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Analytic Newvectors for $\mathrm{GL}_n(\mathbb{R})$ and Applications

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Abstract

We introduce an analytic archimedean analogue of some aspects of the classical non-archimedean newvector theory formulated by Casselman and Jacquet–Piatetski-Shapiro–Shalika. We relate the analytic conductor of a generic irreducible representation of $\mathrm{GL}_n(\mathbb{R})$ to the invariance properties of some special vectors in that representation, which we name *analytic newvectors*.

We also provide a few natural applications of analytic newvectors to some analytic questions concerning automorphic forms for $\mathrm{GL}_n(\mathbb{Z})$ in the archimedean analytic conductor aspect. We prove an orthogonality result of the Fourier coefficients, a density estimate for the non-tempered forms, an equidistribution result for the Satake parameters with respect to the Sato–Tate measure, as well as a second moment estimate for the central L-values as strong as Lindelöf on average. We also verify the random matrix prediction concerning the distribution of the low-lying zeros of the Langlands L-functions in the analytic conductor aspect.

Zusammenfassung

Wir stellen ein analytisches archimedisches Analogon einiger Aspekte der klassischen nicht-archimedischen Neuvektortheorie vor, die Casselman und Jacquet– Piatetski-Shapiro–Shalika formuliert haben. Wir entwickeln eine Verbindung zwischen dem analytischen Konduktor einer generischen irreduziblen Darstellung von $\mathrm{GL}_n(\mathbb{R})$ und den Invarianzeigenschaften einiger spezieller Vektoren in dieser Darstellung, die wir *analytic newvectors* nennen.

Wir präsentieren auch einige natürliche Anwendungen der *analytic newvectors* für verschiedene analytische Fragen zu automorphen Formen für $\mathrm{GL}_n(\mathbb{Z})$ im archimedischen Aspekt des analytischen Konduktors. Wir beweisen ein Orthogonalitätsergebnis der Fourierkoeffizienten, eine Dichteschätzung der nicht temperierten Formen, ein Gleichverteilungsergebnis der Satake Parameter in Bezug auf das Sato–Tate Maß, sowie eine Abschätzung der zweiten Momente der zentralen L-Werte von derselben Qualität wie eine gemittelte Version der Lindelöfschen Vermutung. Wir beweisen auch die Verteilung der tief liegenden Nullstellen der Langlandschen L-Funktion im analytischen Konduktor-Aspekt, die von der Theorie der Zufallsmatrizen vorhergesagt wird.

Preface

The work on the existence of analytic newvectors appeared as a joint effort with Paul D. Nelson in the preprint [38]. Applications of analytic newvectors appeared in the preprint [37]. Both articles have been submitted to journals to be considered for publication.

Keywords: Newvector, Whittaker function, L-function, Automorphic form

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Contents

Chapter 1

Introduction

The motivating questions leading to this thesis include two directions distinguished by *local* and *global* features, which are described in the following sections.

1.1 Local motivation

1.1.1 Classical/non-archimedean newvectors

Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$, maximal ideal $\mathfrak{p}$, and cardinality of the residue field $\#\mathfrak{o}/\mathfrak{p} = q$. Let χ be a unitary character on $\mathrm{GL}_1(F) = F^\times$. There is a standard notion of a *conductor* of χ in terms of the invariance properties of χ, which we recall in the following.

Consider a filtration of open-compact subgroups in $\mathfrak{o}$ as follows:

$$\mathfrak{o}^\times \supseteq 1 + \mathfrak{p} \supseteq 1 + \mathfrak{p}^2 \supseteq \ldots \supseteq 1 + \mathfrak{p}^M \supseteq \ldots.$$

Because of the continuity of χ there exists an open compact subgroup $1 + \mathfrak{p}^M$ such that

$$\chi\mid_{1+\mathfrak{p}^M} = \text{trivial}.$$

Let N be the minimal non-negative integer M such that the above invariance occurs. Then, N is called the *conductor exponent* of χ, denoted by $c(\chi)$, and the real number q^N is called the *conductor* of χ, denoted by $C(\chi) = q^{c(\chi)}$.

The open-compact subgroup $1 + \mathfrak{p}^M$ can be defined analytically by

$$1 + \mathfrak{p}^M := \{x \in \mathfrak{o} \mid \|x - 1\|_F \leq q^{-M}\}.$$

On the other hand, an analytic approach is available for defining the conductor of a unitary character. A *local ϵ-factor* can be attached to χ, which is an entire function on $\mathbb{C}$. The conductor of χ can then be defined using the equation

$$\epsilon(1/2 - s, \chi) = \epsilon(1/2, \chi)C(\chi)^s.$$

Then, the invariance property of χ can be reformulated as described in the following theorem.

Toy Theorem 1. *If χ is a unitary character of $\mathrm{GL}_1(F)$ with conductor $C(\chi)$ (defined* *analytically), then*

$$\chi(x) = 1, \quad \forall x \in \mathfrak{o} : \|x-1\|_F \leq C(\chi)^{-1}.$$

Moreover, the relation $\chi(x) = 1$ *is not true in general if* $\|x-1\|_F > C(\chi)^{-1}$.

A natural question is if a similar theory of *conductor and invariance* is available in higher rank groups, e.g., in $\mathrm{GL}_n(F)$ for $n \geq 2$. The answer is yes, and the analogous theory was pioneered by Casselman [13] for $n = 2$ and Jacquet–Piatetski-Shapiro–Shalika [30] for $n \geq 2$.

We first describe the ingredients needed to state the theorem of Casselman, Jacquet–Piatetski-Shapiro–Shalika. The analogue of a unitary character on $\mathrm{GL}_1(F)$ is a *generic* (see §4.1.2) irreducible unitary representation π of $\mathrm{GL}_n(F)$. Second, we must fix a filtration of some open-compact subgroups in $\mathrm{GL}_n(\mathfrak{o})$. We define the congruence subgroup $K_1(\mathfrak{p}^M)$ of level M by the subgroup of matrices in $\mathrm{GL}_n(\mathfrak{o})$ such that the last rows of these matrices are congruent to $(0,\ldots,0,1) \mod \mathfrak{p}^M$, analytically expressed as

$$K_1(\mathfrak{p}^M) := \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : \begin{matrix} \|g_{nj}\|_F \leq q^{-M} \text{ for } 1 \leq j < n, \\ \|g_{nn}-1\|_F \leq q^{-M} \end{matrix} \right\}$$
$$= \begin{pmatrix} \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{o} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^M & \cdots & \mathfrak{p}^M & 1+\mathfrak{p}^M \end{pmatrix} \cap \mathrm{GL}_n(\mathfrak{o}). \quad (1.1.1)$$

We define a filtration analogous to the $GL(1)$ case above, as follows.

$$\mathrm{GL}_n(\mathfrak{o}) \supseteq K_1(\mathfrak{p}) \supseteq \ldots \supseteq K_1(\mathfrak{p}^M) \supseteq \ldots.$$

We can also define the congruence subgroup $K_0(\mathfrak{p}^M)$ by requiring that the last rows of the matrices are congruent to $(0,\ldots,0,*) \mod \mathfrak{p}^M$, i.e., analytically,

$$K_0(\mathfrak{p}^M) := \left\{ g \in \mathrm{GL}_n(\mathfrak{o}) : \|g_{nj}\|_F \leq q^{-M} \text{ for } 1 \leq j < n \right\}$$
$$= \begin{pmatrix} \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{o} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{o} & \cdots & \mathfrak{o} & \mathfrak{o} \\ \mathfrak{p}^M & \cdots & \mathfrak{p}^M & \mathfrak{o} \end{pmatrix} \cap \mathrm{GL}_n(\mathfrak{o}). \quad (1.1.2)$$

Consequently, we define a similar filtration of $K_0(\mathfrak{p}^M)$.

Next, we analytically define the conductor $C(\pi)$ of a generic irreducible representation π of $\mathrm{GL}_n(F)$ as before, i.e., through the ϵ-factor attached to π using the equation

$$\epsilon(1/2-s,\pi) = \epsilon(1/2,\pi)C(\pi)^s. \quad (1.1.3)$$

The conductor $C(\pi)$ can be checked as being of the form $q^{c(\pi)}$ where $c(\pi) \in \mathbb{Z}_{\geq 0}$, which we call, analogously, the *conductor exponent* of π.

Finally, we state the main theorem of the non-archimedean *newvector theory* as follows.

Theorem 1 ([13] for $n = 2$, [30] for $n \geq 2$)**.** *For every generic irreducible representation π of $\mathrm{GL}_n(F)$ with conductor $C(\pi) := q^{c(\pi)}$, there is a nonzero vector $v \in \pi$ that is invariant by the subgroup $K_1(\mathfrak{p}^{c(\pi)})$. Moreover, the vector is unique up to scalar multiplication.*

We call such a vector a *newvector* of π.

1.1.2 Archimedean analogue

The discussion in the previous section naturally motivates the question, *is there an analogous theory at the archimedean place?*

At least two immediate obstacles exist that would trivialize this question if we generalize Theorem 1 verbatim at an archimedean place. First,

- there is *no* open compact subgroup of $\mathrm{GL}_n(\mathbb{R})$ unlike the non-archimedean case.

We may relax this assumption and only work with an *open subset* of $\mathrm{GL}_n(\mathbb{R})$. But, even then,

- *very few* vectors in a representation of $\mathrm{GL}_n(\mathbb{R})$ will be invariant by any open subset.

As an example, for $\mathrm{GL}_1(\mathbb{R})$, the trivial and the sign characters satisfy such an invariance property.

To make the problem interesting and useful, we modify the question and only work with an *approximate* analogue of Theorem 1. First, we select an open neighbourhood $K \subset \mathrm{GL}_n(\mathbb{R})$ of the identity that is an *approximate analogue* of $K_1(\mathfrak{p}^M) \subset \mathrm{GL}_n(F)$. Then, we consider if there are representations of $\mathrm{GL}_n(\mathbb{R})$ and vectors contained in it that are *approximately invariant* by K. A more rigorous question is presented below, but first, we gather the relevant ingredients to formulate a statement, as in the non-archimedean case.

Given a generic irreducible representation π of $\mathrm{GL}_n(\mathbb{R})$, an *analytic conductor* $C(\pi)$ can be attached to π, similar to the non-archimedean case (cf. (1.1.3) the analytic definition of conductor), but in terms of the *local γ-factor* attached to π, which is a meromorphic function in $\mathbb{C}$. We characterize $C(\pi)$ as the leading term asymptotic of $\gamma(s, \pi)$ through the equation

$$\gamma(1/2 - s, \pi) = \gamma(1/2, \pi) C(\pi)^s + O_\pi(s),$$

as $s \to 0$. Also, the formula may be written more explicitly for the analytic conductor as $C(\pi) := \prod_j (1 + |\mu_j|)$, where $\{\mu_1, \ldots, \mu_n\}$ is the set of (*Langlands*) parameters of π characterized by the relation $L(s, \pi) = \prod_j \Gamma_\mathbb{R}(s + \mu_j)$, where $\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma(s/2)$ (see [28]). Here, $L(s, \pi)$ denotes the *local L-factor* attached to π. In practice, minor variants of this definition serve the same purpose. For instance, occasionally, the factors occur as $(1 + |\mu_j|)$ replaced by $(3 + |\mu_j|)$, so that $\log C(\pi)$ is bounded uniformly away from zero.

Next, we describe an approximate analogue of $K_0(\mathfrak{p}^M)$. Let $X > 1$ be large and tending off to infinity and $\tau > 0$ be arbitrarily small but fixed. We define

$$K_1(X,\tau) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R}) \;\middle|\; a \in \mathrm{GL}_{n-1}(\mathbb{R}), d \in \mathbb{R}^\times, \begin{matrix} |a - 1_{n-1}| < \tau, & |b| < \tau, \\ |c| < \tau/X, & |d-1| < \tau/X \end{matrix} \right\}.$$

Here, the various $|.|$ denote arbitrary fixed norms on the various spaces of matrices and 1_r denotes the identity element in $\mathrm{GL}_r(\mathbb{R})$. The term X is the analogue of the real number q^M for $K_1(\mathfrak{p}^M)$, and $K_0(X,\tau)$ can be similarly defined as an analogue of $K_0(\mathfrak{p}^M)$ by only requiring $|d-1| < \tau$ in the above definition of $K_1(X,\tau)$.

Finally, we rigorously describe the what we intuitively mean by *approximate invariance*. We formally re-write the question proposed at the beginning of this subsection as the following.

Question: *For every $\epsilon > 0$ does there exist a number $\tau > 0$ such that for every irreducible generic unitary representation π of $\mathrm{GL}_n(\mathbb{R})$ there exists a vector $v \in \pi$ with $\|v\|_\pi = 1$, so that*

$$\|\pi(g)v - v\|_\pi < \epsilon,$$

for every $g \in K_1(C(\pi),\tau)$?

We prove in Theorem 2 that if π is also assumed to be *tempered* (more generally, if π satisfies a non-trivial bound towards Ramanujan), then the answer to this question is affirmative, which proves a stronger and more useful version of this theorem in Theorem 3. By analogy, we call such vectors *analytic newvectors* for $\mathrm{GL}_n(\mathbb{R})$[i].

We complete this section by providing an example where we state an archimedean version of the Toy Theorem 1 for $\mathrm{GL}_1(\mathbb{R})$. We know that the set of unitary characters of $\mathrm{GL}_1(\mathbb{R})$ can be parametrized by the map

$$i\mathbb{R} \times \{0,1\} \ni (s,\delta) \mapsto \{\mathbb{R}^\times \ni y \mapsto \chi_{s,\delta}(y) := |y|^s \mathrm{sgn}(y)^\delta\}.$$

One can calculate that $C(\chi_{s,\delta}) \asymp (1+|s|)$, and for $\mathrm{GL}_1(\mathbb{R})$, we can set

$$K_1(X,\tau) := \{y \in \mathbb{R}^\times \mid |y-1| < \tau/X\}.$$

We now state the corresponding toy theorem in the archimedean case.

Toy Theorem 2. *For $\epsilon > 0$, there exists $\tau > 0$ such that for all (s,δ)*

$$|\chi_{s,\delta}(y) - 1| < \epsilon,$$

for all $y \in K_1(1+|s|,\tau)$.

Proof. For small enough τ, we can take $y > 0$. Let $s = it$ with $t \in \mathbb{R}$. Then,

$$|y^s - 1| = |e^{it\log y} - 1| \ll |t \log y| \ll |t(y-1)|,$$

[i]See Remark 4 for the reason we use the adjective *analytic*.

where all implied constants are absolute. We conclude using $y \in K_1(1+|t|,\tau)$. □

1.2 Global motivation

Given a finite family $\mathcal{F}$ of automorphic representations π of a reductive group G, several important questions can be formulated in terms of the asymptotic behaviour of the average

$$|\mathcal{F}|^{-1}\sum_{\pi\in\mathcal{F}} A(\pi), \tag{1.2.1}$$

as the size of the family $|\mathcal{F}|$ tends off to infinity. Here, $A(\pi)$ can be chosen to be interesting arithmetic objects attached to π, e.g., Fourier coefficients λ_π or powers of central L-values $|L(1/2,\pi)|^k$. Often acquiring knowledge about the statistical behaviour of $A(\pi)$ by making such averages is easier than understanding $A(\pi)$ individually due to the availability of strong analytic techniques, e.g., (relative) trace formulae. The families usually are defined in terms of some intrinsic attributes of the automorphic representations π, such as their levels (non-archimedean), weights, spectral parameters, Laplace eigenvalues (archimedean), and analytic conductors. A plethora of works has been done in these aspects on various higher rank and higher dimensional arithmetic locally symmetric spaces, among which we refer to [2–6, 9].

Consider the family $\mathcal{F}_X$ defined by

$$\{\text{automorphic representation } \pi \text{ for } \mathrm{PGL}_n(\mathbb{Z}) \mid C(\pi) < X\}.$$

Here, $C(\pi)$ denotes the analytic conductor of π, i.e., $C(\pi) = C(\pi_\infty)$ where π_∞ is the archimedean component of π and $C(\pi_\infty)$ is defined as in the previous section. Obtaining an asymptotic formula or estimating the size of (1.2.1) when $\mathcal{F} = \mathcal{F}_X$ as $X \to \infty$ for various $A(\pi)$ is another motivating problem of this thesis. For instance, if $A(\pi) = |L(1/2,\pi)|^k$, then an asymptotic formula for (1.2.1) for large enough k would yield a *sub-convex* bound for $L(1/2,\pi)$ in terms of $C(\pi)$. This seems to be a potential approach to understand the infamous *conductor-drop issue*. We refer to [42] for a survey on other averaging problems over $\mathcal{F}_X$.

There are several instances [5, 47, 60] in the literature where the invariance property of the classical non-archimedean newvectors has been (sometimes implicitly) used to construct projectors on the automorphic representations with a given level. These projectors are well-compatible with the relative trace formulae, which, consequently, helps with the understanding of the averages in (1.2.1) in the level (equivalently, non-archimedean conductor) aspect. So far, the similar problem in the archimedean analytic conductor aspect cannot be addressed because of unavailability of the right notion of invariance or newvectors at the archimedean place.

In the application part of this thesis, our goal is to initiate an understanding of a smoothened version of the averages as in (1.2.1) in the archimedean analytic conductor aspect by constructing an approximate projector on $\mathcal{F}_X$ using *analytic newvectors* and subsequently analysing the averages by means of the Kuznetsov trace formula. We provide a few averaging applications of this kind that (including their proofs) are mostly influenced by the recent works [3–6, 23, 24, 46, 55].

Chapter 2

Main Theorems

2.1 Analytic newvectors

We recall the definition of the approximate congruence subsets in $\mathrm{GL}_{n+1}(\mathbb{R})$. Let $X \geq 1$ (considered as tending off to ∞) and let $\tau \in (0,1)$ (considered as small but fixed, or, perhaps, very slowly tending to 0). We define the following archimedean analogues of the standard p-adic congruence subgroup (1.1.2):

$$K_0(X,\tau) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_{n+1}(\mathbb{R}) \middle| a \in \mathrm{GL}_n(\mathbb{R}), d \in \mathrm{GL}_1(\mathbb{R}), \begin{array}{ll} |a - 1_n|< \tau, & |b|< \tau, \\ |c|< \dfrac{\tau}{X}, & |d-1|< \tau \end{array} \right\}. \tag{2.1.1}$$

Here, the various $|.|$ denote arbitrary fixed norms on the various spaces of matrices. We define $K_1(X,\tau)$ similarly, but with the stronger constraint of $|d-1|< \tau/X$.

While the sets $K_*(X,\tau)$ are not groups, they feature some group-like properties in the limit as $\tau \to 0$. For instance, it is easy to see that if τ' is small enough with respect to τ, then $g_1g_2 \in K_0(X,\tau)$ for all $g_1, g_2 \in K_0(X,\tau')$.

We also recall the usual notion of θ-temperedness. Let $\theta \geq 0$, and by the Langlands classification, we know that any unitary irreducible representation π of $\mathrm{GL}_n(\mathbb{R})$ is a Langlands quotient of an isobaric sum of the form

$$\sigma_1 \otimes |\det|^{s_1} \boxplus \cdots \boxplus \sigma_r \otimes |\det|^{s_r}.$$

where the underlying Levi of this induced representation is attached to a partition of n by 2s and 1s. Here, each σ_i is either a discrete series of $\mathrm{GL}_2(\mathbb{R})$ or a character of $\mathrm{GL}_1(\mathbb{R})$ of the form $\mathrm{sgn}^\delta|.|^{\mu_i}$ for some $\delta \in \{0,1\}$ and $\mu_i \in i\mathbb{R}$. We say that π is *θ-tempered* if all such s_i have real parts in $[-\theta,\theta]$. By [49], the local component at any real place of any cuspidal automorphic representation of $\mathrm{GL}(n)$ over a number field is θ-tempered with $\theta = 1/2 - 1/(1+n^2) < 1/2$.

We next describe the main theorems regarding the existence of analytic newvectors. Theorem 2 provides a simple sense in which the analytic conductor controls the invariance properties of vectors. Theorem 3 is a more powerful, yet more technical, result with additional features that we expect to be useful in applications.

Theorem 2. *Fix $n \in \mathbb{Z}_{\geq 1}$ and $\theta \in [0,1/2)$. For each $\delta > 0$, there exists $\tau > 0$ with the following property: For each generic irreducible θ-tempered unitary representation Π of*

$\mathrm{GL}_{n+1}(\mathbb{R})$, *there exists a unit vector* $v \in \Pi$ *such that for all* $g \in K_0(C(\Pi), \tau)$,

$$\|\Pi(g)v - \omega_\Pi(d_g)v\|_\Pi < \delta.$$

Here, ω_Π *denotes the central character of* Π, *and* d_g *is the lower-right entry of* g.

We fix a generic additive character $\tilde{\psi}$ of the standard maximal unipotent subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$, consisting of upper-triangular unipotent matrices. We choose $\tilde{\psi}$ to be defined in a similar way as in (4.1.1), and we denote $\mathcal{W}\left(\Pi, \tilde{\psi}\right)$ by the Whittaker model of a generic irreducible representation Π of $\mathrm{GL}_{n+1}(\mathbb{R})$ (see §4.1.2 for details).

Theorem 3. *Fix* $n \in \mathbb{Z}_{\geq 1}$ *and* $\theta \in [0, 1/2)$, *let* Ω *be a bounded open subset of* $\mathrm{GL}_n(\mathbb{R})$, *and let* $\iota > 0$ *be small enough in terms of* n *and* Ω. *For each* $\delta > 0$, *there exists* $\tau > 0$ *with the following property:*

For each generic irreducible θ*-tempered unitary representation* Π *of* $\mathrm{GL}_{n+1}(\mathbb{R})$, *there exists an element* $V \in \mathcal{W}\left(\Pi, \tilde{\psi}\right)$ *of its Whittaker model satisfying*

- *the normalization* $\|V\|_{\mathcal{W}(\Pi,\tilde{\psi})} = 1$, *with the norm taken in the Kirillov model (see §4.1.2),*
- *the lower bound* $V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \geq \iota$ *for all* $h \in \Omega$, *and*
- *the invariance properties:*
  1. *for all* $g \in K_0(C(\Pi), \tau)$,
  $$\|\Pi(g)V - \omega_\Pi(d_g)V\|_{\mathcal{W}(\Pi,\tilde{\psi})} < \delta,$$
  2. *and for* $h \in \Omega$,
  $$\left| V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} g\right] - \omega_\Pi(d_g) V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right]\right| < \delta.$$

Here, ω_Π *and* d_g *are as in Theorem 2.*

Informally, Theorem 3 asserts that if τ is small enough, then there are nonzero vectors in Π satisfying a form of approximate invariance under $K_0(C(\Pi), \tau)$, both in the sense of the norm and as quantified by the Whittaker functional.

Remark 1. *Recording the formulation of Theorem 3 in terms of sequences is instructive: for each bounded open* $\Omega \subseteq \mathrm{GL}_n(\mathbb{R})$ *and sequence* Π_j *of generic irreducible* θ*-tempered unitary representations of* $\mathrm{GL}_{n+1}(\mathbb{R})$ *there is a corresponding sequence* $V_j \in \mathcal{W}(\Pi_j, \tilde{\psi})$ *of Whittaker model elements satisfying*

- $\|V_j\| = 1$,

- $\inf_{h\in\Omega}\inf_j |V_j(h)| > 0$, *and*
- *for any sequence* τ_j *of positive numbers tending to zero and any sequence of the group elements* $g_j \in K_0(C(\Pi_j), \tau_j)$,

$$\lim_{j\to\infty} \|\Pi(g_j)V_j - \omega_\Pi(d_{g_j})V_j\| = 0$$

and

$$\lim_{j\to\infty} \inf_{h\in\Omega} |V_j(hg_j) - \omega_\Pi(d_{g_j})V_j(h)| = 0.$$

Remark 2. *The proof is constructive (see §3.1) and shows that we may take* V *to be a fixed bump function in the Kirillov model.*

Remark 3. *The assumption of* θ*-temperedness (or even unitarity) may seem artificial because it is not required in the non-archimedean setting. However, it is used in the proof to ensure that* $\gamma(1/2 - s, \Pi)$ *remains holomorphic for* $\Re(s) \geq 0$ *during a contour shift argument (see §3.1). This assumption is satisfied in our intended applications.*

Remark 4. *The newvector defined by Jacquet–Piatetski-Shapiro–Shalika in [30] is defined differently; a newvector is a Whittaker function* V *on* $\mathrm{GL}_{n+1}(F)$ *such that for any spherical representation* π *of* $\mathrm{GL}_n(F)$ *containing the spherical vector* W_0 *with* $W_0(1) = 1$, *the local zeta integral (see §4.1.5) of* V *and* W_0 *equals the* L*-factor of the Rankin-Selberg convolution* $\Pi \otimes \pi$.

The newvectors at an archimedean place can also be defined as test vectors of the Rankin-Selberg zeta integral, for which Popa [52] introduced such a theory for $\mathrm{GL}_2(\mathbb{R})$, *while the case of* $\mathrm{GL}_n(\mathbb{R})$ *is the subject of an ongoing work of P. Humphries. Such test vectors can be thought of as algebraic analogues at the archimedean place of the classical newvectors in [30]. The analytic newvectors considered here are "analytic test vectors" (i.e., the zeta integral enjoys a quantitative lower bound instead of being merely non-vanishing) for "analytically unramified" representations (i.e., for those with sufficiently small analytic conductors). The source of this dichotomy between algebraic and analytic is related to the question: What is the analogue of* $\mathrm{GL}_n(\mathbb{Z}_p) \subseteq \mathrm{GL}_n(\mathbb{Q}_p)$ *inside* $\mathrm{GL}_n(\mathbb{R})$*? An algebraic analogue is* $\mathrm{O}(n)$ *(a maximal compact subgroup), while an analytic analogue is a small balanced neighborhood of the identity. Algebraic newvectors transform nicely under* $\mathrm{O}(n)$, *while analytic newvectors transform nicely under suitable neighbourhoods of the identity.*

Remark 5. *We expect, by analogy to the non-archimedean theory, that Theorems 2 and 3 feature a "converse" to the effect that their conclusion fails if* $K_0(C(\Pi), \tau)$ *is replaced by* $K_0(X, \tau)$ *for* X *substantially smaller than* $C(\Pi)$. *To make these assertions precise, let* f_X *be an* L^1*-normalized smoothened characteristic function of* $K_0(X, \tau)$. *Because* $K_0(X, \tau)$ *behaves like a group, we may assume that* f_X *is a self-convolution, so that the integral operator* $\Pi(f_X)$ *is positive-definite. The trace* $d_X(\Pi)$ *of that operator is then an analytic proxy for "the dimension of the space of* $V \in \Pi$ *approximately invariant by* $K_0(X, \tau)$*." (In some applications, an alternative proxy* $J_X(\Pi)$ *defined using the Bessel distribution is* *more relevant, see §4.2.5 for details.) Theorem 2 implies that* $d_X(\Pi) \gg 1$ *for* $X \geq C(\Pi)$ *and* $\tau > 0$ *is small but fixed. Conversely, we expect that for any fixed* N*, we have* $d_X(\Pi) \ll_N (C(\Pi)/X)^{-N}$ *for* $X \ll C(\Pi)$*. Thus,* $d_X(\Pi)$ *is small when* X *is substantially smaller compared to* $C(\Pi)$*. In the transition regime of* $X \asymp C(\Pi)$*, our estimates show that such an upper bound is sharp if true, suggesting an analogue of the "multiplicity one" property of non-archimedean newvectors, as well as an asymptotic characterization of the analytic conductor in terms of the invariance properties of vectors.*

2.2 Applications

Let $\mathbb{X} := \mathrm{PGL}_n(\mathbb{Z})\backslash\mathrm{PGL}_n(\mathbb{R})$, and by $\hat{\mathbb{X}}$ we denote the isomorphism class of irreducible unitary *standard* automorphic representations of $\mathrm{PGL}_n(\mathbb{R})$ in $L^2(\mathbb{X})$ that are unramified at all finite places. Here, by standard, we mean the automorphic representations that appear in the spectral decomposition of $L^2(\mathbb{X})$. Similarly, by $\hat{\mathbb{X}}_{\mathrm{gen}}$, we denote the subclass of *generic* representations in $\hat{\mathbb{X}}$, i.e., the class of representations that have (unique) Whittaker models. Let $d\mu_{\mathrm{aut}}$ be the automorphic Plancherel measure on $\hat{\mathbb{X}}$ compatible with the G-invariant probability measure on $\mathbb{X}$ (see [21, Chapter 11.6] for details). Then, let $C(\pi)$ be the analytic conductor of $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ (see §4.2.2).

2.2.1 Orthogonality of Fourier coefficients

The Fourier coefficients of automorphic forms on $\mathrm{GL}(n)$ for $n \geq 2$ behave similarly to the characters on $\mathrm{GL}(1)$. In particular, the Fourier coefficients are expected to satisfy an orthogonality relation when averaged over a sufficiently large family, similar to characters. In [63, Conjecture 1.1], an orthogonality conjecture of the Fourier coefficients is formulated, which loosely states that

$$\lim_{T\to\infty} \frac{\sum_{\varphi \text{ cuspidal}, \, \nu_\varphi \leq T} \lambda_\varphi(l)\overline{\lambda_\varphi(m)} L(1,\varphi,\mathrm{Ad})^{-1}}{\sum_{\varphi \text{ cuspidal}, \, \nu_\varphi \leq T} L(1,\varphi,\mathrm{Ad})^{-1}} = \delta_{l=m}. \tag{2.2.1}$$

Here, φ are spherical (i.e., also unramified at infinity) cusp forms on $\mathbb{X}$, and $l, m \in \mathbb{N}^{n-1}$. Also, λ_φ and ν_φ are the Fourier coefficients and the Laplace eigenvalues of φ, respectively. For detailed notations, see §4.2.

In Theorem 4, we prove a variant of the conjecture in (2.2.1) in the analytic conductor aspect (i.e., the Laplace eigenvalue ν_φ replaced by the analytic conductor of φ) for general n.

Theorem 4. *Let* $l, m \in \mathbb{N}^{n-1}$ *and* $X > 1$ *be a large number tending off to infinity, such that*

$$\min(l_1^{n-1} \dots l_{n-1} m_1 \dots m_{n-1}^{n-1}, m_1^{n-1} \dots m_{n-1} l_1 \dots l_{n-1}^{n-1}) \ll X^n$$

with a sufficiently small implied constant. Then, there exists $J_X : \hat{\mathbb{X}}_{\mathrm{gen}} \to \mathbb{R}_{\geq 0}$ *with*

- $J_X(\pi) \gg 1$ *whenever* π *is cuspidal with* $C(\pi) < X$*,*

- $\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) \asymp X^{n-1}$,

such that

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \overline{\lambda_\pi(l)}\lambda_\pi(m)\frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) = \delta_{l=m}\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi),$$

where $\ell(\pi)$ *is a positive number defined in* (4.2.2) *that depends only on the non-archimedean data attached to* π.

In [8], the conjecture in (2.2.1) is first proved for $n = 2$. Recently, in [3, Theorem, 5], [6, Theorem 5], [23] for $n = 3$ and in [24] for $n = 4$, the conjecture (in the Laplace eigenvalue aspect) has been settled. On the other hand, a variant of the conjecture in (2.2.1) without the harmonic weights $L(1, \text{Ad}, \pi)$ has been proved in [46, Theorem 1.5] for general n. As an easy corollary of Theorem 4, we obtain the following weighted counting of cusp forms for $\text{PGL}_n(\mathbb{Z})$.

Theorem 5. *Let* $L(s, \pi, \text{Ad})$ *be the adjoint* L*-function. Then,*

$$\sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} \frac{1}{L(1, \pi, \text{Ad})} \ll X^{n-1},$$

as X *tends off to infinity.*

Remark 6. *In Theorem 4, unlike* (2.2.1)*, we averaged over not just the cuspidal spectrum but also included the continuous spectrum. However, the proof of Theorem 4 can be modified to have an orthogonality result over a subset of the cuspidal spectrum only, by killing off the contribution from the continuous spectrum using a projector attached to a matrix coefficient of a supercuspidal representation* σ *of* $\text{PGL}_n(\mathbb{Q}_p)$*. We illustrate such an example in Theorem 11 that loosely describes a statement of the following spirit.*
Theorem 11. *If* $l, m \in \mathbb{N}^{n-1}$ *are coprime with* p *for a fixed prime* p*, then*

$$\lim_{X\to\infty} X^{1-n} \sum_{\substack{C(\pi_\infty)<X, \pi_p=\sigma \\ \pi \text{ cuspidal}}} \frac{\overline{\lambda_\pi(m)}\lambda_\pi(l)}{L^{(p)}(1, \text{Ad}, \pi)} = c_\sigma \delta_{m=l},$$

where $L^{(p)}$ *is the partial* L*-function excluding the* p*-adic Euler factor,* c_σ *is a constant depending on* σ*, and* π_p *and* π_∞ *are the* p*-adic and infinity components of an automorphic representation* π*, respectively.*

Remark 7. *We only show that the cut-off function* $J_X(\pi)$ *is non-negative over the relevant spectrum and large if the analytic conductor is bounded by* X*. Although we do not show that* $J_X(\pi)$ *is negligible if the analytic conductor is large, we expect this is true (see Remark 5).*

2.2.2 Vertical Sato-Tate

For a finite prime p and $\pi \in \hat{\mathbb{X}}_{\text{gen}}$, we denote the complex n-tuple

$$\mu^p(\pi) := (\mu_1^p(\pi), \dots, \mu_n^p(\pi)), \quad \sum_i \mu_i^p(\pi) = 0 \mod 2\pi i / \log p$$

to be the Langlands parameters attached to π at p. Langlands parameters are invariant under the action of the Weyl group W. Let T and T_0 be the standard maximal tori in $\mathrm{SL}_n(\mathbb{C})$ and $\mathrm{SU}(n)$, respectively. We identify $\mu^p(\pi)$ as an element of T/W by $\mu^p(\pi) \mapsto p^{\mu^p(\pi)} := \mathrm{diag}(p^{\mu_1^p(\pi)}, \dots, p^{\mu_n^p(\pi)})$. As a tuple, $p^{\mu^p(\pi)}$ are the *Satake parameters* attached to π at p. Let μ_{ST} be the push-forward of the Haar measure on $\mathrm{SU}(n)$ on T_0/W, which is also called the *Sato–Tate* measure attached to the group $\mathrm{GL}(n)$.

The Ramanujan conjecture at a finite prime p for $\mathrm{GL}(n)$ predicts that all the Satake parameters will be purely imaginary. In other words, $p^{\mu^p(\pi)}$ will be in T_0/W for all $\pi \in \hat{\mathbb{X}}_{\text{gen}}$. Although the Ramanujan conjecture remains open, its truth can be verified on an average as in Theorem 6. For details, we refer to [6, 63]. Theorem 6 can also be regarded as a weighted equidistribution result of the Satake parameters with respect to the Sato–Tate measure.

Theorem 6. *Let f be a continuous function on T/W. Let p be a finite prime and $p^{\mu^p(\pi)} \in T/W$ be the Satake parameters attached to π at p. Then,*

$$\frac{\int_{\hat{\mathbb{X}}_{\text{gen}}} f(p^{\mu^p(\pi)}) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi)}{\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi)} \to \int_{T_0/W} f(z) d\mu_{\mathrm{ST}}(z)$$

as $X \to \infty$.

A variant of Theorem 6 in the Laplace eigenvalue aspect that assumes the conjecture in (2.2.1) is proved as in [63] for general n. Unconditionally, the same variant is proved in [41, Theorem 10.2] for $n = 2$ and in [6, Theorem 3] for $n = 3$. In [46, Theorem 1.4] as well as in [55], the authors proved Sato–Tate equidistribution over a cuspidal spectrum without the harmonic weights $\ell(\pi)$ for $\mathrm{PGL}(r)$ and for general reductive group, respectively. In particular, both [46, 55] use the Arthur–Selberg trace formula while we use the Kuznetsov trace formula. Also, Theorem 6 can be formulated over a cuspidal spectrum following a similar technique as in Theorem 11, see Remark 6.

2.2.3 Density estimates

We denote $\theta^p(\pi) := \max_i |\Re(\mu_i^p(\pi))|$, and call π to be tempered at p, i.e., π satisfies the Ramanujan conjecture at p, if $\theta^p(\pi) = 0$. Analogous to the archimedean case, we call π to be *θ-tempered* at p, if $\theta^p(\pi) \leq \theta$. From [32], we know that $\mu^p(\pi) \in T_1/W$ for cuspidal π in $\hat{\mathbb{X}}_{\text{gen}}$, where $T_0 \subset T_1 \subset T$ is defined by the subset in T containing $\mu^p(\pi)$ with $\theta^p(\pi) \leq 1/2$ (Luo–Rudnick–Sarnak improved this bound to $1/2 - 1/(1+n^2)$, see [49]).

In [54, p.465], Sarnak conjectured that for a nice enough finite family $\mathcal{F}$ of unitary irreducible cuspidal automorphic representations for $\mathrm{PGL}_n(\mathbb{Z})$, the number of representations $\pi \in \mathcal{F}$ for which $\theta^p(\pi) > \theta$ at a fixed place p is essentially of size $|\mathcal{F}|^{1-\frac{2\theta}{r-1}}$. We refer to [2] for further motivation and details. In Theorem 7, we prove Sarnak's conjecture in the analytic conductor aspect at a finite place p for $\mathrm{PGL}_n(\mathbb{Z})$.

Theorem 7. *Let p be a fixed prime. Then,*

$$\frac{1}{X^{n-1}}\#\{\pi \text{ cuspidal} \mid C(\pi) < X,\, \theta^p(\pi) > \eta\} \ll_\epsilon X^{-2\eta+\epsilon}$$

as X tends off to infinity.

Many variants of the density estimate, as in Theorem 7, of similar strength are available in the literature. We refer to [3, Theorem 2] for $n = 3$ in the spectral parameter aspect, [6, Theorem 1, Theorem 2] for $n = 3$ in the Laplace eigenvalue aspect, [5, Theorem 4, Theorem 5] for $n = 3$ in the level aspect, and, more recently, [4, Theorem 1] in the level aspect for general n. In [46, Corollary 1.8] (also see [19], which discusses this for general reductive group), a density bound is obtained using the Arthur–Selberg trace formula for general n. However, the bound is weaker than Sarnak's density hypothesis in [54].

Remark 8. *We mention that [4, Theorem 1] proves a stronger estimate in the non-archimedean aspect than the non-archimedean variant of Sarnak's density hypothesis [54]. The analogous estimate in our setting would be $\ll_\epsilon X^{-4\eta+\epsilon}$. To obtain a stronger bound, an estimate is needed of a certain double unipotent orbital integral that arises in the geometric side of the Kuznetsov formula (as in the second term of the RHS of the equation in Proposition 4.2.1) similar to what is achieved in the non-archimedean case in [4, Theorem 3], see discussion after Theorem 2 in [4]). Such estimates might be achievable by a delicate stationary phase analysis of the unipotent orbital integrals, which is not in our grasp currently for general n and we hope to return to this in the future.*

2.2.4 Statistics of low-lying zeros and symmetry type

Strong evidence is available in support of Hilbert and Pólya's suggestion that there might be a spectral interpretation of the distribution of the zeros of the Riemann Zeta function in terms of eigenvalues of random matrices, e.g., Montgomery's pair correlation of high zeros of the Riemann zeta function. Katz and Sarnak [39] predicted that given a family of L-function for a reductive group, a symmetry type (e.g., orthogonal, unitary, or symplectic) could be associated to that family, which is given by the associated random matrix ensemble that conjecturally determines the distribution of the low-lying zeros of the L-functions in the family. We refer to [55, §1.3-§1.6] for a detailed overview with motivations.

We start by assuming *Langands strong functoriality principle*, as in [55, Hypothesis 10.1]. Note that, Langlands L-group of G is $\mathrm{SL}_r(\mathbb{C})$.

Conjecture 1 (Langlands Functoriality). *For every $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ and every d-dimensional L-homomorphism $\rho : \mathrm{SL}_r(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$, there exists an irreducible automorphic representation $\rho_*\pi$ for $\mathrm{GL}_d(\mathbb{Z})$, called the functorial lift of π under ρ, such that*

$$\Lambda(s, \rho_*\pi, \mathrm{Standard}) = \Lambda(s, \pi, \rho),$$

and for every $p \leq \infty$ the p-component $(\rho_*\pi)_p = \rho_*^p \pi_p$, where ρ_*^p in the RHS denotes the induced functorial transfer for the local representations at p.

To exclude the case of ρ being trivial, we also assume that $d > 1$. It is known that Conjecture 1 implies the *Generalized Ramanujan conjecture* for G.

We define the *average conductor* $C_{\rho,X}$ of the family of automorphic representations $\pi \in \hat{\mathbb{X}}$ with analytic conductor $C(\pi) < X$ by the equation

$$\log C_{\rho,X} \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) := \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \log C(\rho_*\pi) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi).$$

Let ψ be the Fourier transform of a smooth function on $\mathbb{R}$ supported on the interval $[-\delta, \delta]$ for some $\delta > 0$. Then, ψ is a Paley–Wiener type (or Schwartz class) function.

We write the zeros of $\Lambda(s, \pi, \rho)$ in the critical strip, i.e., $\Re(s) \in [0, 1]$, as $1/2 + i\gamma_{\rho_*\pi}$, i.e., under the GRH we have $\gamma_{\rho_*\pi} \in \mathbb{R}$. The *low-lying zeros* of Λ are the zeros with $\gamma_{\rho_*\pi}$ bounded by $(\log C_{\rho,X})^{-1}$. We define a *weighted* 1*-level density* statistic $D_{\rho,X}$ for low-lying zeros of $\Lambda(s, \rho_*\pi)$ with $C(\pi) < X$ by the equation

$$D_{\rho,X}(\psi) \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) := \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \sum_{\gamma_{\rho_*\pi}} \psi\left(\gamma_{\rho_*\pi} \frac{\log C_{\rho,X}}{2\pi}\right) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi). \quad (2.2.2)$$

Similarly, we denote the poles of $\Lambda(s, \rho_*\pi)$ in the critical strip by $1/2 + i\tau_{\rho_*\pi}$, and define

$$D^{\mathrm{pole}}_{\rho,X}(\psi) \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) := \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \sum_{\tau_{\rho_*\pi}} \psi\left(\tau_{\rho_*\pi} \frac{\log C_{\rho,X}}{2\pi}\right) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi). \quad (2.2.3)$$

From Katz–Sarnak's [39] random matrix heuristics about the distribution of the zeros of Λ, a limiting 1-level density can be predicted for the low-lying zeros. That is,

$$\lim_{X \to \infty} D_{\rho,X}(\psi) = \int_{\mathbb{R}} \psi(x) \mathfrak{W}(x) dx,$$

where $\mathfrak{W}$ is determined by the *Frobenius–Schur indicator* $\mathfrak{s}(\rho)$ of the representation ρ. In particular, a family is (even) orthogonal, (unitary) symplectic, or unitary if the value of $\mathfrak{s}$ is $-1, 1$, or 0, respectively. Shin–Templier [55, Theorem 1.5] showed that this prediction is true for certain families of automorphic L-functions in the level and weight aspects for a general reductive group, assuming some hypotheses on the size of the average conductor and the number of poles of the L-function. In Theorem 8, we prove a weighted version of [55, Theorem 1.5] for the group G, assuming similar hypotheses as in [55], as described below.

We make a conjecture about the size of the average conductor as in [55, Hypothesis 11.4], namely that $\log C_{\rho,X} \asymp_\rho \log X$.

Conjecture 2. *For all L-homomorphisms ρ, there exist nonnegative constants $\mathfrak{c}(\rho)$ and $\mathfrak{C}(\rho)$, such that*

$$X^{\mathfrak{c}(\rho)} \ll C_{\rho,X} \ll X^{\mathfrak{C}(\rho)}$$

for all large enough $X > 1$.

Conjecture 2 should not be difficult to establish upon an asymptotic expansion of $J_X(\pi)$ for various ranges of $C(\pi)$ (see Remark 5) and using local Langlands correspondence for G to describe the archimedean L-parameters of $\rho_*\pi$. Heuristically, at least when ρ is the standard representation, we expect

$$\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \log C(\pi) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) \approx \log X \int_{C(\pi)\asymp X} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) \asymp X^{r-1} \log X.$$

For the general d-dimensional representation ρ, proving a trivial bound $C(\rho_*\pi) \leq C(\pi)^d$ is straightforward (however, this bound is far from being sharp when $\rho_*\pi$ has a conductor drop). Correspondingly, a crude bound of $\mathfrak{C}(\rho)$ may be obtained as discussed in the heuristic above.

We assume Langlands functoriality in Conjectures 1 and 2 about the size of the average conductor and state the following weighted version of the density estimate of the low-lying zeros of $\Lambda(s, \pi, \rho)$ over the family $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ with $C(\pi) < X$.

Theorem 8. *Let ρ be a d-dimensional irreducible representation of ${}^LG = \mathrm{SL}_r(\mathbb{C})$ with highest weight $\theta := (\theta_1, \dots, \theta_r)$ that is a dominant element in $\mathbb{Z}^r/\mathbb{Z}(1, \dots, 1)$. Let $\mathfrak{s}(\rho)$ and $\mathfrak{C}(\rho)$ be as in the above discussion. Let ψ be a Schwartz class function with Fourier transform $\hat{\psi}$ that is supported on $[-\delta, \delta]$. Then,*

$$D_{\rho,X}(\psi) - D^{\mathrm{pole}}_{\rho,X}(\psi) = \hat{\psi}(0) - \frac{\mathfrak{s}(\rho)}{2}\psi(0) + O\left(\frac{1}{\log X}\right)$$

for all $\delta < \frac{1}{\mathfrak{C}(\rho)(\theta_1 - \theta_r)}$.

Remark 9. *The D^{pole} term in Theorem 8 typically should be negligible. As shown in [55], the corresponding D^{pole} term is negligible upon assuming [55, Hypothesis 11.2], which heuristically predicts that $\Lambda(s, \rho_*\pi)$ is entire for* most *π. Remark 11 provides additional details.*

Remark 10. *We use the Ramanujan Conjecture in full strength, which is implied by Langlands functoriality principle in Conjecture 1), in the proof. However, we only need a bound towards Ramanujan. Thus, alternatively, functoriality can be assumed for, not all ρ, but a subclass of these and prove a bound towards Ramanujan to prove a stronger version of Theorem 8.*

Apart form [55], a few results exist on the distribution of the zeros of various families of L-functions for $r = 2$ in the weight and level aspect, e.g., [29, 62], for $r = 3$ in the Laplace eigenvalue aspect in [23], and for general r in the dilated Plancherel ball aspect in [46].

2.2.5 A large sieve inequality

In the next application, we prove a large sieve inequality in the analytic conductor aspect for the Fourier coefficients of $\mathrm{PGL}(n)$. This result is in the spirit of the celebrated large sieve inequalities in [17] in the spectral parameter aspect for $n = 2$.

Theorem 9. *Let all the notations be as in Theorem 4. Let $\alpha(m)_{m\in\mathbb{N}^{n-1}}$ be any sequence of complex numbers and $M := m_1 \dots m_{n-1}$. Then,*

$$\sum_{\substack{C(\pi)<X\\ \pi \text{ cuspidal}}} L(1,\pi,\mathrm{Ad})^{-1} \left| \sum_{M\ll X} \alpha(m)\lambda_\pi(m) \right|^2 \ll X^{n-1} \sum_{M\ll X} |\alpha(m)|^2 .$$

Here, the implicit constant in the condition $M \ll X$ is assumed to be sufficiently small.

Theorem 9 is of similar strength as [4, Theorem 4] in the level aspect. We also mention previous works on large Sieve inequalities in [3, Theorem 3] in the spectral parameter aspect, [5, Theorem 2, Theorem 3] in level aspect for $n = 3$, and [18, 59] in the level aspect for general n.

2.2.6 Second moment of the central L-values

Finally, we provide a corollary of Theorem 9 to the best possible, i.e., Lindelöf on average, second moment estimate of the central L-values.

Theorem 10. *Let $L(s,\pi)$ be the automorphic L-function attached to π. Then,*

$$\sum_{\substack{C(\pi)<X\\ \pi \text{ cuspidal}}} \frac{|L(1/2,\pi)|^2}{L(1,\pi,\mathrm{Ad})} \ll_\epsilon X^{n-1+\epsilon}$$

as X tends off to infinity.

A level aspect variant of Theorem 10 hasrecently been proved in [4, Corollary 5]. The harmonic weight $L(1,\mathrm{Ad},\pi)$ can be eliminated in the above second moment estimate (similarly, also in Theorem 5 and Theorem 9) by using an upper bound of $L(1,\mathrm{Ad},\pi)$, as in (6.4.4) from [44].

Chapter 3

Sketches and Discussions

3.1 Sketch for the proof of Theorem 3

In this section, we assume that Π is tempered (instead of θ-tempered) and has trivial central character. We construct the vector $V \in \Pi$ by specifying that it be given by a fixed bump function in the Kirillov model (see (5.1.1)). The key step in the proof of Theorem 3 is to verify that $V(g)$ approximates $V(1)$ for all $g \in K_0(C(\Pi), \tau)$ with τ small, from which the remaining assertions are deduced fairly easily. The main difficulties in the proof are present in the special case that g is lower-triangular unipotent, so for the purposes of this sketch, we restrict to this case. Proving the following quantitative refinement of the conclusion of Theorem 3 will be sufficient: for all small $1 \times n$ row vectors c,

$$V\left[\begin{pmatrix} 1 & \\ c/C(\Pi) & 1 \end{pmatrix}\right] - V(1) \ll |c|. \tag{3.1.1}$$

From this, we first expand the LHS of (3.1.1) using the Whittaker–Plancherel formula (4.1.2). We next apply the $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ local functional equation (4.1.4) and attempt to analyze the resulting integral. We indicate this progress first in the simplest case $n = 1$, and then describe the modifications necessary for general n, along with the technical difficulties in these cases.

3.1.1 Proof for $n = 1$

In this case, Π is a representation of $\mathrm{GL}_2(\mathbb{R})$. We define the Whittaker model $\mathcal{W}(\Pi)$ using the additive character of the unipotent radical in $\mathrm{GL}_2(\mathbb{R})$ defined as in (4.1.1). We recall that, by the theory of the Kirillov model, there is for each $f \in C_c^\infty(\mathbb{R}^\times)$ a unique element $V \in \mathcal{W}(\pi)$ of the Whittaker model of Π for which

$$V\left[\begin{pmatrix} y & \\ & 1 \end{pmatrix}\right] = f(y) \tag{3.1.2}$$

for all $y \in \mathbb{R}^\times$. We recall the local functional equation (4.1.5) for $\mathrm{GL}_2 \times \mathrm{GL}_1$: for $s \in \mathbb{C}$, the zeta integral

$$\int_{\mathbb{R}^\times} V\left[\begin{pmatrix} t & \\ & 1 \end{pmatrix}\right] |t|^s \, d^\times t$$

converges absolutely for $\Re(s) > -1/2$ and extends to a meromorphic function on the complex plane, where it satisfies the relation

$$\int_{\mathbb{R}^\times} V\left[\begin{pmatrix} t & \\ & 1 \end{pmatrix}\right] |t|^s \, d^\times t = \frac{1}{\gamma(\Pi, 1/2+s)} \int_{\mathbb{R}^\times} V\left[\begin{pmatrix} 1 & \\ & t \end{pmatrix} w\right] |t|^s \, d^\times t.$$

Here $w := \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ denotes the Weyl element and γ the local γ-factor, with properties we recall in greater detail in §4.1.5. The meromorphic function

$$\Theta(s, \Pi) := \frac{C(\Pi)^{-s}}{\gamma(1/2+s, \Pi)}$$

is holomorphic for $\Re(s) > -1/2$ and non-vanishing for $\Re(s) < 1/2$. The normalization is such that $\Theta(s, \Pi)$ is approximately of size 1 for bounded s, uniformly in Π. More precisely, we have

$$(1 + |\Im(s)|)^{-2|\Re(s)|} \ll \Theta(s, \Pi) \ll (1 + |\Im(s)|)^{2|\Re(s)|} \tag{3.1.3}$$

for s of the bounded real part and a fixed positive distance away from the poles of $\Theta(s, \Pi)$, and with the implied constant uniform in Π (see Lemma 4.1.1).

Let us assume in this sketch that Π belongs to the discrete series, so that in the Kirillov model of Π, the subspace of functions that vanish off the group $\mathbb{R}_+^\times$ of positive reals is invariant by the group of positive-determinant elements of $\mathrm{GL}_2(\mathbb{R})$, which allows us to simplify the exposition slightly because the character group of $\mathbb{R}_+^\times$ is slightly simpler than that of $\mathbb{R}^\times$.

We fix a test function $f \in C_c^\infty(\mathbb{R}_+^\times)$ satisfying the normalization $\|f\|_2 = 1$. We extend f by zero to an element of $C_c^\infty(\mathbb{R}^\times)$. We construct V using the theory of the Kirillov model by requiring that (3.1.2) holds for this choice of f. The goal, then, is to verify the estimate (3.1.1). We first apply the Mellin inversion, giving for any $c \in \mathbb{R}$ the identity

$$V\left[\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right] = \int_{(0)} \left(\int_{t \in \mathbb{R}_+^\times} V\left[\begin{pmatrix} t & \\ & 1 \end{pmatrix} \begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right] |t|^s \, d^\times t \right) ds.$$

Here and in the rest of the thesis, we use the $2\pi i$-normalized Lebesgue measure (e.g., ds) on any vertical line in $\mathbb{C}$. We then apply the local functional equation to the inner integral, and after some matrix multiplication and appeal to the left-N-equivariance of V, we obtain

$$V\left[\begin{pmatrix} 1 & \\ c & 1 \end{pmatrix}\right] = \int_{(0)} \frac{1}{\gamma(\Pi, 1/2+s)} \left(\int_{t \in \mathbb{R}_+^\times} e(-c/t) V\left[\begin{pmatrix} 1 & \\ & t \end{pmatrix} w\right] |t|^s \, d^\times t \right) ds.$$

We next substitute $c \mapsto c/C(\Pi)$, apply the change of variables $t \mapsto t/C(\Pi)$, and subtract the corresponding identity for $c = 0$, resulting in

$$V\left[\begin{pmatrix} 1 & \\ \frac{c}{C(\Pi)} & 1 \end{pmatrix}\right] - V(1) = \int_{(0)} \Theta(s,\Pi) \int_{t\in\mathbb{R}_+^\times} (e(-c/t) - 1) \times V\left[\begin{pmatrix} C(\Pi) & \\ & t \end{pmatrix} w\right] |t|^s \, d^\times t \, ds.$$

We now claim that if t is small, then $V\left[\begin{pmatrix} C(\Pi) & \\ & t \end{pmatrix} w\right]$ is negligible. More precisely, we claim that for any fixed integers $M, N \geq 0$,

$$(t\partial_t)^N V\left[\begin{pmatrix} C(\Pi) & \\ & t \end{pmatrix} w\right] \ll \min(1, t^M). \tag{3.1.4}$$

From this, it follows that $e(-c/t) \approx 1$ on the "essential support" of the inner integral eventually leads to the required estimate (3.1.1).

We focus on the case $N = 0$ of the claim, and suppose that t is small; we must show that

$$V\left[\begin{pmatrix} C(\Pi) & \\ & t \end{pmatrix} w\right] \ll t^M \tag{3.1.5}$$

for any fixed M. As before, we once again apply a Mellin inversion and appeal to the local functional equation, which gives

$$V\left[\begin{pmatrix} C(\Pi) & \\ & t \end{pmatrix} w\right] = \int_{(0)} t^s \Theta(s,\Pi) \tilde{f}(s) \, ds, \tag{3.1.6}$$

with $\tilde{f}(s) := \int_{t\in\mathbb{R}_+^\times} f(t) t^{-s} \, d^\times t$. By the construction of f, the Mellin transform $\tilde{f}$ is entire and of rapid decay in vertical strips. The crux of the argument is to now shift the integration in (3.1.6) to the line $\Re(s) = M$ for some fixed, large, and positive M; the properties of Θ summarized above imply that

$$\Theta(s,\Pi)\tilde{f}(s) \ll |s|^{-2}, \tag{3.1.7}$$

say, for such s, which leads to the required estimate (3.1.5).

In summary, the proof of the case $n = 1$ follows readily from two applications of the local functional equation and a straightforward Paley–Wiener type analysis of the Mellin integral representation (3.1.6).

3.1.2 Difficulties in generalizing to $n \geq 2$

We choose V in a similar manner to the $n = 1$ case, such that $V\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right]$ is given by a unipotent equivariant bump function on $\mathrm{GL}_n(\mathbb{R})$ (see (5.1.1) for details) and show, as before, that for small $1 \times n$ row vectors c,

$$V\left[\begin{pmatrix} 1 & \\ c/C(\Pi) & 1 \end{pmatrix}\right] - V(1) \ll |c|,$$

where the matrix entries are written in the evident block form. We first appeal to the local functional equation, much like in the $n = 1$ case, reducing in this way to proving estimates slightly more general than the following generalization of (3.1.5) (see Proposition 5.2.1 for details): if $a = \mathrm{diag}(a_1, \dots, a_n)$ is a diagonal matrix with positive entries and a_1 is small, then

$$V\left[\begin{pmatrix} C(\Pi) & \\ & a \end{pmatrix} w\right] \ll_N \delta^{1/2}(a) a_1^N. \tag{3.1.8}$$

Here, $\delta(a) := \prod_{j<k} |a_j/a_k|$ is the modular character of the upper-triangular Borel in GL_n. Some similarities exist between the present task and what is accomplished in the non-archimedean analogue by [30, §5 Lemme].

For the proof of (3.1.8), we begin as in the $n = 1$ case by expanding the function $\mathrm{GL}_n(\mathbb{R}) \ni h \mapsto V\left[\begin{pmatrix} C(\Pi) & \\ & h \end{pmatrix} w\right]$ using the Whittaker–Plancherel formula for $\mathrm{GL}_n(\mathbb{R})$ and applying the local functional equation for $\mathrm{GL}_{n+1} \times \mathrm{GL}_n$ (see (5.3.5)). We arrive in this way at the following generalization of (3.1.6) with

$$V\left[\begin{pmatrix} C(\Pi) & \\ & a \end{pmatrix} w\right] = \int_{(0)^n} W_\mu(a) \Theta(\mu, \Pi) \langle f, W_\mu \rangle \frac{d\mu}{|c(\mu)|^2}. \tag{3.1.9}$$

Here,

- $\int_{(0)^n}$ denotes an integral over $\mu \in \mathbb{C}^n$ with $\Re(\mu_1) = \cdots = \Re(\mu_n) = 0$. Here (and elsewhere in this paper), $d\mu$ denotes $\prod_i d\mu_i$ where $d\mu_i$ is the Lebesgue measure on the vertical line $\Re(s) = 0$ normalized by $2\pi i$.
- W_μ is the spherical Whittaker function normalized so that $W_\mu(1) \asymp 1$ (see 4.1.12 for details).
- $\Theta(\mu, \Pi) := C(\Pi)^{-\mu_1 - \cdots - \mu_n} \gamma(\Pi \otimes \tilde{\pi}_\mu, 1/2)$ is holomorphic for $\Re(\mu_i) > -1/2$ and has properties analogous to those of $\Theta(s, \mu)$ mentioned above in the $n = 1$ case, and
- $c(\mu)$ is a product of Γ-functions, related to the Plancherel density.
- $\langle f, W_\mu \rangle := \int_{N \backslash \mathrm{GL}_n(\mathbb{R})} f(g) \overline{W_\mu(g)} dg$, where N is the unipotent subgroup of the upper triangular matrices in $\mathrm{GL}_n(\mathbb{R})$.

The product $\Theta(\mu, \Pi) \langle f, W_\mu \rangle$ features strong estimates analogous to (3.1.7) as it extends to a meromorphic function in μ, holomorphic in $\Re(\mu_i) > -1/2$ and of rapid decay in vertical strips, and uniformly in Π.

Similar to the standard proof of any Paley–Wiener-type statement, we use the rapid decay of $\langle f, W_\mu \rangle$ in μ to ensure the convergence of the integral (3.1.9), and the source of the decay in the a_1 direction (i.e., as $a_1 \to 0$, as required) is contour shifts. However, the Paley–Wiener-type argument, as simple as it is in the $n = 1$ case, cannot be directly applied in this case because any shift of the μ contour that avoids polar hyperplanes of $\Theta(\mu, \Pi)$ is insufficient to achieve the required bound $\ll \min(1, a_1^N)$.

The reason for this obstruction is that W_μ has asymptotic expansion of the form

$$\delta^{1/2}(a) \sum_{w \in S_n} a^{w\mu} M_{w\mu}(a), \quad a^\mu = \prod_{i=1}^{n} a_i^{\mu_i}.$$

Here, S_n is the Weyl group of $\mathrm{GL}(n)$ and $M(\mu, a)$, which we informally call an M-Whittaker function, is an infinite series of the form

$$\sum_{k \in Z_{\geq 0}^{n-1}} c_\mu(k) \prod_{i=1}^{n-1} (a_i/a_{i+1})^{2k_i}.$$

For example, on GL(2), the spherical Whittaker function $W_\mu(a)$ is given by the K-Bessel function as

$$\begin{aligned} &(a_1/a_2)^{1/2}(a_1a_2)^{(\mu_1+\mu_2)/2} K_{(\mu_1-\mu_2)/2}(2\pi a_1/a_2) \\ &= a_1^{\mu_1+1/2} a_2^{\mu_2-1/2} \sum_{k=0}^{\infty} c_\mu(k)(a_1/a_2)^{2k} + a_1^{\mu_2+1/2} a_2^{\mu_1-1/2} \sum_{k=0}^{\infty} d_\mu(k)(a_1/a_2)^{2k}. \end{aligned}$$

for some complex coefficients $c_\mu(k)$ and $d_\mu(k)$. The infinite sums are essentially I-Bessel functions, and are exponentially increasing in a_1/a_2. If we shift the contour of μ_1 to right side. e.g.. to $\Re(\mu_1) = N$ for some large positive N, we do not cross any pole of $\Theta(\mu, \Pi)$. If a_2 is very large compared to a_1 (e.g., $a_2 = a_1^{-1}$ and $a_1 \to 0$), then this contour shift does not yield the required bound $\ll a_1^N$. That is why we first need to decompose the Whittaker functions into finitely many M-Whittaker functions (see Lemma 5.3.6), and for each summand M-Whittaker function we shift the contour to the relevant direction and obtain the required bound. For example, in this case of the first summand, we shift μ_1 and in the second, we shift μ_2 to the right side.

However, one technical issue remains. Each M-Whittaker function, like the I-Bessel functions, is exponentially increasing in the positive roots (e.g., for GL(2) it exponentially increases in a_1/a_2), which is why we can effectively apply the technique of contour shifting only when the positive roots are bounded. For example, on GL(2), we decompose $W_\mu(\mathrm{diag}(a_1, a_2))$ into relevant an$M$-Whittaker function and shift the contour only when $a_1 < a_2$, as in the discussion above. On the other hand, in GL(2), the Whittaker function $W_\mu(a)$ decays rapidly as $a_2/a_1 \to 0$. Such decay is enough to treat the complimentary case $a_1 > a_2$.

Considering that we should decompose only when a is in the positive Weyl chamber is tempting. If a does not lie in a positive Weyl chamber, then at least one root is large, and we may expect that the rapid decay of the Whittaker function will save the day.

Although this works when $n = 2$, it fails for general n. For example, there are diagonal elements in $\mathrm{GL}(n)$ that barely fail to be in a positive Weyl chamber. In $\mathrm{GL}(3)$, the element $Y := \mathrm{diag}(y, -y/\log y, e^{1/y})$ as $y \to 0$ logarithmically fails to be in the positive chamber. For this element, the rapid decay estimate of the Whittaker function yields only a logarithmic decay

$$W(Y) \ll |\log y|^{-N},$$

not a polynomial decay $\ll y^N$, so does not meet our requirement.

To deal with this issue, we need to treat the elements like Y as if they are in the positive chamber. We do this by dividing the set of diagonal matrices into two classes whether they satisfy a property **pop** or not (see Definition 5.3.1). The **pop** refers to if the tuple $(a_1, \ldots, a_n)$ of a diagonal element $a = \mathrm{diag}(a_1, \ldots, a_n)$ has a partial ordering of the form *all of* $a_1, \ldots, a_s$ *are smaller than all of* $a_{s+1}, \ldots, a_n$. For instance, the element Y above has **pop** property for $s = 2$, as $y \to 0$.

In Lemma 5.3.1 we demonstrated that for the elements a that do not satisfy **pop**, the rapid decay of the Whittaker function implies the required bound. The remainder of the section §5.3 is devoted to the case when a satisfies **pop**(s) for some s. In this case, we decompose the Whittaker function W into the M-Whittaker functions. However, we cannot do a *full* decomposition, as described above in the $\mathrm{GL}(2)$ case, because a may not lie in the positive Weyl chamber and M might exponentially blow up. To make sure we have control on the exponential increment, we only *partially* decompose, so that the M-Whittaker functions have an exponential increment only in the roots of the form a_i/a_j with $1 \le i \le s$ and $s+1 \le j \le n$. Therefore, M does not blow up as a satisfies **pop**(s). Loosely speaking, a partial decomposition corresponds to the Levi in $\mathrm{GL}(n)$ attached to the partition of n of the form $n = s + 1 + \cdots + 1$, and a full decomposition is the same with $s = 1$. Such full decompositions of the spherical Whittaker function have appeared in the literature, such as in [11, 26].

3.1.3 Sketch for the proof of $\mathrm{PGL}_{n+1}(F)$, where F is non-archimedean

We re-establish the invariance result [30, §5 Théoréme] along the same lines we proved in the archimedean case, but [30] instead proved it using the test vector property of the newvector. However, the proof in the p-adic case remains easier due to the existence of more straightforward explicit algebraic formulas. Also, the corresponding M-Whittaker functions do not exponentially increase in the non-archimedean case.

We only concentrate on proving the analogue of the crucial part, Proposition 5.2.1. Let F be a non-archimedean local field with ring of integers $\mathfrak{o}$ and uniformizer ϖ. Let Π be a generic irreducible tempered unitary representation of $\mathrm{GL}_{n+1}(F)$. Let V be the vector in $\mathcal{W}\left(\Pi, \tilde{\psi}\right)$ given in the Kirillov model by

$$V\left[\begin{pmatrix} nak & \\ & 1 \end{pmatrix}\right] = \psi(n)\mathrm{char}_{\mathfrak{o}^\times}(a), \tag{3.1.10}$$

where $n, a := \mathrm{diag}(a_1, \ldots, a_n)$ are the unipotent and diagonal elements, respectively, and

$k \in \mathrm{GL}_n(\mathfrak{o})$.

Proposition 3.1.1. *Let w be the long Weyl element in* $\mathrm{GL}(n+1)$*. Then,*

$$V\left[\begin{pmatrix} C(\Pi) & \\ & a \end{pmatrix} w\right] \neq 0 \implies |a_1| \gg 1.$$

We emphasize the similarity of this proposition with [30, §5 Lemme]. For simplicity, in the sketch of the proof, we assume that Π is supercuspidal.

Sketch of the proof. Our point of departure is, as in the archimedean case, the p-adic Kontorovich-Lebedev-Whittaker transform (for a proof in the case of $F = \mathbb{Q}_p$, see [25], and for the general Whittaker–Plancherel formula, we refer to [16]) and $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$ local functional equation. We obtain

$$V\left[\begin{pmatrix} C(\Pi) & \\ & a \end{pmatrix} w\right] = \int_\pi W_\pi(a)\gamma(1/2, \Pi\otimes\bar{\pi})\omega_\pi^{-1}(C(\Pi))\langle \mathrm{char}_{\mathfrak{o}^\times}, W_\pi\rangle d\mu_p(\pi).$$

Here, π runs over the spherical tempered dual of $\mathrm{GL}_n(F)$, and $d\mu_p$ is the Plancherel measure on it. Let $m \in \mathbb{Z}^n$ and $a = \mathrm{diag}(\varpi^m)$, i.e., $a_i = \varpi^{m_i}$. Let $\alpha \in (S^1)^n$ be the Langlands parameters of π, and W_π is the spherical Whittaker function of π described by Shintani's formula [56] below.

$$W_\pi(a) = \begin{cases} \delta^{1/2}(a)\dfrac{\det\left((\alpha_j^{m_i+n-i})_{i,j}\right)}{\prod_{i<j}(\alpha_i-\alpha_j)}, \text{ if } m_1 \geq \ldots \geq m_n, \\ 0, \text{ if otherwise.} \end{cases} \tag{3.1.11}$$

Thus, we may restrict a to be of the form $\mathrm{diag}(\varpi^m)$ with $m_1 \geq \ldots \geq m_n$. Inserting this formula for W_π and explicating the Plancherel density (see [25]), we rewrite $V\left[\begin{pmatrix} C(\Pi) & \\ & a \end{pmatrix} w\right]$ as

$$\delta^{1/2}(a)\int_{(S^1)^n} \det\left((\alpha_j^{m_i+n-i-1})_{i,j}\right)\gamma(1/2, \Pi\otimes\bar{\pi})\omega_\pi^{-1}(C(\Pi))\prod_{i>j}(\alpha_i-\alpha_j)d\alpha. \tag{3.1.12}$$

As Π is supercuspidal and π is unitary unramified, $\gamma(s, \Pi\otimes\bar{\pi})$ is the same as $\epsilon(s, \Pi\otimes\bar{\pi})$ and

$$\omega_\pi^{-1}(C(\Pi))\epsilon(1/2, \Pi\otimes\bar{\pi})$$

is independent of π as well as bounded. To see this, recall that if π has Langlands parameters $\{\alpha_i\}_{i=1}^n$ with $\alpha_i := p^{\beta_i}$, then

$$\epsilon(1/2, \Pi\otimes\bar{\pi}) = \prod_{i=1}^n \epsilon(1/2-\beta_i, \Pi).$$

From (1.1.3), we obtain

$$\epsilon(1/2-\beta_i,\Pi)=C(\Pi)^{\beta_i}\epsilon(1/2,\Pi).$$

From the above formula and the unitarity of $\epsilon(1/2,\Pi)$, the claim is immediate.

We next proceed along with the sketch we provided in the previous subsection for the archimedean case. The analogue of "decomposing the spherical Whittaker function into finitely many M-Whittaker function" is expanding the determinant $\det\Big((\alpha_j^{m_i+n-i})_{i,j}\Big)$ into various monomials depending on α_i, i.e., the analogue of the M-Whittaker function is a monomial of the form $\prod_j \alpha_j^{n_j}$. We distribute the integral over this decomposition, which includes a generic term that looks like

$$\delta^{1/2}(a)\int_{(S^1)^n}\alpha_j^{m_1}H_j(\alpha)d\alpha,$$

where $H_j(\alpha)$ is a meromorphic function in $\{\alpha \mid |\alpha_j|\leq 1\}$ with poles at most at $\alpha_i=0$, such that the order of the pole at $\alpha_j=0$ is bounded (i.e., does not depend on m). Thus, by making m_1 sufficiently positive, we compute the α_j integral to be zero. Hence, we conclude. □

Chapter 4

Basic Notations and Background

In the section, we develop some basic notations and recall well-known tools from representation theory of $\mathrm{GL}_n(\mathbb{R})$, which we describe in the following subsections. We use π to denote a local representation of $\mathrm{GL}_n(\mathbb{R})$ and also an automorphic representation for $\mathrm{PGL}_n(\mathbb{Z})$, which should be clear from the contexts.

4.1 Local preliminaries

Let $G := \mathrm{GL}_n(\mathbb{R})$. We use the Iwasawa decomposition of $G = NAK$ with N being the maximal unipotent subgroup of upper triangular matrices, A being the subgroup of the positive diagonal matrices, and $K = \mathrm{O}(n)$. We fix the Haar measures dg, dn, dk on G, N, K, respectively, such that the volume of K is one with respect to dk, and

$$dg = dn\frac{da}{\delta(a)}dk, \quad g = nak,$$

where da on $A \ni \mathrm{diag}(a_1, \dots a_n)$ is given by $\prod_i d^\times a_i$ and δ is the modular character on NA. We use similar Haar measures on $\mathrm{GL}(r)$ (and its subgroups) for any r without mentioning this explicitly.

We next introduce a modified Vinogradov notation. Let $\epsilon > 0$ be a fixed small quantity (say, $< n^{-10}$). In this thesis, we abbreviate the inequality

$$\varphi_1(a, \dots) \ll_{\epsilon,\dots} \varphi_2(a, \dots) \prod_{i=1}^{n} (a_i + a_i^{-1})^\epsilon$$

by

$$\varphi_1(a, \dots) \prec_{\dots} \varphi_2(a, \dots),$$

where φ_i are some functions on $a, \dots$.

4.1.1 Additive character

Recall the maximal unipotent $N < G$ of the upper triangular matrices. We define an additive character ψ_α of N for $\alpha \in \mathbb{R}^{n-1}$, given by

$$\psi_\alpha(n(x)) = e\left(\sum_{i=1}^{n-1} \alpha_i x_{i,i+1}\right), \quad n(x) := (x_{i,j}) \in N, \tag{4.1.1}$$

where $e(z) := \exp(2\pi i z)$. For $\alpha := (1, \dots 1)$, we abbreviate ψ_α by ψ. By $\tilde{\psi}$, we denote the similarly defined character of N_{n+1}, which is the maximal unipotent subgroup of $\mathrm{GL}_{n+1}(\mathbb{R})$.

4.1.2 Whittaker and Kirillov models

For details of this subsection, we refer to [34, Chapter 3]. Let Π be a generic irreducible unitary representation of $\mathrm{GL}_{n+1}(\mathbb{R})$. Let $\tilde{\psi}$ be the character of $N_{n+1} < \mathrm{GL}_{n+1}(\mathbb{R})$, as in the previous subsection. Recall that Π is generic if

$$\mathrm{Hom}_{\mathrm{GL}_{n+1}(\mathbb{R})}(\Pi, \mathrm{Ind}_{N_{n+1}}^{\mathrm{GL}_{n+1}(\mathbb{R})} \tilde{\psi}) \neq 0.$$

If Π is generic, then the above space is one dimensional. Let λ be a nonzero element in this Hom-space. Then, the Whittaker model $\mathcal{W}\left(\Pi, \tilde{\psi}\right)$ of Π is the image of λ. One writes

$$W_v(g) = \lambda(\Pi(g)v), \quad g \in \mathrm{GL}_{n+1}(\mathbb{R}), v \in \Pi.$$

If Π is unitary, then the corresponding unitary structure on $\mathcal{W}\left(\Pi, \tilde{\psi}\right)$ is given by

$$\langle W_1, W_2 \rangle_{\mathcal{W}(\Pi, \tilde{\psi})} = \int_{N\backslash G} W_1\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] \overline{W_2\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right]} dg,$$

for $W_1, W_2 \in \mathcal{W}\left(\Pi, \tilde{\psi}\right)$.

Let $C_c^\infty(N\backslash G, \psi) \ni \phi$ be the set of smooth functions on G compactly supported mod N, such that

$$\phi(ng) = \psi(n)\phi(g), \quad n \in N, g \in G.$$

The theory of the Kirillov model states that [34, Proposition 5] there exists a unique $W_\phi \in \mathcal{W}\left(\Pi, \tilde{\psi}\right)$ such that

$$W_\phi\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] = \phi(g),$$

and the map $\phi \mapsto W_\phi$ is continuous.

4.1.3 Langlands parameters

For a local representation π of $\mathrm{GL}_m(\mathbb{R})$, let us write its L-factor

$$L(s,\pi)=\prod_{i=1}^{m}\Gamma_{\mathbb{R}}(s+\mu_i(\pi)),$$

where $\mu_i(\pi)\in\mathbb{C}$ are the Langlands Parameters attached to π. In this case, we define the *analytic conductor* of π to be

$$C(\pi)=\prod_{i=1}^{m}(1+|\mu_i(\pi)|).$$

By μ we will denote a complex n-tuple $(\mu_1,\dots,\mu_n)$. We define a quantity

$$c(s,\mu):=\prod_{i<j}\Gamma_{\mathbb{R}}\left(s+\mu_i-\mu_j\right).$$

We abbreviate $c(0,\mu)$ as $c(\mu)$.

4.1.4 Whittaker–Plancherel formula

For a general discussion on the Whittaker–Plancherel theorem, we refer to [61, Chapter 15]. Let $\hat{G}$ be the set of isomorphism classes of the generic irreducible tempered unitary representations of G. Let $\hat{G}_0\subseteq\hat{G}$ be the isomorphism classes of spherical representations, which are representations that contain a right K-invariant vector. The general Whittaker–Plancherel formula for G can be written as follows. Let $F\in L^2(N\backslash G,\psi)$ be continuous, then

$$F(g)=\int_{\hat{G}}\sum_{W\in\mathcal{B}(\pi)}W(g)\langle F,W\rangle d\mu_p(\pi), \tag{4.1.2}$$

where $d\mu_p$ is the Plancherel measure on $\hat{G}$, and

$$\langle F,W\rangle:=\int_{N\backslash G}F(g)\overline{W(g)}dg.$$

Here, $\mathcal{B}(\pi)$ is an orthonormal basis of π, and the above sum does not depend on a choice of $\mathcal{B}(\pi)$.

We may choose a basis

$$\mathcal{B}(\pi):=\cup_{\tau\in\hat{K}}\{W_\tau^i\mid 1\le i\le n_\tau\}$$

consisting of K-isotypic vectors. Here, $\{W_\tau^i\}_{i=1}^{n_\tau}$ is an orthonormal basis of τ-type. We also know that $n_\tau=1$ if τ is the trivial representation. We now produce a simplified version of the Whittaker–Plancherel formula for spherical functions, i.e., functions that are right K-invariant. First, if $F \in L^2(N\backslash G, \psi)^K$, then

$$\langle F, W\rangle = 0, \quad \text{for all } W \in \mathcal{B}(\pi) \setminus \pi^K.$$

Therefore, for spherical F, only the spherical representations contribute to the right hand side of (4.1.2). Let $W_\pi \in \pi^K$ with $\|W_\pi\|= 1$, then (4.1.2) reduces to

$$F(g) = \int_{\hat{G}_0} W_\pi(g)\langle F, W_\pi\rangle d\mu_p(\pi). \tag{4.1.3}$$

4.1.5 Local functional equation

Let ω_π be the central character of π. Then, for any $W \in \mathcal{W}(\pi, \bar{\psi})$, the local functional equation is (see [35, 36])

$$\begin{aligned}\int_{N\backslash G} V\left[w\begin{pmatrix} g^{-t} & \\ & 1\end{pmatrix}\right] W(w'g^{-t})|\det(g)|^{-s}dg \\ = \omega_\pi(-1)^n\gamma(1/2+s, \Pi\otimes\pi)\int_{N\backslash G} V\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\right] W(g)|\det(g)|^{s}dg,\end{aligned} \tag{4.1.4}$$

where w, w' are long Weyl elements of GL_{n+1} and G, respectively, and

$$\gamma(s, \Pi\otimes\pi) := \epsilon(s, \Pi\otimes\pi)\frac{L(1-s, \widetilde{\Pi}\otimes\widetilde{\pi})}{L(s, \Pi\otimes\pi)},$$

and $\epsilon(s, .)$ is the epsilon factor attached to π. The ϵ-factors are entire in s and have absolute value one, thus, are constants $\epsilon(\pi)$. Changing the variable $g \mapsto w'g^{-t}w'$, we can also rewrite (4.1.4) as

$$\begin{aligned}\int_{N\backslash G} V\left[\begin{pmatrix} 1 & \\ & g\end{pmatrix} w\right] W(gw')|\det(g)|^{s}dg \\ = \omega_\pi(-1)^n\gamma(1/2+s, \Pi\otimes\pi)\int_{N\backslash G} V\left[\begin{pmatrix} g & \\ & 1\end{pmatrix}\right] W(g)|\det(g)|^{s}dg.\end{aligned} \tag{4.1.5}$$

If π is unitary, then one has $\epsilon(\tilde{\pi}) = \overline{\epsilon(\pi)}$ and consequently

$$\gamma(1/2, \tilde{\pi}) = \overline{\gamma(1/2, \pi)}, \quad |\gamma(1/2, \pi)|= 1.$$

4.1.6 Spherical tempered dual

The term $\hat{G}_0$ can be parametrized by

$$\{\mu := (\mu_1, \dots, \mu_n) \mid \mu_i \in i\mathbb{R}, 1 \le i \le n\},$$

where a purely imaginary n-tuple μ corresponds with the induced representation

$$\pi_\mu := \mathrm{Ind}_B^G \chi, \quad \chi(na) = \prod_{i=1}^n |a_i|^{\mu_i}, n \in N,$$

where $a := \mathrm{diag}(a_1, \ldots, a_n)$ because any tempered spherical representation of G is of the above form. For our later purposes, for $\mu \in \mathbb{C}^n$, we define the quantity

$$d(\pi_\mu) := d(\mu) := 1 + \sum_{j=1}^n |\Im(\mu_j)|^2. \tag{4.1.6}$$

4.1.7 Conductors and gamma-factors

Recall the definition of the γ-factor from §4.1.5 and the definition of the analytic conductor from §4.1.3.

Lemma 4.1.1. *Let Π and π be generic irreducible unitary representations of $\mathrm{GL}_{n+1}(\mathbb{R})$ and G, respectively. Then,*

1. *For $s \in \mathbb{C}$ of bounded real part and a fixed positive distance away from any pole of $\gamma(1/2 - s, \pi)$, we have*

$$\gamma(1/2 - s, \pi) \asymp C(\pi \otimes |\det|^{\Im(s)})^{\Re(s)}.$$

2. $\frac{C(\Pi)^n}{C(\pi)^{n+1}} \leq C(\Pi \otimes \pi) \leq C(\Pi)^n C(\pi)^{n+1}$.

Proof. (1) is standard and follows from the Stirling approximation of the L-factors, e.g., see [10]. The second inequality of (2) is derived in [27, Appendix A], and the first inequality can be proved in the same way as the other. As in [27, Appendix A], one can appeal tothe Langlands classification of the admissible dual and reduce to the case of the representations Φ and ϕ of the Weil-Deligne group of $\mathbb{R}$. For instance, let both Φ and ϕ be one-dimensional representations with the Langlands parameters $(\mu, 0)$ and $(\nu, 0)$, respectively. Then, the parameter of $\Phi \otimes \phi$ can be given by $(\mu + \nu, 0)$, and the first inequality follows from

$$(1 + |\mu|) \leq (1 + |\mu + \nu|)((1 + |\nu|).$$

The remainder of the cases follows similarly. □

For brevity, we define $\Theta : \hat{G} \to \mathbb{C}$ by

$$\pi \mapsto \Theta(\pi, \Pi) := \omega_\pi^{-1}(C(\Pi))\gamma(1/2, \Pi \otimes \bar{\pi}),$$

where ω_π is the central character of π. If π is the spherical representation π_μ for some $\mu \in \mathbb{C}^n$, then we, by abuse of notation, denote $\Theta(\pi_\mu, \Pi)$ by $\Theta(\mu, \Pi)$. We record that it follows from Lemma 4.1.1

$$\Theta(\mu + 2M, \Pi) \ll_M \prod_{i=1}^{n} (1 + |\mu_i|)^{O_M(1)}. \tag{4.1.7}$$

for $M \in \mathbb{Z}_{\geq 0}^n$ fixed and $\mu \in \mathbb{C}^n$ with $0 \leq \Re(\mu) \ll 1$. If Π is θ-tempered for $0 \leq \theta < 1/2$, then $\Theta(\mu, \Pi)$ is holomorphic for $\Re(\mu_i) \geq 0$.

4.1.8 Explicit Plancherel measure

We describe the Plancherel measure explicitly in the case of the spherical Whittaker–Plancherel transform (4.1.3). From [22], we obtain that if $\pi_\mu \in \hat{G}_0$ for some $\mu \in i\mathbb{R}^n$, then

$$d\mu_p(\pi_\mu) = \left| \frac{c(1, \mu)}{c(0, \mu)} \right|^2 d\mu_1 \dots d\mu_n \tag{4.1.8}$$

where $d\mu_i$ are the Lebesgue measures on $i\mathbb{R}$ normalized by $2\pi i$.

4.1.9 Differential operator and Sobolev norm

Let $\{X_i\}$ be a basis of $\mathfrak{g} := \mathrm{Lie}(G)$. We define, for each $M \geq 0$, a second order differential operator by

$$\mathcal{D}_M := M + 1 - \sum_{i=1}^{n^2} (X_i^2). \tag{4.1.9}$$

We abbreviate $\mathcal{D}_0$ as $\mathcal{D}$, and define a Sobolev norm on the space of $\pi \in \hat{G}$ by

$$S_d(v) := \|\mathcal{D}^d v\|_\pi. \tag{4.1.10}$$

A similar type of Sobolev norm has been used in [47].

Lemma 4.1.2. *Let $\mathcal{D}_M$ be the differential operator in* (4.1.9).

1. *$\mathcal{D}_M$ is self-adjoint and positive definite on unitary representations of G. Eigenvalues of $\mathcal{D}_M$ are at least $M + 1$.*
2. *If C_G and C_K denote the Casimir elements for the groups G and K, respectively then,*
$$\mathcal{D}_M = M - (1 + C_G) + 2(1 + C_K).$$
3. *C_G acts on π_μ by the scalar $\lambda(\pi_\mu) := -T + \|\mu\|^2$, where $\|\mu\|^2 := \sum_{i=1}^n |\mu_i|^2$ and $T > 0$ is an absolute constant depending only on n.*
4. *The eigenvalue of the spherical vector in π_μ under $\mathcal{D}_M$ is of size $\asymp 1 + \|\mu\|^2$.*

Proof. (1) is standard and follows from the fact that $\sum_{i=1}^{n^2} X_i^2$ is self-adjoint and negative definite, which can be found in [50]. To prove (2), note that $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$ where $\mathfrak{p}$ and $\mathfrak{k}$

are Lie algebras of NA and K, respectively. We fix the bases $\{X_i\}_i$ of $\mathfrak{k}$ and $\{Y_j\}_j$ of $\mathfrak{p}$. Thus, from the definitions of the standard Cartan involution and Killing form [40, Chapter VIII], we obtain

$$C_G = -\sum_{X_i \in \mathfrak{k}} X_i^2 + \sum_{Y_i \in \mathfrak{p}} Y_i^2, \quad C_K = -\sum_{X_i \in \mathfrak{k}} X_i^2.$$

Thus, from the definition of (4.1.2) we obtain

$$\mathcal{D}_M = M + 1 - \sum_{X_i \in \mathfrak{k}} X_i^2 - \sum_{Y_i \in \mathfrak{p}} Y_i^2 = M - (1 + C_G) + 2(1 + C_K).$$

(3) is standard in the literature (see [7, p.2]). (4) follows from (3) as C_K act trivially on the spherical vector in π_μ. □

Lemma 4.1.3. *Let S_d be the Sobolev norm defined in* (4.1.10)*. Then, for $d_1, d_2 > 0$, there exists $L := L(d_1, d_2) > 0$ such that*

$$\int_{\hat{G}} C(\pi)^{d_1} \sum_{W \in \mathcal{B}(\pi)} S_{d_2}(W) S_{-L}(W) d\mu_p(\pi)$$

is convergent. Here, $\mathcal{B}(\pi)$ is an orthonormal basis of π consisting of eigenvectors of $\mathcal{D}$.

Proof. Let $\mathcal{B}(\pi) = \{W_i\}_{i \in \mathbb{N}}$ with eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}}$, correspondingly. From (1) of Lemma 4.1.2, we obtain that $\lambda_i \geq 1$. Thus,

$$\sum_{W \in \mathcal{B}(\pi)} S_{d_2}(W) S_{-L}(W) = \text{Trace} |_\pi (\mathcal{D}^{d_2 - L}).$$

There exists an element P_{d_1} in the center of the universal enveloping algebra of G such that

$$C(\pi)^{d_1} \ll \lambda_\pi(P_{d_1}),$$

where $\lambda_\pi(P)$ is the scalar by which P acts on π. Thus, the integral in the question can be bounded by

$$\int_{\hat{G}} \text{Trace}_\pi(P_{d_1} \mathcal{D}^{d_2 - L}) d\mu_p(\pi).$$

From [51, §8.5, Lemma 2], we know that for large enough A, the operator $P_{d_1}\mathcal{D}^{-A}$ is bounded. Finally, the integral $\int_{\hat{G}} \text{Trace}_\pi(\mathcal{D}^{-B}) d\mu_p(\pi)$ is convergent for sufficiently large B. A proof of this result can be found in [51, §A.4.2, Lemma (ii)]. We conclude our proof by making L large enough. □

4.1.10 Spherical Whittaker functions

In this subsection, we work out a relevant analysis of the spherical Whittaker function on G. The general references for spherical Whittaker functions are [21, Chapter 5], [57, 58], and Jacquet's work [33]. Let $\mu \in \mathbb{C}^n$. We define π_μ to be the spherical principal series representation with the Langlands parameters μ. Let W_μ be the spherical vector in π_μ defined by the following normalization of the Jacquet's integral.

$$W_\mu(g) := c(1,\mu)d_n \int_N I_\mu(wng)\overline{\psi(n)}dn, \quad g \in G, \Re(\mu_i - \mu_{i+1}) > 0, \tag{4.1.11}$$

where

$$I_\mu(nak) := \delta^{1/2}(a) \prod_{i=1}^{n} a_i^{\mu_i}, \quad n \in N, a \in A, k \in K.$$

Jacquet in [33] showed that W_μ has an analytic continuation to $\mathbb{C}^n$ and is invariant under the action of the Weyl group on μ. Here, d_n is an absolute constant such that when μ is purely imaginary,

$$\|W_\mu\|^2 = |c(1,\mu)|^2 \tag{4.1.12}$$

by Stade's formula [58, Theorem 1.1]. We record two types of bounds of W_μ to be used at various stages of the proofs.

Lemma 4.1.4. *Let μ be purely imaginary. Then, for any $M \in \mathbb{Z}_{\geq 0}^{n-1}$*

$$\frac{W_\mu(a)}{c(1,\mu)} \prec_M \delta^{1/2}(a) d(\mu)^{O_M(1)} \prod_{j=1}^{n-1} \min(1, a_{j+1}/a_j)^{M_j},$$

where $O_M(1)$ denotes a bounded quantity depending on M.

Proof. This result was previously proved in a similar form in [7, Theorem 1]. However, as we are happy with a polynomial dependency on μ, we infer from the more general result in Lemma 5.2.2. □

Lemma 4.1.5. *Let $\mu \in \mathbb{C}^n$, such that $\Re(\mu_i)$ are non-negative, distinct, and small enough (e.g., $< 1/100$). Then, for any $k \in \mathbb{Z}_{\geq 0}^n$*

$$W_{\mu+k}(a) \prec c(1, -\sigma(\mu+k))\delta^{1/2}(a) \prod_{i=1}^{n} a_i^{\Re((\sigma(\mu+k))_i}},$$

where $\sigma \in S_n$, such that $\Re(\mu_{\sigma(1)} + k_{\sigma(1)}) \leq \ldots \leq \Re(\mu_{\sigma(n)} + k_{\sigma(n)})$.

Proof. Let $\mu' := \mu + k$. Using the Weyl group invariance of $W_{\mu'}$, we assume, without loss of generality, that $\Re(\mu'_1) \geq \ldots \geq \Re(\mu'_n)$. Also, the assumption on the real parts of μ forces the above ordering to be strict. In fact, there is an $\epsilon > 0$ such that $\min\{\Re(\mu'_i - \mu'_j) \mid i < j\} \geq \epsilon$. Now, we perform a change of variable within the integral of (4.1.11) to see that

$$W_{\mu'}(a) = d_n c(1,\mu')\delta^{1/2}(a)\prod_{i=1}^{n} a_i^{(w\mu')_i}\int_N I_{\mu'}(wn)\psi(-ana^{-1})dn$$
$$\prec |c(1,\mu')|\delta^{1/2}(a)\prod_{i=1}^{s} a_i^{\Re((w\mu')_i)}.$$

The last integral is absolutely convergent, as $\min\{\Re(\mu_i'-\mu_j') \mid i<j\}\geq \epsilon$ (see [33]). Moreover, we can bound the last integral uniformly in μ'. A proof of this is accomplished in the proof of the absolute convergence of the Jacquet's integral in [21, Chapter 5.8]. From this proof, it can be inductively seen that

$$\int_N |I_{\mu'}(wn)|dn \ll \int_N I_\epsilon(wn)dn \ll_\epsilon 1,$$

where I_ϵ is the spherical section in the principal series with real parameters ν such that $\min\{(\nu_i-\nu_j) \mid i<j\}\geq\epsilon$. □

We record a rapid decay estimate of the Whittaker transform of the test function f in the following, which we require later in the proof.

Lemma 4.1.6. *Let $\mu\in\mathbb{C}^n$ with $\Re(\mu_i)\geq 0$ and $\sum_{i=1}^n \Re(\mu_i)^2\leq R$ for all i and for some $R\geq 0$. Let p be a fixed, sufficiently large natural number. Then, for each fixed $f\in C_c^\infty(N\backslash G,\psi)$,*

$$\langle f, W_\mu\rangle := \int_{N\backslash G} f(g)\overline{W_\mu(g)}dg \ll_{R,p} d(\mu)^{-p}|c(1,\Im(\mu))|.$$

Proof. Recall from Lemma 4.1.2 that $\mathcal{D}_R = 1+R-C_G+2C_K$. As W_μ is right K-invariant, $C_K W_\mu = 0$. Thus,

$$\mathcal{D}_R W_\mu = (1+R+T-\sum_{j=1}^n \mu_j^2)W_\mu.$$

We confirm that

$$|1+R+T-\sum_{j=1}^n \mu_j^2|\geq T+\sum_{j=1}^n |\Im(\mu_j)|^2 \asymp d(\mu).$$

Let Z denote the center of G as identified with $\mathbb{R}^\times$ in the usual way. Integrating by parts with respect to $\mathcal{D}^p$, we obtain

$$\langle f, W_\mu\rangle = (1+R+T-\sum_{j=1}^n \bar{\mu}_j^2)^{-p}\int_{N\backslash G}\overline{W_\mu(h)}\mathcal{D}^p f(h)dh.$$

We use coordinates $h := z\mathrm{diag}(h',1)k$ for $N\backslash G$ where h' lies in the diagonal subgroup A_{n-1} of $\mathrm{GL}_{n-1}(\mathbb{R})$, $z \in \mathbb{R}^\times$, and $k \in K$. Correspondingly, we write $dh = d^\times z \frac{dh'}{\delta(h')|\det(h')|} dk$. We next rewrite the last integral as

$$\int_{A_{n-1}} W_\mu \left[\begin{pmatrix} h' & \\ & 1 \end{pmatrix}\right] \int_K \int_Z \mathcal{D}^p f \left[z \begin{pmatrix} h' & \\ & 1 \end{pmatrix} k\right] |z|^{\sum \bar{\mu}_i} |\det(h')|^{-1} d^\times z dk \frac{dh'}{\delta(h')}.$$

We apply Cauchy–Schwarz on the h' integral and use (4.1.12) and $f \in C_c^\infty$ to obtain that the last integral is

$$\ll_{p,R,f} d(\mu)^{-p} \|W_\mu\|^2 \asymp d(\mu)^{-p} \prod_{i,j} \Gamma_{\mathbb{R}}(1 + \mu_i + \overline{\mu_j}).$$

Finally, we apply Stirling's estimate to obtain

$$\begin{aligned} &\prod_{i,j} \Gamma_{\mathbb{R}}(1 + \mu_i + \overline{\mu_j}) \asymp_R \prod_{i \neq j} \Gamma_{\mathbb{R}}(1 + \Im(\mu_i) - \Im(\mu_j) + \Re(\mu_i) + \Re(\mu_j)) \\ &\asymp_R \prod_{i \neq j} |\Im(\mu_i) - \Im(\mu_j)|^{R'} \Gamma_{\mathbb{R}}(1 + \Im(\mu_i) - \Im(\mu_j)) \ll_{R,p} d(\mu)^{R'} |c(1, \Im(\mu))|^2, \end{aligned}$$

where R' is a bounded constant depending on R and n. By making p sufficiently large, we conclude the proof. □

4.2 Global preliminaries

In this section, we change and reuse several notations for different purposes. Let $G := \mathrm{PGL}_n(\mathbb{R})$, $\Gamma := \mathrm{PGL}_n(\mathbb{Z})$ and $\mathbb{X} := \Gamma \backslash G$. Let $\hat{\mathbb{X}}$ (resp. $\hat{\mathbb{X}}_{\mathrm{gen}}$) be the isomorphism class of irreducible unitary automorphic (resp. generic) representations appearing in the spectral decomposition of $L^2(\mathbb{X})$ with automorphic Plancherel measure $d\mu_{\mathrm{aut}}$ compatible with the G-invariant probability measure on $\mathbb{X}$. In this section, π is an element in $\hat{\mathbb{X}}$ or $\hat{\mathbb{X}}_{\mathrm{gen}}$, i.e., a global automorphic representation for Γ. Correspondingly, $L(s, \pi)$ denotes the global L-function of π.

We mention next the Langlands description for $\hat{\mathbb{X}}_{\mathrm{gen}}$. We take a partition $n = n_1 + \cdots + n_k$, and let π_j be a unitary cuspidal automorphic representation for $\mathrm{GL}_{n_j}(\mathbb{Z})$ (if $n_j = 1$, then we take π_j to be a unitary character). Consider the unitary induction Π from the Levi $\mathrm{GL}(n_1) \times \ldots \times \mathrm{GL}(n_k)$ to $\mathrm{PGL}(n)$ of the tensor product $\pi_1 \otimes \ldots \otimes \pi_k$. There exists a unique irreducible constituent of Π, which we denote by the isobaric sum $\pi_1 \boxplus \cdots \boxplus \pi_k$. Then, Langlands classification states that every element in $\hat{\mathbb{X}}_{\mathrm{gen}}$ is isomorphic to such an isobaric sum. We refer to [14, Section 5], [48] for details.

4.2.1 Automorphic Forms

For details and a general discussion, we refer to [21]. For any $\pi \in \hat{\mathbb{X}}_{\text{gen}}$ and $\varphi \in \pi$ we denote its Whittaker functional by

$$W_\varphi(g) := \int_{[N]} \varphi(xg)\overline{\psi(x)}dx. \tag{4.2.1}$$

As π is generic, there is a G-equivariant isomorphism between π and its Whittaker model $\mathcal{W}(\pi,\psi) := \{W_\varphi \mid \varphi \in \pi\}$, where G acts on $\mathcal{W}(\pi,\psi)$ by right translation.

We fix the usual G-invariant inner product on π, e.g., if π is cuspidal, then it is induced from the inner product underlying $L^2(\mathbb{X})$. A unitary structure on $\mathcal{W}(\pi,\psi)$ can be given as in §4.1.2. By Schur's Lemma, there exists a positive constant $\ell(\pi)$ such that

$$\|\varphi\|_\pi^2 = \ell(\pi)\|W_\varphi\|^2_{\mathcal{W}(\pi,\psi)}. \tag{4.2.2}$$

When π is cuspidal, then $\ell(\pi) \asymp L(1,\pi,\mathrm{Ad})$ where the underlying constant in $\asymp$ is absolute (coming from the residue of a maximal Eisenstein series at 1, see [2, p. 617]).

We denote the m'th Fourier coefficient, for $m \in \mathbb{Z}^{n-1}$, attached to π by $\lambda_\pi(m)$. So, for $\varphi \in \pi$ and $m \in \mathbb{N}^{n-1}$, we can write

$$\int_{[N]} \varphi(xg)\overline{\psi_m(x)}dx = \frac{\lambda_\pi(m)}{\delta^{1/2}(\tilde{m})}W_\varphi(\tilde{m}g), \tag{4.2.3}$$

where

$$\tilde{m} = \mathrm{diag}(m_1 \dots m_{n-1}, m_1 \dots m_{r-2}, \dots, m_1, 1).$$

Here, $\lambda_\pi(m)$ is normalized so that $\lambda_\pi(1,\dots,1) = 1$ and the Ramanujan conjecture implies that $\lambda_\pi(m) \asymp 1$.

4.2.2 L-functions and conductor

A global L-function can be attached to a $\pi \in \hat{\mathbb{X}}_{\text{gen}}$, denoted by $L(s,\pi)$, that can be given by a Dirichlet series in some right half plane by

$$L(s,\pi) := \sum_{m=1}^{\infty} \frac{\lambda_\pi(m,1,\dots,1)}{m^s},$$

and mermorphically continued to all $s \in \mathbb{C}$. Every such L-function satisfies a functional equation as

$$L(1/2+s,\pi) = \gamma(1/2+s,\pi)L(1/2-s,\tilde{\pi}), \tag{4.2.4}$$

where $\tilde{\pi}$ is the contragradient of π and γ is the local gamma factor attached to the archimedean data attached to π, as defined in §4.1.5.

4.2.3 Kloosterman sum

We refer to [21, Chapter 11] and [20] for a detailed discussion of Kloosterman sum on GL(r). For a tuple of nonzero integers $c := (c_1, \ldots, c_{n-1})$, we denote the diagonal matrix $\mathrm{diag}(1/c_{n-1}, c_{n-1}/c_{n-2}, \cdots_2/c_1, c_1)$ by c^*. For $w \in W$ an Weyl element, let $\Gamma_w := \Gamma \cap G_w$, where G_w is the Bruhat cell of G attached to w.

If, for $l, m \in \mathbb{N}^{n-1}$ and $w \in W$, the *compatibility condition*

$$\psi_m(c^* w x w^{-1} c^{*-1}) = \psi_l(x), \quad x \in w^{-1} N w \cap N,$$

Holds, then the Kloosterman sum attached to l, m and moduli c is defined by

$$S_w(l, m; c) := \sum_{\substack{\gamma \in \Gamma_N \backslash \Gamma_w / (w^{-1} \Gamma_N^t w \cap \Gamma_N) \\ \gamma = b_1 c^* w b_2}} \psi_m(b_1) \psi_l(b_2),$$

where $\gamma = b_1 c^* w b_2$ denotes its Bruhat decomposition (see [21, Chapter 10]). The following result is due to Friedberg, and a proof can be found in [20, p.175].

Lemma 4.2.1. *Let $w \in W$ be any Weyl element. The compatibility condition in the definition of the Kloosterman sum is satisfied only if w is of the form*

$$\begin{pmatrix} & & I_{d_1} \\ & \iddots & \\ I_{d_k} & & \end{pmatrix}, \quad d_1 + \cdots + d_k = n,$$

where I_d is the identity matrix of rank d, i.e., $S_w(l, m; c)$ is nonzero only for this type of Weyl element.

4.2.4 Bessel distribution

In this subsection, we let π be an abstract generic irreducible representation of G. Let $\mathcal{B}(\pi)$ be an orthonormal basis of $\mathcal{W}(\pi, \psi)$. We define the Bessel distribution J_π attached to π by

$$J_F(\pi) := \sum_{W \in \mathcal{B}(\pi)} \pi(F) W(1) \overline{W(1)}, \tag{4.2.5}$$

for some $F \in C_c^\infty(G)$. We refer to [1, 15, 43] and references therein for general discussions about Bessel functions attached to generic representations.

Lemma 4.2.2. *The RHS of* (4.2.5) *is well-defined and does not depend on the choice of the orthonormal basis $\mathcal{B}(\pi)$. Thus, if F is a self-convolution of some $f \in C_c^\infty(G)$, i.e.,*

$$F(g) = \int_G f(gh) \overline{f(h)} dh,$$

then

$$J_F(\pi) = \sum_{W \in \mathcal{B}(\pi)} |\pi(f)W(1)|^2,$$

for $\mathcal{B}(\pi)$ some orthonormal basis of π.

Proof. The first assertion follows from [1, Lemma 23.1, Lemma 23.3]. For the second assertion, we see that the RHS of (4.2.5) is

$$\sum_{W \in \mathcal{B}(\pi)} \int_G \int_G f(gh)W(g)\overline{f(h)W(1)}dgdh = \sum_{W \in \mathcal{B}(\pi)} \int_G f(g)W(g)dg \int_G \overline{f(h)W(h)}dh,$$

after changing the basis to $\{\pi(h)W\}_{W \in \mathcal{B}(\pi)}$, and we conclude. □

4.2.5 Pre-Kuznetsov formula

In this subsection, we record a soft version of the Kuznetsov formula for $\mathrm{GL}(n)$, as in [15].

Recall that $d\mu_{\mathrm{aut}}$ is the automorphic Plancherel measure on $\hat{\mathbb{X}}$ compatible with the G-invariant measure of $\mathbb{X}$. We fix a test function $f \in C_c^\infty(G)$. We define F to be the self-convolution of f, as in Lemma 4.2.2. We consider the sum

$$\sum_{\gamma \in \Gamma} F(x_1^{-1}\gamma x_2), \quad x_1, x_2 \in G.$$

As a function x_2, this sum is left Γ-invariant, hence lives in $L^2(\mathbb{X})$. So, we can spectrally decompose the sum as

$$\sum_{\gamma \in \Gamma} F(x_1^{-1}\gamma x_2) = \int_{\hat{\mathbb{X}}} \sum_{\varphi \in \mathcal{B}(\pi)} \overline{\pi(\bar{F})\varphi(x_1)}\varphi(x_2)d\mu_{\mathrm{aut}}(\pi), \tag{4.2.6}$$

where $\mathcal{B}(\pi)$ is an orthonormal basis of π.

We fix $l, m \in \mathbb{N}^{n-1}$ and integrate both sides of (4.2.6) against $\psi_m(x_1)\overline{\psi_l(x_2)}$ over $x_1, x_2 \in [N]$. As the non-generic part of the spectrum automatically vanishes, we obtain using (4.2.3) and (4.2.2) that

$$\begin{aligned}\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \ell(\pi)^{-1} \frac{\overline{\lambda_\pi(m)}\lambda_\pi(l)}{\delta^{1/2}(\tilde{m}\tilde{l})} \sum_{W \in \mathcal{B}(\pi)} \overline{\pi(\bar{F})W(\tilde{m})}W(\tilde{l})d\mu_{\mathrm{aut}}(\pi) \\ = \sum_{\gamma \in \Gamma} \int_{[N]^2} F(x_1^{-1}\gamma x_2)\psi_m(x_1)\overline{\psi_l(x_2)}dx_1 dx_2. \end{aligned} \tag{4.2.7}$$

Here, we identified π with its Whittaker model $\mathcal{W}(\pi)$ for generic π.

We replace F in (4.2.7) by $F_{l,m}$, where

$$F_{l,m}(g) := F(\tilde{m}g\tilde{l}^{-1}).$$

Using Lemma 4.2.2, we obtain

$$\sum_{W\in\mathcal{B}(\pi)} \overline{\pi(\bar{F}_{l,m})W(\tilde{m})}W(\tilde{l}) = J_{\bar{F}}(\pi).$$

We next replace F by $F_{l,m}$ in the RHS of (4.2.7) as well and simplify. Doing a folding-unfolding over the γ-sum and x_1-integral, we obtain the RHS as

$$\sum_{\gamma\in\Gamma_N\backslash\Gamma} \int_N \int_{[N]} F(\tilde{m}x_1\gamma x_2\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_2)}dx_1dx_2.$$

The identity term in the above sum simplifies to

$$\int_N F(\tilde{m}x_1\tilde{l}^{-1})\overline{\psi_m(x_1)}dx_1 \int_{[N]} \overline{\psi_l(x_2)}\psi_m(x_2)dx_2 = \frac{\delta_{m=l}}{\delta(\tilde{m})}\int_N F(x)\overline{\psi(x)}.$$

On the other hand, we do a Bruhat decomposition of $\Gamma - \Gamma_N$. For $\gamma \in \Gamma_w$, which is the Bruhat cell attached to $w \in W$, and write the x_2-integral over $[N]$ as a product of the $x_{2,1}$-integral over $[N_w]$ and the $x_{2,2}$-integral over $[\bar{N}_w]$, where

$$\bar{N}_w := w^{-1}Nw \cap N, \quad N_w := w^{-1}N^tw \cap N.$$

$$\begin{aligned}
&\sum_{\gamma\in\Gamma_N\backslash(\Gamma-\Gamma_N)} \int_N \int_{[N]} F(\tilde{m}x_1\gamma x_2\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_2)}dx_1dx_2 \\
&= \sum_{1\neq w\in W} \sum_{c\in\mathbb{Z}^{n-1}_{\neq 0}} \sum_{\substack{\gamma\in\Gamma_N\backslash\Gamma_w/\Gamma_{N_w} \\ \gamma=b_1c^*wb_2}} \sum_{\theta\in\Gamma_{N_w}} \int_N \int_{[N_w]} \int_{[\bar{N}_w]} F(\tilde{m}x_1b_1c^*wb_2\theta x_{2,2}x_{2,1}\tilde{l}^{-1}) \\
&\qquad \times \overline{\psi_m(x_1)\psi_l(x_{2,1})\psi_l(x_{2,2})}dx_{2,2}dx_{2,1}dx_1.
\end{aligned}$$

Again, unfolding over the θ-sum and $x_{2,2}$ integral, changing variables, and recalling the definition of the Kloosterman sum, we obtain the summand above corresponding to $w \in W$ and $c \in \mathbb{Z}^{n-1}_{\neq 0}$ is

$$S_w(l,m;c)\int_N \int_{N_w} \int_{[\bar{N}_w]} F(\tilde{m}x_1c^*wx_{2,1}x_{2,2}\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_{2,2})\psi_l(x_{2,1})}dx_{2,1}dx_{2,2}dx_1.$$

Now, writing $x_{2,1} = x_{2,1}(c^*w)^{-1}(c^*w) \in N$ and using the compatibility condition of the Kloosterman sum, the above equals

$$S_w(l,m;c)\int_N \int_{N_w} F(\tilde{m}x_1c^*wx_2\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_2)}dx_2dx_1.$$

Finally, changing variables $x_1 \mapsto \tilde{m}x_1\tilde{m}^{-1}$ and $x_2 \mapsto \tilde{l}x_2\tilde{l}^{-1}$, we obtain the RHS of (4.2.7) as

$$\sum_{1\neq w\in W}\sum_{c\in\mathbb{Z}^{n-1}_{\neq 0}}\frac{S_w(l,m;c)}{\delta(\tilde{m})\delta_w(\tilde{l})}\int_N\int_{N_w}F(x_1\tilde{m}cw\tilde{l}^{-1}x_2)\overline{\psi(x_1)\psi(x_2)}dx_2dx_1.$$

Here, $\delta_w(\tilde{l})$ is the Jacobian arising from the change of variable $x_2\mapsto\tilde{l}x_2\tilde{l}^{-1}$. We define, for $F\in C_c^\infty(G)$, the function

$$W_F(g):=\int_N F(xg)\overline{\psi(x)}dx \tag{4.2.8}$$

that lies in $C_c^\infty(N\backslash G,\psi)$. The RHS of (4.2.8) can be verified as being absolutely convergent. Thus, we obtain a version of the Kuznetsov trace formula.

Proposition 4.2.1. *Let $F\in C_c^\infty(G)$ and $l,m\in\mathbb{N}^{n-1}$. Then,*

$$\begin{aligned}\int_{\hat{\mathbb{X}}_{\mathrm{gen}}}\overline{\lambda_\pi(m)}\lambda_\pi(l)\frac{J_{\bar{F}}(\pi)}{\ell(\pi)}d\mu_{\mathrm{aut}}(\pi)&=\delta_{m=l}W_F(1)\\&+\sum_{1\neq w\in W}\frac{\delta^{1/2}(\tilde{l})}{\delta^{1/2}(\tilde{m})\delta_w(\tilde{l})}\sum_{c\in\mathbb{Z}^{n-1}_{\neq 0}}S_w(l,m;c)\int_{N_w}W_F(\tilde{m}c^*w\tilde{l}^{-1}x)\overline{\psi(x)}dx,\end{aligned}$$

where W_F and J_F are as in (4.2.8) *and* (4.2.5), *respectively.*

Chapter 5

Proof of Existence of Analytic Newvectors

We again return to the local setting where $G := \mathrm{GL}_n(\mathbb{R})$ and π denotes a local representation of G. We adopt all other notations of §4.1. We also employ the standard convention from analytic number theory of writing ϵ (or τ) for a small positive fixed quantity whose precise value that we allow to change from one line to the next.

5.1 Reduction of the proof of the main results

Let $\Omega \subseteq G$ be the bounded neighborhood of the identity element and ι be as in Theorem 3, and recall ψ from (4.1.1). We first construct $V \in \mathcal{W}\left(\Pi, \tilde{\psi}\right)$ using the theory of the Kirillov model. We denote $C_c^\infty(N\backslash G, \psi)$ as the space of smooth functions $f : G \to \mathbb{C}$ satisfying $f(ng) = \psi(n)f(g)$ for all $n \in N$ and for which the support of f has compact image in $N\backslash G$. We choose an element $f \in C_c^\infty(N\backslash G, \psi)$ with the following properties:

- f is right K-invariant
- $f(h) \geq \iota$ for all $h \in \Omega$
- $\int_{N\backslash G} |f|^2 \, dg = 1$

We now define V by requiring that

$$V\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] := f(g), \tag{5.1.1}$$

In this scenario, the sphericality assumption of f is not essential, and we refer to the discussion in §5.3.4 for details.

Proposition 5.1.1. *Let Π be as in Theorem 3. For every $\delta > 0$ and h in a fixed bounded neighbourhood around the identity in G, there exists a $\tau > 0$ such that*

$$\left| V\left[\begin{pmatrix} h & \\ c/C(\Pi) & 1 \end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \right| < \delta,$$

for $c \in \mathbb{R}^n$ with $|c| < \tau$. Here, V is the vector chosen in (5.1.1).

We now assume Proposition 5.1.1 and prove Theorems 3, where Proposition 5.1.1 is proven in the next two sections. In the following Lemma, we first prove a weaker version of Theorem 3.

Lemma 5.1.1. *For every $\delta > 0$ and h in a fixed bounded neighbourhood around the identity in G, there exists a $\tau > 0$ such that*

- *the normalization $\|V\|_{\mathcal{W}(\Pi,\tilde{\psi})} = 1$, with the norm taken in the Kirillov model (see §4.1.2),*
- *the lower bound $V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \geq \iota$ for all $h \in \Omega$, and*
- *for all $g \in K_0(C(\Pi), \tau)$ and for $h \in \Omega$,*

$$\left| V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} g\right] - \omega_\Pi(d_g) V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \right| < \delta.$$

Here, V is the vector chosen in (5.1.1) *with ω_Π and d_g being the same as in Theorem 2.*

Proof. We choose V as in (5.1.1). The first two requirements of V are automatically satisfied by the choice in (5.1.1). To prove the invariance property of V, we claim the following.

For every $\delta > 0$ and h in a given fixed bounded set, there exists $\tau > 0$ such that for all $\begin{pmatrix} g' & \\ c & 1 \end{pmatrix} \in K_0(C(\Pi), \tau)$,

$$\left| V\left[\begin{pmatrix} hg' & \\ c & 1 \end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \right| < \delta.$$

This claim is sufficient. To see that, first we note in the following Iwahori-type decomposition that for $d \neq 0$

$$\mathrm{GL}_{n+1}(\mathbb{R}) \ni \begin{pmatrix} A & b \\ c & d \end{pmatrix} = d \begin{pmatrix} 1_n & b/d \\ & 1 \end{pmatrix} \begin{pmatrix} A/d - bc/d^2 & \\ & 1 \end{pmatrix} \begin{pmatrix} 1_n & \\ c/d & 1 \end{pmatrix}.$$

Hence we can assume that any $g \in K_0(C(\Pi), \tau)$ is of the form

$$g = d_g \begin{pmatrix} 1_n & b \\ & 1 \end{pmatrix} \begin{pmatrix} g' & \\ c & 1 \end{pmatrix},$$

with $g' \in G$ such that $\|g' - 1\| \ll \tau$, $|b|, |d-1|, and C(\Pi)|c| < \tau$. Therefore,

$$V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} g\right] = \omega_\Pi(d_g) \tilde{\psi}\left[\begin{pmatrix} 1_n & hb \\ & 1 \end{pmatrix}\right] V\left[\begin{pmatrix} hg' & \\ c & 1 \end{pmatrix}\right].$$

Using the unitarity of ω_Π and $\tilde{\psi}$,

$$\begin{aligned}&\left|V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix} g\right] - \omega_\Pi(d_g)V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right]\right| \\ &= \left|(e(hb.e_n)-1)V\left[\begin{pmatrix} hg' & \\ c & 1\end{pmatrix}\right] + V\left[\begin{pmatrix} hg' & \\ c & 1\end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right]\right|.\end{aligned}$$

Now, using the claim above and the fact that

$$V\left[\begin{pmatrix} hg' & \\ c & 1\end{pmatrix}\right] \ll 1,$$

which also follows from the same claim, we obtain the required invariance property by making b small enough.

Now, we prove the claim. From (5.1.1), for h in a compact set in G, there exists τ small enough with $\|g'-1\|< \tau$ such that

$$\left|V\left[\begin{pmatrix} hg' & \\ & 1\end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right]\right| < \delta.$$

By applying a triangle inequality in the following

$$\begin{aligned}&V\left[\begin{pmatrix} hg' & \\ c & 1\end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right] \\ &= V\left[\begin{pmatrix} hg' & \\ c & 1\end{pmatrix}\right] - V\left[\begin{pmatrix} hg' & \\ & 1\end{pmatrix}\right] + V\left[\begin{pmatrix} hg' & \\ & 1\end{pmatrix}\right] - V\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right],\end{aligned}$$

along with Proposition 5.1.1, we prove the claim. □

The subsets $K_*(X,\tau)$ for $* \in \{0,1\}$ feature the following Følner-type property, the verification of which we leave to the reader.

Lemma 5.1.2. *Let $* \in \{0,1\}$. For all $\tau_0, \delta \in (0,1)$, there exists $\tau_1 > 0$ so that for all $X \geq 1$ and all $g \in K_*(X,\tau_1)$, the set $A := K_*(X,\tau_0)$ features the following approximate invariance property under translation by g:*

$$\frac{\operatorname{vol}(gA\triangle A)}{\operatorname{vol}(A)} < \delta.$$

Here, $A\triangle B := (A\setminus B)\cup(B\setminus A)$ denotes the symmetric difference and vol *is taken with respect to any fixed Haar measure.*

Proofs of Theorem 2 and Theorem 3 assuming Proposition 5.1.1. First, we use a similar technique as in Lemma 5.1.1 and reduce to the case $g \in K_1(C(\Pi),\tau)$ using the unitarity of ω_Π as

$$|\Pi(g)V - \omega_\Pi(d_g)V| = |\Pi(g/d_g)V - V|.$$

Set $\delta_0 := \min(\delta, 1)\iota/2$ where δ and ι are as in the statement of Theorem 3. Let $V_0 \in \mathcal{W}(\Pi, \tilde{\psi})$ and $\tau_0 > 0$ be as in (5.1.1), so that

$$V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \geq \iota,$$

and for all $g \in K_1(C(\Pi), \tau_0), h \in \Omega$,

$$\left| V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} g \right] - V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \right| \leq \delta_0.$$

The existence of such V_0 follows from Lemma 5.1.1 and the toy theorem 2.

Let ξ be the L^1-normalized characteristic function of $K_1(C(\Pi), \tau_1)$. There exists $\tau_2 > 0$ such that for $g \in K_1(C(\Pi), \tau_2)$,

$$\| g * \xi - \omega_\Pi(d_g)\xi \|_{L^1} \leq \| g * \xi - \xi \| + |\omega_\pi(d_g) - 1| \leq \delta_0,$$

which follows from Lemma 5.1.2 and the toy theorem 2. Here, $g * \xi(h) := \xi(g^{-1}h)$ so that $\Pi(g)\Pi(\xi) = \Pi(g * \xi)$. Set $V_1 := \Pi(\xi)V_0$, and because $\|V_0\| = 1$, we have by the triangle inequality that for $g \in K_1(C(\Pi), \tau_1)$,

$$\|\Pi(g)V_1 - \omega_\Pi(d_g)V_1\| \leq \|g * \xi - \omega_\Pi(d_g)\xi\|_{L^1(G)} \leq \delta_0,$$

and for τ_1 sufficiently small in terms of τ_0

$$\left| V_1 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} g \right] - V_1 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \right| \\ \leq \int_{K_1(C(\Pi), \tau_1)} \xi(t) \left| V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} gt \right] - V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} t \right] \right| dg \leq 2\delta_0.$$

Also, for $h \in \Omega$, we have

$$V_1 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] - V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] = \int_{K_1(C(\Pi), \tau_0)} \xi(t) \left(V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} t \right] - V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \right) dt,$$

Hence,

$$\left| V_1 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] - V_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \right| \leq \iota/2,$$

so in particular,

$$V_1 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \geq \iota - \iota/2 = \iota/2 > 0,$$

thus, $\|V_1\| \asymp 1$. Then, the vector $V_1/\|V_1\| \in \mathcal{W}\left(\Pi, \tilde{\psi}\right)$ and its image in Π satisfy the required conclusions of Theorem 3 and Theorem 2, respectively. □

5.2 Proof of Proposition 5.1.1

To prove Proposition 5.1.1, we need a uniform bound of a Weyl element shifted newvector, as in Proposition 5.2.1. Proving this proposition is the heart and most difficult part of this thesis.

Proposition 5.2.1. *Let $l \in \mathbb{N}$. Let $g = ak$ with $a := \operatorname{diag}(a_1, \dots, a_n) \in A$, $k \in K$. Then,*

$$\mathcal{D}^l V\left[\begin{pmatrix} C(\Pi) & \\ & g \end{pmatrix} w\right] \prec_{l,N} \delta^{1/2}(a) \min(a_1^N, 1),$$

where w is the long Weyl element in $\mathrm{GL}_{n+1}(\mathbb{R})$ and $\mathcal{D}$ is the differential operator defined in (4.1.9).

We prove the above proposition in the next section. First, we see how to derive Proposition 5.1.1 from Proposition 5.2.1.

Lemma 5.2.1. *Let $p \in \mathbb{N}$ and $\mathcal{D}$ as in* (4.1.9)*. Then,*

1. *For $\sigma \in \mathbb{C}$ small, we have $\mathcal{D}^p |\det(g)|^\sigma \asymp_p |\det(g)|^\sigma$.*
2. *As a function of h, we have $\mathcal{D}^p(e(cw'h^{-1}e_1) - 1) \ll_p \sum_{r=1}^{2p+1} |c|^r |h^{-1}e_1|^r$.*

Proof. Recall the definition of $\mathcal{D}$. It is straightforward to check that

$$\mathcal{D}^p |\det(g)|^\sigma = (1 - n\sigma^2)^p |\det(g)|^\sigma,$$

which proves (1).

Let $x := 2\pi i c w'$, so

$$e(cw'h^{-1}e_1) - 1 = \sum_{r=1}^{\infty} \frac{(xh^{-1}e_1)^r}{r!}.$$

Let $f_r := \frac{(xh^{-1}e_1)^r}{r!}$, then it is straightforward to check that[i]

$$\mathcal{D} f_r = f_r - (xh^{-1}e_1) f_{r-1} - |x|^2 |h^{-1}e_1|^2 f_{r-2}.$$

The following can also easily be checked that

$$\mathcal{D}(|x|^2 |h^{-1}e_1|^2) = -2n|x|^2 |h^{-1}e_1|^2.$$

Therefore, inducting on p and summing over r, we conclude (2). □

Lemma 5.2.2. *Let π be a tempered representation of G and $W \in \mathcal{W}(\pi, \psi)$ be any L^2-normalized vector. Let $a = \operatorname{diag}(a_1, \dots, a_n) \in A$, $k \in K$, and $L > 0$. For any η small*

[i] The computation will be intuitive following the similar computation in the $n = 1$ case, where $\mathcal{D}$ becomes the simpler differential operator $1 - (h\partial_h)^2$.

enough and N large enough,

$$\mathcal{D}^{-L}W(ak) \ll_{\eta,N} \delta^{1/2-\eta}(a)\prod_{i=1}^{n-1}\min(1,(a_{i+1}/a_i)^N)S_{p-L}(W),$$

where p depends on N and the group G only.

Proof. $W(ak)=\pi(k)W(a)$ and $S_p(W)\asymp_K S_p(\pi(k)W)$ is sufficient to prove the Lemma for $k=1$. We next prove this inducting on n. For $n=2$, this is proved in [47, Proposition 3.2.3]. We generalize this idea for a proof for general n by inducting on n. First, there is a $Y\in\mathfrak{g}$ such that for any $W\in\mathcal{W}(\pi,\psi)$,

$$d\pi(Y)W(a)=(a_{n-1}/a_n)W(a).$$

Let ω_π be the central character of π.

For a generic irreducible tempered representation π' of $\mathrm{GL}_{n-1}(\mathbb{R})$, let $\mathcal{B}(\pi'):=\{W'\}$ be an orthonormal basis of π' consisting of eigenvectors of $\mathcal{D}'$ by diagonalizing it, where $\mathcal{D}'$ is the analogous differential operator on $\mathrm{GL}_{n-1}(\mathbb{R})$, as in (4.1.9). We obtain using the Whittaker–Plancherel formula (4.1.2) on $\mathrm{GL}(n-1)$

$$\begin{aligned}(a_{n-1}/a_n)^N\mathcal{D}^{-L}W(a)&=\omega_\pi(a_n)d\pi(Y^N)\mathcal{D}^{-L}W(a/a_n)\\&=\omega_\pi(a_n)\int_{\widehat{\mathrm{GL}_{n-1}}}\sum_{W'\in\mathcal{B}(\pi')}W'(a'/a_n)Z(W')d\mu_p(\pi'),\end{aligned}$$

where $a':=\mathrm{diag}(a_1,\dots,a_{n-1})$, and

$$Z(W'):=\int_{N_{n-1}\backslash\mathrm{GL}_{n-1}}d\pi(Y^N)\mathcal{D}^{-L}W\left[\begin{pmatrix}h&\\&1\end{pmatrix}\right]\overline{W'(h)}dh.$$

We change the variable $\pi'\mapsto\pi'\otimes|\det|^s$ to obtain

$$\begin{aligned}(a_{n-1}/a_n)^N\mathcal{D}^{-L}W(a)=\omega_\pi(a_n)\int_{\widehat{\mathrm{GL}_{n-1}}}\sum_{W'\in\mathcal{B}(\pi')}&W'(a'/a_n)\\&\times|\det\big(a'/a_n\big)|^sZ(W'\otimes|\det|^s)d\mu_p(\pi').\end{aligned}$$

Using the induction hypothesis, we obtain

$$W'(a'/a_n)\ll_{\eta,N}\delta^{1/2-\eta}(a')\prod_{i=1}^{n-2}(1,(a_{i+1}/a_i)^N)S_q(W'),$$

for some q depending on N. From the local functional equation (4.1.5) for $\mathrm{GL}(n)\times\mathrm{GL}(n-1)$, we confirm that $Z(W'\otimes|\det|^s)$ is holomorphic for $\Re(s)<1/2$ (as both π and π' are tempered), and the defining integral of Z is absolutely convergent. We choose $s=1/2-\eta$ for some small $\eta>0$, and $|\det(a'/a_n)|^{1/2-\eta}\delta^{1/2-\eta}(a')=\delta^{1/2-\eta}(a)$.

Now, in the integral defining $Z(W' \otimes |\det|^{1/2-\eta/2})$, we perform integration by parts sufficiently (say, d) many times using $\mathcal{D}'$ to prove that

$$Z(W' \otimes |\det|^{1/2-\eta/2}) \ll S_{-d}(W')S_{d'-L}(W),$$

for all large d and d' depending only on d. By applying Lemma 4.1.3 and making d large enough, we conclude. □

Proof of Proposition 5.1.1 assuming Proposition 5.2.1. Recall that Π is θ-tempered as in Theorem 3. For every $\pi \in \hat{G}$, we choose an orthonormal basis $\mathcal{B}(\pi) := \{W\}$ of π consisting of eigenvectors of $\mathcal{D}$ by diagonalizing it. Applying the Whittaker–Plancherel formula (4.1.2) of G, we get for $0 < \sigma < 1/2 - \theta$ that

$$\begin{aligned}
&|\det(g)|^{-\sigma} V\left[\begin{pmatrix} g & \\ c & 1 \end{pmatrix}\right] = |\det(g)|^{-\sigma} \Pi\left[\begin{pmatrix} 1_n & \\ c & 1 \end{pmatrix}\right] V\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] \\
&= \int_{\hat{G}} \sum_{W \in \mathcal{B}(\pi)} W(g) \int_{N\backslash G} \Pi\left[\begin{pmatrix} 1_n & \\ c & 1 \end{pmatrix}\right] V\left[\begin{pmatrix} h & \\ & 1 \end{pmatrix}\right] \overline{W(h)} |\det(h)|^{-\sigma} dh d\mu_p(\pi).
\end{aligned}$$

The inner integral can be checked to be absolutely convergent by Lemma 5.2.2. We apply the $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ local functional equation (4.1.5) and the N-equivariance for V to obtain the inner integral above as

$$\gamma(1/2-\sigma, \Pi \otimes \bar{\pi})^{-1} \int_{N\backslash G} e(cw'h^{-1}e_1) V\left[\begin{pmatrix} 1 & \\ & h \end{pmatrix} w\right] \overline{W(hw')} |\det(h)|^{-\sigma} dh.$$

By changing variable $h \mapsto C(\Pi)^{-1}h$ in the latter integral, we conclude that

$$\begin{aligned}
&V\left[\begin{pmatrix} g & \\ c/C(\Pi) & 1 \end{pmatrix}\right] - V\left[\begin{pmatrix} g & \\ & 1 \end{pmatrix}\right] \\
&= \int_{\hat{G}} C(\Pi)^{n\sigma} \gamma(1/2-\sigma, \Pi \otimes \bar{\pi})^{-1} \sum_{W \in \mathcal{B}(\pi)} W(g) |\det(g)|^{\sigma} \\
&\qquad \int_{N\backslash G} (e(cw'h^{-1}e_1) - 1) V\left[\begin{pmatrix} C(\Pi) & \\ & h \end{pmatrix} w\right] \overline{W(hw')} |\det(h)|^{-\sigma} dh d\mu_p(\pi).
\end{aligned} \tag{5.2.1}$$

From Lemma 4.1.1,

$$C(\Pi)^{n\sigma} \gamma(1/2-\sigma, \Pi \otimes \bar{\pi})^{-1} \ll C(\pi)^{(n+1)\sigma}.$$

In the last integral of (5.2.1), we integrate by parts by $\mathcal{D}$ as

$$\int_{N\backslash G} \mathcal{D}^L \left((e(cw'h^{-1}e_1) - 1) V\left[\begin{pmatrix} C(\Pi) & \\ & h \end{pmatrix} w\right] |\det(h)|^{-\sigma} \right) \mathcal{D}^{-L} \overline{W(hw')} dh.$$

We write $N\backslash G$ with (a, k) coordinates. Using Lemma 5.2.1, Proposition 5.2.1, and

Lemma 5.2.2, we estimate the last integral by

$$\prec S_{p-L}(W)\sum_{1\le r\ll L}|c|^r\int_A a_1^{-r}\prod_{i=1}^{n-1}\min(1,(a_{i+1}/a_i)^M)\min(1,a_1^N)\frac{da}{|\det(a)|^\sigma},$$

where p depends on M and n. The above A-integral can be checked to be absolutely convergent and, hence, (5.2.1) is bounded by

$$\ll_{p,L}|c|\int_{\hat{G}}C(\pi)^{(n+1)\sigma}\sum_{W\in\mathcal{B}(\pi)}|W(g)||\det(g)|^\sigma S_{p-L}(W)d\mu_p(\pi).$$

As g varies in a compact set Ω, we use Lemma 5.2.2 to bound

$$W(g)|\det(g)|^\sigma\ll_\Omega S_q(W)$$

for all $W\in\mathcal{B}(\pi)$ and for some fixed q. Finally, making L sufficiently large and appealing to Lemma 4.1.10, we conclude. □

5.3 Proof of Proposition 5.2.1

For $1\le s\le n$, we define a property $\mathbf{pop}(s)$, which stands for *Partial Ordering at the Pivot* s, of the elements $a\in A$.

Definition 5.3.1. *We say an element $a\in A$ satisfies the property $\mathbf{pop}(s)$ for some $1\le s\le n$ if*

$$\max\{a_1,\dots,a_s\}\le\min\{1,\min\{a_{s+1},\dots,a_n\}\}.$$

The proof of Proposition 5.2.1 spans two cases, depending on if a satisfies $\mathbf{pop}(s)$ for some s. When a does not satisfy $\mathbf{pop}(s)$ for any s, the proof becomes relatively easier, as can be seen at the end of this section. Here, we prove the required bound of W_μ, which is the spherical vector in π_μ as defined in (4.1.11), on the elements a that do not satisfy $\mathbf{pop}(s)$ for any s.

Lemma 5.3.1. *Let $a\in A$ does not satisfy $\mathbf{pop}(s)$ for any $1\le s\le n$, then*

$$\frac{W_\mu(a)}{c(1,\mu)}\prec_N\delta^{1/2}(a)\min(1,a_1^N)d(\mu)^{O_N(1)}$$

for N large enough.

Proof. If $a_1\ge 1$, then Lemma 4.1.4 with $M=0$ immediately implies this lemma. So, we assume that $a_1<1$. As we do not have $\mathbf{pop}(1)$, there exists $1<l'\le n$ such that $a_{l'}<a_1$. Also, as we do not have $\mathbf{pop}(n)$, there exists $1<r'\le n$ such that $a_{r'}>1$. Let

$$l:=\max\{l'\mid a_{l'}<a_1\},\quad r:=\min\{r'\mid a_{r'}>1\}.$$

If $r = n$, then we have $\mathbf{pop}(n-1)$, so $r < n$. If $l > r$, then from Lemma 4.1.4, we estimate that

$$\frac{W_\mu(a)}{c(1,\mu)} \prec_N \delta^{1/2}(a) d(\mu)^{O_N(1)} \prod_{j=r}^{l-1} (a_{j+1}/a_j)^N \leq \delta^{1/2}(a) a_1^N d(\mu)^{O_N(1)},$$

and we conclude. Thus, we may assume that $l < r$ (note that $l \neq r$ as $a_l < a_1 < 1 < a_r$). Now, we define

$$a_L := \max\{a_1, \ldots, a_l\}, \quad a_R := \min\{a_r, \ldots, a_n\}.$$

Here, $L < l$ and $R > r$, as if $R = r$, then we have $\mathbf{pop}(r-1)$. Also, if $a_L \geq a_R$, then

$$\frac{W_\mu(a)}{c(1,\mu)} \prec_N \delta^{1/2}(a) d(\mu)^{O_N(1)} \prod_{j=L}^{l-1} (a_{j+1}/a_j)^N \prod_{i=r}^{R-1} (a_{i+1}/a_i)^N \leq \delta^{1/2}(a) a_1^N d(\mu)^{O_N(1)}.$$

Thus, we may now assume that $a_L < a_R$. Hence, we have $1 < l < r < n$ such that

$$\max\{a_1, \ldots, a_l\} \leq \min\{1, \min(a_r, \ldots, a_n)\}. \tag{5.3.1}$$

Next, we define

$$l_1 := \max\{l' \mid a_{l'} \leq a_L\}, \quad r_1 := \min\{r' \mid a_{r'} \geq a_R\}.$$

Here, $l_1 \geq l$ and, further, $l_1 > l$, otherwise, we have $\mathbf{pop}(l)$. Similarly, $r_1 \leq r$, and, further, $r_1 < r$, otherwise, we have $\mathbf{pop}(r-1)$. Also, $l_1 \neq r_1$ as $a_{l_1} \leq a_L < a_R \leq a_{r_1}$. But, if $l_1 > r_1$, then

$$\begin{aligned}\frac{W_\mu(a)}{c(1,\mu)} &\prec_N \delta^{1/2}(a) d(\mu)^{O_N(1)} \prod_{j=L}^{l-1} (a_{j+1}/a_j)^N \prod_{i=r_1}^{l_1-1} (a_{i+1}/a_i)^N \prod_{k=r}^{R-1} (a_{i+1}/a_i)^N \\ &\leq \delta^{1/2}(a) a_1^N d(\mu)^{O_N(1)}.\end{aligned}$$

Thus, we may assume that $l_1 < r_1$, as $l_1 = r_1$ would have $\mathbf{pop}(l_1 - 1)$. We now define

$$a_{L_1} := \max\{a_1, \ldots, a_{l_1}\}, \quad a_{R_1} := \min\{a_{r_1}, \ldots, a_n\}.$$

If $a_{L_1} = a_{l_1}$, then we have $\mathbf{pop}(l_1)$, so $L_1 < l_1$. Similarly, if $a_{R_1} = a_{r_1}$, then we have $\mathbf{pop}(r_1 - 1)$, so $R_1 > r_1$. Also, if $a_{L_1} \geq a_{R_1}$, then

$$\begin{aligned}\frac{W_\mu(a)}{c(1,\mu)} &\prec_N \delta^{1/2}(a) d(\mu)^{O_N(1)} \prod_{j=L}^{l-1} (a_{j+1}/a_j)^N \prod_{p=L_1}^{l_1-1} (a_{p+1}/a_p)^N \prod_{i=r_1}^{R_1-1} (a_{i+1}/a_i)^N \prod_{k=r}^{R-1} (a_{i+1}/a_i)^N \\ &\leq \delta^{1/2}(a) a_1^N d(\mu)^{O_N(1)}.\end{aligned}$$

So, we may assume that $a_{L_1} < a_{R_1}$. Thus, we obtained nested pairs $1 < l < l_1 < r_1 <$

$r < n$ such that

$$\max\{a_1, \ldots, a_{l_1}\} \leq \min\{1, \min(a_{r_1}, \ldots, a_n\}\}. \tag{5.3.2}$$

Proceeding in this way, we eventually, as there are only finitely, say P, many steps, obtain $l_P = r_P$ or $l_P = r_P - 1$ with similar properties as in (5.3.1) or (5.3.2). In either case, we arrive at **pop**, thus, a contradiction. □

5.3.1 A few auxiliary notations

In the remainder of this section, we prove the required decomposition of W_μ into partial M-Whittaker functions as well as the required bounds. Here, by N_r, A_r, K_r... etc., we denote the maximal unipotent subgroup of upper triangular matrices, the positive diagonal subgroup, and the maximal compact $\mathrm{O}(r)$... in $\mathrm{GL}_r(\mathbb{R})$, respectively. For $\mu \in \mathbb{C}^r$, by W_μ (suppressing r) we denote the spherical Whittaker function (with our chosen normalization as in (4.1.12)) on $\mathrm{GL}_r(\mathbb{R})$ with parameters μ. For $a := \mathrm{diag}(a_1, \ldots, a_n) \in A$, by a^r, we denote the element $\mathrm{diag}(a_1, \ldots, a_r) \in A_r$. Let us denote $(\alpha)^s := (\alpha_1, \ldots, \alpha_s)$ for $s \leq n$ and $\sum \alpha := \alpha_1 + \ldots \alpha_n$ for any $\alpha \in \mathbb{C}^n$. We abbreviate the condition $\Re(\alpha_i) \geq 0$ (resp. > 0) for all i by $\Re(\alpha) \geq 0$ (resp. > 0). By a *regular* α, we mean that the coordinates α_j are distinct. By S_r, we will denote the symmetric group of r letters, which is isomorphic to the Weyl group of $\mathrm{GL}(r)$.

We record that the residue of $\Gamma_\mathbb{R}(s)$ at $s = -2n$ for any $n \in \mathbb{Z}_{\geq 0}$ is $2\frac{(-\pi)^n}{n!} = \frac{2(-1)^n}{\Gamma_\mathbb{R}(2n+2)}$. Let $\nu \in \mathbb{C}^r$ and $\nu' \in \mathbb{C}^{r'}$ with $r > r'$. Let

$$\{\nu_i - \nu'_j \mid 1 \leq i \leq r, 1 \leq j \leq r'\} = A(\nu, \nu') \cup B(\nu, \nu'),$$

where $A(\nu, \nu')$ is the set of elements that are of form $2\mathbb{Z}_{\leq 0}$, i.e., a possible pole of $\Gamma_\mathbb{R}$ and $B(\nu, \nu')$ is the compliment. We define

$$L(\nu, \nu') := \prod_{a \in A(\nu,\nu')} \mathrm{res}_{s=a}\Gamma_\mathbb{R}(s) \prod_{b \in B(\nu,\nu')} \Gamma_\mathbb{R}(b). \tag{5.3.3}$$

We use these notations in the rest of the section.

5.3.2 Integral representation of the spherical Whittaker function

By applying the Whittaker–Plancherel formula (4.1.2) and the $\mathrm{GL}(n+1) \times \mathrm{GL}(n)$ local functional equation (4.1.5), we arrive at

$$V\left[\begin{pmatrix} C(\Pi) & \\ & g \end{pmatrix} w\right] = \int_{\hat{G}} \omega_\pi(-1)^n \Theta(\pi, \Pi) \sum_{W \in \mathcal{B}(\pi)} W(gw')\langle f, W\rangle d\mu_p(\pi),$$

We choose $\mathcal{B}(\pi) := \{W\}$ containing an ONB of π of K-types. Recall that f is spherical (see (5.1.1)), so we conclude that for all $W \in \mathcal{W}(\pi, \psi)$ with non-trivial K-type, $\langle f, W\rangle =$

0. Applying $\mathcal{D}^l$, we obtain that

$$\mathcal{D}^l V\left[\begin{pmatrix} C(\Pi) & \\ & g\end{pmatrix} w\right] = \int_{\hat{G}_0} \Theta(\pi,\Pi)\mathcal{D}^l W_\pi(g)\langle f, W_\pi\rangle d\mu_p(\pi), \tag{5.3.4}$$

where W_π is an L^2-normalized spherical vector in π. Thus, we can also conclude that $V\left[\begin{pmatrix} C(\Pi) & \\ & g\end{pmatrix} w\right]$ is spherical. Hence, this is enough to prove Proposition 5.2.1 for $g = a \in A$. Finally, choosing $W_{\pi_\mu} := \frac{W_\mu}{\|W_\mu\|}$, using (4.1.3), (4.1.12), and (4.1.8) along with the description of the tempered spherical dual of G, we rewrite (5.3.4) as

$$\mathcal{D}^l_M V\left[\begin{pmatrix} C(\Pi) & \\ & a\end{pmatrix} w\right] = \int_{(0)^n} \Theta(\mu,\Pi)\mathcal{D}^l_M W_\mu(a)\langle f, W_\mu\rangle \frac{d\mu}{|c(\mu)|^2}. \tag{5.3.5}$$

We record the following integral representation of the spherical Whittaker function that is a corollary of Stade's formula.

Lemma 5.3.2. *Let $\nu \in \mathbb{C}^{r+1}$ with $\Re(\nu_i) \geq 0$. Then,*

$$W_\nu(a^r) = \kappa_r a_r^{\sum \nu} \int_{(0)^{r-1}} W_{\nu'}(a^{r-1}/a_r)\frac{L(\frac{1}{2},\pi_\nu\otimes\pi_{-\nu'})}{c(\nu')c(-\nu')} d\nu'$$

for some absolute constant κ_r (depending only on r).

The unspecified constant appears because of our chosen normalization in (4.1.12). From now on, we denote any unspecified constant that only depends on $n, \ldots$ by $\kappa_{n,\ldots}$, i.e., $\kappa_{n,\ldots}$ may vary from line to line.

Proof. Using (4.1.3) for $\mathrm{GL}_{r-1}(\mathbb{R})$ and proceeding as before, we can write

$$\begin{aligned} W_\nu(a^r) &= a_r^{\sum\nu} W_\nu\left[\begin{pmatrix} a^{r-1}/a_r & \\ & 1\end{pmatrix}\right] \\ &= a_r^{\sum\nu}\int_{(0)^{r-1}} W_{\nu'}(a^{r-1}/a_r)\int_{N_{n-1}\backslash\mathrm{GL}_{n-1}} W_\nu\left[\begin{pmatrix} t & \\ & 1\end{pmatrix}\right]\overline{W_{\nu'}(t)}dt\frac{d\nu'}{|c(\nu')|^2}. \end{aligned}$$

We conclude the proof noting that the inner integral is a constant multiple of $L(\frac{1}{2},\pi_\nu\otimes\overline{\pi_{\nu'}})$ by Stade's formula [57, Theorem 3.4]. □

We abbreviate $\delta^{-1/2}W$, for any GL_r Whittaker function W, with W'. If $\Re(\nu) > 0$, then shifting the contour $\nu' \mapsto \nu'+1/2$ in the ν' integral in Lemma 5.3.2 (without crossing any pole), we obtain

$$W'_\nu(a) = \kappa_r a_r^{\sum\nu}\int_{(0)^{r-1}} W'_{\nu'}(a^{r-1}/a_r)\frac{L(\nu,\nu')}{c(\nu')c(-\nu')}d\nu'. \tag{5.3.6}$$

From now on, we work with W' instead of W[ii].

We fix $1 \leq s \leq n$ for the remainder. Let $\Re(\mu) > 0$ be small enough. In (5.3.6) we shift the contours of the ν integrals to some positive quantity so that the integrand does not cross any polar hyperplanes. For instance, we may choose $\Re(\nu) > 0$ such that $\max_j \Re(\nu_j) < \min_i \Re(\mu_i)$. We obtain

$$W'_\mu(a) = \kappa_n a_n^{\sum\mu} \int W'_\nu(a^{n-1}/a_n) \frac{L(\mu,\nu)}{c(\nu)c(-\nu)} d\nu,$$

where the contours are the vertical lines with real parts, as stated above. In this section, we will typically not specify these types of contours explicitly. If the contours are unspecified, then we implicitly assume that the contours are vertical lines on the left of all possible poles and very close to the vertical lines with real parts being zero, as described above. In the RHS, we expand W'_ν using (5.3.6) exactly the same as before to obtain

$$W'_\mu(a) = \kappa_n a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\nu)c(-\nu)} \int W'_{\nu'}(a^{n-2}/a_{n-1}) \frac{L(\nu,\nu')}{c(\nu')c(-\nu')} d\nu' d\nu.$$

Proceeding in this way, we arrive at an iterated integral representation of W' as

$$W'_\mu(a) = \kappa_n a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\nu)c(-\nu)} \cdots \int W'_\tau(a^{s+1}/a_{s+2}) \frac{L(\gamma,\tau)}{c(\tau)c(-\tau)} d\tau d\gamma \dots d\nu. \tag{5.3.7}$$

5.3.3 Decomposition of the spherical Whittaker function

Now, we prepare for the decomposition of the Whittaker function. We define some power series that are analogues of the I-Bessel function on GL(2) (see the relevant discussion in §3.1.2). Let $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau) > 0$ small enough, and $k \in \mathbb{Z}^s_{\geq 0}$. We define

$$P_k(\tau) := \frac{L(\tau, (\tau)^s + 2k)}{c(\tau)c(-\tau)c((\tau)^s + 2k)c(-(\tau)^s - 2k)}, \tag{5.3.8}$$

and

$$M_\tau(a^{s+1}) := \sum_{k \in \mathbb{Z}^s_{\geq 0}} P_k(\tau) W'_{(\tau)^s + 2k}(a^s/a_{s+1}). \tag{5.3.9}$$

In the next four lemmata, we prove the decomposition of W' into M inductively (see the statement we aim for also in Lemma 5.3.6). For ease of the reader, we describe the themes of these technical lemmata. In Lemma 5.3.3, we prove the base case of the induction. We being with the innermost integral in (5.3.7) and shift all the contours to infinity. The integrand crosses finitely many families of infinitely many poles. We collect the residues and construct the M-Whittaker functions (more precisely, the function $M^1 = M$,

[ii] This is because if we work with W , then the normalization by $\delta^{-1/2}$ will appear in every equation, somewhat unimportantly.

according to the definition in (5.3.9) and (5.3.10)) as power series. Then, we replace the inner integral with such an M-Whittaker function. We do the similar contour shifting process for the second innermost integral of (5.3.7), and by similarly collecting residues and making power series, we obtain similar M-Whittaker functions (more precisely, M^2, as in the definition in (5.3.10)). Inductively, we define certain partial M-Whittaker functions in (5.3.10). Finally, we prove the inductive step of the decomposition in the proof of Lemma 5.3.6.

A similar decomposition in the case of $s = 1$ appeared in [11, 26]. In both articles, the authors proved the results by the method of the differential equation, while we prove it by spectral analysis and the zeta integrals.

Lemma 5.3.3. *For $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau) > 0$ and small enough,*

$$\frac{1}{c(\tau)c(-\tau)} W'_\tau(a^{s+1}) = \kappa_s a_{s+1}^{\sum \tau} \sum_{\sigma \in S_{s+1}} M_{\sigma\tau}(a^{s+1}),$$

such that M_τ are entire in τ.

Proof. We prove the equality for regular τ so that the result will follow by analyticity. By using (5.3.6), we write

$$W'_\tau(a^{s+1}) = \kappa_s a_{s+1}^{\sum \tau} \int_{(0)^s} W'_z(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz.$$

We want to shift the contour of z_1 to ∞. To justify this, we first shift the z_1 contour to $\Re(z_1) = 2N+1$ to collect residues at the poles and estimate the following shifted integral

$$\int_{(0)^{s-1}} \int_{(2N+1)} W'_{z_1,z'}(a^s/a_{s+1}) \frac{L(\tau, z')L(\tau, z_1)}{c(z')c(-z') \prod_{i=2}^{s} \Gamma_\mathbb{R}(z_1 - z_s)\Gamma_\mathbb{R}(z_s - z_1)} dz_1 dz'.$$

We perturb the z' contour a little bit so that we can apply Lemma 4.1.5 to obtain (dropping the entries of $W_{z_1,z'}$)

$$W'_{z_1,z'} \ll_a |c(1, -z')| \prod_{i=2}^{s} |\Gamma_\mathbb{R}(1 + z_1 - z_s)|.$$

Using Stirling's approximation,

$$\frac{c(1, -z')}{c(-z')} \prod_{i=2}^{s} \frac{\Gamma_\mathbb{R}(1 + z_1 - z_s)}{\Gamma_\mathbb{R}(z_1 - z_s)} \ll \prod_{i=2}^{s} (1 + |z_i|)^{O(1)} \prod_{i=2}^{s} |z_1 - z_i|^{O(1)}.$$

On the other hand,

$$L(\tau, z_1) \ll (N!)^{-s-1} \prod_{i=1}^{s+1} \Gamma_\mathbb{R}(\tau_j - 1/2 - \Im(z_1)).$$

Writing z_1 with $\Re(z_1) = 2N+1$ as $z_1 + 2N + 1$ with $\Re(z_1) = 0$ we obtain, by Stirling's estimate, that the integral is bounded by

$$\ll_\tau \frac{N^{O(1)}}{(N!)^{s+1}} \int E_\tau(z) \prod_{i=2}^{s} (1+|z_i|)^{O(1)} |z_1 - z_i|^{O(1)} \prod_{k=1}^{N} |z_i - z_1 - k|,$$

where

$$E_\tau(z) := \exp\left[-\sum_{i,j} |\Im(\tau_i - z_j)| + \sum_{i \neq j} |\Im(z_i - z_j)| \right] \ll_\tau \exp\left[-\sum_i |\Im(z_i)| \right].$$

We then estimate

$$(1+|z_i|)^{O(1)} |z_1 - z_i|^{O(1)} \prod_{k=1}^{N} |z_i - z_1 - k| \ll z_1^{O(1)} z_i^{O(1)} \sum_{k_i \leq N; r} k_1 \dots k_r |z_1 - z_i|^{N-r}.$$

Upon integrating this against $\exp[-\sum_i |\Im(z_i)|]$, we obtain that the integral is bounded by $\frac{N^{O(1)}}{(N!)^{s+1}} \prod_{i=1}^{s} (N + O(1))! \ll \frac{N^{O(1)}}{N!}$, which tends to zero as $N \to \infty$.

Next, we denote $z' := (z_2, \dots, z_s)$ and shift the z_1 contour to infinity. We cross (simple) poles and gather the corresponding residues to obtain

$$\begin{aligned}
&\int_{(0)^s} W'_z(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz \\
&= \int_{(0)^{s-1}} \frac{\prod_{i=1}^{s+1} \prod_{j=2}^{s} \Gamma_\mathbb{R}(\tau_i - z_j)}{|c(z_2, \dots, z_s)|^2} \int_{(0)} W'_z(a^s/a_{s+1}) \frac{\prod_{i=1}^{s+1} \Gamma_\mathbb{R}(\tau_i - z_1)}{\prod_{j=2}^{s} \Gamma_\mathbb{R}(z_1 - z_j) \Gamma_\mathbb{R}(z_j - z_1)} dz_1 dz' \\
&= \sum_{i=1}^{s+1} \int_{(0)^{s-1}} \frac{\prod_{i=1}^{s+1} \prod_{j=2}^{s} \Gamma_\mathbb{R}(\tau_i - z_j)}{|c(z_2, \dots, z_s)|^2} \sum_{k_1=0}^{\infty} \frac{W'_{\tau_i + 2k_1, z_2, \dots, z_s}(a^s/a_{s+1}) L(\tau, \tau_i + 2k_1)}{\prod_{j=2}^{s} \Gamma_\mathbb{R}(\tau_i - z_j + 2k_1) \Gamma_\mathbb{R}(z_j - \tau_i - 2k_1)} dz'.
\end{aligned}$$

We record the functional equation for $\Gamma_\mathbb{R}$ that $\forall m \in \mathbb{Z}$ as

$$\Gamma_\mathbb{R}(s)\Gamma_\mathbb{R}(-s) = (-1)^{m-1} \frac{2\pi}{s} \Gamma_\mathbb{R}(s + 2m) \Gamma_\mathbb{R}(2 - s - 2m).$$

Using this, we can specify that

$$\prod_{j=2}^{s} \frac{\Gamma_\mathbb{R}(\tau_i - z_j)}{\Gamma_\mathbb{R}(\tau_i - z_j + 2k_1) \Gamma_\mathbb{R}(z_j - \tau_i - 2k_1)} = \prod_{j=2}^{s} \frac{(-1)^{k_1} (\tau_i - z_j + 2k_1) \Gamma_\mathbb{R}(\tau_i - z_j)}{2\pi \Gamma_\mathbb{R}(\tau_i - z_j) \Gamma_\mathbb{R}(z_j - \tau_i + 2)}.$$

Hence, in the i'th summand above, any z_2 only has family of poles at τ_j for $j \neq i$. Now, we shift the z_2 contour to ∞ (upon similar justification as in the case of z_1). Proceeding in this way, we obtain that for any s-tuple τ' consisting of distinct elements from $\{\tau_1, \ldots, \tau_{s+1}\}$

$$\int_{(0)^s} W'_z(a^s/a_{s+1}) \frac{L(\tau, z)}{c(z)c(-z)} dz = \sum_{\tau'} \sum_{k \in Z^s_{\geq 0}} C_k(\tau) W'_{\tau'+2k}(a^s/a_{s+1}),$$

where

$$C_k(\tau, \tau') := \frac{L(\tau, \tau' + 2k)}{c(\tau' + 2k)c(-\tau' - 2k)}.$$

If $\sigma \in S_{s+1}$ such that $(\sigma\tau)^s = \tau'$, then

$$P_k(\sigma\tau) = \frac{C_k(\tau, \tau')}{c(\tau)c(-\tau)}.$$

Thus, we conclude the proof of the decomposition. To confirm that M_τ is holomorphic, it is enough to check that $P_k(\tau)$ are holomorphic (W' is entire in its parameters, see [33]) and the series defining M_τ in (5.3.9) is locally uniformly convergent. Here, we only check that $P_k(\tau)$ are holomorphic. Later in Lemma 5.3.7, we estimate P_k that, along with Lemma 4.1.5, implies locally uniform convergence. To check holomorphicity, we first note that

$$c(\tau)c(-\tau) = c((\tau)^s)c(-(\tau)^s) \prod_{j=1}^{s} \Gamma_{\mathbb{R}}(\tau_{s+1} - \tau_j)\Gamma_{\mathbb{R}}(\tau_j - \tau_{s+1}).$$

and, thus, $P_k(\tau)$ equals

$$\prod_{j=1}^{s} \frac{(-\pi)^{k_j}}{k_j!} \prod_{i>j} \frac{\Gamma_{\mathbb{R}}(\tau_i - \tau_j - 2k_j)}{\Gamma_{\mathbb{R}}(\tau_j - \tau_i)\Gamma_{\mathbb{R}}(\tau_i - \tau_j)} \prod_{i<j} \frac{\Gamma_{\mathbb{R}}(\tau_i - \tau_j - 2k_j)}{\Gamma_{\mathbb{R}}(\tau_j + 2k_j - \tau_i - 2k_i)\Gamma_{\mathbb{R}}(\tau_i + 2k_i - \tau_j - 2k_j)}.$$

We confirm that each factor in this expression of $P_k(\tau)$ is holomorphic, thus, we conclude. □

Now using Lemma 5.3.3, we can expand the inner most integral in (5.3.7) and rewrite (5.3.7) as

$$W'_\mu(a) = \kappa_{s,n} a_n^{\sum \mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum \nu} \frac{L(\mu, \nu)}{c(\nu)c(-\nu)} \cdots$$
$$\cdots \int_{(0)^{s+1}} \left(\frac{a_{s+1}}{a_{s+2}}\right)^{\sum \tau} L(\gamma, \tau) \sum_{\sigma \in S_{s+1}} M_{\sigma\tau}(a^{s+1}) d\tau d\gamma \ldots d\nu.$$

We interchange the innermost integral with the finite sum over S_{s+1}. We do a change of variable $\sigma\tau \mapsto \tau$ and rewrite as

$$W'_\mu(a) = \kappa_{s,n} a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\nu)c(-\nu)} \dots (s+1)! \int_{(0)} \left(\frac{a_{s+1}}{a_{s+2}}\right)^{\tau_{s+1}} L(\gamma, \tau_{s+1}) \\ \times \int_{(0)^s} \left(\frac{a_{s+1}}{a_{s+2}}\right)^{\sum(\tau)^s} L(\gamma, (\tau)^s) M_\tau(a^{s+1}) d(\tau)^s d\tau_{s+1} d\gamma \dots d\nu.$$

Now, we shift contours of the last integral to ∞. The family of the poles occurs at $(\tau)^s_j = \gamma_i + 2\mathbb{Z}_{\geq 0}$ for all $1 \leq j \leq s$ and some $1 \leq i \leq s+1$. The residues are of the form

$$\left(\frac{a_{s+1}}{a_{s+2}}\right)^{\sum\gamma'+2l} L(\gamma, \gamma'+2l) M_{\gamma'+2l,\tau_{s+1}}$$

for $l \in Z^s_{\geq 0}$ and $\gamma' \in \mathbb{C}^s$ with $\gamma'_j \in \{\gamma_1, \dots, \gamma_{s+2}\}$. However, some of these residues do not occur in the asymptotic expansion of W', such as the residues with a parameter γ' with $\gamma'_1 = \gamma'_2$. In fact, the terms in the asymptotic expansion come from the residues where the coordinates of γ' are distinct.[iii] But, due to Lemma 5.3.4 where we show below that the residues other than

$$\left(\frac{a_{s+1}}{a_{s+2}}\right)^{\sum(\sigma\gamma)^s+2l} L(\gamma, (\sigma\gamma)^s + 2l) M_{(\sigma\gamma)^s+2l,\tau_{s+1}}$$

for $\sigma \in S_{s+2}$, vanish identically. In other words, the family of poles (of form $\gamma' + 2\mathbb{Z}^s$) of the integrand only occur at γ, which has distinct coordinates.

We follow the same method of collecting residues and construct the power series. We continue this process in (5.3.7) until the outer most integral, and recursively define

$$M^1_\tau(a^{s+1}) := M_\tau(a^{s+1})$$

for $r \geq 1$, and for $\alpha \in \mathbb{C}^{s+r}$

$$M^r_\alpha(a^{s+r}) := \int_{(0)^{r-1}} \left(\frac{a_{s+r-1}}{a_{s+r}}\right)^{\sum z} L(\alpha, z) \\ \sum_{l\in\mathbb{Z}^s_{\geq 0}} \left(\frac{a_{s+r-1}}{a_{s+r}}\right)^{\sum(\alpha)^s+2l} \frac{L(\alpha, (\alpha)^s+2l)}{c(\alpha)c(-\alpha)} M^{r-1}_{(\alpha)^s+2l,z}(a^{s+r-1}) dz. \quad (5.3.10)$$

In each stage of the contour shifting, we must prove that the integrand is holomorphic at the *non-regular* points, as discussed above. We show this, inductively, in Lemma 5.3.4, which is the base case, and in Lemma 5.3.5, through which we prove the inductive step. These lemmata can be thought of as higher rank analogues of the fact that $I_n = I_{-n}$ for any natural number n, where I denotes the classical I-Bessel function.

[iii] We assume that γ is regular.

Lemma 5.3.4. *Let* $\tau \in \mathbb{C}^{s+1}$ *such that* $\Re(\tau) \geq 0$ *and* $\tau_a \equiv \tau_b \mod 2\mathbb{Z}$ *for* $1 \leq a \neq b \leq s$. *Then,* M_τ *is identically zero.*

Proof. Let $1 \leq a < b \leq s$. Expanding out the definition (5.3.8) of P_k gives

$$P_k(\tau) = \frac{\Gamma_{\mathbb{R}}(\tau_a - \tau_b - 2k_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a - 2k_a)\mathrm{res}_{s=-2k_a}\Gamma_{\mathbb{R}}(s)\mathrm{res}_{s=-2k_b}\Gamma_{\mathbb{R}}(s)}{\Gamma_{\mathbb{R}}(\tau_a - \tau_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a)\Gamma_{\mathbb{R}}(\tau_a - \tau_b + 2k_a - 2k_b)\Gamma_{\mathbb{R}}(\tau_b - \tau_a + 2k_b - 2k_a)}$$
$$\times \frac{\prod_{j\neq a,b} L(\tau, \tau_j + 2k_j) \prod_{i\neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_a - 2k_a)\Gamma_{\mathbb{R}}(\tau_i - \tau_b - 2k_b)}{\prod_{(i,j)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\tau_i - \tau_j)\Gamma_{\mathbb{R}}(\tau_i - \tau_j + 2k_i - 2k_j)}.$$

Computing the residues, we obtain that the above equals

$$\frac{\pi^{-4}(\tau_a - \tau_b)(\tau_b - \tau_a + 2k_b - 2k_a)}{\Gamma_{\mathbb{R}}(2k_a + 2)\Gamma_{\mathbb{R}}(2k_b + 2)\Gamma_{\mathbb{R}}(\tau_a + 2k_a - \tau_b + 2)\Gamma_{\mathbb{R}}(\tau_b + 2k_b - \tau_a + 2)}$$
$$\times Q_k(\tau) \times \frac{\prod_{j\neq a,b} L(\tau, \tau_j + 2k_j)}{\prod_{(i,j)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\tau_i - \tau_j)},$$

where

$$Q_k(\tau) := \frac{\prod_{i\neq a,b} \Gamma_{\mathbb{R}}(\tau_i - \tau_a - 2k_a)\Gamma_{\mathbb{R}}(\tau_i - \tau_b - 2k_b)}{\prod_{(i,j)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\tau_i - \tau_j + 2k_i - 2k_j)}.$$

Without loss of generality, suppose that $\tau_a - \tau_b = 2l \geq 0$. Note that $\Gamma_{\mathbb{R}}(\tau_b - \tau_a + 2k_b + 2)^{-1} = 0$ for $k_b < l$, by changing $k_b \mapsto k_b + l$ and replacing $\tau_b + 2l = \tau_a$ only in the first two factors, we obtain from (5.3.9) that

$$M_\tau = \sum_{k\in\mathbb{Z}^s_{\geq 0}} W'_{\ldots,\tau_a+2k_a,\ldots,\tau_a+2k_b,\ldots} \times Q_k(\ldots, \tau_a, \ldots, \tau_a, \ldots) \times \frac{\prod_{j\neq a,b} L(\tau, \tau_j + 2k_j)}{\prod_{(i,j)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\tau_i - \tau_j)}$$
$$\times \frac{\pi^{-4}(2l)(k_b - k_a)}{\Gamma_{\mathbb{R}}(2k_a + 2)\Gamma_{\mathbb{R}}(2k_b + 2l + 2)\Gamma_{\mathbb{R}}(2k_a + 2l + 2)\Gamma_{\mathbb{R}}(2k_b + 2)}.$$

Let σ_{ab} be the element in the Weyl group that transposes the ath and bth elements of τ and fixes everything else. Then, by performing a similar calculation, we can check that

$$M_{\sigma_{ab}\tau} = \sum_{k\in\mathbb{Z}^s_{\geq 0}} W'_{\ldots,\tau_a+2k_a,\ldots,\tau_a+2k_b,\ldots} \times Q_k(\ldots, \tau_a, \ldots, \tau_a, \ldots) \times \frac{\prod_{j\neq a,b} L(\tau, \tau_j + 2k_j)}{\prod_{(i,j)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\tau_i - \tau_j)}$$
$$\times \frac{\pi^{-4}(-2l)(k_b - k_a)}{\Gamma_{\mathbb{R}}(2k_a + 2l + 2)\Gamma_{\mathbb{R}}(2k_b + 2)\Gamma_{\mathbb{R}}(2k_a + 2)\Gamma_{\mathbb{R}}(2k_b + 2l + 2)}$$
$$= -M_\tau.$$

On the other hand, the following can be easily checked from (5.3.8) that

$$P_{\sigma_{ab}k}(\sigma_{ab}\tau) = P_k(\tau).$$

Thus, using the fact that the Whittaker function is invariant under the Weyl group action on its parameters, we also conclude that

$$M_\tau = M_{\sigma_{ab}\tau},$$

hence, the conclusion. $\square$

Lemma 5.3.5. *Let* $\alpha \in \mathbb{C}^{s+r}$ *such that* $\Re(\alpha) \geq 0$ *and* $\alpha_a \equiv \alpha_b \mod 2\mathbb{Z}$ *for* $1 \leq a \neq b \leq s$*. Then,* M_α^r *is identically zero.*

Proof. We prove by inducting on r.Note that the base case $r = 1$ is proved in Lemma 5.3.4, we assume the claim is also true for $r \geq 1$. We consider the inner most sum of M_α^{r+1}, which is

$$\sum_{l \in \mathbb{Z}_{\geq 0}^s} \left(\frac{a_{s+r}}{a_{s+r+1}} \right)^{\sum(\alpha)^s + 2l} \frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)} M^r_{(\alpha)^s+2l,z}(a^{s+r-1}). \tag{5.3.11}$$

We take a similar proof path as in Lemma 5.3.4. Suppose that σ_{ab} is the element in the Weyl group that only transposes ath and bth elements of α, . Clearly, $M^{r+1}_{\sigma_{ab}\alpha} = M^{r+1}_\alpha$, as before. We show that when $\alpha_a - \alpha_b \in 2\mathbb{Z}$, then $M^{r+1}_{\sigma_{ab}\alpha} = -M^{r+1}_\alpha$, which yields the claim. The coefficient

$$\begin{aligned} &\frac{L(\alpha, (\alpha)^s + 2l)}{c(\alpha)c(-\alpha)} \\ &= \frac{4\pi^{-2}\Gamma_\mathbb{R}(\alpha_a - \alpha_b - 2l_b)\Gamma_\mathbb{R}(\alpha_b - \alpha_a - 2l_a)(-1)^{l_a+l_b}}{\Gamma_\mathbb{R}(\alpha_a - \alpha_b)\Gamma_\mathbb{R}(\alpha_b - \alpha_a)\Gamma_\mathbb{R}(2l_a+2)\Gamma_\mathbb{R}(2l_b+2)} \frac{\prod_{i,j \notin \{a,b\}} L(\alpha_i, \alpha_j + 2l_j)}{\prod_{(e \neq f) \neq (a,b),(b,a)} \Gamma_\mathbb{R}(\alpha_e - \alpha_f)} \\ &\times \prod_{i \neq a,b} \Gamma_\mathbb{R}(\alpha_i - \alpha_a - 2l_a)\Gamma_\mathbb{R}(\alpha_i - \alpha_a - 2l_b) \prod_{j \neq a,b} \Gamma_\mathbb{R}(\alpha_a - \alpha_j - 2l_j)\Gamma_\mathbb{R}(\alpha_b - \alpha_j - 2l_j) \end{aligned}$$

has i and j varying over $1, \ldots, r+s$ and $1, \ldots, s$, respectively. Using the functional equation for $\Gamma_\mathbb{R}$, we obtain that the above becomes

$$\begin{aligned} &= \frac{2\pi^{-3}(\alpha_b - \alpha_a)\Gamma_\mathbb{R}(\alpha_b - \alpha_a - 2l_a)(-1)^{l_a}}{\Gamma_\mathbb{R}(\alpha_b - \alpha_a + 2l_b + 2)\Gamma_\mathbb{R}(2l_a+2)\Gamma_\mathbb{R}(2l_b+2)} \frac{\prod_{i,j \notin \{a,b\}} L(\alpha_i, \alpha_j + 2l_j)}{\prod_{(e \neq f) \neq (a,b),(b,a)} \Gamma_\mathbb{R}(\alpha_e - \alpha_f)} \\ &\times \prod_{i \neq a,b} \Gamma_\mathbb{R}(\alpha_i - \alpha_a - 2l_a)\Gamma_\mathbb{R}(\alpha_i - \alpha_a - 2l_b) \prod_{j \neq a,b} \Gamma_\mathbb{R}(\alpha_a - \alpha_j - 2l_j)\Gamma_\mathbb{R}(\alpha_b - \alpha_j - 2l_j). \end{aligned}$$

Without loss of generality, suppose that $a < b$, $\alpha_a - \alpha_b = 2t \geq 0$, and $t \in \mathbb{Z}$. As $M_\alpha^r = 0$ by the induction hypothesis, by the above computation, the summand vanishes for $l_b < t$. In the sum (5.3.11), we change the variable $l_b \mapsto l_b + t$ and obtain that (5.3.11) equals

$$\begin{aligned} &\sum_{l \in \mathbb{Z}_{\geq 0}^s} \left(\frac{a_{s+r}}{a_{s+r+1}} \right)^{2t + \sum(\alpha)^s + 2l} \times Q_l(\alpha) \times \frac{2\pi^{-3}(\alpha_b - \alpha_a)(-1)^{l_a}}{\Gamma_\mathbb{R}(2l_b+2)\Gamma_\mathbb{R}(2l_a+2)\Gamma_\mathbb{R}(2l_b+2t+2)} \\ &\times \lim_{\alpha_a - \alpha_b \to 2t} \Gamma_\mathbb{R}(\alpha_b - \alpha_a - 2l_a) M^r_{\ldots,\alpha_a+2l_a,\ldots,\alpha_b+2l_b+2t,\ldots,z}(a^{s+r-1}), \end{aligned}$$

where

$$Q_l(\alpha) := \frac{\prod_{i,j\notin\{a,b\}} L(\alpha_i, \alpha_j + 2l_j)}{\prod_{(e\neq f)\neq(a,b),(b,a)} \Gamma_{\mathbb{R}}(\alpha_e - \alpha_f)} \prod_{j\neq a,b} \Gamma_{\mathbb{R}}(\alpha_a - \alpha_j - 2l_j)\Gamma_{\mathbb{R}}(\alpha_b - \alpha_j - 2l_j)$$
$$\times \prod_{i\neq a,b} \Gamma_{\mathbb{R}}(\alpha_i - \alpha_a - 2l_a)\Gamma_{\mathbb{R}}(\alpha_i - \alpha_a - 2l_b).$$

We compute the last limit by letting $\beta := \alpha_b - \alpha_a + 2t$, which then equals

$$\begin{aligned}&\operatorname{res}_{s=-2t-2l_a}\Gamma_{\mathbb{R}}(s) \lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a,\dots,\alpha_a+\beta+2l_b,\dots,z}(a^{s+r-1})\\ &= \frac{2(-1)^{l_a+t}}{\pi\Gamma_{\mathbb{R}}(2l_a+2t+2)} \lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a,\dots,\alpha_a+\beta+2l_b,\dots,z}(a^{s+r-1}).\end{aligned}$$

Thus, we obtain that (5.3.11) equals

$$\sum_{l\in\mathbb{Z}^s_{\geq 0}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{2t+\sum(\alpha)^s+2l} \times \frac{4\pi^{-4}(-1)^t Q_l(\alpha)}{\Gamma_{\mathbb{R}}(2l_b+2)\Gamma_{\mathbb{R}}(2l_a+2)\Gamma_{\mathbb{R}}(2l_b+2t+2)\Gamma_{\mathbb{R}}(2l_a+2t+2)}$$
$$\times (\alpha_b - \alpha_a) \lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a,\dots,\alpha_a+2l_b+\beta,\dots,z}(a^{s+r-1}).$$

By performing a similar computation, we obtain that the relevant sum, as in (5.3.11), in the corresponding expression of $M_{\sigma_{ab}\alpha}$ equals

$$\sum_{l\in\mathbb{Z}^s_{\geq 0}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{2t+\sum(\alpha)^s+2l} \times \frac{4\pi^{-4}(-1)^t Q_l(\alpha)}{\Gamma_{\mathbb{R}}(2l_b+2)\Gamma_{\mathbb{R}}(2l_a+2)\Gamma_{\mathbb{R}}(2l_b+2t+2)\Gamma_{\mathbb{R}}(2l_a+2t+2)}$$
$$\times (\alpha_a - \alpha_b) \lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a-\beta,\dots,\alpha_a+2l_b,\dots,z}(a^{s+r-1}).$$

Thus, the proof is complete if we can show that

$$\lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a-\beta,\dots,\alpha_a+2l_b,\dots,z}(a^{s+r-1}) = \lim_{\beta\to 0} \beta^{-1} M^r_{\dots,\alpha_a+2l_a,\dots,\alpha_a+2l_b+\beta,\dots,z}(a^{s+r-1}).$$

Both limits exist by the induction hypothesis (M^r has a zero at a non-regular point). The above equality follows from twisting the quantity in the limit by $|\det|^\beta$ and letting $\beta \to 0$. □

Finally, we prove the inductive step of the decomposition of W' into an M-Whittaker function, whose base case is proven in Lemma 5.3.3.

Lemma 5.3.6. *Let $\mu \in E_n(\epsilon)$ for some $\epsilon > 0$. Then, there exists an absolute constant $\kappa_{s,n}$ such that*

$$\frac{1}{c(\mu)c(-\mu)} W'_\mu(a) = \kappa_{s,n} a_n^{\sum\mu} \sum_{\sigma\in S_n} M^{n-s}_{\sigma\mu}(a),$$

and M_μ is holomorphic in $\Re(\mu) \geq 0$.

Proof. From the definition (5.3.10) and estimates in Lemma 5.3.7. holomorphicity of M_μ follows. To prove the equality in the statement, we induct on r. The base case $r = 1$ was proven in Lemma 5.3.3. At the rth intermediate stage, the expression of $W'_\mu(a)$ looks like

$$\begin{aligned}
&\frac{1}{c(\mu)c(-\mu)} W'_\mu(a) \\
&= \kappa_{s,r,n} a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\mu)c(-\mu)} \cdots \int_{(0)^{s+r}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum\theta} \frac{L(\eta,\theta)}{c(\eta)c(-\eta)} M^r_\theta(a^{s+r}) d\theta \ldots d\nu \\
&= \kappa_{s,r,n} a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\mu)c(-\mu)} \cdots \int_{(0)^{r}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum\theta'} L(\eta,\theta') \\
&\times \int_{(0)^s} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum(\theta)^s} \frac{L(\eta,(\theta)^s)}{c(\eta)c(-\eta)} M^r_\theta(a^{s+r}) d(\theta)^s d\theta' \ldots d\nu,
\end{aligned}$$

where $\theta' := (\theta_{s+1}, \ldots, \theta_{s+r})$. Now, we shift the contours to infinity in the last integral. Thus, employing Lemma 5.3.5, by collecting residues, we obtain for some constant $d' = d_{r,s}$ that

$$\begin{aligned}
&\frac{1}{c(\mu)c(-\mu)} W'_\mu(a) \\
&= \kappa_{s,r,n} a_n^{\sum\mu} \int \left(\frac{a_{n-1}}{a_n}\right)^{\sum\nu} \frac{L(\mu,\nu)}{c(\mu)c(-\mu)} \cdots \int_{(0)^{r}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum\theta'} L(\eta,\theta') \\
&\sum_{\sigma\in S_{s+r}} \sum_{l\in Z^s_{\geq 0}} \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum(\sigma\eta)^s+2l} \frac{L(\eta,(\sigma\eta)^s+2l)}{c(\eta)c(-\eta)} M^r_{(\sigma\eta)^s+2l,\theta'}(a^{s+r}) d\theta' \ldots d\nu.
\end{aligned}$$

By recognizing the symmetries of the variables inside the integrals and recalling (5.3.10), we can conclude. □

Now we estimate the function M^{n-s} inductively. The following lemma is used to prove the required bound in Proposition 5.2.1 for those $a \in A$ that are in the complementary case of what we considered in Lemma 5.3.1, i.e., a satisfies $\mathbf{pop}(s)$ (see definition in (5.3.1)). We loosely mention that the M^{n-s}-Whittaker functions are exponentially increasing in a_i/a_j for $1 \leq i \leq s$ and $s+1 \leq j \leq n$ (see §3.1.2), which is where we use $\mathbf{pop}(s)$ to control the increment.

Lemma 5.3.7. *Let $1 \leq s \leq n$ and a satisfies $\mathbf{pop}(s)$. Also, let $\mu \in \mathbb{C}^n$ such that $\Re(\mu) > 0$ be small enough. We define $\mu' := (\mu_1 + 2N, \ldots, \mu_s + 2N, \mu_{s+1}, \ldots, \mu_n)$ for some fixed large integer N. Then,*

$$M^{(n-s)}_{\mu'}(a) \prec_{N,\epsilon} \frac{d(\mu)^{O(1)}}{c(\Im(\mu))} \frac{a_1^{2N}}{a_n^{2sN}},$$

here $O(1)$ in the exponent of $d(\mu)$ that denotes a bounded constant depending on N and n.

Proof. We prove by inducting on r using the inductive definition in (5.3.10). The required bound of M_α^r can be sufficiently proven for $\alpha := (\mu_1+2N_1, \ldots, \mu_s+2N_s, \alpha')$ where $N_i \geq N$ and $\alpha' \in \mathbb{C}^r$ with small positive real parts. For $\alpha \in \mathbb{C}^r$, by $\iota(\alpha)$, we denote the reordering of the coordinates of α such that $\Re(\iota(\alpha)_1)) \leq \ldots \leq \Re(\iota(\alpha)_r)$. We frequently apply the Stirling approximation, and for $s \in \mathbb{C}$ having a small real part and $k \geq 0$, we have

$$|\Gamma(s-k)| \ll |\Gamma(s)|.$$

Let us first prove the claimed bound for M^1. From the definition (5.3.8), for $\tau \in \mathbb{C}^{s+1}$ with $\Re(\tau_j) \geq 2N$ for $1 \leq j \leq s$, we estimate

$$c(1, -\iota((\tau)^s+2k))P_k(\tau) := \frac{L(\tau, (\tau)^s+2k)c(1, -\iota((\tau)^s+2k))}{c(\tau)c(-\tau)c((\tau)^s+2k)c(-(\tau)^s-2k)}$$
$$\asymp \prod_{i=1}^s \frac{\pi^{k_i}}{k_i!} \frac{\Gamma_{\mathbb{R}}(\tau_{s+1}-\tau_i-2k_i)}{\Gamma_{\mathbb{R}}(\tau_{s+1}-\tau_i)\Gamma_{\mathbb{R}}(\tau_i-\tau_{s+1})} \frac{\prod_{1\leq i\neq j\leq s}\Gamma_{\mathbb{R}}(\tau_i-\tau_j-2k_j)}{\prod_{i\neq j}\Gamma_{\mathbb{R}}(\tau_j-\tau_i)c(\iota((\tau)^s+2k))} \frac{c(1, -\iota((\tau)^s+2k))}{c(-\iota((\tau)^s+2k))}.$$

The last factor in the above quantity can be bounded by $(1+\|k\|)^{O(1)}d((\tau)^s)^{O(1)}$. The second factor is $\ll |\Gamma_{\mathbb{R}}(\tau_i-\tau_{s+1})|^{-1}$. In the third factor, suppose that $\Gamma_{\mathbb{R}}(\tau_p+2k_p-\tau_q-2k_q)$ appears in $c(\iota((\tau)^s+2k))$ for some $1 \leq p \neq q \leq s$. Then,

$$\frac{\Gamma_{\mathbb{R}}(\tau_p-\tau_q-2k_q)\Gamma_{\mathbb{R}}(\tau_q-\tau_p-2k_p)}{\Gamma_{\mathbb{R}}(\tau_p-\tau_q)\Gamma_{\mathbb{R}}(\tau_q-\tau_p)\Gamma_{\mathbb{R}}(\tau_p+2k_p-\tau_q-2k_q)} \ll |\Gamma_{\mathbb{R}}(\tau_q-\tau_p)|^{-1}.$$

We can perform a similar estimate for each sub-factor in the third factor. If we define

$$\tilde{c}(\tau) := \prod_{i=1}^s \Gamma_{\mathbb{R}}(\tau_i-\tau_{s+1}) \prod_{1\leq i\neq j\leq s} \min\{|\Gamma_{\mathbb{R}}(\tau_i-\tau_j)|, |\Gamma_{\mathbb{R}}(\tau_j-\tau_i)|\},$$

then we have obtained that

$$c(1, -\iota((\tau)^s+2k))P_k(\tau) \ll |\tilde{c}(\tau)|^{-1}|\prod_{i=1}^s \frac{\pi^{k_i}}{k_i!}(1+\|k\|)^{O(1)}d(\tau)^{O(1)}.$$

Finally, using (5.3.9) and Lemma 4.1.5, we estimate

$$\begin{aligned} M_\tau(a^{s+1}) &= \sum_{k\in\mathbb{Z}^s_{\geq 0}} P_k(\tau)W'_{(\tau)^s+2k}(a^s/a_{s+1}) \\ &\prec \frac{d(\tau)^{O(1)}}{\tilde{c}(\tau)} \frac{a_1^{2N}}{a_{s+1}^{\Re\sum(\tau)^s}} \sum_{k\in\mathbb{Z}^s_{\geq 0}} \frac{\pi^{k_i}}{k_i!}(1+\|k\|)^{O(1)} \\ &\ll \frac{d(\tau)^{O(1)}}{\tilde{c}(\tau)} \frac{a_1^{2N}}{a_{s+1}^{\Re\sum(\tau)^s}}, \end{aligned}$$

where in the first inequality we employed the **pop**(s) assumption.

Now, we make an inductive hypothesis that for $r \geq 1$

$$M^r_\alpha(a^{s+r}) \prec \frac{d(\alpha)^{O(1)}}{\tilde{c}(\alpha)} \frac{a_1^{2N}}{a_{r+s}^{\Re\sum(\alpha)^s}},$$

from which, through motivation by the previous computation, we define

$$\begin{aligned} \tilde{c}(\alpha) := \prod_{i=1}^{s}\prod_{j=1}^{r} \Gamma_\mathbb{R}(\alpha_i - \alpha_{s+j}) \prod_{1\leq i<j\leq r} \Gamma_\mathbb{R}(\alpha_{s+i}-\alpha_{s+j}) \\ \times \prod_{1\leq i\neq j\leq s} \min\{|\Gamma_\mathbb{R}(\alpha_i-\alpha_j)|, |\Gamma_\mathbb{R}(\alpha_j-\alpha_i)|\}. \end{aligned}$$

We start with (5.3.10) and integrate term by term, and claim that

$$\begin{aligned} &\frac{L(\alpha,(\alpha)^s+2l)}{c(\alpha)c(-\alpha)} \int_z \left(\frac{a_{s+r}}{a_{s+r+1}}\right)^{\sum z} L(\alpha,z)\frac{d((\alpha)^s+2l)^{O(1)}}{\tilde{c}((\alpha)^s+2l,z)} d(z)^{O(1)} \\ &\prec \frac{a_1^{2N}}{a_{s+r}^{\Re\sum(\alpha)^s+2l}} |\tilde{c}(\alpha)|^{-1} \prod_{i=1}^{s} \frac{\pi^{l_i}}{l_i!}(1+\|l\|)^{O(1)} d((\alpha)^s)^{O(1)} d(\alpha_{s+1},\dots\alpha_{s+r+1})^{O(1)}. \end{aligned}$$

Employing the inductive hypothesis, we yield the proof, as in the base case.

To prove the claim,

$$\begin{aligned} &\frac{L(\alpha,(\alpha)^s+2l)L(\alpha,z)}{c(\alpha)c(-\alpha)\tilde{c}((\alpha)^s+2l,z)} \prod_{i=1}^{s}\prod_{j=1}^{r+1}\Gamma_\mathbb{R}(\alpha_i-\alpha_{s+j}) \prod_{1\leq i<j\leq r+1}\Gamma_\mathbb{R}(\alpha_{s+i}-\alpha_{s+j}) \\ &\asymp \prod_{i=1}^{s}\frac{\pi_i^l}{l_i!} \frac{\prod_{i=1}^{r+1}\prod_{j=1}^{r}\Gamma_\mathbb{R}(\alpha_{s+i}-z_j)}{c(z)\prod_{1\leq i<j\leq r+1}\Gamma_\mathbb{R}(\alpha_{s+j}-\alpha_{s+i})} \prod_{i=1}^{s}\prod_{j=1}^{r}\frac{\Gamma_\mathbb{R}(\alpha_i-z_j)}{\Gamma_\mathbb{R}(\alpha_i+2l_i-z_j)} \\ &\qquad \times \frac{\prod_{1\leq i\neq j\leq s}\Gamma_\mathbb{R}(\alpha_i-\alpha_j-2l_j)}{c((\alpha)^s)\tilde{c}((\alpha)^s+2l)} \prod_{i=1}^{r+1}\prod_{j=1}^{s}\frac{\Gamma_\mathbb{R}(\alpha_{s+i}-\alpha_j-2l_j)}{\Gamma_\mathbb{R}(\alpha_{s+i}-\alpha_j)}, \end{aligned}$$

where, as in the $r=1$ case, the fifth and third factors are $\ll 1$, and the fourth factor is $\ll \tilde{c}((\alpha)^s)$. Thus, this is sufficient to prove that the integral

$$\int_{(0)^r}\left|\frac{\prod_{i=1}^{r+1}\prod_{j=1}^{r}\Gamma_{\mathbb{R}}(\alpha_{s+i}-z_j)}{c(z)\prod_{1\leq i<j\leq r+1}\Gamma_{\mathbb{R}}(\alpha_{s+j}-\alpha_{s+i})}\right|d(z)^{O(1)}dz\ll d(\alpha_{s+1},\dots\alpha_{s+r+1})^{O(1)},$$

along with

$$d((\alpha)^s+2l)^{O(1)}\ll d((\alpha)^s)^{O(1)}(1+\|l\|)^{O(1)}$$

proves the claim. To see the estimate of the last integral, we follow the same path as in [3, Proposition 1]. We assume that $0<\Re(\alpha_{s+j})<\epsilon$, and write $\alpha_{s+j}=\Re(\alpha_{s+j})+i\beta_j$ and $z_j=it_j$. We apply the Stirling approximation to obtain that the integral is bounded by

$$d(\beta)^{O(1)}\int_{\mathbb{R}^r}\prod_{i,j}(1+|\beta_i-t_j|)^{-1/2+O(\epsilon)}\prod_{i\neq j}|t_i-t_j|^{1/2}d(t)^{O(1)}E(t,\beta)dt,$$

where E is the exponential factor given by

$$E(t,\beta):=\exp\left[-\frac{\pi}{4}\left(\sum_{i,j}|\beta_i-t_j|-\sum_{i<j}|t_i-t_j|-\sum_{i<j}|\beta_i-\beta_j|\right)\right].$$

By fixing an order among β_i, this can be checked elementarily that (as in the proof of [3, Proposition 1]) the quantity inside exp is always non-positive. Thus, the essential supports of t in this integral are bounded by polynomials in β. The integrand, other than the exponential factor, also being a polynomial in t and β, causes the integral to be bounded by some polynomial in β, so is $d(\beta)^{O(1)}$.

Finally, we conclude the proof with

$$\tilde{c}(\mu)\gg c(\Im(\mu))d(\mu)^{O(1)},$$

where $O(1)$ in the exponent means a fixed non-negative exponent that depends on N. □

We now have all the ingredients needed to prove Proposition 5.2.1, and we do so by dividing the argument into the two cases of if a has the property **pop**(s) for some $1\leq s\leq n$ or not.

Proof of Proposition 5.2.1. The primary ingredient of the proof is to shift the contours in the integral of (5.3.5). We separate the proof into two cases.

Case I: We assume that a does not satisfy **pop**(s) for any $1\leq s\leq n$. Let $d_0(\mu)$ be the eigenvalue of W_μ under $\mathcal{D}$ (recall Definition (4.1.9)). We have $d_0(it)\ll d(it)$ that follows from (4.1.6) and Lemma 4.1.2. We see that, using (4.1.7) for $M=0$, Lemma 4.1.6 for $R=0$, and Lemma 5.3.1, the RHS of (5.3.5) is bounded by

$$\mathcal{D}^lV\left[\begin{pmatrix}C(\Pi) & \\ & a\end{pmatrix}w\right]\prec_{N,p}\delta^{1/2}(a)\min(1,a_1^N)\int_{\mathbb{R}^n}d(it)^{l-p}\frac{|c(1,it)|^2}{|c(it)|^2}dt.$$

Using Stirling, $\frac{c(1,it)}{c(it)} \ll d(it)^{l'}$ for some absolute l'. Thus, the integral in the RHS above is convergent if p is sufficiently large. Hence, the proof of this case concludes.

Case II: We assume that a satisfies $\mathbf{pop}(s)$ for some given s. We use Lemma 5.3.6 in the RHS of (5.3.5), and exchange the finite sum with an integral over μ and change the variable $w\mu \mapsto \mu$. We obtain that for some explicit constant κ_n,

$$\mathcal{D}^l V\left[\begin{pmatrix} C(\Pi) & \\ & a\end{pmatrix} w\right] = \kappa_n \delta^{1/2}(a)\int_{(0)^n} a_n^{\sum\mu}\Theta(\mu,\Pi)d_0(\mu)\langle f, W_\mu\rangle M_\mu^{n-s}(a)d\mu.$$

Now, we shift the contour of $\mu_i \mapsto \mu_i + 2N$ for $1 \le i \le s$ for some natural number N. The integrand does not cross any pole, so by employing Lemma 5.3.7, the bound of Θ in (4.1.7), and Lemma 4.1.6 with $R = nN + 1$, we conclude that

$$\mathcal{D}^l V\left[\begin{pmatrix} C(\Pi) & \\ & a\end{pmatrix} w\right] \prec_{N,p} \delta^{1/2}(a)a_1^{2N}\int_{\mathbb{R}^n} \frac{d(it)^{O(1)}}{d(it)^p}\frac{|c(1,it)|}{|c(it)|}dt,$$

where $O(1)$ in the exponent of $d(it)$ that depends at most on l, N. We argue as in the previous case and, thus, conclude. □

5.3.4 Remarks on the sphericality assumption of the chosen newvector

In the following, we discuss the choice of the newvector in (5.1.1), which is motivated by the construction of the newvectors in [30, 45]. In the non-archimedean case, the newvectors are spherical in the Kirillov model, i.e., $\mathrm{GL}_n(\mathbb{Z}_p)$-invariant. We analogously choose our analytic newvectors to be $\mathrm{O}(n)$-invariant, but this feature of our construction is not essential. The purpose of this assumption is to make the presentations and proof of Proposition 5.2.1 simpler, and . In fact, any $f \in C_c^\infty(N\backslash G, \psi)$ serves the purpose, as in (5.1.1). We provide a brief description about the essential modification needed in the case when f is not spherical.

The proof of Proposition 5.2.1 only needs modification because the sphericality of f is used only in this proof. If we do not choose f in (5.1.1) to be spherical, then for the Whittaker–Plancherel expansion, we must use (4.1.2) instead of (4.1.3). Therefore, (5.3.5) changes to the following for $l = 0$, as an example,

$$\int_{\hat{G}} \langle \lambda(g)f, J_\pi\rangle d\mu_p(\pi), \tag{5.3.12}$$

where $\lambda(g)f(h) := f(hg)$ and J_π is the relative character of π, also known as the (long Weyl) Bessel distribution attached to π, defined as a distribution on G by

$$J_\pi(g) := \sum_{W\in\mathcal{B}(\pi)} W(g)\overline{W(1)}.$$

Here, $\mathcal{B}$ is an orthonormal basis of the Whittaker model of π. A reference on the Bessel distribution can be found in [1, 15, 34].

Spectral analysis can prove that J_π satisfies the following recursion (compare with (5.3.6)) of

$$J_\pi\left[\begin{pmatrix}1 & \\ & g\end{pmatrix} w\right] = \int_{\widehat{\mathrm{GL}_{n-1}(\mathbb{R})}} \gamma(1/2, \pi\otimes\bar{\sigma})\omega_\sigma(-1)^{n-1} J_\sigma(gw')d\mu_p(\sigma), \tag{5.3.13}$$

where ω_σ is the central character of σ and J_σ is the Bessel distribution attached to σ. A decomposition of J_π, analogous to the decomposition of the spherical Whittaker function, can be obtained using (5.3.13) (for GL(2) see, for instance, [15, chapter 6]). The Plancherel density, in this case, is also not holomorphic, unlike the spherical case. So, while we shift the contour, we likely cross some polar hyperplanes coming from the Plancherel density. However, the residues cancel with some part of the integral over π. For instance, in GL(2), the Plancherel integral over $\widehat{\mathrm{GL}(2)}$ can be decomposed as a sum of contour integrals over principal series, and sum over discrete series. Such residues can be verified that from the integrals over the principal series they cancel out some summands in the sum over the discrete series.

Chapter 6

Proof of Applications

In this chapter, we adopt the notations from §4.2, including $G := \mathrm{PGL}_n(\mathbb{R})$ and π denote an automorphic representation for $\mathrm{PGL}_n(\mathbb{Z})$.

6.1 A few auxiliary lemmata

We start by recalling the *approximate subgroup* $K_0(X,\tau)$ of G (analogously defined), as in (2.1.1), where $X > 1$ tending off to infinity and $\tau > 0$ is sufficiently small but fixed.

Let f_X be a smoothened L^1-normalized non-negative characteristic function of f_X and F_X be the self-convolution of f_X, as in Lemma 4.2.2. We abbreviate J_{F_X} and W_{F_X} as J_X and W_X, respectively (see (4.2.5) and (4.2.8)).

Lemma 6.1.1. *The function J_X, as in* (4.2.5)*, is non-negative. For $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ cuspidal, if the analytic conductor $C(\pi)$ is smaller than X, then $J_X(\pi) \gg 1$.*

Proof. The non-negativity of $J_X(\pi)$ follows from Lemma 4.2.2. Let π be cuspidal. We know that generic irreducible unitary cuspidal automorphic representations of $\mathrm{PGL}_n(\mathbb{R})$ are θ-tempered for $\theta < 1/2 - 1/(n^2+1)$ [49]. We apply Theorem 3 to state that for each $\epsilon > 0$ there exists a $\tau > 0$ such that for each generic irreducible unitary standard automorphic representation π of G there exists an *analytic newvector* $W \in \pi$ with

- $|W(g) - W(1)| < \epsilon$ for all $g \in K_0(C(\pi), \tau)$,
- and $W(1) \gg 1$.

Thus, if $C(\pi) < X$, i.e., $K_0(C(\pi), \tau) \supseteq K_0(X, \tau)$,

$$|\pi(f_X)W(1) - W(1)| = |\int_{K_0(X,\tau)} f_X(g)(W(g) - W(1))dg| \leq \epsilon,$$

hence, $\pi(f_X)W(1) \gg 1$.

We choose an orthonormal basis $\mathcal{B}(\pi)$ containing the above analytic newvector W of π. Thus, by dropping all but the summand corresponding to W in the expression of $J_X(\pi)$, as in Lemma 4.2.2, we conclude that $J_X(\pi) \gg 1$. ☐

Lemma 6.1.2. *Let $m, l \in \mathbb{N}^{n-1}$ and $x_1, x_2 \in [N]$. Let $\gamma \in \Gamma$ such that $\tilde{m}x_1\gamma x_2\tilde{l}^{-1} \in K_0(X,\tau)$ for small enough τ. If*

$$\min(l_{n-1}^{n-1} \dots l_1 m_1^{n-1} \dots m_{n-1}, m_{n-1}^{n-1} \dots m_1 l_1^{n-1} \dots l_{n-1}) \ll X^n$$

with a sufficiently small implied constant, then the last row of γ is $(0,\dots,0,1)$.

Proof. We write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $n-1,1$ block decomposition, i.e., (c,d) is the bottom row of γ. Let $N := (l_{n-1}^{n-1}\dots l_1 m_1^{n-1}\dots m_{n-1})^{1/n} \ll X$ with a sufficiently small constant. From the definition of $K_0(X,\tau)$ in (2.1.1), we obtain that the last row of $\det\left(\tilde{m}\tilde{l}^{-1}\right)^{1/n}\tilde{m}x_1\gamma x_2\tilde{l}^{-1}$ is

$$|\det\left(\tilde{m}\tilde{l}^{-1}\right)^{1/n}(Xc,d)x_2\tilde{n}^{-1}-(0,1)|< \tau.$$

Thus, $|c|\ll \tau N/X \ll \tau$ with a sufficiently small implied constant. As coordinates of c are integers for sufficiently small τ, we obtain $c=0$ and, consequently, $d=1$, as γ lies in $\mathrm{PGL}_n(\mathbb{Z})$.

However, $\tilde{l}x_2^{-1}\gamma^{-1}x_1^{-1}\tilde{m}^{-1}\in K_0(X,\tau)$. By interchanging m and l in the above argument, we obtain that if $m_{n-1}^{n-1}\dots m_1 l_1^{n-1}\dots l_{n-1}\ll X^r$ with a sufficiently small constant, then the last row of γ^{-1} is $(0,\dots,0,1)$, and the same holds for γ, so we conclude the proof. □

Lemma 6.1.3. *Let l,m,x_1,x_2 be as in Lemma 6.1.2. Then,*

$$\sum_{\gamma\in\Gamma-\Gamma_N}\int_{[N]^2}F_X(\tilde{m}x_1\gamma x_2\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_2)}dx_1dx_2=0,$$

where F_X is as defined after (2.1.1).

Proof. From the proof of Proposition 4.2.1, we obtain that

$$\sum_{\gamma\in\Gamma-\Gamma_N}\int_{[N]^2}F_X(\tilde{m}x_1\gamma x_2\tilde{l}^{-1})\overline{\psi_m(x_1)\psi_l(x_2)}dx_1dx_2$$
$$=\sum_{1\neq w\in W}\frac{\delta^{1/2}(\tilde{l})}{\delta^{1/2}(\tilde{m})\delta_w(\tilde{l})}\sum_{c\in\mathbb{Z}_{\neq 0}^{n-1}}S_w(l,m;c)\int_{N_w}W_X(\tilde{m}c^*w\tilde{l}^{-1}x)\overline{\psi(x)}dx.$$

Hence, from Lemma 4.2.1 we conclude that it is enough to consider $\gamma\in\Gamma-\Gamma_N$ in the Bruhat cell attached to the Weyl elements w of the form $\begin{pmatrix} & & I_{d_1} \\ & \iddots & \\ I_{d_k} & & \end{pmatrix}$ with $d_k<n$, which implies that the last row of γ is not of the form $(0,\dots,0,1)$. Therefore, from Lemma 6.1.2, we obtain the support condition that $\tilde{m}x_1\gamma x_2\tilde{l}^{-1}\notin K_0(X,\tau)$ for small enough τ. We conclude by noting that the support of F_X is $K_0(X,\tau)$, which follows from the definition of F_X. □

6.2 Proof of Theorems 4, 5, and 9

Proof of Theorem 4. Proposition 4.2.1 and Lemma 6.1.3 imply that

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{\lambda_\pi(m)\overline{\lambda_\pi(l)}}{\ell(\pi)} J_X(\pi) d\mu_{\text{aut}}(\pi) = \delta_{m=l} W_X(1).$$

Choosing $m = l = (1, \ldots, 1)$ in the Proposition 4.2.1 we obtain

$$\int_{\hat{X}_{\text{gen}}} J_X(\pi) d\mu_{\text{aut}}(\pi) = W_X(1).$$

Finally, noting that

$$W_X(1) \asymp \text{Vol}(K_0(X, \tau)) \asymp X^{n-1},$$

we conclude the proof. □

Proof of Theorem 5. We use Lemma 6.1.1 to drop all the terms but cuspidal π with $C(\pi) < X$ from the LHS of

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) \asymp X^{n-1},$$

which is obtained in Theorem 4 and use $J_X(\pi) \gg 1$ for these π. We conclude the proof using the fact that $\ell(\pi) = L(1, \pi, \text{Ad})$ for cuspidal π. □

Proof of Theorem 9. Using Lemma 6.1.1, (4.2.2), and artificially adding the continuous spectrum, we write

$$\sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} L(1, \pi, \text{Ad})^{-1} \left| \sum_{M \ll X} \alpha(m)\lambda_\pi(m) \right|^2 \ll \int_{\hat{\mathbb{X}}_{\text{gen}}} \left| \sum_{M \ll X} \alpha(m)\lambda_\pi(m) \right|^2 \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi).$$

We open the square above, and let $L := l_1, \ldots l_{n-1}$ for $l \in \mathbb{N}^{n-1}$ to obtain the RHS as

$$\sum_{L,M \ll X} \overline{\alpha(m)}\alpha(l) \int_{\hat{\mathbb{X}}_{\text{gen}}} \overline{\lambda_\pi(m)}\lambda_\pi(l) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi).$$

For $L, M \ll X$ with sufficiently small implied constant yields

$$\min(l_{n-1}^{n-1} \ldots l_1 m_1^{n-1} \ldots m_{n-1}, m_{n-1}^{n-1} \ldots m_1 l_1^{n-1} \ldots l_{n-1}) \ll X^n$$

with a sufficiently small implied constant. If not, then there would be a sequence of m, l such that

$$(m_1 \ldots m_{n-1} l_1 \ldots l_{n-1})^n \gg X^{2n},$$

with large implied constants, contradicting the assumptions on M, L.

Thus, we can apply Theorem 4 to obtain

$$\sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} L(1,\pi,\mathrm{Ad})^{-1} \left| \sum_{M \ll X} \alpha(m)\lambda_\pi(m) \right|^2 \ll \sum_{L,M \ll X} \overline{\alpha(m)}\alpha(l)\delta_{m=l} W_X(1).$$

We conclude the proof by recalling that $W_X(1) \asymp X^{n-1}$ from the proof of Theorem 4. □

6.3 Proof of Theorem 10

In this subsection, we slightly change the notation. For $m \in \mathbb{N}$, we denote the Fourier coefficient $\lambda_\pi(m,1,\dots,1)$ by $\lambda_\pi(m)$. Following lemma is a standard exercise using the approximate functional equation of the L-value, so we include the argument for the sake of completeness.

Lemma 6.3.1. *Let π be cuspidal with $C(\pi) < X$. Then,*

$$|L(1/2,\pi)|^2 \ll_\epsilon X^\epsilon \int_{|t| \ll X^\epsilon} \left| \sum_{m \ll X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{m^{1/2+\epsilon+it}} \right|^2 dt + O(X^{-A})$$

for all large $A > 0$.

Proof. We start by proving the approximate functional equation as in [26], and first define

$$H_X(s,\pi) := \frac{1}{2}X^{s/2} + \frac{1}{2}X^{-s/2}\gamma(1/2,\pi)\gamma(1/2-s,\tilde{\pi}).$$

Here, $H_X(s,\pi)$ is holomorphic as $\Re(s) > 0$ and $H_X(0,\pi) = 1$. From (4.2.4), we can write the global functional equation as

$$L(1/2+s,\pi)H_X(s,\pi) = \gamma(1/2,\pi)L(1/2-s,\tilde{\pi})H_X(-s,\tilde{\pi}).$$

The estimate in Lemma 4.1.1 implies that

$$H_X(s,\pi) \ll_{\Re(s)} X^{\Re(s)/2} \left(1 + (C(\pi)/X)^{\Re(s)} (1+|s|)^{O(1)}\right).$$

We choose an entire function h with $h(0) = 1$ and $h(s) = h(-s)$. For π cuspidal, $L(1/2+s,\pi)H_X(s,\pi)$ is entire. Then, by Cauchy's theorem and applying $s \mapsto -s$ with the above version of the global functional equation in the second equality, we obtain

$$\begin{aligned} L(1/2,\pi) &= \int_{(1)} L(1/2+s,\pi)H_X(s,\pi)h(s)\frac{ds}{s} - \int_{(-1)} L(1/2+s,\pi)H_X(s,\pi)h(s)\frac{ds}{s} \\ &= \int_{(1)} L(1/2+s,\pi)H_X(s,\pi)h(s)\frac{ds}{s} + \gamma(1/2,\pi)\int_{(1)} L(1/2+s,\tilde{\pi})H_X(s,\tilde{\pi})h(s)\frac{ds}{s} \\ &= \sum_m \frac{\lambda_\pi(m)}{\sqrt{m}}V_1(m) + \gamma(1/2,\pi)\sum_m \frac{\overline{\lambda_\pi(m)}}{\sqrt{m}}V_2(m), \end{aligned}$$

where we used the Dirichlet series of $L(s,\pi)$ in the last equality, as in §4.2.2. Here,

$$V_1(m) = V_1(m;X,\pi) := \int_{(1)} m^{-s}H_X(s,\pi)\tilde{h}(s)\frac{ds}{s},$$

and V_2 has a similar definition.

Let $C(\pi) < X$ from now on. Thus, $H_X(s,\pi) \ll X^{\Re(s)/2}(1+|s|)^{O(1)}$. We first claim that the above sums can be truncated at $m \le X^{1/2+\epsilon}$ with an error $O(X^{-\infty})$. To see this, we shift the contour in the defining integral of V_1 to K for some large $K > 0$. Thus, we obtain

$$V_1(m) \ll_K m^{-K}X^{K/2}.$$

Consequently, using $\lambda_\pi(m) \ll n^c$ for some fixed c (following from, e.g., [32]),

$$\sum_{m>X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{\sqrt{m}}V_1(m) \ll_K X^{K/2}\sum_{m>X^{1/2+\epsilon}} m^{-1/2-K+c} \ll_B X^{-B}$$

for all large B.

We also note that, for $\epsilon > 0$ small,

$$\begin{aligned} V_1(m) &= \int_{(\epsilon)} m^{-s}H_X(s,\pi)\tilde{h}(s)\frac{ds}{s} \\ &= \int_{|t|<X^\epsilon} m^{-\epsilon-it}H_X(\epsilon+it,\pi)\tilde{h}(\epsilon+it)\frac{dt}{\epsilon+it} + O\left(m^{-\epsilon}X^{\epsilon/2}\int_{|t|\ge X^\epsilon}|t|^{-K}\right) \\ &= \int_{|t|<X^\epsilon} m^{-\epsilon-it}H_X(\epsilon+it,\pi)\tilde{h}(\epsilon+it)\frac{dt}{\epsilon+it} + O(m^{-\epsilon}X^{-B}). \end{aligned}$$

But,

$$\sum_{m\ll X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{m^{1/2+\epsilon}} \ll X^{(1/2+\epsilon)(1/2+c-\epsilon)}.$$

Thus, we obtain that

$$\sum_m \frac{\lambda_\pi(m)}{\sqrt{m}} V_1(m) = \int_{|t|\ll X^\epsilon} \sum_{m \ll X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{m^{1/2+\epsilon+it}} H_X(\epsilon+it,\pi) \frac{\tilde{h}(\epsilon+it)}{\epsilon+it} dt + O(X^{-B})$$

for all large B.

We can perform a similar analysis for the V_2 summand, and by using that $|\gamma(1/2,\pi)|=1$, we arrive at

$$|L(1/2,\pi)|^2 \ll \left| \int_{|t|\ll X^\epsilon} \sum_{m \ll X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{m^{1/2+\epsilon+it}} H_X(\epsilon+it,\pi) \frac{\tilde{h}(\epsilon+it)}{\epsilon+it} dt \right|^2 + O(X^{-B}).$$

Finally,

$$\int_{|t|\ll X^\epsilon} \left| H_X(\epsilon+it,\pi) \frac{\tilde{h}(\epsilon+it)}{\epsilon+it} \right|^2 dt \ll X^\epsilon.$$

Thus, applying a Cauchy-Schwarz in the first term of the estimate of $|L(1/2,\pi)|^2$ above, we conclude the proof. □

Proof of Theorem 10. Lemma 6.3.1 and Theorem 5 imply that

$$\sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} \frac{|L(1/2,\pi)|^2}{L(1,\pi,\mathrm{Ad})} \ll_\epsilon X^\epsilon \int_{|t|\ll X^\epsilon} \sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} L(1,\mathrm{Ad},\pi)^{-1} \times \left| \sum_{m \ll X^{1/2+\epsilon}} \frac{\lambda_\pi(m)}{m^{1/2+\epsilon+it}} \right|^2 dt + O(X^{-A}).$$

Using Lemma 9 with $\alpha(m,1,\dots,1) = m^{-1/2-\epsilon-it}$, for $n \ll X^{1/2+\epsilon}$, $\alpha = 0$. Otherwise, we obtain that the first sum in the RHS is bounded by $X^{n-1+\epsilon}$ and, thus, we conclude. □

6.4 Proof of Theorems 6 and 7

The multiplicative properties the Fourier coefficients of automorphic forms for $\mathrm{PGL}_n(\mathbb{Z})$ can be understood in terms of the standard character theory of $\mathrm{SU}(n)$ by the work of Shintani and Casselman–Shalika. The irreducible representations of $\mathrm{SU}(n)$ are in bijection with the dominant elements (i.e., the elements with non-increasing coordinates) of $\mathbb{Z}^n/\mathbb{Z}(1,\dots,1)$, which are in bijection with the elements of $\mathbb{Z}_{\geq 0}^{n-1}$ by the map

$$\iota : \mathbb{Z}^n/\mathbb{Z}(1,\dots,1) \ni \alpha \mapsto (\alpha_1-\alpha_2,\dots,\alpha_{n-1}-\alpha_n) \in \mathbb{Z}_{\geq 0}^{n-1}.$$

For $\alpha \in \mathbb{Z}_{\geq 0}^{r-1}$, a dominant vector, let χ_α be the character of the highest weight representation of $\mathrm{SU}(n)$ attached to α, given by a Schur polynomial. The following Lemma, which is a theorem by Shintani and Casselman–Shalika [12, 56], gives a formula of the Fourier coefficients of π in terms of the Satake parameters.

Lemma 6.4.1. *Let* $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ *with the Satake parameters* $\mu^p(\pi)$ *at a finite prime* p. *Then,*

$$\chi_\alpha(\mathrm{diag}(p^{\mu^p(\pi)})) = \lambda_\pi(p^{\iota(\alpha)}),$$

where λ_π *is defined in* (4.2.3).

Lemma 6.4.1 helps to linearize the monomials of the Fourier coefficients. By decomposing the tensor product of the highest weight representations in a direct sum of irreducible representations attached to various α, we can compute that for any multi-indices $\alpha_1, \ldots, \alpha_k \in \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$ and nonnegative integers $s_1, \ldots, s_k$, we may write

$$\prod_i \chi_{\alpha_i}^{s_i} = \sum_\beta c_\beta \chi_\beta$$

for some coefficients c_β. Correspondingly, using Lemma 6.4.1, we obtain

$$\prod_i \lambda_\pi(p^{\iota(\alpha_i)})^{s_i} = \sum_\beta c_\beta \lambda(p^{\iota(\beta)}), \tag{6.4.1}$$

where the sums in the RHS are finite.

Proof of Theorem 6. We proceed exactly as in [63]. Using Lemma 6.4.1, we can embed the Satake parameters into the space of Fourier coefficients by mapping under elementary symmetric polynomials by

$$\rho : p^{\mu^p(\pi)} \mapsto (e_1(p^{\mu^p(\pi)}), \ldots, e_{n-1}(p^{\mu^p(\pi)})),$$

where e_j is the jth elementary symmetric polynomial. It can be checked that

$$e_j(p^{\mu^p(\pi)}) = \lambda_\pi(\underbrace{p, \ldots, p}_{j \text{ times}}, 1 \ldots, 1).$$

Thus, we may identify the space of continuous functions on T_1/W as the same on the image of ρ as a compact subset of $\mathbb{C}^{n-1}$. Hence, to prove Theorem 6, it suffices to prove that for any continuous $\tilde{f}$ (e.g., composing f with ρ^{-1}) on the image of ρ,

$$\lim_{X\to\infty} \frac{\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \tilde{f}(e_1(p^{\mu^p(\pi)}), \ldots, e_{n-1}(p^{\mu^p(\pi)})) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi)}{\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi)} \to \int_{T_0/W} \tilde{f} \circ \rho(z) d\mu_{\mathrm{ST}}(z), \tag{6.4.2}$$

We can approximate $\tilde{f}$, by Stone–Weierstrauss, using polynomials in $z := (z_1, \ldots, z_{n-1})$ and $\bar{z}$. Thus, by linearity, it is enough to prove (6.4.2) for $\tilde{f} = z^{\alpha_1} \bar{z}^{\alpha_2}$ for some multi-indices α_1, α_2.

For any multi-indices $\alpha, \beta \in \mathbb{Z}^n/\mathbb{Z}(1, \ldots, 1)$, from Theorem 4, we obtain

$$\lim_{X\to\infty} \frac{\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \lambda_\pi(p^{\iota(\alpha)})\overline{\lambda_\pi(p^{\iota(\beta)})}\frac{J_X(\pi)}{\ell(\pi)}d\mu_{\mathrm{aut}}(\pi)}{\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \frac{J_X(\pi)}{\ell(\pi)}d\mu_{\mathrm{aut}}(\pi)} = \delta_{\iota(\alpha)=\iota(\beta)} = \int_{T_0/W} \chi_\alpha\overline{\chi_\beta}(z)d\mu_{\mathrm{ST}}(z),$$

where the last equality follows by Schur's orthogonality of characters. Finally, using (6.4.1) (and the corresponding relation for χ_α), we conclude (6.4.2). □

Proof of Theorem 7. We follow the proof of [4, Theorem 1]. From [4, Lemma 4], we obtain that for $n > r$

$$\sum_{j=0}^{n-1}|\lambda_\pi(p^{r-j}, 1, \ldots, 1)|^2 \geq (2p^{\theta^p(\pi)})^{2(1-n)}p^{2r\theta^p(\pi)}. \tag{6.4.3}$$

We choose r large enough such that $p^r \asymp X$ with a small enough implied absolute constant (admissible by Theorem 4). Then, for $\theta^p(\pi) > \eta$, we obtain from (6.4.3) that

$$\sum_{j=0}^{n-1}|\lambda_\pi(p^{r-j}, 1, \ldots, 1)|^2 \gg X^{2\eta}.$$

On the other hand,

$$\#\{\pi \text{ cuspidal} \mid C(\pi) < X,\, \theta^p(\pi) > \eta\} \ll X^{-2\eta} \sum_{\substack{C(\pi)<X \\ \pi \text{ cuspidal}}} \sum_{j=0}^{n-1}|\lambda_\pi(p^{r-j}, 1, \ldots, 1)|^2.$$

Using the fact that (see [44])

$$L(1, \pi, \mathrm{Ad}) \ll C(\pi)^\epsilon, \tag{6.4.4}$$

adding the similar contributions from the continuous spectrum, and using Lemma 6.1.1, we obtain that

$$\#\{\pi \text{ cuspidal} \mid C(\pi) < X,\, \theta^p(\pi) > \eta\}$$
$$\ll_\epsilon X^{-2\eta+\epsilon} \sum_{j=0}^{n-1} \int_{\hat{\mathbb{X}}_{\mathrm{gen}}} |\lambda_\pi(p^{r-j}, 1, \ldots, 1)|^2 \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi).$$

From Theorem 4, we conclude that the RHS is $\ll_\epsilon X^{n-1-2\eta+\epsilon}$. □

6.5 Proof of Theorem 8

6.5.1 Explicit formula

Recall that ${}^LG = \mathrm{SL}_n(\mathbb{C})$. For the L-group homomorphism $\rho : \mathrm{SL}_n(\mathbb{C}) \to \mathrm{GL}_d(\mathbb{C})$, we abbreviate $\rho_*\pi$ as Π for $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$. Let $\{\mu^p(\Pi)\}_{i=1}^d$ be the local Satake (resp. Langlands) parameters for $p < \infty$ (resp. $p = \infty$).

We start by recalling the functional equation of $L(s, \Pi)$ as in §4.2.2. We write the global L-function

$$\Lambda(s, \Pi) = L(s, \Pi)L_\infty(s, \Pi),$$

where $L_\infty(s, \Pi) := \prod_{i=1}^d \Gamma_\mathbb{R}(1/2 + s + \mu_i^\infty(\Pi))$. Then, (4.2.4) reads as

$$\Lambda(s, \Pi) = \epsilon_\infty(\Pi)\Lambda(1 - s, \tilde{\Pi}).$$

Let ψ_0 be an entire function with a rapid decay in the vertical strips. Using Cauchy's argument principle and the global functional equation, we write

$$\sum_{\rho \text{ zeros}} \psi_0(\rho) - \sum_{\tau \text{ poles}} \psi_0(\tau) = \int_{(2)} \psi_0(s)\frac{\Lambda'}{\Lambda}(s, \Pi)ds + \int_{(2)} \psi_0(1 - s)\frac{\Lambda'}{\Lambda}(s, \tilde{\Pi})ds, \quad (6.5.1)$$

where the zeros and poles are of $\Lambda(., \Pi)$ in the critical strip, i.e., $\Re(s) \in [0, 1]$, counted with multiplicity.

Let $\Re(s) = 2$. We employ the Euler product

$$L(s, \Pi) = \prod_p \prod_{j=1}^d (1 - p^{\mu_j^p(\Pi) - s})^{-1}$$

to obtain

$$\frac{\Lambda'}{\Lambda}(s, \Pi) = \sum_{j=1}^d \frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}(s + \mu_j^\infty(\Pi)) - \sum_{p<\infty} \log p \sum_{j=1}^d \sum_{k=1}^\infty p^{k(\mu_j^p(\Pi) - s)}.$$

We exchange $\int_{(2)}$ and $\sum_{p<\infty}$, which is justified by the absolute convergence of the Euler product and the rapid decay of ψ_0 in the vertical direction. We then shift each contour to the vertical line $\Re(s) = 1/2$. Thus, we obtain the RHS of (6.5.1) as

$$\int_{(1/2)} \psi_0(s) \left(\sum_{j=1}^d \frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}(s + \mu_j^\infty(\Pi)) - \sum_{p<\infty} \log p \sum_{j=1}^d \sum_{k=1}^\infty p^{k(\mu_j^p(\Pi) - s)} \right) ds$$
$$+ \int_{(1/2)} \psi_0(1 - s) \left(\sum_{j=1}^d \frac{\Gamma_\mathbb{R}'}{\Gamma_\mathbb{R}}(s + \overline{\mu_j^\infty(\Pi)}) - \sum_{p<\infty} \log p \sum_{j=1}^d \sum_{k=1}^\infty p^{k(\overline{\mu_j^p(\Pi)} - s)} \right) ds.$$

We use that the parameters of $\tilde{\Pi}$ are the complex conjugate of parameters of Π.

Recall the average conductor $C_{\rho,X}$ and the test function ψ from the statement of Theorem 8. We choose

$$\psi_0(s) := \psi\left((s-1/2)\frac{\log C_{\rho,X}}{2\pi i}\right),$$

which is admissible as ψ is (the analytic continuation of) a Fourier transform of a smooth function supported on $[-\delta,\delta]$. Then,

$$\int_{(1/2)} \psi_0(s)x^{-s}ds = \frac{x^{-1/2}}{\log C_{\rho,X}}\hat{\psi}\left(\frac{\log x}{\log C_{\rho,X}}\right),$$

and

$$\int_{(1/2)} \psi_0(1-s)x^{-s}ds = \frac{x^{-1/2}}{\log C_{\rho,X}}\hat{\psi}\left(\frac{-\log x}{\log C_{\rho,X}}\right),$$

where $\hat{\psi}$ is the Fourier transform of ψ. The Langlands functoriality in Conjecture 1 implies that $\mu_j^\infty(\Pi)$ are away from the poles of $\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}(s+\mu_j^\infty(\Pi))$. We use Stirling's estimate for $s\in i\mathbb{R}$ to obtain

$$\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}(1/2+s+\mu) = \frac{1}{2}\log|1/2+s+\mu|+O(1) = \log(1+|\mu|)+O(\log(1+|s|))$$

uniformly in μ. Thus,

$$\begin{aligned}&\int_{(1/2)} \psi_0(s)\sum_{j=1}^{d}\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}(s+\mu_j^\infty(\Pi))ds\\ &= \frac{1}{2}\tilde{\psi}_0(1)\log C(\Pi) + O\left(\int_\mathbb{R}|\psi_0(1/2+it)|\log(1+|t|)dt\right)\\ &= \hat{\psi}(0)\frac{\log C(\Pi)}{2\log C_{\rho,X}} + O((\log X)^{-2}).\end{aligned}$$

Performing a similar computation, we obtain

$$\int_{(1/2)} \psi_0(1-s)\sum_{j=1}^{d}\frac{\Gamma'_\mathbb{R}}{\Gamma_\mathbb{R}}(s+\overline{\mu_j^\infty(\Pi)})ds = \hat{\psi}(0)\frac{\log C(\Pi)}{2\log C_{\rho,X}} + O((\log X)^{-2}).$$

We also define the moment of the Satake parameters by

$$\beta_k^p(\Pi) := \sum_{j=1}^{d} p^{k\mu_j^p(\Pi)}, \quad \beta_k(\tilde{\Pi}) := \sum_{j=1}^{d} p^{k\overline{\mu_j^p(\Pi)}} = \overline{\beta_k^p(\Pi)}.$$

As before, we write the zeros of $\Lambda(.,\Pi)$ as $1/2+i\gamma_\Pi$ and poles as $1/2+i\tau_\Pi$. We rewrite

(6.5.1) to obtain the *explicit formula*

$$\sum_{\gamma_\Pi} \psi\left(\gamma_\Pi \frac{\log C_{\rho,X}}{2\pi}\right) - \sum_{\tau_\Pi} \psi\left(\tau_\Pi \frac{\log C_{\rho,X}}{2\pi i}\right) = \hat{\psi}(0)\frac{\log C(\Pi)}{\log C_{\rho,X}} + O((\log X)^{-2})$$
$$-\sum_{k=1}^{\infty}\sum_{p<\infty} \frac{p^{-k/2}\log p}{\log C_{\rho,X}}\left(\beta_k^p(\Pi)\hat{\psi}\left(\frac{k\log p}{\log C_{\rho,X}}\right) + \overline{\beta_k^p(\Pi)}\hat{\psi}\left(\frac{-k\log p}{\log C_{\rho,X}}\right)\right), \quad (6.5.2)$$

The Langlands functoriality in Conjecture 1 implies that

$$|p^{\mu_j^p(\Pi)}| = 1$$

to truncate the last k-sum at $k \leq 2$ with an error of $O((\log X)^{-1})$. Finally, summing over the spectrum $\pi \in \hat{\mathbb{X}}_{\mathrm{gen}}$ in (6.5.2) with the weight $\frac{J_X(\pi)}{\ell(\pi)}$, we obtain an expression for the 1-*level zero density* statistic

$$\left(D_{\rho,X}(\psi) - D_{\rho,X}^{\mathrm{pole}}\right)\int_{\hat{\mathbb{X}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) = \hat{\psi}(0)\int_{\hat{\mathbb{X}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi)$$
$$-\frac{1}{\log C_{\rho,X}}\sum_{k=1}^{2}\sum_{p<\infty}\frac{\log p}{p^{k/2}}\hat{\psi}\left(\frac{k\log p}{\log C_{\rho,X}}\right)\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \beta_k^p(\rho_*\pi)\frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi)$$
$$-\frac{1}{\log C_{\rho,X}}\sum_{k=1}^{2}\sum_{p<\infty}\frac{\log p}{p^{k/2}}\hat{\psi}\left(\frac{-k\log p}{\log C_{\rho,X}}\right)\int_{\hat{\mathbb{X}}_{\mathrm{gen}}} \overline{\beta_k^p(\rho_*\pi)}\frac{J_X(\pi)}{\ell(\pi)} d\mu_{\mathrm{aut}}(\pi) + O\left(\frac{X^{n-1}}{\log X}\right). \quad (6.5.3)$$

The error bound follows from Theorem 4.

6.5.2 Moments of Satake parameters

We use standard representation theory of $\mathrm{SU}(n)$ to understand the quantities $\beta_k^p(\rho_*\pi)$ and their averages over the representations π.

Lemma 6.5.1. *Let $\rho, \theta, \mathfrak{s}$ be as in Theorem 8 and $k \in \mathbb{N}$. There exists coefficients c_α with $\alpha := (\alpha_1, \ldots, \alpha_{n-1}) \in \mathbb{Z}_{\geq 0}^{n-1}$ such that*

$$\beta_k^p(\rho_*\pi) = c_0(k,\theta) + \sum_{0<\sum \alpha_i \leq k(\theta_1-\theta_n)} c_\alpha(k,\theta)\lambda_\pi(p^\alpha),$$

with $c_0(1,\theta) = 0$ and $c_0(2,\theta) = \mathfrak{s}(\rho)$.

Proof. We have,

$$\beta_k^p(\rho_*\pi) = \mathrm{Tr}(\rho^k(\nu)) = \mathrm{Tr}(\rho(\nu^k)) = \chi_\theta(\nu^k),$$

where ν is the diagonal matrix in $\mathrm{SL}_n(\mathbb{C})$ with entries $\nu_i^p(\pi)$ such that $\{\nu_i^p(\pi)\}_{i=1}^n$ are the Satake parameters of π at p, and χ_θ is the character of the highest weight representation of SU(n) attached to θ. Any symmetric polynomial in $\{\nu_j\}_j$ can be written as a finite linear combination of characters attached to the highest weight representations (the Schur polynomials), evaluated at $\{\nu_j\}_j$. Thus,

$$\chi_\theta(\nu^k) = \sum_\gamma c_\gamma(k,\theta)\chi_\gamma(\nu),$$

where γ runs over dominant elements in $\mathbb{Z}^n/\mathbb{Z}(1,\ldots,1)$. We write every weight $\gamma := (\gamma_1,\ldots,\gamma_n)$ appearing in the above equation in $(\gamma_1-\gamma_n,\ldots,\gamma_{n-1}-\gamma_n,0) \in \mathbb{Z}^r_{\geq 0}$. The lexicographically highest weight of the polynomial expression in the LHS is $k(\theta_1-\theta_n,\ldots,\theta_{n-1}-\theta_n,0)$. By comparing the highest weights in the polynomial expansion of the both sides, we can rewrite the last display as

$$\chi_\theta(\nu^k) = \sum_{\gamma:\gamma_1-\gamma_n\leq k(\theta_1-\theta_n)} C_\gamma(k,\theta)\chi_\gamma(\nu).$$

We calculate the summand corresponding to the zero weight using orthogonality of characters of SU(n) with respect to its probability Haar measure as

$$c_0(k,\theta) = \int_{\mathrm{SU}(n)} \chi_\theta(g^k)dg.$$

Thus, $c_0(1,\theta)=0$ and $c_0(2,\theta)=\mathfrak{s}(\rho)$, and we conclude the proof by writing $\alpha=\iota(\gamma)$ and the relation between λ and χ, as in Lemma 6.4.1. □

Lemma 6.5.2. *Let ψ,ρ,θ be as in Theorem 8 and $c_0(k,\theta)$ as in Lemma 6.5.1. Then,*

$$\frac{1}{\log C_{\rho,X}}\sum_{k=1}^{2} c_0(k,\theta)\sum_{p<\infty}\frac{\log p}{p^{k/2}}\left(\hat{\psi}\left(\frac{k\log p}{\log C_{\rho,X}}\right)+\hat{\psi}\left(\frac{-k\log p}{\log C_{\rho,X}}\right)\right) = \frac{\mathfrak{s}(\rho)}{2}\psi(0)+O\left(\frac{1}{\log X}\right),$$

as X tends off to infinity.

Proof. The summand for $k=1$ vanishes by Lemma 6.5.1. By employing the value of $c_0(2,\theta)$ from Lemma 6.5.1, it is sufficient to show that

$$\frac{1}{\log C_{\rho,X}}\sum_{p<\infty}\frac{\log p}{p}\hat{\psi}\left(\frac{2\log p}{\log C_{\rho,X}}\right) = \frac{1}{2}\int_0^\infty \hat{\psi}(t)dt + O\left(\frac{1}{\log X}\right),$$

as changing $\log p$ to $-\log p$ in the entry of $\hat{\psi}$, the main term in the RHS above would be $\int_{-\infty}^0 \hat{\psi}(t)dt$. Adding these two terms, we obtain

$$\int_{-\infty}^\infty \hat{\psi}(t)dt = \psi(0),$$

as required.

Let $\Psi(x) := \sum_{n\le x} \Lambda(n)$ be the Chebyshev's function where Λ is the Von Mangdolt function (not to be confused with the completed L-function Λ). A strong form of prime number theorem implies that

$$\Psi(x) = x(1 + O((1 + \log x)^{-2})).$$

We write the sum as a Stieltjes integral,

$$\frac{1}{\log C_{\rho,X}} \sum_{p<\infty} \frac{\log p}{p} \hat{\psi}\left(\frac{2\log p}{\log C_{\rho,X}}\right) = \frac{1}{\log C_{\rho,X}} \int_1^\infty \hat{\psi}\left(\frac{2\log x}{\log C_{\rho,X}}\right) \frac{d\Psi(x)}{x}.$$

Performing an integration by parts and a change of variable, the RHS of the above can be written as

$$\frac{1}{2}\int_0^\infty \hat{\psi}(t)(1 + (1+t)^{-2})dt - \frac{1}{\log C_{\rho,X}} \int_0^\infty \hat{\psi}'(t)(1 + O(1+t)^{-2})dt,$$

which can be verified to equal

$$\frac{1}{2}\int_0^\infty \hat{\psi}(t)dt + O(1/\log X),$$

hence, the claim follows. □

Proof of Theorem 8. We calculate the RHS of (6.5.3). Using Lemma 6.5.1, we write

$$\int_{\hat{\mathbb{X}}} \beta_k^p(\pi) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) = c_0(k,\theta) \int_{\hat{\mathbb{X}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi)$$
$$+ \sum_{0<\sum \alpha_i \le k(\theta_1-\theta_r)} c_\alpha(k,\theta) \int_{\hat{\mathbb{X}}} \lambda_\pi(p^\alpha) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi).$$

If $p^{k(\theta_1-\theta_r)} = o(X)$, then using Theorem 4, we conclude the second summand in the RHS above vanishes. However, $\hat{\psi}$ is supported on $[-\delta,\delta]$. Conjecture 2 and the size of δ, as in the statement of Theorem 8, imply that

$$p^{k(\theta_1-\theta_n)} \le C_{\rho,X}^{\delta(\theta_1-\theta_n)} \ll_\rho X^{\delta\mathfrak{C}(\rho)(\theta_1-\theta_n)} = o(X).$$

Hence, the second summand in the RHS of (6.5.3) is

$$\sum_{k=1}^{2} \sum_{p<\infty} \frac{\log p}{p^{k/2}} \hat{\psi}\left(\frac{k\log p}{\log C_{\rho,X}}\right) \int_{\hat{\mathbb{X}}_{\text{gen}}} \beta_k^p(\rho_*\pi) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi)$$
$$= \sum_{k=1}^{2} c_0(k,\theta) \sum_{p<\infty} \frac{\log p}{p^{k/2}} \hat{\psi}\left(\frac{k\log p}{\log C_{\rho,X}}\right) \int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi).$$

Applying a similar analysis with the third summand in the RHS of (6.5.3), we conclude from (6.5.3) that $D_{\rho,X}(\psi) - D^{\text{pole}}_{\rho,X}(\psi)$ equals

$$\hat{\psi}(0) - \frac{1}{\log C_{\rho,X}} \sum_{k=1}^{2} c_0(k,\theta) \sum_{p<\infty} \frac{\log p}{p^{k/2}} \left(\hat{\psi}\left(\frac{k \log p}{\log C_{\rho,X}}\right) + \hat{\psi}\left(\frac{-k \log p}{\log C_{\rho,X}}\right)\right) + O\left(\frac{1}{\log X}\right).$$

We conclude the proof by applying Lemma 6.5.2. □

Remark 11. *We expand the Remark after Theorem 8 that D^{pole} is typically negligible. Following from the conjecture below, as in [55, Hypothesis 11.2] (also, see the subsequent comment), $\Lambda(., \rho_* \pi)$ is entire for almost all π. Quantitatively, we conjecture the following weighted version.*

Conjecture 3. *Let $\mathcal{P}$ denote the characteristic function on $\hat{\mathbb{X}}_{\text{gen}}$ such that $\Lambda(s, \rho_* \pi)$ has a pole in the critical strip. Then,*

$$\int_{\hat{\mathbb{X}}_{\text{gen}}} \mathcal{P}(\pi) \frac{J_X(\pi)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) \ll X^{n-1-\eta(\rho)}$$

for some $\eta(\rho) > 0$.

Due to the Langlands functoriality in Conjecture 1, considering the poles of $\Lambda(., \rho_* \pi)$ is reasonable, and Conjecture 3 is most likely true. In fact, if ρ is the standard representation, then we know that $\Lambda(s, \pi)$ has a pole only if π is non-cuspidal. A standard Weyl law for a non-cuspidal spectrum of G supports Conjecture 3.

Lemma 6.5.3. *Let $\delta < \frac{4\pi\eta(\rho)}{\mathfrak{C}(\rho)}$, and we assume Conjecture 2 and 3. Then,*

$$\mathcal{D}^{\text{pole}}_{\rho,X}(\psi) \ll_N (\log X)^{-N}$$

for all large $N > 0$.

Proof. $\Lambda(s, \rho_* \pi)$ has a bounded number of poles in the critical strip. We write these poles as $1/2 + i\tau$ so that $|\Im(\tau)| < 1/2$. We recall that $\hat{\psi}$ is supported in $[-\delta, \delta]$. Performing an integration by parts sufficiently many times results in

$$\sum_{\tau} \psi\left(\tau \frac{\log C_{\rho,X}}{2\pi i}\right) \ll \max_{\tau} \left| \psi\left(\tau \frac{\log C_{\rho,X}}{2\pi}\right)\right|$$
$$\ll_{N,\psi} C_{\rho,X}^{\delta \frac{\Im(\tau)}{2\pi}} (1 + |\tau| \log C_{\rho,X})^{-N} \ll \frac{X^{\delta \mathfrak{C}(\rho)/4\pi}}{(1 + \log X)^N}.$$

The last estimate follows from Conjecture 2. Finally, Conjecture 3 immediately yields the claim if $\delta \mathfrak{C}(\rho) < 4\pi\eta(\rho)$. □

Thus, if we further assume Conjecture 3, then the assertion of Theorem 8 can be rewritten as

$$D_{\rho,X}(\psi) = \hat{\psi}(0) - \frac{\mathfrak{s}(\rho)}{2}\psi(0) + O\left(\frac{1}{\log X}\right),$$

if $\delta < \frac{1}{\mathfrak{C}(\rho)} \min\left(\frac{1}{\theta_1 - \theta_n}, 4\pi\eta(\rho)\right)$.

6.6 The orthogonality conjecture over a cuspidal spectrum

We provide a variant of Theorem 4 (that is, the conjecture in (2.2.1)) over a cuspidal spectrum as discussed in Remark 6. We also mention that a similar technique may be applied to prove variants of Theorem 6 over a cuspidal spectrum.

In this section, we work in the S-arithmetic setting. We fix p to be a finite prime, and re-define our notations. Let $G := \mathrm{PGL}_n(\mathbb{R}) \times \mathrm{PGL}_n(\mathbb{Q}_p)$, $\Gamma := \mathrm{PGL}_n(\mathbb{Z}[1/p])$ diagonally embedded in G, and $\mathbb{X} := \Gamma\backslash G$. The strong approximation theorem on G implies that

$$\mathbb{X} \cong \mathrm{PGL}_n(\mathbb{Q})\backslash\mathrm{PGL}_n(\mathbb{A})/\prod_{v\neq p}\mathrm{PGL}_n(\mathbb{Z}_v),$$

where $\mathbb{A}$ is the ring of adeles over $\mathbb{Q}$. For any irreducible automorphic representation π of G, let π_p and π_∞ be p-adic and infinite components of π, respectively.

Let N be the unipotent subgroup in G and $\psi_m := \psi_{m,\infty}\otimes\psi_{m,p}$ for $m \in \mathbb{Z}[1/p]^{n-1}_{\neq 0}$ is an additive character on N, where $\psi_{m,\infty}$ is defined as in (4.1.1) and $\psi_{\cdot,p}$ is induced similarly from an unramified character ψ_0 of $\mathbb{Q}_p$. We abbreviate $\psi_v = \psi_{m,v}$ for $m = (1,\ldots,1)$, $v = p,\infty$.

We can realize a Whittaker model $\mathcal{W}(\pi_\infty,\psi_\infty)$ of π_∞. We define $J_F(\pi_\infty)$ (similarly, $J_{F_X}(\pi_\infty) = J_X(\pi_\infty)$ for $F = F_X$ as defined after (2.1.1)) as in (4.2.5). J_X, as usual, features similar properties as in Lemma 6.1.1. We also define W_F (similarly, $W_{F_X} = W_X$ for $F = F_X$) as in (4.2.8) with respect to the additive character ψ_∞ of $N(\mathbb{R})$. We provide a similar orthogonality statement as in Theorem 4 varying only over the cusp forms, as follows.

Theorem 11. *Let X be large. Let σ be any fixed supercuspidal representation of* $\mathrm{PGL}_n(\mathbb{Q}_p)$. *Then, for $l,m \in \mathbb{N}^{n-1}$ coprime to p (i.e., all the coordinates are coprime to p) with*

$$\min(l_1^{n-1}\ldots l_{n-1}m_1\ldots m_{n-1}^{n-1}, m_1^{n-1}\ldots m_{n-1}l_1\ldots l_{n-1}^{n-1}) \ll X^n$$

with a sufficiently small implied constant, we have

$$\sum_{\substack{\pi \text{ cuspidal}\\ \pi_p=\sigma}} \overline{\lambda_\pi(m)}\lambda_\pi(l)\frac{J_X(\pi_\infty)}{L^{(p)}(1,\pi,\mathrm{Ad})} = \delta_{m=l}\gamma_p(1,\sigma\otimes\tilde{\sigma})W_X(1),$$

where γ_p is the p-adic γ-factor, $W_X(1) \asymp X^{n-1}$, and $L^{(p)}$ denotes the partial L-function excluding the Euler factor at p.

6.6.1 Proof of Theorem 11

As before, for a subgroup $H < G$, we define $\Gamma_H := \Gamma \cap H$. For a generic irreducible unitary automorphic representation $\pi \in \hat{\mathbb{X}}_{\text{gen}}$ and a factorizable $\varphi \in \pi$, we may attach a Whittaker function $W_\varphi := W_{\varphi,p} W_{\varphi,\infty}$ to φ, such that

$$\int_{[N]} \varphi(x(g_\infty, g_p))\overline{\psi_m(x)}dx = \frac{\lambda_\pi(m_o)}{\delta^{1/2}(\tilde{m})\delta^{1/2}(\tilde{m}_p)} W_{\varphi,p}(\tilde{m}g_p) W_{\varphi,\infty}(\tilde{m}g_\infty), \tag{6.6.1}$$

where m_p and m_o are the p-adic part and the p-coprime part of $\tilde{m}$, respectively[i]. A similar formula as in (4.2.2) holds in this case as well, where in this case (see [43, §2.2])

$$\|W_\varphi\| = \|W_{\varphi,p}\|_{\mathcal{W}(\pi_p,\psi_p)} \|W_{\varphi,\infty}\|_{\mathcal{W}(\pi_\infty,\psi_\infty)}, \tag{6.6.2}$$

and $\ell(\pi) \asymp L^{(p)}(1, \pi, \mathrm{Ad})$ with an absolute implied constant, if π is cuspidal.

Let Φ_σ be a matrix coefficient of a supercuspidal representation σ of $\mathrm{PGL}_n(\mathbb{Q}_p)$ with the conductor exponent of σ being 1. Also, recall F_X, a smoothened L^1-normalized characteristic function of $K_0(X, \tau)$, as in (2.1.1) and the discussion afterwards. Then, a similar argument from the proof of Proposition 4.2.1 yields

$$\begin{aligned}
&\int_{\hat{\mathbb{X}}_{\text{gen}}} \frac{\overline{\lambda_\pi(m_o)}\lambda_\pi(l_o)}{\delta^{1/2}(\tilde{m}_p)\delta^{1/2}(\tilde{l}_p)} \frac{J_X(\pi_\infty)J_\sigma(\pi_p)}{\ell(\pi)} d\mu_{\text{aut}}(\pi) \\
&\quad = \delta_{m=l} W_X(1) W_\sigma(1) + \sum_{1\neq w \in W} \frac{\delta^{1/2}(\tilde{l})}{\delta^{1/2}(\tilde{m})\delta_w(\tilde{l})} \sum_{c \in \mathbb{Z}[1/p]^{r-1}_{\neq 0}} S_w(l, m; c) \\
&\quad \times \int_{N_w(\mathbb{R})} W_X(\tilde{m}c^* w\tilde{l}^{-1}x)\overline{\psi_\infty(x)}dx \int_{N_w(\mathbb{Q}_p)} W_\sigma(\tilde{m}c^* w\tilde{l}^{-1}x)\overline{\psi_p(x)}dx,
\end{aligned} \tag{6.6.3}$$

where, similar to (4.2.5),

$$J_\sigma(\pi_p) = \sum_{W \in \mathcal{B}(\pi_p)} \pi_p(\bar{\Phi}_\sigma) W(1) \overline{W(1)} \tag{6.6.4}$$

for some orthonormal basis of $\mathcal{B}(\pi_p)$ of π_p. Similarly, as in (4.2.8), we define[ii]

$$W_\sigma(g) = \int_N \Phi_\sigma(xg)\overline{\psi_p(x)}dx. \tag{6.6.5}$$

[i] That is, if $m = (m_1, \ldots, m_{n-1})$ and $m_i = p^{k_i} m_i'$, then $m_o := (m_1', \ldots, m_{n-1}'$ and $m_p := (p^{k_1}, \ldots, p^{k_r-1})$.

[ii] We hope that readers do not confuse W_σ and Whittaker functions W.

6.6.2 Local p-adic computation

Let the matrix coefficient Φ_σ of the supercuspidal representation σ be defined by

$$\Phi_\sigma(g) = \langle \sigma(g)W_0, W_0\rangle_{\mathcal{W}(\sigma,\psi_p)},$$

where W_0 is an L^2-normalized vector in the Whittaker model $\mathcal{W}(\sigma, \psi_p)$ of σ with respect to ψ_p, with $W_0(1) \neq 0$.[iii]

Proposition 6.6.1. *Recall the definitions in* (6.6.4) *and* (6.6.5). *Then,*

$$J_\sigma(\pi_p) = \delta_{\pi_p \cong \sigma}\gamma(1, \sigma \otimes \tilde{\sigma})W_\sigma(1),$$

where γ denotes the local gamma factor and $\tilde{\sigma}$ is the contrgradient of σ.

We first prove the following lemma about a Fourier transform of the matrix coefficient to prepare the proof of Proposition 6.6.1. The following lemma appeared in $\mathrm{GL}_2(\mathbb{Q}_p)$ in [53, Lemma 3.5] as well as in [47, Lemma 3.4.2].

Lemma 6.6.1. *Let Φ_σ be the matrix coefficient defined above. Then,*

$$\int_{N(\mathbb{Q}_p)} \Phi_\sigma(xg)\overline{\psi_p(x)}dx = W_0(g)\overline{W_0(1)}.$$

Proof. As σ is supercuspidal, Φ_σ is compactly supported in $\mathrm{PGL}_n(\mathbb{Q}_p)$. Thus, the integral is absolutely convergent. It is sufficient to prove that

$$\int_{N(\mathbb{Q}_p)}\int_{N_{n-1}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_p)} W_1\left[\begin{pmatrix} h & \\ & 1\end{pmatrix} x\right]\overline{W_2\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right]}\overline{\psi_p(x)}dhdx = W_1(1)\overline{W_2(1)}$$

for $W_1, W_2 \in \mathcal{W}(\sigma, \psi_p)$.

We write $x = \begin{pmatrix} I_{n-1} & v \\ & 1\end{pmatrix}\begin{pmatrix} u & \\ & 1\end{pmatrix}$ for $u \in N_{n-1}(\mathbb{Q}_p)$ and $v \in \mathbb{Q}_p^{n-1}$. Correspondingly, we write $dx = dudv$ and $\psi_p(x) = \psi_p(u)\psi_0(e_{n-1}v)$, where e_{n-1} is the row vector $(0, \ldots, 0, 1)$ and $\psi_p(u)$ (denoted with an abuse of notation) is the restriction of ψ_p on $N_{n-1}(\mathbb{Q}_p)$. We use the unipotent equivariance of W_1 to re-write the last integral as

$$\int_{N_{n-1}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \overline{W_2\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right]}\int_{N_{n-1}(\mathbb{Q}_p)} W_1\left[\begin{pmatrix} hu & \\ & 1\end{pmatrix}\right]\overline{\psi_p(u)}du \times \int_{\mathbb{Q}_p^{n-1}} \psi_0(e_{n-1}hv)\overline{\psi_0(e_{n-1}v)}dxdh.$$

By Fourier inversion the inner-most integral evaluates to $\delta_{e_{n-1}h=e_{n-1}}$ (as a distribution)

[iii] Such a vector exists, e.g., a newvector of σ.

and, thus, we can re-write the above as

$$\int_{N_{n-2}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n-2}(\mathbb{Q}_p)} \overline{W_2\left[\begin{pmatrix} h' & \\ & I_2\end{pmatrix}\right]} \int_{N_{n-1}(\mathbb{Q}_p)} W_1\left[\begin{pmatrix}\begin{pmatrix} h' & \\ & 1\end{pmatrix} & u \\ & 1\end{pmatrix}\right] \overline{\psi_p(u)} du dh'.$$

Proceeding similarly with the u-integral and inducting on r, we conclude the proof. □

Lemma 6.6.2. *Let $W_0 \in \mathcal{W}(\sigma, \psi_p)$ with $\|W_0\|_{\mathcal{W}(\sigma,\psi_p)}= 1$. Then,*

$$\int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} |W_0(g)|^2 dg = \gamma(1, \sigma \otimes \tilde{\sigma}),$$

where γ is the local gamma factor.

Proof. Let P be the maximal parabolic, attached to the partition $n = (n - 1) + 1$ in $\mathrm{PGL}(n)$. Let Ψ be a characteristic function of $\mathbb{Z}_p^n$. We define a function in $\mathrm{PGL}_n(\mathbb{Q}_p)$ by

$$f_{s,\Psi}(g) := \int_{\mathbb{Q}_p^\times} \Psi(te_n g)|\det(tg)|^s d^\times t.$$

We normalize Ψ so that $f_{s,\Psi}(1) = 1$, so $f_{s,\Psi}$ is spherical (i.e., $\mathrm{PGL}_n(\mathbb{Z}_p)$ invariant) and

$$f_{s,\Psi}\left[\begin{pmatrix} h & \\ & 1\end{pmatrix}\right] = |\det(h)|^s, \quad h \in \mathrm{GL}_{n-1}(\mathbb{Q}_p).$$

We use the $\mathrm{PGL}(n) \times \mathrm{PGL}(n)$ local functional equation as in [31, 2.5 Theorem] to obtain

$$\int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} |W_0(g)|^2 f_{s,\Psi}(g)dg = \gamma(1-s, \sigma\otimes\tilde{\sigma}) \int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} |\tilde{W}_0(g)|^2 f_{1-s,\hat{\Psi}}(g)dg, \tag{6.6.6}$$

where $\tilde{W}_0$ is the contragradient of W_0 and $\hat{\Psi}$ is the Fourier transform of Ψ, which also equals Ψ.

Using the Iwasawa decomposition, we write $N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p) \ni g = \begin{pmatrix} a & \\ & 1\end{pmatrix} k$ where $a \in A_{n-1}(\mathbb{Q}_p)$ is the subgroup of the diagonal matrices in $\mathrm{GL}_{n-1}(\mathbb{Q}_p)$ and $k \in \mathrm{PGL}_n(\mathbb{Z}_p)$. Correspondingly, we write $dg = \frac{d^\times a}{\delta(a)|\det(a)|}dk$. Thus, we can write the RHS of (6.6.6) as

$$\int_{A_{n-1}(\mathbb{Q}_p)\times \mathrm{PGL}_n(\mathbb{Z}_p)} \left|\tilde{W}_0\left[\begin{pmatrix} a & \\ & 1\end{pmatrix} k\right]\right|^2 |\det(a)|^{1-s} \frac{da}{\delta(a)|\det(a)|} dk.$$

We change variable $k \mapsto \begin{pmatrix} k' & \\ & 1\end{pmatrix} k$ with $k' \in \mathrm{GL}_{n-1}(\mathbb{Z}_p)$ and average over $\mathrm{GL}_{n-1}(\mathbb{Z}_p)$ (we normalize the measures so that $\mathrm{vol}(\mathrm{GL}_{r'}(\mathbb{Z}_p)) = 1$ for $1 \le r' \le n$) to obtain that the above equals

$$\int_{\mathrm{PGL}_n(\mathbb{Z}_p)} \int_{N_{n-1}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_p)} \left| \tilde{W}_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} k \right] \right|^2 |\det(h)|^{-s} dh dk.$$

Thus, for $s = 0$, using the invariance of the unitary product, the above equals $\|\tilde{W}_0\|^2_{\mathcal{W}(\tilde{\sigma},\overline{\psi_p})}$.

On the other hand, performing a similar computation as above, we can obtain that, for $s = 0$, the LHS of (6.6.6) is

$$\int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} |W_0(g)|^2 dg.$$

Finally, from $\|\tilde{W}_0\|^2_{\mathcal{W}(\tilde{\sigma},\overline{\psi_p})} = \|W_0\|_{\mathcal{W}(\sigma,\psi_p)} = 1$, we conclude the proof. □

Proof of Proposition 6.6.1. We first note that Lemma 6.6.1 implies that

$$W_\sigma(1) = |W_0(1)|^2.$$

For $W \in \mathcal{W}(\pi_p, \psi_p)$, we consider the $\mathrm{GL}_n(\mathbb{Q}_p)$-invariant pairing between π_p and σ by

$$\int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} W(g)\overline{W_0(g)} dg, \tag{6.6.7}$$

where dg is the projection of the Haar measure on $\mathrm{GL}_n(\mathbb{Q}_p)$. The integral in (6.6.7) is absolutely convergent as W_0 is compactly supported in $\mathrm{PGL}_n(\mathbb{Q}_p)$ mod $N(\mathbb{Q}_p)$ [12, Corollary 6.5].

Schur's lemma implies that (6.6.7) is non-zero only if $\pi_p \cong \sigma$, in which case (6.6.7) is then proportional to the invariant unitary product in σ given by

$$\int_{N_{n-1}(\mathbb{Q}_p)\backslash \mathrm{GL}_{n-1}(\mathbb{Q}_p)} W \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right] \overline{W_0 \left[\begin{pmatrix} h & \\ & 1 \end{pmatrix} \right]} dh.$$

Recalling (6.6.4) and folding the PGL_n integral over N, we can write

$$J_\sigma(\pi_p) = \sum_{W \in \mathcal{B}(\pi_p)} \overline{W(1)} W_0(1) \int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} W(g) \int_{N(\mathbb{Q}_p)} \overline{\Phi_\sigma(ng)} \psi(n) dn dg.$$

Using Lemma 6.6.1, we obtain

$$J_\sigma(\pi_p) = \sum_{W \in \mathcal{B}(\pi_p)} \overline{W(1)} W_0(1) \int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} W(g)\overline{W_0(g)} dg.$$

Hence, $J_\sigma(\pi_p) = 0$ if π_p is not isomorphic to σ, and if $\pi_p = \sigma$, then by choosing an orthonormal basis $\mathcal{B}(\sigma) \ni W_0$, we obtain

$$J_\sigma(\sigma) = |W_0(1)|^2 \int_{N(\mathbb{Q}_p)\backslash \mathrm{PGL}_n(\mathbb{Q}_p)} |W_0(g)|^2 dg.$$

We conclude the proof using Proposition 6.6.2. □

Proof of Theorem 11. We apply (6.6.3) for l, m coprime with p. A similar argument as in Theorem 4 implies that all terms in the geometric side of (6.6.3) corresponding to non-trivial Weyl terms vanish. We conclude the proof after applying Lemma 6.6.1. □

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