



Journal Article

Monadic second order definable relations on the binary tree

Author(s):

Läuchli, Hans; Savioz, Christian

Publication Date:

1987

Permanent Link:

<https://doi.org/10.3929/ethz-b-000423098> →

Originally published in:

The Journal of Symbolic Logic 52(1), <http://doi.org/10.2307/2273878> →

Rights / License:

[In Copyright - Non-Commercial Use Permitted](#) →

This page was generated automatically upon download from the [ETH Zurich Research Collection](#). For more information please consult the [Terms of use](#).

MONADIC SECOND ORDER DEFINABLE RELATIONS ON THE BINARY TREE

HANS LÄUCHLI AND CHRISTIAN SAVIOZ

Abstract. Let S2S [WS2S] respectively be the strong [weak] monadic second order theory of the binary tree T in the language of two successor functions. An S2S-formula whose free variables are just individual variables defines a relation on T (rather than on the power set of T). We show that S2S and WS2S define the same relations on T , and we give a simple characterization of these relations.

§1. The infinite binary tree T is given by the set $\{0, 1\}^*$ of all finite $(0, 1)$ -words, called the nodes of the tree. Every node x has two successor nodes, $s_0(x) := x0$ and $s_1(x) := x1$.

S2S is the monadic second order theory of (T, s_0, s_1) in the language of two successor functions: In addition to the first order theory there are set variables ranging over subsets of T , existential and universal quantifier over set variables and the membership relation. WS2S is the corresponding monadic *weak* second order theory: Set variables range only over *finite* subsets of T .

An S2S-formula with free set variables defines a relation on $P(T)$, the power set of T , while an S2S-formula with just free individual variables defines a relation on T . The following results are due to M. O. Rabin (see [7] and [8]):

(I) *There are S2S-definable even one-place relations on $P(T)$ which are not WS2S-definable.*

(II) *A subset of T is S2S-definable iff it is regular; in particular, S2S and WS2S define the same one-place relations on T .*

A slightly simpler proof of (II), based on [4], is given in [10].

In this paper we give a simple characterization of the S2S-definable relations on T . In particular, we prove

THEOREM 1. *For $n \in \omega$ and $R \subset T^n$, R is S2S-definable iff it is WS2S-definable.*

The corresponding result for S1S, the monadic second order theory of the natural numbers with successor function, is due to J. R. Büchi [2] and answers a question raised by R. M. Robinson in [9]. Monadic second-order definability and weak monadic second-order definability are known to be equivalent even for relations on $P(\omega)$ (see W. Thomas [11]).

Our proof is based on Rabin's characterization of S2S-definability in terms of finite tree automata.

Received October 17, 1985; revised May 9, 1986.

©1987, Association for Symbolic Logic
0022-4812/87/5201-0021/\$01.80

A natural question (raised by Rabin upon communication of our result) is this: Does Theorem 1 generalize to the case where free variables are allowed to range over paths in T ? We do not know the answer.

§2. We start by giving a characterization of the binary S2S-definable relations on T and some examples.

We use small letters a, b, \dots, x, y, \dots for elements of T and capital letters A, B, \dots, X, Y, \dots for subsets of T . Concatenation of words a, b is written ab . λ is the empty word, Λ the empty set. $AB := \{ab \mid a \in A \text{ and } b \in B\}$, $aB := \{a\}B = \{ab \mid b \in B\}$. Thus, $\lambda B = B$ and $\Lambda B = \Lambda$. $A^0 := \{\lambda\}$, $A^{n+1} := A^nA$, and $A^* := \bigcup_{n \in \omega} A^n$.

The regular subsets of T (in the sense of Kleene [5]) are given by the following formation rules:

- a) Every finite subset of T is regular.
- b) If A and B are regular subsets of T , then so are $A \cup B$, AB , and A^* .

For later use we state the following well-known fact.

PROPOSITION 1. *The class of regular sets is closed under Boolean operations; if aB is regular, then so is B .*

A relation $R \subset T^2$ is said to be special if $R = \{(ab, ac) \mid a \in A, b \in B, c \in C\}$ for some regular subsets A, B, C of T .

THEOREM 2. *For $R \subset T^2$, R is S2S-definable iff it is a finite union of special relations.*

EXAMPLES. Let us use the abbreviation $[A, B, C]$ for $\{(ab, ac) \mid a \in A, b \in B, c \in C\}$.

1. $[T, \{\lambda\}, T]$ is the partial order \leq by initial segments, $[T, \{\lambda\}, T] \cup [T, 0T, 1T]$ is the lexicographical ordering, and $[T, \{\lambda\}, T \setminus \{\lambda\}] \cup [T, T \setminus \{\lambda\}, \{\lambda\}] \cup [T, 0T, 1T] \cup [T, 1T, 0T]$ is inequality.

2. The relation “ $xy = z$ ” is not S2S-definable, not even the relation “ $x = 0y$ ”. Otherwise, the relation $x = 0y \wedge 1 \leq y$ could be represented as $\bigcup_k [A_k, B_k, C_k]$. This implies $A_k = \{\lambda\}$ and, since the relation is one-to-one, the B_k 's and C_k 's are singletons, which makes the relation finite, a contradiction.

3. “ x and y are of the same length” is not S2S-definable.

It is even known that the theory WS2S (T, s_0, s_1, P) is undecidable if P is one of the predicates “ $x = 0y$ ” or “ x and y are of the same length”. (See Savioz [10] and Buszkowski [3]. For the strong second order case, the following simple undecidability proof was pointed out to us by the referee: The domino problem on a quadrant of the plane can be formulated using “ $x = 0y$ ”, together with “ $x = y1$ ”, as grid successor functions on 0^*1^* .)

Incidentally, the reader who is familiar with the terminology of [1] will observe that the class of S2S-definable relations $R \subset T^2$ is properly included in the class of “rational” relations and properly contains the class of “recognizable” relations. (The relation “ $x = 0y$ ” is rational but not S2S-definable, while the relation “ $x \leq y$ ” is S2S-definable but not recognizable.)

§3. In order to state our result in more generality we need some additional notation and terminology.

If U is a word in $\{1, 2, \dots, m\}^*$ and $a_k, k = 1, 2, \dots, m$, are words in $T = \{0, 1\}^*$, then $U[\vec{a}]$ denotes the word in T obtained from U by substitution $k \rightarrow a_k$.

A finite sequence of words in $\{1, 2, \dots, m\}^*$ is said to be *admissible*, if it can be obtained according to the following rules:

- a) The one-term sequence (k) whose entry is the one-letter word k is admissible.
- b) If (\tilde{U}, V) is admissible (\tilde{U} possibly empty) and h, k do not occur in any word of this sequence and $h \neq k$, then (\tilde{U}, Vh, Vk) is admissible.
- c) Any permutation of an admissible sequence is admissible.

EXAMPLE. $(371, 32, 374)$ is admissible; $(13, 23)$ is not.

If (U_1, U_2, \dots, U_n) is an admissible sequence of words in $\{1, 2, \dots, m\}^*$ and A_1, A_2, \dots, A_m are regular subsets of T , then

$$R = \{(U_1[\tilde{a}], U_2[\tilde{a}], \dots, U_n[\tilde{a}]) \mid a_i \in A_i, i = 1, \dots, m\}$$

is a *special* (n -ary) relation on T .

Let Th be the *first-order* theory of T in the following language and interpretation. Language: A constant λ , a binary function symbol \wedge and, for each regular subset $A \subset T$, a binary predicate P_A . Interpretation: λ is the empty word, $x \wedge y$ is the maximal common initial segment of the words x and y , and $P_A(x, y)$ holds iff $x \in yA$ (that is, $x = ya$ for some $a \in A$). Thus the atomic formulas of Th are $P_A(t, s)$, where t, s are \wedge -terms built from individual variables and λ . $P_A(x, \lambda)$ means $x \in A$.

THEOREM. Let $n \geq 1$. For relations $R \subset T^n$, the following are equivalent:

- (i) R is S2S-definable.
- (ii) R is WS2S-definable.
- (iii) R is Th -definable by a finite disjunction of finite conjunctions of atomic formulas of Th .
- (iv) R is a finite union of special relations.

We note as a corollary:

COROLLARY. Th admits quantifier elimination.

We first prove the easy implications (ii) \rightarrow (i), (iii) \rightarrow (ii) and (iv) \rightarrow (ii).

(ii) \rightarrow (i). It is well known that the notion “ X is finite” is S2S-definable.

(iii) \rightarrow (ii). It is well known that λ and $x \wedge y$ are WS2S-definable. As to P_A : If A is finite, then $x \in yA$ if $\bigvee_{a \in A} (x = ya)$; for fixed a , ya is given by an (s_0, s_1) -term (the reader is reminded that s_0 and s_1 are the successor functions on T). Furthermore,

$$\begin{aligned} x \in y(A \cup B) & \text{ iff } (x \in yA \vee x \in yB), \\ x \in y(AB) & \text{ iff } \exists z(z \in yA \wedge x \in zB), \\ x \in yA^* & \text{ iff } \forall X^{\text{finite}} [(x \in X \wedge \forall u \forall v (u \in X \wedge u \vee vA \rightarrow v \in X)) \rightarrow y \in X]. \end{aligned}$$

(iv) \rightarrow (ii). By way of example, if

$$R = \{(ab, acd, ace) \mid a \in A, b \in B, c \in C, d \in D, e \in E\},$$

then $(x, y, z) \in R$ iff

$$\exists u [(x, u) \in \{(ab, ac) \mid a \in A, b \in B, c \in C\} \wedge y \in uD \wedge z \in uE]$$

iff

$$\exists u [\exists v (v \in A \wedge x \in vB \wedge u \in vC) \wedge y \in uD \wedge z \in uE],$$

which is WS2S-definable according to the last paragraph.

Next, we prove a simple proposition and state the main lemma, which settles the remaining implications (i) → (iii) and (i) → (iv).

For $n \geq 2$, $i, j \leq n$, $i \neq j$, let $\delta_{i,j}^n(x_1, \dots, x_n)$ and $\varepsilon_{i,j}^n(x_1, \dots, x_n)$ be S2S-formulas expressing the following:

$$\begin{aligned} \delta_{i,j}^n(\vec{x}): x_i \leq x_j \wedge \bigwedge_{k \neq i,j} \neg(x_i \leq x_k), \\ \varepsilon_{i,j}^n(\vec{x}): (x_i \wedge x_j)0 \leq x_i \wedge (x_i \wedge x_j)1 \leq x_j \\ \wedge \bigwedge_{k \neq i,j} \neg[(x_i \wedge x_j)0 \leq x_k \vee (x_i \wedge x_j)1 \leq x_k]. \end{aligned}$$

We write $T \models \varphi(\vec{x})$ if φ is identically true in the structure (T, s_0, s_1) .

PROPOSITION 2. For $n \geq 2$,

$$T \models \bigvee_{i \neq j} [x_i = x_j \vee \delta_{i,j}^n(\vec{x}) \vee \varepsilon_{i,j}^n(\vec{x})].$$

PROOF (by induction). The assertion holds for $n = 2$. Given (\vec{x}, x_{n+1}) , $n \geq 2$, assume that the x_i 's are pairwise distinct, and, for instance, assume $\delta_{1,2}^n(\vec{x})$. If $\neg(x_1 \leq x_{n+1})$, then $\delta_{1,2}^{n+1}(\vec{x}, x_{n+1})$. If $x_1 < x_{n+1}$, then one of $\delta_{2,n+1}^{n+1}(\vec{x}, x_{n+1})$, $\delta_{n+1,2}^{n+1}(\vec{x}, x_{n+1})$, $\varepsilon_{2,n+1}^{n+1}(\vec{x}, x_{n+1})$ or $\varepsilon_{n+1,2}^{n+1}(\vec{x}, x_{n+1})$ holds. The case $\varepsilon_{1,2}^n(\vec{x})$ is analogous.

To avoid subscripts we consider $(n + 2)$ -tuples (x, y, \vec{z}) and write $\delta(x, y, \vec{z})$ for $\delta_{1,2}^{n+2}(x, y, \vec{z})$, where x is x_1 and y is x_2 .

MAIN LEMMA. Let $\varphi(x, y, \vec{z})$ be an S2S-formula with $n + 2$ free individual variables.

a) If $T \models \varphi(x, y, \vec{z}) \rightarrow \delta(x, y, \vec{z})$, then there are regular sets B_k and S2S-formulas $\varphi_k(x, \vec{z})$ with $n + 1$ free individual variables, $k = 1, \dots, m$, such that

$$T \models \varphi(x, y, \vec{z}) \leftrightarrow \bigvee_k [\varphi_k(x, \vec{z}) \wedge y \in xB_k].$$

b) If $T \models \varphi(x, y, \vec{z}) \rightarrow \varepsilon(x, y, \vec{z})$, then there are regular sets A_k, B_k and formulas $\varphi_k(u, \vec{z})$ with $n + 1$ variables such that

$$T \models \varphi(x, y, \vec{z}) \leftrightarrow \bigvee_k [\varphi_k(x \wedge y, \vec{z}) \wedge x \in (x \wedge y)0A_k \wedge y \in (x \wedge y)1B_k].$$

For the following, call a formula $\varphi(\vec{x})$ nice if for some $i, j \leq n$, $i \neq j$, either $T \models \varphi(\vec{x}) \rightarrow x_i = x_j$ or $T \models \varphi(\vec{x}) \rightarrow \delta_{i,j}^n(\vec{x})$ or $T \models \varphi(\vec{x}) \rightarrow \varepsilon_{i,j}^n(\vec{x})$ holds.

Proof of (i) → (iii). We have to show that every S2S-formula $\varphi(x_1, \dots, x_n)$ is equivalent to a disjunction of conjunctions of atomic formulas of Th. We do this by induction. For $n = 1$, $T \models \varphi(x) \leftrightarrow x \in A$ for some regular set A (Theorem (II)). “ $x \in \lambda A$ ” is an atomic formula of Th.

Induction step. By Proposition 2, every formula is equivalent to a disjunction of nice formulas. The induction step for nice formulas is accomplished by the main lemma and by the obvious reduction: If $T \models \varphi(x, y, \vec{z}) \rightarrow x = y$, then

$$T \models \varphi(x, y, \vec{z}) \leftrightarrow (\varphi(x, x, \vec{z}) \wedge y \in x\{\lambda\}).$$

Proof of (i) → (iv) (by induction). For $n = 1$, again by Theorem (II), $\varphi(x)$ iff $x \in A = \{(a) \mid a \in A\}$, a special relation (we just define one-tuples this way: $(a) = a$).

Induction step. It again suffices to consider nice formulas.

Case 1. $T \models \varphi(x, y, \bar{z}) \rightarrow x = y$. By the induction hypothesis, $\varphi(x, x, \bar{z})$ iff $(x, \bar{z}) \in \bigcup_k R_k$ with R_k special. Thus, $\varphi(x, y, \bar{z})$ iff $\bigvee [(x, \bar{z}) \in R_k \text{ and } y = x]$. But, by way of example,

$$(x, z) \in \{(ab, ac) \mid a \in A, b \in B, c \in C\} \quad \text{and} \quad y = x$$

iff

$$(x, y, z) \in \{(abd, abe, ac) \mid a \in A, b \in B, c \in C, d \in \{\lambda\}, e \in \{\lambda\}\}.$$

Case 2. $T \models \varphi(x, y, \bar{z}) \rightarrow \delta(x, y, \bar{z})$. By part a) of the main lemma and the induction hypothesis (and distributivity),

$$\varphi(x, y, \bar{z}) \quad \text{iff} \quad \bigvee_{h,k} [(x, \bar{z}) \in R_{hk} \wedge y \in xB_k].$$

By way of example,

$$(x, z) \in \{(uv, uw) \mid u \in U, v \in V, w \in W\} \quad \text{and} \quad y \in xB$$

iff

$$(x, y, z) \in \{(uva, uwb, uw) \mid u \in U, v \in V, w \in W, a \in \{\lambda\}, b \in B\}.$$

Case 3. $T \models \varphi(x, y, \bar{z}) \rightarrow \varepsilon(x, y, \bar{z})$. Then, by part b) of the main lemma and the induction hypothesis,

$$\begin{aligned} \varphi(x, y, \bar{z}) \quad \text{iff} \quad \bigvee_{h,k} [(x \wedge y, \bar{z}) \in R_{h,k} \quad \text{and} \quad x \in (x \wedge y)0A_k \\ \text{and} \quad y \in (x \wedge y)1B_k]. \end{aligned}$$

But, $(x \wedge y, z) \in \{(uv, uw) \mid u \in U, v \in V, w \in W\}$ and $x \in (x \wedge y)0A$ and $y \in (x \wedge y)1B$ iff $(x, y, z) \in \{(uva, uwb, uw) \mid u \in U, v \in V, w \in W, a \in 0A, b \in 1B\}$.

§4. In this section we prove the main lemma.

DEFINITION. An *n-automaton* is a system $\mathfrak{A} = (S, M, S_0, F)$, where S is a finite set, the set of states, $S_0 \subset S$, the set of initial states, $F \subset P(S)$, the set of designated subsets of S , and $M \subset S \times \{0, 1\}^n \times S \times S$, the transition relation.

A *path* Π of T is a maximal (initial-segment-) totally ordered subset of T .

For a mapping $r: \Pi \rightarrow S$, define

$$\text{In}(r) := \{s \in S \mid r^{-1}(s) \text{ is infinite}\}.$$

For an n -tuple $\vec{A} = (A_1, \dots, A_n) \in P(T)^n$, define the *characteristic function* $\chi_{\vec{A}}: T \rightarrow \{0, 1\}^n$ by

$$\chi_{\vec{A}}(x)(i) = 1 \quad \text{iff} \quad x \in A_i.$$

DEFINITION. Given an n -automaton $\mathfrak{A} = (S, M, S_0, F)$, an n -tuple $\vec{A} \in P(T)^n$ and a mapping $r: T \rightarrow S$. The pair (\mathfrak{A}, r) *accepts* \vec{A} if 1) $r(\lambda) \in S_0$, 2) $\text{In}(r \upharpoonright \Pi) \in F$ for every path $\Pi \subset T$, and

3) $(r(x), \chi_{\vec{A}}(x), r(x_0), r(x_1)) \in M$ for all $x \in T$.

We say \mathfrak{A} *accepts* \vec{A} , if there is an r such that (\mathfrak{A}, r) accepts \vec{A} .

The following theorem is due to Rabin [6].

(III) *Given an S2S-formula $\varphi(X_1, \dots, X_n)$, there is an n -automaton \mathfrak{A} such that for all $\vec{A} \in P(T)^n$, \mathfrak{A} accepts \vec{A} iff $\varphi(\vec{A})$ holds.*

In this case, the automaton \mathfrak{A} is said to *represent* the formula φ . Individual variables are identified with singletons: \mathfrak{A} is said to represent $\varphi(x, \dots)$ if it represents the formula $\psi(X, \dots) := \exists x(X = \{x\} \wedge \varphi(x, \dots))$, and \mathfrak{A} accepts (a, \dots) if it accepts $(\{a\}, \dots)$.

The following lemma is a mild version of Rabin’s grafting technique (see [7] or [8]).

LEMMA 1. *Let \mathfrak{A} be an $(n + 2)$ -automaton accepting only tuples of the form (a, aB, \vec{C}) , where $C_i \cap aT = A$, $i = 1, \dots, n$. Suppose (\mathfrak{A}, r) accepts (a, aB, \vec{C}) , (\mathfrak{A}, r') accepts $(a', a'B', \vec{C}')$ and $r(a) = r'(a')$. Then \mathfrak{A} accepts (a, aB', \vec{C}) .*

PROOF. Define the run \bar{r} by $\bar{r}(x) := r(x)$ if $x \notin aT$ and $\bar{r}(ay) := r'(a'y)$. Then it is easy to see that (\mathfrak{A}, \bar{r}) accepts (a, aB', \vec{C}) .

LEMMA 2. *Let $\alpha(x, Y, \vec{Z}_1, \dots, \vec{Z}_n)$ be an S2S-formula such that*

$$(1) \quad T \models \alpha(x, Y, \vec{Z}) \rightarrow Y \subset xT \wedge \bigwedge (Z_i \cap xT = A)$$

and

$$(2) \quad T \models \alpha(x, Y, \vec{Z}) \wedge \alpha(x, Y', \vec{Z}) \rightarrow Y = Y'.$$

Then there are finitely many regular sets $B_k \subset T$, $k = 1, \dots, m$, such that

$$T \models \alpha(x, Y, \vec{Z}) \rightarrow \bigvee_k (Y = xB_k).$$

PROOF. α is represented by an $(n + 2)$ -automaton \mathfrak{A} satisfying the hypothesis of Lemma 1 (because of (1)). Let S be the set of states of \mathfrak{A} . Let $\phi(s, a, B, \vec{C})$ be the following statement: $s \in S$ and there is a run $r: T \rightarrow S$ such that $r(a) = s$ and (\mathfrak{A}, r) accepts (a, aB, \vec{C}) . In particular, $\phi(s, a, B, \vec{C})$ implies $\alpha(a, aB, \vec{C})$. If $\phi(s, a, B, \vec{C})$ and $\phi(s, a', B', \vec{C}')$, then, by Lemma 1, \mathfrak{A} accepts (a, aB', \vec{C}) , so $\alpha(a, aB', \vec{C})$ holds. By (2) we get $aB = aB'$; hence $B = B' =: B_s$. If $\alpha(a, D, \vec{C})$, then $D = aB$ and $\phi(s, a, B, \vec{C})$ for some s and B ; that is, $D = aB_s$ for some $s \in S$.

It remains to show that the B_s ’s are regular. Fix s, a , and \vec{C} such that $\alpha(a, aB_s, \vec{C})$ holds. The formula $\psi(Y) := \exists \vec{Z} \alpha(a, Y, \vec{Z})$ defines a finite relation on $P(T)$, and the set aB_s belongs to it.

Choose “discriminators” $d_i, e_j \in T$ such that

$$T \models \psi(Y) \wedge \bigwedge (d_i \in Y) \wedge \bigwedge \neg (e_j \in Y) \leftrightarrow Y = aB_s.$$

By (II), aB_s is regular; hence, by Proposition 1, B_s is regular.

LEMMA 3. *Let $\psi(u, Y_0, Y_1, \vec{z})$ be an S2S-formula such that*

$$(1') \quad T \models \psi(u, Y_0, Y_1, \vec{z}) \rightarrow Y_0 \subset u0T \wedge Y_1 \subset u1T \\ \wedge \bigwedge \neg (z_i \in (u0T \cup u1T))$$

and

$$(2') \quad T \models \psi(u, Y_0, Y_1, \vec{z}) \wedge \psi(u, Y'_0, Y'_1, \vec{z}) \rightarrow (Y_0 = Y'_0 \leftrightarrow Y_1 = Y'_1).$$

Then there are finitely many regular sets $A_k, B_k, k = 1, \dots, m$, such that

$$T \models \psi(u, Y_0, Y_1, \vec{z}) \rightarrow \bigvee_k (Y_0 = u0A_k \wedge Y_1 = u1B_k).$$

PROOF. Let $\alpha(x, Y_0, Y_1, \bar{z})$ be the formula

$$\exists u[x = u0 \wedge \psi(u, Y_0, Y_1, \bar{z})].$$

Then

$$(1) \quad T \models \alpha(x, Y_0, Y_1, \bar{z}) \rightarrow Y_0 \subset xT \wedge (Y_1 \cap xT = A) \\ \wedge \bigwedge (\{z_i\} \cap xT = A)$$

and

$$(2) \quad T \models \alpha(x, Y_0, Y_1, \bar{z}) \wedge \alpha(x, Y'_0, Y_1, \bar{z}) \rightarrow Y_0 = Y'_0.$$

By Lemma 2, there are regular sets C_r such that $\alpha(x, Y_0, Y_1, \bar{z})$ implies $\bigvee (Y_0 = xC_r)$. Since $\psi(u, Y_0, Y_1, \bar{z})$ implies $\alpha(u0, Y_0, Y_1, \bar{z})$, we have

$$T \models \psi(u, Y_0, Y_1, \bar{z}) \rightarrow \bigvee (Y_0 = u0C_r).$$

By symmetry, there are regular sets D_s such that

$$T \models \psi(u, Y_0, Y_1, \bar{z}) \rightarrow \bigvee (Y_1 = u1D_s).$$

This concludes the proof: Just let k run over the pairs (r, s) and let $A_{(r,s)} := C_r$ and $B_{(r,s)} := D_s$.

PROOF OF THE MAIN LEMMA. a) Assume $T \models \varphi(x, y, \bar{z}) \rightarrow \delta(x, y, \bar{z})$. Let

$$\alpha(x, Y, \bar{z}) := Y \neq A \wedge \forall v[v \in Y \leftrightarrow \varphi(x, v, \bar{z})].$$

Then α satisfies hypotheses (1) and (2) of Lemma 2 (since $\alpha(x, Y, \bar{z})$ implies $Y \neq A$, it implies $\neg(x \leq z_i)$, i.e. $\{z_i\} \cap xT = A$). Therefore, $T \models \alpha(x, Y, \bar{z}) \rightarrow \bigvee (Y = xB_k)$ for some regular B_k 's. We conclude that

$$T \models \varphi(x, y, \bar{z}) \leftrightarrow \exists Y[\alpha(x, Y, \bar{z}) \wedge y \in Y] \\ \leftrightarrow \bigvee_k [\alpha(x, xB_k, \bar{z}) \wedge y \in xB_k].$$

We are done with $\varphi_k(x, \bar{z}) := \alpha(x, xB_k, \bar{z})$.

b) Assume $T \models \varphi(x, y, \bar{z}) \rightarrow \varepsilon(x, y, \bar{z})$. Let $\psi(u, Y_0, Y_1, \bar{z})$ be the following formula:

$$Y_0 \neq A \wedge Y_1 \neq A \wedge Y_0 \subset u0T \wedge Y_1 \subset u1T \\ \wedge \forall x \in Y_0 \forall y \in Y_1 \varphi(x, y, \bar{z}) \\ \wedge \forall x \in (u0T \setminus Y_0) \exists y \in Y_1 \neg \varphi(x, y, \bar{z}) \\ \wedge \forall y \in (u1T \setminus Y_1) \exists x \in Y_0 \neg \varphi(x, y, \bar{z}).$$

Then ψ satisfies hypotheses (1') and (2') of Lemma 3.

(1'). Assume $\psi(u, Y_0, Y_1, \bar{z})$. Then, by definition, $Y_0 \subset u0T$ and $Y_1 \subset u1T$. Y_0 and Y_1 are nonempty. Let $x \in Y_0$ and $y \in Y_1$. Then $\varphi(x, y, \bar{z})$, and therefore $\varepsilon(x, y, \bar{z})$ holds, and $u = x \wedge y$. Thus

$$\bigwedge \neg(z_i \in (u0T \cup u1T)).$$

(2'). Assume $\psi(u, Y_0, Y_1, \bar{z})$, $\psi(u, Y'_0, Y'_1, \bar{z})$, $Y_0 = Y'_0$ and, for a contradiction, $y \in Y'_1 \setminus Y_1$. Then there is $x \in Y_0$ with $\neg \varphi(x, y, \bar{z})$, contradicting $\psi(u, Y'_0, Y'_1, \bar{z})$.

By Lemma 3, there are regular sets A_k and B_k such that

$$(*) \quad T \models \psi(u, Y_0, Y_1, \bar{z}) \rightarrow \bigvee_k (Y_0 = u0A_k \wedge Y_1 = u1B_k).$$

Next, we show

$$(**) \quad T \models \varphi(x, y, \bar{z}) \leftrightarrow \exists Y_0 \exists Y_1 [\psi(x \wedge y, Y_0, Y_1, \bar{z}) \wedge x \in Y_0 \wedge y \in Y_1].$$

Assume $\varphi(x, y, \bar{z})$. Let $Y_0 := \{v \in (x \wedge y)0T \mid \varphi(v, y, \bar{z})\}$ and $Y_1 := \{w \in (x \wedge y)1T \mid \varphi(v, w, \bar{z}) \text{ for all } v \in Y_0\}$. Then $\psi(x \wedge y, Y_0, Y_1, \bar{z})$ and $x \in Y_0$ and $y \in Y_1$. The converse implication is trivial.

By (*) and (**) we conclude that

$$T \models \varphi(x, y, \bar{z}) \leftrightarrow \bigvee [\psi(x \wedge y, (x \wedge y)0A_k, (x \wedge y)1B_k, \bar{z}) \\ \wedge x \in (x \wedge y)0A_k \wedge y \in (x \wedge y)1B_k].$$

Setting $\varphi_k(u, \bar{z}) := \psi(u, u0A_k, u1B_k, \bar{z})$, we are done.

REFERENCES

- [1] J. BERSTEL, *Transductions and context-free languages*, Teubner, Stuttgart, 1979.
- [2] J. R. BÜCHI, *On a decision method in restricted second order arithmetic*, *Proceedings of the international conference on logic, methodology and philosophy of science, 1960*, Stanford University Press, Stanford, California, 1962, pp. 1–11.
- [3] W. BUSZKOWSKI, *Logical complexity of some classes of tree languages generated by multiple-tree-automata*, *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, vol. 26 (1980), pp. 41–49.
- [4] Y. GUREVICH and L. HARRINGTON, *Trees, automata and games*, *Proceedings of the fourteenth ACM symposium on the theory of computing (San Francisco, 1982)*, ACM, New York, 1982, pp. 60–65.
- [5] S. C. KLEENE, *Representation of events in nerve nets and finite automata*, *Automata studies*, Annals of Mathematics Studies, no. 34, Princeton University Press, Princeton, New Jersey, 1956, pp. 3–41.
- [6] M. O. RABIN, *Decidability of second order theories and automata on infinite trees*, *Transactions of the American Mathematical Society*, vol. 141 (1969), pp. 1–35.
- [7] ———, *Weakly definable relations and special automata*, *Mathematical logic and foundations of set theory (proceedings of an international colloquium, Jerusalem, 1968)*, North-Holland, Amsterdam, 1970, pp. 1–23.
- [8] ———, *Automata on infinite objects and Church's problem*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, no. 13, American Mathematical Society, Providence, Rhode Island, 1972.
- [9] R. M. ROBINSON, *Restricted set-theoretical definitions in arithmetic*, *Proceedings of the American Mathematical Society*, vol. 9 (1958), pp. 238–242.
- [10] CH. SAVIOZ, *Extensions décidables et indécidables de la théorie monadique du deuxième ordre de deux fonctions de successeur*, Dissertation No. 7898, ETH, Zürich, 1985.
- [11] W. THOMAS, *On the bounded monadic theory of well-ordered structures*, this JOURNAL, vol. 45 (1980), pp. 334–338.

DEPARTMENT OF MATHEMATICS
ETH
CH-8092 ZÜRICH, SWITZERLAND