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Abstract. Let X be a globally symmetric space of noncompact type and rank greater that one, and $\Gamma \subset \mathrm{Isom}(X)$ a Schottky group of axial isometries. Then $M := X/\Gamma$ is a locally symmetric Riemannian manifold of infinite volume. The goal of this note is to give an asymptotic estimate for the number of primitive closed geodesics in M modulo free homotopy with period less than t.

1. Introduction

Let M be a complete Riemannian manifold of nonpositive sectional curvature, and denote by $P(t)$ the number of primitive closed geodesics in M of period less than t modulo free homotopy. If M is compact with volume entropy h, there are various results describing the asymptotic behavior of this function $P(t)$: The most remarkable early result due to Margulis [17,18] states that if M has pinched negative curvature, then

$$\lim_{t\to\infty} P(t) \cdot ht \cdot e^{-ht} = 1.$$

Later, Knieper [11–13] obtained a slightly weaker analogon of this result for geometric rank one manifolds: He proved the existence of constants $a > 1$ and $t_0 > 0$ such that

$$\frac{1}{a\,t}\, e^{ht} \le P(t) \le \frac{a}{t}\, e^{ht}$$

for $t > t_0$ [13, Theorem 5.6.2].

For compact rank one symmetric spaces of noncompact type, Margulis' result has been improved by giving error terms [5]. Moreover, several authors obtained asymptotic equivalents of $P(t)$ for quotients of noncompact manifolds with pinched negative curvature by convex cocompact and certain geometrically finite discrete isometry groups (see e.g. [7,14,19]). In this case, the exponential growth rate of $P(t)$ is no longer governed by the volume entropy h but instead by the critical

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exponent of the deck transformation group Γ of the Riemannian universal covering $X \to M$

$$\delta(\Gamma) := \inf \left\{ s > 0 \;\middle|\; \sum_{\gamma \in \Gamma} e^{-sd(x,\gamma y)} < \infty \right\},$$

where x and y are arbitrary points in X. Recently, Roblin [21,22] proved similar results in the even more general context of CAT(-1)-spaces . The common approach to obtain an asymptotic equivalent for $P(t)$ consists in establishing an asymptotic equivalent for the orbit counting function $N_\Gamma(R) := \#\{\gamma \in \Gamma : d(x,\gamma y) < R\}$; in negative curvature, when the geodesic flow is Anosov, $N_\Gamma(R)$ and $P(t)$ are intimately related.

However, in higher rank symmetric spaces the knowledge of the asymptotic behavior of $N_\Gamma(R)$ does not allow in general to obtain an asymptotic estimate for $P(t)$. This is due to the fact that there are uncountably many closed geodesics in each free homotopy class. Although for finite volume locally symmetric spaces of rank $r \geq 2$ an asymptotic equivalent for $N_\Gamma(R)$ of the form $R^{(r-1)/2} e^{\delta(\Gamma)R}$ is known ([9, Theorem 1.4] combined with [12, Theorem 6.2]), not even an analogon of Knieper's result could be proved so far. In this note we treat the case where M is a locally symmetric space of noncompact type with a Schottky group Γ (in the sense of Benoist [2]) as deck transformation group. For such groups Jean-François Quint [20] recently established an asymptotic equivalent for $N_\Gamma(R)$ of the form constant times $e^{\delta(\Gamma)R}$ using symbolic dynamics. Here we describe a geometric method to deduce from his result an asymptotic estimate for $P(t)$ as follows:

Main theorem. *If M is a locally symmetric space of rank $r \geq 2$ with deck transformation group Γ as above, there exist constants $a > 1$ and $t_0 > 0$ such that for any $t > t_0$*

$$\frac{1}{a\, t^r}\, e^{\delta(\Gamma)t} \leq P(t) \leq \frac{a}{t}\, e^{\delta(\Gamma)t}.$$

We remark that in the case of rank one symmetric spaces, Schottky groups are known to be convex cocompact. Hence our result is covered by a special case of one of the above mentioned results due to Lalley [14]: If M is a quotient of a Hadamard manifold with pinched negative curvature by a convex cocompact discrete isometry group Γ, then $P(t)$ is asymptotically equivalent to a constant times $e^{\delta(\Gamma)t}/t$. For a precise statement in an even more general setting we refer the reader to Theorem 5.1.1 in [22].

The paper is organized as follows: In Sect. 2 we recall some basic facts about symmetric spaces of noncompact type and decompositions of semisimple Lie groups. Section 3 describes some important concepts concerning closed geodesics in a locally symmetric space. In Sect. 4 we introduce the Schottky groups we will be concerned with and state some results about their limit set and the exponential growth rate of certain orbit points. Section 5 is devoted to the proof of the key step in our main theorem, namely to give an upper bound for the number of isometries corresponding to the same free homotopy class of closed geodesics in the quotient. Finally, in Sect. 6, we restate the main theorem and complete its proof.

2. Preliminaries on symmetric spaces

The purpose of this section is to introduce some terminology and notation, and to summarize some basic results about symmetric spaces of noncompact type (see also [3,8,10]) which we shall need.

Let X be a simply connected symmetric space of noncompact type with base point $o \in X$, $G = \mathrm{Isom}^o(X)$ the connected component of the identity, and $K \subset G$ the isotropy subgroup of o in G. Then X is a manifold of nonpositive curvature, G a semisimple Lie group with trivial center, K a maximal compact subgroup of G, and we may write $X = G/K$. If $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K, then the geodesic symmetry of X at o determines a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is identified with the tangent space T_oX of X at o. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subalgebra, and $\mathfrak{a}^+ \subset \mathfrak{a}$ an open Weyl chamber with closure $\overline{\mathfrak{a}^+}$. A flat in X is a totally geodesic submanifold of the form $ge^{\mathfrak{a}}o$, $g \in G$. The decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ is called the Cartan decomposition of G.

Definition 2.1. For $x, y \in X$ the unique vector $H \in \overline{\mathfrak{a}^+}$ with the property $x = go$ and $y = ge^H o$ for some $g \in G$ is called the Cartan vector of the ordered pair of points $(x, y) \in X \times X$ and will be denoted $H(x, y)$.

Notice that the length of the Cartan vector $H(x, y)$ is exactly the Riemannian distance between x and y. If $\mathrm{rank}(X) := \dim \mathfrak{a} = 1$, then the Cartan vector of a pair of points is simply this number.

Let Σ be the set of restricted roots of the pair $(\mathfrak{g}, \mathfrak{a})$, and $\Sigma^+ \subset \Sigma$ the set of positive roots determined by the Weyl chamber $\mathfrak{a}^+$. We denote $\mathfrak{g}_\alpha$ the root space of $\alpha \in \Sigma$, $\mathfrak{n}^+ := \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$, and N^+ the Lie group exponential of the nilpotent Lie algebra $\mathfrak{n}^+$. The decomposition $G = KAN^+$ is called the Iwasawa decomposition associated to the Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$. If M denotes the centralizer of $\mathfrak{a}$ in K, the Iwasawa decomposition induces a natural projection

$$\begin{aligned} \pi^I : G &\to K/M \\ g = kan &\mapsto kM. \end{aligned}$$

Let M^* be the normalizer of $\mathfrak{a}$ in K, and $W = M^*/M$ the Weyl group of the pair $(\mathfrak{g}, \mathfrak{a})$. We will denote $w_* \in W$ the unique element such that $\mathrm{Ad}(m_{w_*})(-\mathfrak{a}^+) = \mathfrak{a}^+$ for any representative m_{w_*} of w_* in M_*.

The geometric boundary ∂X of X is defined as the set of equivalence classes of asymptotic geodesic rays endowed with the cone topology. This boundary is homeomorphic to the unit tangent space of an arbitrary point in X (compare [1, Chap. II]), and $\overline{X} := X \cup \partial X$ is homeomorphic to a closed ball in $\mathbb{R}^N$, where $N := \dim X$.

Let $\mathfrak{a}_1 \subset \mathfrak{a}$ be the set of unit vectors in $\mathfrak{a} \subset \mathfrak{p} \cong T_oX$. For $x \in X$ and $z \in \overline{X}\backslash\{x\}$ we denote $\sigma_{x,z}$ the unique unit speed geodesic ray emanating from x containing z. The direction $H \in \overline{\mathfrak{a}_1^+}$ of a point $\xi \in \partial X$ is defined as the Cartan vector of the ordered pair $(o, \sigma_{o,\xi}(1))$, i.e. $H = H(o, \sigma_{o,\xi}(1))$. If $k \in K$ is such that ξ belongs to the class of the geodesic ray σ given by $\sigma(t) := ke^{Ht}o, t \geq 0$, we write $\xi = (k, H)$. Notice that k is only determined up to right multiplication by an element in the centralizer of H in K.

If the rank of X is greater than one, then the regular boundary ∂X^{reg} is defined as the set of classes with Cartan projection $H \in \mathfrak{a}_1^+$. If rank$(X) = 1$, then $\overline{\mathfrak{a}_1^+}$ is a point and we use the convention $\partial X^{reg} := \partial X$. We will further need the continuous projection

$$\pi^B : \partial X^{reg} \to K/M$$
$$(k, H) \mapsto kM,$$

which is a homeomorphism if and only if rank$(X) = 1$.

The isometry group of X has a natural action by homeomorphisms on the geometric boundary. If $g \in G, \xi = (k, H) \in \partial X$ and $k' \in K$ is such that $\pi^I(gk) = k'M$, then $g\cdot(k, H) = (k', H)$ (see [15, Lemma 2.2]). In particular, the G-action preserves the directions of boundary points, hence G acts transitively on the geometric boundary if and only if rank$(X) = 1$. However, the projection π^B induces a transitive action of G by homeomorphisms on the Furstenberg boundary $K/M = \pi^B(\partial X^{reg})$. So if $\xi = (k, H) \in \partial X^{reg}$, then $g\pi^B(\xi) = \pi^B(g\xi) = k'M$.

Moreover, for $\xi \in \partial X$ we denote $\mathrm{Vis}^\infty(\xi)$ the set of points in the geometric boundary which can be joined to ξ by a geodesic, i.e.

$$\mathrm{Vis}^\infty(\xi) := \{\eta \in \partial X | \exists \text{ geodesic } \sigma \text{ such that } \sigma(-\infty) = \xi,\ \sigma(\infty) = \eta\}.$$

Notice that for rank one symmetric spaces we have $\mathrm{Vis}^\infty(\xi) = \partial X \setminus \{\xi\}$ for all $\xi \in \partial X$. The Bruhat visibility set of a point $\xi \in \partial X^{reg}$

$$\mathrm{Vis}^B(\xi) := \pi^B\left(\mathrm{Vis}^\infty(\xi)\right)$$

will play an important role in the sequel. It is a dense and open submanifold of the Furstenberg boundary K/M and corresponds to a Bruhat cell of maximal dimension (compare [15, Sect. 2.3]).

3. Conjugacy classes and closed geodesics

Let M be a locally symmetric space of noncompact type with universal Riemannian covering manifold X, and $\Gamma \subset \mathrm{Isom}(X)$ the group of deck transformations of the covering projection $X \to M$. It is well-known that Γ is a discrete and torsion free group isomorphic to the fundamental group of M. If $G = \mathrm{Isom}^o(X)$ is the connected component of the identity, we fix a Cartan decomposition $G = Ke^{\overline{\mathfrak{a}^+}}K$ and the associated Iwasawa decomposition $G = KAN^+$. Let $o \in X$ be the unique point stabilized by K.

In order to describe closed geodesics in M, we will need to work with the following kind of isometries of X:

Definition 3.1. An isometry $\gamma \neq \mathrm{id}$ of X is called axial, if there exists a constant $l > 0$ and a unit speed geodesic $\sigma \subset X$ such that $\gamma(\sigma(t)) = \sigma(t + l)$ for all $t \in \mathbb{R}$.

We call $L(\gamma) := H(\sigma(0), \sigma(l)) \in \overline{\mathfrak{a}^+}\backslash\{0\}$ the translation vector, and $l(\gamma) := \|L(\gamma)\| > 0$ the translation length of γ. The boundary point $\gamma^+ := \sigma(\infty)$ is called the attractive fixed point, and $\gamma^- := \sigma(-\infty)$ the repulsive fixed point of γ. We say that γ is regular axial if $\gamma^+ \in \partial X^{reg}$.

For an axial isometry γ we define the set $\mathrm{Ax}(\gamma) := \{x \in X \mid d(x, \gamma x) = l(\gamma)\}$ which consists of the union of parallel geodesics translated by γ. Then $\overline{\mathrm{Ax}(\gamma)} \cap \partial X$ is exactly the set of fixed points of γ.

If γ is regular axial and $x \in \mathrm{Ax}(\gamma)$, then there exists $g \in G$ such that $x = go$ and $\gamma x = ge^{L(\gamma)}o$. Moreover, $\pi^B(\gamma^+) = \pi^I(g)$, $\pi^B(\gamma^-) = \pi^I(gw_*)$, and the set of fixed points in the Furstenberg boundary $\mathrm{Fix}^B(\gamma)$ is exactly the finite set

$$\mathrm{Fix}^B(\gamma) = \{\pi^I(gw) | w \in W\}.$$

If $\gamma \in \Gamma$ is axial and $\sigma, \sigma' \subset \mathrm{Ax}(\gamma) \subset X$ are geodesics translated by γ, then σ and σ' project to freely homotopic closed geodesics of the same period $\leq l(\gamma)$ in the quotient M. Since $\mathrm{Ax}(\gamma)$ consists of an uncountable union of parallel geodesics translated by γ, there are uncountably many closed geodesics in each free homotopy class. In order to describe these free homotopy classes, we will make use of the following

Definition 3.2. $\gamma, \gamma' \in \Gamma$ are said to be equivalent if and only if there exist $n, m \in \mathbb{Z}$ and $\varphi \in \Gamma$ such that $(\gamma')^m = \varphi\gamma^n\varphi^{-1}$. An element $\gamma_0 \in \Gamma$ is called primitive if it cannot be written as a proper power $\gamma_0 = \varphi^n$, where $\varphi \in \Gamma$ and $n \geq 2$.

Each equivalence class can be represented as

$$[\gamma] = \{\varphi\gamma_0^k\varphi^{-1} | \gamma_0 \in \Gamma,\ \gamma_0 \text{ primitive},\ k \in \mathbb{Z},\ \varphi \in \Gamma\}.$$

It is easy to see that the set of equivalence classes of axial elements in Γ is in one to one correspondence with the set of geometrically distinct closed geodesics modulo free homotopy. If $\gamma \in \Gamma$ is axial, we put

$$l([\gamma]) := \min\{l(\varphi) | \varphi \in [\gamma]\}$$

and notice that if γ_0 is a primitive isometry representing $[\gamma]$, then $l([\gamma]) = l(\gamma_0)$. Moreover,

$$P(t) := \#\{[\gamma] | \gamma \in \Gamma \text{ axial},\ l([\gamma]) < t\}$$

counts the number of geometrically distinct closed geodesics of period less than t modulo free homotopy.

We will see that $P(t)$ is intimately related to the number

$$N_\Gamma(R) := \#\{\gamma \in \Gamma | d(o, \gamma o) < R\},$$

which allows the following alternative characterization of the critical exponent of Γ:

$$\delta(\Gamma) = \limsup_{R\to\infty} \left(\frac{1}{R} \log N_\Gamma(R)\right).$$

In the sequel, we will further need the following

Definition 3.3. If $\Gamma \subset \mathrm{Isom}(X)$ is a discrete group, its limit set L_Γ is defined by $L_\Gamma := \overline{\Gamma \cdot o} \cap \partial X$. We denote $K_\Gamma := \pi^B(L_\Gamma \cap \partial X^{reg})$, and $P_\Gamma := \{H \in \overline{\mathfrak{a}_1^+} \,|\, \exists\, k \in K$ such that $(k, H) \in L_\Gamma\}$.

4. Schottky groups

We next introduce and recall some properties of the Schottky groups we will be concerned with in the sequel. As in the previous section, X is a globally symmetric space of noncompact type, $G = K e^{\overline{\mathfrak{a}^+}} K$ a Cartan decomposition of $G = \mathrm{Isom}^o(X)$, and $o \in X$ the unique point stabilized by K.

Let $h_1, h_2, \dots, h_l$ be regular axial isometries with the following property: If we denote $\xi_{2m} := h_m^+$ and $\xi_{2m-1} := h_m^-$, $1 \le m \le l$, then

$$\pi^B(\xi_i) \in \bigcap_{\substack{n=1\\ n\neq i}}^{2l} \mathrm{Vis}^B(\xi_n) \quad \forall\, i \in \{1, 2, \dots, 2l\}.$$

Up to replacing h_1 and h_2 by approximate elements, we may assume that the group generated by $h_1, h_2, \dots, h_l$ is Zariski dense in G (compare [20, Sect. 4.2]). Furthermore, by Lemma 4.5.15 in [8] there exist neighborhoods $U_n' \subset K/M$ of $\pi^B(\xi_n)$, $1 \le n \le 2l$, such that for all $i \in \{1, 2, \dots, 2l\}$ and every point $\eta \in \partial X^{reg}$ with $\pi^B(\eta) \in U_i'$ we have

$$\bigcup_{\substack{n=1\\ n\neq i}}^{2l} U_n' \subset \mathrm{Vis}^B(\eta). \tag{1}$$

After possibly replacing $h_1, h_2, \dots, h_l$ by sufficiently large powers, we may assume that there exist sets $(U_n, W_n)_{1\le n\le 2l}$ in K/M, $U_n \subset U_n'$, such that conditions (i), (ii), (iii) before Proposition 4.4 in [20] are satisfied with $b_i^+ = U_{2i}$, $b_i^- = U_{2i-1}$, $B_i^+ = W_{2i}$, $B_i^- = W_{2i-1}$, $1 \le i \le l$, for some $\varepsilon \in (0, 1)$. In particular,

$$\bigcap_{n=1}^{2l} W_n \neq \emptyset, \quad U_{2i} \subset \bigcap_{\substack{n=1\\ n\neq 2i-1}}^{2l} W_n, \quad U_{2i-1} \subset \bigcap_{\substack{n=1\\ n\neq 2i}}^{2l} W_n, \quad \text{and}$$
$$h_i(W_{2i}) \subset U_{2i}, \quad h_i^{-1}(W_{2i-1}) \subset U_{2i-1} \qquad \text{for all} \quad i \in \{1, 2, \dots, l\}.$$

Hence by Klein's criterion (see [6]) the Zariski dense subgroup $\Gamma := \langle h_1, h_2, \dots, h_l\rangle$ of G is free and discrete, and we have the following well-known

Proposition 4.1 ([2], [15, Sect. 4.5]). *The limit set of Γ is contained in the regular boundary and splits as a product $L_\Gamma \cong K_\Gamma \times P_\Gamma$. Moreover, K_Γ is a minimal closed set for the action of Γ, $K_\Gamma \subset \bigcup_{i=1}^{2l} \overline{U_i}$ and $P_\Gamma \subset \mathfrak{a}_1^+$ is a closed convex cone. Every element $\gamma \in \Gamma$ is regular axial and satisfies $L(\gamma)/l(\gamma) \in P_\Gamma$.*

Moreover, Theorem 5.1 in [20] applies to Γ, hence there exist constants $b > 1$ and $R_0 > 0$ such that for all $R > R_0$

$$\frac{1}{b}e^{\delta(\Gamma)R} \le N_\Gamma(R) \le be^{\delta(\Gamma)R}. \tag{2}$$

As one of the main ingredients in the proof of the lower bound will serve Proposition 4.3, a stronger version of Eq. (2). The idea of proof is originally due to Roblin [21], but, due to the necessity of dealing both with the geometric boundary and the Furstenberg boundary, is more technical in our situation. For this reason the following definitions will be convenient.

If $C \subset K/M$, $z \in \overline{X}$, we put

$$\angle_o(z, C) := \inf\{\angle_o(z,\eta) | \eta \in \partial X^{reg} \text{ such that } \pi^B(\eta) \in C\}, \tag{3}$$

for $A, B \subset K/M$ we denote

$$N_\Gamma(R; A, B) := \#\{\gamma \in \Gamma | d(o, \gamma o) < R,\ \angle_o(\gamma o, A) = 0,\ \angle_o(\gamma^{-1}o, B) = 0\}.$$

If $V \subset K/M$ and $\varepsilon > 0$ we define a subset of the regular boundary by

$$H^\varepsilon_\Gamma(V) := \{\eta \in \partial X^{reg} | \exists\, \xi = (k, H) \in \partial X^{reg} \text{ with } kM \in V \text{ and } H \in P_\Gamma \text{ such that } \angle_o(\eta, \xi) < \varepsilon\}. \tag{4}$$

If $C \subset K/M$ is an open set containing the closure $\overline{V}$ of V, we further put

$$\varepsilon_\Gamma(V, C) := \sup\{\varepsilon > 0 | \pi^B(H^\varepsilon_\Gamma(V)) \subset C\}. \tag{5}$$

Moreover, the following easy lemma will be necessary in the proof of Proposition 4.3.

Lemma 4.2. *Let $V \subset C \subset K/M$ be as above and put $\varepsilon := \varepsilon_\Gamma(\overline{V}, C)$. Then for all $z \in \overline{X}$ with $\angle_o(z, C) > 0$ and $\angle(H(o,z), P_\Gamma) < \varepsilon/3$ we have*

$$\angle_o(z, \xi) > \frac{\varepsilon}{3} \qquad \forall \xi \in \partial X^{reg} \textit{ with } \ \pi^B(\xi) \in \overline{V}.$$

Proof. Suppose $\xi \in \partial X^{reg}$ satisfies $\pi^B(\xi) \in \overline{V}$ and $\angle_o(z,\xi) \le \varepsilon/3$. If $H \in \mathfrak{a}_1^+$ denotes the direction of ξ, then $\angle(H(o,z), H) \le \angle_o(z,\xi) \le \varepsilon/3$. Hence $\angle(H(o,z), P_\Gamma) < \varepsilon/3$ implies

$$\angle(H, P_\Gamma) \le \angle(H, H(o,z)) + \angle(H(o,z), P_\Gamma) < \frac{2}{3}\varepsilon.$$

In particular, if $\xi = (k, H)$ and $H' \in P_\Gamma$ such that $\angle(H, P_\Gamma) = \angle(H, H')$, then $\eta := (k, H') \in \partial X^{reg}$ satisfies $\angle_o(\xi, \eta) = \angle(H, H') < 2\varepsilon/3$. We conclude that

$$\angle_o(z,\eta) \le \angle_o(z,\xi) + \angle_o(\xi,\eta) < \varepsilon.$$

Moreover, since $\pi^B(\eta) = \pi^B(\xi) \in \overline{V}$ and $H' \in P_\Gamma$ we have $\sigma_{o,z}(\infty) \in H^\varepsilon_\Gamma(\overline{V})$, hence by choice of $\varepsilon = \varepsilon_\Gamma(\overline{V}, C)$ $\pi^B\left(\sigma_{o,z}(\infty)\right) \in C$, in contradiction to $\angle_o(\sigma_{o,z}(\infty), C) = \angle_o(z, C) > 0$. $\square$

Proposition 4.3. *If $A, B \subset K/M$ are open sets with $\pi^B(\gamma^+) \in A$ and $\pi^B(\gamma^-) \in B$ for some element $\gamma \in \Gamma$, then there exist constants $a > 1$ and $R_0 > 0$ such that for all $R > R_0$*

$$\frac{1}{a}\, e^{\delta(\Gamma)R} \le N_\Gamma(R; A, B) \le a\, e^{\delta(\Gamma)R}.$$

Proof. Let $U, V \subset K/M$ be open sets with $\pi^B(\gamma^+) \in U, \pi^B(\gamma^-) \in V, \overline{U} \subset A$ and $\overline{V} \subset B$.

We first prove the claim for $N_\Gamma(R; A) := \#\{\gamma \in \Gamma \mid d(o, \gamma o) < R, \angle_o(\gamma o, A) = 0\}$. Since K_Γ is compact, and $K_\Gamma = \pi^B(\overline{\Gamma\cdot\gamma^+})$ by Proposition 4.1, there exist $g_1, g_2, \dots, g_m \in \Gamma$ such that $K_\Gamma \subseteq \bigcup_{i=1}^m g_i U$. Let $Y \subseteq \bigcup_{i=1}^m g_i U$ be an open set which contains K_Γ. Then $\angle_o(\gamma o, Y) = 0$ for all but finitely many $\gamma \in \Gamma$ by the definition of the limit set and (3), hence $N_\Gamma(R; Y) \ge N_\Gamma(R) - M$ for some constant $M \in \mathbb{N}$.

Fix $g \in \{g_1, g_2, \dots, g_m\}$ and suppose $\angle_o(\gamma o, gU) = 0$ and $\angle_o(g^{-1}\gamma o, A) > 0$ for infinitely many $\gamma \in \Gamma$. Let $(\gamma_j) \subset \Gamma$ be a sequence with this property. Passing to a subsequence if necessary, we may assume that $\gamma_j o$ converges to a point $\eta \in L_\Gamma \subset \partial X^{reg}$, and we have $\pi^B(\eta) \in g\overline{U}$. Let $\varepsilon := \varepsilon_\Gamma(\overline{U}, A)$ and $N_0 \in \mathbb{N}$ such that for all $j \ge N_0$ we have

$$\angle_o(g^{-1}\gamma_j o, g^{-1}\eta) = \angle_{go}(\gamma_j o, \eta) < \varepsilon/4. \tag{6}$$

On the other hand, by choice of (γ_j) and since $\pi^B(g^{-1}\eta) \in \overline{U}$, Lemma 4.2 implies $\angle_o(g^{-1}\gamma_j o, g^{-1}\eta) > \varepsilon/3$ for all j sufficiently large, in contradiction to (6).

We conclude that

$$c(g) := \#\{\gamma \in \Gamma \mid \angle_o(\gamma o, gU) = 0 \ \text{ and } \ \angle_o(g^{-1}\gamma o, A) > 0\}$$

is finite and $N_\Gamma(R; gU) \le N_\Gamma(R + d(o, go); A) + c(g)$. With $d := \max_{1\le i\le m} d(o, g_i o)$ and $c := \sum_{i=1}^m c(g_i)$ we obtain

$$\begin{aligned} N_\Gamma(R-d; A) - M &\le N_\Gamma(R-d) - M \le N_\Gamma(R-d; Y) \le \sum_{i=1}^m N_\Gamma(R-d; g_i U) \\ &\le \sum_{i=1}^m \big(N_\Gamma(R-d+d(o, g_i o); A) + c(g_i)\big) \le m N_\Gamma(R; A) + c, \end{aligned}$$

which proves the first assertion.

Now again by compactness of K_Γ and the fact that $K_\Gamma = \pi^B(\overline{\Gamma\cdot\gamma^+})$, there exist $g_1, g_2, \dots, g_n \in \Gamma$ such that $K_\Gamma \subseteq \bigcup_{i=1}^n g_i V$. By the same argument as above the numbers

$$c(g) := \#\{\gamma \in \Gamma \mid \angle_o(\gamma^{-1}o, gV) = 0 \quad \text{and} \quad \angle_o(g^{-1}\gamma^{-1}o, B) > 0\}$$

are finite for all $g \in \{g_1, g_2, \dots, g_n\}$, and $N_\Gamma(R; A, gV) \le N_\Gamma(R + d(o, go); A, B) + c(g)$. We put $d := \max_{1\le i\le n} d(o, go)$ and $c := \sum_{i=1}^n c(g_i)$ and conclude

$$N_\Gamma(R-d; A, B) \le N_\Gamma(R-d; A) \le \sum_{i=1}^n N_\Gamma(R-d; A, g_i V) \le n N_\Gamma(R; A, B) + c,$$

which yields the assertion. $\square$

5. The key step

The most difficult task when trying to obtain a lower bound for $P(t)$ is to give an upper bound for the number of elements in the same conjugacy class. The main problem compared to the situation in [12] or [16] arises from the fact that in the higher rank setting, a cyclic group generated by an axial isometry does not act cocompactly on its invariant set. In the case of Schottky groups as in Sect. 4, however, we are able to show that for every isometry $\gamma \in \Gamma$, the number of orbit points in X close to such an unbounded fundamental set for the action of $\langle\gamma\rangle$ on $\mathrm{Ax}(\gamma)$ is bounded from above by a function of $l(\gamma)$. This will finally allow to obtain the desired estimate.

We start with a couple of preliminary lemmata. For the remainder of this section, $\Gamma = \langle h_1, h_2, \dots, h_l\rangle \subset \mathrm{Isom}(X)$ will be a Schottky group acting on a globally symmetric space X of rank $r > 1$ as in the previous section. Recall Definition 3.3 and put

$$\alpha_\Gamma := \sup_{H,H'\in P_\Gamma} \angle(H, H').$$

Since P_Γ is a closed convex cone strictly included in $\mathfrak{a}_1^+$, there exists $\delta \geq 0$ such that for any $H \in \mathfrak{a}$

$$\angle(H, P_\Gamma) := \inf_{H'\in P_\Gamma} \angle(H, H') \leq \delta \quad \text{implies } H \in \mathfrak{a}^+. \tag{7}$$

From the fact that $\max\{\angle(H, H') | H, H' \in \overline{\mathfrak{a}^+}\} \leq \pi/2$ (see i.e. [10, Theorem X.3.6 (iii)]) we conclude that $\alpha_\Gamma + \delta$ is strictly smaller than $\pi/2$.

Given $\gamma \in \Gamma$, we denote x_γ the orthogonal projection of o to $\mathrm{Ax}(\gamma)$. Let $g \in G$ such that $x_\gamma = go$ and $\gamma x_\gamma = ge^{L(\gamma)}o$, and put

$$F(\gamma) := \{ge^H o | H \in \mathfrak{a} \text{ such that } \langle H, \frac{L(\gamma)}{l(\gamma)}\rangle \in [0, l(\gamma)]\},$$

$$C_\Gamma^\delta(\gamma) := \{ge^H o \mid H \in \mathfrak{a} \text{ such that } \angle(H, P_\Gamma) \leq \delta\} \subset ge^{\mathfrak{a}^+}o.$$

Notice that $F(\gamma), C_\Gamma^\delta(\gamma) \subset \mathrm{Ax}(\gamma)$ are closed sets, and $\langle\gamma\rangle \cdot F(\gamma) = \mathrm{Ax}(\gamma)$. We further have the following

Lemma 5.1. *There exists a constant $a = a(\Gamma, \delta, r) > 0$ such that for all $\gamma \in \Gamma$*

$$\mathrm{vol}\left(F(\gamma) \cap C_\Gamma^\delta(\gamma)\right) \leq a \cdot l(\gamma)^r.$$

Furthermore, $F(\gamma) \cap C_\Gamma^\delta(\gamma)$ is compact for any $\gamma \in \Gamma$.

Proof. For $\gamma \in \Gamma$ we put

$$\mathfrak{a}(\gamma) := \{H \in \mathfrak{a}^+ | \langle H, L(\gamma)\rangle \leq l(\gamma)^2, \angle(H, P_\Gamma) \leq \delta\}.$$

Then $\mathrm{vol}\left(F(\gamma) \cap C_\Gamma^\delta(\gamma)\right) = \int_{\mathfrak{a}(\gamma)} dH$.

We substitute $\hat{H} = H/\|H\|$, $t = \|H\|$, and remark that $\angle(H, P_\Gamma) \le \delta$ and $L(\gamma)/l(\gamma) \in P_\Gamma$ imply

$$\angle(H, L(\gamma)) \le \alpha_\Gamma + \delta < \frac{\pi}{2}, \quad \text{hence} \quad \langle \hat{H}, L(\gamma)\rangle \ge l(\gamma)\cdot\cos(\alpha_\Gamma + \delta) > 0.$$

In particular, $H \in \mathfrak{a}(\gamma)$ satisfies the condition $\|H\| \cdot \cos(\alpha_\Gamma + \delta) \le l(\gamma)$. We summarize

$$\int\limits_{\mathfrak{a}(\gamma)} dH \le \int\limits_0^{l(\gamma)/\cos(\alpha_\Gamma+\delta)} \underbrace{\left(\int_{\angle(\hat{H}, P_\Gamma)\le\delta} d\hat{H}\right)}_{=:v_\Gamma(\delta)} t^{r-1} dt = \frac{v_\Gamma(\delta)}{r\cdot(\cos(\alpha_\Gamma+\delta))^r}\cdot l(\gamma)^r,$$

hence the assertion follows with $a = v_\Gamma(\delta)/(r\cdot\cos^r(\alpha_\Gamma + \delta))$.

Moreover, $F(\gamma)\cap C_\Gamma^\delta(\gamma)$ is compact since $\mathfrak{a}(\gamma)$ is closed and the norm of every element in $\mathfrak{a}(\gamma)$ is bounded by $l(\gamma)/\cos(\alpha_\Gamma + \delta) < \infty$. □

Using the notation from the previous section we recall that $K_\Gamma \subset \bigcup_{i=1}^{2l} \overline{U_i}$, and $\overline{U_i} \subset U_i'$ for all $1 \le i \le 2l$. We put

$$\Gamma' := \{\gamma \in \Gamma \mid \pi^B(\gamma^-) \in U_1 \text{ and } \pi^B(\gamma^+) \in U_2\}, \text{ and}$$
$$K' := \{kM \in K/M \mid \exists\, \gamma \in \Gamma' \text{ such that } kM \in \mathrm{Fix}^B(\gamma)\backslash\{\pi^B(\gamma^-), \pi^B(\gamma^+)\}\}.$$

K' describes the set of fixed points in the Furstenberg boundary of elements in Γ' which do not correspond to attractive or repulsive fixed points. Since for all $\gamma \in \Gamma$ we have $\mathrm{Vis}^B(\gamma^-)\cap\mathrm{Fix}^B(\gamma) = \{\pi^B(\gamma^+)\}$ and $\mathrm{Vis}^B(\gamma^+)\cap\mathrm{Fix}^B(\gamma) = \{\pi^B(\gamma^-)\}$, condition (1) implies that $K' \subset K/M\backslash\bigcup_{i=1}^{2l} U_i' \subseteq K/M\backslash K_\Gamma$.

Moreover, by $\overline{U_i} \subseteq U_i'$ for $1 \le i \le 2l$, the number

$$\min_{1\le i\le 2l} \varepsilon_\Gamma(\overline{U_i}, U_i')$$

is positive. We fix $\varepsilon \in (0, \delta)$ strictly smaller than this minimum and put

$$L_\Gamma^\varepsilon := \{\eta \in \partial X^{reg} \mid \exists\, \xi \in L_\Gamma \text{ such that } \angle_o(\xi, \eta) < \varepsilon\} \subseteq H_\Gamma^\varepsilon(K_\Gamma) \subseteq \bigcup_{i=1}^{2l} H_\Gamma^\varepsilon(\overline{U_i}).$$

Then by choice of ε we have $\pi^B(L_\Gamma^\varepsilon) \cap \overline{K'} = \emptyset$, hence there can be only finitely many orbit points close to K'. More precisely, we have

Lemma 5.2. *If $\varepsilon \in (0, \delta)$ is as above, then $N_\varepsilon := \#\{\gamma \in \Gamma \mid \angle_o(\gamma o, K') < \varepsilon/4\} < \infty$.*

Proof. Suppose N_ε is infinite. Then there exists a sequence $(\gamma_j) \subset \Gamma$ such that $\angle_o(\gamma_j o, K') < \varepsilon/4$. Let $(\eta_j) \subset \partial X^{reg}$, $\pi^B(\eta_j) \in K'$, such that $\angle_o(\gamma_j o, \eta_j) < \varepsilon/2$ for all $j \in \mathbb{N}$. Passing to subsequences if necessary, we may assume that $\gamma_j o \to \xi \in L_\Gamma \subset \partial X^{reg}$ and $\eta_j \to \eta \in \partial X$. Hence for j sufficiently large we have $\angle_o(\gamma_j o, \xi) < \varepsilon/4$ and $\angle_o(\eta_j, \eta) < \varepsilon/4$, and we conclude

$$\angle_o(\xi, \eta) \le \angle_o(\xi, \gamma_j o) + \angle_o(\gamma_j o, \eta_j) + \angle_o(\eta_j, \eta) < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon.$$

If $H_\xi \in P_\Gamma$, $H_\eta \in \overline{\mathfrak{a}_1^+}$ are the directions of ξ and η, this implies $\angle(H_\xi, H_\eta) \le \angle_o(\xi,\eta) < \varepsilon < \delta$, hence by choice of δ we have $\eta \in \partial X^{reg}$. From $\pi^B(\eta_j) \in K'$ and the definition of L_Γ^ε we conclude $\pi^B(\eta) \in \overline{K'} \cap \pi^B(L_\Gamma^\varepsilon)$, a contradiction to $\pi^B(L_\Gamma^\varepsilon) \cap \overline{K'} = \emptyset$. □

We next put

$$c_0 := \sup\{d(o, \mathrm{Ax}(g)) \mid g \in \mathrm{Isom}(X) \text{ regular axial with } \pi^B(g^-) \in U_1,\ \pi^B(g^+) \in U_2\}$$

and recall Definition 2.1 for the Cartan vector of a pair of points in X. Let

$$\rho := \frac{1}{4} \min_{\gamma\in\Gamma} \left(\inf_{x\in B_o(c_0)} d(x, \gamma x) \right) \tag{8}$$

and choose $R = R(\varepsilon, c_0, \rho) > 0$ such that the following three conditions hold:

(i) $\forall\, x, y \in B_o(c_0)\ \forall\, \gamma\in\Gamma : d(y,\gamma x) > R - c_0 - \rho \Longrightarrow \angle(H(y,\gamma x), P_\Gamma) < \varepsilon/8$,
(ii) $\forall\, x, p, z \in X\ :\ d(x,p) > R - c_0,\ d(p,z) < \rho + c_0 \Longrightarrow \angle_x(p,z) < \varepsilon/8$,
(iii) $\forall\, x, p \in X\ :\ d(o,p) > R,\ d(o,x) < c_0 \Longrightarrow \angle_o(p, \sigma_{x,p}(\infty)) < \varepsilon/8$.

Our first observation is the following

Lemma 5.3. *Let $x \in B_o(c_0)$, $h \in \Gamma'$ and $x_h \in Ax(h)$ the orthogonal projection of o to $Ax(h)$. Then for any $p \in Ax(h)$ with $d(o,p) > R$ and $B_p(\rho) \cap \Gamma\cdot x \neq \emptyset$ we have $\angle(H(x_h, p), P_\Gamma) < \varepsilon/4$.*

Proof. Since $B_p(\rho)\cap\Gamma\cdot x \neq \emptyset$, there exists $\gamma\in\Gamma$ such that $d(p,\gamma x) < \rho$. By choice of $c_0 > 0$ we have $d(o, x_h) < c_0$, hence the triangle inequality implies

$$d(x_h, \gamma x) \ge d(o,p) - d(o,x_h) - d(p,\gamma x) > R - c_0 - \rho.$$

Condition (i) above then gives $\angle(H(x_h,\gamma x), P_\Gamma) < \varepsilon/8$, and $d(x_h, p) > R - c_0$ and $d(p,\gamma x) < \rho$ imply $\angle_{x_h}(p,\gamma x) < \varepsilon/8$ by condition (ii). We estimate

$$\angle(H(x_h,p), H(x_h,\gamma x)) \le \angle_{x_h}(p,\gamma x) < \varepsilon/8$$

and conclude $\angle(H(x_h,p), P_\Gamma) < \varepsilon/4$. □

We will now combine these lemmata in order to obtain the desired estimate.

Proposition 5.4. *Let $\varepsilon \in (0,\delta)$, $c_0 > 0$ and $\rho > 0$ as above. Then there exist constants $b > 0$ and $t_0 > 0$ such that for any $x \in \overline{B_o(c_0)}$, every primitive element $h \in \Gamma'$ and all $t > t_0$*

$$\#\{\gamma = \varphi h^k \varphi^{-1} \mid \varphi\in\Gamma,\ k\in\mathbb{Z},\ \pi^B(\varphi h^-) \in U_1,\ \pi^B(\varphi h^+) \in U_2,\ \varphi Ax(h) \cap B_x(\rho) \cap B_o(c_0) \neq \emptyset,\ l(\gamma) \le t\} \le b\cdot t^r.$$

Proof. We first remark that if $\gamma = \varphi h^k \varphi^{-1}$ with $l(\gamma) \le t$, then $t \ge l(h^k) = |k| \cdot l(h)$, hence $|k| \le t/l(h)$.

Now if $k \in \mathbb{Z}\setminus\{0\}$ is fixed, then $\varphi h^k \varphi^{-1} \ne \beta h^k \beta^{-1}$ implies that $\beta^{-1}\varphi$ does not belong to the centralizer $Z_\Gamma(h)$ of h in Γ, in particular $\varphi \mathrm{Ax}(h) \ne \beta \mathrm{Ax}(h)$. If

$$\Gamma(h) := \{\gamma \in \Gamma \mid \pi^B(\gamma h^-) \in U_1,\ \pi^B(\gamma h^+) \in U_2,\\ \gamma \mathrm{Ax}(h) \cap B_x(\rho) \cap B_o(c_0) \ne \emptyset\},$$

then the number of different conjugates $\gamma = \varphi h^k \varphi^{-1}$ with $\varphi \in \Gamma, \pi^B(\varphi h^-) \in U_1,\ \pi^B(\varphi h^+) \in U_2$ and $\varphi \mathrm{Ax}(h) \cap B_x(\rho) \cap B_o(c_0) \ne \emptyset$ equals the cardinality of $\Gamma(h)$ modulo the centralizer $Z_\Gamma(h)$ of h in Γ.

Denote $x_h \in \mathrm{Ax}(h)$ the projection of o to $\mathrm{Ax}(h)$, let $g_h \in G$ such that $x_h = g_h o$ and $h x_h = g_h e^{L(h)} o$, and

$$F(h) = \{g_h e^H o \mid H \in \mathfrak{a} \text{ such that } \langle H, L(h)\rangle \in [0, l(h)^2]\} \subset \mathrm{Ax}(h)$$

as defined before Lemma 5.1. If $\varphi, \beta \in \Gamma(h)$, $\beta^{-1}\varphi \notin Z_\Gamma(h) = \langle h \rangle$, there exist $p, q \in F(h)$ and $n, m \in \mathbb{Z}$ such that $\varphi h^n p, \beta h^m q \in B_x(\rho) \cap B_o(c_0)$. Furthermore $g := \varphi h^n \ne \beta h^m =: f$ implies $d(p, q) \ge 2\rho$ since

$$\begin{aligned} 2\rho &\ge d(gp, fq) \ge d(gp, fp) - d(fp, fq)\\ &= d(gf^{-1}gp, gp) - d(p, q) \overset{(8)}{\ge} 4\rho - d(p, q). \end{aligned}$$

So if we assign to each element $\varphi \in \Gamma(h)/\langle h \rangle$ a unique point $p = p(\varphi) \in F(h)$ as above, then the balls $B_{p(\varphi)}(\rho)$, $\varphi \in \Gamma(h)/\langle h\rangle$, are pairwise disjoint.

For $R > 0$ as before Lemma 5.3 we are going to bound the cardinality of

$$Y_R(h) := \{p(\varphi) \mid \varphi \in \Gamma(h)/\langle h \rangle,\ d(o, p(\varphi)) > R\} \subset F(h).$$

If $p \in Y_R(h)$, then $p \in F(h)$ and we may write $p = g_h e^H o$ with $H \in \mathfrak{a}$, $\langle H, L(h)\rangle \in [0, l(h)^2]$. Furthermore, $H = \mathrm{Ad}(w) H(x_h, p)$ for some $w \in W$, and we have $w \ne w_*$ by the fact that $\langle -H', L(h)\rangle < 0$ for all $H' \in \mathfrak{a}^+ \setminus \{0\}$. Hence the following two cases may occur:

1. *Case*: $H = H(x_h, p) \in \mathfrak{a}^+$ or, equivalently, $w = \mathrm{id}$:
   By Lemma 5.3 we have $\angle(H, P_\Gamma) < \varepsilon/4 < \delta$, hence $p \in C_\Gamma^\delta(h)$. Since the balls $B_{p(\varphi)}(\rho)$, $\varphi \in \Gamma(h)$, are pairwise disjoint, and $\mathrm{vol}(B_p(\rho) \cap \mathrm{Ax}(h)) = \omega_r \cdot \rho^r$ for any $p \in \mathrm{Ax}(h)$, there are at most
   $$\frac{\mathrm{vol}\left(F(h) \cap C_\Gamma^\delta(h)\right)}{\omega_r \cdot \rho^r} \le \frac{a \cdot l(h)^r}{\omega_r \cdot \rho^r}$$
   different points $p(\varphi) \in Y_R(h)$ with $p(\varphi) = g_h e^{H(x_h, p(\varphi))} o$.
2. *Case*: $H = \mathrm{Ad}(w) H(x_h, p)$ for some $w \in W \setminus \{\mathrm{id}, w_*\}$:
   We have $p = g_h w e^{H(x_h, p)} o$ and put $\xi := \sigma_{x_h, p}(\infty) \in \partial X^{reg}$. Then $\pi^B(\xi) \in \mathrm{Fix}^B(h) \setminus \{\pi^B(h^+), \pi^B(h^-)\}$, hence $h \in \Gamma'$ implies $\pi^B(\xi) \in K'$. Let $\gamma \in \Gamma$ such that $\gamma x \in B_p(\rho)$. By condition (ii) above, $d(o, p) > R$ and $d(p, \gamma o) \le d(p, \gamma x) + d(\gamma x, \gamma o) < \rho + c_0$ imply $\angle_o(p, \gamma o) < \varepsilon/8$. In particular,
   $$\angle_o(\gamma o, K') \le \angle_o(\gamma o, \xi) \le \angle_o(\gamma o, p) + \angle_o(p, \xi) \overset{\text{(iii)}}{<} \frac{\varepsilon}{8} + \frac{\varepsilon}{8} = \frac{\varepsilon}{4}.$$

This shows that there are at most N_ε points $p(\varphi) \in Y_R(h)$ of the form $p(\varphi) = g_h w e^{H(x_h, p(\varphi))} o$ with $w \neq \mathrm{id}$.

We summarize $\#Y_R(h) \leq a \cdot l(h)^r/(\omega_r \cdot \rho^r) + N_\varepsilon$, hence

$$\#\,\Gamma(h)/\langle h\rangle \leq N_\Gamma(R+\rho) + \frac{a}{\omega_r \cdot \rho^r} \cdot l(h)^r + N_\varepsilon.$$

Since $N_\Gamma(R+\rho)$ and N_ε are finite, the assertion follows from the inequality

$$\#\{k \in \mathbb{Z} \,|\, |k| \leq t/l(h)\} \leq 2t/l(h)$$

□

Corollary 5.5. *There exist constants $b > 0$ and $t_0 > 0$ such that for any $h \in \Gamma'$ primitive and all $t > t_0$*

$$\#\{\gamma = \varphi h^k \varphi^{-1} \mid \varphi \in \Gamma,\ k \in \mathbb{Z},\ \pi^B(\varphi h^-) \in U_1,\ \pi^B(\varphi h^+) \in U_2,\\ l(\gamma) \leq t\} \leq b \cdot t^r.$$

Proof. We use the notation from the previous lemma and notice that by choice of $c_0 > 0$ the conditions $\pi^B(\varphi h^-) \in U_1$ and $\pi^B(\varphi h^+) \in U_2$ imply that $\varphi \mathrm{Ax}(h) \cap B_o(c_0) \neq \emptyset$.

Since $\overline{B_o(c_0)} \subset X$ is compact, there exist finitely many points $\{x_1, x_2, \ldots, x_m\} \subseteq \overline{B_o(c_0)}$ such that the balls $B_{x_i}(\rho)$, $1 \leq i \leq m$, cover $\overline{B_o(c_0)}$. Hence if $\pi^B(\varphi h^-) \in U_1$ and $\pi^B(\varphi h^+) \in U_2$, there exists $j \in \{1, 2, \ldots, m\}$ such that $\varphi \mathrm{Ax}(h) \cap \big(B_{x_j}(\rho) \cap B_o(c_0)\big) \neq \emptyset$. We conclude

$$\#\{\gamma = \varphi h^k \varphi^{-1} \mid \varphi \in \Gamma,\ k \in \mathbb{Z},\ \pi^B(\varphi h^-) \in U_1,\ \pi^B(\varphi h^+) \in U_2,\ l(\gamma) \leq t\}\\ \leq m \cdot a \cdot t^r.$$

□

6. The proof of the main theorem

We will now state two more lemmata in order to obtain the upper bound and to relate $P(t)$ to $N_\Gamma(R; A, B)$ for appropriate sets $A, B \subset K/M$. We use the notation from sections 4 and 5 and fix

$$\varepsilon < \min\{\ \delta,\ \varepsilon_\Gamma(\{\pi^B(h_1^-)\}, U_1),\ \varepsilon_\Gamma(\{\pi^B(h_1^+)\}, U_2)\ \}.$$

Lemma 6.1. *There exists $T > 0$ such that for all $\gamma \in \Gamma$ with*

$$d(o, \gamma o) \geq T,\ \angle_o(\gamma o, \pi^B(h_1^+)) < \varepsilon/6 \ \ and \ \ \angle_o(\gamma^{-1} o, \pi^B(h_1^-)) < \varepsilon/6$$

either $g := \gamma$ or $g := \gamma^{-1}$ satisfies $\pi^B(g^-) \in U_1$ and $\pi^B(g^+) \in U_2$.

Proof. If $\gamma \in \Gamma$ satisfies $\angle_o(\gamma o, \pi^B(h_1^+)) < \varepsilon/6$ and $\angle_o(\gamma^{-1} o, \pi^B(h_1^-)) < \varepsilon/6$, then there exist $\xi^-, \xi^+ \in \partial X^{reg}$ with $\pi^B(\xi^-) = \pi^B(h_1^-), \pi^B(\xi^+) = \pi^B(h_1^+)$, $\angle_o(\gamma^{-1} o, \xi^-) < \varepsilon/4$ and $\angle_o(\gamma o, \xi^+) < \varepsilon/4$. Moreover, there exists $R > 0$ with the property that for every such γ with $d(o, \gamma o) \geq R$, and for all $x \in \overline{B_o(c_0)}$ we have

$$\angle_x(\gamma o, \xi^+) < \varepsilon/3, \ \angle_o(\gamma^{-1} x, \xi^-) < \varepsilon/3, \ \angle_x(\gamma^{-1} o, \xi^-) < \varepsilon/3 \\ \text{and} \quad \angle_o(\gamma x, \xi^+) < \varepsilon/3.$$

Let $kM \in \overline{U_1} \subset U_1'$ arbitrary, and $\zeta \in \mathrm{Vis}^\infty(\xi^+)$ such that $\pi^B(\zeta) = kM$. Denote $x \in X$ the orthogonal projection of o to the unique flat containing both ζ and ξ^+ in its boundary. Then $d(o, x) < c_0$ by definition of c_0, and we have $\angle_x(\gamma o, \zeta) = \pi - \angle_x(\gamma o, \xi^+) > \pi - \varepsilon/3$. From $\angle_{\gamma o}(\zeta, x) + \angle_x(\gamma o, \zeta) \leq \pi$ we therefore obtain

$$\angle_o(\gamma^{-1}\zeta, \gamma^{-1} x) = \angle_{\gamma o}(\zeta, x) \leq \pi - \angle_x(\gamma o, \zeta) < \varepsilon/3,$$

hence $\angle_o(\gamma^{-1}\zeta, \xi^-) \leq \angle_o(\gamma^{-1}\zeta, \gamma^{-1} x) + \angle_o(\gamma^{-1} x, \xi^-) < 2\varepsilon/3$.

We now let $T \geq R$ such that every $\gamma \in \Gamma$ with $d(o, \gamma o) \geq T$ satisfies $\angle(H(o, \gamma o), P_\Gamma) < \varepsilon/12$. We write $\xi^- = (k^-, H)$, where $k^- M = \pi^B(h_1^-)$ and $H \in \mathfrak{a}_1^+$ is the direction of ξ^-. If $H \in P_\Gamma$, then $\gamma^{-1}\zeta \in H_\Gamma^\varepsilon(\{\pi^B(h_1^-)\})$, hence by choice of ε we have $\pi^B(\gamma^{-1}\zeta) = \gamma^{-1} kM \in U_1$.

If $H \notin P_\Gamma$ we choose $H' \in P_\Gamma$ such that $\angle(H(o, \gamma^{-1} o), P_\Gamma) = \angle(H(o, \gamma^{-1} o), H')$. Then for $\eta^- := (k^-, H')$ and $\gamma \in \Gamma$ with $d(o, \gamma o) \geq T$ we estimate

$$\begin{aligned} \angle_o(\xi^-, \eta^-) &= \angle(H, H') \leq \angle(H, H(o, \gamma^{-1} o)) + \angle(H(o, \gamma^{-1} o), H') \\ &\leq \angle_o(\xi^-, \gamma^{-1} o) + \angle(H(o, \gamma^{-1} o), P_\Gamma) < \frac{\varepsilon}{4} + \frac{\varepsilon}{12} = \frac{\varepsilon}{3}. \end{aligned}$$

We conclude that $\angle_o(\gamma^{-1}\zeta, \eta^-) \leq \angle_o(\gamma^{-1}\zeta, \xi^-) + \angle_o(\xi^-, \eta^-) < 2\varepsilon/3 + \varepsilon/3 = \varepsilon$, hence also $\pi^B(\gamma^{-1}\zeta) = \gamma^{-1} kM \in U_1$. This proves $\gamma^{-1}\overline{U_1} \subseteq U_1$.

Analogously we can show that $\gamma \overline{U_2} \subseteq U_2$. Since both U_1 and U_2 are contractible, Brouwer's fixed point theorem implies that γ possesses fixed points in U_1 and U_2. Since every element in Γ is regular axial by Proposition 4.1, the attractive and repulsive fixed point γ^+, γ^- are contained in the regular boundary. The assertion now follows from the fact that

$$\mathrm{Fix}^B(\gamma) \cap \bigcup_{i=1}^{2l} U_i = \{\pi^B(\gamma^+), \pi^B(\gamma^-)\}.$$

□

The following lemma will be crucial for the upper bound in our main theorem. It holds for any finitely generated free and discrete isometry group of a Hadamard manifold, hence in particular for our Schottky groups acting on a symmetric space of arbitrary rank. Notice that if $\Gamma = \langle h_1, h_2, \ldots, h_l \rangle$ is a free group, then every element $\gamma \in \Gamma$ can be written uniquely as $\gamma = s_1 s_2 \ldots s_n$ with letters $s_i \in S := \{h_1, h_1^{-1}, \ldots, h_l, h_l^{-1}\}$ such that $s_{i+1} \neq s_i^{-1}$ for $1 \leq i \leq n-1$. We say that γ is cyclically reduced if $s_n \neq s_1^{-1}$.

Lemma 6.2. *Put* $d := \max\{d(o, h_i o) \mid 1 \le i \le l\}$. *Then for any* $\gamma \in \Gamma$ *there exist at least* $[l(\gamma)/d - 2]$ *different cyclically reduced conjugates of* γ.

Proof. We write $\gamma \in \Gamma$ as above, i.e. $\gamma = s_1' s_2' \cdots s_m'$ with $s_i' \in S$ and $s_{i+1}' \neq (s_i')^{-1}$ for $1 \le i \le m-1$. If $s_1' = (s_m')^{-1}$, we conjugate γ by $(s_1')^{-1}$ and get $\gamma' = s_2' s_3' \cdots s_{m-1}'$. Repeating this procedure as long as possible, we obtain a conjugate $\gamma(1) = s_1 s_2 \cdots s_n$ of γ such that $s_1 \neq s_n^{-1}$, $n \le m$.

For $2 \le j \le n$ we put $\gamma(j) := s_j s_{j+1} \cdots s_n s_1 \cdots s_{j-1}$. Then $\gamma(j)$ is conjugate to $\gamma(1)$ by $u(j) := (s_1 s_2 \cdots s_{j-1})^{-1}$, and $\gamma(j)$ is different from $\gamma(i)$ for $i \neq j$ because there are no relations in Γ. Since all elements $\gamma(j)$, $1 \le j \le n$, are cyclically reduced by construction, there exist at least $n-1$ different cyclically reduced conjugates of γ.

In order to relate n to $l(\gamma)$, we remark that

$$\begin{aligned} l(\gamma) = l(\gamma(1)) &\le d(o, \gamma(1)o) \\ &\le d(o, s_1 o) + d(s_1 o, s_1 s_2 o) + \cdots + d(s_1 s_2 \cdots s_{n-1} o, \gamma(1) o) \\ &= d(o, s_1 o) + d(o, s_2 o) + \cdots + d(o, s_n o) \le n \cdot \max_{s \in S} d(o, so) = n \cdot d, \end{aligned}$$

hence $n \ge l(\gamma)/d$. □

The following statement due to Y. Benoist will finally give the upper bound. Recall Definition 2.1 and the definition of the translation vector of an axial isometry from the beginning of Sect. 3.

Proposition 6.3 ([2, Sect. 4.1]). *Let* $\Gamma < Isom^o(X)$ *be a Schottky group of a globally symmetric space* X *as described in Sect. 4. Then there exists a constant* $M \ge 0$ *such that every cyclically reduced element* $\gamma \in \Gamma$ *satisfies* $\|L(\gamma) - H(o, \gamma o)\| \le M$.

Since $\|H(o, \gamma o)\| = d(o, \gamma o)$ and $\|L(\gamma)\| = l(\gamma)$, this proposition implies in particular that every cyclically reduced element $\gamma \in \Gamma$ satisfies

$$l(\gamma) \le d(o, \gamma o) \le \|H(o, \gamma o) - L(\gamma) + L(\gamma)\| \le M + l(\gamma). \tag{9}$$

Theorem 6.4. *Let* M *be a locally symmetric space of noncompact type and rank* $r \ge 1$ *with universal Riemannian covering manifold* X *and fundamental group isomorphic to a Schottky group* $\Gamma \subset Isom(X)$ *as in Sect. 4. Then there exist constants* $a > 1$ *and* $t_0 > 0$ *such that for all* $t > t_0$

$$\frac{1}{a \cdot t^r} \cdot e^{\delta(\Gamma) t} \le P(t) \le \frac{a}{t} \cdot e^{\delta(\Gamma) t}.$$

Proof. For the proof of the upper bound, we let $t \ge 2 + 4d$ with $d > 0$ as in Lemma 6.2 and consider a conjugacy class $[\gamma]$ with $t - 1 \le l([\gamma]) < t$. By Lemma 6.2 there exist at least $\frac{t}{2d}$ different cyclically reduced elements representing $[\gamma]$ with translation length equal to $l([\gamma])$, hence

$$\begin{aligned} \Delta P(t) := P(t) - P(t-1) \le \frac{2d}{t} \cdot \#\{\gamma \in \Gamma | \gamma \text{ cyclically reduced}, \\ l(\gamma) \in [t-1, t)\}. \end{aligned}$$

Moreover, the estimate (9) implies

$$\#\{\gamma \in \Gamma | \gamma \text{ cyclically reduced, } l(\gamma) \in [t-1, t)\} \le N_\Gamma(t+M) \le b e^{\delta(\Gamma)(t+M)}$$

by (2). We summarize

$$\Delta P(t) \le \frac{b'}{t} e^{\delta(\Gamma)t} \qquad \text{for some constant } b' > 0.$$

Now Lemma 3.2 in [4] states that for all $n \in \mathbb{N}$ and $a > 0$

$$\sum_{k=1}^{n} \frac{e^{ak}}{k} \le \text{const} \cdot \frac{e^{an}}{n}.$$

So for $n \in \mathbb{N}$ with $n \ge 2 + 4d$ we have

$$P(n) = \sum_{k=1}^{n} \Delta P(k) \le b' \sum_{k=1}^{n} \frac{e^{\delta(\Gamma)k}}{k} \le \text{const} \cdot \frac{e^{\delta(\Gamma)n}}{n},$$

which implies the upper bound.

To obtain the lower bound, we let $U_1, U_2 \subset K/M$ be the neighborhoods of $\pi^B(h_1^-), \pi^B(h_1^+)$ as in Sect. 4. Then by Lemma 6.1, there exist $\varepsilon > 0$ and $T > 0$ such that for any $t > T$ we have

$$\begin{aligned}&\#\{\gamma \in \Gamma | T \le d(o, \gamma o) < t,\ \angle_o(\gamma o, \pi^B(h_1^+)) < \varepsilon,\ \angle_o(\gamma^{-1} o, \pi^B(h_1^-)) < \varepsilon\}\\ &\ \le \#\{\gamma \in \Gamma | d(o, \gamma o) < t,\ \pi^B(\gamma^-) \in U_1,\ \pi^B(\gamma^+) \in U_2\} \le \#\{\gamma \in \Gamma' | l(\gamma) \le t\},\end{aligned}$$

where we applied the inequality $l(\gamma) \le d(o, \gamma o)$, $\gamma \in \Gamma$, in the last step. Hence using the notation of (4) and putting

$$A := \pi^B(H_\Gamma^\varepsilon(\{\pi^B(h_1^+)\})) \subset U_2, \quad B := \pi^B(H_\Gamma^\varepsilon(\{\pi^B(h_1^-)\})) \subset U_1,$$

we obtain

$$\#\{\gamma \in \Gamma' | l(\gamma) \le t\} \ge N_\Gamma(t; A, B) - N_\Gamma(T; A, B).$$

From Corollary 5.5 and Proposition 4.3 we conclude that for $t > T_0 := \max\{T, R_0\}$

$$\begin{aligned}P(t) &\ge \#\{[\gamma] | \gamma \in \Gamma' \text{ with } l([\gamma]) \le t\} \ge \frac{1}{b \cdot t^r} \cdot \#\{\gamma \in \Gamma' | l(\gamma) \le t\}\\ &\ge \frac{1}{b \cdot t^r} \left(\frac{1}{a} \cdot e^{\delta(\Gamma)t} - a \cdot e^{\delta(\Gamma)T_0} \right),\end{aligned}$$

which proves that the lower bound holds for t sufficiently large. □

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