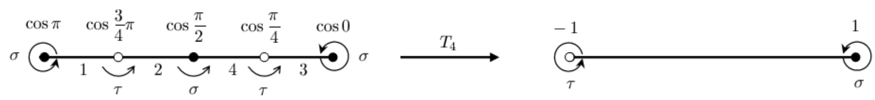
In any case we could look at the monodromy representation of $\pi_1(\mathbb{C}\backslash\{-1, 1\})$ which is a rank 2 free group generated by σ and τ with σ respectively τ represented by small loops counter-clockwise around -1 respectively 1. Concerning the monodromy $\rho\colon \langle\sigma, \tau\rangle \to S_n$ we claim that if $n = 2k$ then

$$\begin{aligned} \rho(\sigma) &= (2,\ 2k)\,(3,\ 2k-1)\,\cdots\,(k,\ k+2) \\ \rho(\tau) &= (2,\ 1)\,(3,\ 2k)\,\cdots\,(k+1,\ k+2) \end{aligned}$$

and if $n = 2k+1$ then

$$\begin{aligned} \rho(\sigma) &= (2,\ 2k+1)\,(3,\ 2k)\,\cdots\,(k+1,\ k+2) \\ \rho(\tau) &= (2,\ 1)\,(3,\ 2k+1)\,\cdots\,(k+1,\ k+3). \end{aligned}$$

For instance, they can be read as the hypercatographic groups of the corresponding dessins d'enfants of T_n.



$1^\sigma = 1, \quad 2^\sigma = 4, \quad 3^\sigma = 3, \quad 1^\tau = 2, \quad 4^\tau = 3$

Figure 1.1: Chebyshev representation.

Proof of the claim. This fact is well-known, therefore we only verify it in the case $n = 4$. It is easily checked that the T_4-preimage of the real interval $[-1, 1]$

is $[-1,1]$, of -1 is $\{\cos 3\pi/4, \cos\pi/4\}$ and of 1 is $\{\cos\pi, \cos\pi/2, \cos 0\}$. We mark the 4 copies of the T_4-preimage of open interval $(-1,1)$ with $1,2,3$ and 4 as tagged in Figure 1.1 and then it is clear that 1 goes to 2 under the action of τ, 2 goes to 1, 3 goes to 4 and 4 goes to 3. This gives $\rho(\tau) = (1,2)(3,4)$ and similarly $\rho(\sigma) = (2,4)$. $\square$

We write $\mathscr{F}_r$ for the free group of rank r and call a group homomorphism $\rho\colon \mathscr{F}_2 = \langle\sigma,\tau\rangle \to S_n$ a *Chebyshev representation* if it agrees with the one described as above.

From the viewpoint of topology a finite map can be uniquely recovered from its monodromy by means of "Schere und Kleister" surgery [119, p.41] and this leads to the following restatement of Theorem 1.3.1.

Theorem 1.3.4 (Riemann's existence theorem)**.** *Let $\mathfrak{N}$ be a Riemann surface, $\mathfrak{s}$ a discrete subset in $\mathfrak{N}$, $q \notin \mathfrak{s}$ a point in $\mathfrak{N}$ and $\rho\colon \pi_1(\mathfrak{N}\backslash\mathfrak{s}, q) \to S_n$ a transitive representation. There exist a unique pointed Riemann surface $(\mathfrak{M}, p)$ associated with a finite map $f\colon (\mathfrak{M}, p) \to (\mathfrak{N}, q)$ with the monodromy of f given by ρ.*

The uniqueness of $(f, \mathfrak{M}, p)$ implies that if finite maps $f\colon (\mathfrak{M}, p) \to (\mathfrak{N}, q)$ and $g\colon (\mathfrak{R}, r) \to (\mathfrak{N}, q)$ have the same monodromy then there exists a biholomorphic map ε: $(\mathfrak{M}, p) \to (\mathfrak{R}, r)$ making the diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[rr, "\varepsilon"] \arrow[dr, "f"'] & & \mathfrak{R} \arrow[dl, "g"] \\ & \mathfrak{N} & \end{tikzcd}$$

commutative.

Let $f\colon \mathfrak{M} \to \mathfrak{N}$ be a finite map with $f(p) = q$. Regarded as permutation groups, the monodromy group of f is isomorphic to the image of the action of $\pi_1(\mathfrak{N}\backslash\mathfrak{d}_f, q)$ on the coset space $\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{d}_f), p) \backslash \pi_1(\mathfrak{N} \backslash \mathfrak{d}_f, q)$, for which we refer to [119, p.41]. It is also isomorphic to the image of the action of $\mathrm{Gal}(K/\mathbb{C}(\mathfrak{N}))$ on the coset space $\mathrm{Gal}(K/\mathbb{C}(\mathfrak{M}))\backslash\mathrm{Gal}(K/\mathbb{C}(\mathfrak{N}))$, as explained in [131, Theorem 5.14], where K is any Galois extension of $\mathbb{C}(\mathfrak{N})$ which contains $\mathbb{C}(\mathfrak{M})$.

We call a Riemann surface $\mathfrak{M}$ *finite* if $\pi_1(\mathfrak{M})$ is finitely generated. Thanks to Ahlfors finiteness theorem [1], this is equivalent to saying that $\mathfrak{M}$ is homeomorphic to a compact Riemann surface with finitely many disks and points deleted. We shall make use of the following version of Riemann-Hurwitz formula.

Lemma 1.3.5. *If $\mathfrak{N}$ is a finite Riemann surface and if f is a finite map from $\mathfrak{M}$ to $\mathfrak{N}$ with $\mathfrak{d}_f$ a finite set then $\mathfrak{M}$ is also finite and*

$$\deg \mathfrak{D}_f = \deg f \cdot \chi(\mathfrak{N}) - \chi(\mathfrak{M}). \tag{1.2}$$

We shall prove Lemma 1.3.5 by Schreier's Index Formula applied to fundamental groups, which might be new and gives an example in explaining that fundamental groups detect morphisms.

Theorem 1.3.6 (Schreier's Index Formula)**.** *If $\mathscr{G}$ is a subgroup of $\mathscr{F}_r$ with index i then $\mathscr{G}$ is a free group with rank*

$$i(r-1)+1.$$

Proof of Lemma 1.3.5. Choose a non-empty subset $\mathfrak{s}$ of $\mathfrak{N}$ containing $\mathfrak{d}_f$ and let n be its cardinality. We shall calculate $\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{s}), p)$ in two different ways. By elementary topology $\pi_1(\mathfrak{N}\backslash\mathfrak{s}, q)$ is a free group of rank $n+1-\chi(\mathfrak{N})$ and from Theorem 1.3.1 we know that $\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{s}), p)$ is a subgroup of $\pi_1(\mathfrak{N}\backslash\mathfrak{s}, q)$ with index $\deg f$. Schreier's index formula implies that

$$\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{s}), p) \cong \mathscr{F}_a \tag{1.3}$$

where $a = \deg f(n-\chi(\mathfrak{N}))+1$. The group homomorphism i_* from $\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{s}))$ to $\pi_1(\mathfrak{M})$ obtained from the inclusion map i is surjective, and so $\mathfrak{M}$ is also finite. Elementary topology again gives

$$\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{s}), p) \cong \mathscr{F}_b \tag{1.4}$$

where $b = n\deg f - \deg \mathfrak{D}_f + 1 - \chi(\mathfrak{M})$. Using the main theorem of finitely generated abelian groups we see that $\mathscr{F}_s \cong \mathscr{F}_t$ implies that $s = t$. Comparing (1.3) and (1.4) leads to $a = b$ which gives the desired identity. □

Riemann defines his surface as a fibration over the complex plane or over the Riemann sphere, and Grothendieck regards an algebraic variety defined over k as a fibration over Spec$k$. In Riemann's story a finite analytic map can be recovered from the action of fundamental groups, as seen in for instance

Theorem 1.3.1. Grothendieck's story is contained in his birational anabelian conjectures, and as an example we refer to [90] and [45] for the following

Theorem 1.3.7. *Let $\mathbb{K}$ be a finite extension of $\mathbb{Q}_p$, X and Y smooth geometrically irreducible projective curves over $\mathbb{K}$, $\pi_1(X)$ and $\pi_1(Y)$ their algebraic fundamental groups and*

$$\alpha\colon \pi_1(X) \to \pi_1(Y)$$

an open homomorphism of extensions of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})$. *Then α is induced from a unique dominant morphism of curves $X \to Y$.*

1.4 Transcendental maps

The idea of constructing a universal covering surface originated with Schwarz. His idea was adopted by Klein and Poincaré and then led to the theory of uniformization. Comparing to finite maps which are tied up with subgroups of finite index of the fundamental group, the theory of uniformization is an issue involving describing the full fundamental group. Finite maps are somehow algebraic objects, and the uniformization produces instead transcendental maps. Though in literature transcendence was only implicitly attached to universal covering spaces, it should not be surprising that the theory of transcendence will be closely related to the uniformization theory.

As pointed out by Kollár in [77], one hopes that there are many interesting connections between the meromorphic function theory of a variety and the holomorphic function theory of its universal cover. Indeed, Hermite and Lindemann's breakthrough work on the transcendence of values of the exponential function is basically related to the uniformization of $\mathbb{G}_m(\mathbb{C})$. Although their original proof does not really reflect Kollár's viewpoint, the subsequent achievements in transcendence rely heavily on constructing suitable auxiliary polynomials of automorphic functions on the complex Euclidean spaces. This technique was extensively developed by Siegel, Gelfond, Schneider and Baker. As an example we recall Lang's version of a very general theorem of Schneider which covers almost all known transcendence results up to his time.

Theorem 1.4.1 (Schneider's Criterion). *Let $\mathbb{K} \subset \mathbb{C}$ be a number field and let $f_1, \cdots, f_N$ be meromorphic functions of order $\leq \rho$. Assume that the field $\mathbb{K}(f) = \mathbb{K}(f_1, \cdots, f_N)$ has transcendence degree ≥ 2 over K and that the derivative $\nabla = d/dt$ maps the ring $\mathbb{K}[f] = \mathbb{K}[f_1, \cdots, f_N]$ into itself. If $\mathcal{S}$ is a set of points in $\mathbb{C}$ such that $f_i(w) \in \mathbb{K}$ for all $w \in \mathcal{S}$ then $|\mathcal{S}| \leq 20\rho[\mathbb{K} : \mathbb{Q}]$.*

From the point of view of uniformization, the complex Euclidean spaces $\mathbb{C}^n$ are the higher dimensional generalizations of the complex plane. In his beautiful paper [19], Bombieri successfully obtained a criterion of Schneider's type to this general context which applies to analytic subgroups of group varieties [22]. Restricted to the exponential maps of algebraic groups the most general result is due to Wüstholz' analytic subgroup theorem which covers most of the transcendental results up to date. Let G be a commutative algebraic group defined over $\mathbb{Q}$ with Lie algebra $\mathfrak{g}$. Further, let $\mathfrak{b}$ be a subalgebra of $\mathfrak{g}$ and put $B = \exp_G(\mathfrak{b} \bigotimes_{\mathbb{Q}} \mathbb{C})$. Wüstholz's theorem [136] now reads as follows

Theorem 1.4.2 (Analytic subgroup theorem). *There exists a point $z \neq 0$ in $\mathfrak{b}(\overline{\mathbb{Q}})$ such that $\exp_G(z) \in G(\overline{\mathbb{Q}})$ if and only if there exists a non-trivial algebraic subgroup $H \subseteq G$ defined over $\overline{\mathbb{Q}}$ such that $H \subseteq B$.*

This theorem doesn't cover the full version of Bombieri's work, and even in the case of the exponential maps of algebraic groups there remains open problems such as Schanuel's conjecture as well as its generalized version formulated in [137].

In general we expect that if a universal covering maps an algebraic object to another algebraic object then it admits special reasons. Therefore a large part of transcendence theory amounts to laying out suitable theories on the universal covering space. There even exists no such one, as on the complex plane Schneider's criterion or on the complex Euclidian spaces Bombieri and Wüstholz's theories, on the unit disk. Nevertheless we have several interesting examples which support the existence of such a theory. Classical results of Kronecker-Weber and of Schneider tell us that z and $j(z)$ are both algebraic if and only if z is quadratic. Wüstholz verifies that a period on curves defined over $\overline{\mathbb{Q}}$ is either 0 or transcendental. Lang's conjecture in [80], partially proved by Wolfart-Wüstholz [134], predicts that if $\pi: \mathbb{E} \to X$ is the universal covering map of an algebraic curve X defined over $\overline{\mathbb{Q}}$ and if $\pi(0)$ lies in $X(\overline{\mathbb{Q}})$ then $\pi'(0)$

is transcendental. Those results, inherently attached to the unit disk, are all proved by Schneider's criterion or by Wüstholz's analytic subgroup theorem.

In the theory of uniformization the higher dimensional model manifolds which generalize the unit disk are bounded symmetric domains. In this context Schneider's theorem on the singular moduli was generalized by Shiga-Wolfart [120] and Tretkoff [34] to Shimura varieties of abelian type. The proof again relies heavily on the analytic subgroup theorem. More recently Ullmo and Yafaev have also obtained related interesting results in [128].

We would like to point out that Borel's density theorem and Ratner's theory agree with the spirit of our discussions and might be of transcendental interest.

Now we switch to general transcendental maps rather than the universal covering. In this case we have to give up fundamental groups, and instead to adopt the theory of value distribution as the major technique.

The value distribution theory dates back to at least the fundamental theorem of algebra which tells us that a complex polynomial assume every complex value equally many times. This remains the case for finite maps between Riemann surfaces as shown by Radó [102] and for flat finite morphisms in the higher dimensional cases.

General holomorphic or meromorphic functions on the complex plane are much more complicated than polynomials, for instance the exponential map exp never assumes 0. The situation is not too bad as Picard's little theorem concludes that a non-constant holomorphic function assumes the whole complex plane except for at most one point. It might be interesting to notice that the original proof of such a statement for general functions relies on the uniformization associated with $\Gamma(2)$. Picard's little theorem together with his big one give a qualitative transcendental analogue of the fundamental theorem of algebra.

E. Borel realized that growth is essential to an understanding of Picard's theorem, and this motivated Nevanlinna to establish his theory which gives a quantitative version of the fundamental theorem of algebra. Although the theory has been generalized to many other spaces in the remaining of this section we will stick to the case of holomorphic or meromorphic functions.

For real α we denote by $\log^+ \alpha$ the maximum of the numbers $\log \alpha$ and 0 and $n(r, \infty)$ the number of poles of f in the closed disk $|z| \leq r$. Then the standard functions in Nevanlinna theory are given by

$$\begin{aligned}
M(r, f) &= \max_{|z|=r} |f(z)| \\
N(r, f) = N(r, \infty) &= \int_0^r \frac{n(t, \infty) - n(0, \infty)}{t} dt + n(0, \infty) \log r \\
m(r, f) = m(r, \infty) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\varphi})| d\varphi \\
N(r, a) &= N(r, \frac{1}{f - a}) \\
m(r, a) &= m(r, \frac{1}{f - a}) \\
T(r, f) &= m(r, \infty) + N(r, \infty).
\end{aligned}$$

We will write them simply $M(r), m(r), N(r), T(r)$ when no confusion can arise, and we call $M(r, f)$ *maximum modulus functions* as well as $T(r, f)$ *Nevanlinna characteristic functions*. Write D_a for the divisor $f^{-1}(a)$ then $N(r, a)$ characterizes the growth of D_a. For meromorphic functions we have in Nevanlinna theory the following principal elements.

‡ *Nevanlinna inequality*: the growth of D_a does not exceed the the growth of f.

$$N(r, a) \leq T(r, f) + O(1).$$

‡ *Crofton Formula*: the growth of f equals the average of growthes of D_a.

$$T(r, f) = \int_{a \in \mathbb{P}^1} N(r, a)\, da.$$

‡ *Hadamard's solution to Weierstrass Problem*: Given a divisor D of $\mathbb{C}$ there exist a holomorphic function f such that $D_0 = D$ and that

$$N(r, 0) = T(r, f) + O(1).$$

‡ *Defect relation*: the set of a with small growth of D_a is sparse.

$$\sum_{a \in \mathbb{P}} \liminf_{r \to \infty} \frac{m(r, a)}{T(r)} \leq 2.$$

The first three elements there belong to the context of the First Main Theorem, and the last one is contained in the Second Main Theorem which is the deepest part of Nevanlinna theory. The analogy between the Second Main Theorem and Diophantine approximation, as discovered by Osgood, Lang and Vojta, is well-known and remains a subject of intense interest. Here we would like to point out the connection between the First Main theorem and the transcendence theory, for which we sketch the proof of a theorem of Schneider on the transcendence of values of Weierstrass $\wp$-function [115]. Crofton's Formula implies that elliptic functions are of finite order which guarantees the moderate growth of auxiliary polynomials, Hadamard's theorem makes it possible to pass from meromorphic functions to entire functions without the increase of growth, and Nevanlinna's inequality underlies the analysis of auxiliary polynomials.

1.5 Endomorphisms of bounded symmetric domains

There is a duality between Hermitian symmetric manifolds X of noncompact type and those of the compact type $X^\vee$, as deduced from the Borel embedding theorem, in such way that biholomorphisms of X extend to to bilolomorphisms of $X^\vee$. This fact fits into the GAGA principle and remains the case for finite endomorphisms of polydisks according to the work of Fatou, Remmert, Stein and Rischel.

The unit disk carries the same topological structure as the complex plane, but their complex structures are different and this leads to quite different function theories. Finite endomorphisms of the complex plane are non-constant polynomials, and finite endomorphisms of the unit disk were characterized by Fatou. In a first step he showed in [48] that

Theorem 1.5.1 (Fatou). *If f is a finite endomorphism of the unit disk then it extends to a rational function and it satisfies the following functional equation:*

$$f\left(\frac{1}{\bar{z}}\right) = \overline{\left(\frac{1}{f(z)}\right)}. \tag{1.5}$$

Three years later he gave in [49] a more precise version of the theorem involving a product formula.

Theorem 1.5.2 (Fatou)**.** *If f is a finite endomorphism of the unit disk then*

$$f(z) = \varrho \prod_{i=1}^{n} \frac{z - a_i}{1 - \overline{a_i}\, z}$$

with ϱ an element in $\mathbb{T}$, n a positive integer and a_i points in $\mathbb{E}$.

These products were first introduced by Blaschke in [16] and they are of great importance in transcendence as employed by Baker-Wüstholz in [7] and[8, p.141]. We shall regard a finite Blaschke product as an endomorphism of the unit disk $\mathbb{E}$, the unit circle $\mathbb{T}$, the Riemann sphere $\mathbb{P}^1$ or the mirror image of the unit disk $\overline{\mathbb{E}}^c$, depending on corresponding contexts. By a generalized Blaschke product we mean a rational function of the above form but without the assumption that a_i are contained in $\mathbb{E}$.

The theory of complex dynamics where mappings are iterated is very different in the case of polynomials from the case of finite Blaschke products. The case of Blaschke products goes back at least as far as Fatou [47], where the whole third chapter was devoted to the problem of classifying finite Blaschke products by their dynamics. This subject was later taken up by many other authors. Herman gave in [67] the first example of a special type of Fatou set, which are now called Herman ring, by studying generalized Blaschke products. In the famous dictionary between Kleinian groups and rational maps as promoted by Sullivan, finitely-generated Fuchsian groups goes to finite Blaschke products. For recent research on illustrating this viewpoint and for research on the restricted circle dynamics, we refers the reader to a series of papers by McMullen for instance [92], [93] and [94].

Now we come to the case of polydisks. We start by recalling Rischel's result in [107] which generalizes Remmert-Stein's original work in [104].

Theorem 1.5.3 (Remmert-Stein-Rischel)**.** *If $\Omega_1, \cdots, \Omega_n, D_1, \cdots, D_n$ are bounded domains in $\mathbb{C}$ and if f is a proper map from $\Omega_1 \times \cdots \times \Omega_n$ to $D_1 \times \cdots \times D_n$ then there exists a permutation σ in S_n and proper maps f_j from Ω_{j^σ} to D_j such that*

$$f(z_1, \cdots, z_n) = (f_1(z_{1^\sigma}), \cdots, f_n(z_{n^\sigma})).$$

Fatou's theorem together with Rischel's version [107] of Remmert-Stein's theorem [104] are able to classify finite endomorphisms of polydisks.

Theorem 1.5.4 (Fatou-Remmert-Stein-Rischel). *If f is a finite map from $\mathbb{E}^n$ to $\mathbb{E}^n$ then there exists a permutation σ in S_n and finite Blaschke products f_j such that*

$$f(z_1, \cdots, z_n) = (f_1(z_{1^\sigma}), \cdots, f_n(z_{n^\sigma})).$$

This theorem implies the famous theorem of Cartan's famous on automorphism groups of polydisks.

For any point $\mathfrak{p}$ in the unit disk we shall denote by $\iota_{\mathfrak{p}}$ the linear Blaschke product given by $z \mapsto (z+\mathfrak{p})/(1+\overline{\mathfrak{p}}z)$. By a *totally ramified* Blaschke product we mean a rational function of the form $\epsilon \circ z^n \circ \varepsilon$ where ϵ and ε are elements in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$.

1.6 Änderung à la Hurwitz

Für meine Unterauchungen war es wesentlich, die Riemann'sche Fläche als ein rein topologisch erklärtes Gebilde, also ganz unabhängig von den auf ihr verlaufenden Functionen, aufzufassen.

Hurwitz ([72] 1891)

There are many different points of view on the concept of Riemann surface including the altas principle à la Klein, Weierstrass' analytic configurations and the sheaf principle introduced by Weyl, among which we shall adopt Riemann's original covering principle. He introduced in [105] and [106] his surface in the form of a topological multi-sheeted surface spread out *a priori* over the complex plane (or the Riemann sphere). This multi-sheeted surface is then reflected by its ramification data at the underlying complex plane. In other words, as remarked by Klein in [76, p.545], sie mögen selbst mit mehreren Blaättern überdeckt sein, die unter sich durch Verzweigungspunkte, beziehungsweise Verzweigungsschnitte zusammenhängen. Hurwitz studied in great details the "Änderung" of Riemann surfaces according to the "Bewegung" of ramified points in his fundamental paper [72], and actually the consideration of Bewegung had already appeared in Riemann's famous count [106]. Here Hurwitz's Änderung is used in such a unusual way that we will move the complex structure of the underlying space rather than the location of ramified points. The object we are looking at is a finite Blaschke product f which makes the unit disk a topological multi-sheeted surface X spreads over an underlying unit disk. In other words, f is regarded as an analytic representative of the topological object X. We may give the underling unit disk a new complex structure which makes it to be the complex plane, and then X is a topological surface spreads over $\mathbb{C}$ but not the unit disk any more. This gives X a new analytic representative which turns out to be a polynomial. By such an operation we have successfully passed from a finite Blaschke product to a polynomial, and conversely we may also pass from a polynomial to a finite Blaschke product in a similar way.

To be in accord with the modern language we shall make use of the following version of Riemann's covering principle as given in [2, p.119-120]. Here a Riemann surface is a pair $(\mathfrak{R}, \mathfrak{r})$ with $\mathfrak{R}$ a connected Hausdorff space and $\mathfrak{r}$ a complex structure, see [2, p.144]. However we shall simply write $\mathbb{E}$ and $\mathbb{C}$ when $\mathfrak{r}$ is canonical.

Theorem 1.6.1 (Riemann's covering principle)**.** *If $f\colon \mathfrak{R}_1 \to \mathfrak{R}_2$ is a covering surface and if $\mathfrak{r}_2$ is a complex structure on $\mathfrak{R}_2$. Then there exists a unique complex structure $\mathfrak{r}_1$ on $\mathfrak{R}_1$ such that $f\colon (\mathfrak{R}_1, \mathfrak{r}_1) \to (\mathfrak{R}_2, \mathfrak{r}_2)$ is holomorphic.*

Let $f\colon \mathbb{E} \to \mathbb{E}$ be a finite map and $i_0\colon \mathbb{E} \to \mathbb{C}$ a homeomorphism. The canonical complex structure on $\mathbb{C}$ induces a new complex structure $\mathfrak{r}_0$ on $\mathbb{E}$ and we obtain a new Riemann surface $(\mathbb{E}, \mathfrak{r}_0)$. By Theorem 1.6.1 applied to $f\colon \mathbb{E} \to (\mathbb{E}, \mathfrak{r}_0)$ there exists a Riemann surface $(\mathbb{E}, \mathfrak{r}_1)$ such that $f\colon (\mathbb{E}, \mathfrak{r}_1) \to (\mathbb{E}, \mathfrak{r}_0)$ is holomorphic. Consequently there exists a holomorphic map $(i_1, i_0)_* f$ from $(\mathbb{E}, \mathfrak{r}_1)$ to $\mathbb{C}$ which makes the following diagram

$$\begin{tikzcd} \mathbb{E} \arrow[r, "f"] \arrow[d, "i_1"] & \mathbb{E} \arrow[d, "i_0"] \\ (\mathbb{E}, \mathfrak{r}_1) \arrow[r, "(i_1,i_0)_* f"] & \mathbb{C} \end{tikzcd}$$

commutative, where i_1 is the topological identity map. We shall call i_1 a *f-lifting* of i_0 and $(i_1, i_0)_* f$ a (i_1, i_0)-*descent* of f.

The uniqueness part of Theorem 1.6.1 shows that if i_1, i_1' are two f-liftings of i_0 then there exists a holomorphic isomorphism σ between $(\mathbb{E}, \mathfrak{r}_1)$ and $(\mathbb{E}, \mathfrak{r}_1')$ such that $\sigma \circ i_1 = i_1'$. The classical uniformization theorem for simply connected Riemann surfaces together with Lemma 1.3.2 shows that $(\mathbb{E}, \mathfrak{r}_1)$ and $(\mathbb{E}, \mathfrak{r}_1')$ must be biholomorphic to $\mathbb{C}$. To sum up we may state the following

Corollary 1.6.2. *Let f be a finite map from $\mathbb{E}$ to $\mathbb{E}$, i_0 a homeomorphism from $\mathbb{E}$ to $\mathbb{C}$ and i_1, i_1' homeomorphisms from $\mathbb{E}$ to the complex plane f-liftings of i_0. There exists σ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ such that $i_1' = \sigma \circ i_1$ and $(i_1', i_0)_* f \circ \sigma = (i_1, i_0)_* f$.*

This can be illustrated by the following diagram

$$\begin{tikzcd}
 & \mathbb{E} \arrow[r, "f"] \arrow[dl, "i_1'"'] \arrow[d, "i_1"] & \mathbb{E} \arrow[d, "i_0"] \\
\mathbb{C} \arrow[r, "z \overset{\sigma}{\mapsto} az+b"] & \mathbb{C} \arrow[r, "(i_1, i_0)_* f"] & \mathbb{C}
\end{tikzcd}$$

Notice that $(i_1, i_0)_* f$ is a finite map from $\mathbb{C}$ to $\mathbb{C}$ and therefore are given by polynomials. The next proposition shows that our construction of liftings is functorial.

Proposition 1.6.3. *Let f_1, f_2 be finite maps from $\mathbb{E}$ to $\mathbb{E}$, i_0 a homeomorphism from $\mathbb{E}$ to $\mathbb{C}$ and $f = f_1 \circ f_2$. If i_1 is a f_1-lifting of i_0 and if i_2 is a f_2-lifting of i_1 then i_2 is a f-lifting of i_0 and $(i_2, i_0)_* f = (i_1, i_0)_* f_1 \circ (i_2, i_1)_* f_2$ is a composition of polynomials.*

$$\begin{tikzcd}
\mathbb{E} \arrow[r, "f_2"] \arrow[d, "i_2"] & \mathbb{E} \arrow[r, "f_1"] \arrow[d, "i_1"] & \mathbb{E} \arrow[d, "i_0"] \\
\mathbb{C} \arrow[r, "(i_2, i_1)_* f_2"] & \mathbb{C} \arrow[r, "(i_1, i_0)_* f_1"] & \mathbb{C}
\end{tikzcd}$$

Proof. We conclude from Theorem 1.6.1 that both $i_0 \circ f_1 \circ i_1^{-1}$ and $i_1 \circ f_2 \circ i_2^{-1}$ are holomorphic. Again by Theorem 1.6.1 it suffices to show $i_0 \circ f \circ i_2^{-1}$ is holomorphic. This follows from $i_0 \circ f \circ i_2^{-1} = (i_0 \circ f_1 \circ i_1^{-1}) \circ (i_1 \circ f_2 \circ i_2^{-1})$. □

Similarly we have

Proposition 1.6.4. *Let f_1, f_2 be finite maps from $\mathbb{C}$ to $\mathbb{C}$, k_0 a homeomorphism from $\mathbb{C}$ to $\mathbb{E}$ and $f = f_1 \circ f_2$. If k_1 is a f_1-lifting of k_0 and if k_2 is a f_2-lifting of k_1 then k_2 is also a f-lifting of k_0 and $(k_2, k_0)_* f =$*

$(k_1, k_0)_* f_1 \circ (k_2, k_1)_* f_2$ *is a composition of two finite Blaschke products.*

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f_2"] \arrow[d, "k_2"] & \mathbb{C} \arrow[r, "f_1"] \arrow[d, "k_1"] & \mathbb{C} \arrow[d, "k_0"] \\
\mathbb{E} \arrow[r, "(k_2,k_1)_* f_2"] & \mathbb{E} \arrow[r, "(k_1,k_0)_* f_1"] & \mathbb{E}
\end{tikzcd}$$

Now we derive as a corollary of Proposition 1.6.3 the following

Corollary 1.6.5. *Let* $f, \alpha_1, \alpha_2, \beta_1, \beta_2$ *be finite maps from* $\mathbb{E}$ *to* $\mathbb{E}$, i_0 *a homeomorphism from* $\mathbb{E}$ *to* $\mathbb{C}$ *and* $f = \alpha_2 \circ \alpha_1 = \beta_2 \circ \beta_1$. *If* i_1 *is a* α_2*-lifting of* i_0, i_2 *is a* α_1*-lifting of* i_1 *and* j_1 *is a* β_2*-lifting of* i_0 *then* i_2 *is also a* β_1*-lifting of* j_1 *and we have the following decompositions of polynomials*

$$(i_2, i_0)_* f = (i_1, i_0)_* \alpha_2 \circ (i_2, i_1)_* \alpha_1 = (j_1, i_0)_* \beta_2 \circ (i_2, j_1)_* \beta_1.$$

Proof. The argument similar to that in the proof of Proposition 1.6.3 applies. □

Compared with finite Blaschke products polynomials are generally easier to be handled with since much more algebraic techniques (such as the place at infinity) are available.

1.7 Weierstrassian elliptic functions

The theory of elliptic function finds its root in the study of elliptic integrals. Let g_2 and g_3 be complex numbers with $g_2^3 - 27g_3^2 \neq 0$ and consider the integral of the Weierstrass form $u = \int_y^\infty ds/\sqrt{4s^3 - g_2 s - g_3}$. Here u is a multi-valued function of the lower limit y, but its inverse y is single-valued. One can even show that y is elliptic, and $y = \wp(u)$ is called *Weierstrassian elliptic function.* Notice that $y = \wp(u)$ is independent of the choice of the branch of the Weierstrass form because $\wp$ is an even function. We collect the following elementary properties of $\wp$.

$$\begin{aligned}
\wp(z) &= \wp(-z), && (1.6)\\
\wp'^2 &= 4\wp^3 - g_2\wp - g_3, && (1.7)\\
2\wp'' &= 12\wp^2 - g_2, && (1.8)\\
\wp(u+v) &= -\wp(u) - \wp(v) + \frac{1}{4}\left(\frac{\wp'(u)-\wp'(v)}{\wp(u)-\wp(v)}\right)^2. && (1.9)
\end{aligned}$$

The addition formula (1.9) leads to

$$\begin{aligned}
\wp(u+v)+\wp(u-v) &= \frac{(\wp(u)+\wp(v))\cdot(2\wp(u)\cdot\wp(v)-g_2/2)-g_3}{(\wp(u)-\wp(v))^2}, && (1.10)\\
\wp(u+v)\cdot\wp(u-v) &= \frac{(\wp(u)\cdot\wp(v)+g_2/4)^2+g_3(\wp(u)+\wp(v))}{(\wp(u)-\wp(v))^2}. && (1.11)
\end{aligned}$$

If Λ is the lattice of periods of $\wp$ then we have the following Mittag-Leffler expansion:

$$\wp(z) = \wp(z;\Lambda) := \frac{1}{z^2} + \sum_{\omega\in\Lambda,\ \omega\neq 0}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right).$$

Let ω_1, ω_2 be a basis of Λ, $\omega_3 = \omega_1 + \omega_2$ and $e_i = \wp\left(\frac{\omega_i}{2}\right)$. We recall

$$\begin{aligned}
4\wp^3 - g_2\wp - g_3 &= 4(\wp - e_1)(\wp - e_2)(\wp - e_3), && (1.12)\\
e_1 + e_2 + e_3 &= 0. && (1.13)
\end{aligned}$$

We have defined the Weierstrass function by elliptic integrals, and obtained expressions for them as infinite sums. This procedure of course can be reversed, and then we fall into the so called Mittag-Leffler approach. Given a lattice Λ of $\mathbb{C}$ one may attach it with an elliptic function $\wp(z;\Lambda)$, and the numbers g_2 and g_3 can be uniquely expressed in terms of the lattice.

Following usual notations given $\Lambda_{\omega_1,\omega_2} = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ we write $\wp(z;\omega_1,\omega_2) = \wp(z;\Lambda_{\omega_1,\omega_2})$ and moreover given $\tau \in \mathbb{H}$ we write $\wp(z;\tau) = \wp(z;1,\tau)$. Furthermore we shall write E_{ω_1,ω_2} and E_τ for the elliptic curves $\mathbb{C}/\Lambda_{\omega_1,\omega_2}$ and $E_{1,\tau}$ respectively.

1.8 Jacobian elliptic functions

If we consider the integral of a Jacobi form $w = \int_0^x dt/\sqrt{(1-t^2)(1-k^2t^2)}$ with *elliptic modulus* k in $\mathbb{P}^1\backslash\{0,-1,1,\infty\}$ rather than of a Weierstrass form then the theory of elliptic functions is quite different. Here we choose the branch such that $\sqrt{(1-t^2)(1-k^2t^2)} = 1$ at $t = 0$. The integral w is a multi-valued function of x, but its inverse x is single-valued and elliptic and this function $x = \mathrm{sn}(w, k)$ is called *Jacobian sin function*.

It is often convenient to develop the theory by means of theta functions. Given a point τ in $\mathbb{H}$ we write $q = e^{\pi i\tau}$ where the branch of $q^{1/4}$ is the one for which $q^{1/4}$ assumes $e^{-\pi/4}$ at $\tau = i$ and introduce the four theta functions following the notation of Tannery-Molk:

$$\begin{aligned}
\vartheta_1(u,\tau) &= \sum_{n=-\infty}^{\infty} i^{2n-1} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)ui}, \\
\vartheta_2(u,\tau) &= \sum_{n=-\infty}^{\infty} q^{\left(n+\frac{1}{2}\right)^2} e^{(2n+1)ui}, \\
\vartheta_3(u,\tau) &= \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nui}, \\
\vartheta_0(u,\tau) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nui}.
\end{aligned}$$

We shall write simply $\vartheta_1(v)$ instead of $\vartheta_1(v,\tau)$ when no ambiguity arises, and this convention applies to many other functions. We introduce following special functions

$$\begin{aligned}
\omega_1(\tau) &= \pi\,\vartheta_3^2(0,\tau),\\
\omega_2(\tau) &= \pi\tau\,\vartheta_3^2(0,\tau),\\
k\,(\tau) &= \frac{\vartheta_2^2(0,\tau)}{\vartheta_3^2(0,\tau)},\\
k^{\frac{1}{2}}(\tau) &= \frac{\vartheta_2\,(0,\tau)}{\vartheta_3\,(0,\tau)},\\
k'\,(\tau) &= \frac{\vartheta_0^2(0,\tau)}{\vartheta_3^2(0,\tau)},\\
k'^{\frac{1}{2}}(\tau) &= \frac{\vartheta_0\,(0,\tau)}{\vartheta_3\,(0,\tau)},\\
\lambda\,(\tau) &= \frac{\vartheta_2^4(0,\tau)}{\vartheta_3^4(0,\tau)}.
\end{aligned}$$

If τ is purely imaginary then $k^{\frac{1}{2}}$ and ω_1 are positive real numbers. We will write $\omega_{i,\tau}$ for $\omega_i(\tau)$. As a hypergeometric function on the $\lambda = k^2$ domain we have $\omega_{1,\tau} = 2\mathcal{K}(\lambda)$ where $\mathcal{K}$ is Legendre's complete elliptic integral of the first kind $\mathcal{K}(\lambda) = \int_0^1 dt/\sqrt{(1-t^2)(1-\lambda t^2)}$.

Following Jacobi [74, p.512] his elliptic functions can be defined by

$$\begin{aligned}
\operatorname{sn} u &= \frac{1}{\sqrt{k}} \cdot \frac{\vartheta_1(u/\omega_1)}{\vartheta_0(u/\omega_1)},\\
\operatorname{cn} u &= \frac{\sqrt{k'}}{\sqrt{k}} \cdot \frac{\vartheta_2(u/\omega_1)}{\vartheta_0(u/\omega_1)},\\
\operatorname{dn} u &= \sqrt{k'} \cdot \frac{\vartheta_3(u/\omega_1)}{\vartheta_0(u/\omega_1)}.
\end{aligned}$$

The elliptic function sn takes $2\omega_1$ and ω_2 as a pair of primitive periods and satisfies

$$\operatorname{sn}\left(\frac{\pm\,\omega_1}{2},\,\tau\right) = \pm\,1 \tag{1.14}$$

as well as

$$\begin{aligned}
\operatorname{sn}(\omega_1 - u) &= \operatorname{sn} u, &(1.15)\\
\operatorname{dn}^2 u - k^2 \operatorname{cn}^2 u &= k'^2. &(1.16)
\end{aligned}$$

The Jacobian elliptic functions can be expressed as infinite products ([74, p.145]):

$$\vartheta_0(u) = c \prod_{n=1}^{\infty} \left(1 - q^{2n-1} e^{2\pi i u}\right) \left(1 - q^{2n-1} e^{-2\pi i u}\right), \tag{1.17}$$

$$\vartheta_1(u) = 2\, c\, q^{\frac{1}{4}} \,\sin \pi u \prod_{n=1}^{\infty} \left(1 - q^{2n} e^{2\pi i u}\right) \left(1 - q^{2n} e^{-2\pi i u}\right)$$

where $c = \prod_{n\geq 1}(1 - q^{2n})$, and consequently

$$\operatorname{sn} u = 2k^{-\frac{1}{2}} q^{\frac{1}{4}} \,\sin \pi u/\omega_1 \prod_{n=1}^{\infty} \frac{(1 - q^{2n} e^{2\pi i u/\omega_1})(1 - q^{2n} e^{-2\pi i u/\omega_1})}{(1 - q^{2n-1} e^{2\pi i u/\omega_1})(1 - q^{2n-1} e^{-2\pi i u/\omega_1})}.$$

From the addition formulae

$$\operatorname{sn}(u+v) = \frac{\operatorname{sn} u \operatorname{cn} v \operatorname{dn} v + \operatorname{sn} v \operatorname{cn} u \operatorname{dn} u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

$$\operatorname{cn}(u+v) = \frac{\operatorname{cn} u \operatorname{cn} v - \operatorname{dn} u \operatorname{dn} v \operatorname{sn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v},$$

$$\operatorname{dn}(u+v) = \frac{\operatorname{dn} u \operatorname{dn} v - k^2 \operatorname{cn} u \operatorname{cn} v \operatorname{sn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}$$

we deduce that (see for instance [74, p.475] and [74, p.468])

$$\operatorname{cn}(u+v)\operatorname{cn}(u-v) = \frac{\operatorname{cn}^2 u - \operatorname{sn}^2 v \operatorname{dn}^2 u}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{1.18}$$

$$\operatorname{cn}(u+v)\operatorname{dn}(u-v) = \frac{\operatorname{cn} u \operatorname{cn} v \operatorname{dn} u \operatorname{dn} v - k'^2 \operatorname{sn} u \operatorname{sn} v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}, \tag{1.19}$$

$$\operatorname{dn}(u+v)\operatorname{dn}(u-v) = \frac{\operatorname{cn}^2 v \operatorname{dn}^2 u + k'^2 \operatorname{sn}^2 v}{1 - k^2 \operatorname{sn}^2 u \operatorname{sn}^2 v}. \tag{1.20}$$

In 1882 Glaisher introduced nine other elliptic functions in [60, p.86] among which $\operatorname{cd} u := \operatorname{cn} u/\operatorname{dn} u$ will be of particular importance in the sequel. By (1.16) we have

$$k^2 \operatorname{cd}^2 u + k'^2/\operatorname{dn}^2 u = 1 \tag{1.21}$$

and by the addition formula of sn we have

$$\operatorname{cd} u = \operatorname{sn}(u + \omega_1/2).$$

Both cn and dn are even functions and as a result the same is true for cd

$$\operatorname{cd} u = \operatorname{cd}(-u). \tag{1.22}$$

Which also takes $2\omega_1$ and ω_2 as a pair of primitive periods. In contrast to the Weierstrass $\wp$ function the Jacobian function cd has two remarkable identities which involve half-periods:

$$\operatorname{cd}(u+\omega_1) = -\operatorname{cd} u, \tag{1.23}$$

$$\operatorname{cd}\Big(u+\frac{\omega_2}{2}\Big) = \frac{1}{k\operatorname{cd} u}. \tag{1.24}$$

The equality (1.23) suggests that it is more reasonable to regard cd as a counterpart of trigonometric cos function among elliptic functions because they are both even and take the half-period as a quasi-period, i.e. $\cos(z+\frac{2\pi}{2}) = -\cos(z)$. The equality (1.24) is very similar to Schwarz reflection principle, which provides an intuitive reason for the connection between Jacobi's products and Blaschke products.

By addition formulas of cn and dn we have

$$\operatorname{cd}(u+v) = \frac{\operatorname{cd} u\operatorname{cd} v - \operatorname{sn} u\operatorname{sn} v}{1-k^2\operatorname{cd} u\operatorname{cd} v\operatorname{sn} u\operatorname{sn} v} \tag{1.25}$$

and by (1.21), (1.19) and (1.20) we have

$$\begin{aligned}
\operatorname{cd}(u{+}v)+\operatorname{cd}(u{-}v) &= \frac{\operatorname{cn}(u{+}v)\operatorname{dn}(u{-}v)+\operatorname{cn}(u{-}v)\operatorname{dn}(u{+}v)}{\operatorname{dn}(u{+}v)\operatorname{dn}(u{-}v)}\\
&= \frac{2\operatorname{cn} u\operatorname{cn} v\operatorname{dn} u\operatorname{dn} v}{\operatorname{cn}^2 v\operatorname{dn}^2 u+k'^2\operatorname{sn}^2 v}\\
&= \frac{2\operatorname{cd} u\operatorname{cn} v\operatorname{dn} v}{\operatorname{cn}^2 v+\operatorname{sn}^2 v\,(1-k^2\operatorname{cd}^2 u)}\\
&= \frac{2\operatorname{cn} v\operatorname{dn} v\operatorname{cd} u}{1-k^2\operatorname{sn}^2 v\operatorname{cd}^2 u}.
\end{aligned} \tag{1.26}$$

Moreover we deduce from (1.18), (1.20) and (1.21) that

$$\begin{aligned}
\operatorname{cd}(u+v)\operatorname{cd}(u-v) &= \frac{\operatorname{cn}(u+v)\operatorname{cn}(u-v)}{\operatorname{dn}(u+v)\operatorname{dn}(u-v)}\\
&= \frac{\operatorname{cn}^2 u-\operatorname{sn}^2 v\operatorname{dn}^2 u}{\operatorname{cn}^2 v\operatorname{dn}^2 u+k'^2\operatorname{sn}^2 v}\\
&= \frac{\operatorname{cd}^2 u-\operatorname{sn}^2 v}{\operatorname{cn}^2 v+\operatorname{sn}^2 v\,(1-k^2\operatorname{cd}^2 u)}\\
&= \frac{\operatorname{cd}^2 u-\operatorname{sn}^2 v}{1-k^2\operatorname{sn}^2 v\operatorname{cd}^2 u}.
\end{aligned} \tag{1.27}$$

If X is an abelian variety then the quotient of X by its inherent involution will be denoted by K_X, the *Kummer variety* of X. The canonical projection $\pi\colon X \to K_X$ will be called the *Kummer map*. Elliptic functions $\wp$ and cd are even and of order 2. This implies that for any τ chosen from the upper half plane $\wp_\tau$ is an analytic representative of the Kummer map of E_τ, and this remains the same for cd with E_τ replaced by $E_{2\omega_{1,\tau},\omega_{2,\tau}}$.

It follows readily from transformation formulae of theta functions as given in [101] that

$$k\left(\frac{a\tau+b}{c\tau+d}\right) = \frac{i^{2b-2bc}\,\vartheta^2_{1-c,1-d}\,(0,\tau)}{i^{ab}\,\vartheta^2_{1-c-a,1-b-d}(0,\tau)}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

where $\theta_{11} = i\,\theta_1, \theta_{10} = \theta_2, \theta_{00} = \theta_3$ and $\theta_{01} = \theta_4$. In particular if $a \equiv d \equiv 1, b \equiv 0 \pmod 2$ then we have

$$k\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} (-1)^{\frac{b}{2}}\,k(\tau) & \text{if } \ c \equiv 0 \pmod 2; \\ \\ (-1)^{\frac{b}{2}}/k(\tau) & \text{if } \ c \equiv 1 \pmod 2. \end{cases} \tag{1.28}$$

This implies that

$$\lambda\left(\frac{a\tau+b}{c\tau+d}\right) = \begin{cases} \lambda(\tau) & \text{if } \ c \equiv 0 \pmod 2; \\ \\ \lambda(\tau)^{-1} & \text{if } \ c \equiv 1 \pmod 2. \end{cases} \tag{1.29}$$

2

Fundamental Groups and Ritt's Theory

The structure of endomorphism monoid of a space, where the binary operation is given by composition, might be an interesting problem. Given an elliptic curve E without complex multiplication its endomorphism(as a Lie group) monoid $\mathrm{End}(E)$ is presented by $\langle p_1, p_2, p_3, \dots | p_i p_j = p_j p_i \rangle$, and the structure of $\mathrm{End}(E)$ for elliptic curves with complex multiplication goes to the factorization theory of orders which is much more complicated.

We write $(\mathrm{End}(X), \circ)$ for the monoid of finite endomorphisms of an algebraic or an analytic space X, where a finite map refers to a map which is quasifinite and proper. It is clear that $\mathrm{End}(X)$ is generated by units and irreducible elements, where the set of units consists of automorphisms $\mathrm{Aut}(X)$ and an irreducible element refers to an endomorphism f in $\mathrm{End}(X)\backslash\mathrm{Aut}(X)$ satisfying that there do not exist $\psi_1, \psi_2 \in \mathrm{End}(X)\backslash\mathrm{Aut}(X)$ for which $f = \psi_1 \circ \psi_2$. In other words we have

$$(\mathrm{End}(X), \circ) = \langle\, \text{Units, Irreducible elements} \,|\, \text{Relations} \,\rangle\,.$$

Hopefully, all relations admit simple geometric reasons.

In a fundamental paper [109] Ritt described the relations of $(\mathrm{End}(\mathbb{C}), \circ)$, and it turns out that all these rations do arise from rather simple reasons. The main result of this chapter is Theorem 2.1.4 which describes relations of

$(\mathrm{End}(\mathbb{E}), \circ)$, where $\mathbb{E}$ is the unit disk, and we may give here that theorem a slightly different presentation.

Theorem 2.0.1. *The finite endomorphism monoid* $(\mathrm{End}(\mathbb{E}), \circ)$ *of the unit disk is presented by*

$$\langle S \mid R \rangle$$

where S consists of linear and of prime finite Blaschke product and R consists of the following relations

(i) $\iota \circ f = g$ *or* $f \circ \iota = g$ *where* $\iota \in \mathrm{Aut}(\mathbb{E})$*;*

(ii) $z^r g(z)^k \circ z^k = z^k \circ z^r g(z^k)$*;*

(iii) $f_{p,\,qt} \circ f_{q,\,t} = f_{q,\,pt} \circ f_{p,\,t}$ *with p, q primes and t a positive real number*

where $f_{n,t}$ are associated to isogenies of elliptic curves.

2.1 Introduction

We call a non-constant complex polynomial $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ *prime* if there do not exist complex polynomials $\psi_1 \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ and $\psi_2 \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ for which $f = \psi_1 \circ \psi_2$. Otherwise f is called *composite* or *factorized*. A representation of f in the form $f = \psi_1 \circ \cdots \circ \psi_k$ is a *factorization* or *decomposition* of f and a maximal factorization of f into prime polynomials only is called a *prime factorization* of f. The *length* of f, with respect to a given prime factorization, is defined to be the number of prime polynomials present in that prime factorization. In 1922 J.F. Ritt [109] proved three fundamental results on factorizations of complex polynomials.

Let $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ be a non-constant complex polynomial. He first gives a necessary and sufficient condition for f to be composite and shows that f is composite if and only if its monodromy group is imprimitive (Ritt I), and that the length of f is independent of its prime factorizations (Ritt II). The third result of Ritt tells us how to pass from one prime factorization to another one.

Theorem 2.1.1 (Ritt III)**.** *Given two prime factorizations of a non-constant complex polynomial* $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$*, one can pass from one prime factorization to the other one by repeatedly uses of the following operations:*

(i) $h \circ f = (h \circ \iota^{-1}) \circ (\iota \circ f)$ *with* $\iota \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$, h *and* f *prime;*

(ii) $T_m \circ T_n = T_n \circ T_m$ *with* T_k *the Chebyshev polynomial of degree* k*;*

(iii) $z^r g(z)^k \circ z^k = z^k \circ z^r g(z^k)$ *with* r, k *positive integers and* g *a non-constant complex polynomial.*

Ritt's theory was later studied by Engstrom, Levi, Dorey, Whaples, Fried, Zannier, Müller, Beardon, Ng, Pakovich, Zieve and many others. All their work are based on algebraic techniques although Ritt's original work is simply topological in nature. We shall adopt Ritt's topological point of view and explore the theory by means of topological fundamental groups. This enable us to put Ritt's theory in a more general analytic setting and the main goal of this paper is to develop a version of Ritt's theory for the unit disk. In the context of finite maps between Riemann surfaces Ritt's first two theorems can be reformulated as follows.

Theorem 2.1.2 (Ritt I′)**.** *If* f *is a nonlinear finite map from* $\mathfrak{M}$ *to* $\mathfrak{N}$ *then it is composite if and only if its monodromy group is imprimitive.*

For our version of (Ritt II) we need an additional hypothesis, which is satisfied for all finite maps with a totally ramified point, in particular for polynomial maps.

Theorem 2.1.3 (Ritt II′)**.** *If* $\alpha\colon [0,1] \to \mathfrak{N}$ *is a closed cycle over which* f *is unramified and if the monodromy of* α *acts transitively then the length of* f *is independent of the prime factorizations.*

The proofs are only slight technical modifications of the original proofs to deal with the more general situation. We shall apply these two theorems when the Riemann surfaces $\mathfrak{M}$ and $\mathfrak{N}$ are unit disks $\mathbb{E}$ and carefully develop a complete version of Ritt's theory on $\mathbb{E}$. Since Chebyshev polynomials play an important role in Ritt's theory, it is natural to find their counterparts in the unit disk case. We solve this central problem by introducing in Section 2.3 a new class of finite Blaschke products, which we call Chebyshev-Blaschke products $f_{n,t}$ for any positive integer n and positive number t.

Theorem 2.1.4. *Let* f *be a finite endomorphisms of* $\mathbb{E}$*. Let*

$$\mathbb{E} \xrightarrow{\varphi_1} \mathfrak{T}_1 \xrightarrow{\varphi_2} \mathfrak{T}_2 \to \quad \cdots \quad \to \mathfrak{T}_{r-1} \xrightarrow{\varphi_r} \mathbb{E}$$

and

$$\mathbb{E} \xrightarrow{\psi_1} \mathfrak{R}_1 \xrightarrow{\psi_2} \mathfrak{R}_2 \rightarrow \quad \cdots \quad \rightarrow \mathfrak{R}_{s-1} \xrightarrow{\psi_s} \mathbb{E}$$

be decompositions of f into a product of prime finite maps. We can pass from the first decomposition to the second by applying repeatedly the following operations:

(i) $h \circ g = (h \circ \iota^{-1}) \circ (\iota \circ g)$ *where h, g are finite endomorphisms of $\mathbb{E}$ and ι is a biholomorphic map from $\mathbb{E}$ to another Riemann surface;*

(ii) $(\iota \circ f_{m,nt}) \circ (f_{n,t} \circ \jmath) = (\iota \circ f_{n,mt}) \circ (f_{m,t} \circ \jmath)$ *with m, n positive integers, t a positive real number and $\iota, \jmath$ in* $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$*;*

(iii) $(\iota \circ z^r g(z)^k) \circ (z^k \circ \jmath) = (\iota \circ z^k) \circ (z^r g(z^k) \circ \jmath)$ *with r, k positive integers, g a finite endomorphism of $\mathbb{E}$ and $\iota, \jmath$ in* $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$.

2.2 Modular lattices

In this section we give a proof of Theorem 2.1.2 and of Theorem 2.1.3. Even though our proof carries no essentially new ingredients compared with Ritt's original work [109], we present it with the aim to clarify that Ritt's original ideas extend to the more general category. Moreover, a number of consequences which results from the proofs are needed to prove our Main Theorem 5.1.3. Notice that a topological version of Theorem 2.1.2 was already discussed in [78, p.65].

Proof of Theorem 2.1.2. Choose $q \notin \mathfrak{d}_f$ in $\mathfrak{N}$ and $p \in \mathfrak{M}$ with $f(p) = q$ then we deduce from Corollary 1.3.3 that f is prime if and only if $\pi_1(\mathfrak{M} \backslash f^{-1}(\mathfrak{d}_f), p)$ is a maximal subgroup of $\pi_1(\mathfrak{N} \backslash \mathfrak{d}_f, q)$ and this is equivalent to $\pi_1(\mathfrak{N} \backslash \mathfrak{d}_f)$ acting primitively on $\pi_1(\mathfrak{M} \backslash f^{-1}(\mathfrak{d}_f)) \backslash \pi_1(\mathfrak{N} \backslash \mathfrak{d}_f)$ and now by the equivalence remarked in Section 2.3 we may deduce the desired conclusion. □

We shall recall some basic lattice theory and we shall follow the notations $x < y$, $x \prec y$, $x \vee y$ and $x \wedge y$ as described in [18]. A lattice $\mathfrak{L}$ is said to satisfy the *Jordan-Dedekind chain condition* if the length of maximal proper chains depends only on the endpoints. We say that $\mathfrak{L}$ is of *locally finite* if every interval of $\mathfrak{L}$ is of finite length. We call $\mathfrak{L}$ *modular* if

$$x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z \quad \forall\ y \in \mathfrak{L}.$$

The following modular lattices play an important role in Ritt's theory.

Example 2.2.1. *Let* $\mathfrak{L}_n = \{t \in \mathbb{N} : t \mid n\}$ *so that* $i \leq j$ *if and only if* $i \mid j$. *Then* $(\mathfrak{L}_n; \leq)$ *is a lattice and* $x \vee y = \mathrm{lcm}(x, y)$, $x \wedge y = \gcd(x, y)$. *This lattice of devisors of* n *is modular and any sublattice* $\mathfrak{F}$ *of* $(\mathfrak{L}_n; \leq)$ *is also modular.*

If a locally finite lattice $\mathfrak{L}$ is modular, then it satisfies the Jordan-Dedekind chain condition. Furthermore there is a *dimension function* $d \colon \mathfrak{L} \to \mathbb{Z}$ such that $x \prec y$ if and only if $x < y$ and $d(y) = d(x) + 1$ for all x, y in $\mathfrak{L}$. In addition we have $d(x) + d(y) = d(x \vee y) + d(x \wedge y)$. Ritt [109] proved an important property for sublattices of $(\mathfrak{L}_n; \leq)$. We shall extend Ritt's result to general modular lattices.

Proposition 2.2.2. *Let* $\mathfrak{L}$ *be a locally finite modular lattice,* $a, b \in \mathfrak{L}$ *with* $a \leq b$ *and* $\mathcal{C}, \mathcal{C}'$ *maximal proper chains of* $\mathfrak{L}$ *with the same endpoints* a *and* b. *There exists* $m \geq 0$ *and a sequence of maximal proper chains* $\mathcal{C}_i$, $0 \leq i \leq m$, *with endpoints* a *and* b *such that* $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_m = \mathcal{C}'$ *and* $\mathcal{C}_i$ *and* $\mathcal{C}_{i+1}$ *differ in only one element.*

Proof. We first of all write $\mathcal{C}$ and $\mathcal{C}'$ as

$$\begin{aligned} \mathcal{C} &: a = x_0 \prec x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_n = b, \\ \mathcal{C}' &: a = y_0 \prec y_1 \prec y_2 \prec y_3 \prec \cdots \prec y_n = b, \end{aligned}$$

then choose a dimension function d and will prove the claim by induction. If $n = 2$ nothing requires a proof. Assume the claim holds for all $2 \leq n \leq k-1$, we will prove it for $n = k$.

If $x_1 = y_1$ we apply the induction assumption to $\mathcal{B} : x_1 \prec x_2 \prec x_3 \prec \cdots \prec x_k = b$ and $\mathcal{B}' : y_1 \prec y_2 \prec y_3 \prec \cdots \prec y_k = b$ and this proves the proposition in the case $n < k$ or $n = k, x_1 = y_1$.

It remains the case that $n = k, y_1 \neq x_1$. We first show that $y_1 \nleq x_1$. If not then $y_1 < x_1$ and this leads to $d(a) < d(y_1) < d(x_1)$, a contradiction to $d(x_1) = d(a) + 1$. Since $y_1 \leq x_k = b$, there exists $1 \leq i \leq k-1$ such that $y_1 \nleq x_i,\ y_1 \leq x_{i+1}$. If $i = 1$ we put $\mathcal{C}_0 = \mathcal{C}$ and define $\mathcal{C}_1 : x_0 \prec y_1 \prec x_2 \prec x_3 \prec \cdots \prec x_k$. To go from $\mathcal{C}_1$ to $\mathcal{C}'$ we note that here the case where $n = k$ and $x_1 = y_1$ applies and we are done.

Assume the proposition holds when $1 \leq i \leq l-1$ and we shall prove it for $i = l$.

Since

$$\begin{aligned} d(y_1 \vee x_{l-1}) &= d(y_1) + d(x_{l-1}) - d(y_1 \wedge x_{l-1}) \\ &= d(y_1) + d(x_{l-1}) - d(x_0) \\ &= d(x_{l-1}) + 1 \end{aligned}$$

and since $y_1 \leq x_{l+1}$ implies $x_{l-1} \leq y_1 \vee x_{l-1} \leq x_{l+1}$, we conclude that $x_{l-1} \prec y_1 \vee x_{l-1} \prec x_{l+1}$. This shows that we can choose $\mathfrak{C}_0 = \mathfrak{C}$ and $\mathfrak{C}_1 = x_0 \prec x_1 \prec x_2 \prec \cdots \prec x_{l-1} \prec y_1 \vee x_{l-1} \prec x_{l+1} \prec \cdots \prec x_k$. To go from $\mathfrak{C}_1$ to $\mathfrak{C}'$ we see that the case $i = l-1$ applies and we are done. $\square$

As an explicit example we give the following

Example 2.2.3. *Let $\mathfrak{F}$ be a sublattice of $\mathfrak{L}_n$ and $\mathfrak{C}, \mathfrak{C}'$ maximal proper chains of $\mathfrak{F}$ with endpoints* 1 *and n. There exists a positive integer m and a sequence of maximal proper chains $\mathfrak{C}_i$, $0 \leq i \leq m$, with endpoints* 1 *and n such that $\mathfrak{C}_0 = \mathfrak{C}$, $\mathfrak{C}_m = \mathfrak{C}'$ and any two consecutive ones $\mathfrak{C}_j$ and $\mathfrak{C}_{j+1}$ differ only in one element. This means that we can write $\mathfrak{C}_j$ as $\cdots \prec a_i \prec a_{i+1} \prec a_{i+2} \prec \cdots$ and $\mathfrak{C}_{j+1}$ as $\cdots \prec a_i \prec a'_{i+1} \prec a_{i+2} \prec \cdots$ respectively. As both two chains are proper and $a_{i+1} \neq a'_{i+1}$, we have*

$$\left(\frac{a_{i+1}}{a_i}, \frac{a'_{i+1}}{a_i}\right) = 1, \quad \left(\frac{a_{i+2}}{a_{i+1}}, \frac{a_{i+2}}{a'_{i+1}}\right) = 1$$

or equivalently

$$\frac{a_{i+1}}{a_i} = \frac{a_{i+2}}{a'_{i+1}}, \ \frac{a_{i+2}}{a_{i+1}} = \frac{a'_{i+1}}{a_i}, \ \left(\frac{a_{i+1}}{a_i}, \frac{a_{i+2}}{a_{i+1}}\right) = 1. \tag{2.1}$$

For the proof of Theorem 2.1.3 we need the following lemma, for which we refer to [110].

Theorem 2.2.4 (Dedekind's Modular Law**).** *Let G be a group and let $H \leq K, L$ be its subgroups. Then we have $(LH) \cap K = (L \cap K)H$.*

Proof of Theorem 2.1.3. We write $n = \deg f$, $\alpha(0) = q$ and choose $f^{-1}(q) = \{p = p_1, p_2, \ldots, p_n\}$. According to Corollary 1.3.3 it suffices to prove that the lattice $\mathfrak{L}$ consisting of all intermediate groups between $G = \pi_1(\mathfrak{N} \backslash \mathfrak{d}_f, q)$ and $H = \pi_1(\mathfrak{M} \backslash f^{-1}(\mathfrak{d}_f), p)$ is modular. Writing K_α for $K \cap \langle \alpha \rangle$ we consider the following map

$$K \in \mathfrak{L} \mapsto^g K_\alpha \in \langle \alpha \rangle ,$$

and it is then clear that $G_\alpha = \langle\alpha\rangle$. By the transitivity of the monodromy action of α on $f^{-1}(q)$ we have

$$\begin{aligned} H_\alpha &= H \cap \langle\alpha\rangle \\ &\overset{(1.1)}{=} \{\beta \in \langle\alpha\rangle : p_1 = p = p^\beta = p_1^\beta = p_{1^\beta}\} \\ &= \{\beta \in \langle\alpha\rangle : 1^\beta = 1\} \\ &= \langle\alpha^n\rangle . \end{aligned}$$

The fact that $H \cap \langle\alpha\rangle = \langle\alpha^n\rangle$ immediately leads to $\alpha^i H \neq \alpha^j H$ for all $0 \leq i < j \leq n-1$. This together with $[G : H] = n$ leads to $G = H \cup \alpha H \cup \cdots \cup \alpha^{n-1} H$ and in particular $G = \langle\alpha\rangle\, H$. Since $\mathfrak{L}_n$ is isomorphic to the lattice consisting of intermediate groups between $\langle\alpha\rangle$ and $\langle\alpha^n\rangle$, we shall treat them equally and consequently $g\colon \mathfrak{L} \to \mathfrak{L}_n$ is a lattice morphism. By Dedekind's Modular Law and by $G = \langle\alpha\rangle\, H$ we have

$$K_\alpha H = (\langle\alpha\rangle \cap K) H = \langle\alpha\rangle\, H \cap K = G \cap K = K. \tag{2.2}$$

This implies immediately that g is injective. Since K is a group we deduce from (2.2) that $K_\alpha H$ is also a group and this leads to

$$K_\alpha H = H K_\alpha. \tag{2.3}$$

To prove that g is a lattice morphism it suffice to verify that $K_\alpha \cap M_\alpha = (K \cap M)_\alpha$ and $\langle K, M\rangle_\alpha = \langle K_\alpha, M_\alpha\rangle$ for all K, M in $\mathfrak{L}$. The former is trivial since

$$\begin{aligned} K_\alpha \cap M_\alpha &= K \cap \langle\alpha\rangle \cap M \cap \langle\alpha\rangle \\ &= (K \cap M) \cap \langle\alpha\rangle \\ &= (K \cap M)_\alpha. \end{aligned}$$

By $\langle K, M\rangle = \langle K_\alpha H, M_\alpha H\rangle \overset{(2.3)}{=} H K_\alpha M_\alpha$ and $K_\alpha M_\alpha = M_\alpha K_\alpha$ the latter follows from

$$\begin{aligned} \langle K, M\rangle_\alpha &= \langle K, M\rangle \cap \langle\alpha\rangle = H K_\alpha M_\alpha \cap \langle\alpha\rangle \\ &= (H \cap \langle\alpha\rangle) K_\alpha M_\alpha \\ &= K_\alpha M_\alpha \end{aligned}$$

where the second last equality relies on Dedekind's modular law. We have proved that g is an injective lattice morphism and this gives that $\mathfrak{L}\simeq g(\mathfrak{L})$ and the latter is a sublattice of $\mathfrak{L}_n$. We conclude from Example 2.2.1 that $\mathfrak{L}$ is modular. □

A much shorter proof exists, but our proof gives more information. In particular it implies that $\mathfrak{L}$ is a sublattice of $\mathfrak{L}_n$. According to (2.1), we can pass from one maximal factorization of f to another with each step given by a solution $(\phi_i, \phi_{i+1}, \phi'_i, \phi'_{i+1})$ to the two finite maps equation

$$\phi_i\circ\phi_{i+1}=\phi'_i\circ\phi'_{i+1}, \deg\phi_i=\deg\phi'_{i+1}, \ (\deg\phi_i, \deg\phi_{i+1})=1. \tag{2.4}$$

This functional equation in polynomials is a major difficulty solved in [109] and we shall solve this functional equation in finite Blaschke products.

2.3 Chebyshev representations

In this section we shall construct *Chebyshev-Blaschke products* using the geometric monodromy action. If $a\neq b$ are points in $\mathbb{E}$ then the group $\pi_1(\mathbb{E}\backslash\{a,b\})$ can be generated by two elements σ and τ with σ and τ represented by closed paths around a and b with counterclockwise orientation.

Lemma 2.3.1. *Given $n\in\mathbb{N}$ there exists a finite map $f_{n,a,b}: \mathbb{E}\to\mathbb{E}$ of degree n with*

(i) $\mathfrak{d}_{f_{n,a,b}}=\{a,b\}$ *(if $n>2$) or* $\{a\}$ *(if $n=2$) or* $\emptyset$ *(if $n=1$),*

(ii) the monodromy representation $\rho: \langle\sigma,\tau\rangle\to S_n$ is a Chebyshev representation.

In addition $f_{n,a,b}$ is unique up to composition on the right with an element in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$.

Proof. Theorem 1.3.4 gives a finite map f from some Riemann surfaces $\mathfrak{M}$ to $\mathbb{E}$ which satisfies the monodromy condition. By Lemma 1.3.5 a direct calculation leads to $\chi(\mathfrak{M})=1$ and it follows from elementary topology that $\mathfrak{M}$ is either $\mathbb{C}$ or $\mathbb{E}$. Liouville's Theorem rules out the possibility of $\mathbb{C}$ and the uniqueness part of Theorem 1.3.4 completes the proof. □

We will call those $f_{n,a,b}$ *Chebyshev-Blaschke products*. Any annulus $\mathcal{A}$ is conformal to $\mathcal{A}(r,t)=\{z : r<|z|<t\}$ with $0\leq r<t\leq\infty$. The *modulus* of $\mathcal{A}$,

denoted by $\mu(\mathcal{A})$, is defined to be $\ln(t/r)$. In order to describe normalized forms of $f_{n,a,b}$, we denote by $\gamma(t)$ for any positive real number t the unique number in $(0,1)$ such that $\mu(\mathbb{E}\backslash[-\gamma(t),\gamma(t)]) = t$. The function γ is closely related to the elliptic modulus k

$$\gamma(t) = k^{\frac{1}{2}}(4ti/\pi) \tag{2.5}$$

Given $t > 0$ and $n \in \mathbb{N}$ we take $a = -\gamma(nt)$ and $b = \gamma(nt)$.

Proposition 2.3.2. *For all positive number t and for all positive integer n, there is a finite map $f\colon \mathbb{E} \to \mathbb{E}$ of degree n which satisfies*

(i) $\mathfrak{d}_f = \{a, b\}$ *(if $n > 2$) or* $\{a\}$ *(if $n = 2$) or* $\emptyset$ *(if $n = 1$);*

(ii) $f^{-1}[-\gamma(nt), \gamma(nt)] = [-\gamma(t), \gamma(t)]$ *and* $f(\gamma(t)) = \gamma(nt)$*;*

(iii) the monodromy representation $\rho\colon \langle\sigma,\tau\rangle \to S_n$ is a Chebyshev representation.

Before the proof we recall some geometry and topology. The isometry group $\mathrm{Is}(\mathbb{E}, ds)$ of $\mathbb{E}$ with respect to the Poincaré metric ds is given by the semidirect product $\mathrm{Aut}_{\mathbb{C}}\mathbb{E} \rtimes \langle i\rangle$, where i is complex conjugation. We write $\mathrm{Is}^+(\mathbb{E}, ds)$ for the set of holomorphic automorphisms and $\mathrm{Is}^-(\mathbb{E}, ds)$ for the antiholomorphic ones. The fixed point set $\mathrm{Fix}(\iota)$ of an element ι in $\mathrm{Is}(\mathbb{E}, ds)$ is either empty, a point, a geodesic line or $\mathbb{E}$. Let f be a finite map from $\mathfrak{M}$ to $\mathfrak{N}$, ϵ a homeomorphism from $\mathfrak{N}$ to $\mathfrak{N}$, $q \notin \mathfrak{d}_f$ a point in $\mathfrak{N}$ and p_1, p_2 points in $\mathfrak{M}$ with $f(p_i) = q$. Elementary topology shows that the map ϵ lifts to a homeomorphism $\varepsilon\colon (\mathfrak{M}, p_1) \to (\mathfrak{M}, p_2)$ making the following diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, "\varepsilon"] \arrow[d, "f"] & \mathfrak{M} \arrow[d, "f"] \\ \mathfrak{N} \arrow[r, "\epsilon"] & \mathfrak{N} \end{tikzcd}$$

commutative if and only if first of all ϵ restricts to a bijection on $\mathfrak{d}_f$ and secondly $(\epsilon\circ f)_*\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{d}_f), p_1) = f_*\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{d}_f), p_2)$.

Proof of Proposition 2.3.2. Lemma 2.3.1 gives a finite map $f\colon \mathbb{E} \to \mathbb{E}$ which satisfies (1) and (3). Moreover if we can prove that $f^{-1}[-\gamma(nt), \gamma(nt)]$ is a geodesic segment then (2) is immediately fulfilled by composing f with an element in $\mathrm{Aut}_{\mathbb{C}}\mathbb{E}$. We only verify this fact for $n=2k$, since similar considerations apply to $n=2k+1$.

By condition f is an unramified map from $\mathbb{E}\backslash f^{-1}[-\gamma(nt),\gamma(nt)]$ to an annulus $\mathbb{E}\backslash[-\gamma(nt),\gamma(nt)]$. This implies that $\mathbb{E}\backslash f^{-1}[-\gamma(nt),\gamma(nt)]$ is an annulus and therefore $f^{-1}[-\gamma(nt),\gamma(nt)]$ is connected.

Choose $q \in (-\gamma(nt),\gamma(nt))$ and write $f^{-1}(q) = \{p_1, p_2, \dots, p_{2k}\}$ with the numbering i chosen such that $p_i^{\alpha} = p_{i\rho(\alpha)}$ for all $1 \le i \le 2k$ and $\alpha \in \pi_1(\mathbb{E}\backslash\{a,b\},q)$. We show now that there is a commutative diagram

$$\begin{array}{ccc} (\mathbb{E},p_1) & \xrightarrow{\iota} & (\mathbb{E},p_1) \\ \downarrow f & & \downarrow f \\ (\mathbb{E},q) & \xrightarrow{i} & (\mathbb{E},q) \end{array}$$

with an isometry ι in $\mathrm{Is}(\mathbb{E})$ such that $f^{-1}[-\gamma(nt),\gamma(nt)] \subset \mathrm{Fix}(\iota)$. As a consequence $f^{-1}[-\gamma(nt),\gamma(nt)]$ will be a geodesic segment. By the remark before it suffices to show that i restricts to a bijection on $\{a,b\}$ and that $(i\circ f)_*\pi_1(\mathbb{E}\backslash f^{-1}\{a,b\},p_1) = f_*\pi_1(\mathbb{E}\backslash f^{-1}\{a,b\},p_1)$.

The involution i restricted on $\mathbb{E}\backslash\{a,b\}$ induces a map $i_*\colon \pi_1(\mathbb{E}\backslash\{a,b\},q) \to \pi_1(\mathbb{E}\backslash\{a,b\},q)$. The base point q of σ and τ is on the interval $(-\gamma(nt),\gamma(nt))$ and therefore the action of i_* on σ and τ simply changes the orientation, and this means that

$$i_*(\sigma) = \sigma^{-1}, \qquad i_*(\tau) = \tau^{-1}.$$

By condition that ρ is a Chebyshev representation we have both $\rho(\sigma)$ and $\rho(\tau)$ are of order two and therefore $\rho(i_*(\sigma)) = \rho(\sigma)$ as well as $\rho(i_*(\tau)) = \rho(\tau)$. This gives $\rho\circ i_* = \rho$ on $\langle\sigma,\tau\rangle = \pi_1(\mathbb{E}\backslash\{a,b\},q)$, and equivalently for all α in $\pi_1(\mathbb{E}\backslash\{a,b\},q)$ we have

$$\rho(i_*(\alpha)) = \rho(\alpha). \tag{2.6}$$

By (1.1) we have $i_*f_*\pi_1(\mathbb{E}\backslash f^{-1}(\{a,b\}),p_1) = i_*\{\alpha\in\pi_1(\mathbb{E}\backslash\{a,b\},q) : 1^{\rho(\alpha)} = 1\}$. Observe that β to be in the group on the right is equivalent to $i_*^{-1}(\beta)$ to be in the group $\{\alpha\in\pi_1(\mathbb{E}\backslash\{a,b\},q) : 1^{\rho(\alpha)} = 1\}$. Therefore the right hand side of the last equality equals $\{\alpha\in\pi_1(\mathbb{E}\backslash\{a,b\},q) : 1^{\rho(i_*^{-1}(\alpha))} = 1\}$. Using (2.6) we find that this is the same as $\{\alpha\in\pi_1(\mathbb{E}\backslash\{a,b\},q) : 1^{\rho(\alpha)} = 1\}$ which is exactly $f_*\pi_1(\mathbb{E}\backslash f^{-1}\{a,b\},p_1)$. This shows that the involution i lifts to a homeomorphism $\iota\colon (\mathbb{E},p_1) \to (\mathbb{E},p_1)$ and from the diagram we deduce with elementary topology that for all $\alpha \in \pi_1(\mathbb{E}\backslash\{a, b\}, q) = \langle\sigma, \tau\rangle$ we have

$$\iota(p_1^\alpha) = p_1^{i_*(\alpha)}.$$

In particular $\iota(p_1^\tau)=p_1^{i_*(\tau)}$ and therefore $\iota(p_2)=p_2$. Similar arguments show that $\iota(p_j)=p_j$ for all $1\leq j\leq 2k$ and we get $f^{-1}(q)\subset\mathrm{Fix}(\iota)$. We differentiate the equation $f(\iota(z)) = \overline{f(z)}$ which follows from the diagram. This implies that $\partial\iota/\partial z = 0$ which means that ι is an antiholomorphic homeomorphism of the unit disk. Consequently $\iota\in\mathrm{Is}^-(\mathbb{E}, ds)$ is an isometry and therefore it suffice to prove that $f^{-1}[-\gamma(nt), \gamma(nt)]\subset\mathrm{Fix}(\iota)$. The paths σ and τ, the preimage p_i and the lift ι vary continuously if q varies continuously in $(-\gamma(nt), \gamma(nt))$. In addition for given f and i the equation $i\circ f = f\circ\iota$ has only finitely many solutions ι in $\mathrm{Is}^-(\mathbb{E}, ds)$.Indeed choose a fixed point $x \in \mathbb{E}$ then any solution ι takes values at x in a finite set $f^{-1}(i(f(x)))$ and since ι is an antiholomorphic automorphism it is uniquely determined by the image at two distinct points. This shows that there are only finitely many possibilities. We conclude that ι is locally constant and therefore independent of q. This shows that $f^{-1}[-\gamma(nt), \gamma(nt)] \subset \mathrm{Fix}(\iota)$ as claimed. □

Proposition 2.3.3. *For all positive real number t and positive integer n there exists a unique finite endomorphism $f_{n,t}$ of the unit disk $\mathbb{E}$ with the property that $f^{-1}[-\gamma(nt), \gamma(nt)] = [-\gamma(t), \gamma(t)]$ and $f(\gamma(t)) = \gamma(nt)$.*

Proof. The existence of $f_{n,t}$ comes from Proposition 2.3.2 and therefore it suffices to prove that any two such maps f_1 and f_2 coincide. As a first step we show that $\mathfrak{d}_f \subset \{-\gamma(nt), \gamma(nt)\}$ for f in $\{f_1, f_2\}$.

The map f restricts to finite maps from the annulus $\mathbb{E}\backslash f^{-1}[-\gamma(nt), \gamma(nt)]$ which is $\mathbb{E}\backslash[-\gamma(t), \gamma(t)]$ to the annulus $\mathbb{E}\backslash[-\gamma(nt), \gamma(nt)]$. Such a map is unramified and this means that $\mathfrak{d}_f \subset [-\gamma(nt), \gamma(nt)]$. Moreover the moduli of these annuli differ by a factor n and this shows that $\deg f = n$.

Taking $q\in(-\gamma(nt), \gamma(nt))$ and p a point in $f^{-1}(q)\subset(-\gamma(t), \gamma(t))$ we deduce that the preimage of an open neighborhood of q in $(-\gamma(nt), \gamma(nt))$ is an open neighborhood of p in $(-\gamma(t), \gamma(t))$. Consequently the preimage of two trajectories in $(-\gamma(nt), \gamma(nt))$ at q consists of two trajectories in $(-\gamma(t), \gamma(t))$ at p and this implies that f is unramified at p. This gives that f is unramified over any point q in $(-\gamma(nt), \gamma(nt))$ showing that $\mathfrak{d}_f\subset\{\pm\gamma(nt)\}$ as stated.

To continue with the proof we distinguish between two cases.

Case $n = 2k$.

Because f is an unramified cover of $(-\gamma(nt), \gamma(nt))$ the preimage of $(-\gamma(nt), \gamma(nt))$ under f is a disjoint union of n real 1-dimensional connected curves in $[-\gamma(t), \gamma(t)]$. As such they have to be open intervals of the form (a_i, b_i) or (b_i, a_{i+1}) for $i = 1, \ldots, k$ with $a_1 < b_1 < a_2 < b_2 < \cdots < b_k < a_{k+1}$, $f(a_i) = \gamma(nt)$, $f(b_i) = -\gamma(nt)$ and $\{a_2, \ldots, a_k\} \cup \{b_1, \ldots, b_k\}$ the critical points. This leads to a picture similar to Figure 1.1, and an argument similar to that given in the proof of the statement there shows that the monodromy representation $\rho\colon \langle\sigma, \tau\rangle \to S_n$ is a Chebyshev representation. The uniqueness part of Theorem 1.3.4 leads to the existence of ι in $\mathrm{Aut}_{\mathbb{C}}\mathbb{E}$ with $f_1 = f_2 \circ \iota$. Taking inverse images and using that $f_i^{-1}[-\gamma(nt), \gamma(nt)] = [-\gamma(t), \gamma(t)]$ leads to $\iota[-\gamma(t), \gamma(t)] = [-\gamma(t), \gamma(t)]$ whence $\iota(\pm\gamma(t)) = \pm\gamma(t)$ or $\iota(\pm\gamma(t)) = \mp\gamma(t)$. In the former case $\iota = \mathrm{id}$ and therefore $f_1 = f_2 \circ \mathrm{id} = f_2$. In latter case $\iota = -\mathrm{id}$, therefore $f_1 = f_2 \circ (-\mathrm{id})$ and finally to conclude $f_1 = f_2$ it suffices to prove $f_2(z) = f_2(-z)$.

Choose $q \in (-\gamma(nt), \gamma(nt))$ and write $f_2^{-1}(q) = \{p_i\colon 1 \le i \le 2k\}$ with the numbering i chosen such that $p_i^{\alpha} = p_{i\rho(\alpha)}$ for all $\alpha \in \langle\sigma, \tau\rangle$. Similar to the proof of Proposition 2.3.2, the map $\mathrm{id}\colon (\mathbb{E}, q) \to (\mathbb{E}, q)$ lifts to a map $\iota\colon (\mathbb{E}, p_1) \to (\mathbb{E}, p_{k+1})$ different from the identity in $\mathrm{Aut}_{\mathbb{C}}\mathbb{E}$ such that $f_2 \circ \iota = id \circ f_2$ and again $\iota[-\gamma(t), \gamma(t)] = [-\gamma(t), \gamma(t)]$. This together with the property that $\iota \neq id$ implies that $\iota(z) = -z$ and therefore $f_2(z) = f_2(-z)$ as desired.

Case $n = 2k + 1$.

The preimage of $(-\gamma(nt), \gamma(nt))$ is a disjoint union of $n = 2k + 1$ open intervals of the form (a_i, b_i) for $1 \le i \le k+1$ or (b_j, a_{j+1}) for $1 \le j \le k$ with $f(a_i) = \gamma(nt)$, $f(b_i) = -\gamma(nt)$ and $\mathfrak{D}_f = (a_2) + \cdots + (a_{k+1}) + (b_1) + \cdots + (b_k)$. Similar considerations to that as above proceed up to there exists ι in $\mathrm{Aut}_{\mathbb{C}}\mathbb{E}$ such that $f_1 = f_2 \circ \iota$ and $\iota(\pm\gamma(t)) = \pm\gamma(t)$. The latter identity implies $\iota = id$ and therefore $f_1 = f_2$ as desired. $\square$

If $n \ge 3$ and if $f_{n,a,b}$ is the Chebyshev-Blaschke product constructed in Lemma 2.3.1 then there exist uniquely an element ϵ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and a positive real number t such that $\epsilon(a) = -\gamma(nt)$ and $\epsilon(b) = \gamma(nt)$ and now $\epsilon \circ f_{n,a,b}$ has the same monodromy as the function $f_{n,t}$ constructed in Proposition 2.3.3. Therefore there exists ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $\epsilon \circ f_{n,a,b} \circ \varepsilon = f_{n,t}$. The maps $f_{n,t}$

obtained in this way is called *normalized Chebyshev-Blaschke products.* We sum up with the following corollary

Corollary 2.3.4. *If f is a finite map from $\mathbb{E}$ to $\mathbb{E}$ with degree at least three and if its monodromy representation is Chebyshev representation then there exist a positive number t and ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that*

$$f_{n,a,b} = \epsilon \circ f_{n,t} \circ \varepsilon$$

and this factorization is unique.

Chebyshev Blaschke products have the following special nesting property.

Theorem 2.3.5. *For any positive number t and positive integers m and n we have*

$$f_{mn,t} = f_{m,nt} \circ f_{n,t}.$$

Proof. Direct calculation leads to

$$\begin{aligned}(f_{m,nt} \circ f_{n,t})^{-1}[-\gamma(mnt), \gamma(mnt)] &= f_{n,t}^{-1}(f_{m,nt}^{-1}[-\gamma(mnt), \gamma(mnt)]) \\ &= f_{n,t}^{-1}[-\gamma(nt), \gamma(nt)] \\ &= [-\gamma(t), \gamma(t)]\end{aligned}$$

and from Proposition 2.3.3 we deduce that $f_{mn,t} = f_{m,nt} \circ f_{n,t}$. □

The topological nature of $f_{n,t}$ may be illustrated by Riemann's 'Schere und Kleister' surgery applied to copies of the unit disk. As an example we take $f_{6,t}$ and get the following picture:

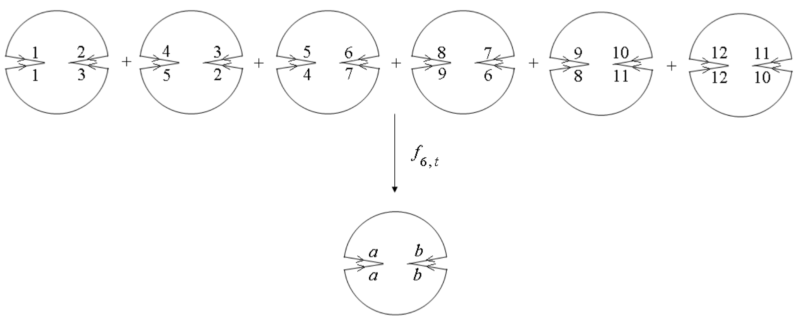
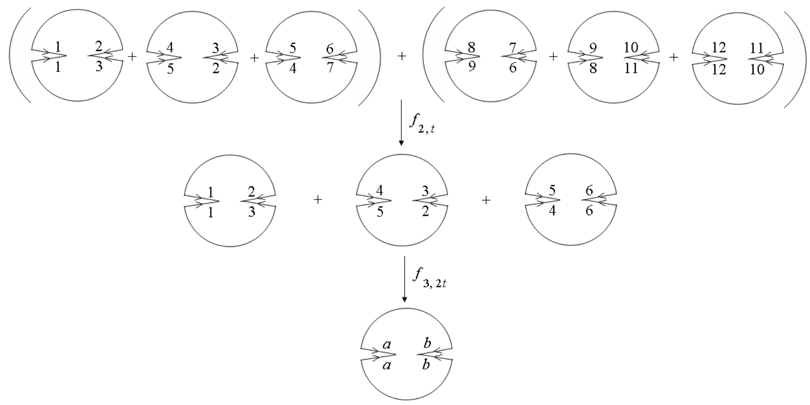


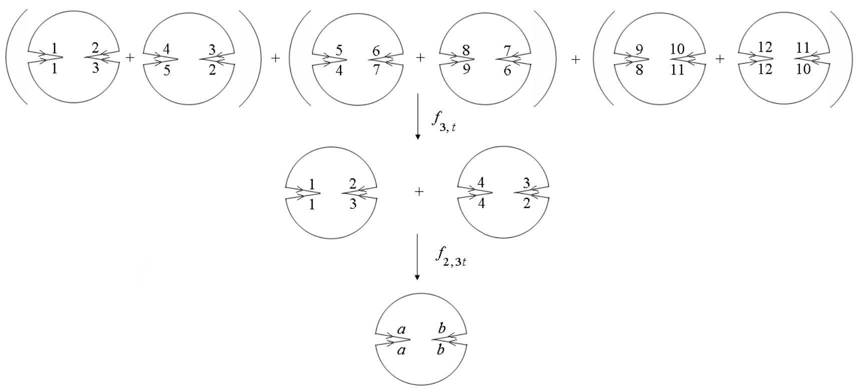
Figure 2.1: The topology of $f_{6,t}$.

Figure 2.2 illustrates the factorization $f_{6,t}{=}f_{3,2t}{\circ}f_{2,t}$ and Figure 2.3 illustrates the factorization $f_{6,t}{=}f_{2,3t}{\circ}f_{3,t}$.



$f_{2,t}$: $7 \to 6, 8 \to 5, 9 \to 4, 10 \to 3, 11 \to 2, 12 \to 1.$

Figure 2.2: The first factorization of $f_{6,t}$.



$f_{3,t}$: $5 \to 4, 6 \to 3, 7 \to 2, 8 \to 1, 9 \to 1, 10 \to 2, 11 \to 3, 12 \to 4$

Figure 2.3: The second factorization of $f_{6,t}$.

2.4 Complete theory on the unit disk

In this section we give a detailed study of the factorization properties of finite endomorphisms of the unit disk. If $f{\notin}\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ is a finite endomorphism of the unit disk then the following two Propositions follow directly from Theorem 2.1.2 and Theorem 2.1.3.

Proposition 2.4.1. *The finite map f is composite if and only if its monodromy group is imprimitive.*

In the introduction we introduced the length of f with respect to a prime factorization as the number of its factors. As a corollary of Theorem 2.1.3 we have

Proposition 2.4.2. *The length of f is independent of prime factorizations.*

Proof. We choose a path α sufficiently close to $\mathbb{T}$ and apply Theorem 2.1.3 to get the assertion. □

Lemma 2.4.3. *If f and g are finite Blaschke products and if $z^n \circ g = f \circ z^n$ then f takes the form $f(z) = z^m h(z)^n$ where $m = ord_0 f$ and h is a finite Blaschke product.*

Proof. It suffices to prove that for any nonzero p in $\mathbb{E}$ we have $ord_p f \equiv 0 \bmod n$. We denote by $p^{1/n}$ any nth root of p and using the functional equation we obtain

$$\mathrm{ord}_p f \equiv \mathrm{ord}_{p^{1/n}}(f \circ z^n) \equiv \mathrm{ord}_{p^{1/n}}(z^n \circ g) \equiv 0 \pmod n$$

as desired. □

Proof of Theorem 2.1.4. By Proposition 2.4.2 the length of f is independent of a given prime factorization. Moreover if

$$\mathbb{E} \xrightarrow{\varphi_1} \mathfrak{T}_1 \xrightarrow{\varphi_2} \mathfrak{T}_2 \to \cdots \to \mathfrak{T}_{r-1} \xrightarrow{\varphi_r} \mathbb{E}$$

is a decomposition of f into a product of finite maps then in particular for any $1 \le i \le r-1$ the map $\varphi_i \circ \varphi_{i-1} \circ \cdots \circ \varphi_1$ from $\mathbb{E}$ to $\mathfrak{T}_i$ is finite. This together with Lemma 1.3.2 implies that $\mathfrak{T}_i$ is biholomorphically equivalent to the unit disk. After taking finitely many operations of the first kind as described in the theorem our problem amounts to describe how one passes from one prime factorization

$$\mathbb{E} \xrightarrow{\varphi_1} \mathbb{E} \xrightarrow{\varphi_2} \mathbb{E} \to \cdots \to \mathbb{E} \xrightarrow{\varphi_r} \mathbb{E}$$

to another decomposition

$$\mathbb{E} \xrightarrow{\psi_1} \mathbb{E} \xrightarrow{\psi_2} \mathbb{E} \rightarrow \cdots \rightarrow \mathbb{E} \xrightarrow{\psi_r} \mathbb{E}$$

with all Riemann surfaces being unit disks. Furthermore by Example 1.5.2 (Fatou) all φ_i and all ψ_i are finite Blaschke products.

Let $\mathfrak{d}_f \subset \mathbb{E}$ be the set of critical values of f, $n = \deg f$ and $\mathfrak{L}$ the lattice of groups lying between $\pi_1(\mathbb{E}\backslash\mathfrak{d}_f)$ and $\pi_1(\mathbb{E}\backslash f^{-1}(\mathfrak{d}_f))$. If we write $G_i = \pi_1(\mathbb{E}\backslash(\varphi_r \circ \cdots \circ \varphi_i)^{-1}(\mathfrak{d}_f))$ and $K_i = \pi_1(\mathbb{E}\backslash(\psi_r \circ \cdots \circ \psi_i)^{-1}(\mathfrak{d}_f))$ then we have $G_1 = K_1 = \pi_1(\mathbb{E}\backslash f^{-1}(\mathfrak{d}_f))$ as well as $G_{r+1} = K_{r+1} = \pi_1(\mathbb{E}\backslash\mathfrak{d}_f)$ and as an application of Corollary 1.3.3 with $\Sigma = \mathfrak{d}_f$, $q \notin \Sigma$ and $f(p) = q$ we deduce that our prime decompositions of f induce maximal chains

$$G_1 \leq G_2 \leq \cdots \leq G_r \leq G_{r+1}$$

and

$$K_1 \leq K_2 \leq \cdots \leq K_r \leq K_{r+1}$$

with G_i, K_i in $\mathfrak{L}$. We apply Theorem 2.1.3 to $\mathfrak{M} = \mathfrak{N} = \mathbb{E}$ and f and therefore we know from the proof of Theorem 2.1.3 that $\mathfrak{L}$ is a sublattice of $\mathfrak{L}_n$ which is in particular modular. By Proposition 2.2.2 we may pass inductively from the first chain to the second with only one change at each step. This gives a topological description of our algorithm using fundamental groups. Corollary 1.3.3 allows us to write down the algorithm in terms of explicit analytic maps as listed in the theorem. As explained at the end of Section 2.2 this boils down to solve the functional equation

$$\alpha_2 \circ \alpha_1 = h = \beta_2 \circ \beta_1 \tag{2.7}$$

where α_i and β_i are prime Blaschke products with $\deg \alpha_1 = \deg \beta_2 = l$, $\deg \alpha_2 = \deg \beta_1 = k$ and $\gcd(k, l) = 1$. Our strategy is to get first a polynomial solution to this equation and then, using the monodromy representations given by such a solution, to transform the polynomial solution into a solution expressed in terms of Blaschke products.

Proposition 1.6.3 applied to (2.7) for some homeomorphism $i_0 = j_0 \colon \mathbb{E} \rightarrow \mathbb{C}$ which induces other homeomorphisms $i_1, j_1, i_2 = j_2$ from $\mathbb{E}$ to $\mathbb{C}$ leads to polynomial decompositions of $(i_2, i_0)_* h$ as

$$(i_1, i_0)_* \alpha_2 \circ (i_2, i_1)_* \alpha_1 \quad \text{and as} \quad (j_1, i_0)_* \beta_2 \circ (i_2, j_1)_* \beta_1.$$

This gives a solution to the two polynomial equation

$$\alpha_2 \circ \alpha_1 = h = \beta_2 \circ \beta_1$$

where α_i and β_i are prime polynomials with $\deg \alpha_1 = \deg \beta_2 = l, \deg \alpha_2 = \deg \beta_1 = k$ and $\gcd(k, l) = 1$. The polynomial solutions to this equation can be written out by Ritt's work [109]. Accordingly there exist linear polynomials ι_i such that one of the identities

(i) $\iota_1 \circ (i_2, i_1)_* \alpha_1 \circ \iota_2 = \iota_3 \circ (j_1, i_0)_* \beta_2 \circ \iota_4 = z^l$;

(ii) $\iota_1 \circ (i_1, i_0)_* \alpha_2 \circ \iota_2 = \iota_3 \circ (i_2, j_1)_* \beta_1 \circ \iota_4 = z^k$;

(iii) $\iota_1 \circ (i_2, i_1)_* h \circ \iota_2 = T_{lk}$

is satisfied. In case (1) of the list above α_1 and β_2 are totally ramified maps from $\mathbb{E}$ to $\mathbb{E}$. After finitely many operations of the first kind we may assume that $\alpha_1 = \beta_2 = z^l$. Then the functional equation (2.7) reduces to $\alpha_2 \circ z^l = z^l \circ \beta_1$ and Lemma 2.4.3 gives the solution as desired. Similar considerations apply to case (2).

If we are in case (3) the monodromy of h is a Chebyshev representation and therefore h is a Chebyshev-Blaschke product as explained in Lemma 2.3.1. After another finitely many operations of the first kind we may assume that $h, \alpha_1, \alpha_2, \beta_1, \beta_2$ are all normalized Chebyshev-Blaschke products and we are done. □

One crucial step in the above proof is to solve the following functional equation

$$a \circ b = c \circ d, \ \deg a = \deg d, \ (\deg a, \deg b) = 1. \tag{2.8}$$

Following [139] we call a solution of (2.8) in prime finite Blaschke products a *Ritt move* and a solution in arbitrary finite Blaschke products a *generalized Ritt move.* Indeed we have obtained the following

Theorem 2.4.4. *If (a, b, c, d) is a generalized Ritt move in finite Blaschke products then either there exist ι_i in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, m and n positive integers and t*

a positive real number such that (a,b,c,d) *equals*

$$\left(\iota_1\circ\mathfrak{T}_{m,\,nt}\circ\iota_2^{-1},\,\iota_2\circ\mathfrak{T}_{n,\,t}\circ\iota_3^{-1},\,\iota_1\circ\mathfrak{T}_{n,\,mt}\circ\iota_4^{-1},\,\iota_4\circ\mathfrak{T}_{m,\,t}\circ\iota_3^{-1}\right)$$

or there exist ι_i *in* $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, h *either the constant map* $h\equiv 1$ *or a finite Blaschke product with* $h(0)\neq 0$ *and* n,s *positive integers such that either* (a,b,c,d) *or* (c,d,a,b) *equals*

$$\left(\iota_1\circ z^n\circ\iota_2^{-1},\,\iota_2\circ z^s h(z^n)\circ\iota_3^{-1},\,\iota_1\circ z^s h(z)^n\circ\iota_4^{-1},\,\iota_4\circ z^n\circ\iota_3^{-1}\right).$$

Lastly we would like to point out that

Lemma 2.4.5. *Let* h *be a finite Blaschke product with* $h(0)\neq 0$ *and* s,n *positive integers with* n *at least 2. Then neither* $z^s h(z)^n$ *nor* $z^s h(z^n)$ *is totally ramified.*

Proof. We assume that $f=z^s h(z)^n\colon\mathbb{E}\to\mathbb{E}$ is totally ramified. Given any $\mathfrak{p}$ for which $h(\mathfrak{p})=0$, it is clear from $n\geq 2$ that f is ramified at $\mathfrak{p}$. This implies that f is ramified over 0, and therefore is totally ramified over 0. However we have $\mathfrak{p}\neq 0$ and $f(\mathfrak{p})=f(0)=0$, which is a contradiction.

Now consider the case $f=z^s h(z^n)$ and assume f is totally ramified. One can check if $\mathfrak{w}\neq 0$ is a point in $\mathfrak{D}_f$ then for any nth root ζ of unity $\zeta\mathfrak{w}$ is also contained in $\mathfrak{D}_f$, which together with $n\geq 2$ contradicts the totally ramified assumption. Therefore f is totally ramified at 0, which contradicts to $h(0)\neq 0$. □

2.5 Polydisks

In this section we sketch how to extend our results to the case of polydisks. Firstly we recall from Example 1.5.2 and Rischel's version [107] of Remmert-Stein's theorem [104] the following famous classification result.

Theorem 1.5.4 together with the results proved in Section 2.4 shows that *if* f *is a nonlinear finite map from* $\mathbb{E}^d$ *to* $\mathbb{E}^d$ *then it is composite if and only if its monodromy group is imprimitive.* In addition *the length of a nonlinear finite map* $f\colon\mathbb{E}^d\to\mathbb{E}^d$ *is independent of prime factorizations* and one sees that this leads without any difficulty to a higher dimensional generalization of our

Theorem 2.1.4

Theorem 2.5.1. *Given two prime factorizations of a nonlinear finite map $f\colon \mathbb{E}^d \to \mathbb{E}^d$, one can pass from one to the other by repeatedly uses of explicit operations.*

3

Elliptic Rational Functions

This chapter as a technical part of this thesis is devoted to elliptic rational functions which are analogues of Chebyshev polynomials in the formal dictionary between elliptic functions and trigonometric functions.

We have constructed in Section 2.3 Chebyshev-Blaschke products $\mathcal{T}_{n,t}(t > 0)$ from the monodromy representation of fundamental groups, and we shall prove in the next section that $\mathcal{T}_{n,t}$ are closely related to elliptic rational functions. Elliptic rational functions were originally constructed by Zolotarev by descent of cyclic isogenies. Later we shall slightly extend Zolotarev's construction and obtain a more general family $\mathcal{T}_{n,\tau}(\tau \in \mathbb{H})$, all of which we also call elliptic rational functions. If we write

$$\begin{aligned}
\mathcal{T}_b(n) &= \text{Chebyshev-Blaschke product of degree n} = \{\mathcal{T}_{n,t} | t > 0\}, \\
\mathcal{T}_z(n) &= \text{Zolotarev's original fractions}, \\
\mathcal{T}_{gz}(n) &= \text{Generalized Zolotarev fractions} = \{\mathcal{T}_{n,\tau} | \tau \in \mathbb{H}\}
\end{aligned}$$

then it turns out that

$$\begin{aligned}
\mathcal{T}_{gz}(n) &= Y_0(4n)(\text{Theorem 3.2.3}), \\
\mathcal{T}_z(n) &= \text{purely imaginary points} \hookrightarrow Y_0(4n), \\
\mathcal{T}_b(n) &= \mathcal{T}_z(n)(\text{Corollary 3.1.7}).
\end{aligned}$$

Notice that $\mathcal{T}_b(n)$ is constructed from monodromy, and $\mathcal{T}_z(n)$ is constructed from isogeny. The proof that two constructions agree is not trivial, which is carried out in the next section. Our starting point of the study of those functions is the Chebyshev representation $\rho_T\colon F_2 \to S_n$, and we summarize here our whole research scope on the dictionary between Chebyshev polynomials and Chebyshev-Blaschke products.

	Chebyshev polynomials	Chebyshev-Blaschke products
Topology	$\rho_T\colon \mathbb{C}\backslash\{2\text{pts}\} \to S_n$	$\rho_T\colon \mathbb{E}\backslash\{2\text{pts}\} \to S_n$
Geometry	1pt	purely imaginary points $\hookrightarrow Y_0(4n)$
Functions	trigonometric	elliptic
Groups	multiplicative group	elliptic curves
Arithmetic	∞ integral points	∞ rational points
Beyond	Siegel	Faltings

In other words, with the same topological condition we have obtained completely different stories in geometry, functions theory, algebraic groups and arithmetic. The basic points behind the whole story are the following simple facts in the uniformization theory:

(a) $\mathbb{C}$ agrees with $\mathbb{E}$ in topology;

(b) $\mathbb{P}^1$ is a double of $\mathbb{E}$.

Besides our viewpoint of monodromy, there are many other points of view on those functions. For instance, Zolotarev's was mostly interested in the optimization properties of those functions [140]. Bogatyrev also did many studies on Zolotarev's fractions, and his focus on the Pell equation in [24] seems interesting. He has also realized in [25] that those functions derived from isogeny give relations of $(\text{End}(\mathbb{P}^1), \circ)$, and compared with that what we have proved in Theorem 2.0.1 (which is essentially the main theorem of [132] submitted in 2007) is that those functions derived from monodromy give almost "all" relations of $(\text{End}(\mathbb{E}), \circ)$. The proof of our assertion that they give "all" relations was based on the study of monodromy. Similar argument by monodromy will appear again and again in the second half of this chapter.

3.1 Jacobi's products and Blaschke products

The contents of chapter 2 together with those of this section will appear in Forum Mathematicum. We have constructed in Section 2.3 Chebyshev Blaschke products $f_{n,t}$. In this section we shall write $\mathcal{T}_{n,t} = f_{n,t}$ and explain how they are related to theta functions. If f is a finite Blaschke product then $|f(z)|$ takes 1 for all z on the unit circle $\mathbb{T}$. This follows readily from

$$\left|z-a\right| = \left|1-\overline{a}z\right|, \text{ for all } \ a\in\mathbb{E}, z\in\mathbb{T}^1$$

which applies to each of factors of f. By similar but more involved arguments we use Jacobi products to prove

Proposition 3.1.1. *If τ is a purely imaginary point of the upper half plane $\mathbb{H}$ and if there exists a integer m such that $\frac{i\,\mathrm{Im}\,v}{\tau} = \frac{2m+1}{4}$ then $|\vartheta_0(v)| = |\vartheta_1(v)|$.*

Proof. The elliptic function $\frac{\vartheta_1(v)}{\vartheta_0(v)}$ has primitive periods 2 and τ and therefore it suffices to prove the claim under the assumption $\frac{i\,\mathrm{Im}\,v}{\tau} = \frac{1}{4}$ or $\frac{i\,\mathrm{Im}\,v}{\tau} = \frac{3}{4}$. We shall only verify this in the case $\frac{i\,\mathrm{Im}\,v}{\tau} = \frac{1}{4}$ since similar arguments apply in the remaining case. By the product formulae (1.17) and by the trivial fact $v = \mathrm{Re}\,v + \tau/4$ we have

$$\vartheta_0(v) = c\prod_{n=1}^{\infty}\left(1-e^{(2n-1/2)\pi i\tau+2\pi i\,\mathrm{Re}\,v}\right)\left(1-e^{(2n-3/2)\pi i\tau-2\pi i\,\mathrm{Re}\,v}\right),$$

$$\vartheta_1(v) = c\,e^{\frac{\pi i\tau}{4}}\,2\sin\pi v\prod_{n=1}^{\infty}\left(1-e^{(2n+1/2)\pi i\tau+2\pi i\,\mathrm{Re}\,v}\right)\left(1-e^{(2n-1/2)\pi i\tau-2\pi i\,\mathrm{Re}\,v}\right)$$

and we have to show that both terms have the same absolute value. Our assumption that τ is purely imaginary gives $e^{(2n\pm1/2)\pi i\tau}$ are real numbers and then leads to

$$\overline{1-e^{(2n\pm1/2)\pi i\tau+2\pi i\,\mathrm{Re}\,v}} = 1-e^{(2n\pm1/2)\pi i\tau-2\pi i\,\mathrm{Re}\,v}.$$

We use this identity to compare the infinite products above and see that for the proof of the proposition it suffices to verify that

$$\left|1-e^{\pi i\tau/2-2\pi i\,\mathrm{Re}\,v}\right| = \left|2e^{\pi i\tau/4}\sin\pi v\right|.$$

This follows from

$$\begin{aligned}\left|1-e^{\pi i\tau/2-2\pi i\operatorname{Re}v}\right| &= \left|1-e^{\pi i\tau/2+2\pi i\operatorname{Re}v}\right|\\ &= \left|1-e^{2\pi iv}\right|\\ &= \left|1-\cos 2\pi v-i\sin 2\pi v\right|\\ &= \left|2\sin^2\pi v-2i\sin\pi v\cos\pi v\right|\\ &= \left|2e^{i(\pi v-\pi/2)}\sin\pi v\right|\\ &= \left|2e^{-\pi\operatorname{Im}v}\sin\pi v\right|\\ &= \left|2e^{\pi i\tau/4}\sin\pi v\right|\end{aligned}$$

and completes the proof. □

Corollary 3.1.2. *Let τ be purely imaginary and let m be an integer. We have*

(i) $|\vartheta_0(v)|<|\vartheta_1(v)|$ *if* $\frac{4m+1}{4}<\frac{i\operatorname{Im}v}{\tau}<\frac{4m+3}{4}$;

(ii) $|\vartheta_0(v)|>|\vartheta_1(v)|$ *if* $\frac{4m-1}{4}<\frac{i\operatorname{Im}v}{\tau}<\frac{4m+1}{4}$.

Proof. The elliptic function $\varphi=\vartheta_1/\vartheta_0$ is of order 2 and takes $2,\tau$ as a pair of primitive periods. We take the parallelogram with vertex $0,2,2+\tau,\tau$ as a fundamental domain. By Proposition 3.1.1 each of the φ-images of $\left\{z:\frac{i\operatorname{Im}z}{\tau}=\frac{4m+1}{4},m\in\mathbb{Z}\right\}$ and of $\left\{z:\frac{i\operatorname{Im}z}{\tau}=\frac{4m+3}{4},m\in\mathbb{Z}\right\}$ covers $\mathbb{T}$. Together with the fact that $\deg\varphi=2$ this leads to $\varphi^{-1}(\mathbb{T})=\left\{z:\frac{i\operatorname{Im}z}{\tau}=\frac{2m+1}{4},m\in\mathbb{Z}\right\}$. If our second claim is not true then there exists w such that $-\frac{1}{4}<\frac{i\operatorname{Im}w}{\tau}<\frac{1}{4}$ and $|\vartheta_0(w)|\leq|\vartheta_1(w)|$. Moreover by $\varphi(0)=0$ we have $|\vartheta_0(0)|\geq 0=|\vartheta_1(0)|$ and by continuity there exists z such that $-\frac{1}{4}<\frac{i\operatorname{Im}z}{\tau}<\frac{1}{4}$ and $|\varphi(z)|=1$. This contradicts our previous conclusion on $\varphi^{-1}(\mathbb{T})$ and proves our second claim. The first assertion is obtained in a similar way. □

Corollary 3.1.3. *If τ is purely imaginary and if m is an integer then we have*

(i) $|\operatorname{sn}w|=k^{-1/2}$ *if* $\frac{i\operatorname{Im}w}{\omega_2}=\frac{2m+1}{4}$;

(ii) $|\operatorname{sn}w|<k^{-1/2}$ *if* $\frac{4m-1}{4}<\frac{i\operatorname{Im}w}{\omega_2}<\frac{4m+1}{4}$;

(iii) $|\operatorname{sn}w|>k^{-1/2}$ *if* $\frac{4m+1}{4}<\frac{i\operatorname{Im}w}{\omega_2}<\frac{4m+3}{4}$.

This remains the case with sn *replaced by* cd.

Proposition 3.1.4. *If τ is purely imaginary then*

$$\operatorname{sn}^{-1}[-1,1]=\{w:i\operatorname{Im}w=m\,\omega_2,m\in\mathbb{Z}\}$$

and this remains the case with sn *replaced by* cd.

Proof. First of all we recall that as remarked in Section 1.8 the assumption $-i\tau > 0$ implies that q, $k^{\frac{1}{2}}$ and ω_1 are all positive real numbers. If w is a real number then the quotient $v = w/\omega_1$ and

$$\operatorname{sn} w = \frac{1}{\sqrt{k}} \frac{2q^{1/4}\sin\pi v - 2q^{9/4}\sin 3\pi v + 2q^{25/4}\sin 5\pi v - \cdots}{1 - 2q\cos 2\pi v + 2q^4\cos 4\pi v - 2q^9\cos 6\pi v + \cdots}$$

are also real. The elliptic function sn takes $2\omega_1, \omega_2$ as a pair of primitive periods, and we have seen in Section 1.8 that $\operatorname{sn}\frac{2\omega_1 \pm \omega_1}{2} = \pm 1$. In a period-parallelogram spanned by the vectors $2\omega_1$ and ω_2, the critical points of sn are $\left\{\frac{\omega_1}{2}, \frac{3\omega_1}{2}, \frac{\omega_1+\omega_2}{2}, \frac{3\omega_1+\omega_2}{2}\right\}$. These facts imply that the sn-image of $[0, 2\omega_1]$ covers $[-1, 1]$ twice and we conclude that the preimage of $[-1, 1]$ in our period-parallelogram by the twofold covering sn is $[0, 2\omega_1]$ which leads to the desired statement. □

For any τ in $\mathbb{H}$ and for any positive integer n there are two natural isogenies. One is from E_τ to $E_{n\tau}$ given by $z \mapsto nz$ and the other one is from $E_{2\omega_{1,\tau},\omega_{2,\tau}}$ to $E_{2\omega_{1,n\tau},\omega_{2,n\tau}}$ given by $z \mapsto nz\omega_{1,n\tau}/\omega_{1,\tau}$.

Lemma 3.1.5. *The isogeny* $[n]\colon E_\tau \to E_{n\tau}$ *descends through* $\wp$ *to a rational function* n_τ *as obtained by the following commutative diagram.*

$$\begin{tikzcd} E_\tau \arrow[r, "{[n]}"] \arrow[d, "\wp_\tau"] & E_{n\tau} \arrow[d, "\wp_{n\tau}"] \\ \mathbb{P}^1 \arrow[r, "n_\tau"] & \mathbb{P}^1 \end{tikzcd}$$

Proof. The map given by the function $\wp$ is an analytic representation of the Kummer map. Obviously $z_1 \equiv \pm z_2 \pmod{\mathbb{Z} + \mathbb{Z}\tau}$ *implies that* $nz_1 \equiv \pm nz_2 \pmod{\mathbb{Z} + \mathbb{Z}n\tau}$ and this shows that the map $[n]$ is invariant under the action given by the involution. Therefore by the theory of descent it induces a rational map n_τ as stated. □

Similarly the isogeny $[n]\colon E_{2\omega_{1,\tau},\omega_{2,\tau}} \to E_{2\omega_{1,n\tau},\omega_{2,n\tau}}$ descends through cd to a rational function $\mathcal{T}_{n,\tau}$ as obtained by the following commutative diagram.

$$\begin{tikzcd} E_{2\omega_{1,\tau},\omega_{2,\tau}} \arrow[r, "{[n]}"] \arrow[d, "\mathrm{cd}"] & E_{2\omega_{1,n\tau},\omega_{2,n\tau}} \arrow[d, "\mathrm{cd}"] \\ \mathbb{P}^1 \arrow[r, "\mathcal{T}_{n,\tau}"] & \mathbb{P}^1 \end{tikzcd}$$

We call a rational function f *elliptic* if there exist ϵ, ι in $\mathrm{Aut}_{\mathbb{C}}\mathbb{E}$ such that $f = \epsilon \circ n_\tau \circ \iota$ for some positive integer n and for some τ in $\mathbb{H}$. The function $\mathcal{T}_{n,\tau}$ is linearly equivalent to $n_{\tau/2}$ and therefore is elliptic. We shall use the following formula repeatedly in the sequel:

$$\mathrm{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z,\ k(n\tau)\right) = \mathcal{T}_{n,\tau}(\mathrm{cd}(z,\ k(\tau))) \tag{3.1}$$

The concept of elliptic rational function is rarely found in the mathematical literature, but it is of central importance for advanced filter design. A nice treatment of elliptic rational functions in engineering can be found in [88, Chapert 12]. Here we have considered more generally elliptic rational functions in a universal family $\mathcal{T}_{n,\tau}$ parameterized by τ ranging on the upper half plane $\mathbb{H}$. This will be more satisfactory in mathematics.

Corollary 3.1.3 and 3.1.4 applied to $f(z) = k^{\frac{1}{2}}(n\tau)\mathcal{T}_{n,\tau}(z/k^{\frac{1}{2}}(\tau))$ gives

Proposition 3.1.6. *If τ is purely imaginary then f is a finite Blaschke product with $f(k^{\frac{1}{2}}(\tau)) = k^{\frac{1}{2}}(n\tau)$ and $f([-k^{\frac{1}{2}}(n\tau),\ k^{\frac{1}{2}}(n\tau)]) = [-k^{\frac{1}{2}}(\tau),\ k^{\frac{1}{2}}(\tau)]$.*

This together with Proposition 2.3.3 and (2.5) leads to

Corollary 3.1.7. *The Blaschke products $f_{n,t}(z)$ are elliptic with respect to $\tau = 4ti/\pi$; in other words we have*

$$f_{n,t}(z) = k^{\frac{1}{2}}(4nti/\pi)\,\mathcal{T}_{n,4ti/\pi}(z/k^{\frac{1}{2}}(4ti/\pi)).$$

Given $\tau \in \mathbb{H}$ we let $e_0(\tau) = \infty, e_1(\tau) = \wp_\tau(1/2), e_2(\tau) = \wp_\tau(\tau/2)$ and $e_3(\tau) = \wp_\tau((1+\tau)/2)$ be the image of the 2-torsion points on the projective line. If $n \geq 3$ then $\sigma_{n_\tau} = \{e_0(\tau), e_1(\tau), e_2(\tau), e_3(\tau)\}$ and we shall need

Lemma 3.1.8. *If τ is a point in $\mathbb{H}$ and if n is an integer greater than two then*

$$\mathfrak{o}_{n_\tau} = \wp_{n\tau}(E_{n\tau}[2]) \quad \textit{and} \quad n_\tau^{-1}(\mathfrak{o}_{n_\tau}) \setminus \mathrm{supp}\,\mathfrak{O}_{n_\tau} = \wp_\tau(E_\tau[2]).$$

Proof. This follows from a calculation of local ramification degree. □

We shall show that any elliptic rational function n_τ carries a closed cycle along which the monodromy action of n_τ is transitive.

Lemma 3.1.9. *Let $f : \mathfrak{M} \to \mathfrak{N}$ be a finite map and let α be a closed cycle on $\mathfrak{N}$ along which f is unramified. If $f^{-1}(\alpha)$ is connected then the monodromy* *action of f along α is transitive.*

Proof. It is almost the definition. $\square$

We write C_τ for the Jordan curve on $\mathbb{P}^1$ which is given by $\wp_\tau(\{z : \operatorname{Im} z = \operatorname{Im}\tau/4\})$.

Proposition 3.1.10. *Given any $\tau\in\mathbb{H}$ and given any positive integer n, there exists a closed cycle α on $\mathbb{P}^1$ along which n_τ is unramified and the monodromy action of n_τ is transitive.*

Proof. By definition we have $n_\tau^{-1}(C_{n_\tau}) = C_\tau$, and our previous lemma applies. $\square$

Lastly we would like to point out that from the above commutative diagrams the following nesting properties are obvious.

Proposition 3.1.11 (Nesting Property)**.** *Given positive integers m and n and given any $\tau\in\mathbb{H}$ we have*

$$\begin{aligned}(mn)_\tau &= m_{n\tau}\circ n_\tau,\\ \mathcal{T}_{mn,\tau} &= \mathcal{T}_{m,n\tau}\circ\mathcal{T}_{n,\tau}.\end{aligned}$$

3.2 Modular curves

This section is devoted to the study of the space of elliptic rational functions.

Theorem 3.2.1. *Given τ_1,τ_2 in $\mathbb{H}$ and given a rational integer $n\geq 3$. There exist ϵ,ε in $\operatorname{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ such that $\epsilon\circ n_{\tau_1}\circ\varepsilon^{-1}=n_{\tau_2}$ if and only if $\Gamma_0(n)\tau_2=\Gamma_0(n)\tau_1$, where*

$$\Gamma_0(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z})\,\middle|\, c\equiv 0 \pmod n\right\}$$

is the modular group.

Proof. First of all we show that for any positive integer n, τ in $\mathbb{H}$ and $0\leq i\leq 3$ there exist ι,ϵ in $\operatorname{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ such that $n_\tau=\epsilon\circ n_\tau\circ\iota^{-1}$ and $\iota(e_i(\tau))=e_0(\tau)$. We only verify this claim for $i=1$ since similar arguments apply to other situations. Notice that the map $\bar\iota\colon E_\tau\to E_\tau$ defined by $\bar\iota(z)=z+1/2$ descends to an element ι in $\operatorname{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ such that $\iota\circ\wp_\tau=\wp_\tau\circ\bar\iota$ and moreover the map $\bar\epsilon\colon E_{n\tau}\to E_{n\tau}$ given by $\overline{\epsilon}(w)=w+n/2$ descends to an element ϵ in $\mathrm{Aut}_{\mathbb{C}}\mathbb{P}^1$ such that $\epsilon\circ\wp_{n\tau}=\wp_{n\tau}\circ\overline{\epsilon}$.

$$\begin{tikzcd}
& E_\tau \arrow[rr, "{[n]}"] \arrow[dd] & & E_{n\tau} \arrow[dd] \\
E_\tau \arrow[ur, "\bar\iota"] \arrow[rr, "{[n]}"] \arrow[dd] & & E_{n\tau} \arrow[ur, "\bar\epsilon"'] \arrow[dd] & \\
& \mathbb{P}^1 \arrow[rr, "n_\tau"] & & \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[ur, "\iota"] \arrow[rr, "n_\tau"] & & \mathbb{P}^1 \arrow[ur, "\epsilon"] &
\end{tikzcd}$$

One checks easily that $\epsilon^{-1}\circ n_\tau\circ\iota=n_\tau$ and $\iota(e_0)=e_1$ which proves the desired claim.

By our construction we have $n_{\tau_1}\circ\wp_{\tau_1}=\wp_{n\tau_1}\circ[n]$ where $[n]$ maps E_{τ_1} to $E_{n\tau_1}$ and $n_{\tau_2}\circ\wp_{\tau_2}=\wp_{n\tau_2}\circ[n]$ where $[n]$ maps E_{τ_2} to $E_{n\tau_2}$. If there exist ϵ,ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ such that $\epsilon\circ n_{\tau_1}\circ\varepsilon^{-1}=n_{\tau_2}$ then ϵ induces a bijection between $\wp_{n\tau_1}(E_{n\tau_1}[2])$ and $\wp_{n\tau_2}(E_{n\tau_2}[2])$ since if $n\geq 3$ then $\mathfrak{o}_{n_{\tau_i}}=\wp_{n\tau_i}(E_{n\tau_i}[2])$. Moreover since ε^{-1} induces a bijection between $n_{\tau_2}^{-1}(\mathfrak{o}_{n_{\tau_2}})$ and $n_{\tau_1}^{-1}(\mathfrak{o}_{n_{\tau_1}})$ as well as a bijection between $\mathrm{supp}\,\mathfrak{O}_{n_{\tau_2}}$ and $\mathrm{supp}\,\mathfrak{O}_{n_{\tau_1}}$ we deduce from Lemma 3.1.8 that ε^{-1} also induces a bijection between $\wp_{\tau_2}(E_{\tau_2}[2])$ and $\wp_{\tau_1}(E_{\tau_1}[2])$ and by the claim made in the previous paragraph we may assume that $\varepsilon^{-1}(e_0(\tau_2))=e_0(\tau_1)$. The monodromy representation of a small loop around any critical value of $\wp$ is an involution, and consequently the map $\varepsilon\colon\mathbb{P}^1\to\mathbb{P}^1$ lifts to an isomorphism $\overline{\varepsilon}\colon E_{\tau_1}\to E_{\tau_2}$ such that $\wp_{\tau_2}\circ\overline{\varepsilon}=\varepsilon\circ\wp_{\tau_1}$ and $\overline{\varepsilon}(0)=0$. Now $\overline{\varepsilon}^{-1}(z)=\gamma z$ for some γ in $\mathbb{C}^*$ and $\overline{\varepsilon}^{-1}$ gives an bijection between Λ_{1,τ_2} and Λ_{1,τ_1} as well as a bijection between $([n]\circ\wp_{\tau_2})^{-1}(E_{n\tau_2}[2])=\Lambda_{\frac{1}{2n},\frac{\tau_2}{2}}$ and $([n]\circ\wp_{\tau_1})^{-1}(E_{n\tau_1}[2])=\Lambda_{\frac{1}{2n},\frac{\tau_1}{2}}$. We now write $\gamma\tau_2=a\tau_1+b$ and $\gamma=c\tau_1+d$ with $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL_2(\mathbb{Z})$ and we must have $\gamma\Lambda_{\frac{1}{2n},\frac{\tau_2}{2}}=\Lambda_{\frac{1}{2n},\frac{\tau_1}{2}}$ which clearly leads to $\frac{c\tau_1+d}{2n}\in\Lambda_{\frac{1}{2n},\frac{\tau_1}{2}}$ and therefore $n|c$. This verifies that $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(n)$.

Conversely if $\tau_2=\begin{pmatrix}a&b\\c&d\end{pmatrix}\tau_1$ with $\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(n)$ we shall verify that n_{τ_2} is linearly equivalent to n_{τ_1}. Let $\gamma=c\tau_1+d$ then the map $\overline{\varepsilon}\colon z\mapsto z/\gamma$ is an isomorphism between E_{τ_1} and E_{τ_2} and descends to an element $\varepsilon\colon\mathbb{P}^1\to\mathbb{P}^1$ such that $\wp_{\tau_2}\circ\overline{\varepsilon}=\varepsilon\circ\wp_{\tau_1}$. Moreover $\overline{\epsilon}\colon z\mapsto z/\gamma$ is an isomorphism between $E_{n\tau_1}$ and $E_{n\tau_2}$(this follows from $n|c$) and descends to an element $\epsilon\colon\mathbb{P}^1\to\mathbb{P}^1$ such that

$\wp_{n\tau_2}\circ\overline{\epsilon}=\epsilon\circ\wp_{n\tau_1}$.

$$\begin{tikzcd}
 & E_{\tau_2} \arrow[rr, "{[n]}"] \arrow[dd] & & E_{n\tau_2} \arrow[dd] \\
E_{\tau_1} \arrow[ur, "\overline{\varepsilon}"] \arrow[rr, "{[n]}"'] \arrow[dd] & & E_{n\tau_1} \arrow[ur, "\overline{\epsilon}"'] \arrow[dd] & \\
 & \mathbb{P}^1 \arrow[rr, "n_{\tau_2}"] & & \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[ur, "\varepsilon"] \arrow[rr, "n_{\tau_1}"] & & \mathbb{P}^1 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

One may check easily that $\epsilon\circ n_{\tau_1}=n_{\tau_2}\circ\varepsilon$ as desired. □

The first part of our proof could be much shorter with the help of the classical result on the moduli of cyclic isogeny of elliptic curves, and we sketch this as follows. If there exist ϵ,ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ such that $\epsilon\circ n_{\tau_1}\circ\varepsilon^{-1}=n_{\tau_2}$ then it induces, according to Lemma 3.1.8, bijections $\epsilon\colon\wp_{n\tau_1}(E_{n\tau_1}[2])\to\wp_{n\tau_2}(E_{n\tau_2}[2])$ and $\varepsilon\colon\wp_{\tau_1}(E_{\tau_1}[2])\to\wp_{\tau_2}(E_{\tau_2}[2])$. This implies that ϵ respectively ε lift to holomorphic isomorphisms $\overline{\varepsilon}\colon E_{n\tau_1}\to E_{n\tau_2}$ respectively $\overline{\epsilon}\colon E_{\tau_1}\to E_{\tau_2}$. By the claim made at the beginning of the above proof we may assume that $\overline{\varepsilon}(0)=0$ and $\overline{\epsilon}(0)=0$. One checks readily that $\overline{\epsilon}\circ n_{\tau_1}\circ\overline{\varepsilon}^{-1}=n_{\tau_2}$, and consequently that the isogeny $[n]\colon E_{\tau_1}\to E_{n\tau_1}$ is equivalent to $[n]\colon E_{\tau_2}\to E_{n\tau_2}$. This together with the classical theory gives $\Gamma_0(n)\tau_2=\Gamma_0(n)\tau_1$. The next corollary is important for our later discussion.

Corollary 3.2.2. *If n is an integer greater than two and if t_1,t_2 are positive real numbers then elliptic Blaschke product $\mathfrak{T}_{n,t_1}$ and $\mathfrak{T}_{n,t_2}$ are linearly equivalent to each other if and only if $t_1=t_2$.*

Proof. This follows immediately form Corollary 3.1.7 and Theorem 3.2.1 since the map $i\,\mathbb{R}_+\hookrightarrow Y(n)=\Gamma_0(n)\backslash\mathbb{H}$ is injective. □

If f is a rational function which is linearly equivalent to n_τ for some τ in $\mathbb{H}$ and some integer greater n than two then we call $\chi(f):=\tau\in\Gamma_0(n)\backslash\mathbb{H}$ the *character* of f. This is well-defined according to Theorem 3.2.1. For all $n\geq 3$ and for all $\tau\in\mathbb{H}$ we have $\chi(\mathfrak{T}_{n,\tau})=\tau/2$. For all $n\geq 3$ and for all $t>0$ we have $\chi(\mathfrak{T}_{n,t})=4ti/\pi$.

Elliptic rational functions those used in the theory of filter designs are only $\mathfrak{T}_{n,it}$ with $t>0$, and here we have considered a slightly more general family

$\mathcal{T}_{n,\tau}$ parameterized by τ ranging on the upper half plane $\mathbb{H}$. We write

$$\begin{aligned}\mu_{0,\tau} &= 0,\\ \mu_{1,\tau} &= \omega_{1,\tau},\\ \mu_{2,\tau} &= \frac{\omega_{2,\tau}}{2},\\ \mu_{3,\tau} &= \frac{2\omega_{1,\tau}+\omega_{2,\tau}}{2}\end{aligned}$$

and we shall prove

Theorem 3.2.3. *Given τ_1, τ_2 in $\mathbb{H}$ and given a rational integer $n \geq 3$. Elliptic rational functions $\mathcal{T}_{n,\tau_1}$ and $\mathcal{T}_{n,\tau_2}$ are equal if and only if $\Gamma_0(4n)\frac{\tau_1}{2} = \Gamma_0(4n)\frac{\tau_2}{2}$.*

Proof. We begin with recalling the following facts under the assumption that $n \geq 3$:

(a) The set of two torsion points $E_{2\omega_{1,\tau},\omega_{2,\tau}}[2]$ consists of $\mu_{i,\tau}$ for all $0 \leq i \leq 3$;

(b) Under the isogeny $[n]\colon E_{2\omega_{1,\tau},\omega_{2,\tau}} \to E_{2\omega_{1,n\tau},\omega_{2,n\tau}}$ we have that if n is odd then

$$\mu_{i,\tau} \mapsto \mu_{i,n\tau}$$

and if n is even then

$$\mu_{i,\tau} \mapsto \mu_{[i/2],n\tau}.$$

(c) Under the Kummer map $\mathrm{cd}\colon E_{2\omega_{1,\tau},\omega_{2,\tau}} \to \mathbb{P}^1$ we have

$$\begin{aligned}\mu_{0,\tau} &\mapsto +1,\\ \mu_{1,\tau} &\mapsto -1,\\ \mu_{2,\tau} &\mapsto +k(\tau)^{-1},\\ \mu_{3,\tau} &\mapsto -k(\tau)^{-1}.\end{aligned}$$

(d) Concerning critical values and critical points of $\mathcal{T}_{n,\tau}$ we deduce from Lemma 3.1.8 and from facts (a),(c) that

$$\begin{aligned}\mathfrak{o}_{\mathcal{T}_{n,\tau}} &= \{\pm 1, \pm k(n\tau)^{-1}\},\\ \mathcal{T}_{n,\tau}^{-1}(\mathfrak{o}_{\mathcal{T}_{n,\tau}}) \setminus \operatorname{supp} \mathfrak{D}_{\mathcal{T}_{n,\tau}} &= \{\pm 1, \pm k(\tau)^{-1}\}.\end{aligned}$$

(e) Under the map $\mathcal{T}_{n,\tau}\colon \mathbb{P}^1 \to \mathbb{P}^1$ we deduce from facts (b), (c) that if n is odd then

$$\pm 1 \mapsto \pm 1,\ \pm k(\tau)^{-1} \mapsto \pm k(n\tau)^{-1}$$

and if n is even then

$$\pm 1 \mapsto +1,\ \pm k(\tau)^{-1} \mapsto +k(n\tau)^{-1}.$$

Firstly we prove the "only if" part, i.e. assuming $\mathcal{T}_{n,\tau_1} = \mathcal{T}_{n,\tau_2}$ we will prove that $\Gamma_0(4n)\frac{\tau_1}{2} = \Gamma_0(4n)\frac{\tau_2}{2}$. We write for simplicity E'_τ for the elliptic curve $E_{2\omega_{1,\tau},\omega_{2,\tau}}$ and consider the identity $id \circ \mathcal{T}_{n,\tau_1} = \mathcal{T}_{n,\tau_2} \circ id$. It follows from a similar argument to that used in the proof of Theorem 3.2.1 that one can lift these two ids by Kummer maps, i.e. there exist group isomorphisms $\overline{\varepsilon}\colon E'_{\tau_1} \to E'_{\tau_2}$ and $\overline{\epsilon}\colon E'_{n\tau_1} \to E'_{n\tau_2}$ making the following diagram

$$\begin{tikzcd}
& E'_{\tau_2} \arrow[rr, "{[n]}"] \arrow[dd] & & E'_{n\tau_2} \arrow[dd] \\
E'_{\tau_1} \arrow[ur, "\overline{\varepsilon}"] \arrow[rr, "{[n]}"'] \arrow[dd] & & E'_{n\tau_1} \arrow[ur, "\overline{\epsilon}"'] \arrow[dd] & \\
& \mathbb{P}^1 \arrow[rr, "\mathcal{T}_{n,\tau_2}"] & & \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[ur, "id"] \arrow[rr, "\mathcal{T}_{n,\tau_1}"] & & \mathbb{P}^1 \arrow[ur, "id"] &
\end{tikzcd}$$

commutative, where all the vertical projections are given by the Kummer map cd. By an argument similar to that used in the proof of Theorem 3.2.1 we have

$$\overline{\varepsilon}(\omega_{2,\tau_1}) = a \cdot \omega_{2,\tau_2} + b \cdot 2\omega_{1,\tau_2}$$
$$\overline{\varepsilon}(2\omega_{1,.\tau_1}) = c \cdot \omega_{2,\tau_2} + d \cdot 2\omega_{1,\tau_2}$$

for some $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$, and in particular $\frac{\tau_1}{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\frac{\tau_2}{2}$. Using the commutative relation $\overline{\epsilon} \circ [n] = [n] \circ \overline{\varepsilon}$ we easily obtain

$$\overline{\epsilon}(\omega_{2,n\tau_1}) = a \cdot \omega_{2,n\tau_2} + nb \cdot 2\omega_{1,n\tau_2}$$
$$\overline{\epsilon}(2\omega_{1,n\tau_1}) = \frac{c}{n} \cdot \omega_{2,n\tau_2} + d \cdot 2\omega_{1,n\tau_2}.$$

Because $id \circ \mathrm{cd} = \mathrm{cd} \circ \overline{\epsilon}$ we deduce from fact (c) that $\overline{\epsilon}(\mu_{1,n\tau_1}) = \mu_{1,n\tau_2}$. This together with $\overline{\epsilon}(\mu_{1,n\tau_1}) = c\omega_{2,n\tau_2}/2n + d\omega_{1,n\tau_2}$ implies that $2n|c$. The relation $\frac{\tau_1}{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{\tau_2}{2}$ leads to $n\tau_1 = \begin{pmatrix} a & 2nb \\ c/2n & d \end{pmatrix} n\tau_2$. Using the identity $id \circ \mathrm{cd} = \mathrm{cd} \circ \overline{\epsilon}$

and fact (c) we deduce that either

$$\overline{\epsilon}(\mu_{2,n\tau_1})=\mu_{2,n\tau_2},\ \overline{\epsilon}(\mu_{3,n\tau_1})=\mu_{3,n\tau_2},\ k(n\tau_1)=k(n\tau_2)$$

or

$$\overline{\epsilon}(\mu_{2,n\tau_1})=\mu_{3,\tau_2},\ \overline{\epsilon}(\mu_{3,n\tau_1})=\mu_{2,n\tau_2},\ k(n\tau_1)=-k(n\tau_2).$$

Simple calculation leads to $\overline{\epsilon}(\mu_{2,n\tau_1}) = \frac{a}{2}\omega_{2,\tau_2} + nb\omega_{1,\tau_2}$, and therefore if nb is even (or odd) then we fall into the former (or latter) case. Equivalently we have

$$k(n\tau_1)=(-1)^{nb}k(n\tau_2). \tag{3.2}$$

This implies in particular that $\lambda(n\tau_1)=\lambda(n\tau_2)$. However the transformation formula (1.29) of λ applies and leads to

$$\lambda(n\tau_1) = \begin{cases} \lambda(n\tau_2) & \text{if } \ \frac{c}{2n}\equiv 0 \pmod 2; \\ \\ \lambda(n\tau_2)^{-1} & \text{if } \ \frac{c}{2n}\equiv 1 \pmod 2, \end{cases}$$

and therefore either $c/(2n)$ is even or $c/(2n)$ is odd but with $\lambda(n\tau_1)=\lambda(n\tau_2)=-1$. If we are in the latter case then we have $k^2(n\tau_1)=k^2(n\tau_2)=-1$, and by (1.28)

$$k(n\tau_1)=(-1)^{nb}/k(n\tau_2)=(-1)^{nb+1}k(n\tau_2)$$

which contradicts to (3.2). Therefore we have $4n|c$ which leads to our assertion that $\Gamma_0(4n)\frac{\tau_1}{2}=\Gamma_0(4n)\frac{\tau_1}{2}$.

Secondly we will prove $\mathfrak{T}_{n,\tau_1}=\mathfrak{T}_{n,\tau_2}$ under the assumption that $\Gamma_0(4n)\frac{\tau_1}{2}=\Gamma_0(4n)\frac{\tau_1}{2}$. By this assumption there exists $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ in $\Gamma_0(4n)$ such that $\frac{\tau_1}{2}=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\frac{\tau_2}{2}$. Considering the following group isomorphisms

$$\begin{aligned} \overline{\varepsilon}\colon z\in E'_{\tau_1} &\mapsto \frac{(c\omega_{2,\tau_2}+2d\omega_{1,\tau_2})z}{2\omega_{1,\tau_1}}\in E'_{\tau_2}, \\ \overline{\epsilon}\colon z\in E'_{n\tau_1} &\mapsto \frac{(c\omega_{2,n\tau_2}/n+2d\omega_{1,n\tau_2})z}{2\omega_{1,n\tau_1}}\in E'_{n\tau_2}. \end{aligned}$$

These maps together with their descends $\varepsilon,\epsilon\colon\mathbb{P}^1\to\mathbb{P}^1$ make the following diagram

$$\begin{tikzcd}
& E'_{\tau_2} \arrow[rr, "{[n]}"] \arrow[dd] & & E'_{n\tau_2} \arrow[dd] \\
E'_{\tau_1} \arrow[ur, "\overline{\varepsilon}"] \arrow[rr, "{[n]}"] \arrow[dd] & & E'_{n\tau_1} \arrow[ur, "\overline{\epsilon}"'] \arrow[dd] & \\
& \mathbb{P}^1 \arrow[rr, "\mathfrak{T}_{n,\tau_2}"] & & \mathbb{P}^1 \\
\mathbb{P}^1 \arrow[ur, "\varepsilon"] \arrow[rr, "\mathfrak{T}_{n,\tau_1}"] & & \mathbb{P}^1 \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

commutative, where all the vertical projections are given by the Kummer map cd. It suffices to prove that $\varepsilon = id$ and $\epsilon = id$, since this together with $\epsilon \circ \mathfrak{T}_{n,\tau_1} = \mathfrak{T}_{n,\tau_2} \circ \varepsilon$ leads to $\mathfrak{T}_{n,\tau_1} = \mathfrak{T}_{n,\tau_2}$. We deduce from $d \equiv 1 \pmod 2$ that $\overline{\varepsilon}(\mu_{1,\tau_1}) = \mu_{1,\tau_2}$ and $\overline{\epsilon}(\mu_{1,n\tau_1}) = \mu_{1,n\tau_2}$. Then our fact (c) leads to $\varepsilon(\pm 1) = \pm 1$ and $\epsilon(\pm 1) = \pm 1$. Consequently the fact that ϵ, ε are identities will follow from equalities $\varepsilon(1/k(\tau_1)) = 1/k(\tau_1)$ and $\epsilon(1/k(n\tau_1)) = 1/k(n\tau_1)$.

Indeed by the direct calculation we have $\varepsilon(\mu_{2,\tau_1}) = \frac{a\omega_{2,\tau_2} + 2b\omega_{1,\tau_2}}{2}$. Therefore if b is even then $\overline{\varepsilon}(\mu_{2,\tau_1}) = \mu_{2,\tau_2}$ and if b is odd then $\overline{\varepsilon}(\mu_{2,\tau_1}) = \mu_{3,\tau_2}$. This together with fact (c) implies that

$$\varepsilon(1/k(\tau_1)) = (-1)^b / k(\tau_2). \tag{3.3}$$

Moreover applying the first part of (1.28) we deduce that

$$k(\tau_1) = (-1)^b k(\tau_2) \tag{3.4}$$

The equality (3.3) together with (3.4) gives that $\varepsilon(1/k(\tau_1)) = 1/k(\tau_1)$, and therefore ε is the identity map.

The relation $\frac{\tau_1}{2} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{\tau_2}{2}$ leads to $\frac{n\tau_1}{2} = \begin{pmatrix} a & nb \\ c/n & d \end{pmatrix} \frac{n\tau_2}{2}$ and $n\tau_1 = \begin{pmatrix} a & 2nb \\ c/(2n) & d \end{pmatrix} n\tau_2$. Because $4n|c$ a similar argument similar to that used in previous paragraph applies and gives that

$$\begin{aligned} \epsilon(1/k(n\tau_1)) &= (-1)^{nb} / k(n\tau_2) \\ k(n\tau_1) &= (-1)^{nb} k(n\tau_2) \end{aligned}$$

which leads to $\epsilon(1/k(n\tau_1)) = 1/k(n\tau_1)$, and finally that ϵ is also the identity. $\square$

To sum up the space $\mathrm{Aut}(\mathbb{P}^1) \backslash \{$elliptic rational functions of degree n$\} / \mathrm{Aut}(\mathbb{P}^1)$ is $Y_0(n)$, and the space $\{\mathfrak{T}_{n,\tau} | \tau \in \mathbb{H}\}$ is $Y_0(4n)$.

3.3 Siegel factors and Faltings factors

Given finite Blaschke products f and g, we will use the fact that the curve $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ admits a anti-holomorphic involution. Indeed it is a double of $\mathbb{E} \times_{f,g} \mathbb{E}$ which will follow from the functional equation (1.5). Recalling that $\mathbb{T}$ is the unit circle and letting $\overline{\mathbb{E}}^c$ be the complement of the closed unit disk in the projective line, it is clear that

$$\mathbb{P}^1 \times_{f,g} \mathbb{P}^1 = \mathbb{E} \times_{f,g} \mathbb{E} \cup \mathbb{T} \times_{f,g} \mathbb{T} \cup \overline{\mathbb{E}}^c \times_{f,g} \overline{\mathbb{E}}^c.$$

We write $\mathfrak{R}$ for $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$, $\mathfrak{R}'$ for $\mathbb{E} \times_{f,g} \mathbb{E}$ and $\mathfrak{R}''$ for $\overline{\mathbb{E}}^c \times_{f,g} \overline{\mathbb{E}}^c$.

Lemma 3.3.1. *Let f, g be two finite Blaschke products and $\mathfrak{R}'$, $\mathfrak{R}$ and $\mathfrak{R}''$ defined as above. There is a one-one correspondence between irreducible components of $\mathfrak{R}'$ and irreducible components of $\mathfrak{R}$ such that a component $\mathfrak{A}'$ of $\mathfrak{R}'$ homeomorphic to an affine curve of type (g,d) corresponds to a component $\mathfrak{A}$ of $\mathfrak{R}$ of type $(2g+d-1, 0)$.*

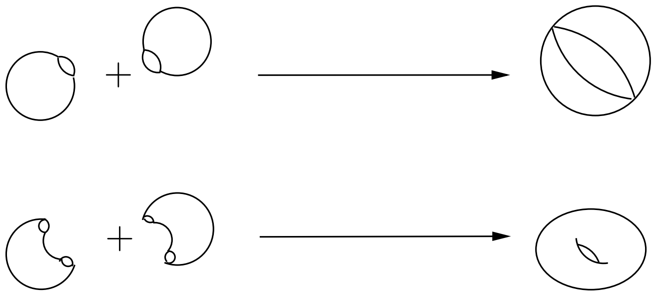
Proof. We write $\mathfrak{R}^0 = \mathbb{T} \times_{f,g} \mathbb{T}$ and we find that $\mathfrak{R}^0$ consists of finite many real closed smooth curves in $\mathfrak{R}$ and $\mathfrak{R} = \mathfrak{R}' \cup \mathfrak{R}^0 \cup \mathfrak{R}''$. If $\mathfrak{A}'$ is an irreducible component of $\mathfrak{R}'$ then $\mathfrak{A}'' = \{(x,y) | (1/\overline{x}, 1/\overline{y}) \in \mathfrak{A}'\}$ is an irreducible component of $\mathfrak{R}''$ by

$$(x,y) \in \mathfrak{R}' \Leftrightarrow (1/\overline{x}, 1/\overline{y}) \in \mathfrak{R}'' \tag{3.5}$$

as follows from the Schwarz reflection principle. The irreducible component $\mathfrak{A}$ of $\mathfrak{R}$ which contains $\mathfrak{A}'$ is given by $\overline{\mathfrak{A}'} \cup \overline{\mathfrak{A}''}$ and the correspondence given by $\mathfrak{A}' \to \mathfrak{A}$ is easily checked to be a bijection as desired. If $\mathfrak{A}'$ is of type (g,d) then we shall prove $\mathfrak{A}$ is of type $(2g+d-1, 0)$. The correspondence (3.5) implies that $\mathfrak{A}'$ is homeomorphic to $\mathfrak{A}''$ and we have $\chi(\mathfrak{A}') = \chi(\mathfrak{A}'') = 2-2g-d$. Moreover the contribution of $\mathfrak{A} \backslash \mathfrak{A}' \cup \mathfrak{A}''$ which consists of finitely many circles to the Euler characteristic is 0, therefore $\chi(\mathfrak{A}) = 2(2-2g-d)$ and finally the genus of $\mathfrak{A}$ is $2g+d-1$. □

We use notations and results from Section 1.6. Let $i: \mathbb{E} \to \mathbb{C}$ be a homeomorphism, choose $j_1: \mathbb{E} \to \mathbb{C}$ to be a f-lifting of i and $j_2: \mathbb{E} \to \mathbb{C}$ to be a g-lifting of i. We write $\mathfrak{R}_* = \mathbb{C} \times_{(j_1,i)_* f, (j_2,i)_* g} \mathbb{C}$ and get

Proposition 3.3.2. *There is a one-one correspondence between irreducible*



Glue of Siegel factors of $\mathbb{E} \times_{f,g} \mathbb{E}$ and $\overline{\mathbb{E}}^c \times_{f,g} \overline{\mathbb{E}}^c$

Faltings factor of $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$

Figure 3.1: Curves defined by Blaschke products

components of $\mathfrak{R}$ *of type* $(0,0)$ *and irreducible components of* $\mathfrak{R}_*$ *of type* $(0,1)$ *as well as a one-one correspondence between irreducible components of* $\mathfrak{R}$ *of type* $(1,0)$ *and irreducible components of* $\mathfrak{R}_*$ *of type* $(0,2)$.

Proof. By Lemma 3.3.1 if $\mathfrak{R}$ has a component of type $(0,0)$ then $\mathfrak{R}'$ has a component which is homeomorphic to an affine curve of type (g,d) such that $2g+d-1=0$. This gives $(g,d)=(0,1)$ and therefore $\mathfrak{R}_*$ has a component of type $(0,1)$ since $\mathfrak{R}_*$ topologically equals to $\mathfrak{R}'$. Similar arguments apply to the case $\mathfrak{R}$ has a component of type $(1,0)$. □

By Proposition 3.3.2 there is a one-one correspondence between Faltings factors of $\mathfrak{R}$ and Siegel factors of $\mathfrak{R}_*$ as illustrated in Figure 3.1. The crucial point behind the proof of this fact is that *an algebraic curve* X *is abelian if and only if* $\chi(X) \geq 0$. The existence of a symmetry above implies that our curve $\mathfrak{R}$ is defined over $\mathbb{R}$, and $\mathfrak{R}$ is the *double* of $\mathfrak{R}'$ according to the theory of Fuchian groups.

3.4 Davenport-Lewis-Schinzel's formula

Davenport-Lewis-Schinzel derived in [37, p.305] a very interesting formula for the factorization of $T_n(x)+T_n(y)$ where T_n denotes the Chebyshev polynomial of degree n and similar formula for $T_n(x)-T_n(y)$ was found for instance in [112]. The formulae are very important in the work of Bilu-Tichy [15] and of Avanzi-Zannier [5]. We collect them as follows,

Proposition 3.4.1 (Davenport-Lewis-Schinzel). *If n is even then we have*

$$T_n(x)+T_n(y) = 2^{n-1}\prod_{k=1}^{\frac{n}{2}}\left(x^2-2xy\cos\frac{(2k-1)\pi}{n}+y^2-\sin^2\frac{(2k-1)\pi}{n}\right)$$

and if n is odd then we have

$$\frac{T_n(x)+T_n(y)}{x+y} = 2^{n-1}\prod_{k=1}^{\frac{n-1}{2}}\left(x^2-2xy\cos\frac{(2k-1)\pi}{n}+y^2-\sin^2\frac{(2k-1)\pi}{n}\right).$$

By applying the method of Davenport-Lewis-Schinzel one may easily deduce another well-known formula: *if n is odd then we have*

$$\frac{T_n(x)-T_n(y)}{x-y} = 2^{n-1}\prod_{k=1}^{\frac{n-1}{2}}\left(x^2-2\cos\frac{2k\pi}{n}xy+y^2-\sin^2\frac{2k\pi}{n}\right)$$

and if n is even then we have

$$\frac{T_n(x)-T_n(y)}{x^2-y^2} = 2^{n-1}\prod_{k=1}^{\frac{n-2}{2}}\left(x^2-2\cos\frac{2k\pi}{n}xy+y^2-\sin^2\frac{2k\pi}{n}\right).$$

Therefore Chebyshev polynomials are very "exceptional" in the sense that both $\mathbb{C}\times_{T_n,T_n}\mathbb{C}$ and $\mathbb{C}\times_{T_n,-T_n}\mathbb{C}$ have a large number of components which in addition are abelian. The exceptionality of T_n was further clarified in for instance the work of Bilu-Tichy ([14] and [15]) and the work of Avanzi-Zannier ([5]) together with applications to arithmetic problems.

In this section we shall prove elliptic analogues of the formulae and to achieve this we employ Davenport-Lewis-Schinzel's original idea but with the addition formulae for trigonometric functions used there replaced by addition formulae for elliptic functions. We start with a description of $\mathbb{P}^1\times_{n_\tau,n_\tau}\mathbb{P}^1$ and then we shall investigate $\mathbb{P}^1\times_{\mathcal{T}_{n,\tau},\mathcal{T}_{n,\tau}}\mathbb{P}^1$ and $\mathbb{P}^1\times_{\mathcal{T}_{n,\tau},-\mathcal{T}_{n,\tau}}\mathbb{P}^1$. The non existence of an analogous result for $\mathbb{P}^1\times_{n_\tau,-n_\tau}\mathbb{P}^1$ again explains that in the formal dictionary between trigonometric functions and elliptic functions cd should be regarded as a counterpart of cos and $\mathcal{T}_{n,\tau}$ should be regarded as a counterpart of T_n. Given a point $([z_0:z_1],[w_0:w_1])$ in $\mathbb{P}^1\times\mathbb{P}^1$ we let $u=z_0/z_1$ and $v=w_0/w_1$ be the affine coordinates of its projection images. We write for short $e_i = e_i(\tau)$ and $\Delta_{\mathbb{P}^1} = \{(u,\, v) : u = v\}$.

Proposition 3.4.2. *If n is odd then we have*

$$\mathbb{P}^1 \times_{n_\tau,\, n_\tau} \mathbb{P}^1 = \Delta_{\mathbb{P}^1} \cup \mathfrak{A}_1 \cup \mathfrak{A}_2 \cup \cdots \cup \mathfrak{A}_{(n-1)/2}$$

and if n is even then we have

$$\mathbb{P}^1 \times_{n_\tau,\, n_\tau} \mathbb{P}^1 = \Delta_{\mathbb{P}^1} \cup \mathfrak{A}_0 \cup \mathfrak{A}_1 \cup \cdots \cup \mathfrak{A}_{(n-2)/2}$$

where $\mathfrak{A}_0$ is a rational curve defined by

$$u = \frac{e_1 v + e_2 e_3 + e_1^2}{v - e_1}$$

and $\mathfrak{A}_i (i \geq 1)$ is a genus one curve defined by

$$u^2 = \frac{\left(v + \wp_\tau\left(\frac{i}{n}\right)\right)\left(2\wp_\tau\left(\frac{i}{n}\right) v - \frac{g_2}{2}\right) - g_3}{\left(v - \wp_\tau\left(\frac{i}{n}\right)\right)^2} u - \frac{\left(\wp_\tau\left(\frac{i}{n}\right) v + \frac{g_2}{4}\right)^2 + g_3\left(v + \wp_\tau\left(\frac{i}{n}\right)\right)}{\left(v - \wp_\tau\left(\frac{i}{n}\right)\right)^2}.$$

Proof. We note that

$$n_\tau(\wp_\tau(z)) = \wp_{n\tau}(nz).$$

Setting $u = \wp_\tau(z)$ and $v = \wp_\tau(z')$ then we have

$$n_\tau(u) - n_\tau(v) = \wp_{n\tau}(nz) - \wp_{n\tau}(nz').$$

This gives easily that for any integer i the equality

$$\wp_\tau(z) - \wp_\tau(z' + i/n) = 0 \tag{3.6}$$

leads to $n_\tau(u) = n_\tau(v)$. Using (1.10) and (1.11) the product of $\wp_\tau(z) - \wp_\tau(z' + i/n)$ and $\wp_\tau(z) - \wp_\tau(z' - i/n)$ equals

$$\wp_\tau(z)^2 - \frac{\left(\wp_\tau(z') + \wp_\tau(i/n)\right)\left(2\wp_\tau(z')\wp_\tau(i/n) - g_2/2\right) - g_3}{\left(\wp_\tau(z') - \wp_\tau(i/n)\right)^2} \wp_\tau(z)$$
$$+ \frac{\left(\wp_\tau(z')\wp_\tau(i/n) + g_2/4\right)^2 + g_3\left(\wp_\tau(z') + \wp_\tau(i/n)\right)}{\left(\wp_\tau(z') - \wp_\tau(i/n)\right)^2}$$

and therefore

$$u^2 - \frac{\left(v + \wp_\tau\left(\frac{i}{n}\right)\right)\left(2\wp_\tau\left(\frac{i}{n}\right) v - \frac{g_2}{2}\right) - g_3}{\left(v - \wp_\tau\left(\frac{i}{n}\right)\right)^2} u + \frac{\left(\wp_\tau\left(\frac{i}{n}\right) v + \frac{g_2}{4}\right)^2 + g_3\left(v + \wp_\tau\left(\frac{i}{n}\right)\right)}{\left(v - \wp_\tau\left(\frac{i}{n}\right)\right)^2} = 0$$

is an algebraic factor of $\mathbb{P}^1 \times_{n_\tau,\, n_\tau} \mathbb{P}^1$.

If $i/n=1/2$ then the factor (3.6) is already an algebraic one. Indeed by using (1.9), (1.12) and (1.13) we have

$$\begin{aligned}
\wp_\tau(z) &= \wp_\tau\left(z'+\frac{1}{2}\right)\\
&= -\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)+\frac{1}{4}\frac{\wp'_\tau(z')^2}{\left(\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)\right)^2}\\
&= -\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)+\frac{\left(\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)\right)\left(\wp_\tau(z')-\wp_\tau\left(\frac{\tau}{2}\right)\right)\left(\wp_\tau(z')-\wp_\tau\left(\frac{1+\tau}{2}\right)\right)}{\left(\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)\right)^2}\\
&= \frac{\wp_\tau\left(\frac{1}{2}\right)\wp_\tau(z')+\wp_\tau\left(\frac{\tau}{2}\right)\wp_\tau\left(\frac{1+\tau}{2}\right)+\wp_\tau^2\left(\frac{1}{2}\right)}{\wp_\tau(z')-\wp_\tau\left(\frac{1}{2}\right)}
\end{aligned}$$

and therefore for any even number n the rational curve $u(v-e_1)=e_1v+e_2e_3+e_1^2$ is a factor of $\mathbb{P}^1\times_{n_\tau,n_\tau}\mathbb{P}^1$. □

Proposition 3.4.3. *Given $\tau\in\mathbb{H}$ if n is odd then we have*

$$\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},\mathfrak{T}_{n,\tau}}\mathbb{P}^1=\Delta_{\mathbb{P}^1}\cup\mathfrak{A}_1\cup\mathfrak{A}_2\cup\cdots\cup\mathfrak{A}_{(n-1)/2}$$

and if n is even then we have

$$\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},\mathfrak{T}_{n,\tau}}\mathbb{P}^1=\Delta_{\mathbb{P}^1}\cup\mathfrak{A}_0\cup\mathfrak{A}_1\cup\cdots\cup\mathfrak{A}_{(n-2)/2}$$

where $\mathfrak{A}_0$ is a rational curve defined by

$$u+v=0$$

and $\mathfrak{A}_r(r\geq 1)$ is a genus one curve defined by

$$u^2=\frac{2\,\mathrm{cn}\left(\frac{2r\omega_1}{n}\right)\,\mathrm{dn}\left(\frac{2r\omega_1}{n}\right)v}{1-k^2\,\mathrm{sn}^2\left(\frac{2r\omega_1}{n}\right)\,v^2}u-\frac{v^2-\mathrm{sn}^2\left(\frac{2r\omega_1}{n}\right)}{1-k^2\,\mathrm{sn}^2\left(\frac{2r\omega_1}{n}\right)\,v^2}.$$

Proof. We start from recalling the equality (3.1),

$$\mathfrak{T}_{n,\tau}(\mathrm{cd}(z,\,k(\tau)))=\mathrm{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z,\,k(n\tau)\right).$$

Setting $u=\mathrm{cd}(z,k(\tau))$ and $v=\mathrm{cd}(z',k(\tau))$ we have

$$\mathfrak{T}_{n,\tau}(u)-\mathfrak{T}_{n,\tau}(v)=\mathrm{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z,\,k(n\tau)\right)-\mathrm{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z',\,k(n\tau)\right).$$

This implies easily that for any integer r the equality

$$\operatorname{cd}(z,\ k(\tau))-\operatorname{cd}\left(z'+\frac{2\omega_{1,\tau}r}{n},\ k(\tau)\right)=0 \tag{3.7}$$

leads to $\mathfrak{T}_{n,\tau}(u)=\mathfrak{T}_{n,\tau}(v)$. It follows from (1.26) and (1.27) that the product of $\operatorname{cd}z-\operatorname{cd}\left(z'+\frac{2r\omega_1}{n}\right)$ and $\operatorname{cd}z-\operatorname{cd}\left(z'-\frac{2r\omega_1}{n}\right)$ equals

$$\operatorname{cd}^2 z-\operatorname{cd} z\,\frac{2\operatorname{cn}\left(\frac{2r\omega_1}{n}\right)\,\operatorname{dn}\left(\frac{2r\omega_1}{n}\right)\,\operatorname{cd} z'}{1-k^2\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)\,\operatorname{cd}^2 z'}+\frac{\operatorname{cd}^2 z'-\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)}{1-k^2\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)\,\operatorname{cd}^2 z'}.$$

Therefore

$$u^2-\frac{2\operatorname{cn}\left(\frac{2r\omega_1}{n}\right)\,\operatorname{dn}\left(\frac{2r\omega_1}{n}\right)v}{1-k^2\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)\,v^2}u+\frac{v^2-\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)}{1-k^2\operatorname{sn}^2\left(\frac{2r\omega_1}{n}\right)\,v^2}=0$$

is an algebraic factor of $\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},\,\mathfrak{T}_{n,\tau}}\mathbb{P}^1$.

If $r/n = 1/2$ then the factor (3.7) is already an algebraic one defined by $u+v=0$ as follows from (1.23). □

Proposition 3.4.4. *Given $\tau\in\mathbb{H}$ if n is even then we have*

$$\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},\,-\mathfrak{T}_{n,\tau}}\mathbb{P}^1=\mathfrak{A}_1\cup\mathfrak{A}_2\cup\cdots\cup\mathfrak{A}_{n/2}$$

and if n is odd then we have

$$\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},\,-\mathfrak{T}_{n,\tau}}\mathbb{P}^1=\mathfrak{A}_0\cup\mathfrak{A}_1\cup\cdots\cup\mathfrak{A}_{(n-1)/2}$$

where $\mathfrak{A}_0$ is a rational curve defined by

$$u+v=0$$

and $\mathfrak{A}_r(r\geq 1)$ is a genus one curve defined by

$$u^2=\frac{2\operatorname{cn}\left(\frac{(2r+1)\omega_1}{n}\right)\operatorname{dn}\left(\frac{(2r+1)\omega_1}{n}\right)v}{1-k^2\operatorname{sn}^2\left(\frac{(2r+1)\omega_1}{n}\right)v^2}u-\frac{v^2-\operatorname{sn}^2\left(\frac{(2r+1)\omega_1}{n}\right)}{1-k^2\operatorname{sn}^2\left(\frac{(2r+1)\omega_1}{n}\right)v^2}.$$

Proof. We start from recalling the equality (3.1),

$$\mathfrak{T}_{n,\tau}(\operatorname{cd}(z,\ k(\tau)))=\operatorname{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z,\ k(n\tau)\right).$$

Setting $u=\operatorname{cd}(z,k(\tau))$ and $v=\operatorname{cd}(z',k(\tau))$ we have

$$\mathfrak{T}_{n,\tau}(u)+\mathfrak{T}_{n,\tau}(v)=\operatorname{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z,\ k(n\tau)\right)+\operatorname{cd}\left(\frac{n\omega_{1,n\tau}}{\omega_{1,\tau}}z',\ k(n\tau)\right).$$

This implies easily that the equality

$$\mathrm{cd}(z,k(\tau))-\mathrm{cd}\left(z'+\frac{(2r+1)\omega_{1,\tau}}{n},k(\tau)\right)=0 \tag{3.8}$$

leads to $\mathcal{T}_{n,\tau}(u)+\mathcal{T}_{n,\tau}(v)=0$. It follows from (1.26) and (1.27) that the product of $\mathrm{cd}z-\mathrm{cd}\big(z'+\frac{(2r+1)\omega_1}{n}\big)$ and $\mathrm{cd}z-\mathrm{cd}\big(z'-\frac{(2r+1)\omega_1}{n}\big)$ equals

$$\mathrm{cd}^2z-\mathrm{cd}\,z\,\frac{2\mathrm{cn}\big(\frac{(2r+1)\omega_1}{n}\big)\,\mathrm{dn}\big(\frac{(2r+1)\omega_1}{n}\big)\,\mathrm{cd}\,z'}{1-k^2\,\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)\,\mathrm{cd}^2z'}+\frac{\mathrm{cd}^2z'-\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)}{1-k^2\,\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)\,\mathrm{cd}^2z'}.$$

Therefore

$$u^2-\frac{2\,\mathrm{cn}\big(\frac{(2r+1)\omega_1}{n}\big)\,\mathrm{dn}\big(\frac{(2r+1)\omega_1}{n}\big)v}{1-k^2\,\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)v^2}u+\frac{v^2-\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)}{1-k^2\,\mathrm{sn}^2\big(\frac{(2r+1)\omega_1}{n}\big)v^2}=0$$

is a factor of $\mathbb{P}^1\times_{\mathcal{T}_{n,\tau},-\mathcal{T}_{n,\tau}}\mathbb{P}^1$ as desired.

If $2r+1=n$ then the factor (3.8) is already an algebraic one defined by $u+v=0$ as follows immediately from (1.23). $\square$

3.5 The generalized Schur Problem

If $f\colon\mathfrak{M}\to\mathfrak{N}$ is a finite map between algebraic curves then f is called *exceptional* if and only if $\mathfrak{M}\times_{f,f}\mathfrak{M}$ has at least three components. Motivated by Schur's paper [118] Fried raised the following question in [53].

Question 3.5.1 (Generalized Schur Problem for rational functions)**.** *Classify exceptional rational functions.*

For polynomials this question was answered by Fried in [51].

Theorem 3.5.2 (Fried)**.** *If f is a polynomial then it is exceptional if and only if up to linear compositions $f(z)=z^p$ for a prime number p greater than 2 or $f=T_p$ for a prime number p greater than* 3*.*

To verify that power maps and Chebyshev polynomials are exceptional we notice that if $f(z)=z^p$ then

$$(f(x)-f(y))/(x-y)=\prod_{j=1}^{p-1}(x-e^{\frac{2\pi j}{p}i}y)$$

and if $f(z)=T_p(z)$ then the corresponding factorization formula is of

Davenport-Lewis-Schinzel's type as described in Section 3.4. On the basis of our Section 3.3 we are able to deduce the hyperbolic analogue of Fried's theorem. It turns out that a new class of exceptional functions comes from the elliptic rational functions.

Theorem 3.5.3. *If f is a finite Blaschke product then it is exceptional if and only if up to linear compositions $f(z)=z^p$ for a prime number p greater than 2 or $f=\mathcal{T}_{p,t}$ for a positive real number t and a prime number p greater than 3.*

Proof. We follow the notation of Section 3.3. Taking $g=f$ we may assume that $j_1=j_2=j$. By Lemma 3.3.1 we deduce that if f is exceptional then $\overline{f}:=(j,\,i)_*f$ is also exceptional. Now we obtain from Theorem 3.5.2 that up to linear compositions either $\overline{f}(z)=z^p$ with p a prime number other than 2 or $\overline{f}=T_p$ with p a prime number greater than 3.

In the former case the map $f\colon\mathbb{E}\to\mathbb{E}$ is also totally ramified and therefore up to linear compositions $f(z)=z^p$.

In the latter case the monodromy representation of $f\colon\mathbb{E}\to\mathbb{E}$ is a Chebyshev representation. By the uniqueness part of Riemann existence theorem and by Lemma 2.3.1 it must be an elliptic Blaschke product and then we conclude from Corollary 2.3.4 that up to linear compositions $f=\mathcal{T}_{p,t}$. □

Using the classification of finite simple groups Guralnick-Müller-Saxl have made a monumental progress on the generalized Schur Problem in [65] but with neither explicit expressions of exceptional functions nor explicit factorizations. Our next Theorem 3.5.3 should be an easy consequence of one of the main theorems of loc. cit. . However we proved it by different and relatively elementary method, and our previous Proposition 3.4.3 gives explicit factorizations.

3.6 The generalized Fried Problem

Diophantine equations defined by two polynomials $f(x)=g(y)$ were studied by many authors. Qualitative finiteness results were investigated by Siegel [121], Davenport-Lewis-Schinzel [37], Fried [52], Schinzel [111], Avanzi-Zannier

[4] [5] and many others. Effective analysis on superelliptic curves were started by Baker [6] which is an important progress in mathematics. Although the general effectivity is still open, the ultimate qualitative results on integral points is due to Bilu-Tichy [15]. Instead of Bilu-Tichy's much stronger version for an arbitrary field of characteristic zero we only employ the following simpler version for polynomials defined over $\mathbb{C}$.

Theorem 3.6.1 (Bilu-Tichy)**.** *Let f and g be nonlinear polynomials. If $\mathbb{C}\times_{f,g}\mathbb{C}$ has a Siegel factor then f and g admit the following decompositions into polynomials*

$$f=e\circ f_1\circ\varepsilon, \quad g=e\circ g_1\circ\epsilon$$

where ε, ϵ lie in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ and there exist positive integers m, n together with a non vanishing polynomial p such that $\{f_1, g_1\}$ falls into one of the following cases:

(i) $\{z^m, z^r p(z)^m\}$ *with* $r \geq 0$ *and* $\gcd(r, m) = 1$*;*

(ii) $\{z^2, (z^2+1)p(x)^2\}$*;*

(iii) $\{T_m, T_n\}$ *with* $\gcd(m, n) = 1$*;*

(iv) $\{T_m, -T_n\}$ *with* $\gcd(m, n) > 1$*;*

(v) $\{(z^2-1)^3, 3z^4-4z^3\}$.

Bilu-Tichy's theorem completely answers Fried's Problem in [52]. To study rational points it is natural to reformulate Fried's original problem as follows,

Question 3.6.2 (Generalized Fried Problem)**.** *Classify the rational function pairs $\{f, g\}$ such that $\mathbb{P}^1\times_{f,g}\mathbb{P}^1$ has a Faltings factor.*

In this section we shall prove

Theorem 3.6.3. *Let f and g be finite Blaschke products. If $\mathbb{P}^1\times_{f,g}\mathbb{P}^1$ has a Faltings factor then f and g admit the following decompositions into finite Blaschke products*

$$f=e\circ f_1\circ\varepsilon, \quad g=e\circ g_1\circ\epsilon$$

where ε, ϵ lie in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and there exist positive integers m, n together with p either a finite Blaschke product or $p \equiv 1$ such that $\{f_1, g_1\}$ falls into one of the following cases:

(i) $\{z^m, z^r p(z)^m\}$ *with* $r \geq 0$ *and* $\gcd(r, m) = 1$*;*

(ii) $\{z^2, z(z-a)/(1-\overline{a}z)p(z)^2\}$ *with* $a \neq 0$ *in* $\mathbb{E}$*;*

(iii) $\{\mathfrak{T}_{m,nt}, \mathfrak{T}_{n,mt}\}$ *with* $t>0$ *and* $\gcd(m,n)=1$;

(iv) $\{\mathfrak{T}_{m,nt}, -\mathfrak{T}_{n,mt}\}$ *with* $t>0$ *and* $\gcd(m,n)>1$;

(v) $\{((z^2-a^2)/(1-\overline{a}^2z^2))^3, z^3(z-b)/(1-\overline{b}z)\}$ *where* a, b *are points in* $\mathbb{E}$ *and* $a, b, \overline{a}, \overline{b}$ *satisfy an algebraic relation.*

The converse remains true.

Proof. In the following we use the notation used in Section 3.3 and write further that $\overline{f} := (j_1, \imath)_* f$, $\overline{g} := (j_2, \imath)_* g$. By definition we have $f = \imath^{-1} \circ \overline{f} \circ j_1$, $g = \imath^{-1} \circ \overline{g} \circ j_2$ and this shows that j_1^{-1} is a $\overline{f}$-lifting of $\imath^{-1}$. If $\mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ has a Faltings factor then by Proposition 3.3.2 the curve $\mathbb{C} \times_{\overline{f},\overline{g}} \mathbb{C}$ has a Siegel factor and then by Bilu-Tichy's Criterion we may assume that there exist $\overline{\varepsilon}, \overline{\epsilon}$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ such that $\overline{f}$, $\overline{g}$ admit one of the following decompositions into polynomials:

(1) $\overline{f} = \overline{e} \circ z^m \circ \overline{\varepsilon}$, $\quad \overline{g} = \overline{e} \circ z^r \overline{p}(z)^m \circ \overline{\epsilon}$.

Let i_1 be a $\overline{e}$-lifting of $\imath^{-1}$ and i_2 a z^m-lifting of i_1. Proposition 1.6.4 together with an induction argument implies that j_1^{-1} is also a $\overline{\varepsilon}$-lifting of i_2, and consequently $f = e \circ f_1 \circ \varepsilon$ is a composition of finite Blaschke products where e, f_1 and ε are obtained by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{\varepsilon}"'] \arrow[rrr, bend left, "\overline{f}"] \arrow[d, "j_1^{-1}"] & \mathbb{C} \arrow[r, "z^m"'] \arrow[d, "i_2"] & \mathbb{C} \arrow[r, "\overline{e}"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "\imath^{-1}"] \\
\mathbb{E} \arrow[r, "\varepsilon"] \arrow[rrr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_1"] & \mathbb{E} \arrow[r, "e"] & \mathbb{E}
\end{tikzcd}$$

Similarly if i_2' is a $z^r\overline{p}(z)^m$-lifting of i_1 then $g = e \circ g_1 \circ \epsilon$ is also a decomposition of finite Blaschke products according to the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{\epsilon}"'] \arrow[rrr, bend left, "\overline{g}"] \arrow[d, "j_2^{-1}"] & \mathbb{C} \arrow[r, "z^r\overline{p}(z)^m"'] \arrow[d, "i_2'"] & \mathbb{C} \arrow[r, "\overline{e}"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "\imath^{-1}"] \\
\mathbb{E} \arrow[r, "\epsilon"] \arrow[rrr, bend right, "g"'] & \mathbb{E} \arrow[r, "g_1"] & \mathbb{E} \arrow[r, "e"] & \mathbb{E}
\end{tikzcd}$$

Write $\mathfrak{p} = i_1(0), \mathfrak{r} = i_2(0)$ and $\mathfrak{q} = i_2'(0)$. The map $f_1 \colon \mathbb{E} \to \mathbb{E}$ is totally ramified over $\mathfrak{p}$ with critical point $\mathfrak{r}$, and $(g_1)_{\mathfrak{p}} \equiv r(\mathfrak{q}) \pmod{m}$. Choosing suitable ι_i in

$\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and making the following replacement

$$\begin{aligned} e &\mapsto e\circ\iota_1^{-1}, \\ \varepsilon &\mapsto \iota_2\circ\varepsilon, \\ \epsilon &\mapsto \iota_3\circ\epsilon, \\ f_1 &\mapsto \iota_1\circ f_1\circ\iota_2^{-1}, \\ g_1 &\mapsto \iota_1\circ g_1\circ\iota_3^{-1} \end{aligned} \tag{3.9}$$

we may assume that $\mathfrak{p}=\mathfrak{r}=\mathfrak{q}=0$, and this implies our desired assertion.

(2) $\overline{f}=\overline{e}\circ z^2\circ\overline{\varepsilon}, \quad \overline{g}=\overline{e}\circ(z^2+1)p(z)^2\circ\overline{\epsilon}$.

By arguments similar to that in the proof of case 1 we may obtain the following composition of finite Blaschke products $f=e\circ f_1\circ\varepsilon$ and $g=e\circ g_1\circ\epsilon$, and in addition $f_1\colon\mathbb{E}\to\mathbb{E}$ is totally ramified over some $\mathfrak{p}$ and $(g_1)_{\mathfrak{p}}\equiv(\mathfrak{q})+(\mathfrak{r})$ (mod 2) for some distinct points $\mathfrak{q},\mathfrak{r}$ in $\mathbb{E}$. Choosing suitable ι_i in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and making the replacement as in (3.9) we may assume that $\mathfrak{p}=\mathfrak{q}=0, \mathfrak{r}=a$, and this implies our desired assertion.

(3) $\overline{f}=\overline{e}\circ T_m\circ\overline{\varepsilon}, \quad \overline{g}=\overline{e}\circ T_n\circ\overline{\epsilon}$ with $(m,n)=1$.

By arguments similar to that in the proof of case 1 we may obtain the following composition of finite Blaschke products $f=e\circ f_1\circ\varepsilon$ and $g=e\circ g_1\circ\epsilon$ where $f_1,g_1\colon\mathbb{E}\to\mathbb{E}$ are both unramified outside $\{\mathfrak{p},\mathfrak{q}\}$ for some distinct points $\mathfrak{p},\mathfrak{q}$ in $\mathbb{E}$ and in addition their monodromy representations are Chebyshev representation. We conclude from Corollary 2.3.4 that after making a replacement as in (3.9) for suitable ι_i chosen from $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we will have $f_1=\mathfrak{T}_{m,nt}$ and $g_1=\mathfrak{T}_{n,mt}$ as desired.

(4) $\overline{f}=\overline{e}\circ T_m\circ\overline{\varepsilon}, \quad \overline{g}=\overline{e}\circ -T_n\circ\overline{\epsilon}$ with $\gcd(m,n)>1$.

We may apply arguments similar to that in the proof of Case 3.

(5) $\overline{f}=\overline{e}\circ(z^2-1)^3\circ\overline{\varepsilon}, \quad \overline{g}=\overline{e}\circ(3z^4-4z^3)\circ\overline{\epsilon}$.

We first notice that $(z^2-1)^3$ takes -1 and 0 as critical values, ±1 ramified points over 0 and 0 ramified point over -1 with ramification index $e_{\pm1}=3$ and $e_0=2$. Moreover the polynomial $3z^4-4z^3$ takes also -1 and 0 as critical values, 0 ramified points over 0 and 1 ramified point over -1 with $e_0=3$ and $e_1=2$. By arguments similar to that in the proof of case 1 we obtain the following composition of finite Blaschke products $f=e\circ f_1\circ\varepsilon$, $g=e\circ g_1\circ\epsilon$, where $f_1\colon \mathbb{E}\to\mathbb{E}$ admits two points $\mathfrak{q},\mathfrak{r}$ ramified over some point $\mathfrak{p}$ in $\mathbb{E}$ with $e_{\mathfrak{q}}=e_{\mathfrak{r}}=3$ and the finite map $f_2\colon\mathbb{E}\to\mathbb{E}$ admits a point $\mathfrak{s}$ ramified over $\mathfrak{p}$ with $e_{\mathfrak{s}}=3$. After making a replacement as in (3.9) for well-chosen ι_i in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we may assume that $\mathfrak{p}=\mathfrak{s}=0, \mathfrak{q}=-\mathfrak{r}$ which gives the desired f_1 and g_1. The algebraic relation is given by another critical value of f_1 and of g_1 coincide. □

We make one more remark about this theorem

Remark 3.6.4. *In the first case if $g_1=z^r p(z)^m\colon\mathbb{E}\to\mathbb{E}$ is totally ramified and if $m\geq 2$ then one checks readily that g_1 is ramified over 0 and therefore $g_1(z)=\rho z^t$ for some $\rho\in\mathbb{T}$. Modifying ϵ slightly we may assume we are in the case that $\{f_1,g_1\}=\{z^m,z^t\}$.*

3.7 Irreducibility

If f is a rational function then we will denote by Ω_f the splitting field of $f(z)-t$ over $\mathbb{C}(t)$. Avanzi and Zannier proved in [5] that *if f and g are rational functions with f indecomposable and with $\mathbb{C}\times_{f,g}\mathbb{C}$ reducible then $\Omega_g\supseteq\Omega_f$.* In addition they showed that *if g is also indecomposable then $\Omega_f=\Omega_g$.* In the polynomial case they obtained that *if f and g are indecomposable polynomials with $\mathbb{C}\times_{f,g}\mathbb{C}$ reducible then $\deg f=\deg g$ and $\mathfrak{o}_f=\mathfrak{o}_g$.*

Theorem 3.7.1 (Avanzi-Zannier)**.** *Let f be an indecomposable polynomial and c in $\mathbb{C}\setminus\{0,1\}$. If $\mathbb{C}\times_{f,cf}\mathbb{C}$ is reducible then one of the follows assertions*

(1) *$f(z)=a(z-b)^n$ with $a\in\mathbb{C}^*$, $b\in\mathbb{C}$ and c any number in $\mathbb{C}\setminus\{0,1\}$.*

(2) *$f(z)=aT_n(z-b)$ with $n\equiv 1 \pmod 2$, $a\in\mathbb{C}^*$, $b\in\mathbb{C}$ and $c=-1$.*

(3) *$f(z)=(z-b)^r g((z-b)^d)$ with $b\in\mathbb{C}$, $r\geq 1, d\geq 2, (r,d)=1$, g a complex polynomial and c d-th root of unity.*

is satisfied.

We first show an analogue of the remark made at the beginning of the section for Blaschke products.

Lemma 3.7.2. *If f and g are indecomposable finite Blaschke products with $\mathbb{P}^1\times_{f,g}\mathbb{P}^1$ reducible then $\deg f=\deg g$ and $\mathfrak{o}_f=\mathfrak{o}_g$.*

Proof. We follow the notation in Section 3.3. By Lemma 3.3.1 and by the irreducibility of $\mathfrak{R} = \mathbb{P}^1 \times_{f,g} \mathbb{P}^1$ we deduce that the curve $\mathfrak{R}_* = \mathbb{C} \times_{(j_1,i)_*f,(j_2,i)_*g} \mathbb{C}$ is irreducible. We then conclude from Avanzi-Zannier's results that $\deg (j_1, i)_* f = \deg (j_2, i)_* g$ and therefore $\deg f = \deg g$. By Avanzi-Zannier's result mentioned before we have $\Omega_f = \Omega_g$. Moreover $\mathfrak{a}$ lies in $\mathfrak{o}_f$ (resp. g) if and only if it corresponds to a place of $\mathbb{C}(z)$ which ramifies in Ω_f (resp. Ω_g) and this immediately gives $\mathfrak{o}_f = \mathfrak{o}_g$. □

For the proof of the main result of this section we need a lemma,

Lemma 3.7.3. *Let f be a finite Blaschke product, c a non-zero complex number, and $i: \mathbb{E} \to \mathbb{C}$ a homeomorphism which satisfies $i(c z) = c\, i(z)$. If j is a f-lifting of i then j is also a cf-lifting of i and we have*

$$(j,\, i)_* cf = c(j,\, i)_* f.$$

This applies in particular to the case when c lies in $\mathbb{T}$ and $i(z) = z/(1-|z|)$.

Proof. A simple calculations gives

$$\begin{aligned}(j,\, i)_* cf(z) &= i \circ cf \circ j^{-1}(z)\\ &= ci \circ f \circ j^{-1}(z)\\ &= c(j,\, i)_* f(z),\end{aligned}$$

and this is a holomorphic function as desired. □

Theorem 3.7.4. *Let f be an indecomposable finite Blaschke product and let c be in $\mathbb{C}\backslash\{0,1\}$. If $\mathbb{P}^1 \times_{f,cf} \mathbb{P}^1$ is reducible then one of the following assertions*

(1) $f(z) = a((z-b)/(1-\bar{b}z))^p$ *with* $a \in \mathbb{T}$, $b \in \mathbb{E}$ *and* $c \in \mathbb{C}\backslash\{0,1\}$.

(2) $f(z) = a\mathcal{T}_{p,t}((z-b)/(1-\bar{b}z))$ *with* $p \geq 3$ *prime,* $a \in \mathbb{T}$, $b \in \mathbb{E}$ *and* $c = -1$.

(3) $f(z) = ((z-b)/(1-\bar{b}z))^r g(((z-b)/(1-\bar{b}z))^d)$, *with* $r \geq 1$, $d \geq 2$, $\gcd(r,d) = 1$, $b \in \mathbb{E}$, *g a finite Blaschke product and c a d-th root of unity.*

is satisfied.

Proof. By Lemma 3.7.2 we have $\mathfrak{o}_f = \mathfrak{o}_{cf} = c\mathfrak{o}_f$. If $\mathfrak{o}_f$ is supported on $\{0, \infty\}$ then we fall into the first case. If $\mathfrak{o}_f$ is not supported on $\{0,\infty\}$ then it follows from $\mathfrak{o}_f = c\mathfrak{o}_f$ that $|c| = 1$. Following the notation of Section 3.3 we now employ Lemma 3.7.3 by taking $f = f, g = cf$, $i(z) = z/(1-|z|)$ and $j_1 = j_2 = j$, and we write $\overline{f} := (j,\, i)_* f$. By the reducibility of $\mathbb{P}^1 \times_{f,cf} \mathbb{P}^1$ and by Lemma 3.3.1 the curve $\mathbb{C}\times_{(j,i)_*f,(j,i)_*cf}\mathbb{C}$ is reducible. This together with Lemma 3.7.3 leads to the reducibility of $\mathbb{C}\times_{\overline{f},c\overline{f}}\mathbb{C}$ and as a corollary we obtain from Theorem 3.7.1 that either $\overline{f}(z)=\mathfrak{a}T_n(z-\mathfrak{b})$ with $c=-1$ or $\overline{f}(z)=(z-\mathfrak{b})^r g((z-\mathfrak{b})^d)$ with $c^d=-1$.

In the former case we have $\mathfrak{o}_f=\hat{\imath}^{-1}(\mathfrak{o}_{\overline{f}})=\hat{\imath}^{-1}\{\pm\mathfrak{a}\}=\{\pm\mathfrak{a}/(1+|\mathfrak{a}|)\}$ and the monodromy representation of the finite map $f\colon\mathbb{E}\to\mathbb{E}$ is a Chebyshev representation. We conclude from Corollary 2.3.4 that there exist a positive real number t and an element ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $f=(\mathfrak{a}/|\mathfrak{a}|)\mathfrak{T}_{n,t}\circ\iota$ which falls into the second case as claimed in the theorem.

In the latter case we have $(f)_0=(\hat{\imath}^{-1}\circ\overline{f}\circ j)_0\equiv r(j)_{\mathfrak{b}}\pmod d$ and consequently we fall into the last case of our assertion with $b=(j)_{\mathfrak{b}}$. □

The reader may read quickly out of Proposition 3.4.4 the explicit factorization of $\mathbb{P}^1\times_{\mathfrak{T}_{n,\tau},-\mathfrak{T}_{n,\tau}}\mathbb{P}^1$. By Corollary 3.1.7 one may derive from this quickly the factorization of $\mathbb{P}^1\times_{\mathfrak{T}_{p,t},-\mathfrak{T}_{p,t}}\mathbb{P}^1$.

3.8 On curves $\mathbb{P}^1\times_{f,cf}\mathbb{P}^1$

This section is devoted to obtain a hyperbolic analogue of Avanzi-Zannier's work in [5]. The authors introduced there the following polynomials: $p_1(z;l,m):=z^l(z+1)^m$ with $\gcd(l,m)=1$ and $l+m\geq 4$; $p_2(z):=z(z+a)^2(z+b)^2$ with $9a^2-2ab+9b^2=0$ and $ab\neq 0$; $p_3(z):=z(z+a)^3(z+b)^3$ with $a^2-5ab+8b^2=0$ and $ab\neq 0$. We now point out that if f is a polynomial of the above type then the map $f\colon\mathbb{C}\to\mathbb{C}$ is non-exceptional and the nontrivial factor of $\mathbb{C}\times_{f,f}\mathbb{C}$ has at least 3 points at infinity.

Proof. Those polynomials are indecomposable and they cannot be linearly equivalent to cyclic or Chebyshev polynomials (see for instance [5, Remark4.4]). Therefore as a corollary of Theorem 3.5.2 those polynomials are not exceptional. Moreover the cardinality of the infinity of the nontrivial factor of $\mathbb{C}\times_{f,f}\mathbb{C}$ is $\deg f-1$ which is at least 3 as claimed. □

By this remark we now extract a small part of Theorem 1 of [5] as follows,

Theorem 3.8.1 (Avanzi-Zannier). *If for a polynomial f the curve $\mathbb{C}\times_{f,f}\mathbb{C}$ has a nontrivial Siegel factor then there exist an integer $m\geq 2$, $\iota\in\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$,* *polynomials g and h, a* m*-th root of unity ζ other than one such that $\mathbb{C}\times_{g,\,\zeta g}\mathbb{C}$ admits a Siegel factor and*

$$f = h\circ z^m \circ g \quad \text{or} \quad f = h\circ T_m \circ \iota.$$

Our hyperbolic analogue is as follows

Theorem 3.8.2. *If for a finite Blaschke product f the curve $\mathbb{P}^1\times_{f,\,f}\mathbb{P}^1$ has a nontrivial Faltings factor then there exist an integer $m \geq 2$, a positive real number t, ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$, finite Blaschke products g and h, a* m*-th root of unity ζ other than one such that $\mathbb{P}^1\times_{g,\,\zeta g}\mathbb{P}^1$ admits a Faltings factor and*

$$f = h\circ z^m \circ g \quad \textit{or} \quad f = h\circ \mathfrak{T}_{m,\,t} \circ \iota.$$

Proof. Following the notation of Section 3.3, since in our case $g = f$ we assume $j_1 = j_2 = j$ and we write $\overline{f}$ for $(j,\, i)_* f$. By Proposition 3.3.2 if $\mathbb{P}^1\times_{f,\,f}\mathbb{P}^1$ has a Faltings factor then $\mathbb{P}^1\times_{\overline{f},\,\overline{f}}\mathbb{P}^1$ has a Siegel factor. To which we apply Theorem 3.8.1 and obtain immediately the following compositions of polynomials: either

$$\overline{f} = \overline{h}\circ z^m \circ \overline{g}$$

and there exists a m-th root of unity $\xi \neq 1$ which satisfies $\mathbb{C}\times_{\overline{g},\,\xi\overline{g}}\mathbb{C}$ has a Siegel factor or

$$\overline{f} = \overline{h}\circ T_n \circ \overline{\iota}$$

with $\overline{\iota} \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$. By definition j^{-1} is always a $\overline{f}$-lifting of i^{-1}.

In the former case we let i_1 be a $\overline{h}$-lifting i^{-1} and i_2 a z^m-lifting of i_1. By Proposition 1.6.4 the homeomorphism j^{-1} is also a $\overline{g}$-lifting of i_2, and consequently $f = h\circ f_1 \circ g$ is a composition of finite Blaschke products where h, f_1 and g are obtained from the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{g}"'] \arrow[rrr, bend left, "\overline{f}"] \arrow[d, "j^{-1}"] & \mathbb{C} \arrow[r, "z^m"'] \arrow[d, "i_2"] & \mathbb{C} \arrow[r, "\overline{h}"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "i^{-1}"] \\
\mathbb{E} \arrow[r, "g"] \arrow[rrr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_1"] & \mathbb{E} \arrow[r, "h"] & \mathbb{E}
\end{tikzcd}$$

The map $f_1\colon \mathbb{E} \to \mathbb{E}$ is totally ramified over some point $\mathfrak{p}$ in $\mathbb{E}$ and therefore after a similar replacement to (3.9) we may assume that $\mathfrak{p} = 0$ and $f_1(z) = z^m$.

It remains to show that there exists a m-th root of unity ζ other than one such that $\mathbb{P}^1\times_{g,\zeta g}\mathbb{P}^1$ has a Faltings factor. On the one hand we apply Proposition 3.3.2 to $\mathbb{C}\times_{z^m\circ\overline{g},\, z^m\circ\overline{g}}\mathbb{C}$ and then deduce that Siegel factors of

$$\mathbb{C}\times_{z^m\circ\overline{g},\, z^m\circ\overline{g}}\mathbb{C} = \mathbb{C}\times_{\overline{g},\, \zeta_m^0\overline{g}}\mathbb{C} \cup \mathbb{C}\times_{\overline{g},\, \zeta_m^1\overline{g}}\mathbb{C} \cup\cdots\cup \mathbb{C}\times_{\overline{g},\, \zeta_m^{n-1}\overline{g}}\mathbb{C}$$

are one to one corresponding to Faltings factors of

$$\mathbb{P}^1\times_{z^m\circ g,\, z^m\circ g}\mathbb{P}^1 = \mathbb{P}^1\times_{g,\, \zeta_m^0 g}\mathbb{P}^1 \cup \mathbb{P}^1\times_{g,\, \zeta_m^1 g}\mathbb{P}^1 \cup\cdots\cup \mathbb{P}^1\times_{g,\, \zeta_m^{m-1} g}\mathbb{P}^1.$$

On the other hand we apply Proposition 3.3.2 to $\mathbb{C}\times_{\overline{g},\overline{g}}\mathbb{C}$ and then deduce that Siegel factors of $\mathbb{C}\times_{\overline{g},\overline{g}}\mathbb{C}$ are one to one corresponding to Faltings factors of $\mathbb{P}^1\times_{g,g}\mathbb{P}^1$. Comparing these two facts it is clear that $\mathbb{C}\times_{\overline{g},\xi\overline{g}}\mathbb{C}$ has a Siegel factor for some m-th root of unity $\xi\neq 1$ if and only if $\mathbb{P}^1\times_{g,\zeta g}\mathbb{P}^1$ has a Faltings factor for some m-th root of unity $\zeta\neq 1$ and we are done.

In the latter case we let i_1 be a $\overline{h}$-lifting i^{-1} and i_2 a T_n-lifting of i_1. By Proposition 1.6.4 the homeomorphism j^{-1} is also a $\overline{\iota}$-lifting of i_2, and consequently $f = h\circ f_1\circ\iota$ is a composition of finite Blaschke products where h, f_1 and ι are obtained from the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{\iota}"'] \arrow[rrr, bend left, "\overline{f}"] \arrow[d, "j^{-1}"] & \mathbb{C} \arrow[r, "T_n"'] \arrow[d, "i_2"] & \mathbb{C} \arrow[r, "\overline{h}"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "i^{-1}"] \\
\mathbb{E} \arrow[r, "\iota"] \arrow[rrr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_1"] & \mathbb{E} \arrow[r, "h"] & \mathbb{E}
\end{tikzcd}$$

The monodromy of f_1 is Chebyshev representation and therefore we conclude from Corollary 2.3.4 that after a similar replacement to (3.9) we are in the case that $f_1 = \mathcal{T}_{n,t}$ for some positive integer n and some positive real number t as claimed in our assertion. $\square$

It remains to classify the case that the curve $\mathbb{P}^1\times_{g,\zeta g}\mathbb{P}^1$ admits a Faltings factor. Before doing that we recall another small part of Theorem 2 of [5].

Theorem 3.8.3 (Avanzi-Zannier). *We assume that the curve $\mathbb{C}\times_{f,cf}\mathbb{C}$ with f a polynomial of degree at leat 2 and with c a complex number different from 0 and 1 has a Siegel factor. Then there exists a polynomial decomposition $f = f_0\circ f_1$ with f_0 prime and with one of the assertions*

(1) $f_0(z) = z^p$ *and there exists a p-th root ζ of c such that $\mathbb{C}\times_{f_1,\zeta f_1}\mathbb{C}$ admits*

a Siegel factor.

(2) $f_0(z) = \alpha T_p(z)$ *with* $p \geq 3$, $c = -1$ *and* $\mathbb{C}\times_{f_1, -f_1}\mathbb{C}$ *admitting a Siegel factor.*

(3) $f_0(z) = z^r g(z^d)$ *with* $g(0) \neq 0, \deg g \geq 1, \deg f_0 \geq 4, r \geq 1, d \geq 2, (r, d) = 1$ *and* $c^d = 1$. *Moreover for any positive integer* s *with* $rs \equiv 1 \pmod{d}$ *the curve* $\mathbb{C} \times_{f_1, c^s f_1} \mathbb{C}$ *admits a Siegel factor.*

satisfied.

In the last case if we set $v = \bar{c}^{r'} w$ then

$$\mathbb{C}\times_{f_0(u), cf_0(v)}\mathbb{C} = \mathbb{C}\times_{f_0(u), f_0(w)}\mathbb{C} = \{(u, w) : u = w\} \cup \mathfrak{C}_1$$

for some curve $\mathfrak{C}_1$. We conclude from Theorem 3.5.2 that the curve $\mathfrak{C}_1$ is irreducible with $\deg f_0 - 1 \geq 3$ points at infinity. This gives that $\mathbb{C}\times_{f_0(u), cf_0(v)}\mathbb{C}$ has a unique Siegel factor $\{(u, v) : u = c^{r'} v\}$ and therefore there is bijection between Siegel factors of $\mathbb{C}\times_{f_0 \circ f_1, cf_0 \circ f_1}\mathbb{C}$ and Siegel factors of $\mathbb{C}\times_{f_1, c^{r'} f_1}\mathbb{C}$.

The following theorem answers when does $\mathbb{P}^1 \times_{g, \xi g} \mathbb{P}^1$ admit a Faltings factor.

Theorem 3.8.4. *We assume that the curve* $\mathbb{P}^1\times_{f, cf}\mathbb{P}^1$ *with* f *a finite Blaschke product of degree at least 2 and with* c *a complex number of modulus 1 has a Faltings factor. Then there exists a Blaschke product decomposition* $f = f_0 \circ f_1$ *with* f_0 *prime and with one of the assertions*

(1) $f_0(z) = z^p$ *and there exists a* p*-th root* ζ *of* c *such that* $\mathbb{P}^1\times_{f_1, \zeta f_1}\mathbb{P}^1$ *admits a Faltings factor.*

(2) $f_0(z) = \alpha \mathcal{T}_{p, t}(z)$ *with* $p \geq 3, \alpha \in \mathbb{T}, t > 0, c = -1$ *and* $\mathbb{P}^1\times_{f_1, -f_1}\mathbb{P}^1$ *admitting a Faltings factor.*

(3) $f_0 = z^r g(z^d)$ *with* $g(0) \neq 0, \deg g \geq 1, r \geq 1, d \geq 2, (r, d) = 1$ *and* $c^d = 1$. *Moreover for any positive integer* r' *with* $rr' \equiv 1 \pmod{d}$ *the curve* $\mathbb{P}^1\times_{f_1, c^{r'} f_1}\mathbb{P}^1$ *has a Faltings factor.*

satisfied.

Proof. Taking $i(z) = z/(1 - |z|)$ and using the assumption that $|c| = 1$, we deduce from Lemma 3.7.3 that there exists a homeomorphism $j : \mathbb{E} \to \mathbb{C}$ serving as a f-lifting of i as well as a cf-lifting of i. By Proposition 3.3.2 the curve $\mathbb{C}\times_{(j, i)_* f, (j, i)_* cf}\mathbb{C}$ has a Siegel factor. Writing $\overline{f}$ for $(j, i)_* f$, Lemma 3.7.3 gives that

$$\mathbb{C}\times_{\overline{f},\,c\overline{f}}\mathbb{C}$$

admits a Siegel factor. According to Theorem 3.8.3 $\overline{f}$ admits one of the following decompositions into polynomials:

1, $\overline{f} = z^p \circ \overline{f_1}$ and there exists a p-th root ξ of c such that the curve $\mathbb{C}\times_{\overline{f_1},\,\xi\overline{f_1}}\mathbb{C}$ has a Siegel factor.

By definition j^{-1} is a $\overline{f}$-lifting of i^{-1}. Choosing i_1 to be a z^p-lifting of i^{-1}, we deduce from Proposition 1.6.4 that j^{-1} is also a $\overline{f_1}$-lifting of i_1. As a consequence we obtain a composition of finite Blaschke products $f = f_0 \circ f_1$ with f_0 and f_1 determined by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{f_1}"'] \arrow[rr, bend left, "\overline{f}"] \arrow[d, "j^{-1}"] & \mathbb{C} \arrow[r, "z^p"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "i^{-1}"] \\
\mathbb{E} \arrow[r, "f_1"] \arrow[rr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_0"] & \mathbb{E}
\end{tikzcd}$$

The map $f_0 \colon \mathbb{E} \to \mathbb{E}$ is totally ramified over 0, and therefore after the following replacement

$$f_0 \mapsto f_0 \circ \iota^{-1}, \quad f_1 \mapsto \iota \circ f_1$$

for some ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we are in the case that $f_0(z) = z^p$. It remains to verify that there exists a p-th root of unity ζ of c such that the curve $\mathbb{P}^1\times_{f_1,\,\zeta f_1}\mathbb{P}^1$ has a Faltings factor. Indeed by applying Proposition 3.3.2 there is a bijection between Siegel factors of $\mathbb{C}\times_{\overline{f},\,c\overline{f}}\mathbb{C} = \cup_{\zeta^p = c}\mathbb{C}\times_{\overline{f_1},\,\zeta\overline{f_1}}\mathbb{C}$ and Faltings factors of $\mathbb{P}^1\times_{f,\,cf}\mathbb{P}^1 = \cup_{\zeta^p=c}\mathbb{P}^1\times_{f_1,\,\zeta f_1}\mathbb{P}^1$, and this together with the existence of Siegel factor of $\mathbb{C}\times_{\overline{f_1},\,\xi\overline{f_1}}\mathbb{C}$ gives the desired claim.

2, $\overline{f} = \overline{\alpha} T_p \circ \overline{f_1}$ with p a prime number other than 2 and $c = -1$. In addition the curve $\mathbb{C}\times_{\overline{f_1},\,-\overline{f_1}}\mathbb{C}$ admits a Siegel factor.

Let i_1 be a $\overline{\alpha} T_p$-lifting of i^{-1}. By arguments similar to that in the discussion of previous case we obtain a composition of finite Blaschke products $f = f_0 \circ f_1$

with f_0 and f_1 determined by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{f_1}"'] \arrow[rr, bend left, "\overline{f}"] \arrow[d, "j^{-1}"] & \mathbb{C} \arrow[r, "\overline{\alpha}T_p"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "i^{-1}"] \\
\mathbb{E} \arrow[r, "f_1"] \arrow[rr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_0"] & \mathbb{E}
\end{tikzcd}$$

The monodromy of $f_0\colon \mathbb{E} \to \mathbb{E}$ is a Chebyshev representation and $\mathfrak{o}_{f_0} = i^{-1}\{\pm\overline{\alpha}\}$. We conclude from Corollary 2.3.4 that after the following replacement

$$f_0 \mapsto f_0 \circ \iota^{-1}, \quad f_1 \mapsto \iota \circ f_1$$

for some ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we are in the case that $f_0 = \alpha\mathfrak{T}_{p,t}$ for some α on the unit circle and some positive number t.

By Proposition 3.4.1 and by Proposition 3.4.4 the curve $\mathbb{C}\times_{\overline{\alpha}T_p,\,-\overline{\alpha}T_p}\mathbb{C}$ has a unique component $\mathfrak{A}$ of $(0,1)$ type defined by $u+v=0$ and the curve $\mathbb{P}^1\times_{f_0,\,-f_0}\mathbb{P}^1$ has a unique component $\mathfrak{B}$ of $(0,0)$ type defined by $u+v=0$, where u and v are affine coordinates of $\mathbb{C}\times\mathbb{C}$ or of $\mathbb{P}^1\times\mathbb{P}^1$ as described before Proposition 3.4.2. According to Proposition 3.3.2 the map $i_1 \times i_1\colon \mathbb{C}\times\mathbb{C} \to \mathbb{E}\times\mathbb{E} \subset \mathbb{P}^1\times\mathbb{P}^1$ takes $\mathfrak{A}$ to $\mathfrak{B}$, or equivalently

$$i_1(-z) = -i_1(z).$$

Using Lemma 3.7.3 we have $(j^{-1}; i_1)_*(-\overline{f_1}) = -(j^{-1}; i_1)_*\overline{f_1} = -f_1$. By Proposition 3.3 applied to $\mathbb{C}\times_{\overline{f_1},-\overline{f_1}}\mathbb{C}$ and $\mathbb{P}^1\times_{f_1,-f_1}\mathbb{P}^1$ and by the existence of Siegel factors of $\mathbb{C}\times_{\overline{f_1},-\overline{f_1}}\mathbb{C}$ we obtain the existence of Faltings factors of $\mathbb{P}^1\times_{f_1,-f_1}\mathbb{P}^1$.

3, $\overline{f} = z^r\overline{g}(z^d)\circ\overline{f_1}$ with properties as stated in Theorem 3.8.3.

Let i_1 be a $z^r\overline{g}(z^d)$-lifting of i^{-1}. By arguments similar to that in the discussion of previous case we obtain a composition of finite Blaschke products $f = f_0 \circ f_1$ with f_0 and f_1 determined by the following commutative diagram.

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\overline{f_1}"'] \arrow[rr, bend left, "\overline{f}"] \arrow[d, "j^{-1}"] & \mathbb{C} \arrow[r, "z^r\overline{g}(z^d)"'] \arrow[d, "i_1"] & \mathbb{C} \arrow[d, "i^{-1}"] \\
\mathbb{E} \arrow[r, "f_1"] \arrow[rr, bend right, "f"'] & \mathbb{E} \arrow[r, "f_0"] & \mathbb{E}
\end{tikzcd}$$

Write $\mathfrak{q}$ for $i_1(0)$ and then it follows from the above diagram that

$$(f_0)_0 \equiv r(\mathfrak{q}) \pmod{d},$$

and therefore after the following replacement

$$f_0 \mapsto f_0 \circ \iota^{-1}, \quad f_1 \mapsto \iota \circ f_1$$

for some ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we are in the case that $f_0 = z^r g(z^d)$. On the one hand Proposition 3.3 gives a bijection between Siegel factors of $\mathbb{C}\times_{\overline{f},\overline{f}}\mathbb{C}$ and Faltings factors of $\mathbb{P}^1\times_{f,f}\mathbb{P}^1$. On the other hand by the remark made after Theorem 3.8.3 all Siegel factors of $\mathbb{C}\times_{\overline{f},\overline{f}}\mathbb{C}$ arises from $\mathbb{C}\times_{\overline{f_1},c^{r'}\overline{f_1}}\mathbb{C}$, and by a similar reason all Faltings factors of $\mathbb{P}^1\times_{f,f}\mathbb{P}^1$ arises from $\mathbb{P}^1\times_{f_1,c^{r'}f_1}\mathbb{P}^1$. Consequently the existence of Siegel factors of $\mathbb{C}\times_{\overline{f_1},c^{r'}\overline{f_1}}\mathbb{C}$ leads to the existence of Faltings factors of $\mathbb{P}^1\times_{f_1,c^{r'}f_1}\mathbb{P}^1$. $\square$

4

Dynamical Mordell-Lang

The purpose of this chapter is to prove Theorem 4.1.5 which says that if two finite endomorphisms of the unit disk have orbits with infinitely many intersections then they have a common iterate. This implies as a corollary Theorem 4.1.2 which is a dynamical analogue of Mordell-Lang and André-Oort type for finite endomorphisms of polydisks. Based on Theorem 3.6.3, we reduce the proof of Theorem 4.1.5 to hard analysis of endomorphism monoid $\mathrm{End}(\mathbb{E})$ of the unit disk.

4.1 Introduction

The first breathtaking result relating number theory and geometry might be Siegel's theorem on the integral points [122], and this striking phenomenon reached its culmination in Faltings' proof of Mordell's conjecture [42]. Based on techniques from diophantine approximation Faltings was able to prove later in [43] and [44] for an abelian variety X the following stronger

Conjecture 4.1.1 (Mordell-Lang Conjecture)**.** *Let X be a semiabelian variety over $\mathbb{C}$, V a subvariety of X, and Γ a finitely generated subgroup of $X(\mathbb{C})$. The intersection set $V(\mathbb{C}) \cap \Gamma$ is the union of finitely many cosets of subgroups of Γ.*

This conjecture, which relates number theory and geometry, was proved earlier than Faltings by Laurent ([84]) for a torus X and extended later by Vojta ([129]) to all semiabelian varieties.

This intimate relation between number theory and geometry appears again in the context of Shimura varieties. Here the first major result might be the theorem of Schneider [115] which together with a result of Kronecker-Weber implies that an algebraic point τ in $\mathbb{H}$ arises from an elliptic curve with complex multiplication if and only if and its projection $j(\tau)$ in $Y(1)$ is also algebraic. From this the geometric property of τ is characterized in terms of arithmetical property of $j(\tau)$. Based on Wüstholz's subgroup theorem [136], Schneider's result was generalized by Shiga-Wolfart [120] and Tretkoff [34] to Shimura varieties of abelian type, but the general case remains open. Concerning higher dimensional subvarieties rather than points, the similar statement is predicted by the André-Oort conjecture which was promoted among others by Wüstholz (see for instance [35] and [8, Section 8.4]) in the last decade and remains an object of intensive study.

In a fundamental case André-Oort's conjecture expects that if a subcurve of certain quotient of $\mathbb{H}^n$ contains infinitely many special points then it is a special curve. We shall work on the uniformization space $\mathbb{H}^n$ instead of its quotients and prove the following dynamical analogue of the conjectues of Mordell-Lang and André-Oort type.

Theorem 4.1.2. *Let d be an integer greater than one, F a finite endomorphism of the polydisk $\mathbb{E}^d$, L in $\mathbb{E}^d$ a complex geodesic line and p a point of $\mathbb{E}^d$. If $\mathcal{O}_F(p) \cap L$ is infinite then there exists a positive integer k such that $F^k(L) = L$.*

Similar to subcurves of a product of modular curves for which the André-Oort conjecture reduces easily to the case of $\mathbb{C}^2$, our above theorem follows quickly from the following theorem concerning bidiscs together with a classification theorem of Remmer-Stein and an argument on height.

Theorem 4.1.3. *Let X be a simply connected open Riemann surface, f and g finite endomorphisms of X with degree greater than one and x, y points in X. If $\mathcal{O}_f(x) \cap \mathcal{O}_g(y)$ is infinite then f and g have a common iterate.*

The above theorem is a combination of two theorems. The first one is on the complex plane:

Theorem 4.1.4 (Ghioca-Tucker-Zieve [57], [58]). *Let x, y be points in the complex plane $\mathbb{C}$ and let f, g be complex polynomials of degree greater than 1. If $\mathcal{O}_f(x) \cap \mathcal{O}_g(y)$ is infinite then f and g have a common iterate.*

The second one is on the unit disk

Theorem 4.1.5. *Let x, y be points in $\mathbb{P}^1(\mathbb{C})$ and let $f, g \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be finite Blaschke products. If $\mathcal{O}_f(x) \cap \mathcal{O}_g(y)$ is infinite then f and g have a common iterate.*

By carrying out more tedious analysis but without any essentially new techniques other than those required for the proof of Theorem 4.1.2 one may deduce

Theorem 4.1.6. *Let p be a point in $\mathbb{E}^d$, L in $\mathbb{E}^d$ a complex geodesic line, e a positive integer, $F_i \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E}^d)$ for $1 \leq i \leq e$ pairwise commutative finite endomorphisms of $\mathbb{E}^d$ and S the semigroup generated by F_i. The intersection $L \cap \mathcal{O}_S(p)$ can be written as $\mathcal{O}_T(p)$ where T is a union of at most finitely many cosets of subsemigroups of S.*

This work follows the strategy of Ghioca-Tucker-Zieve which will be summarized in Section 4.2, but our work carries several new features. Firstly we have to look at rational points instead of integral points and therefore we shall employ Faltings' theorem instead of Siegel's theorem. Secondly we shall prove a hyperbolic version of Bilu-Tichy's theorem. Thirdly we need Ritt's theory on the unit disk which was established in [132] and [98]. Moreover we have to investigate the family of elliptic rational functions $\mathcal{T}_{n,\tau}$ for $n \in \mathbb{N}$ and for $\tau \in \mathbb{H}$. Functions $\mathcal{T}_{n,\tau}$ are descents of cyclic isogenies of elliptic curves by Jacobian cd functions and they provide a large part of non-trivial exceptional cases to the reach of Faltings' theorem. Furthermore, we shall invoke a recent work of M. Baker [9] on the theory of heights. Lastly a classical result of Remmert-Stein [104] and Rischel [107] make it possible to manage all finite endomorphisms of polydisks. The basic new idea of this work is to adopt the topological viewpoint and to make use of a simple fact, that topologically the Poincaré unit disk agrees with the Gaussian complex plane, in several different respects.

Concerning Zhang's dynamical conjectures and the dynamical Mordell-Lang conjecture complemented by Ghioca-Tucker we refer to [138] and [58, Question 1.5]. For recent progress in the dynamical Mordell-Lang problem in various contexts, we refer the reader to [11], [13], [55] and [56]. Regarding another similar problem, the so called "unlikely intersection" which was originated from Bombieri-Masser-Zannier in [23] and developed by Habegger, Maurin, Rémond, Viada and many others, we refer to [66] and references there for a recent account.

4.2 The strategy of Ghioca-Tucker-Zieve

We sketch briefly in this section Ghioca-Tucker-Zieve's strategy in their proof of Theorem 4.1.4. Their method is based on the following two facts.

- Speciality: The existence of infinitely many intersections gives, for any positive integers i and j, infinitely many integral points on the curve $\mathbb{C}\times_{f^i,g^j}\mathbb{C}$. As a consequence of Bilu-Tichy's theorem the polynomials f^i and g^j admit very special decompositions.
- Rigidity: By Ritt's theory the decomposition of polynomials is very rigid.

Now the existence of special decompositions and the rigidity of polynomial decompositions are kind of incompatible and this makes it possible to continue by lengthy discussions to reach the desired goal.

Ghioca-Tucker-Zieve's work explains how dynamics could be controlled by arithmetics and therefore by the underlying geometric structure. We shall proceed a bit more with the adoption of this idea to touch the following Conjecture which is formalized by Chioca-Tucker.

Conjecture 4.2.1 (Dynamical Mordell-Lang Conjecture)**.** *Let X be a complex algebraic variety, F a finite endomorphism of X and V a closed subvariety in X. If there is a point p in X such that $\overline{V\cap \mathcal{O}_F(p)} = V$ then V is periodic under F in the sense that there exists a positive integer k such that F^k maps V to itself.*

In light of Ghioca-Tucker-Zieve's method one might try to argue as follows. Firstly we can assume that all objects are defined over a field k of finite type over $\mathbb{Q}$. Moreover we write Γ_{F^n} for the graph of F^n in $X\times X$ and $\Gamma_{F^n,V}$ for the intersection of Γ_{F^n} with $X\times V$. The canonical projection π_n from $\Gamma_{F^n,V}$ to V is finite, and if a projective embedding of $X(\mathbb{C})$ is fixed then as remarked in Section 1.1 that given complex subvariety C of X the notation $C(k)$ causes no confusion even if C is not defined over k. Now we have

Lemma 4.2.2. *With notations and assumptions as above there exists an irreducible component C_n of $\Gamma_{F^n,V}(\mathbb{C})$ such that $\overline{C_n(k)} = C_n$.*

Proof. By assumption given a positive integer n there exists x_n in $X(k)$ such that $\overline{\mathcal{O}_{F^n}(x_n)\cap V} = V$. This means that there exists an increasing sequence of positive integers r_i for all $i \in \mathbb{N}$ such that $F^{nr_m}(x_n)$ lies in V and $\overline{\{F^{nr_m}(x_n) : m \in \mathbb{N}\}} = V$. Taking y_k as $(F^{n(r_m-1)}(x_n), F^{nr_m}(x_n))$ which lies in $\Gamma_{F^n,V}$ we obtain

$$\overline{\pi_n(\{y_m : m \in \mathbb{N}\})} = \overline{\{F^{nr_m}(x_n) : m \in \mathbb{N}\}} = V.$$

The map π_n is finite and as a consequence there exists an irreducible component C_n of $\Gamma_{F^n,V}$ such that $\overline{\{y_m : m\in\mathbb{N}\}\cap C_n} = C_n$. It is clear that the sequence $y_m (m\in\mathbb{N})$ are defined over k and therefore we conclude that $\overline{C_n(k)} = C_n$ as desired. □

We shall give several simple examples to explain the effect of arithmetics in dynamics and we begin with

Corollary 4.2.3. *Let X be a complex algebraic variety, V a subvariety in X and F a finite endomorphism of X. If V is mordellic then for an arbitrary point p in X the intersection $\mathcal{O}_F(p)\cap V$ is finite.*

The above argument generalizes easily to the following correspondence.

Corollary 4.2.4. *Let X be a complex algebraic variety, V its subvariety, $f_i (1 \le i \le d)$ finite endomorphisms of X and F a correspondence in $X\times X$ which is given by $(x, y) \in F$ if and only if there exists $1 \le i \le d$ such that $y = f_i(x)$. If a subvariety V of X is mordellic then for an arbitrary point $p\in X$ the intersection $\mathcal{O}_F(p)\cap V$ is finite.*

Now we deduce a small part of Hurwitz's theorem ([73]) from the point of view of arithmetic and of dynamics.

Proposition 4.2.5. *If X is an irreducible curve of genus greater than one and if f is a finite endomorphism of X then f is an automorphism of finite* *order. In other words there exists a positive integer n such that $f^n = id$.*

Proof. We assume that X and f are defined over a field k of finite type over $\mathbb{Q}$. If f is not of finite order then all preperiodic points of f are $\overline{k}$-rational points since they are discrete solutions of equations defined over k. Choose any point $x \in X(\mathbb{C})\backslash X(\overline{K})$. Now x is not preperiodic and this means that $|\mathcal{O}_f(x) \cap X(k(x))| = \infty$ which contradicts the mordellicity of X. $\square$

The major difficulty in applying Lemma 4.2.2 is the large open part of Bombieri-Lang's conjecture which was only verified by Faltings in the case when X is a subvariety of a abelian variety in [42] and [43]. Fortunately this is not an issue for curves since Diophantine geometry in dimension one is well-understood. Indeed in the context of curves $\mathbb{C} \times_{f,g} \mathbb{C}$ defined by polynomials it seems reasonable to expect the following decomposition principle, suggested by the work of Bilu-Tichy and of Avanzi-Zannier, that exceptional arithmetical properties of $\mathbb{C} \times_{f,g} \mathbb{C}$ leads to special decompositions of f and of g.

To adopt the strategy of Ghioca-Tucker-Zieve in new cases, on the one hand given exceptional arithmetic properties we need a criterion of Bilu-Tichy's type to obtain special decompositions and on the other hand we need a counterpart of Ritt's type to deduce the rigidity of decompositions.

We now explain briefly how can we adopt the strategy of Ghioca-Tucker-Zieve in our context.

- Rigidity: The decomposition property is encoded in the monodromy action of fundamental groups and therefore is topological in nature. This enables us to put Ritt's original theory on the Gaussian complex plane to the context of the Poincaré unit disk since $\mathbb{E} \cong \mathbb{C}$ as topological spaces. Many details was already carried out in Chapter 2 and this leads to the rigidity of decompositions of finite maps between the unit disk.

- Speciality: The arithmetic of curves as summarized in Theorem 1.2.1 is also topological in nature. This together with the topological nature of decompositions enable us to obtain many results, including a hyperbolic version of Bilu-Tichy's criterion, on curves defined by finite Blaschke products which fits into the decomposition principle. This is based on the topological fact that $\mathbb{E} \cong \mathbb{C}$ and was already discussed in Chapter 3.

Under the assumption of Theorem 4.1.5 we now have at hand both speciality and rigidity of decompositions of f^i as well as of g^j for all positive integers i and j. If we continue with the proof along this way we have to handle explicit analytic functions and this issue is not of topological nature any more. In the complex plane case Ghioca-Tucker-Zieve have to deal with Chebyshev polynomials which are connected to trigonometric functions. However in the unit disk case we have to play with elliptic rational functions which arise from elliptic functions and fortunately the theory was already sufficiently touched in the previous chapter.

4.3 Linear equations in finite Blaschke products

We begin with recalling a well-known fact, and for which we refer to [123, p.1–p.14].

Lemma 4.3.1. *Any $\iota \neq id$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ is parabolic, hyperbolic or elliptic. Moreover if ι is parabolic or hyperbolic then it is fixed point free in $\mathbb{E}$ and there exists p in $\mathbb{T}$ such that*

$$\lim_{n\to\infty} f^n(z) = p$$

for all z in $\mathbb{E}$. If ι is elliptic then it admits a unique fixed point $\mathfrak{p}$ in $\mathbb{E}$ and there exists $\xi \in \mathbb{T}$ such that

$$\iota_{\mathfrak{p}} \circ \iota \circ \iota_{\mathfrak{p}}^{-1}(z) = \xi z.$$

Lemma 4.3.2. *Let $\Omega \neq \emptyset$ be a finite subset of $\mathbb{E}$ and $\iota \neq id$ an element in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If $\iota(\Omega) = \Omega$ then there exist $\mathfrak{p} \in \mathbb{E}$ and $\zeta \in \mathbb{T}$ such that $\iota = \iota_{\mathfrak{p}} \circ \zeta \circ \iota_{\mathfrak{p}}^{-1}$. Moreover if $\deg \Omega \geq 2$ then ζ is a root of unity.*

Proof. For the first claim it suffices to show that ι is elliptic. Indeed if ι is hyperbolic or parabolic then we deduce from Lemma 4.3.1 that ι cannot fix any non-empty finite subset of $\mathbb{E}$, and this contradicts our assumption. The set Ω is finite and therefore there exists a positive integer n such that $\iota^n|_{\Omega} = id$. If $\deg \Omega \geq 2$ then we must have $\iota^n = id$ and therefore $\zeta^n = 1$ as desired. □

The following lemma is a hyperbolic analogue of Lemma 2.4 in [57].

Lemma 4.3.3. *Let f be a finite Blaschke product which is not totally ramified. If $\epsilon\circ f = f\circ\varepsilon$ for ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ then there exist $\mathfrak{p}$ and $\mathfrak{q}$ in $\mathbb{E}$, nonnegative integers r and s, a s-th root of unity ζ and a finite Blaschke product $\bar{f}$ such that*

$$\varepsilon = \iota_{\mathfrak{q}}\circ\zeta z\circ\iota_{-\mathfrak{q}}, \quad \epsilon = \iota_{\mathfrak{p}}\circ\zeta^r z\circ\iota_{-\mathfrak{p}}, \quad f = \iota_{\mathfrak{p}}\circ z^r\bar{f}(z^s)\circ\iota_{-\mathfrak{q}}.$$

Proof. By assumption that the finite map $f\colon\mathbb{E}\to\mathbb{E}$ is not totally ramified it is clear that $\deg\mathfrak{d}_f \geq 2$ and $\deg\operatorname{supp}\mathfrak{D}_f \geq 2$. Moreover we deduce from the functional equation $\epsilon\circ f = f\circ\varepsilon$ that $\epsilon(\mathfrak{d}_f) = \mathfrak{d}_f$ and $\varepsilon(\operatorname{supp}\mathfrak{D}_f) = \operatorname{supp}\mathfrak{D}_f$. Lemma 4.3.2 implies that there exist points $\mathfrak{p}, \mathfrak{q}$ in $\mathbb{E}$ and roots of unit ζ, γ such that $\varepsilon = \iota_{\mathfrak{q}}\circ\zeta\circ\iota_{-\mathfrak{q}}$, $\epsilon = \iota_{\mathfrak{p}}\circ\gamma\circ\iota_{-\mathfrak{p}}$ and consequently $\iota_{\mathfrak{p}}\circ\gamma\circ\iota_{-\mathfrak{p}}\circ f = f\circ\iota_{\mathfrak{q}}\circ\zeta\circ\iota_{-\mathfrak{q}}$. If we write $h = \iota_{-\mathfrak{p}}\circ f\circ\iota_{\mathfrak{q}}$ then we have $\gamma\circ h = h\circ\zeta$ and this gives $(h)_0 = \zeta(h)_0$. If ζ is a primitive s-th root of unity then it follows immediately that there exists a finite Blaschke product $\bar{f}$ such that $h(z) = z^r\bar{f}(z^s)$ and therefore $f = \iota_{\mathfrak{p}}\circ z^r\bar{f}(z^s)\circ\iota_{-\mathfrak{q}}$ as desired. ☐

Proposition 4.3.4. *If f and g are finite Blaschke products and if at least one of them is not totally ramified then the functional equation $\epsilon\circ f = g\circ\varepsilon$ has only finitely many solutions (ϵ, ε) in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$.*

Proof. We assume that f is not totally ramified and consequently the degree of the support $\operatorname{supp}\mathfrak{D}_f$ of $\mathfrak{D}_f$ and the degree of $\mathfrak{d}_f$ are at least two. If two elements ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ gives a solution to $\epsilon\circ f = g\circ\varepsilon$ then ϵ induces a bijection from $\mathfrak{d}_f$ to $\mathfrak{d}_g$ and ε induces a bijection from $\operatorname{supp}\mathfrak{D}_f$ to $\operatorname{supp}\mathfrak{D}_g$. The natural map

$$\{(\epsilon, \varepsilon) \mid \epsilon\circ f = g\circ\varepsilon\} \to \mathrm{Hom}(\mathfrak{d}_f, \mathfrak{d}_g)\times\mathrm{Hom}(\operatorname{supp}\mathfrak{D}_f, \operatorname{supp}\mathfrak{D}_g)$$

is injective since the only element in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ which fixes at least 2 points must be the identity map and this clearly induces the desired conclusion. ☐

We get immediately the following corollaries.

Corollary 4.3.5. *If f is a finite Blaschke product which is not totally ramified then the functional equation $\epsilon\circ f = f\circ\varepsilon$ has only finitely many solutions (ϵ, ε) in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$.*

Proof. This easily follows from Proposition 4.3.4, but we would like to point out that it also follows from Lemma 4.3.3. To apply Lemma 4.3.3 it is crucial to notice that since $\mathbb{E}$ is a CAT(0) space points $\mathfrak{p}$ and $\mathfrak{q}$ in the conclusion there are uniquely determined by f. Indeed $\mathfrak{p}$ is the center of $\mathfrak{d}_f$ and $\mathfrak{q}$ is the center of $\operatorname{supp}\mathfrak{D}_f$. □

Corollary 4.3.6. *If ι is in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and if there exists a finite Blaschke product which is not totally ramified such that $\iota\circ f = f\circ\iota$ then there exists a positive integer n such that $\iota^n = id$.*

4.4 Dynamical Mordell-Lang for linear maps

Theorem 4.1.2 reduces to the following simple lemma of Skolem-Mahler-Lech type as soon as $F\in\mathrm{Aut}_{\mathbb{C}}(\mathbb{E}^d)$.

Lemma 4.4.1. *Let x, y be points in $\mathbb{E}$ and ϵ, ε elements of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If the intersection of $\mathcal{O}_{\epsilon\times\varepsilon}(x, y)$ with the diagonal $\Delta_{\mathbb{E}}$ of $\mathbb{E}^2$ is infinite then there exists a positive integer n such that $\epsilon^n = \varepsilon^n$.*

Proof. If $\mathfrak{p}$ is a point in $\mathbb{E}$ and if ι is an element in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ then by Lemma 4.3.1 the orbit $\mathcal{O}_\iota(\mathfrak{p})$ either lies on a circle inside $\mathbb{E}$ (when ι is elliptic) or tends to a point $\mathfrak{q}$ in $\mathbb{T}$ (when ι is parabolic or hyperbolic). This together with the assumption that $\mathcal{O}_{\epsilon\times\varepsilon}(x, y)\cap\Delta_{\mathbb{E}}$ is infinite implies that ϵ and ε are both elliptic or both non-elliptic.

In the former case the intersection of two circles is infinite, and this forces that two circles to coincide. Consequently their hyperbolic centers also coincide and by taking a conjugation we may assume that there are $\gamma, \xi\in\mathbb{T}$ such that $\epsilon(z) = \gamma z$ and $\varepsilon(z) = \xi z$. The hypothesis is not affected if we replace (x, y) by $(\epsilon^n(x), \varepsilon^n(y))$ for any positive integer n and therefore we may assume $x = y = \mathfrak{p}$. It is clear that $\mathfrak{p}\neq 0$, otherwise the intersection of the two orbits is a single point which contradicts the assumption. Moreover there exists a positive integer k such that $\epsilon^k(\mathfrak{p}) = \varepsilon^k(\mathfrak{p})$ and this together with $p\neq 0$ leads to $\gamma^k = \xi^k$ which implies $\epsilon^k = \varepsilon^k$ as desired.

In the latter case we shall work on the upper half plane $\mathbb{H}$ instead of the unit disk $\mathbb{E}$. For the same reason as in the previous case we may assume $x = y = \mathfrak{p}$ and moreover we can assume $\mathfrak{q} = \infty$. This implies that both $\mathcal{O}_\epsilon(p)$ and $\mathcal{O}_\varepsilon(\mathfrak{p})$ tend to ∞ and therefore there are real numbers $a > 0, b, c > 0, d$

such that $\epsilon(z) = az+b$ and $\varepsilon(z) = cz+d$. Now by comparing the imaginary parts of $\epsilon^k(\mathfrak{p})$ and $\varepsilon^k(\mathfrak{p})$ we deduce that $a=c$. Neither the hypothesis nor the conclusion is affected if we replace (ϵ, ε) by $(\epsilon^n, \varepsilon^n)$ for any positive integer n and therefore after doing so for a suitable n we may assume that $\epsilon(\mathfrak{p}) = \varepsilon(\mathfrak{p})$. This immediately gives $b=d$ and therefore $\epsilon = \varepsilon$ as desired. $\square$

Lemma 4.4.1 is a special case of the main theorem of [11], where the authors completely proved the dynamical Mordell-Lang conjecture for étale maps by using methods from p-adic analysis. I am grateful to D. Ghioca for this reference.

4.5 Rigidity

The decomposition of finite endomorphisms of the complex plane or of the unit disk is very rigid, which follows from a transitive monodromy action of an element of corresponding topological fundamental groups. The rigidity is encoded in elements of fundamental groups generated by for polynomials paths around the infinity and for finite Blaschke products paths around the unit circle. In general we have

Proposition 4.5.1. *Let f be a finite map from $\mathfrak{M}$ to $\mathfrak{N}$, $\alpha\colon[0,1] \to \mathfrak{N}$ a closed path over which f is unramified and consider finite maps b from $\mathfrak{M}$ to $\mathfrak{A}$, a from $\mathfrak{A}$ to $\mathfrak{N}$, d from $\mathfrak{M}$ to $\mathfrak{R}$, c from $\mathfrak{R}$ to $\mathfrak{N}$ satisfying $a\circ b = c\circ d = f$. If the monodromy action of α is transitive then there exist Riemann surfaces $\mathfrak{T}$ and $\mathfrak{W}$ and finite maps $h\colon\mathfrak{M} \to \mathfrak{T}, \overline{b}\colon\mathfrak{T} \to \mathfrak{A}, \overline{d}\colon\mathfrak{T} \to \mathfrak{R}, \overline{a}\colon\mathfrak{A} \to \mathfrak{W}, \overline{c}\colon\mathfrak{R} \to \mathfrak{W}, g\colon\mathfrak{W} \to \mathfrak{N}$ such that the diagram in Figure 4.1 commutates. In other words*

(i) $g\circ\overline{a}=a,\ g\circ\overline{c}=c,\ \deg g=(\deg a, \deg c)$;

(ii) $\overline{b}\circ h=b,\ \overline{d}\circ h=d,\ \deg h=(\deg b, \deg d)$;

(iii) $\overline{a}\circ\overline{b}=\overline{c}\circ\overline{d}$.

Proof. It is clear from the proof of Theorem 2.1.3 that the lattice of groups intermediate between $\pi_1(\mathfrak{N}\backslash\mathfrak{d}_f)$ and $\pi_1(\mathfrak{M}\backslash f^{-1}(\mathfrak{d}_f))$ is isomorphic to a sublattice of $\mathfrak{L}_n$. By Corollary 1.3.3 it suffices to verify the following: if $\mathfrak{L}$ is a sublattice of $(\mathfrak{L}_n;\leq)$ and contains a and b then it also contains (a,b) and $[a,b]$.

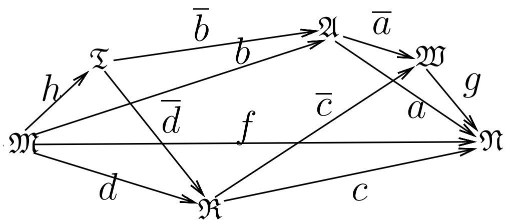


Figure 4.1: Finite maps.

Indeed this follows immediately from the definition of sublattice and it can be illustrated by Figure 4.2 where we use $s \to t$ to denote $s \leq t$ (for lattice

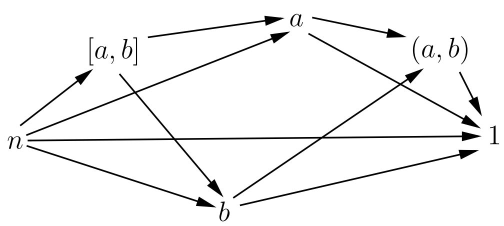


Figure 4.2: Lattice.

structure) or equivalently $t|s$. □

Ritt [109] proved this proposition in the case f is a polynomial and $\deg c = \deg a$. Levi [85] proved Ritt's result for polynomials defined over any field of characteristic zero. For a polynomial f and $\deg c \,|\, \deg a$ or $\deg d \,|\, \deg b$, Proposition 4.5.1 is due to Engstrom [40]. The complete version of Proposition 4.5.1 for polynomials first appeared in [127].

Proposition 4.5.1 applies to finite Blaschke products and gives

Proposition 4.5.2. *If a, b, c, d, f are finite Blaschke products and satisfy $a \circ b = c \circ d = f$ then there exist finite Blaschke products $\overline{a}, \overline{b}, \overline{c}, \overline{d}, h, g$ such that*

(i) $g \circ \overline{a} = a$, $g \circ \overline{c} = c$, $\deg g = (\deg a, \deg c)$;

(ii) $\overline{b} \circ h = b$, $\overline{d} \circ h = d$, $\deg h = (\deg b, \deg d)$;

(iii) $\overline{a} \circ \overline{b} = \overline{c} \circ \overline{d}$.

We give a simple example to explain how to apply the above rigidity properties.

Corollary 4.5.3. *Let f be a finite map from $\mathfrak{M}$ to $\mathfrak{N}$ which satisfies the conditions required for Proposition 4.5.1. If we have two decompositions of f into finite maps $f = a \circ b = c \circ d$ with $\deg a = \deg c$, then there exist biholomorphic maps ι such that*

$$a = c \circ \iota^{-1}, \quad b = \iota \circ d.$$

Proof. Applying Proposition 4.5.1 we obtain $\overline{a}, \overline{b}, \overline{c}, \overline{d}, h$ and g with properties as stated there. Because $\deg a = \deg c$ and $\deg b = \deg d$ it is clear that $\overline{a}, \overline{b}, \overline{c}, \overline{d}$ are all biholomorphic maps. One may choose $\iota = \overline{a}^{-1} \circ \overline{c}$ to fulfill the desired assertion. $\square$

The next corollary is about totally ramified maps.

Corollary 4.5.4. *If f is a finite Blaschke product of degree $s > 1$ and if f^t is totally ramified for some integer $t > 1$, then there exists $\mathfrak{p} \in \mathbb{E}$ and $\rho \in \mathbb{T}$ such that $f = \iota_{\mathfrak{p}} \circ \rho z^s \circ \iota_{-\mathfrak{p}}$.*

Proof. By assumption there exist ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $f^t = \epsilon \circ z^{s^t} \circ \varepsilon$. In other words we have

$$f \circ f^{t-1} = (\epsilon \circ z^s) \circ (z^{s^{t-1}} \circ \varepsilon),$$

which together with Corollary 4.5.3 implies that f is linearly related to $\epsilon \circ z^s$ and is in particular totally ramified. Writing $\mathfrak{p}$ respectively $\mathfrak{q}$ for critical value respectively critical point of f, we mush have $\mathfrak{p} = \mathfrak{q}$ since otherwise f^t cannot be totally ramified. It is then clear that $f = \iota_{\mathfrak{p}} \circ \rho z^s \circ \iota_{-\mathfrak{p}}$ for some $\rho \in \mathbb{T}$. $\square$

Our proposition also applies to elliptic finite Blaschke products.

Corollary 4.5.5. *If f is a finite Blaschke product of degree $s > 1$ and if $n > 1$ is a positive integer then f^n is not elliptic.*

Proof. If there exist ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ and a positive real number t such that $f^n = \epsilon \circ \mathfrak{T}_{s^n, t} \circ \varepsilon$ then it follows from the nesting property Theorem 2.3.5 that

$$f \circ f^{n-2} \circ f = (\epsilon \circ \mathfrak{T}_{s, s^{n-1}t}) \circ \mathfrak{T}_{s^{n-2}, st} \circ (\mathfrak{T}_{s,t} \circ \varepsilon).$$

Because of Proposition 3.1.10 we may apply Proposition 4.5.1 or Corollary 4.5.3 to elliptic rational functions, and consequently the rational function f is equivalent to both $\mathfrak{T}_{s, s^{n-1}t}$ and $\mathfrak{T}_{s,t}$ but this contradicts to Corollary 3.2.2 since $s^{n-1}t$ is greater than t. $\square$

Corollary 4.5.6. *Let f be an elliptic rational function and let $f=a\circ b$ be its decomposition into rational functions. Then a and b are both elliptic.*

Proof. Let $m=\deg a$ and let $n=\deg b$. There exist ϵ,ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{P}^1)$ and $\tau\in\mathbb{H}$ such that $f=\epsilon\circ(mn)_\tau\circ\varepsilon$, and it follows from the nesting property Proposition 3.1.11 that

$$a\circ b=(\epsilon\circ m_{n\tau})\circ(n_\tau\circ\varepsilon).$$

Applying Corollary 4.5.3 directly it is clear that a and b are both elliptic. □

Zieve-Müller discovered in [139] the following property.

Theorem 4.5.7 (Zieve-Müller)**.** *Let a,b and $f\notin\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ be complex polynomials, $n=\deg f$ and e a positive integer. If $a\circ b=f^e$ and if $\iota\circ f\circ\iota^{-1}\neq z^n, T_n$ or T_{-n} for all $\iota\in\mathrm{Aut}_{\mathbb{C}}(\mathbb{C})$ then there exist polynomials $\overline{a},\overline{b}$ and positive integers i,j,k with $k\leq\log_2(n+2)$ such that*

$$a=f^i\circ\overline{a}\quad\textit{and}\quad b=\overline{b}\circ f^j\quad\textit{and}\quad\overline{a}\circ\overline{b}=f^k.$$

We shall show that Zieve-Müller's phenomenon also holds in the Blaschke case.

Theorem 4.5.8. *Let a,b and $f\notin\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be finite Blaschke products, $n=\deg f$, k a positive integer and assume $a\circ b=f^k$. If $\iota\circ f\circ\iota^{-1}\neq z^n$ for all ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and if there does not exist a finite Blaschke product g for which either $a=f\circ g$ or $b=g\circ f$ then $k\leq\max(8,\ 2+2\log_2 n)$.*

The proof of Theorem 4.5.8 follows techniques developed in Zieve-Müller's original work and therefore our arguments are completely similar to that in [139] except the appearance of elliptic rational functions. We will be sketchy at many places and refer to [139] for more details.

Lemma 4.5.9. *If (a,b,c,d) is a generalized Ritt move in finite Blaschke products and if the finite map b(or a)$:\mathbb{E}\to\mathbb{E}$ is neither totally ramified nor elliptic then*

$$\deg a<\deg b\quad(\text{or}\quad\deg b<\deg a).$$

Proof. This follows immediately from Theorem 2.4.4. □

Lemma 4.5.10. *Let h be a finite Blaschke product with $h(0)\neq 0$ and s,n*

coprime positive integers with n at least 2. If an elliptic rational function $f(z)$ is of the form $z^s h(z)^n$ or $z^s h(z^n)$ then we must have $n=2$ and $s=1$

Proof. Firstly consider the case $f = z^s h(z)^n$. By the assumption that f is elliptic any point $\mathfrak{p}$ in $\mathbb{E}$ contributes at most once in $\mathfrak{D}_f$. Choose any $\mathfrak{w}$ in $\mathbb{E}$ for which $h(\mathfrak{w}) = 0$. Then we have $(s-1)(0) \leq \mathfrak{D}_f$ and $(n-1)(\mathfrak{w}) \leq \mathfrak{D}_f$, and this together with our other assumptions on n and s leads to $n=2$ and $s=1$.

Now consider the case $f = z^s h(z^n)$. Again any point $\mathfrak{p}$ in $\mathbb{E}$ contributes at most once in $\mathfrak{D}_f$. It is clear that $(s-1)(0) \leq \mathfrak{D}_f$ and therefore $s \leq 2$. Moreover since f elliptic its critical points $\mathfrak{D}_f$ are contained in a hyperbolic geodesic line. Using the fact that $f = z^s h(z^n)$ one can check if $\mathfrak{w} \neq 0$ is a point in $\mathfrak{D}_f$ then for any n-th root ζ of unity $\zeta\mathfrak{w}$ is also contained in $\mathfrak{D}_f$. These two facts excludes the case that $n \geq 3$, since otherwise $\mathfrak{D}_f$ contains at least two points which determine a hyperbolic geodesic line l preserved by a primitive n-th root of unity ζ. This is impossible and consequently $n \leq 2$ and we are done. □

Corollary 4.5.11. *If (a, b, c, d) is a generalized Ritt move in finite Blaschke products and if b (or a) is elliptic with degree at least three then d (or c) is also elliptic.*

Proof. If b is elliptic with degree at least three and if assume that d is not elliptic then we are in the second case of Theorem 2.4.4. Therefore there exist ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $\epsilon \circ b \circ \varepsilon$ is of the form $z^s h(z^{\deg d})$ and we deduce from Lemma 4.5.10 that $\deg d = 2$. This contradicts to the assumption that a is not elliptic.

If a is elliptic with degree at least three and if b is not elliptic then a similar argument applies. □

Let $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be a finite Blaschke product and $\mathcal{U} = (u_1, \ldots, u_r), \mathcal{V} = (v_1, \ldots, v_r)$ its maximal decompositions. Here the terminology that $\mathcal{U}$ is a maximal decomposition means that $f = u_1 \circ \cdots \circ u_r$ is a decomposition of f into prime finite Blaschke products. By Theorem 2.1.4 we can pass from $\mathcal{U}$ to $\mathcal{V}$, and this gives a unique permutation $\sigma_{\mathcal{U},\mathcal{V}}$ of $\{1, 2, \ldots, r\}$ which satisfies $\deg u_i = \deg v_{\sigma_{\mathcal{U},\mathcal{V}}(i)}$. In addition we have

Lemma 4.5.12. *If $i < j$ and if $\sigma_{\mathcal{U},\mathcal{V}}(i) > \sigma_{\mathcal{U},\mathcal{V}}(j)$ then* $(\deg u_i, \deg u_j) = 1$.

Here we need some notations introduced in [139]. We define

$$\begin{aligned}
\mathcal{LL}(\mathcal{U},\mathcal{V},i,j) &= \{k : k < i, \sigma(k) < \sigma(j)\},\\
\mathcal{LR}(\mathcal{U},\mathcal{V},i,j) &= \{k : k < i, \sigma(k) > \sigma(j)\},\\
\mathcal{RL}(\mathcal{U},\mathcal{V},i,j) &= \{k : k > i, \sigma(k) < \sigma(j)\},\\
\mathcal{RR}(\mathcal{U},\mathcal{V},i,j) &= \{k : k > i, \sigma(k) > \sigma(j)\}
\end{aligned}$$

and we write

$$\begin{aligned}
LL(\mathcal{U},\mathcal{V},i,j) &= \prod_{k\in\mathcal{LL}\,(\mathcal{U},\mathcal{V},i,j)} \deg u_k,\\
LR(\mathcal{U},\mathcal{V},i,j) &= \prod_{k\in\mathcal{LR}\,(\mathcal{U},\mathcal{V},i,j)} \deg u_k,\\
RL(\mathcal{U},\mathcal{V},i,j) &= \prod_{k\in\mathcal{RL}\,(\mathcal{U},\mathcal{V},i,j)} \deg u_k,\\
RR(\mathcal{U},\mathcal{V},i,j) &= \prod_{k\in\mathcal{RR}\,(\mathcal{U},\mathcal{V},i,j)} \deg u_k.
\end{aligned}$$

Let $\mathcal{U}=(u_1,\dots,u_r)$ be a complete decomposition of a finite Blaschke product f and let $u_k\in\mathcal{U}$ be an elliptic rational function with $\deg u_k\geq 3$. The *length* of u_k with respect to $\mathcal{U}$, denoted by $h_{\mathcal{U}}(u_k)$, is defined as $h_{\mathcal{U}}(u_k)=\Pi_{i=1}^{k-1}\deg(u_i)$. If u_i is an elliptic rational function then so is $v_{\sigma_{\mathcal{U},\mathcal{V}}(i)}$. We recall from Corollary 3.2.2 that for elliptic rational functions which are linear equivalent to finite Blaschke products, χ taking values in $\mathbb{R}_{>0}$ is well-defined.

Lemma 4.5.13. *If u_i is elliptic with degree at least three then*

$$h(u_i)\chi(u_i) = h(v_{\sigma_{\mathcal{U},\mathcal{V}}(i)})\chi(v_{\sigma_{\mathcal{U},\mathcal{V}}(i)}).$$

Proof. This follows immediately from Theorem 2.4.4. □

Moreover we also have

Lemma 4.5.14. *If $i<j$ and if u_i, u_j, $u_i\circ u_{i+1}\circ\cdots\circ u_j$ are elliptic finite Blaschke products with degree at least three then*

$$h(u_i)\chi(u_i)=h(u_j)\chi(u_j).$$

Proof. Assuming $u_i\circ u_{i+1}\circ\cdots\circ u_j$ is linearly equivalent to $\mathcal{T}_{n,t}$ and applying Corollary 4.5.3, it is clear that u_j is linearly equivalent to $\mathcal{T}_{\deg u_j,t}$ and u_i is linearly equivalent to $\mathcal{T}_{\deg u_i,\,t\Pi_{k=i+1}^{j}\deg u_k}$. As a consequence $\chi(u_j)=t$ and

$\chi(u_i)=t\Pi_{k=i+1}^{j}\deg u_k$ which immediately gives the desired assertion. □

Proposition 4.5.15. *Let $f\notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be a finite Blaschke product, $\mathcal{U}=(u_1,\dots,u_r)$ and $\mathcal{V}=(v_1,\dots,v_r)$ its maximal decompositions and k an integer between 1 and r. If we write $LL=LL(\mathcal{U},\mathcal{V},k,k)$ and LR, RL, RR analogously then LR, RL and $\deg u_k$ are coprime and there exist finite Blaschke products a with degree LL, d with degree RR, $b,\hat{b},\tilde{b}$ with degree LR, $c,\tilde{c},\bar{c}$ with degree RL and $\hat{u},\tilde{u},\bar{u}$ with degree $\deg u_k$ such that*

(i) $u_1\circ u_2\circ\cdots\circ u_{k-1}=a\circ b$ *and* $u_{k+1}\circ\cdots\circ u_r=c\circ d$;

(ii) $b\circ u_k=\hat{u}\circ\hat{b}$;

(iii) $\hat{u}\circ\hat{b}\circ c=\tilde{c}\circ\tilde{u}\circ\tilde{b}$;

(iv) $u_k\circ c=\bar{c}\circ\bar{u}$.

Proof. Based on Proposition 4.5.2, some analysis similar to that in proof of [139, Proposition 4.2] applies to our case. □

Proof of Theorem 4.5.8. We assume that $k\geq 2$. Choose $\mathcal{U}=(u_1,\dots,u_r)$ to be a complete decomposition of f, then $\mathcal{U}^k=(u_1,\dots,u_{kr})$ is a complete decomposition of f^k where $u_i=u_{i-r}$. Let $\mathcal{V}=(v_1,\dots,v_{kr})$ be a complete decomposition of f^k for which $a=v_1\circ v_2\cdots\circ v_e$ and $b=v_{e+1}\circ\cdots\circ v_{kr}$. By the assumption that $f^k=a\circ b$ and that there does not exist a finite Blaschke product g for which either $a=f\circ g$ or $b=g\circ f$, Proposition 4.5.2 applies and leads to $\deg f\nmid\deg a$ and $\deg f\nmid\deg b$. Therefore there exists $1\leq m\leq r$ such that $\sigma_{\mathcal{U}^k,\mathcal{V}}(m+tr)>e$ for all $0\leq t\leq k-1$ and there exists $1\leq l\leq r$ such that $\sigma_{\mathcal{U}^k,\mathcal{V}}(l+tr)\leq e$ for all $0\leq t\leq k-1$. Otherwise it follows from Proposition 4.5.2 immediately gives a contradiction. Moreover by Lemma 4.5.12 we have $(\deg u_m,\deg u_l)=1$.

Case 1. There exists $1\leq p\leq r$ such that for any ϵ,ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ the composition $\epsilon\circ u_p\circ\varepsilon^{-1}$ is not a power map, elliptic, of the form $z^sh(z^n)$ or of the form $z^sh(z^n)$, where n is an integer greater than one and h a finite Blaschke product.

We claim that $k=2$. Suppose, contrary to our claim, that $k\geq 3$. On the one hand we deduce from Theorem 2.4.4 that u_{p+r} never changes under Ritt moves and therefore $\sigma_{\mathcal{U}^k,\mathcal{V}}(p+r)=p+r$, $\sigma_{\mathcal{U}^k,\mathcal{V}}(i)<p+r$ for all $i<p+r$ and $\sigma_{\mathcal{U}^k,\mathcal{V}}(i)>p+r$ for all $i>p+r$. On the other hand we have $\sigma_{\mathcal{U}^k,\mathcal{V}}(m)>e$ and $\sigma_{\mathcal{U}^k,\mathcal{V}}(l+(k-1)r)\leq e$. Notice that $m<p+r$ and $l+(k-1)r>p+r$, consequently we have $e<p+r$ and $p+r<e$ respectively which is a contradiction.

Case 2. There exists $1 \le p \le r$ and ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that u_p is neither totally ramified nor elliptic and $\epsilon \circ u_p \circ \varepsilon^{-1}$ is of the form $z^s h(z^n)$ or $z^s h(z)^n$, where n is an integer greater than one and h a finite Blaschke product with $h(0) \neq 0$.

There exists $0 \le q \le k-1$ for which $\sigma_{\mathcal{U}^k,\mathcal{V}}(p+qr) \le e$ and $\sigma_{\mathcal{U}^k,\mathcal{V}}(p+(q+1)r) > e$. By Proposition 4.5.15 and by the fact that $\sigma_{\mathcal{U}^k,\mathcal{V}}(m+tr) > e$ for all $0 \le t \le q-1$ we have

$$(\deg u_m)^q | LR(p+qr).$$

Similarly from the fact that $\sigma_{\mathcal{U}^k,\mathcal{V}}(u_{l+tr}) \le e$ for all $q+2 \le t \le k-1$ we have

$$(\deg u_l)^{k-q-2} | RL(p+(q+1)r).$$

By Corollary 4.5.9 and by Proposition 4.5.15 we have

$$(\deg u_m)^q < \deg u_p, \quad (\deg u_l)^{k-q-2} < \deg u_p.$$

This gives $2^{k-2} \le (\deg u_p)^2 \le n^2$ and therefore $k \le 2 + 2\log_2 n$ as desired.

Case 3. All $u_i \colon \mathbb{E} \to \mathbb{E}$ in $\mathcal{U}$ are totally ramified.

If $\mathrm{supp}\, \mathfrak{D}_{u_i} = \mathfrak{d}_{u_{i+1}} = \mathfrak{p}$ holds for all integer i with $1 \le i \le kr-1$ then $\iota_{\mathfrak{p}} \circ f \circ \iota_{-\mathfrak{p}} = \zeta z^n$ for some $\zeta \in \mathbb{T}$, which contradicts to the assumption. As a consequence there exists $1 \le p \le r$ such that $\mathrm{supp}\, \mathfrak{D}_{u_p} \neq \mathfrak{d}_{u_{p+1}}$. It is clear from Theorem 2.4.4 and from Corollary 2.4.5 that any Ritt move in totally ramified finite Blaschke products (a, b, c, d) must satisfy $\mathfrak{d}_a = \mathfrak{d}_c$ and $\mathrm{supp}\, \mathfrak{D}_b = \mathrm{supp}\, \mathfrak{D}_d$. This implies that if $1 \le i \le r+p$ then $\sigma_{\mathcal{U}^k,\mathcal{V}}(i) \le r+p$ and if $(k-2)r+p+1 \le i \le kr-1$ then $\sigma_{\mathcal{U}^k,\mathcal{V}}(i) \ge (k-2)r+p+1$. Recall that $\sigma_{\mathcal{U}^k,\mathcal{V}}(m) > e$ and $\sigma_{\mathcal{U}^k,\mathcal{V}}(l+(k-1)r) \le e$, and notice that $m < r+p$ and $(k-1)r+l \ge (k-2)r+p+1$. Consequently we have $e < p+r$ and $(k-2)r+p+1 \le e$ which gives $k=1$.

Case 4. There exist $1 \le p \le r$ such that u_p is elliptic and is of degree at least three.

We claim that $k \le 8$. If contrary to our claim $k \ge 9$ then either $\sigma_{\mathcal{U}^k,\mathcal{V}}(4r+p) \le e$ or $\sigma_{\mathcal{U}^k,\mathcal{V}}(4r+p) > e$.

In the former case we deduce from Proposition 4.5.15 that there exist finite Blaschke products $a, b, \hat{u}, \overline{b}$ such that $\deg b = \deg \overline{b} = LR(\mathcal{U}^k, \mathcal{V}, p+4r, p+4r) =$

$\hat{n}, \deg \hat{u} = \deg u$ and

$$\begin{aligned} a \circ b &= u_1 \circ u_2 \circ \cdots \circ u_{p+4r-1}, \\ \hat{u} \circ \bar{b} &= b \circ u_{p+4r}. \end{aligned}$$

By Corollary 4.5.11 the finite Blaschke product b is elliptic, and by our assumption that $\sigma_{\mathcal{U}^k,\mathcal{V}}(r+m) > \sigma_{\mathcal{U}^k,\mathcal{V}}(m) > e \geq \sigma_{\mathcal{U}^k,\mathcal{V}}(4r+p)$ we have $\hat{n} \geq 4$. We now follow another critical trick of Zieve-Müller in their proof of [139, Proposition 4.4]. If we write $n = LR(\mathcal{U}^k, \mathcal{V}, p+2r, p+4r)$, $h = u_1 \circ u_2 \circ \cdots \circ u_{p+2r-1}$ and $g = u_{p+2r} \circ u_{p+2r+1} \circ \cdots \circ u_{p+4r-1}$ then apparently

$$h \circ g = a \circ b,$$

and for the same reason to that for $\hat{n}$ we have $n \geq 4$.

By Proposition 4.5.2 there exist finite Blaschke products $\hat{b}, \hat{g}, k, e, \hat{a}$ and $\hat{h}$ such that $\deg k = (\hat{n} = \deg b, \deg g)$, $\deg e = (\deg a, \deg h)$ and

$$\begin{aligned} \hat{b} \circ k &= b, \quad \hat{g} \circ k = g, \\ e \circ \hat{a} &= a, \quad e \circ \hat{h} = h, \\ \hat{h} \circ \hat{g} &= \hat{a} \circ \hat{b}. \end{aligned}$$

We denote $\deg k$ by s and notice that $(\hat{h}, \hat{g}, \hat{a}, \hat{b})$ is a generalized Ritt move. By

$$\hat{n}/n = \prod_{p+2r \leq i \leq p+4r-1,\, \sigma_{\mathcal{U}^k,\mathcal{V}}(i) > \sigma_{\mathcal{U}^k,\mathcal{V}}(p+4r)} \deg u_i$$

and by for all $p+2r \leq i \leq p+4r-1$

$$(\deg u_i, n) > 1 \Rightarrow \sigma_{\mathcal{U}^k,\mathcal{V}}(i) > \sigma_{\mathcal{U}^k,\mathcal{V}}(p+4r)$$

we have

$$\left(\frac{\deg g}{\hat{n}/n}, n \right) = 1. \tag{4.1}$$

Apparently $\hat{n}/n | (\deg g, \hat{n}) = s$ and we may set $s' := s/(\hat{n}/n)$. Moreover we deduce from $s' = n/(\hat{n}/s)$ that $s' | n$ and this together with $s' = s/(\hat{n}/n) | \deg g/(\hat{n}/n)$ and (4.1) leads to $s' = 1$. Consequently $\hat{n} = ns$ and this gives

$$\deg k = \prod_{p+2r \le i \le p+4r-1,\, \sigma_{\mathcal{U}^k,\mathcal{V}}(i) > \sigma_{\mathcal{U}^k,\mathcal{V}}(p+4r)} \deg u_i.$$

The rational function b is elliptic and so is $\hat{b}$. Noticing that $\deg \hat{b} = \deg b / \deg k = \hat{n}/\deg k$ and $\deg k = \hat{n}/n$, we have $\deg \hat{b} = n$. By $\deg \hat{b} = n > 3$ and by $(\hat{h}, \hat{g}, \hat{a}, \hat{b})$ is a generalized Ritt move we may apply Lemma 4.5.11 and deduce that $\hat{g}$ is also elliptic. We now examine

$$g = u_{p+2r} \circ u_{p+2r+1} \circ \cdots \circ u_{p+4r-1} = \hat{g} \circ k$$

and we write $\overline{\mathcal{U}} = (\overline{u}_1 = u_{p+2r}, \ldots, \overline{u}_{2r} = u_{p+4r-1})$ which is a complete decomposition of g. If $\overline{\mathcal{V}} = (v_1, \ldots, v_{2r})$ is a complete decomposition of g for which $\hat{g} = v_1 \circ v_2 \cdots \circ v_o$ and $k = v_{o+1} \circ \cdots \circ v_{2r}$ then we have $\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1) \le o$ and $\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1+r) \le o$. By Lemma 4.5.14 we have

$$h(v_{\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1)})\chi(v_{\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1)}) = h(v_{\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1+r)})\chi(v_{\sigma_{\overline{\mathcal{U}},\overline{\mathcal{V}}}(1+r)}).$$

and then by Lemma 4.5.13 we have

$$h(\overline{u}_1)\chi(\overline{u}_1) = h(\overline{u}_{1+r})\chi(\overline{u}_{1+r}).$$

This is impossible since $\chi(\overline{u}_1) = \chi(\overline{u}_{1+r}) = \chi(u_p)$ and $h(\overline{u}_1) < h(\overline{u}_{r+1})$.

Similar arguments apply to the case $\sigma_{\mathcal{U}^k,\mathcal{V}}(4r+p) > e$. □

Corollary 4.5.16. *Let f, a, b be finite Blaschke products with $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and l a positive integer. If there does exist ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ f \circ \iota^{-1} = z^{\deg f}$ and if $a \circ b = f^l$ then there exist Blaschke products $\overline{a}, \overline{b}$ and nonnegative integers i, j, k with $k \le \max(8, 2+2\log_2 \deg f)$ such that*

$$a = f^i \circ \overline{a}, \qquad b = \overline{b} \circ f^j, \qquad \overline{a} \circ \overline{b} = f^k.$$

Proof. There exists a maximal positive integer i such that $a = f^i \circ \overline{a}$ for some finite Blaschke product $\overline{a}$, and a maximal j such that $b = \overline{b} \circ f^j$ for some finite Blaschke product $\overline{b}$. We have $f^i \circ \overline{a} \circ \overline{b} \circ f^j = f^l$ and therefore $f^i \circ \overline{a} \circ \overline{b} = f^{l-j}$. This together with Corollary 4.5.3 implies that there exists ϵ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which

$$f^i = f^i \circ \epsilon^{-1}, \quad \overline{a} \circ \overline{b} = \epsilon \circ f^{l-i-j}.$$

Replacing $\overline{a}$ by $\epsilon^{-1} \circ \overline{a}$ we have $a = f^i \circ \overline{a}, b = \overline{b} \circ f^j$ and $\overline{a} \circ \overline{b} = f^k$. Lastly the maximality of i, j together with Theorem 4.5.8 leads to $k \leq \max(8, 2+2\log_2 \deg f)$. $\square$

This result tells us that the decomposition of f^l is not flexible, and as a further corollary we have

Corollary 4.5.17. *If $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ is a finite Blaschke product and if there does not exist ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ f \circ \iota^{-1} = z^{\deg f}$ then there is a finite subset $\mathcal{S}$ such that if two finite Blaschke products r and s satisfy $r \circ s = f^d$ then the following assertions*

(i) either there exists a finite Blaschke product h for which $r = f \circ h$ or there exists an element ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $r \circ \iota \in \mathcal{S}$;

(ii) either there exists a finite Blaschke product h for which $s = h \circ f$ or there exists an element ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ s \in \mathcal{S}$.

are satisfied.

Proof. We only prove the first assertion since a similar argument applies to the second one. If we are not in the case that $r = f \circ h$ for some finite Blaschke product h, then according to Corollary 4.5.16 r is a left factor of f^k for some $k \leq \max(8, 2+2\log_2 \deg f)$. Up to linear equivalence this set is finite and we are done. $\square$

4.6 Height

In this section we shall prove Proposition 4.6.4 by comparing the logarithmic naive height and Call-Siverman's canonical height and a key ingredient of our proof is a recent theorem of M. Baker [9], which improves a result of Benedetto [12] in the context of polynomials and which proves a special but important case of a general conjecture of Szpiro-Tucker [126] on the dynamics over function field. We first recall some basic facts. For a general reference on the theory of height, we refer to [21]. Associated with any global field E there is a set M_E of normalized absolute values which satisfies product formula and can be used to define logarithmic height $h(x)$ for all x in $\overline{E}$, i.e. the algebraic closure of E.

Lemma 4.6.1. *If E is a global field and if ι is an element in $\mathrm{Aut}_{\overline{K}}(\mathbb{P}^1)$ then* *there exists a positive constant c such that for all x in $\mathbb{P}^1(\overline{E})$ we have* $|h(\iota(x)) - h(x)| \leq c$.

Let $f \notin \mathrm{Aut}_{\overline{E}}(X)$ be a polarized endomorphism of an algebraic variety X defined over $\overline{K}$. Then the canonical height $\hat{h}_f(z)$ for all z in $X(\overline{E})$ is defined in the sense of Call-Silverman [29] and we shall need the following simple facts.

Lemma 4.6.2. *If E is a global field and if $f \in K(\mathbb{P}^1)$ is a rational function of degree at least two then there exists a constant c such that for any $z \in \mathbb{P}^1(\overline{E})$ and for any positive integer k we have:*

(i) $\hat{h}_f(f^k(z)) = (\deg f)^k \, \hat{h}_f(z)$;

(ii) $|h(z) - \hat{h}_f(z)| \leq c$;

(iii) if E is a number field then z is preperiodic if and only if $\hat{h}_f(z) = 0$.

The last statement of Lemma 4.6.2 fails if E is a function field. Instead we have M. Baker's theorem [9] and to state his result we first recall that a rational function g in $E(x)$ is said to be *isotrivial* if there is a finite extension E' of E and a linear fractional transformation ι in $\mathrm{Aut}_{E'}(\mathbb{P})$ such that $\iota \circ g \circ \iota^{-1}$ is defined over the field of constants of E.

Theorem 4.6.3 (M. Baker)**.** *If E is a function field and if $f \notin \mathrm{Aut}_E(\mathbb{P}^1)$ is a non-isotrivial rational function in $E(\mathbb{P}^1)$ then a point z in $\mathbb{P}^1(\overline{E})$ is preperiodic if and only if $\hat{h}_f(z) = 0$.*

This theorem is crucial for the proof of the following main result of this section,

Proposition 4.6.4. *Let f, g be complex rational functions and x_0, y_0 points $\mathbb{P}^1$. If $\mathcal{O}_{f\times g}(x_0, y_0)$ has infinitely many points on the diagonal then* $\deg f = \deg g$.

I am not sure whether this result is new or not. In [57, p.478] the authors remarked that they can prove Proposition 4.6.4 for polynomials and their proof depends on Benedetto's theorem and many other results from polynomial dynamics. Based on some idea of Ghioca-Tucker-Zieve we shall invoke M. Baker's theorem to prove the above proposition by induction on the transcendental degree of a field of definition of f, g, x_0, y_0 over $\mathbb{Q}$. We start with the following lemma,

Lemma 4.6.5. *Let k be a number field, f, g rational functions in $k(\mathbb{P}^1)$ and*

x_0, y_0 points in $\mathbb{P}^1(k)$. If the orbit $\mathcal{O}_{f\times g}(x_0,\, y_0)$ has infinitely many points on the diagonal then $\deg f = \deg g$.

Proof. It suffices to obtain a contradiction by assuming that $\deg f < \deg g$ and that there exists x in k for which $\mathcal{O}_{f\times g}(x,\, x)$ has infinitely many points on the diagonal. It is clear that x is not a preperiodic point of g since otherwise $\mathcal{O}_{f\times g}(x,\, x)$ has at most finitely many points on the diagonal and this leads to $\hat{h}_g(x) > 0$. On the one hand we deduce from Lemma 4.6.2 and from $\hat{h}_g(x) > 0$ that there exists a positive constant c_1 such that

$$h(g^m(x)) \geq c_1 \deg^m g$$

for all sufficiently large positive integers m. On the other hand if $f \notin \mathrm{Aut}_k(\mathbb{P}^1)$ then there exists a positive constant c_2 such that

$$h(f^m(x)) \leq c_2 \deg^m f$$

for all sufficiently large positive integers m and if f is in $\mathrm{Aut}_k(\mathbb{P}^1)$ then there exists another positive constant c_3 such that

$$h(f^m(x)) \leq c_3\, m$$

for m sufficiently large. By comparing the heights of $f^m(x)$ and $g^m(x)$ we conclude that there are only finitely many m for which $f^m(x) = g^m(x)$. This contradicts our assumption and completes the proof. $\square$

The proof of Proposition 4.6.4 is based on Lemma 4.6.5 and the technique of specialization.

Proof of Proposition 4.6.4. For the same reason as in the proof of Lemma 4.6.5 we may assume $x_0 = y_0 = x$, $\deg f < \deg g$ and x neither a preperiodic point of f nor of g. Objets f, g and x are all defined over a field k of finite type over $\mathbb{Q}$ and we continue with the proof by induction on *tr.deg*$(k/\mathbb{Q})$. If *tr.deg*$(k/\mathbb{Q}) = 0$ then it reduces to Lemma 4.6.5. If s is a positive integer greater than 1 and if we assume that the claim holds if *tr.deg*$(K/\mathbb{Q}) \leq s-1$ then we will prove it for *tr.deg*$(K/\mathbb{Q}) = s$. Choose a subfield k' of k such that *tr.deg*$(k/k') = 1$ and then k is the function field of a curve X defined over k'. Now we restrict our attention to $k \times_{k'} \overline{k'}/\overline{k'}$ instead of k/k'. If g is not isotrivial then we also have $\hat{h}_g(x) > 0$ by M. Baker's theorem and the argument in the proof of Lemma 4.6.5 still works. Now we assume g is isotrivial. After a conjugation by a linear fractional transformation and after a finite extension of $k \times_{k'} \overline{k'}$, we may assume g is defined over $\overline{k'}$. Now we fall into two cases:

Case 1, $x \in \overline{k'}$.

We choose α in $X(\overline{k'})$ at which f has good reduction and consider the reduction triple f_α, $g_\alpha = g, x_\alpha = x$. By assumption x is not preperiodic for g and therefore x_α is not preperiodic for g_α. This means that $\mathcal{O}_{f_\alpha \times g_\alpha}(x_\alpha, x_\alpha)$ has infinitely many points on the diagonal and we are done by the induction assumption.

Case 2, $x \notin \overline{k'}$.

We will give two alternative arguments. For the first proof we notice that x is a function of positive degree d on $X(\overline{k'})$ and therefore $g^m(x)$ is a function of degree $d \deg^m g$ on X. Moreover by induction it follows easily that there exists a natural number e such that for all positive integer m the function $f^m(x)$ is of degree at most $d \deg^m f + em \deg^m f$. We obtain a contradiction by comparing the degrees of $f^m(x)$ and of $g^m(x)$. For the second proof we notice that g is a function in $\overline{k'}(z)$ and there exists q in $\overline{k'}$ such that q is not preperiodic. Let α be a point in $X(\overline{k'})$ for which x_α equals q. We do the reduction at α and then we complete the proof by the induction assumption. $\square$

Although Siu has remarked that for the proof of Theorem 4.1.2 the use of Baker's theorem and of the full version of Proposition 4.6.4 can be easily avoided, a comparison of height of orbits of a linear map and of a non-linear map seems to be inevitable. Moreover we think that the statement of Proposition 4.6.4 should be of independent interest.

4.7 Commensurable case

Finite Blaschke products f, g are called *commensurable* if for any positive integer m there exist finite Blaschke products h_1, h_2 and positive integer n such that

$$f^n = g^m \circ h_1, \quad g^n = f^m \circ h_2.$$

Lemma 4.7.1. *Let $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ be a finite Blaschke product and ι an element in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. If for infinitely many positive integers n there exists ι_n in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $f^n = (f\circ\iota)^n \circ \iota_n$ then one of the following assertions*

(i) there exists a positive integer k for which $f^k = (f\circ\iota)^k$.

(ii) there exist μ, ρ in the unit circle $\mathbb{T}$ and ϵ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $f = \epsilon\circ\mu\, z^d \circ \epsilon^{-1}$ and $\iota = \epsilon\circ\rho\, z\circ\epsilon^{-1}$.

is satisfied.

Proof. By Corollary 4.5.3 applied to $f^n = (f\circ\iota)^n \circ \iota_n$ we deduce that there exist $\epsilon_n, \varepsilon_n$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ which satisfy $f\circ\iota\circ\iota_n = \epsilon_n \circ f$ and $f\circ\iota\circ f\circ\iota\circ\iota_n = \varepsilon_n \circ f^2$.

Case 1: In this case the map $f\colon \mathbb{E} \to \mathbb{E}$ is not totally ramified. We obtain from Proposition 4.3.4 positive integers $n < m$ which satisfy $\iota_n = \iota_m$. This gives $f^m = (f\circ\iota)^m \circ \iota_n = (f\circ\iota)^{m-n} \circ (f\circ\iota)^n \circ \iota_n = (f\circ\iota)^{m-n} \circ f^n$ which leads to $f^{m-n} = (f\circ\iota)^{m-n}$.

Case 2: If either f^2 or $f\circ\iota\circ f$ is not totally ramified, then we use a similar argument as in case 1.

Case 3: The maps f, f^2 and $f\circ\iota\circ f$ are all totally ramified. Firstly we write $\mathfrak{q} = \operatorname{supp} \mathfrak{D}_f$ and $\mathfrak{p} = \mathfrak{d}_f$. In our case f^2 is also totally ramified, and this leads to $\mathfrak{q} = \mathfrak{p}$. Using the fact that $f\circ\iota\circ f$ is also totally ramified, we deduce that $\iota(\mathfrak{p}) = \mathfrak{p}$. Consequently there exist μ, ρ in $\mathbb{T}$ such that $f = \iota_{\mathfrak{p}} \circ \mu\, z^d \circ \iota_{-\mathfrak{p}}$ and $\iota = \iota_{\mathfrak{p}} \circ \rho\, z \circ \iota_{-\mathfrak{p}}$. ☐

Proposition 4.7.2. *If $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and $g \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ are commensurable finite Blaschke products then either f and g have common iterates or there exist ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and μ in $\mathbb{T}$ such that*

$$\iota\circ f\circ\iota^{-1} = \mu z^r, \quad \iota\circ g\circ\iota^{-1} = z^s$$

where $r = \deg f$ and $s = \deg g$.

Proof. By the commensurability assumption for all positive integer m there exists a positive integer n and finite Blaschke products h_1, h_2 such that

$$f^n = g^m \circ h_1, \qquad g^n = f^m \circ h_2.$$

Case 1: In this case there exist positive integers k, t such that $r^k = s^t$. For any positive integer m we choose a positive integer n_m and a finite Blaschke product h_m for which $f^{mk} \circ h_m = g^{n_m}$ or equivalently $f^{mk} \circ h_m = g^{mt} \circ g^{n_m - mt}$.

By the condition $r^k = s^t$ we have $\deg f^{mk} = \deg g^{mt}$, to which Proposition 4.5.2 applies leads to that there exists ι_m in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $g^{mt} = f^{mk} \circ \iota_m$. Consequently for all positive integer m we have $(f^k)^m = (f^k \circ \iota_1)^m \circ \iota_m^{-1}$, and this reduces to Lemma 4.7.1.

Case 2: If there does not exist ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ g \circ \iota^{-1} = z^s$. For any positive integer m we denote by n_m the minimal integer for which there exists a finite Blaschke product h_m such that $g^{n_m} = f^m \circ h_m$. The minimality of n_m implies that there does exist a finite Blaschke product t for which $h_m = t \circ g$ and therefore we deduce from Corollary 4.5.17 that there exist positive integers $m < p$ for which $\deg h_m = \deg h_p$. This leads to $s^{n_p - n_m} = r^{p-m}$ and reduces the problem to the previous case.

Case 3: If there exists ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ g \circ \iota^{-1} = z^s$. For all positive integer m there exist a finite Blaschke product h_m and a positive integer n_m such that $f^m \circ h_m = g^{n_m}$ or equivalently

$$f^m \circ h_m \circ \iota^{-1} = \iota^{-1} \circ z^{r^m} \circ z^{s^{nm}/r^m}.$$

By Proposition 4.5.2 there exists ι_m in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $\iota \circ f^m = z^{r^m} \circ \iota_m$. This implies in particular that f^2 is totally ramified, and therefore by Lemma 4.5.4 there exists μ in $\mathbb{T}$ for which $\iota \circ f \circ \iota^{-1} = \mu z^r$. □

4.8 Non-commensurable case

This section is devoted to the proof of the following

Proposition 4.8.1. *If $f \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and $g \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ are non-commensurable finite Blaschke products and if for all positive integers m and n the curve $\mathbb{P}^1 \times_{f^n, g^m} \mathbb{P}^1$ admits a Faltings factor then there exist an element ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and a complex number μ in $\mathbb{T}$ such that*

$$\iota \circ f \circ \iota^{-1} = z^r, \quad \iota \circ g \circ \iota^{-1} = \mu z^s$$

where $r = \deg f$ and $s = \deg g$.

Proof. By the assumption that f and g are non-commensurable we may assume that there exists a positive integer t such that for any positive integer

n and for any finite Blaschke product h

$$g^n \neq f^t \circ h. \tag{4.2}$$

Given positive integers i and j, we deduce from the existence of Faltings factor of $\mathbb{P}^1 \times_{(f^t)^i, g^j} \mathbb{P}^1$ and from Theorem 3.6.3 that there exist finite Blaschke products a_{ij}, b_{ij}, c_{ij} and elements $\epsilon_{ij}, \varepsilon_{ij}$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that

$$(f^t)^i = a_{ij} \circ b_{ij} \circ \epsilon_{ij}, \quad g^j = a_{ij} \circ c_{ij} \circ \varepsilon_{ij},$$

where the set $\{b_{ij}, c_{ij}\}$ is described in Theorem 3.6.3. We write $\mathcal{S} = \{\deg a_{ij} : (i,j) \in \mathbb{N} \times \mathbb{N}\}$ and consider the following two cases.

Case 1. The cardinality of $\mathcal{S}$ is infinite.

Given any finite Blaschke product h and given any pair of positive integers i, j we must have

$$a_{ij} \neq f^t \circ h. \tag{4.3}$$

Otherwise we have $g^j = a_{ij} \circ c_{ij} \circ \varepsilon_{ij} = f^t \circ (h \circ c_{ij} \circ \varepsilon_{ij})$ and this gives a contradiction to (4.2). Using the assumption that the cardinality of $\mathcal{S}$ is infinite and using (4.3), Corollary 4.5.17 applied to $a_{ij} \circ (b_{ij} \circ \epsilon_{ij}) = (f^t)^i$ shows that there exists ι in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\iota \circ f^t \circ \iota^{-1} = z^{rt}$. In particular f^t is totally ramified, and this together with Lemma 4.5.4 leads to the existence of $\sigma \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\sigma \circ f \circ \sigma^{-1} = z^r$. Neither the hypothesis nor the conclusion are affected if we do the following replacement

$$f \mapsto \sigma \circ f \circ \sigma^{-1}, \quad g \mapsto \sigma \circ g \circ \sigma^{-1}.$$

Therefore we can assume $f(z) = z^r$ and then b_{ij} is a factor of $(f^t)^i = z^{rti}$. This shows that b_{ij} is totally ramified. As a consequently for all positive integers i and$j$ the pair of functions $\{b_{ij}, c_{ij}\}$ falls into either case 1 or case 2 of Theorem 3.6.3. Using this together with the fact b_{ij} is totally ramified, if c_{ij} is not totally ramified then it is clear that $\deg b_{ij} \le \deg c_{ij}$ and if c_{ij} is totally ramified then by Remark 3.6.4 we may assume that $\{b_{ij}, c_{ij}\} = \{z^{\hat{m}}, z^{\hat{r}}\}$. The inequality $\deg b_{ij} \le \deg c_{ij}$ is equivalent to $\deg f^{ti} \le \deg g^j$ which fails for large i and small j. As a result if i is sufficiently larger than j then

$$(z^{tr})^i = a_{ij} \circ b_{ij} \circ \epsilon_{ij}, \quad g^j = a_{ij} \circ c_{ij} \circ \varepsilon_{ij} \tag{4.4}$$

with $b_{ij} = z^{\hat{m}}, c_{ij} = z^{\hat{r}}$ and $\epsilon_{ij}, \varepsilon_{ij} \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. We need a simple fact that if $a \circ b$ with $\min\{\deg a, \deg b\} > 1$ is totally ramified then so are a, b in addition $\mathrm{supp}\, \mathfrak{D}_a = \mathfrak{d}_b$, and vice versa. Our decomposition of $(z^{tr})^i$ implies that a_{ij} is totally ramified as well and $\mathrm{supp}\, \mathfrak{D}_{a_{ij}} = \mathfrak{d}_{b_{ij}}$. This together with $b_{ij} = z^{\hat{m}}$ and $c_{ij} = z^{\hat{r}}$ leads to $\mathrm{supp}\, \mathfrak{D}_{a_{ij}} = \mathfrak{d}_{c_{ij}}$ and therefore g^j, which equals $a_{ij} \circ c_{ij} \circ \varepsilon_{ij}$, is also totally ramified. By Corollary 4.5.4 applied to g^j there exists $\bar{\iota}$ in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ such that $\bar{\iota} \circ g \circ \bar{\iota}^{-1} = z^s$. It is clear from (4.4) that $\mathfrak{d}_{g^j} = \mathfrak{d}_{a_{ij}} = \mathfrak{d}_{z^{tri}} = 0$ and consequently $\mathfrak{d}_g = 0$. This together with $\bar{\iota} \circ g \circ \bar{\iota}^{-1} = z^s$ implies that $g(z) = \mu z^s$ for some μ in $\mathbb{T}$.

Case 2. The cardinality of $\mathcal{S}$ is finite.

If $(r, s) = 1$ then for all positive integers i, j we have $\deg a_{ij} = 1$ and $\{b_{ij}, c_{ij}\}$ falls into case 1 or case 3 of Theorem 3.6.3. We exclude the case when $i = j = 1$. This assumption together with Corollary 4.5.5 implies that not both of f^{ti} and g^j are elliptic, and then $\{b_{ij}, c_{ij}\}$ falls into the first case of Theorem 3.6.3. From the expression there the one of $\{b_{ij}, c_{ij}\}$ with smaller degree must be totally ramified. This remains the case with $\{b_{ij}, c_{ij}\}$ replaced by $\{f^{ti}, g^j\}$ since $\deg a_{ij} = 1$. Because we can choose either i or j to be arbitrary large it follows in particular that both f^{2t} and g^2 are totally ramified. Using Lemma 4.5.4 there exist ϵ, ε in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ for which $\epsilon \circ f \circ \epsilon^{-1} = z^r$ and $\varepsilon \circ g \circ \varepsilon^{-1} = z^s$. For the desired assertion it suffices to show that $\mathfrak{d}_f = \mathfrak{d}_g$. Indeed by Remark 3.6.4 we have $\mathfrak{d}_{b_{ij}} = \mathfrak{d}_{c_{ij}}$ and therefore $\mathfrak{d}_f = \mathfrak{d}_g = a_{ij}(0)$.

If $(r, s) \neq 1$ then for sufficiently large i, j we deduce from the finiteness assumption for $|\mathcal{S}|$ that $(\deg b_{ij}, \deg c_{ij})$ is sufficiently large, and hence for $\{f^{ti}, g^j\}$ the case 4 of Theorem 3.6.3 applies. If we choose sufficiently large positive integers p, n and m with $n+1 < m$ such that $\deg a_{p,n} = \deg a_{p,m}$ then

$$g^n = a_{p,n} \circ c_{p,n} \circ \varepsilon_{p,n}, \quad g^m = a_{p,m} \circ c_{p,m} \circ \varepsilon_{p,m}$$

where $c_{p,n}, c_{p,m}$ are elliptic rational functions. This gives

$$a_{p,n} \circ \big(c_{p,n} \circ \varepsilon_{p,n} \circ g^{m-n}\big) = a_{p,m} \circ \big(c_{p,m} \circ \varepsilon_{p,m}\big).$$

Using Lemma 4.5.3 we obtain that $c_{p,n} \circ \varepsilon_{p,n} \circ g^{m-n}$ is linearly related to the elliptic rational function $c_{p,m}$, and then using Corollary 4.5.6 we conclude that g^{m-n} is elliptic. This contradicts Corollary 4.5.5 and completes the proof. $\square$

4.9 Proof of theorems

We are now ready to prove our main theorems.

Proof of Theorem 4.1.5. The infiniteness assumption on $\mathcal{O}_f(x)\cap\mathcal{O}_g(y)$ implies that for all positive integers m,n there are infinitely many rational points on $\mathbb{P}^1\times_{f^n,g^m}\mathbb{P}^1$ over the absolute field k generated by all coefficients of f,g and x, y. Indeed by assumption there exist pairwise distinct point p_i in $\mathbb{P}^1$ for all $i\geq 1$ and positive integers n_i,m_i such that

$$f^{n_i}(x)=p_i, \quad g^{m_i}(y)=p_i.$$

It is clear from the pairwise distinctness condition of p_i that n_i are also pairwise distinct and therefore tends to infinity as i goes to infinity, and this remains the case for m_i. Therefore for all i sufficiently large, $(f^{n_i-n}(x),g^{m_i-m}(y))$ are k-rational points in $\mathbb{P}^1\times_{f^n,g^m}\mathbb{P}^1$. These points are pairwise distinct, since otherwise x would be preperiodic for f which contradicts to the infiniteness assumption on $\mathcal{O}_f(x)\cap\mathcal{O}_g(y)$.

By Faltings' theorem for finitely generated fields over the rational the curve $\mathbb{P}^1\times_{f^n,g^m}\mathbb{P}^1$ has a Faltings factor. If f and g have no common iterate then by Proposition 4.7.2 and Proposition 4.8.1 we may assume that $f=z^r, g=\mu z^s$ for some μ in $\mathbb{T}$. This case is easily covered by Laurent's theorem. □

The following example shows that it is crucial to require that $f,g\notin\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ in Theorem 4.1.5.For convenience we consider $\mathrm{Aut}_{\mathbb{C}}(\mathbb{H})$ instead of $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and we will construct f,g in $\mathrm{Aut}_{\mathbb{C}}(\mathbb{H})$ and x,y in $\mathbb{P}^1$ for which $\mathcal{O}_f(x_0)\cap\mathcal{O}_g(y_0)$ is infinite but without f,g having a common iterate. We just take $f(z)=z+1$, $g(z)=2z$ and $x=y=1$.

It is of no more essential difficulty to generalize Theorem 4.1.5 to the case of commutative random dynamics.

Corollary 4.9.1. *Let x_0,y_0 be points in $\mathbb{P}^1$, $f_i\notin\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})(1\leq i\leq e)$ a set of pairwise commutative finite Blaschke products, $g_j\notin\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ $(1\leq j\leq k)$ another set of pairwise commutative finite Blaschke products, $\mathcal{S}=\langle f_1,f_2,\ldots,f_e\rangle$, $\mathcal{G}=\langle g_1,g_2,\ldots,g_k\rangle$ and $\alpha\colon\mathbb{N}_0^{e+k}\to\mathcal{S}\times\mathcal{G}$ given by $(i_1,i_2,\ldots,i_{e+k})\mapsto(f_1^{i_1}\circ f_2^{i_2}\circ\cdots\circ f_e^{i_e}, g_1^{i_{e+1}}\circ g_2^{i_{e+2}}\circ\cdots\circ g_k^{i_{e+k}})$. Then*

$$\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0,y_0))\cap\Delta_{\mathbb{P}^1}=\mathcal{O}_{\alpha(E)}((x_0,y_0)),$$

where E is a union of finitely many sets of the form $u+(\mathbb{N}_0^{e+k}\cap H)$ with u in $\mathbb{N}_0^{e+k}$ and H a subgroup of $\mathbb{Z}^{e+k}$.

Proof. It suffices to prove the statement with $\mathcal{S}$ replaced by $\mathcal{S}'$ where $\mathcal{S}'$ is of the form $\langle f_1^{l_1}, f_2^{l_2}\cdots, f_e^{l_e}\rangle$ for any tuple of positive integers l_i.

The proof is by induction on $s=e+k$. Firstly we check the case when $s=2$ which is equivalent to $e=k=1$. If $\mathcal{O}_{f_1}(x_0)\cap\mathcal{O}_{g_1}(y_0)$ is finite then E can be chosen to be finitely many elements of $\mathbb{N}_0^2$ and we are done. Otherwise $\mathcal{O}_{f_1}(x_0)\cap\mathcal{O}_{g_1}(y_0)$ is infinite and we conclude from Theorem 4.1.5 that there exists positive integers m,n for which $f_1^m=g_1^n$. After replacing f_1,g_1 by f_1^m,g_1^n we can assume $f_1=g_1=f$. It is clear that neither x_0 nor y_0 is a preperiodic point of f, otherwise it contradicts the infiniteness assumption of $\mathcal{O}_{f_1}(x_0)\cap\mathcal{O}_{g_1}(y_0)$. Then we claim that

$$\mathcal{O}_{\langle f\rangle\times\langle f\rangle}((x_0,y_0))\cap\Delta_{\mathbb{P}^1}=\mathcal{O}_{\alpha((p,\,q)+\mathbb{N}_0^2\cap\mathbb{Z}\langle 1,1\rangle)}((x_0,y_0))$$

where $(p,\,q)$ is the smallest pair of positive integers for which $f^p(x_0)=f^q(y_0)$. Indeed, let $(p,\,q)$ be the pair of positive integers with the smallest p such that $f^p(x_0)=f^q(y_0)$. It is clear that $f^{p+o}(x_0)=f^{q+o}(y_0)$ for all $o\geq 0$, and it remains to prove that these are all the pairs (m,n) of positive integers that satisfy $f^m(x_0)=f^n(y_0)$. If it is not the case then there according to the minimality of p there exists $(p_1>p, q_1\neq q+p_1-p)$ for which $f^{p_1}(x_0)=f^{q_1}(y_0)$. This implies $f^{q_1}(y_0)=f^{q+p_1-p}(y_0)$ which contradicts the non-preperiodicity of y_0. Assuming that the conclusion holds for $s\leq e+k-1$ we will prove it for $s=e+k$.

Case 1. If there exists a positive integer N for which

$$\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0,y_0))\cap\Delta_{\mathbb{P}^1}=\mathcal{O}_{\alpha(F_N)}((x_0,y_0))\cap\Delta_{\mathbb{P}^1}$$

where $F_N=\{(p_1,\ldots,p_e,q_1,\ldots,q_k)\in\mathbb{N}_0^{e+k}:\min\{p_i,q_j\}\leq N\}$ then equivalently

$$\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0,y_0))\cap\Delta_{\mathbb{P}^1}=\bigcup_{1\leq l\leq e+k,\,0\leq j\leq N}\mathcal{O}_{\alpha\circ i_l(\mathbb{N}_0^{e+k-1})}(\beta_l(j)((x_0,\,y_0))),$$

where i_l is a map from $\mathbb{N}_0^{e+k-1}$ to $\mathbb{N}_0^{e+k}$ given by

$$(n_1,\cdots,n_{e+k-1})\mapsto(n_1,\ldots,n_{l-1},1,n_l,\ldots,n_{e+k-1})$$

and β_l is a map from $\mathbb{N}_0$ to $\mathcal{S}\times\mathcal{G}$ given by $\beta_l(j)=f_l^j$ if $l\leq e$ or $\beta_l(j)=g_{l-e}^j$

if $l > e$. This implies immediately the desired conclusion by the induction hypothesis.

Case 2. If x_0 is preperiodic for $\mathcal{S}$ in the sense that $\alpha(t)(x_0) = \alpha(r)(x_0)$ for some distinct $t = (t_1, \ldots, t_e, 1, \ldots, 1)$ and $r = (r_1, \ldots, r_e, 1, \ldots, 1)$ in $\mathbb{N}_0^{e+k}$ and if we assume $\sum_{i=1}^{e} t_i \geq \sum_{i=1}^{e} r_i$ then we claim that

$$\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0, y_0)) \cap \Delta_{\mathbb{P}^1} = \mathcal{O}_{\alpha(F)}((x_0, y_0)) \cap \Delta_{\mathbb{P}^1}$$

where $F = \{(p_1, \ldots, p_e, q_1, \ldots, q_k) \in \mathbb{N}_0^{e+k} : \min_{1\leq i\leq e}\{p_i\} < \max_{1\leq i\leq e}\{t_i\}\}$. Indeed let $a = (p_1, \ldots, p_l, q_1, \ldots, q_k)$ satisfy $\min_{1\leq i\leq e}\{p_i\} \geq \max_{1\leq i\leq e}\{t_i\}$ and $\mathcal{O}_{\alpha(a)}((x_0, y_0)) \in \Delta_{\mathbb{P}^1}$. Letting $b = \{p_1 - t_1 + r_1, \ldots, p_l - t_l + r_l, q_1, \ldots, q_k\}$ then we have $\mathcal{O}_{\alpha(a)}((x_0, y_0)) = \mathcal{O}_{\alpha(b)}((x_0, y_0))$. By the standard argument of infinite descent we continue with a similar replacement and finally obtain some $c \in F$ satisfying $\mathcal{O}_{\alpha(a)}((x_0, y_0)) = \mathcal{O}_{\alpha(c)}((x_0, y_0))$, which proves our claim. This claim reduces our problem to the previous case and we are done. Our argument here also works in the case that y_0 is preperiodic for $\mathcal{G}$ by symmetry.

Case 3. We continue with assuming e to be at least 2 and excluding the previous cases. If $\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0, y_0)) \cap \Delta_{\mathbb{P}^1}$ is finite then nothing requires a proof. We assume that the intersection is infinite and let k be the absolute field generated by coefficients of f_is, g_is, x_0 and y_0. Given any $1 \leq i \leq e, 1 \leq j \leq k$ and given any pair of positive integers $\{m, n\}$, because we are not in case 1 by a similar argument as that in the proof of Theorem 4.1.5 the curve $\mathbb{P}^1 \times_{f_i^m, g_j^n} \mathbb{P}^1$ admits infinitely many k-rational points and therefore has a Faltings factor. Applying Proposition 4.7.2 and Proposition 4.8.1 we deduce that either f_i and g_j have common iterates or f_i and g_j are both totally ramified, and in the later case we have in addition that $\mathfrak{d}_{f_i} = \mathfrak{d}_{g_j} = \operatorname{supp} \mathfrak{D}_{f_i} = \operatorname{supp} \mathfrak{D}_{g_j}$. This holds for any $1 \leq i \leq e$ and any $1 \leq j \leq k$. Using the fact that $e \geq 2$ it is clear that either all $f_1, \ldots, f_e, g_1, \ldots, g_k$ have common iterates or all $f_1, \ldots, f_e, g_1, \ldots, g_k$ are totally ramified sharing the same critical points and critical values. In the former case x_0 is preperiodic for $\mathcal{S}$ which contradicts our assumption. In the latter case we are reduced to Laurent's theorem. □

Proof of Theorem 4.1.3. It follows immediately from Theorem 4.1.4, Theorem 4.1.5 and the uniformization theorem. □

Proof of Theorem 4.1.2. Neither the hypothesis nor the conclusion is affected if we replace F by F^k for some positive integer k. We assume according to Theorem 1.5.4, by replacing F by some power, that $F(z_1, \ldots, z_d) = (f_1(z_1), \ldots, f_d(z_d))$ for some finite Blaschke products f_i. Moreover we may assume, after taking conjugations, that the complex geodesic L in question is the diagonal $\Delta_{\mathbb{E}}$. Theorem 4.6.4 applied to any pair $\{f_i, f_j\}$ shows that $\deg f_i$ is independent of i. This together with Theorem 4.1.5 gives the desired conclusion if $f_i \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$. In the case when $f_i \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ our Theorem 4.4.1 applies. □

We need one more lemma for the proof of Theorem 4.1.6.

Lemma 4.9.2. *Let x_0, y_0 be points in $\mathbb{E}$, $f_i (1 \le i \le e)$ a set of pairwise commutative finite Blaschke products for which at least one of the f_i is nonlinear, $g_j (1 \le j \le k)$ a set of pairwise commutative finite Blaschke products for which at least one of g_j is nonlinear, $\mathcal{S} = \langle f_1, f_2 \cdots, f_e \rangle$, $\mathcal{G} = \langle g_1, g_2 \cdots, g_k \rangle$ and $\alpha \colon \mathbb{N}_0^{e+k} \to \mathcal{S} \times \mathcal{G}$ the map defined as in Corollary 4.9.1. Then*

$$\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0, y_0)) \cap \Delta_{\mathbb{E}} = \mathcal{O}_{\alpha(E)}((x_0, y_0))$$

where E is a union of finitely many sets of the form $u + (\mathbb{N}_0^{e+k} \cap H)$ with $u \in \mathbb{N}_0^{e+k}$ and H a subgroup of $\mathbb{Z}^{e+k}$.

Proof. With an analysis similar to that in the proof of Proposition 4.9.1, we proceed by induction and can assume that x_0 is not preperiodic for $\mathcal{S}$, y_0 is not preperiodic for $\mathcal{G}$ and $\mathcal{O}_{\mathcal{S}\times\mathcal{G}}((x_0, y_0)) \cap \Delta_{\mathbb{E}}$ is infinite. If all f_i, g_j are of degree greater than 1 then Corollary 4.9.1 gives the desired statement. We assume therefore there exists linear maps in f_i, g_j. If at least one of the f_i and g_j which is not is $\mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ and is not conjugate to the power map z^n, then according to the non-periodicity of x_0, y_0 by Proposition 4.7.2 or by Proposition 4.8.1 so is any nonlinear f_i(or g_j). This together with Corollary 4.3.6 implies that all linear maps among f_i, g_j are of finite order which is in contradiction with the assumption of non-periodicity. It remains to consider the case that all $f_i, g_j \notin \mathrm{Aut}_{\mathbb{E}}$ are conjugate to power maps. This together with Proposition 4.7.2 and Proposition 4.8.1 implies that after suitable conjugations with some $\iota \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ we may assume that all $f_i, g_j \notin \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ are of the form μz^r. Then by the assumption of commutability all $f_i, g_j \in \mathrm{Aut}_{\mathbb{C}}(\mathbb{E})$ are of the form νz. It is clear that we are reduced to Laurent's theorem which completes the proof. □

Proof of Theorem 4.1.6. It suffices to prove the conclusion if F_i are replaced by $F_i^{l_i}$ for any tuple of positive integers l_i. Consequently we shall assume that $F_i(z_1, \dots, z_d) = (f_{i1}(z_1), \dots, f_{id}(z_d))$ for some finite Blaschke products f_{ij}, L is the diagonal and $L \cap \mathcal{O}_{\mathcal{S}}(p)$ is infinite.

Given any j between 1 and d at least one out of the f_{ij} where i runs from 1 to e is of degree greater than 1. This follows from the argument similar to that in the proof of Theorem 4.6.4. The desired conclusion follows immediately from Lemma 4.9.2 and Proposition 2.6 of [59] for which we recall as the next lemma. □

Lemma 4.9.3 (Chioca-Tucker-Zieve)**.** *For any γ_1, γ_2 in $\mathbb{N}_0^r$ and for any subgroups H_1, H_2 in $\mathbb{Z}^r$, the intersection $(\gamma_1 + (H_1 \cap \mathbb{N}_0^r)) \cap (\gamma_2 + (H_2 \cap \mathbb{N}_0^r))$ is a union of at most finitely many cosets of subsemigroups of the form $\beta + (H \cap \mathbb{N}_0^r)$ where $H = H_1 \cap H_2$.*

5

Transcendence and Algebraic Distribution

This chapter is devoted to a generalization of Schneider's theorem in transcendence. The main result here is Theorem 5.1.5, where we have proved that under suitable conditions the image of $f: C \to X_{\mathbb{C}}$, which is a holomorphic map from an algebraic curve to an projective algebraic variety X defined over a number field, assumes finitely many times rational points. This truly generalizes Schneider's criterion, and the proof is along that of Schneider's original work and one of our motivation is to understand an analogy between the first main theorem of Nevanlinna theory and Schneider's criterion.

5.1 Schneider-Lang Criterion

In this chapter we prove a generalization of a theorem of Schneider which can be seen as the counterpart in the theory of transcendental numbers to work of Chern [33] in value distribution theory.

In the past transcendence theory has been largely related to the study of values of analytic functions with additional properties. The first remarkable instance was Hermite's work on e followed by Lindemann's spectacular proof of the transcendence of π as a corollary of his celebrated theorem on the transcendence of e^α for algebraic $\alpha \neq 0$. A further example is the solution of Hilbert's seventh problem by A. O. Gelfond [54] and independently by Th. Schneider [113]. All these results are theorems about holomorphic functions of exponential type which satisfy linear differential equations. In 1936 Schneider [114] extended the results to elliptic functions which again satisfy a system of first order differential equations. In contrast to exponential functions they are only meromorphic and have order of growth 2. Then Schneider [116] realized in 1948 that the methods which were developed by himself and by Gelfond can be used to prove a very general and very conceptual transcendence criterion which included all the results described so far. Schneider pointed out in $3 b) in loc. cit. that Gelfond's proof of the Hilbert problem is not covered by his theorem and he puts this as an open question. In his book S. Lang [80] streamlined the formulation of the theorem by simplifying the hypotheses and got a very elegant criterion, now known as the Schneider-Lang criterion. This has the effect that the theorem becomes less general but has the advantage that its proof becomes slightly simpler and that the criterion is much easier to apply. In particular Lang assumes that the functions satisfy differential equations which is not needed in Schneider's theorem. However it still covers the main applications and even includes Gelfond's proof. It is clear that requiring differential equations is restrictive and schrinks the general applicability of the theorem.

Theorem 5.1.1 (Schneider-Lang Criterion)**.** *Let $K \subset \mathbb{C}$ be a number field and let $f_1, \cdots, f_N$ be meromorphic functions of order $\leq \rho$. Assume that the field $K(f) = K(f_1, \cdots, f_N)$ has transcendence degree ≥ 2 over K and that the derivative $\nabla = d/dt$ maps the ring $K[f] = K[f_1, \cdots, f_N]$ into itself. If $\mathcal{S}$ is a set of points in $\mathbb{C}$ such that*

$$f_i(w) \in K$$

for all $w \in \mathcal{S}$ then $|\mathcal{S}| \leq 20\rho[K : \mathbb{Q}]$.

The elegance of the criterion was the starting point for further spectacular progress. In the same book [80] Lang got a version of the theorem for functions on $\mathbb{C}^n$. However he did not make real use of the much more complicated complex analysis in the case of several variables. Schneider had already pointed out among others in his paper this possibility. Later, in a wonderful paper [19], Bombieri, using very deep techniques in complex analysis of several variables, got the genuine several variables version of Schneider's theorem á la Schneider-Lang.

Theorem 5.1.2 (Bombieri)**.** *Let $K \subset \mathbb{C}$ be a number field and let $f = (f_1, \cdots, f_n)$ be meromorphic functions in $\mathbb{C}^d$ of finite order. Assume that*

(1) *tr deg $K(f) \geq d+1$,*

(2) *the partial derivatives $\partial/\partial z_\alpha, \alpha = 1, \cdots, d$, map the ring $K[f]$ into itself.*

Then the set $\mathcal{S}$ of points $\xi \in \mathbb{C}^n$ where $f(\xi)$ takes values in K^N is contained in an algebraic hypersurface.

It is remarkable that the proofs of the two theorems above are closely related to value distribution theory in the case of functions on $\mathbb{C}$ and $\mathbb{C}^n$ respectively. One main topic in Nevanlinna theory is to understand the relation between the growth of pole divisors and the growth of functions. In the case of rational functions the situation is very simple because the divisors are all finite and then the theory dates back to Gauss and his work on the fundamental theorem of algebra. In the case of meromorphic functions on $\mathbb{C}$ the value distribution theory is highlighted by two Main Theorems which were first given by Nevanlinna, and we refer to his famous book [97].

The beauty of Nevanlinna's theory seduced many mathematicians to try to understand the distribution property of functions in different and more general situations. First it was extended to functions on $\mathbb{C}^n$ by W. Stoll in [125] and by S. S. Chern [32]. Chern's insight into the role played by infinity in Nevanlinna theory lead to an extension of the theory to affine curves which was published in [33]. The general case of affine varieties has been accomplished by Ph. Griffiths and J. King in [30] and [63]. They introduced exhaustion functions to define growth and made use of differential geometric and complex algebrao-geometric methods.

The work of Schneider, Lang and Bombieri in transcendence theory is related to the cases $\mathbb{C}$ and $\mathbb{C}^n$ which were studied by Nevanlinna, Stoll and Chern whereas our work now deals the first time with affine curves.

Let Z be a smooth projective curve of genus g over $\mathbb{C}$ and $\mathcal{P}$ a non-empty set of l points in Z. Then $C := Z - \mathcal{P}$ is an affine algebraic curve. We let X be a non-singular algebraic variety defined over a number field K with tangent sheaf $\mathcal{T}_X$. Further we assume that $\psi : C \to X_{\mathbb{C}}$ is an integral curve of a vector field $\Delta \in \Gamma(X, \mathcal{T}_X \otimes \mathcal{M}_X)$ which by definition acts on the sheaf of meromorphic sections $\mathcal{M}_X$ as derivations. Then for every $c \in C$ there is an analytic local section ∇ at c of the tangent sheaf $\mathcal{T}_C$ which does not vanish at c and such that locally we have $\psi_*(\nabla) = \Delta_{\mathbb{C}}$. The section ∇ acts on analytic functions on C as derivation and the local sections ∇ for varying $c \in C$ glue together and give a nowhere vanishing holomorphic global section $\nabla \in \Gamma(C, \mathcal{T}_C)$.

Our results, in particular the main theorem, and a fortiori the proofs depend on the order of the integral curve ψ. There are several essentially equivalent approaches to a concept of order in our situation. In section 5.5 we shall discuss the different ways to define an order $\rho(\psi)$ of ψ and we shall show that they lead to the same value which coincides with the order in the Schneider-Lang criterion when $C = \mathbb{A}^1(\mathbb{C})$ and $X = \mathbb{A}^n$.

Theorem 5.1.3. *If* $\dim \overline{\psi(C)} \geq 2$ *then*

$$|\psi^{-1}(X(K))| \leq g + 2\,(2\,[K : \mathbb{Q}] + 1)\, l\, \max(\rho(\psi), 2g)$$

In the Main Theorem we start from objects on X. If instead we start with objects on C we get the following theorem which then takes more the form of the Schneider-Lang criterion.

Theorem 5.1.4. *Let $K \subset \mathbb{C}$ be a number field and let $f_1, \cdots, f_N$ be holomorphic functions on C with order $\leq \rho$. We assume that the field $K(f_1, \cdots, f_N)$ has transcendence degree ≥ 2 over K and that ∇ is an analytic section of $\mathcal{T}_C$ which acts as a derivation on $K[f_1, \cdots, f_N]$. If $\mathcal{S}$ is a subset of C such that $f_i(w) \in K$ for all $w \in \mathcal{S}$ and all i then*

$$|\mathcal{S}| \leq g + 2\,(2[K : \mathbb{Q}] + 1)\, \rho\, l. \tag{5.1}$$

We give two proofs of the theorem in the last section. One is based on a Jensen formula which will be discussed in section 5.4 and a second uses the maximum principle.

Before we state our next theorem we discuss how Theorem 5.1.3 and Theorem 5.1.4 are related. Let $f_1, \ldots, f_N$ be functions on C as in Theorem 5.1.4

and let $F\colon C \to \mathbb{C}^N$ be the map $F(z) = (f_1(z), \ldots, f_N(z))$. The derivation ∇ of $K[f_1, \cdots, f_N]$ can be lifted to give a regular algebraic vector field Δ on $\mathbf{A}_K^N$ which can be obtained explicitly. For its construction we express $K[f_1, \cdots, f_N]$ as $K[T_1, \cdots, T_N]/I$ for an ideal I. If $\nabla(f_i) = g_i(f_1, f_2, \ldots, f_N)$ then we take $\Delta = g_i(T_1, \cdots, T_N)\partial/\partial T_i$. Clearly the choice of Δ is not unique since taking $\Delta + h\partial/\partial T_i$ for any $h \in I$ is another possibility to select. It is clear however that their restrictions to $K[f_1, \cdots, f_N]$ coincide.

For $a \in C$ we denote by ∇_a the germ of ∇ at a. We have $dF(\nabla_a)(T_i) = \nabla_a(F^*(T_i))$ by the definition of dF, we have $\nabla_a(F^*(T_i)) = g_i(F(a))$ by hypothesis and we have $g_i(F(a)) = \Delta_{F(a)}(T_i)$ by construction. Since the differentials dT_i give a basis for $\Omega^1_{\mathbb{C}^N}$ at every point in $\mathbb{C}^N$ we deduce that $dF(\nabla_a) = \Delta_{F(a)}$ and we conclude that $F\colon C \to \mathbb{C}^N$ is an integral curve of $\Delta_{\mathbb{C}}$ provided that ∇ does not have any zero on C. This shows that Theorem 5.1.4 is a special case of Main Theorem 5.1.3 when $X = \mathbb{A}_K^N$. However using a result of Griffiths the proof of Main Theorem 5.1.3 can be reduced to a situation as given in Theorem 5.1.4. This will be explained in section 5.6.

At the end of this section, we shall give a variant of the Main Theorem. We consider a holomorphic mapping $f\colon C \to X_{\mathbb{C}}$ where X is a projective variety defined over K. Let ∇ be a vector field in $\Gamma(C, \mathcal{T}_C)$ without zero and assume that ∇ acts as a derivation on the field $f^*(K(X))$. Then ∇ can again be lifted to a rational vector field Δ on X, i.e. $\nabla \in \Gamma(C, \mathcal{T}_C) \cap \Gamma(C, f^{-1}(\mathcal{T}_X \otimes \mathcal{M}_X))$. We call a point $x \in X$ a regular point of Δ if and only if $\Delta(\mathcal{O}_{X,x}) \subset \mathcal{O}_{X,x}$. Otherwise we call x a pole of Δ. The set of regular points of Δ is an open subvariety $\mathcal{U}$ of X. As usual we let $\overline{f(C)}$ be the Zariski closure of $f(C)$.

Theorem 5.1.5. *If* $\dim \overline{f(C)} \geq 2$ *then*

$$|f^{-1}(\mathcal{U}(K))| \leq g + 2\,(2\,[K:\mathbb{Q}] + 1)\,l\,\max(\rho, 2g).$$

A significant hypothesis in our theorems is the existence of differential equations. There are two aspects which should be mentioned in this context. The first concerns the growth. It seems to be possible that in some cases the condition on the growth of f can be replaced by more accessible data related to the differential equations which determine the growth behavior of the solutions to some extend. We intend to come back to this question in the future.

As Schneider taught us differential equations are not necessary. They can also be removed from our work in the spirit of Schneider without making our work obsolete however since expected results without assuming differential equations are of different nature. We shall also come back to this point later.

Finally we should explain why we for the moment kept off from the higher dimensional question á la Bombieri. The main reason is that we intended first to investigate carefully all the possibilities which one has in a very new and unexplored area. For this we chose the most simplest but still generic new situation. The next step would then be to extend the work to mappings from an affine variety into a projective variety along the lines given by Bombieri. The main work here consists of extending the L^2-analysis on $\mathbb{C}^n$ developed by Hörmander in [70] and used by Bombieri in his work to affine varieties. There are no fundamental obstructions to be expected, in particular since much work has been done in this direction so far by quite a number of authors, especially by H. Skoda and J.-P. Demailly.

Since Schneider's original theorem has stayed relatively unattended we decided to state and discuss the theorem in a version more in the style of today. We do this at the end of the paper in an appendix and we shall also discuss possible extensions.

5.2 Standard Estimates

Let K be an algebraic number field of degree d over the rationals $\mathbb{Q}$ and with discriminant $\mathfrak{D}$. For a place v of K we denote by $|\ \ |_v$ the normalized absolute value such that $|p|_v = p^{-[K_v:\mathbb{Q}_p]}$ when $v \mid p$ where K_v is the completion of K at v. For an archimedean place $v \mid \infty$ corresponding to the embedding τ of K into $\mathbb{C}$ we define $|x|_v = |\tau(x)|^{[K_v:\mathbb{R}]}$ where $|\tau(x)|$ is the Euclidean absolute value and where K_v is defined as in the non-archimedean case. Let $\mathbb{P}^n$ be the projective space of dimension n. We define the logarithmic (Weil) height $h(x)$ of a point $x = (x_0 : x_1 : \ldots : x_n) \in \mathbb{P}^n(K)$ as

$$h(x) = \sum_v \log(\max |x_i|_v).$$

and we define the inhomogeneous height of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{C}^n(K)$ as

$$h^+(x) = \sum_v \log^+(\max |x_i|_v).$$

Both sums are taken over all places of K. For $\alpha \in K$ the point $x = (1 : \alpha)$ is in $\mathbb{P}^1(K)$ and its height is defined as $h(\alpha) = h^+(\alpha) = h(x)$. The definition of the height depends on the choice of a field K for which $x \in \mathbb{P}^n(K)$.

Lemma 5.2.1 (Liouville Estimate)**.** *Let ξ be a non-zero element of K and let w be an archimedean absolute value of K. Then we have*

$$\log |\xi|_w \geq -h(\xi). \tag{5.2}$$

Proof. By the product formula we have

$$-\log |\xi|_w = \sum_{v \neq w} \log |\xi|_v \leq \Sigma_v \log^+ |\xi|_v = h(\xi)$$

and the Liouville estimate follows instantly. □

The Weil height can be extended to polynomials in n variables $T_1, \ldots, T_n$ with coefficients in K. Let $P = \sum_i p_i T^i$ be such a polynomial with $i\colon \{1, \ldots, n\} \to \mathbb{N}^n$ a multi-index and $T^i = T_1^{i(1)} \cdot \ldots \cdot T_n^{i(n)}$. Then we define $\|P\|_v = \max_i |p_i|_v$ and let the height of P be

$$h(P) = \sum_v \log \|P\|_v.$$

We shall also use

$$h^+(P) = \sum_v \log^+ \|P\|_v.$$

Now we consider a system of linear equations

$$L_i(T_1, \ldots, T_N) = \sum_{j=1}^{N} l_{i,j} T_j = 0 \tag{5.3}$$

with coefficients in K and $1 \leq i \leq M$.

Lemma 5.2.2 (Siegel's Lemma)**.** *If $N > M$ then (5.3) has a non-trivial solution $x = (x_1, \ldots, x_N) \in \mathcal{O}_K^N$ such that*

$$h^+(x) \leq \frac{1}{2} \log |\mathfrak{D}| + \frac{1}{N-M} \sum_{i=1}^{M} \big(\frac{1}{2}[K : \mathbb{Q}] \log N + h(L_i)\big).$$

Proof. This is a consequence of Corollary 11 of [20] together with an argument as is used in the proof of Lemma 1 of [7]. □

Lemma 5.2.3. *Let Δ be a derivation of $K[T_1, \ldots, T_N]$, $x \in K^N$ and $P_1, P_2, \ldots, P_l$ be polynomials of degree $\leq r$ in $K[T_1, \ldots, T_N]$. Then there exists a positive constant C depending only on Δ, N and x such that*

$$h^+((\Delta^k P_1(x), \ldots, \Delta^k P_l(x))) \leq \sum_v \max_i \log^+ \|P_i\|_v + [K:\mathbb{Q}]k \log(r+k) + C(k+r)$$

for all integers $k \geq 0$.

Proof. At the real places v we refer to lemma 1 in [19] and to [80, p.23] for the inequality

$$\log^+ |\Delta^k P_i(x)|_v \leq \log^+ \|P_i\|_v + k \log(r+k) + C(k+r).$$

Similarly, for complex places v

$$\log^+ |\Delta^k P_i(x)|_v \leq \log^+ \|P_i\|_v + 2k \log(r+k) + C(k+r)$$

and for finite place v

$$\log^+ |\Delta^k P_i(x)|_v \leq \log^+ \|P_i\|_v + C(k+r).$$

Combining the three inequalities gives the estimate stated in the Lemma. □

5.3 Rational Functions on Curves

Let Z be as in the introduction a smooth projective curve of genus g, $\mathcal{P}$ a non-empty finite set of points in Z, $\mathcal{S}$ a finite set of points in $C = Z \setminus \mathcal{P}$ and $\nabla \in \Gamma(C, \mathcal{T}_C)$. We shall construct in this section rational functions with prescribed zero and polar divisors using the classical theory of linear systems and the Riemann-Roch Formula. For a divisor D we write $\mathcal{O}(D)$ for the invertible sheaf associated with D which is $\mathcal{L}(D)$ in Hartshorne's notation. The first lemma is classical but for convenience we give the short proof.

Lemma 5.3.1. *Let Z be a compact Riemann surface of genus g and let D be a divisor on Z with $\deg D \geq 2g$. Then the complete linear system $|D|$ of $\mathcal{O}(D)$ has no base point.*

Proof. For $p \in Z$ the exact sequence

$$0 \longrightarrow \mathcal{O}(D-(p)) \longrightarrow \mathcal{O}(D) \xrightarrow{r_p} L_p \longrightarrow 0$$

induces an exact sequence $H^0(Z,\mathcal{O}(D)) \longrightarrow H^0(Z,L_p) \longrightarrow H^1(Z,\mathcal{O}(D-(p)))$ in cohomology. Since the canonical bundle K_Z on Z has degree $2g-2$ and since $\deg \mathcal{O}(D-(p)) = \deg D - 1 \geq 2g-1$ we have $\deg(K_Z \otimes \mathcal{O}(p) - D) < 0$ and Serre duality gives $H^1(Z,\mathcal{O}(D-(p)) = H^0(Z,K_Z \otimes \mathcal{O}((p)-D)) = 0$. We deduce that the connecting homomorphism $H^0(Z,\mathcal{O}(D)) \longrightarrow H^0(Z,L_p)$ is surjective and this means that p is not a base point and the conclusion of the lemma follows. □

Lemma 5.3.2. *We assume that $|\mathcal{S}| \geq m_1|\mathcal{P}| + 2g$ for some $m_1 \geq 1$. Then there exists a holomorphic map $f\colon Z \to \mathbb{P}^1$ such that $(f)_0 = \sum_{w\in\mathcal{S}}(w)$ and $(f)_\infty \geq \sum_{p\in\mathcal{P}} m_1\,(p)$.*

Proof. We apply the Riemann-Roch formula to the divisor

$$D = \sum_{w\in\mathcal{S}}(w) - \sum_{p\in\mathcal{P}} m_1(p)$$

which has $\deg D = m - m_1 l \geq 2g$ and get $l(D) = i(D) + \deg D + 1 - g \geq g+1$. The linear system $|D|$ is a projective space of dimension g. By Lemma 5.3.1 we know that $|D|$ has no base points. Therefore for $w \in \mathcal{S}$ the space $D_w = \{D' \in |D|; w \in D'\}$ is a hyperplane in $|D|$ and $|D| - \bigcup_{w\in\mathcal{S}} D_w$ is nonempty. Each $D' \in |D| \setminus \bigcup_{w\in\mathcal{S}} D_w$ is effective and has the property that $D \sim D'$. Therefore $\sum_{w\in\mathcal{S}}(w) \sim m_1 P + D'$. By our selection of D' we have $w \in D'$ for $w \in \mathcal{S}$ and this implies that no (w) with $w \in \mathcal{S}$ can be canceled by D'. The difference is linearly equivalent to zero and this means that there is a holomorphic map $f\colon Z \to \mathbb{P}^1$ such that $(f)_0 = \sum_{w\in\mathcal{S}}(w)$ and $(f)_\infty = \sum_{p\in\mathcal{P}} m_1(p) + D' \geq \sum_{p\in\mathcal{P}} m_1(p)$ as stated. □

For each $p \in \mathcal{P}$ we choose a local coordinate z_p in a neighborhood of p. We assume that $|\mathcal{S}| \geq |\mathcal{P}| + 2g$ and then there exist integers $m_1 \geq 1$ and $2g \leq t \leq l + 2g - 1$ such that $|\mathcal{S}| = m_1|\mathcal{P}| + t$.

Lemma 5.3.3. *There exist constants $C_1, C_2 > 0$ such that for all integers $N \geq 1$ there is a holomorphic mapping $\phi_N\colon Z \to \mathbb{P}^1$ such that*

(i) $(\phi_N)_0 = \sum_{w\in\mathcal{S}} N\,(w)$,

(ii) $|\phi_N(z_p)| \geq |C_1/z_p|^{m_1 N}$ *for all $p \in P$ and all z_p sufficiently small,*

(iii) $|\nabla^{(N)}\phi_N(w)| \leq N!\, C_2^N$ *for all* $w \in \mathcal{S}$.

Proof. By Lemma 3.2 there exists a function g such that,

$$\begin{aligned}(g)_0 &= \sum_{w\in\mathcal{S}}(w),\\ (g)_\infty &\geq \sum_{p\in\mathcal{P}} m_1(p).\end{aligned}$$

Now we put $\phi_N = g^N$ and (i) follows. Since g has a pole of order at least m_1 at each p in $\mathcal{P}$ there exists a constant $C_1 > 0$ such that $|g(z_p)| \geq |C_1/z_p|^{m_1}$ for z_p sufficiently small and $p \in \mathcal{P}$. Therefore the function ϕ_N can be estimated from below by $|\phi_N(z_p)| \geq |C_1/z_p|^{m_1 N}$ for z_p sufficiently small and $p \in \mathcal{P}$ which gives (ii). Since ∇ is non-zero at w by hypothesis we find a local parameter t at $w \in \mathcal{S}$ satisfying $t(w) = 0$ such that in a neighborhood of w the derivation ∇ takes the form $\partial/\partial t$. Then $g(t)$ can be written as $t\,\epsilon(t)$ near w for some unit $\epsilon(t)$ and $\phi_N(t)$ as $t^N\epsilon(t)^N$. Therefor $\nabla^N\phi_N(w) = N!\,\epsilon(0)^N$ and if we define C_2 as the maximum of $|\epsilon(0)|$ taken over all $w \in \mathcal{S}$ we get (iii). □

Remark 5.3.4. *The existence of* ϕ_N *in Lemma 5.3.3 is essential for the first proof of Theorem 5.1.4. In classical transcendence proofs there already exist functions which are analogous to our* ϕ_N*. We mention the polynomial* $\prod_{w\in\mathcal{S}}(t-w)^N$ *in [80] or the Blaschke products* $\prod_{w\in\mathcal{S}}(\frac{r^2-tw}{r(w-t)})^N$ *in [8] which play the role of our* ϕ_N *there. All such* ϕ_N *have the property that they take* 0 *up to order* N *at fixed finitely many points and take large values near the boundary.*

In the second proof of Theorem 5.1.4 we need a special exhaustion function for the affine curve $C = Z \setminus \mathcal{P}$. This is provided in the next lemma.

Lemma 5.3.5. *For all* $a \in C$ *and for all integers* $q \geq 0$ *there exists a rational map* $\pi\colon Z \to \mathbb{P}^1$ *such that*

(i) $(\pi)_0 = t\,(a), \quad ql \leq t \leq ql + g,$

(ii) $(\pi)_\infty \geq \sum_{p\in\mathcal{P}} q\,(p).$

The projection π *only depends on* a *and* q*.*

Proof. The divisor $D = (ql+g)(a) - \sum_{p\in\mathcal{P}} q\,(p)$ has degree g and the Riemann-Roch formula gives $l(D) = i(D) + \deg D + 1 - g \geq 1$ which implies the desired conclusion. □

5.4 Jensen's Formula

In this section we shall discuss Jensen's formula which was a starting point for Nevanlinna theory. Although this formula can be stated in very simple and elementary terms it is conceptually better and advantageously to express the formula in terms of the standard functions in Nevanlinna theory. We begin with recalling the definition of the Nevanlinna characteristic function and of the order of a meromorphic function f on $\mathbb{C}$ and we shall state the First Main Theorem (FMT) in classical Nevanlinna theory which shows how the various functions are related.

First Main Theorem ([97, p.166]) *For any meromorphic function f we have*

$$N(r,a) + m(r,a) = T(r,f) + O(1).$$

It is not difficult to see that the First Main Theorem is equivalent to Jensen's formula which we shall present only for holomorphic functions since we need it only in this case. Such an entire function can be expressed as $f(z) = z^\lambda \epsilon(z)$ for some unit $\epsilon(z)$.

Jensen's Formula ([97, p.164]) *We have*

$$\log|\epsilon(0)| + N(r,0) = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|d\theta.$$

Jensen's formula is closely related to the Schwarz Lemma as it is used in transcendence theory. There usually a holomorphic function is constructed with growth and vanishing conditions. The growth conditions are used to give an upper bound for the integral in the formula. The vanishing conditions lead to a lower bound for $N(r,0)$. Arithmetical data enter through the term $\log|c_\lambda|$ on the left and one derives a lower bound of $\log|c_\lambda|$ using Liouville estimates. In this way a proof of Schneider's theorem can be obtained, although there is not too much difference with the standard proof. We shall give one proof of Theorem 5.1.4 along this lines. This needs a Jensen formula in the more general situation of an affine algebraic curve which we shall derive now.

Let f be a meromorphic function on a smooth affine algebraic curve C as in section 5.3. We recall briefly from [63] some aspects of Nevanlinna theory of affine varieties in the case of curves, especially the use of a special exhaustion function. Since Z is a smooth curve we can choose a projection $\pi\colon Z \to \mathbb{P}^1$ such that $\pi^{-1}(\infty) = \sum_{p\in\mathcal{P}} m_p\,(p)$ with $m_p \geq 1$. The projection π gives a special exhaustion function $\tau(z) = \log|\pi(z)|$ on C in the sense of Griffiths and King and for real $r \geq 0$ we put $C[r] = \{z \in C : e^{\tau(z)} = |\pi(z)| \leq r\}$. We consider now the curve C as a Riemann surface and for real $r \geq 0$ we define $\operatorname{Div} C[r]$ to be the free abelian group generated by $C[r]$. Its elements can be expressed as finite suns $\sum_{z\in C[r]} n_z(z)$ with integer coefficients n_z and with a symbol (z) for each $z \in C[r]$. For $r \leq s$ there is a natural surjective group homomorphism $p_{s,r}\colon \operatorname{Div} C[s] \to \operatorname{Div} C[r]$ which maps $\sum_{z\in C[s]} n_z(z)$ to $\sum_{z\in C[r]} n_z(z)$. The family $\{\operatorname{Div} C[r], p_{s,r}\}$ is an inverse system and $\operatorname{Div} C = \varprojlim \operatorname{Div} C[r]$ is defined to be the group of analytic divisor on C. Its elements can be written as

$$D = \sum_{z\in C} D(z)(z)$$

with $D(z) \in \mathbb{Z}$ and zero up to a discrete and countable set of points, the support $\operatorname{supp}(D)$ of D. Let $p_r\colon \operatorname{Div} C \to \operatorname{Div} C[r]$ be the natural projection. Then $D[r] = p_r(D)$ has finite support in $C[r]$ and therefore $n^\pi(D, r) = \int_{D[r]} 1$ is an integer. To characterize the growth of D we define

$$N^\pi(D, r) = \int_0^r \frac{n^\pi(D, t) - n^\pi(D, 0)}{t} dt + n^\pi(D, 0)\log r$$

Let $f\colon C \to \mathbb{P}^1$ be a meromorphic function and $\operatorname{ord}_z f$ its order at $z \in C$. Then

$$(f) = \sum_{z\in C} (\operatorname{ord}_z f)(z)$$

is the divisor of f. To characterize the growth of the function f we define the *Ahlfors-Shimizu* characteristic function by

$$T^\pi_{AS}(r, f) = \int_0^r \int_{C[t]} f^*\left(\frac{i}{2\pi}\frac{dz \wedge d\bar{z}}{(1 + |z|^2)^2}\right)\frac{dt}{t} \tag{5.4}$$

as given by (5.1) in [63] in the case $m = q = 1$. From the definition we easily see that $T^\pi_{AS}(r, f) \geq C\log r$ for some positive C and sufficiently large r if f is non-constant. When we compare the growth of a divisor and the growth of a function, the FMT in Nevanlinna theory (see [33, p.332] and [63, p.184, pp.189-190] for more details) shows us that the error term can be estimated by the proximity function

$$m^{\pi}_{AS}(r,f) = \int_{\partial C[r]} \log\left(1+|f(z)|^2\right) d^c\tau.$$

First Main Theorem*. *Let D_∞ be the polar devisor of f. Then*

$$T^{\pi}_{AS}(r,f) \quad = \quad N^{\pi}(D_\infty, r) + m^{\pi}_{AS}(r,f) + O(1) \tag{5.5}$$

It can be shown that $m^{\pi}_{AS}(\infty, r) = \int_{\partial C[r]} \log^+ |f(z)|^2 d^c\tau + O(1)$ and therefor, if we define the *Nevanlinna's characteristic function* $T^{\pi}(r,f)$ as $N^{\pi}(D_\infty, r) + m^{\pi}(r,f)$, where

$$m^{\pi}(r,f) = \int_{\partial C[r]} \log^+ |f(z)|^2 d^c\tau,$$

we see that the First Main Theorem* implies that the Nevanlinna characteristic function and the Ahlfors-Shimizu characteristic function coincide up to a bounded term.

As we have already mentioned the FMT is equivalent to a formula of Jensen's type. We shall state now a version of a general Jensen Formula as given in [63], Proposition 3.2, that is adapted to our situation.

Jensen's Formula*. *Let f be a meromorphic function on C with divisor D. Then for all real numbers r and r_0 with $r \geq r_0$ we have*

$$N^{\pi}(D,r) - N^{\pi}(D,r_0) + \int_{\partial C[r_0]} \log |f|^2 d^c\tau = \int_{\partial C[r]} \log |f|^2 d^c\tau. \tag{5.6}$$

An explicit form of FMT for functions on affine curves goes back to Chern in [33, p.332]. Later Griffiths and King [63] were able to extend it to affine varieties using special exhaustion functions. Since Chern's result does not depend on any special exhaustion function his result is more general than the result of Griffiths and King in the one variable case. However we still prefer their setting because special exhaustion functions make the formula more applicable. When $n(D,0) = 0$ we can take $r_0 = 0$ and then the above formula becomes

$$N^{\pi}(D,r) + \sum_{\pi(z)=0} \log |f(z)| = \int_{\partial C[r]} \log |f|^2 d^c\tau \tag{5.7}$$

In the second proof of Theorem 5.1.4 we construct a holomorphic function f which satisfies again growth and vanishing conditions. Then from the growth conditions we derive an upper bound for the integral. The vanishing conditions lead to a lower bound for $N(D, r)$ and the sum can be bounded from below by a Liouville estimate. A comparison of the bounds then leads to the stated result. In the next section we discuss the notion of the order of a function which is used to characterize growth conditions.

5.5 Functions of finite order on curves

Since there are at least two possible definitions of the order of functions on curves (see [61] or [63]) we have to discuss the notion carefully. For a clear and detailed exposition on the different indicators of orders for meromorphic functions $f\colon \mathbb{C} \to \mathbb{P}^1$ we refer to [62].

We discuss first the case when f is an entire function. Here the order of f is defined using the maximum modulus function or the Nevanlinna characteristic function. It is given by

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log\log M(r, f)}{\log r}$$

and

$$\rho(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r, f)}{\log r}$$

respectively. The two definitions are equivalent and for a proof we refer to [97, p.216].

When f is a meromorphic function, $M(r, f)$ does not make sense. There are two variants to overcome the difficulty. The first makes use of the well-known fact that a meromorphic function is of order $\leq \rho$ if and only if it can be expressed as $f = h/g$ where h and g are both entire functions and of order $\leq \rho$ (see [97, p.223]). Then $\rho(f)$ is the infimum of $\max(\rho(h), \rho(g))$ with h and g taken over all representations of $f = h/g$ as a quotient of two holomorphic functions. The second variant uses the Nevanlinna characteristic function $T(r, f)$ which is also well- defined for meromorphic functions.

In the case of a meromorphic function on a curve $C = Z \setminus \mathcal{P}$ we again begin with a holomorphic functions f and define the local maximum modulus function and the local order of f at $p \in \mathcal{P}$ by

$$\begin{aligned} M_p(r) &= \max_{|z_p|=1/r} |f(z_p)| \\ \rho_p(f) &= \overline{\lim_{r\to\infty}} \frac{\log\log M_p(r)}{\log r} \end{aligned}$$

Then the order of f is given by $\rho(f) = \max_{p\in\mathcal{P}} \rho_p(f)$ and it is easily seen that $\rho_p(f)$ and therefore $\rho(f)$ are independent of the choice of local coordinates. Moreover f is of order ρ if and only if ρ is maximal with the property that for any $\epsilon > 0$ we always have $\log |f(z_p)| \leq \frac{1}{|z_p|^{\rho+\epsilon}}$ for all z_p sufficiently small. When f is a meromorphic function we define $\rho_p(f)$ as before to be the infimum of $\max(\rho(h), \rho(g))$ taken over all representations of $f = h/g$ in a neighborhood of p and $\rho(f) = \max \rho_p(f)$. This definition was suggested by Griffiths in [61].

Another approach to the growth of functions is to use Nevanlinna's or Ahlfors-Shimizu's characteristic function $T^\pi(r, f)$ and $T^\pi_{AS}(r, f)$ respectively and using a special exhaustion π which was suggested by Griffiths and King in [63]. As already noted both functions coincide up to $O(1)$. Therefor we can use either of them to define the order of growth of a function f. We use Nevanlinna's characteristic function and put

$$\rho^\pi_T(f) = \overline{\lim}_{r\to\infty} \frac{\log T^\pi(r, f)}{\log r}. \tag{5.8}$$

When f is holomorphic we can also use the maximum modulus function

$$M^\pi(r, f) = \max_{z\in\partial C[r]} |f(z)| \tag{5.9}$$

and get

$$\rho^\pi(f) = \overline{\lim}_{r\to\infty} \frac{\log\log M^\pi(r, f)}{\log r}. \tag{5.10}$$

Although it is not the main purpose of this paper we shall prove that $\rho^\pi_T(f)$ and $\rho^\pi(f)$ are equal. This gives a generalization of the classical identity for holomorphic functions on $\mathbb{C}$ described at the beginning of this section. It also provides an example how the exhaustion function of Griffith and King can be applied.

We begin with a simple lemma. Let $r\colon \mathbb{D} \to \mathbb{D}$ be the ramified covering of the open unit disk $\mathbb{D}$ of degree n given by $z \mapsto z^n$ and assume that $u\colon \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ is a subharmonic function on $\mathbb{D}$.

Lemma 5.5.1. *The function* $v\colon \mathbb{D} \to \mathbb{R} \cup \{-\infty\}$ *given by*

$$v(z) = \max_{r(w)=z} u(w)$$

is subharmonic.

Proof. The function v is upper semi-continuous and takes values in $[-\infty, +\infty)$. According to the proof of Theorem 1.6.3 in [71] it suffices to show that for all $w \in \mathbb{D}$ there exists a disk of radius $r > 0$ centered in w and contained in $\mathbb{D}$ such that

$$v(w) \leq \int_0^{2\pi} v(w + se^{i\theta})d\theta. \tag{5.11}$$

When $w \neq 0$ we take a disk $\mathbb{D}_w(s)$ with center in w of radius s which does not contain 0. Then $r^{-1}(\mathbb{D}_w(s))$ is the disjoint union of disks of the form $\zeta\,\mathbb{D}_{w^{1/n}}(s^{1/n})$ taken over all n-th root of unity ζ and for some fixed choice of a n-th root $w^{1/n}$ of w. Then $v(z) = \max_\zeta(u(\zeta z^{1/n}))$ and $u_\zeta(z) := u(\zeta z^{1/n})$ is a subharmonic function on $\mathbb{D}_w(s)$ for all ζ. This shows that $v(z)$ can be expressed as the maximum taken over a finite set of subharmonic functions and is therefor subharmonic.

When $w = 0$, for all $0 < s < 1$, we have

$$\begin{aligned}
v(0) &= u(0) \leq \int_0^{2\pi} u(s^{\frac{1}{n}}e^{i\theta})d\theta \\
&= \sum_{j=1}^{n} \int_{2(j-1)/n}^{2j\pi/n} u(s^{\frac{1}{n}}e^{i\theta})d\theta \\
&= \frac{1}{n}\sum_{j=1}^{n} \int_{2(j-1)\pi}^{2j\pi} u(s^{\frac{1}{n}}e^{\frac{i\alpha}{n}})d\alpha \\
&\leq \frac{1}{n}\sum_{j=1}^{n} \int_{2(j-1)\pi}^{2j\pi} v(se^{i\alpha})d\alpha \\
&= \int_0^{2\pi} v(se^{i\theta})d\theta
\end{aligned}$$

The two cases show that (5.11) holds for all $w \in \mathbb{D}$ in a sufficiently small neighborhood and therefor is subharmonic in $\mathbb{D}$. □

The lemma is applied to yield a global version for finite (i.e. proper and non-constant) mapping between Riemann surfaces.

Lemma 5.5.2. *Let $f\colon X \to Y$ be a finite mapping between Riemann surfaces and u a subharmonic function on X. Then the function $v(z) = \max_{f(w)=z} u(w)$ is a subharmonic function on Y.*

Proof. It is known (see [71], Corollary 1.6.5.) that subharmonicity is a local property. The map f is a finite mapping and if the disk $i\colon \mathbb{D} \hookrightarrow Y$ is small enough the inverse image $f^{-1}(\mathbb{D}) = \mathbb{D} \times_Y X$ of $\mathbb{D}$ under f has only finitely many connected components and all ramification points are in the fiber over the center of the disk. The restriction of f to any of the components then takes the form as described in Lemma 5.5.1 which can be applied now to all the connected components. We conclude that the restriction of v to $\mathbb{D}$ is subharmonic on $\mathbb{D}$ and this proves the Lemma. □

We apply the Lemma in the case when X is an affine curve and when Y is the complex plane $\mathbb{C}$.

Proposition 5.5.3. *We have*

$$\rho^\pi(f) = \rho_T^\pi(f).$$

Proof. Let d be the degree of π. Since f is holomorphic, $N^\pi(D_\infty, r)$ is zero and therefor, by the definition of the Nevanlinna characteristic function, we have

$$T^\pi(r,f) = m^\pi(r,f) = \int_{\partial A[r]} \log^+ |f(z)|^2 d^c\tau \leq d \log^+ M^\pi(r)$$

We know that $\log^+ |f(w)|$ is subharmonic on C for f holomorphic and Lemma 5.5.2 implies that $h(z) = \max_{\pi(w)=z}(\log^+ |f(w)|)$ is a subharmonic function on $\mathbb{C}$. For $z = re^{i\varphi}$ and $r < \varrho$ the Harnack inequality gives

$$h(z) \leq \frac{\varrho + r}{\varrho - r} \int_0^{2\pi} h(\varrho e^{i\theta}) d\theta \leq \frac{\varrho + r}{\varrho - r} \int_{\partial C[\varrho]} \log^+ (|f(z)|^2) d^c\tau = \frac{\varrho + r}{\varrho - r} T^\pi(\varrho, f)$$

and this implies that $\log^+ M^\pi(r,f) \leq \frac{\varrho+r}{\varrho-r} T^\pi(\varrho, f)$. On putting the inequalities together we deduce that

$$\frac{1}{d}T^\pi(r,f) \le \log^+ M^\pi(r,f) \le \frac{\varrho+r}{\varrho-r}T^\pi(\varrho,f).$$

Since f is nontrivial, there exists $C>0$ such that $T^\pi(r,f) \ge C\log r$ (see the remark following (5.4) and notice that $T^\pi(r,f)$ and $T^\pi_{AS}(r,f)$ differ only by a bounded term). Therefore the left hand side of the inequalities shows that $\log^+ M^\pi(r,f) = \log M^\pi(r,f)$ for sufficiently large r. We put $\varrho = r+\epsilon$ for $\epsilon>0$ and then an easy calculation gives

$$\rho^\pi(f) = \overline{\lim}_{r\to\infty}\frac{\log\log^+ M^\pi(r)}{\log r} = \overline{\lim}_{r\to\infty}\frac{\log T^\pi(r,f)}{\log r} = \rho^\pi_T(f).$$

□

The order ρ^π_T depends on the choice of π whereas the order $\rho = \max_p \rho_p$ with $p \in \mathcal{P}$ is independent of any choice. However they can be compared.

Lemma 5.5.4. *We have* $\rho^\pi_T = \max_p(\rho_p/m_p)$.

Proof. Since we shall not use Lemma 5.5.4 later we only verify it for holomorphic f. The function $(\frac{1}{\pi(z)})^{\frac{1}{m_p}}$ gives a local coordinate on a neighborhood U_p of p and then for sufficiently large r we have

$$M_p(r) = \max(|f(z)|; z \in U_p, |\pi(z)| = r^{m_p})$$

Since $M^\pi(r) = \max_{|\pi(z)|=r}(|f(z)|)$ we conclude that $M^\pi(r) = \max_p M_p(r^{\frac{1}{m_p}})$. This implies that

$$\begin{aligned}\rho^\pi_T(f) &= \rho^\pi(f) = \overline{\lim}_{r\to\infty}\frac{\log\log M^\pi(r)}{\log r}\\ &= \max_p\big(\overline{\lim}_{r\to\infty}\frac{\log\log M_p(r^{\frac{1}{m_p}})}{\log r}\big)\end{aligned}$$

and the latter is $\max_p(\rho_p/m_p)$. □

We point out that $\rho(f)$ depends only on some smooth completion of C and therefore is an intrinsic notion in the case of affine curves (since a smooth completion of a smooth affine curve is unique). However the order function ρ^π depends on a special exhaustion function induced by a projection $\pi\colon C \to \mathbb{C}$ and is therefor an extrinsic notion. By Lemma 5.5.4 the order function $\rho(f)$ is finite if and only if $\rho^\pi(f)$ is finite and Griffiths uses both in [61] and [63]. However for our purpose an estimate in terms of $\rho(f)$ is essential.

Let C be an affine algebraic curve and let V be a projective algebraic variety. Let as usual $\mathfrak{R}(V)$ be the field of rational functions on V. A holomorphic map $f: C \to V$ has order $\leq \rho$ if and only if $f^*\mathfrak{R}(V)$ consists of meromorphic functions of order $\leq \rho$. It is easy to see that the latter is equivalent to $\rho(f^*(z_i/z_j)) \leq \rho$.

In the next proposition we shall compare the growth of a positive divisor $D = \sum_{z \in C} D(z)\,(z)$ with the growth of a function. We put

$$n(r, D) = \sum_{z} D(z)$$

where the sum is taken over all $z \in C$ not in the union of the sets $|z_p| \leq r^{-1}$ with $p \in \mathcal{P}$ and

$$\rho(D) = \lim_{r \to \infty} \frac{\int_{r_0}^{r} n(t, D)(dt/t)}{\log r}.$$

The following proposition is taken from [61], Proposition 5.23,

Proposition 5.5.5. *Let D be a positive divisor on C. Then there exists a function $f \in \mathcal{O}(D)$ such that $(f) = D$ and*

$$\rho(f) \leq \max(\rho(D), 2g).$$

and is essential for the proof of the following

Lemma 5.5.6. *Let $f_1, \cdots, f_n$ be meromorphic functions of order $\leq \rho$ on C and let $\mathcal{S} \subset C$ be a finite set. If $f_i(w) \neq \infty$ for $1 \leq i \leq n$ and $w \in \mathcal{S}$ then there exists a holomorphic function h on C such that*

(i) $\rho(h) \leq \max(\rho, 2g)$,

(ii) hf_i *are all holomorphic and* $\rho(hf_i) \leq \max(\rho, 2g)$ *for* $1 \leq i \leq n$,

(iii) $h(w_i) \neq 0$ *for* $1 \leq i \leq m$.

Proof. By (5.5) we have $\rho((f_i)_\infty) \leq \rho$ and this gives

$$\rho((f_1)_\infty + \cdots + (f_n)_\infty) \leq \max_i(\rho(f_i)_\infty) \leq \rho.$$

By Proposition 5.5.5 there is a holomorphic function h of order at most $\max(\rho, 2g)$ on C such that $(h) = (f_1)_\infty + \cdots + (f_n)_\infty$ and such that h satisfies the conditions in the statement of the lemma. $\square$

5.6 Algebraic points and algebraic distributions.

In this section we give the proofs of the theorems. We begin with Theorem 5.1.4 for which we shall give two different proofs. As usual the first step in the proofs is the construction of an auxiliary function F. Here we use Lemma 5.2.2. In the first proof we apply the maximum principle to the function F/ϕ_N with ϕ_N constructed in Lemma 5.3.3. The second proof is based on the Griffiths-King exhaustion function which was constructed in Lemma 5.3.5 and on Jensen's formula associated with the exhaustion function.

Proof of Theorem 5.1.4. Let $f, g \in \{f_1, f_2, \cdots, f_N\}$ be algebraically independent over K. We define $m = |\mathcal{S}|$ and choose integers r, n with n sufficiently large such that $r^2 = 2mn$. For the construction of the auxiliary function we consider the polynomial $P = \sum_{i,j=1}^{r} a_{i,j} S^i T^j$ with undetermined coefficients $a_{i,j}$. They will be chosen in $\mathcal{O}_K$ such that the system of mn linear equations

$$\sum_{i,j=1}^{r} a_{i,j} \nabla^k (f^i g^j)(w) = 0 \tag{5.12}$$

for $0 \leq k < n$ and $w \in \mathcal{S}$ in $r^2 = 2mn$ unknowns $a_{i,j}$ is satisfied. By Lemma 5.2.3 we see that the heights and the inhomogeneous heights of the linear forms

$$L_{k,w} = \sum_{i,j=1}^{r} a_{i,j} \nabla^k (f^i g^j)(w)$$

in the unknowns $a_{i,j}$ can be estimated from above by

$$h(L_{k,w}) \leq h^+(L_{k,w}) \leq [K : \mathbb{Q}] k \log(r + k) + C(k + r).$$

On applying Lemma 5.2.2 and on observing that $r = O(\sqrt{n})$ we find a non-trivial solution $a = (\ldots, a_{i,j}, \ldots) \in \mathcal{O}_K^{r^2}$ such that

$$h^+(P) = h^+(a) \leq ([K : \mathbb{Q}] n/2) \log n + C_3 n,$$

where C_3 only depends on w. Since f, g are algebraically independent over K, the function $F = P(f, g)$ is not identically zero.

From (5.12) we deduce that the holomorphic function F on C satisfies

$(F)_0 \geq \sum_{w \in \mathcal{S}} n\,(w)$. We let $s \geq n$ be the largest integer such that

$$(F)_0 \geq \sum_{w \in \mathcal{S}} s\,(w) \tag{5.13}$$

By definition $\nabla^s F$ does not vanish at some $w \in \mathcal{S}$. By Lemma 5.2.3 we have

$$h^+(\nabla^s F(w)) \leq h^+(P)+([K:\mathbb{Q}]s)\log{(s+2r)}+C(s+2r) \leq (3/2)[K:\mathbb{Q}]s\log s+C_4 s$$

where C_4 only depends on w.

Variant 1. In this variant we obtain a bound which is slightly weaker than the bound stated in the theorem. We write $m = lm_1 + t$ with $m_1 \geq 0$ and $2g \leq t \leq l + 2g - 1$ and we may assume that $m_1 \geq 1$ since otherwise $|\mathcal{S}| \leq 2g + l - 1$. Lemma 5.3.3 gives a function ϕ_s such that

$$E = \frac{F}{\phi_s}$$

is holomorphic. We shall derive an upper bound by the maximum principle and by the Liouville estimate a lower bound for $\log|E(w)|$ and compare the upper and the lower bound. This will eventually give an estimate from above for $|\mathcal{S}|$ which is slightly weaker than (5.1).

We begin with the upper bound. Since f and g are of order $\leq \rho$ we know that for all $\epsilon > 0$, for a sufficiently small positive η and for all $p \in \mathcal{P}$ the inequality

$$\log\max(|f(z_p)|, |g(z_p)|) \leq |z_p|^{-(\rho+\epsilon)}$$

for $|z_p| \leq \eta$ gives

$$\log|F(z_p)| \leq ([K:\mathbb{Q}]n/2)\log n + C_5 n + 2r|z_p|^{-(\rho+\epsilon)}. \tag{5.14}$$

Together with (ii) in Lemma 5.3.3 we conclude that

$$\log|E(z_p)| \leq ([K:\mathbb{Q}]n/2)\log n + C_5 n + 2r|z_p|^{-(\rho+\epsilon)} - m_1 s\log(C_1/|z_p|).$$

From the maximum principle applied to the complement of the union of the discs with radius η around p for $p \in \mathcal{P}$ we get the upper bound

$$\log|E(w)| \leq ([K:\mathbb{Q}]n/2)\log n + C_5 n + 2r\eta^{-(\rho+\epsilon)} - m_1 s\log(C_1/\eta).$$

For the lower bound we observe that $E(w) = \nabla^s F(w)/\nabla^s\phi_s(w)$, that by (iii) in Lemma 5.3.3 we have

$$\log|\nabla^s \phi_s(w)| \leq s\log s + s\log C_2$$

and that

$$\log|\nabla^s F(w)| \geq -h(\nabla^s F(w)) \geq -(3/2)[K:\mathbb{Q}]s\log s - C_4 s$$

by the Liouville estimate. We put the estimates together and obtain the lower bound

$$\log|E(w)| = \log|\nabla^s F(w)| - \log|\nabla^s \phi_s(w)| \geq -((3/2)[K:\mathbb{Q}]+1)s\log s - C_6 s.$$

Since $n \leq s$ a comparison of the upper and the lower bound gives

$$-(3[K:\mathbb{Q}]+2)s\log s \leq [K:\mathbb{Q}]s\log s + 4r\eta^{-(\rho+\epsilon)} - 2m_1\, s\log(C_1/\eta) + C_7 s,$$

where C_7 only depends on w. We relate now η and s by the equation $s\eta^{2\rho+2\epsilon} = 1$ and find that

$$0 \leq 2\,(2[K:\mathbb{Q}]+1)s\log s - \big(m_1/(\rho+\epsilon)\big)s\log s + C_8\, s$$

This can hold for large s only if $2\,(2[K:\mathbb{Q}]+1)(\rho+\epsilon) \geq m_1$ for all $\epsilon > 0$ so that

$$|\mathbb{S}| = m \leq l(m_1+1) + 2g - 1 \leq 2\,(2[K:\mathbb{Q}]+1)\rho\, l + l + 2g - 1.$$

Variant 2. In this variant we use Jensen's formula which needs an exhaustion function. We fix a positive integer q and then Lemma 5.3.5 gives a projection $\pi : Z \to P^1$ with $ql \leq \deg\pi \leq ql + g$. Furthermore the Lemma shows that there exists a divisor $D \geq 0$ such that $(\pi)_\infty = \sum_{p\in\mathcal{P}} q(p) + D$. We put $C' := C \setminus \operatorname{supp} D$ and see that π restricts to a finite covering $\pi' : C' \to \mathbb{C}$ which is totally ramified above 0 and satisfies $\pi^{-1}(0) = t(w)$.

Similar to the definition of E we define the function $G = F^t/\pi^s$ on C' with $t = \deg\pi$ and s the order of F at w. We choose a local parameter z at w such that $z(w) = 0$ and such that ∇ can be written as $\partial/\partial z$. Then we have $F(z)^t/z^{st} = (\nabla^s F(w)/s!)^t + z\epsilon(z)$ and $\pi(z)^s/z^{st} = \epsilon'(z)^s$ near w where ϵ and ϵ' are units. This gives

$$G(w) = \frac{F(w)^t}{\pi(w)^s} = \frac{(\nabla^s F(w)/s!)^t}{\epsilon'(0)^s}$$

and the lower bound

$$\log|G(w)| \geq -((3/2)[K:\mathbb{Q}]+1)\,t\,s\log s - C_9\,ts$$

follows readily. The constant C_9 only depends on w and q.

Now we are ready to apply Jensen's formula (5.7) with C replaced by C' and f replaced by G. Since π is totally ramified in w the sum in Jensen's formula becomes $t\log|G(w)|$ and from the inequality above we obtain the lower bound

$$\sum_{\pi(z)=0} \log|G(z)| \geq -((3/2)[K:\mathbb{Q}]+1)\,t^2\,s\log s - C_9\,t^2\,s. \tag{5.15}$$

For the integral we need an estimate from above for $\|G\|_R := \max|G(z)|$ where the maximum is taken over all $z \in C'$ with $|\pi(z)| = R$. The boundary Γ decomposes into connected components Γ_p, one for each $p \in \mathrm{supp}(\pi)_\infty$ for which we choose local coordinates $z_p = (1/\pi(z))^{1/n_p}$. We have

$$\max_{z\in\Gamma_p} \log|G(z)| = t\max_{z\in\Gamma_p}\log|F(z)| - s\log R.$$

Since $n_p \geq q$ for $p \in \mathcal{P}$ the inequality (5.14) gives

$$\max_{z\in\Gamma_p} \log|G(z)| \leq ([K:\mathbb{Q}]/2)nt\log n + C_5tn + 2rtR^{(\rho+\epsilon)/q} - s\log R$$

for $p \in \mathcal{P}$. This also holds for $p \in \mathrm{supp}(\pi)_\infty \setminus \mathcal{P}$ since f and g are of order 0 at p. We conclude that

$$\int_{\partial C'[R]} \log|G|^2 d^c\tau \leq ([K:\mathbb{Q}]/2)nt^2\log n + C_5t^2n + 2rt^2R^{(\rho+\epsilon)/q} - s\,t\log R \tag{5.16}$$

We also need a lower bound for the zero divisor D_0 of the holomorphic function G on C'. One easily sees from (5.13) and Lemma 5.3.5 that

$$(G)_0 \geq \sum_v s\,t\,(v)$$

where the sum is taken over all $v \neq w$ in $\mathcal{S} \cap C'$. Since $|C \setminus C'| \leq g$ by Lemma 5.3.5 we get

$$n^\pi(D_0, r) = \int_{D[r]} 1 \geq st(m-g-1)$$

for sufficiently large r. Therefor we obtain

$$N^\pi(D_0, R) \geq (m-g-1)s\,t\log(R/C_{10}) \tag{5.17}$$

for some positive constant C_{10} which depends only on w and q provided that R is sufficiently large which is certainly the case for $R = s^{q/(2\rho+2\epsilon)}$. A comparison of the leading terms for n and therefor also for s going to infinity in (5.15), (5.16) and (5.17) shows that

$$(m-g)q \leq 2\,(2\,[K:\mathbb{Q}]+1)(\rho+\epsilon)t$$

for all $q \geq 1$. Since by (i) in Lemma 5.3.5 we have $\lim_{q\to\infty} t/q = l$ the desired inequality for $|\mathcal{S}|$ follows readily. $\square$

Remark 5.6.1. *If we take $r^2 = \lambda mn$ instead of $r^2 = 2mn$ with an extra parameter λ chosen to be a sufficiently large integer then it is possible to derive the better estimate $|\mathcal{S}| \leq g + (3[K:\mathbb{Q}]+2)\,\rho\, l$. Moreover by a suitable modification of Lemma 5.3.5 and of the second proof of Theorem 5.1.4 the statement of Theorem 5.1.4 can be improved to $|\mathcal{S}| \leq g + (3[K:\mathbb{Q}]+2)\sum_{p\in\mathcal{P}}\rho_p$, where $\rho_p = \max_i \rho_p(f_i)$.*

Proof of Theorem 5.1.5. Let $\mathcal{S} \subset C$ be a finite set such that $f(\mathcal{S}) \subset \mathcal{U}(K)$. Let $\pi : \widetilde{X} \to X$ be the blow-up of X in $X \setminus \mathcal{U}$ and let $E = \pi^{-1}(X \setminus \mathcal{U})$ be the exceptional divisor. We choose a hyperplane section H of X which does not meet $f(\mathcal{S})$ and put $\widetilde{H} = \pi^{-1}H$. Then if n is sufficiently large the divisor $n\,\widetilde{H} + E$ is very ample. Therefore $\widetilde{Y} = \widetilde{X} \setminus (\widetilde{H} \cup E)$ takes the form $\operatorname{spec} R$ for some K-algebra R of finite type which can be written as $R = K[y_1, y_2, \ldots, y_n]$. Since the rational vector field Δ is regular on $\widetilde{Y}$ we have $\Delta(\mathcal{O}_{\widetilde{Y},y}) \subseteq \mathcal{O}_{\widetilde{Y},y}$ for all $y \in \widetilde{Y}$ and therefor $\mathcal{O}_{\widetilde{Y},y} = R_{\mathfrak{m}_y}$ where $\mathfrak{m}_y$ denotes the maximal ideal at y. By Theorem 4.7 in [89] we have $R = \cap_{y\in\widetilde{Y}} R_{\mathfrak{m}_y}$. We conclude that $\Delta(R) \subseteq R$ and hence there exist $g_i \in K[y_1, \cdots, y_n]$ such that $\Delta(y_i) = g_i$ on $\widetilde{Y}$. Then the functions $f_i(z) = f^*(y_i)$ satisfy $\nabla(f_i) = g_i[f_1, \cdots, f_n]$. They are meromorphic of order $\leq \rho$ on C and satisfy $f_i(w) \neq \infty$ for $1 \leq i \leq n$ and $w \in \mathcal{S}$. Let h be as in Lemma 5.5.6 and replace in the second proof the function $G = F^t/\pi^s$ of by $G = F^t h^{2rt}/\pi^s$. The required estimate follows then similarly. $\square$

Remark 5.6.2. *Our method still works if the holomorphic tangent vector ∇ has zeroes and we get results similar to Theorem 5.1.4 and Theorem 5.1.5. For instance the conclusion of Theorem5.1.4 holds for $\mathcal{S}$ replaced by $\mathcal{S}_0 = \{w \in \mathcal{S}; \nabla_w \neq 0\}$.*

Proof of the Main Theorem 5.1.3. Our discussion before the statement of Theorem 5.1.3 shows that the hypothesis of Theorem 5.1.5 are satisfied. By Theorem 5.1.5, $|\psi^{-1}(\mathcal{U}(\mathcal{K}))| \leq 2\,(2\,[K:\mathbb{Q}]+1)\max(\rho, 2g)l + g$. However C is integral and this implies that $\psi(C) \subseteq \mathcal{U}$ which implies desired results. $\square$

5.7 Singularities

It is critical to exclude singularities from the statement of Theorem 5.1.5. Otherwise $f^{-1}(X(K))$ might be infinite as we shall see from the following examples. Here we assume that g_2, g_3 are algebraic numbers in K.

Example 5.7.1. *Let f be the map from $\mathbb{C}$ to $\mathbb{P}^3$ given by $f(z) = (z : \wp(z) : \wp'(z) : 1)$ and ∇ a holomorphic vector in $\Gamma(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ given by $\nabla = \frac{\partial}{\partial z}$. If we denote by z_i projective coordinates of $\mathbb{P}^3$ and if we write $z_{ij} = \frac{z_i}{z_j}$ then ∇ lifts to an algebraic vector field Δ on $\mathbb{P}^3$ with the following descriptions:*

$$\begin{aligned}
\Delta(z_{02}) &= z_{32} - \frac{z_{02}z_{12}^2}{z_{32}} + \frac{g_2 z_{02} z_{12}}{2}, \\
\Delta(z_{12}) &= 1 - \frac{6z_{12}^3}{z_{32}} + \frac{g_2 z_{12} z_{32}}{2}, \\
\Delta(z_{32}) &= \frac{g_2 z_{32}^2}{2} - 6z_{12}^2.
\end{aligned}$$

If we let A be the point $(0:0:1:0)$ in $\mathbb{P}^3(K)$ then we shall have $f^{-1}(A) = \Lambda$ which is an infinite set. This doesn't contradict to Theorem 5.1.5 because A is not regular with respect to Δ.

With the same notations z_{ij} as above we take another example:

Example 5.7.2. *Let X be the variety $V(z_2^2 z_3 - 4z_1^3 + g_2 z_1 z_3^2 + g_3 z_3^3)$ in $\mathbb{P}^3$, f a map from $\mathbb{C}$ to X given by $f(z) = (z : \wp(z) : \wp'(z) : 1)$ and ∇ a holomorphic vector in $\Gamma(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ given by $\nabla = \frac{\partial}{\partial z}$. The holomorphic vector ∇ lifts to an algebraic vector field Δ on $\mathbb{P}^3$ with the following descriptions:*

$$\begin{aligned}
\Delta(z_{02}) &= z_{32} - \frac{z_{02}z_{12}^2}{z_{32}} + \frac{g_2 z_{02} z_{12}}{2}, \\
\Delta(z_{12}) &= 1 - \frac{6z_{12}^3}{z_{32}} + \frac{g_2 z_{12} z_{32}}{2}, \\
\Delta(z_{32}) &= \frac{g_2 z_{32}^2}{2} - 6z_{12}^2.
\end{aligned}$$

Now as functions in $\mathcal{O}_{A,X}$, although $\frac{6z_{12}^3}{z_{32}}$ is regular $\frac{z_{02}z_{12}^2}{z_{32}}$ is not. This explains why the set $f^{-1}(A) = \Lambda$ is infinite.

We now change a bit from another way and we shall give a slightly new proof of a theorem of Schneider on the transcendence of periods.

Example 5.7.3. *Let E be the elliptic curve in $\mathbb{P}^2$ given by $V(z_2^2 z_3 - 4z_1^3 + g_2 z_1 z_3^2 + g_3 z_3^3)$, X a variety given by $\mathbb{P}^1 \times E$, f a map from $\mathbb{C}$ to X given by $f(z) = ((z : 1), (\wp(z) : \wp'(z) : 1))$ and ∇ a holomorphic vector in $\Gamma(\mathbb{C}, \mathcal{T}_{\mathbb{C}})$ given by $\nabla = \frac{\partial}{\partial z}$. The holomorphic vector ∇ lifts to an algebraic vector field Δ on $\mathbb{P}^1 \times E$. One can check that for all z in $\mathbb{C}$ the algebraic vector Δ is always regular at $((z : 1), (0 : 1 : 0))$. If an element $w \neq 0$ in Λ is an algebraic number in K, then for all positive integers n we have $f(nw) = ((nw : 1), (0 : 1 : 0))$ lies in $\mathcal{U}(K)$. This contradicts Theorem 5.1.5. Therefore we obtain a classical theorem of Schneider that any $z \neq 0$ in Λ is transcendental.*

5.8 Appendix

In this appendix we shall state Schneider's theorem as announced in the introduction and we shall comment on it and relate it to our work. We make the same geometric assumptions as in Theorem 5.1.4. In particular we fix a vector field ∇.

In section 5.4 we have introduced the notion of an analytic divisor on C as an element of the group $\operatorname{Div} C = \varprojlim \operatorname{Div} C[r]$. We shall extend the notion slightly and define $\widetilde{\operatorname{Div}} C$ using the same direct system and requiring that an element of $\widetilde{\operatorname{Div}} C$ is a family $D = \{D[r] = \sum_{z \in C[r]} D[r](z)\,(z)\}$ which satisfies $p_{r,s}(\sum_{z \in C[s]} D[s](z)\,(z)) \geq \sum_{z \in C[r]} D[r](z)\,(z)$, i.e. with equality replaced by "$\geq$" in the sense of divisors. Clearly we have $\widetilde{\operatorname{Div}}\, C \supseteq \operatorname{Div} C$. The projection $p_r : \operatorname{Div} C \to \operatorname{Div} C[r]$ extends to a projection $\widetilde{p}_r : \widetilde{\operatorname{Div}}\, C \to \operatorname{Div} C[r]$ and maps D to $D[r]$. We identify now the formal divisor $D[r]$ with the (non-reduced) closed scheme $\iota[r] : D[r] \to C$ which is attached to $D[r]$ and note that the collection of morphisms $\{\iota[r]\}_{r \geq 0}$ induce a morphism $\iota : D \to C$. There is a canonical surjective homomorphism of sheaves $\iota[r]^\sharp : \mathcal{O}_C^{an} \to \iota[r]_* \mathcal{O}_{D[r]}$. Here $\mathcal{O}_C^{an}$ is the sheaf of germs of holomorphic functions on C. The kernel of $\iota[r]^\sharp$ is an ideal sheaf $\mathcal{J}$. For $z \in \operatorname{supp} D[r]$ we choose a local coordinate $\tau_z \in \mathcal{J}_z$ at z such that $\nabla = d/d\tau_z$ locally at z. The local coordinate is uniquely determined by this property. We define $R = \bigoplus K[\tau_z]/(\tau_z^{D(z)})$ and then $\Gamma(D[r], \iota[r]_* \mathcal{O}_{D[r]}) = R \otimes_K \mathbb{C}$. There is a canonical identification of R considered as a vector space with $K^{D[r]}$ and this is used to define the height $h^+(f)$ of $f \in R$. The homomorphism $\iota[r]^\sharp$ extends to the sheaf of germs of meromorphic functions on C which are holomorphic on D. We let now $D \in \widetilde{\operatorname{Div}}\, C$ be a divisor and define

$$\delta = \underline{\lim} \frac{\log \deg D[r]}{\log \delta(r)}$$

where $\delta(r)$ is the smallest real number $s \leq r$ such that $D[r] \in C[s]$. Let f be a meromorphic functions on C which is holomorphic on D and which has the property that $\iota[r]^\sharp(f) \in R$ for all r. Then the arithmetic growth of F along D is defined as

$$\mu = \overline{\lim} \frac{\log h^+(\iota[r]^\sharp(f))}{\log \deg D[r]}.$$

The following theorem is an extension of Schneider's Satz III in [116] mentioned in the introduction to Riemann surfaces. Let $D \in \widetilde{\operatorname{Div}} C$ be a divisor such that $D[r](z)/\log D[r](z) \leq \deg D[r]$ for all $z \in \operatorname{supp} D[r]$. We choose meromorphic functions $f_1, f_2, \ldots, f_n$ on C which are holomorphic on D and which satisfy $\iota[r]^\sharp(f_i) \in R$ for $1 \leq i \leq n$ and r sufficiently large. We define $\rho = \max_i(\rho_i)$ and $\mu = \max_i(\mu_i)$ where $\rho_1, \rho_2, \ldots, \rho_n$ and $\mu_1, \mu_2, \ldots, \mu_n$ are the growth and arithmetic growth respectively of the functions.

Theorem 5.8.1. *If we have $\mu \leq \rho/\delta < (1 - (1/n))$ then the image of the mapping*

$$f = (f_1, f_2, \ldots, f_n) : C \to \mathbb{C}^n$$

is contained in an algebraic hypersurface.

The proof of the theorem is very similar to the proof of our main theorems. But for proving this theorem (5.6) is not sufficient, instead we need to use a version of Jensen Formula as formulated in [32, p.332]. As an application of the theorem which is not covered by any of the theorems in the introduction we take a meromorphic function f on C which has the property that $\iota[r]^\sharp(f) \in R$

for all sufficiently large r. Then we put $f_1 = f$, $f_2 = \nabla f, \ldots, f_n = \nabla^{n-1} f$ for n sufficiently large so to satisfy the hypothesis of the theorem. Then the function f satisfies an algebraic differential equation. As has been mentioned in the introduction the theorem does not include Gelfond's proof of the seventh Hilbert Problem and, as a consequence, does not include the Schneider-Lang criterion, even in the case when $C = \mathbb{C}$. It would be very interesting to find a criterion in the style of our theorem above which does include our main theorems and without making the assumption that the functions satisfy differential equations.

All theorems that have been mentioned so far deal only with the transcendence of numbers. As already has been pointed out in the last paragraph of Schneider's paper in loc. cit. the methods are not strong enough to get a criterion about algebraic independence. The only substantial contribution in this very general direction is [135]. The techniques which are applied there are much more involved.Any further progress in the direction opened there would be of highest interest.

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