Diss. ETH No. 19608

Eulerian and Semi-Lagrangian Methods for Advection-Diffusion of Differential Forms

A dissertation submitted to
ETH Zürich

for the degree of
Doctor of Sciences

presented by

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2011

Dank

An dieser Stelle möchte ich mich bei alle denjenigen bedanken die direkt oder indirekt zu dieser Arbeit beigetragen haben.

Mein besonderer Dank gilt meinem Doktorvater Prof. Ralf Hiptmair, der mir bei Fragen stets zur Seite stand, aber auch grossen Freiraum liess für die Entwicklung eigener Ideen.

Weiterhin möchte ich mich bei den beiden Koreferenten Prof. Stefan Kurz und Prof. Siddartha Mishra bedanken. Die ausführlichen Diskussionen mit Stefan Kurz haben die Darstellung der vorliegenden Arbeit wesentlich beeinflusst. Also the many discussion with Siddartha Mishra on Finite Volume schemes and Magnetohydrodynamics were very helpfull and inspiring.

Herzlicher Dank geht auch an Lukas Wampfler, Christian Perret, Carlos Jerez Hankes und Patrick Meury. Patrick Meury war in den Anfängen meiner Doktorarbeit ein geduldiger Lehrer und Diskussionspartner. Die Waldläufe und Bergtouren mit Christian und Lukas, zwei weiteren Kollegen vom Seminar für Angewandte Mathematik, waren ein wichtiger Ausgleich während der Doktorarbeit. Ich danke den beiden für diese Zeit. A Carlos, mi compañero de trabajo con quien compartí el despacho durante las últimos meses de mi tesis doctoral, le agradezco los momentos de distracción y de descanso, los que fueron realmente divertidos.

Meinen Eltern und meinem Bruder Sven bin ich dankbar für die Unterstützung und den Rückhalt in all den Jahren.

Abstract

We consider generalized linear transient advection-diffusion problems for differential forms on bounded domains in $\mathbb{R}^n$. These involve Lie-derivatives with respect to a prescribed smooth vector field. We construct both, new Eulerian and Semi-Lagrangian approaches for the discretization of the Lie-derivative in the context of a Galerkin approximation based on discrete differential forms.

While the discretization of scalar advection-diffusion has attracted immense attention in numerical analysis, there has been little research on the non-scalar case, even though the non-scalar advection-diffusion problems are relevant for numerical modeling. The so-called magnetic advection-diffusion problem in quasistatic electromagnetism, the main motivation of this thesis, is an important example for such a non-scalar advection-diffusion problem.

It is the language of differential forms and in particular the notion of exterior derivatives and Lie derivatives that allows for a unified treatment of many different advection-diffusion problems, including the scalar case and the magnetic advection-diffusion problem. The calculus of differential forms reveals the intrinsic structure of such problems, that might be blurred by the "metric overhead" carried by vector calculus.

Our main interest will be robustness of the methods, that is sustained performance for very small and even vanishing diffusion. Thus, the core part of the thesis is devoted to convergence analysis and numerical studies of the Eulerian and semi-Lagrangian methods for the generalized advection problems. For fully discrete schemes and fixed polynomial degree of discrete forms we prove a priori error estimates in terms of mesh width h and timestep size τ. While for the Eulerian schemes the proofs of the estimates are adapted from the scalar case we present an entirely new approach for the analysis of fully discrete semi-Lagrangian methods. Thereby we can give convergence results that account for all discretization steps involved in the derivation of fully discrete semi-Lagrangian schemes. We even get convergence results for lowest order approximation spaces.

Zusammenfassung

Wir betrachten verallgemeinerte lineare zeitabhänginge Advektions-Diffusions-Probleme für Differentialformen auf beschränkten Gebieten im $\mathbb{R}^n$. Die Formulierung dieser Probleme basiert auf sogenannten Lie-Ableitungen zu einem vorgegebenen glatten Vektorfeld. Wir präsentieren neue Methoden zur Diskretisierung der Lie-Ableitung im Rahmen von Galerkin Approximationen mit diskreten Differentialformen. Diese beinhalten sowohl Eulersche als auch semi-Lagrangesche Ansätze.

Während die Diskretisierung des skalaren Advektions-Diffusions-Problems ein grosses und vielbeachtetes Forschungsgebiet in der Numerik ist und obwohl auch die nicht-skalaren Advektions-Diffusions-Probleme in der numerischen Modellierung wichtig sind, gibt es relativ wenig Forschung zu diesen nicht-skalaren Problemen. Das magnetische Advektions-Diffusions-Problem im quasistatischen Elektromagnetismus, die Hauptmotivation dieser Arbeit, ist ein wichtiges Beispiel für ein solches nicht-skalares Advektions-Diffusions-Problem.

Dank dem Formalismus der Differentialformen, und hier insbesondere dank den Begriffen der äusseren Ableitung und der Lie Ableitung, ist eine einheitliche Behandlung verschiedener Advektion-Diffusion-Probleme, einschließlich des skalaren Problems und des magnetischen Advektions-Diffusions-Problems, möglich. Das Kalkül der Differentialformen verdeutlicht die solchen Problemen gemeinsame innere Struktur, die durch den “metrische Overhead” von Vektorrechnung verwischt werden könnte.

Unser Hauptinteresse gilt der Robustheit der Methoden, d.h. gleichbleibend gute Ergebnisse und gute Effizienz bei Problemen mit sehr kleinem oder sogar verschwindendem Diffusionsterm. Das Kernstück der Arbeit sind daher Konvergenzaussagen und numerische Experimente zu den Eulerschen und semi-Lagrangeschen Methoden für verallgemeinerte Advektions-Probleme. Für komplett diskrete Methoden und Approximationsräume mit festem Polynomgrad beweisen wir a priori Fehlerabschätzungen in Abhängigkeit der Diskretisierungsparameter Gitterweite h und Zeitschrittweite τ. Während bei den Eulerschen-Methoden die Beweise der Abschätzungen eine Verallgemeinerung der Beweise für den skalaren Fall sind, stellen wir bei der Analyse von semi-Lagrangeschen Verfahren einen völlig neuen Ansatz dar. Damit sind wir zum Einen in der Lage sämtliche Diskretisierungsschritte bei semi-Lagrangeschen Methoden in der Konvergenzanalyse explizit zu berücksichtigen. Zum Anderen erhalten wir sogar für Approximationsräume mit niedrigstem Polynomgrad Konvergenzaussagen.

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1 Introduction

The topic of this thesis is the efficient and stable numerical solution of transient generalized advection-diffusion problems. The problem is studied mainly with the intention to solve the so-called magnetic advection-diffusion problem encountered in *magnetohydrodynamics*. Magnetohydrodynamics describes the motion of fluids in electromagnetic fields and has important applications in geophysics, astrophysics and engineering. Magnetohydrodynamic theories describe for example the Earth's magnetic field, astrophysical plasmas or high-voltage circuit-breakers.

In classical scalar advection-diffusion problems a weighted Laplace operator models the diffusion, while a transport operator, a first order differential operator parametrized by a given velocity field, models the advection part. When the scale of the diffusion operator is very small compared to the transport operator, standard numerical methods for elliptic and parabolic problems fail. The stable discretization of such singularly perturbed problems is very challenging and has attracted immense attention in numerical analysis.

We use the language of differential forms to generalize the scalar advection-diffusion problem. In these generalized advection-diffusion problems for differential formsm, the so-called Hodge-Laplacian models the diffusion and the Lie-derivative models the advection. This model not only includes the classical scalar problem but also the magnetic advection-diffusion problem. As in the classical case, the problem type of the generalized problems changes in the limit of vanishing diffusion and numerical algorithms specifically designed for the Hodge-Laplace operators fail.

The intention of placing magnetic convection-diffusion problems in the abstract framework of differential forms is twofold. First we can treat many different problems at once. Second, the formulation in terms of differential forms accentuates the common structure inherent in advection-diffusion models. We can take advantage of the numerous works on the scalar problem to find new methods. One example is the idea of upwinding, which forms the basis of most stable methods for the scalar problem. Due to the unifying language of differential forms we can introduce a natural notion of *upwind* discretization for generalized transport operators [33]. The discontinuous Galerkin method and the so-called semi-Lagrangian time-stepping schemes are other classical techniques that we could extend to the general case. While the formulation and the implementation of such algorithms is straightforward to deduce from the scalar ones, the numerical analysis required the development of novel techniques to prove for example a priori error estimates.

The outline of this thesis is the following: In Chapter 2 we introduce first the basic concepts related to differential forms. The presentation follows basically the presentation in [3] supplemented by detailed explanations on the Lie derivatives and the contraction operators that are related to advection models. Afterwards, we summarize the known results on finite element discrete differential forms spaces, that will later be the major approximation spaces for solving advection-diffusion problems. While the elements of these spaces inherit certain global continuity properties, there is a second important class of approximation spaces, introduced in Section 2.2.2, with elements with no global continuity. We close this chapter with a discussion on so-called constraint preserving finite volume schemes and accentuate the advantages of discrete differential forms spaces for designing such schemes.

The next chapter, Chapter 3, is devoted to the formulation of advection-diffusion of differential forms. We also show that the magnetic advection-diffusion problem, more precisely *the magnetoquasistatic equations in moving conductors*, is such an advection-diffusion problem for differential forms. Then in Section 3.4, we elaborate well-posedness of advection-diffusion problems for various parameter regimes. The chapter concludes with a short review of numerical methods for the scalar problem.

Since the approximation of diffusion of differential forms is standard, in the subsequent two Chapters 4 and 5, the main chapters of this thesis, we focus on the limiting problem, i.e. the advection problem for differential forms.

Chapter 4 deals with the stationary case. We present stabilized Galerkin methods that can be considered as a generalization of standard stabilized discontinuous Galerkin methods for classical scalar advection problems. While the convergence analysis for approximation spaces with no global continuity (Section 4.1.2) is fairly standard, the convergence analysis for the finite element differential forms spaces in Section 4.1.3 requires subtle approximation results. Another interesting method, at least from a theoretical point of view, are the characteristic methods presented thereafter in Section 4.2. For these we can also give a rigorous convergence results for general advection of differential forms.

Finally, Chapter 5 presents Eulerian and semi-Lagrangian methods for the non-stationary advection problem. We prove conditional stability of an explicit and unconditional stability of an implicit timestepping scheme. The fully discrete semi-Lagrangian schemes, defined in Section 5.2.1 for the advection of differential forms, allow for convergence results that reflect explicitly the various approximation parameters. Such results are rare even for the classical scalar problem. Moreover, we present in Theorem 5.2.6 a convergence result for lowest order approximation spaces and for timesteps in the order of the meshsize. For the scalar case, these assertions have been proved only for very special problems, e.g. for constant velocity.

A short discussion on the implication of the obtained results for the magnetic advection-diffusion problem is given in the last Chapter 6.

2 Preliminaries

Large parts of this thesis take advantage of the language of differential forms to present the different methods for solving advection problems. The notion of differential forms enables us to accentuate the main ideas of discretization methods [35, p. 266] in hiding technical details such as partial integration in a unifying notation. By now it is widely appreciated that thinking in terms of co-ordinate free differential forms offers considerable benefits as regards the construction of structure preserving spatial discretizations, *cf.* [64, Sect. 1.2]. The so-called discrete exterior calculus (DEC) [3, 24, 37], or, equivalently, the mimetic finite difference approach [14, 44–46], or discrete Hodge-operators [11, 35] have shed new light on existing discretizations and paved the way for new numerical methods. Therefore, we first give in section 2.1 a short introduction to the concept of differential forms and establish the notation. Table 2.1, 2.2 and 2.3 summarize correspondences of operations on differential forms and operations on scalar or vectorial functions. Afterwards a summary of candidate finite element approximation spaces for differential forms is given in section 2.2. These are piecewise polynomial spaces with different global continuity properties.

2.1 Differential Forms

We refer to the books [17] and [48] for an comprehensive introduction to differential forms. Here, the presentation and the notation is adapted from [3] and [4].

For a vector space V, $\dim V = n$, and a non-negative integer k, $\mathrm{Alt}^k V$ denotes the $\binom{n}{k}$-dimensional set of alternating real-valued k-linear forms on V. For $\omega \in \mathrm{Alt}^j V$ and $\eta \in \mathrm{Alt}^k V$ the wedge product $\omega \wedge \eta \in \mathrm{Alt}^{j+k} V$ is given as:

$$(\omega \wedge \eta)(\mathbf{v}_1, \dots \mathbf{v}_{j+k}) = \sum_\sigma \mathrm{sign}(\sigma)\omega(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(j)})\eta(\mathbf{v}_{\sigma(j+1)} \dots, \mathbf{v}_{\sigma(j+k)}),$$

where the sum runs over all permutations σ of $\{1, \dots j+k\}$, for which $\sigma(1) < \sigma(2) < \dots \sigma(j)$ and $\sigma(j+1) < \sigma(j+2) \dots \sigma(j+k)$. $\mathrm{sign}(\sigma)$ is the sign of permutation σ. The wedge product is anti-commutative in the following sense:

$$\omega \wedge \eta = (-1)^{jk} \eta \wedge \omega, \quad \omega \in \mathrm{Alt}^j V, \eta \in \mathrm{Alt}^k V. \tag{2.1}$$

If V is an oriented vector space with inner product, there exists a unique alternating n-linear form $\mathrm{vol} \in \mathrm{Alt}^n V$, called *volume form*, such that $\mathrm{vol}(\mathbf{e}_1, \dots, \mathbf{e}_n) = 1$ for all orthonormal, positively oriented bases $\{\mathbf{e}_i\}_{i=1}^n$ of V. Further, the inner product on V gives rise to an inner product on $\mathrm{Alt}^k V$:

$$(\omega, \eta) := \sum_\sigma \omega(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)})\eta(\mathbf{v}_{\sigma(1)}, \dots, \mathbf{v}_{\sigma(k)}), \tag{2.2}$$

where the sum is over increasing sequences $\sigma : \{1, \ldots, k\} \mapsto \{1, \ldots n\}$ and $\{\mathbf{v}_i\}_{i=1}^n$ is an arbitrary orthonormal basis. For a fixed $\omega \in \mathrm{Alt}^k V$ the wedge product $\omega \wedge \eta, \eta \in \mathrm{Alt}^{n-k} V$ induces a linear map $\mathrm{Alt}^{n-k} \mapsto \mathbb{R}$. The Riesz representation Theorem then ensures, that there exists a $\star\omega \in \mathrm{Alt}^{n-k} V$ such that:

$$\omega \wedge \eta = (\star\omega, \eta)\,\mathrm{vol}, \quad \forall \eta \in \mathrm{Alt}^{n-k} V. \tag{2.3}$$

The linear map $\omega \mapsto \star\omega$ mapping alternating k-linear forms to alternating $(n-k)$-linear forms is called the *Hodge star operator*. The definitions of the volume form vol and inner product (2.2) give

$$\omega(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(k)}) = \mathrm{sign}(\sigma) \star \omega(\mathbf{e}_{\sigma(k+1)}, \ldots, \mathbf{e}_{\sigma(n)}) \tag{2.4}$$

for positively oriented bases $\{\mathbf{e}_i\}_{i=1}^n$ and permutations σ, thus the Hodge star operator is an isometry. As a consequence we derive

$$\star(\star\omega) = (-1)^{k(n-k)}\omega, \quad \omega \in \mathrm{Alt}^k V. \tag{2.5}$$

Now, let $\Omega \subset \mathbb{R}^n$ be a smooth Riemannian n-dimensional oriented manifold. At each point $x \in \Omega$ the tangent space $T_x\Omega$ is an n-dimensional vector space and we can define alternating forms $\mathrm{Alt}^k T_x\Omega$. A *differential k-form* ω is then the map, assigning to each $x \in \Omega$ an element $\omega_x \in \mathrm{Alt}^k T_x\Omega$. A differential k-form ω is called smooth if the map

$$x \mapsto \omega_x(\mathbf{v}_1(x), \ldots, \mathbf{v}_k(x)) \tag{2.6}$$

is smooth for smooth vector fields $\mathbf{v}_1(x), \ldots \mathbf{v}_k(x)$, with $\mathbf{v}_i(x) \in T_x\Omega$. Then $\Lambda^k(\Omega)$ denotes the set of smooth differential k-forms on Ω, in particular $\Lambda^0(\Omega) = C^\infty(\Omega)$. The definition of the wedge product of alternating forms can be extended to differential forms by a pointwise definition:

$$(\omega \wedge \eta)_x = \omega_x \wedge \eta_x, \quad \omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega) \tag{2.7}$$

Since we assume $\Omega \subset \mathbb{R}^n$ to be a smooth Riemannian manifold, i.e. the spaces $\mathrm{Alt}^k T_x\omega$ are endowed with an inner product, there exists a measure on Ω and we can define integrals of 0-forms $\omega \in \Lambda^0(\Omega)$: $\int_\Omega \omega\,\mathrm{vol}$ and L^2-inner products on $\Lambda^k(\Omega)$:

$$(\omega, \eta)_\Omega = \int_\Omega (\omega_x, \eta_x)\,\mathrm{vol} = \int_\Omega \omega \wedge \star\eta, \quad \omega, \eta \in \Lambda^k(\Omega). \tag{2.8}$$

Completion of $\Lambda^k(\Omega)$ in the norm $\|\omega\|^2_{L^2\Lambda^k(\Omega)} = (\omega, \omega)_\Omega$ yields the Hilbert space $L^2\Lambda^k(\Omega)$.
Due to the assumptions on Ω, we can define the Sobolev spaces $H^m(\Omega)$ and $W^{m,p}(\Omega)$ for functions with $m > 0$ derivatives in $L^2(\Omega)$ and $L^p(\Omega)$ [81, Section 1.3]. Analogously we define Sobolev-spaces $W^{m,p}\Lambda^k(\Omega)$ and $H^m\Lambda^k(\Omega)$ of differential forms by requiring that the map (2.6) is in $W^{m,p}(\Omega)$ and $H^m(\Omega)$. In the following $\|\cdot\|_{W^{m,p}\Lambda^k(\Omega)}$ $(|\cdot|_{W^{m,p}\Lambda^k(\Omega)})$ and $\|\cdot\|_{H^m\Lambda^k(\Omega)}$ $(|\cdot|_{H^m\Lambda^k(\Omega)})$ will denote the corresponding (semi)-norms. Another important family of Hilbert spaces is defined via the notion of the exterior derivatives. For $\Omega \subset \mathbb{R}^n$ the exterior derivative $\mathsf{d}\,\omega$ of a k-form $\omega \in \Lambda^k(\Omega)$ is given as [3, page 15]:

$$\mathsf{d}\,\omega_x(\mathbf{v}_1(x), \ldots \mathbf{v}_{k+1}(x)) = \sum_{j=1}^{k+1} (-1)^j \partial_{\mathbf{v}_j} \omega_x(\mathbf{v}_1(x), \ldots, \hat{\mathbf{v}}_j(x), \ldots, \mathbf{v}_{k+1}(x)), \tag{2.9}$$

where the hat shall indicate a suppressed argument. $\partial_{\mathbf{v}_j}$ is the partial derivative in direction $\mathbf{v}_j$. Then we define the spaces $H\Lambda^k(\Omega)$ containing differential forms $\omega \in L^2\Lambda^k(\Omega)$ with exterior derivatives in $L^2\Lambda^{k+1}(\Omega)$:

$$H\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega), \mathsf{d}\,\omega \in L^2\Lambda^{k+1}(\Omega)\}. \tag{2.10}$$

The spaces $H\Lambda^k(\Omega)$ are Hilbert space with norm $\|\cdot\|^2_{H\Lambda^k(\Omega)} := \|\cdot\|^2_{L^2\Lambda^k(\Omega)} + \|\mathsf{d}\cdot\|^2_{L^2\Lambda^k(\Omega)}$. In particular $H\Lambda^0(\Omega)$ and $H\Lambda^n(\Omega)$ are equal to $H^1\Lambda^0(\Omega)$ and $L^2\Lambda^n(\Omega)$, respectively. For $0 < k < n$ the spaces $H\Lambda^k(\Omega)$ are strictly between the spaces $L^2\Lambda^k(\Omega)$ and $H^1\Lambda^k(\Omega)$ [22]. The exterior derivative satisfies a Leibniz rule with respect to the wedge product:

$$\mathsf{d}(\omega \wedge \eta) = \mathsf{d}\,\omega \wedge \eta + (-1)^j \omega \wedge \mathsf{d}\,\eta, \quad \omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega), \tag{2.11}$$

and the exterior derivative of an exterior derivative vanishes:

$$\mathsf{d} \circ \mathsf{d} = 0. \tag{2.12}$$

This second property ensures that the range of the exterior derivative of differential k-forms is contained in the kernel of the exterior derivative of $k+1$-forms:

$$\mathrm{Range}(\mathsf{d} : \Lambda^k(\Omega) \mapsto \Lambda^{k+1}(\Omega)) \subset \mathrm{Ker}(\mathsf{d} : \Lambda^{k+1}(\Omega) \mapsto \Lambda^{k+2}(\Omega))$$

Therefore the de Rham sequence, i.e. the sequence of mappings:

$$0 \longrightarrow \Lambda^0(\Omega) \xrightarrow{\mathsf{d}} \Lambda^1(\Omega) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} \Lambda^n(\Omega) \longrightarrow 0 \tag{2.13}$$

is a so-called *cochain complex*. For our oriented Riemannian manifold Ω this extends to the L^2 de Rham complex:

$$0 \longrightarrow H\Lambda^0(\Omega) \xrightarrow{\mathsf{d}} H\Lambda^1(\Omega) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} H\Lambda^n(\Omega) \longrightarrow 0. \tag{2.14}$$

The quotient spaces $\mathrm{Ker}(\mathsf{d} : \Lambda^{k+1}(\Omega) \mapsto \Lambda^{k+2}(\Omega)) / \mathrm{Range}(\mathsf{d} : \Lambda^k(\Omega) \mapsto \Lambda^{k+1}(\Omega))$, the de Rham cohomology spaces, are finite dimensional vector spaces, whose dimension is given by the Betti numbers of the domain Ω [81, Section 2.6]. For contractible Ω the quotient spaces vanish.

It is this de Rham cohomology technique, a tool belonging to algebraic and differential topology that has become more and more important also in numerics. It turned out that a rigorous translation to a finite dimensional setting paves the way to superior numerical methods [3, 4]. We will present these ideas in detail in the next chapter.

Another remarkable property of differential k-forms is the possibility to define integration on k-dimensional manifolds without additional structures such as measure or metric. For a continuous differential k-form ω and an oriented, piecewise smooth, compact k-dimensional submanifold $f \subset \Omega$ the integral $\int_f \omega$ is well-defined. Further, if $\phi : \Omega \mapsto \Omega'$ is a smooth map between the manifolds Ω and Ω', the pullback $\phi^* : \Lambda^k(\Omega') \mapsto \Lambda^k(\Omega)$ maps differential forms on Ω' to differential forms on Ω:

$$(\phi^*\omega)_x(\mathbf{v}_1(x), \ldots, \mathbf{v}_k(x)) = \omega_{\phi(x)}(D\phi_x(\mathbf{v}_1(x)), \ldots, D\phi_x(\mathbf{v}_k(x))). \tag{2.15}$$

The Jacobian $D\phi_x$ is a linear map $T_x\Omega \mapsto T_{\phi(x)}\Omega'$. If $(D\phi_x)_{1\leq i,j\leq n} \in \mathbb{R}^{n\times n}$ is the matrix representation of $D\phi_x$ with respect to a basis $\{\mathbf{e}_i\}_{i=1}^n$ of $T_x\Omega$ and a basis $\{\mathbf{e}'_i\}_{i=1}^n$ of $T_{\phi(x)}\Omega'$, and $(D\phi_x)_{\sigma',\sigma} \in \mathbb{R}^{k\times k}$ denotes that submatrix that consists of the rows $\sigma'(1),\ldots,\sigma'(k)$ and the columns $\sigma(1),\ldots,\sigma(k)$ then [77, Page 610]

$$(\phi^*\omega)_x(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(k)}) = \sum_{\sigma'} \det\left((D\phi_x)_{\sigma',\sigma}\right)\omega_{\phi(x)}(\mathbf{e}'_{\sigma'(1)},\ldots,\mathbf{e}'_{\sigma'(k)}), \tag{2.16}$$

where σ and σ' are increasing sequences $\{1,\ldots,k\} \mapsto \{1,\ldots,n\}$. The quantities $\det\left((D\phi_x)_{\sigma',\sigma}\right)$ are refered to as the k-minors of $D\phi_x$. For convenience we introduce for ϕ and x fixed the operator $\mathbf{M}_k(\phi_x) : \operatorname{Alt}^k T_{\phi(x)}\Omega' \mapsto \operatorname{Alt}^k T_x\Omega$ such that for $\omega \in \operatorname{Alt}^k T_{\phi(x)}\Omega'$

$$(\mathbf{M}_k(\phi_x)\omega)(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(k)}) = \sum_{\sigma'} \det\left((D\phi_x)_{\sigma',\sigma}\right)\omega(\mathbf{e}'_{\sigma'(1)},\ldots,\mathbf{e}'_{\sigma'(k)}). \tag{2.17}$$

If ϕ is an orientation-preserving diffeomorphism we have for all oriented, piecewise smooth, k-dimensional submanifolds f:

$$\int_f \phi^*\omega = \int_{\phi(f)} \omega, \quad \omega \in \Lambda^k\left(\Omega'\right). \tag{2.18}$$

For the pullback of the inclusion map $\partial\Omega \mapsto \Omega$, the trace onto $\partial\Omega$, we use the symbol tr, thus Stokes law reads [3, page 16]:

$$\int_\Omega \mathsf{d}\,\omega = \int_{\partial\Omega} \operatorname{tr}\omega. \tag{2.19}$$

For the more general case of inclusion maps $\imath : \Omega' \mapsto \Omega$, with $\dim(\Omega') < \dim(\Omega) - 1$, $\Omega' \subset \Omega$ we introduce the notation

$$\operatorname{tr}_{\Omega,\Omega'} := \imath^*. \tag{2.20}$$

The pullback and in particular the trace respect both the wedge product and exterior derivative:

$$\mathsf{d}(\phi^*\omega) = \phi^*(\mathsf{d}\,\omega), \quad \phi^*(\omega\wedge\eta) = \phi^*\omega\wedge\phi^*\eta, \tag{2.21}$$

and

$$\mathsf{d}(\operatorname{tr}_{\Omega',\Omega}\omega) = \operatorname{tr}_{\Omega',\Omega}(\mathsf{d}\,\omega), \quad \operatorname{tr}_{\Omega',\Omega}(\omega\wedge\eta) = \operatorname{tr}_{\Omega',\Omega}\omega\wedge\operatorname{tr}_{\Omega',\Omega}\eta. \tag{2.22}$$

Further, we need to define the so-called contraction or interior product $\mathsf{i}_{\boldsymbol{\beta}} : \Lambda^{k+1}(\Omega) \mapsto \Lambda^k(\Omega)$ parametrized by a Lipschitz continuous velocity field $\boldsymbol{\beta} : \Omega \mapsto \mathbb{R}^n$, i.e. $\boldsymbol{\beta}(x) \in T_x\Omega$:

$$(\mathsf{i}_{\boldsymbol{\beta}}\,\omega)_x(\mathbf{v}_1(x),\ldots,\mathbf{v}_k(x)) := \omega_x(\boldsymbol{\beta}(x),\mathbf{v}_1(x),\ldots,\mathbf{v}_k(x)). \tag{2.23}$$

A standard density argument shows that $\mathsf{i}_{\boldsymbol{\beta}} : L^2\Lambda^{k+1}(\Omega) \mapsto L^2\Lambda^k(\Omega)$ is bounded map. We use the standard notations $\boldsymbol{W}^{m,p}(\Omega)$, $|\boldsymbol{\beta}|_{\boldsymbol{W}^{m,p}(\Omega)}$ and $\|\boldsymbol{\beta}\|_{\boldsymbol{W}^{m,p}(\Omega)}$ to denote Sobolev spaces, Sobolev semi-norms and Sobolev norms of vector valued functions with $m > 0$

derivatives in $L^p(\Omega)$. The contraction operator together with the exterior derivative forms the so called *Lie derivative* $\mathsf{L}_{\boldsymbol{\beta}} : \Lambda^k(\Omega) \mapsto \Lambda^k(\Omega)$:

$$\mathsf{L}_{\boldsymbol{\beta}}\,\omega := \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega + \mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega. \tag{2.24}$$

Since the contraction $\mathsf{i}_{\boldsymbol{\beta}}$ satisfies similar to the exterior derivative a Leibniz rule:

$$\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \eta) = \mathsf{i}_{\boldsymbol{\beta}}\,\omega \wedge \eta + (-1)^j \omega \wedge \mathsf{i}_{\boldsymbol{\beta}}\,\eta, \quad \omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega), \tag{2.25}$$

we get a Leibniz rule also for Lie derivatives:

$$\mathsf{L}_{\boldsymbol{\beta}}(\omega \wedge \eta) = \mathsf{L}_{\boldsymbol{\beta}}\,\omega \wedge \eta + \omega \wedge \mathsf{L}_{\boldsymbol{\beta}}\,\eta, \quad \omega \in \Lambda^j(\Omega), \eta \in \Lambda^k(\Omega). \tag{2.26}$$

Instead of (2.24) we could equivalently define the Lie derivative by means of the flow function $X_t(x) := X(t,x)$ induced by velocity $\boldsymbol{\beta}$. Here, $X : \Omega \times \mathbb{R} \mapsto \Omega$ is the flow function if

$$\frac{\partial X_t(x)}{\partial t} = \boldsymbol{\beta}(X_t(x)), \quad X_0(x) = x. \tag{2.27}$$

One can then show that [50, p. 142, prop. 5.3]

$$\mathsf{L}_{\boldsymbol{\beta}}\,\omega = \frac{\partial X_t^* \omega}{\partial t}|_{t=0}. \tag{2.28}$$

This definition of the Lie derivative gives rise to a new perspective on the contraction operator. To motivate this we apply (2.28) to some k-form ω and integrate over some k-dimensional manifold M:

$$\begin{aligned}\int_M \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \lim_{\tau \to 0} \frac{1}{\tau} \int_M X_\tau^* \omega - \omega \\ &= \lim_{\tau \to 0} \frac{1}{\tau} \left(\int_{X_\tau(M)} \omega - \int_M \omega \right).\end{aligned}$$

We define a $k+1$-dimensional manifold

$$\operatorname{extr}(M, X_\tau) = \bigcup_{s=0}^{\tau} X_s(M), \tag{2.29}$$

and orient $\operatorname{extr}(M, X_\tau)$ such that the orientation of

$$\partial \operatorname{extr}(M, X_\tau)_{|X_\tau(M)} \text{ and } X_\tau(M)$$

coincide. Then the orientations

$$\begin{gathered}\partial \operatorname{extr}(M, X_\tau)_{|M} \text{ and } M, \\ \partial \operatorname{extr}(M, X_\tau)_{|\operatorname{extr}(\partial M, X_\tau)} \text{ and } \operatorname{extr}(\partial M, X_\tau)\end{gathered}$$

do not coincide and this yields

$$\int_M \mathsf{L}_{\boldsymbol{\beta}}\,\omega = \lim_{\tau \to 0} \frac{1}{\tau} \left(\int_{\operatorname{extr}(M, X_\tau)} \mathsf{d}\,\omega + \int_{\operatorname{extr}(\partial M, X_\tau)} \omega \right).$$

Comparing this with (2.24) we find

$$\int_M \mathsf{i}_{\boldsymbol{\beta}}\, \omega = \lim_{\tau\to 0} \frac{1}{\tau} \int_{\mathrm{extr}(M,X_\tau)} \omega. \tag{2.30}$$

We will later use this characterization in order to derive semi-Lagrangian time stepping schemes.

Since the inner product (2.8) makes the space of differential forms to a Hilbert space, we introduce here also the formal adjoint of the exterior derivative

$$\star\, \delta\, \omega = (-1)^k\, \mathsf{d} \star\omega, \quad \omega \in \Lambda^k(\Omega), \tag{2.31}$$

the formal adjoint of the contraction:

$$\star\, \mathsf{j}_{\boldsymbol{\beta}}\, \omega = (-1)^k\, \mathsf{i}_{\boldsymbol{\beta}} \star\omega, \quad \omega \in \Lambda^k(\Omega), \tag{2.32}$$

and the formal adjoint of the Lie derivative:

$$\star\, (\delta\, \mathsf{j}_{\boldsymbol{\beta}} + \mathsf{j}_{\boldsymbol{\beta}}\, \delta)\omega = \star\, \mathcal{L}_{\boldsymbol{\beta}}\, \omega = -\, \mathsf{L}_{\boldsymbol{\beta}} \star\omega, \quad \omega \in \Lambda^k(\Omega). \tag{2.33}$$

With these definitions we derive the following integration by parts formulas from the Leibniz rules (2.11), (2.25) and (2.26) for $\omega \in \Lambda^j(\Omega)$ and $\eta \in \Lambda^k(\Omega)$:

$$\begin{aligned}
\mathsf{d}(\omega \wedge \star\eta) &= \mathsf{d}\,\omega \wedge \star\eta + (-1)^{j+k} \omega \wedge \star\, \delta\, \eta, \\
\mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta) &= \mathsf{i}_{\boldsymbol{\beta}}\, \omega \wedge \star\eta + (-1)^{j+k} \omega \wedge \star\, \mathsf{j}_{\boldsymbol{\beta}}\, \eta, \\
\mathsf{L}_{\boldsymbol{\beta}}(\omega \wedge \star\eta) &= \mathsf{L}_{\boldsymbol{\beta}}\, \omega \wedge \star\eta - \omega \wedge \star\, \mathcal{L}_{\boldsymbol{\beta}}\, \eta.
\end{aligned}$$

Note that these formulas are valid for $j + n - k > n$, by the convention that $\mathsf{d}\,\omega$ and $\mathsf{i}_{\boldsymbol{\beta}}\,\omega$ are set to zero whenever $\omega \in \Lambda^j(\Omega)$ with $j > n$. Later, the cases $k = j + 1$, $k = j$ and $k = j - 1$ will be of particular importance. For convenience, we write these cases here in terms of bilinear forms $(\cdot,\cdot)_{\partial\Omega,\boldsymbol{\beta}} : \Lambda^k(\Omega) \times \Lambda^k(\Omega) \mapsto \mathbb{R}$ and $(\cdot,\cdot)_\Omega : \Lambda^k(\Omega) \times \Lambda^k(\Omega) \mapsto \mathbb{R}$ and bilinear mappings $(\cdot,\cdot)_{\partial\Omega,\mathrm{tr}} : \Lambda^{k-1}(\Omega) \times \Lambda^k(\Omega) \mapsto \mathbb{R}$.

Proposition 2.1.1. *Let* $\omega \in \Lambda^{k-1}(\Omega), \eta \in \Lambda^k(\Omega)$, *then*

$$(\omega,\eta)_{\partial\Omega,\mathrm{tr}} := \int_{\partial\Omega} \mathrm{tr}(\omega \wedge \star\eta) = (\mathsf{d}\,\omega,\eta)_\Omega - (\omega,\delta\,\eta)_\Omega. \tag{2.34}$$

Let $\omega \in \Lambda^k(\Omega), \eta \in \Lambda^k(\Omega)$, *then*

$$(\omega,\eta)_{\partial\Omega,\boldsymbol{\beta}} := \int_{\partial\Omega} \mathrm{tr}\, \mathsf{i}_{\boldsymbol{\beta}}(\omega \wedge \star\eta) = (\mathsf{i}_{\boldsymbol{\beta}}\,\omega,\eta)_{\partial\Omega,\mathrm{tr}} + \left(\omega,\mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{\partial\Omega,\mathrm{tr}} \tag{2.35}$$

and

$$(\omega,\eta)_{\partial\Omega,\boldsymbol{\beta}} = (\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_\Omega - (\omega,\mathcal{L}_{\boldsymbol{\beta}}\,\eta)_\Omega. \tag{2.36}$$

Let $\omega \in \Lambda^{k+1}(\Omega), \eta \in \Lambda^k(\Omega)$, *then*

$$0 = (\mathsf{i}_{\boldsymbol{\beta}}\,\omega,\eta)_\Omega - \left(\omega,\mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_\Omega. \tag{2.37}$$

k	correspondence	
0	$x \mapsto \omega(x)$	$u(x) := \omega(x)$
1	$x \mapsto \{\mathbf{v} \mapsto \omega(x)(\mathbf{v})\}$	$\mathbf{u}(x) \cdot \mathbf{v} := \omega(x)(\mathbf{v})$
2	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2) \mapsto \omega(x)(\mathbf{v}_1, \mathbf{v}_2)\}$	$\mathbf{u}(x) \cdot (\mathbf{v}_1 \times \mathbf{v}_2) := \omega(x)(\mathbf{v}_1, \mathbf{v}_2)$
3	$x \mapsto \{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \mapsto \omega(x)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\}$	$u(x) \det(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) := \omega(x)(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$

Table 2.1: Correspondences of forms ω with scalar functions u or vectorial functions $\mathbf{u}$.

Proof. The proof follows by direct calculations from the Leibniz rules (2.11), (2.25) and (2.26) and the definitions of the adjoint operators (2.31), (2.32) and (2.33). □

Remark 2.1.2. *The bilinear form* $(\cdot,\cdot)_{\partial\Omega,\boldsymbol{\beta}}$ *introduced in* (2.35) *is in general not positive semidefinite. By defining a velocity* $\hat{\boldsymbol{\beta}}$*:*

$$\hat{\boldsymbol{\beta}}_{|\partial\Omega} := \begin{cases} \left(\boldsymbol{\beta}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_\Omega}{|\boldsymbol{\beta}\cdot\mathbf{n}_\Omega|}\right)_{|\partial\Omega} & \boldsymbol{\beta}\cdot\mathbf{n}_\Omega \neq 0 \\ 0 & \boldsymbol{\beta}\cdot\mathbf{n}_\Omega = 0 \end{cases}$$

with $\mathbf{n}_\Omega(x) \in T_x\Omega$ *outward pointing normal of* $\partial\Omega$*, the bilinear form* $(\cdot,\cdot)_{\partial\Omega,\hat{\boldsymbol{\beta}}}$ *is positive semidefinite. To see that property, we recall that* $\omega\wedge\star\omega$ *is proportional to the volume form* vol *of* Ω *(follows from* (2.8)*) and that the volume form on* $\partial\Omega$ *is given by* $\mathsf{i}_{\mathbf{n}_\Omega} \mathrm{vol}_{|\partial\Omega}$ *[81, p. 26]. Hence linearity of* $\mathsf{i}_{\boldsymbol{\beta}}$ *in* $\boldsymbol{\beta}$ *yields:*

$$(\omega,\omega)_{\partial\Omega,\boldsymbol{\beta}} = \int_{\partial\Omega} \boldsymbol{\beta}\cdot\mathbf{n}_\Omega \operatorname{tr} \mathsf{i}_{\mathbf{n}_\Omega}(\omega\wedge\star\omega).$$

The sign of $(\boldsymbol{\beta}\cdot\mathbf{n}_\Omega)_{|\partial\Omega}$ *determines the sign of* $(\omega,\omega)_{\partial\Omega,\boldsymbol{\beta}}$ *and* $(\hat{\boldsymbol{\beta}}\cdot\mathbf{n}_\Omega)_{|\partial\Omega} \geq 0$.

For completeness we also introduce the Sobolev space for the adjoint exterior derivative δ:

$$H^*\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega),\ \delta\,\omega \in L^2\Lambda^{k-1}(\Omega)\}. \tag{2.38}$$

The space $H^*\Lambda^k(\Omega)$ is a Hilbert space with the norm $\|\cdot\|^2_{H^*\Lambda^k(\Omega)} := \|\cdot\|^2_{L^2\Lambda^k(\Omega)} + \|\delta\,\cdot\|^2_{L^2\Lambda^{k-1}(\Omega)}$. Finally we will need Sobolev spaces of differential forms with prescribed traces:

$$H\Lambda^k(\Omega,\psi) = \{\omega \in H\Lambda^k(\Omega)\ \operatorname{tr}\omega = \psi\},$$
$$H^*\Lambda^k(\Omega,\psi) = \{\omega \in H^*\Lambda^k(\Omega)\ \operatorname{tr}\star\omega = \psi\}.$$

Another important entity will be the operator $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$. This operator is obviously symmetric:

$$(\omega, \mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\,\eta) = (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\,\omega, \eta), \quad \omega,\eta \in \Lambda^k(\Omega). \tag{2.39}$$

In Table 2.2 and 2.3 we listed the correspondences of $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ for the forms in $\mathbb{R}^3$ and $\mathbb{R}^2$. From these representations we easily infer

$$\left|\,(\eta, \mathsf{L}_{\boldsymbol{\beta}}\,\omega + \mathcal{L}_{\boldsymbol{\beta}}\,\omega)_\Omega\right| \leq C\,|\beta|_{\boldsymbol{W}^{1,\infty}(\Omega)}\,\|\omega\|_{L^2\Lambda^k(\Omega)}\,\|\eta\|_{L^2\Lambda^k(\Omega)} \tag{2.40}$$

and it can be shown that this results holds also in the general case of $\mathbb{R}^n$.

k	$H\Lambda^k(\Omega)$	tr	ϕ^*
0	$H^1(\Omega)$	$u(x)$	$u(\phi(x))$
1	$\boldsymbol{H}(\mathbf{curl}, \Omega)$	$\mathbf{n}_\Omega(x) \times \mathbf{u}(x)$	$D\phi(x)^T \mathbf{u}(\phi(x))$
2	$\boldsymbol{H}(\mathrm{div}, \Omega)$	$\mathbf{u}(x) \cdot \mathbf{n}_\Omega(x)$	$\det D\phi(x) D\phi(x)^{-1} \mathbf{u}(\phi(x))$
3	$L^2(\Omega)$		$\det D\phi(x) u(\phi(x))$

k	$\mathsf{d}\,\omega$	$\mathsf{i}_{\boldsymbol{\beta}}\,\omega$	$\delta\,\omega$	$\mathsf{j}_{\boldsymbol{\beta}}\,\omega$
0	$\mathbf{grad}\, u$			$u\boldsymbol{\beta}$
1	$\mathbf{curl}\,\mathbf{u}$	$\boldsymbol{\beta} \cdot \mathbf{u}$	$-\mathrm{div}\,\mathbf{u}$	$-\mathbf{u} \times \boldsymbol{\beta}$
2	$\mathrm{div}\,\mathbf{u}$	$\mathbf{u} \times \boldsymbol{\beta}$	$\mathbf{curl}\,\mathbf{u}$	$\boldsymbol{\beta} \cdot \mathbf{u}$
3		$u\boldsymbol{\beta}$	$-\mathbf{grad}\, u$	

k	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega$	$\mathcal{L}_{\boldsymbol{\beta}}\,\omega$	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega + \mathcal{L}_{\boldsymbol{\beta}}\,\omega$
0	$\boldsymbol{\beta} \cdot \mathbf{grad}\, u$	$-\mathrm{div}(u\boldsymbol{\beta})$	$-u\,\mathrm{div}\,\boldsymbol{\beta}$
1	$\mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) + \mathbf{curl}\,\mathbf{u} \times \boldsymbol{\beta}$	$\mathbf{curl}(\boldsymbol{\beta} \times \mathbf{u}) - \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u}$	$D\boldsymbol{\beta}\mathbf{u} + (D\boldsymbol{\beta})^T\mathbf{u} - \mathbf{u}\,\mathrm{div}\,\boldsymbol{\beta}$
2	$\mathbf{curl}(\mathbf{u} \times \boldsymbol{\beta}) + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u}$	$\boldsymbol{\beta} \times \mathbf{curl}\,\mathbf{u} - \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u})$	$\mathbf{u}\,\mathrm{div}\,\boldsymbol{\beta} - D\boldsymbol{\beta}\mathbf{u} - (D\boldsymbol{\beta})^T\mathbf{u}$
3	$\mathrm{div}(u\boldsymbol{\beta})$	$-\boldsymbol{\beta} \cdot \mathbf{grad}\, u$	$u\,\mathrm{div}\,\boldsymbol{\beta}$

Table 2.2: Correspondences of spaces and operations on forms ω with spaces and operations on scalar functions u or vectorial functions $\mathbf{u}$ in $\mathbb{R}^3$. ϕ is a diffeomorphism and $D\boldsymbol{\beta}$ is the Jacobi matrix. The correspondences of $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ follow from standard vector calculus identities, see e.g. [47, cover page].

k	$H\Lambda^k(\Omega)$	tr	ϕ^*
0	$H^1(\Omega)$	$u(x)$	$u(\phi(x))$
1	$\boldsymbol{H}(\mathrm{div},\Omega)$	$\mathbf{u}(x)\cdot\mathbf{n}_\Omega(x)$	$\det D\phi(x)D\phi(x)^{-1}\mathbf{u}(\phi(x))$
2	$L^2(\Omega)$	not defined	$\det D\phi(x)u(\phi(x))$

k	$\mathsf{d}\,\omega$	$\mathsf{i}_{\boldsymbol{\beta}}\,\omega$	$\delta\,\omega$	$\mathsf{j}_{\boldsymbol{\beta}}\,\omega$
0	$\mathbf{R}\,\mathbf{grad}\,u$			$u\mathbf{R}\boldsymbol{\beta}$
1	$\mathrm{div}\,\mathbf{u}$	$\mathbf{u}\cdot\mathbf{R}\boldsymbol{\beta}$	$\mathrm{div}\,\mathbf{Ru}$	$\boldsymbol{\beta}\cdot\mathbf{u}$
2		$u\boldsymbol{\beta}$	$-\,\mathbf{grad}\,u$	

k	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega$	$\mathcal{L}_{\boldsymbol{\beta}}\,\omega$	$\mathsf{L}_{\boldsymbol{\beta}}\,\omega+\mathcal{L}_{\boldsymbol{\beta}}\,\omega$
0	$\boldsymbol{\beta}\cdot\mathbf{grad}\,u$	$-\,\mathrm{div}(u\boldsymbol{\beta})$	$-u\,\mathrm{div}\,\boldsymbol{\beta}$
1	$\mathbf{R}\,\mathbf{grad}(\mathbf{u}\cdot\mathbf{R}\boldsymbol{\beta})+\boldsymbol{\beta}\,\mathrm{div}\,\mathbf{u}$	$\mathrm{div}(\mathbf{Ru})\mathbf{R}\boldsymbol{\beta}-\mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u})$	$\mathbf{u}\,\mathrm{div}\,\boldsymbol{\beta}-D\boldsymbol{\beta}\mathbf{u}-(D\boldsymbol{\beta})^T\mathbf{u}$
2	$\mathrm{div}(u\boldsymbol{\beta})$	$-\boldsymbol{\beta}\cdot\mathbf{grad}\,u$	$u\,\mathrm{div}\,\boldsymbol{\beta}$

Table 2.3: Correspondences of spaces and operations on forms ω with spaces and operations on scalar functions u or vectorial functions $\mathbf{u}$ in $\mathbb{R}^2$. ϕ is a diffeomorphism, $D\boldsymbol{\beta}$ is the Jacobi matrix and $\mathbf{R}\in\mathbb{R}^{2\times 2}$ is a rotation matrix with entries $(\mathbf{R})_{11}=0$, $(\mathbf{R})_{12}=1$, $(\mathbf{R})_{21}=-1$ and $(\mathbf{R})_{22}=0$.

Proposition 2.1.3. *Let $\boldsymbol{\beta} \in \boldsymbol{W}^{1,\infty}(\Omega)$, then (2.40) holds for any $\omega, \eta \in L^2\Lambda^k(\Omega)$ and $\Omega \subset \mathbb{R}^n$.*

Proof. Let $\eta, \omega \in \Lambda^k(\Omega)$. If $\bar{\omega}$ denotes an extension of ω to $\Lambda^k(\mathbb{R}^n)$ we deduce from (2.28) and (2.18):

$$\begin{aligned}(\eta, (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega)_\Omega &= \lim_{\tau\to 0} \frac{1}{\tau}\int_\Omega \bar{\eta} \wedge \star\left(\bar{\omega} - X^*_{-\tau}\bar{\omega}\right) - \frac{1}{\tau}\int_\Omega \bar{\eta} \wedge \left(\star\bar{\omega} - X^*_{-\tau}\star\bar{\omega}\right)\\ &= \lim_{\tau\to 0} \frac{1}{\tau}\int_\Omega \bar{\eta} \wedge \left(X^*_{-\tau}\star\bar{\omega} - \star X^*_{-\tau}\bar{\omega}\right)\\ &= \lim_{\tau\to 0} \frac{1}{\tau}\left(\int_{X_{-\tau}(\Omega)} X^*_\tau\bar{\eta} \wedge \star\bar{\omega} - \int_\Omega \bar{\eta} \wedge \star X^*_{-\tau}\bar{\omega}\right).\end{aligned}$$

By formulas (2.8) and (2.17) we find for the second term in the last line

$$\int_\Omega \bar{\eta} \wedge \star X^*_{-\tau}\bar{\omega} = \int_\Omega (\bar{\eta}_x, \mathbf{M}_k(X_{-\tau,x})\bar{\omega}_{X_{-\tau}(x)}) \operatorname{vol}.$$

From the expansion $DX_{\tau,x} = \mathsf{id}_x + \tau D\boldsymbol{\beta}_x + O(\tau^2)$, the definition of $\mathbf{M}_k(\cdot)$ and Taylor expansion of det() we infer

$$\int_\Omega \bar{\eta} \wedge \star X^*_{-\tau}\bar{\omega} = \int_\Omega (\bar{\eta}_x, \bar{\omega}_{X_{-\tau}(x)} - \tau\mathbf{M}'_k(X_{-\tau,x})\bar{\omega}_{X_{-\tau}(x)}) \operatorname{vol} + O(\tau^2),$$

where $\mathbf{M}'_k(X_{-\tau,x}) : \operatorname{Alt}^k T_{X_{-\tau}(x)}X_{-\tau}(\Omega) \mapsto \operatorname{Alt}^k T_x\Omega$ is defined for an alternating form $\omega \in \operatorname{Alt}^k T_{X_{-\tau}(x)}X_{-\tau}(\Omega)$ by

$$\left(\mathbf{M}'_k(X_{-\tau,x})\omega\right)(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(k)}) = \sum_{\sigma'} \operatorname{tr}\left(\mathsf{Adj}((I_n)_{\sigma',\sigma})(D\boldsymbol{\beta}_x)_{\sigma',\sigma}\right)\omega(\mathbf{e}'_{\sigma'(1)}, \ldots, \mathbf{e}'_{\sigma'(k)}),$$

with Adj and tr the adjunct and trace operator for matrices, $I_n \in \mathbb{R}^{n\times n}$ unit matrix and σ and σ' increasing sequences $\{1, \ldots, k\} \mapsto \{1, \ldots, n\}$ and $\{\mathbf{e}_i\}_{i=1}^n$ and $\{\mathbf{e}'_i\}_{i=1}^n$ basis of $T_x\Omega$ and $T_{X_{-\tau}(x)}X_{-\tau}(\Omega)$. Similarly we deduce

$$\int_{X_{-\tau}(\Omega)} X^*_\tau\bar{\eta} \wedge \star\bar{\omega} = \int_{X_{-\tau}(\Omega)} (\bar{\eta}_{X_\tau(x)} + \tau\mathbf{M}'_k(X_{\tau,x})\bar{\eta}_{X_\tau(x)}, \bar{\omega}_x) \operatorname{vol} + O(\tau^2).$$

where $\mathbf{M}'_k(X_{\tau,x}) : \operatorname{Alt}^k T_{X_\tau(x)}X_\tau(\Omega) \mapsto \operatorname{Alt} T_x\Omega$ is defined for $\eta \in \operatorname{Alt}^k T_{X_\tau(x)}X_\tau(\Omega)$ by

$$\left(\mathbf{M}'_k(X_{\tau,x})\eta\right)(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(k)}) = \sum_{\sigma'} \operatorname{tr}\left(\mathsf{Adj}((I_n)_{\sigma',\sigma})(D\boldsymbol{\beta}_x)_{\sigma',\sigma}\right)\eta(\mathbf{e}'_{\sigma'(1)}, \ldots, \mathbf{e}'_{\sigma'(k)}).$$

where here $\{\mathbf{e}'_i\}_{i=1}^n$ is a basis of $T_{X_\tau(x)}X_\tau(\Omega)$. Collecting all these results we get:

$$\begin{aligned}
(\eta, (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega)_\Omega &= \lim_{\tau\to 0}\frac{1}{\tau}\left(\int_{X_{-\tau}(\Omega)} (\bar{\eta}_{X_\tau(x)}, \bar{\omega}_x)\,\mathrm{vol} - \int_\Omega (\bar{\eta}_x, \bar{\omega}_{X_{-\tau}(x)})\,\mathrm{vol}\right) \\
&\quad + \lim_{\tau\to 0}\left(\int_{X_{-\tau}(\Omega)} (\mathbf{M}'_k(X_{\tau,x})\bar{\eta}_{X_\tau(x)}, \bar{\omega}_x)\,\mathrm{vol} + \int_\Omega (\bar{\eta}_x, \mathbf{M}'_k(X_{-\tau,x})\bar{\omega}_{X_{-\tau}(x)})\,\mathrm{vol}\right) \\
&= \lim_{\tau\to 0}\frac{1}{\tau}\int_\Omega (\bar{\eta}_x, \bar{\omega}_{X_{-\tau}(x)})(X^*_{-\tau}\,\mathrm{vol} - \mathrm{vol}) \\
&\quad + \left(\int_\Omega (\mathbf{M}'_k(X_{0,x})\eta_x, \omega_x)\,\mathrm{vol} + \int_\Omega (\eta_x, \mathbf{M}'_k(X_{0,x}))\omega_x)\,\mathrm{vol}\right) \\
&= \left(\int_\Omega (\mathbf{M}'_k(X_{0,x}))\eta_x, \omega_x)\,\mathrm{vol} + \int_\Omega (\eta_x, \mathbf{M}'_k(X_{0,x})\omega_x)\,\mathrm{vol}\right) \\
&\quad - \int_\Omega (\eta_x, \omega_x)\mathbf{M}'_n(X_{0,x})\,\mathrm{vol}\,.
\end{aligned}$$

This result holds for any extension of ω and η and the assertion follows by density of $\Lambda^k(\Omega)$ in $L^2\Lambda^k(\Omega)$, since $\mathbf{M}'_k(\cdot)$ depends only on the Jacobian of $\boldsymbol{\beta}$. □

2.2 Discrete Differential Forms

To live up to their name, *discrete differential forms* spaces $\Lambda^k_h(\mathcal{T})$ should inherit the principal mathematical structure of differential forms spaces $H\Lambda^k(\Omega)$, namely the de Rham complex (2.14). The first spaces of discrete differential forms, also called Whitney forms were introduced by Whitney [87] as a tool in algebraic topology. Later, these low order polynomial spaces were rediscovered by many different authors as finite element spaces in computational electromagnetism [1,6,9,10,52,66] or mixed problem formulations [73]. Here, we will stay with the systematic presentation in [3] and [4], giving conforming discrete differential forms spaces, i.e. $\Lambda^k_h(\mathcal{T}) \subset H\Lambda^k(\Omega)$, with arbitrary local polynomial degree. For the sake of completeness we also present some non-conforming approximation spaces, that yield de Rham complexes with approximative exterior derivatives. Such spaces are of particular interest in finite volume schemes.

2.2.1 Conforming Discrete Differential Forms

Let $\mathcal{P}_r(\mathbb{R}^n)$ and $\mathcal{H}_r(\mathbb{R}^n)$ be spaces of polynomials in n variables of degree at most r and of homogeneous polynomial degree r respectively, with the convention that $\mathcal{P}_r(\mathbb{R}^n)$ and $\mathcal{H}_r(\mathbb{R}^n)$ are the empty space for $r < 0$. We then define polynomial differential forms, $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ and $\mathcal{H}_r\Lambda^k(\mathbb{R}^n)$, as those elements $\omega \in \Lambda^k(\mathbb{R}^n)$ such that the map

$$x \mapsto \omega_x(\mathbf{v}_1, \ldots, \mathbf{v}_k), \quad \mathbf{v}_1, \ldots \mathbf{v}_k \in \mathbb{R}^n$$

is in $\mathcal{P}_r(\mathbb{R}^n)$ and $\mathcal{H}_r(\mathbb{R}^n)$, respectively. The L^2 de Rham complex (2.13) extends to polynomial subcomplexes:

$$0 \longrightarrow \mathcal{H}_r\Lambda^0(\mathbb{R}^n) \xrightarrow{d} \mathcal{H}_{r-1}\Lambda^1(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{H}_{r-n}\Lambda^n(\mathbb{R}^n) \longrightarrow 0$$

and

$$0 \longrightarrow \mathcal{P}_r\Lambda^0(\mathbb{R}^n) \xrightarrow{d} \mathcal{P}_{r-1}\Lambda^1(\mathbb{R}^n) \xrightarrow{d} \cdots \xrightarrow{d} \mathcal{P}_{r-n}\Lambda^n(\mathbb{R}^n) \longrightarrow 0.$$

Later, we will use some of these polynomial differential forms for constructing piecewise polynomial finite element spaces on simplicial decompositions that inherit the continuity conditions of piecewise smooth elements in $H\Lambda^k(\Omega)$. However, it has turned out that there is another important space of polynomial differential forms between $\mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n)$ and $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$, for such constructions. To characterize this space we introduce the *Koszul differential* [3, p. 29]$\kappa : \Lambda^{k+1}(\mathbb{R}^n) \mapsto \Lambda^k(\mathbb{R}^n)$:

$$(\kappa\omega)_x(\mathbf{v}_1, \ldots, \mathbf{v}_k) := \omega_x(\mathbf{x}(x), \mathbf{v}_1, \ldots \mathbf{v}_k), \tag{2.41}$$

where $\mathbf{x}(x)$ is a vector of length $|x|$ located at x and pointing opposite to the origin. The Koszul differential is an instance of the contraction operator (2.23). Concrete realization in $\mathbb{R}^3$ can be translated from the corresponding realization in Table 2.1. In contrast to the exterior derivatives d, the Koszul differential increases the polynomial degree and decreases the form degree when applied to homogeneous polynomial forms [3, page 30]:

$$\kappa : \mathcal{H}_r\Lambda^k(\mathbb{R}^n) \mapsto \mathcal{H}_{r+1}\Lambda^{k-1}(\mathbb{R}^n).$$

In particular, a simple calculation with (2.28) gives the identity [3, p.31]:

$$(\mathsf{d}\,\kappa + \kappa\,\mathsf{d})\omega = (r+k)\omega, \quad \omega \in \mathcal{H}_r\Lambda^k(\mathbb{R}^n). \tag{2.42}$$

With this identity it is easy to establish a direct sum decomposition of $\mathcal{H}_r\Lambda^k(\mathbb{R}^n)$ for $r, k \geq 0$ with $r + k > 0$ [3, p. 32]:

$$\mathcal{H}_r\Lambda^k(\mathbb{R}^n) = \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(\mathbb{R}^n) \oplus \mathsf{d}\,\mathcal{H}_{r+1}\Lambda^{k-1}(\mathbb{R}^n). \tag{2.43}$$

But since the space $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ obviously permits the decomposition:

$$\mathcal{P}_r\Lambda^k(\mathbb{R}^n) = \mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n) + \mathcal{H}_r\Lambda^k(\mathbb{R}^n)$$

it is clear that the space

$$\mathcal{P}_r^-\Lambda^k(\mathbb{R}^n) := \mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n) + \kappa\mathcal{H}_{r-1}\Lambda^{k+1}(\mathbb{R}^n)$$

lies between $\mathcal{P}_{r-1}\Lambda^k(\mathbb{R}^n)$ and $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$. Before we present now the finite element spaces based on $\mathcal{P}_r\Lambda^k(\mathbb{R}^n)$ and $\mathcal{P}_r^-\Lambda^k(\mathbb{R}^n)$ we remark two important properties. First both spaces are invariant under pullbacks of affine maps. Second the construction of the reduced spaces applies also to polynomial spaces defined on affine subsets of $\mathbb{R}^n$ [4, p. 331].

Let now $\mathcal{T}$ be a finite set of n-simplices determining a simplicial decomposition of Ω. A set of n-simplices is a simplicial decomposition if the union of all elements is the closure of Ω and if the intersection of any two elements is either empty or a common subsimplex. d-simplices f are the image of affine subsets of $\mathbb{R}^n$ under affine maps, hence we can define polynomial differential forms $\mathcal{P}_r\Lambda^k(f)$ and $\mathcal{P}_r^-\Lambda^k(f)$ on d-simplices. $\Delta_d(T)$ is the set of d-subsimplices of any simplex T and similar $\Delta_d(\mathcal{T})$ the set of all such subsimplices in $\mathcal{T}$. $\Delta(T)$ and $\Delta(\mathcal{T})$ denote the sets of all subsimplices of simplex T and mesh $\mathcal{T}$. Then, we tend to define finite element spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ on the mesh $\mathcal{T}$ in choosing $\mathcal{P}_r\Lambda^k(T)$ and $\mathcal{P}_r^-\Lambda^k(T)$ as the local spaces on all n-simplices T. The only missing part for a proper definition of the finite element space is a choice of degrees of freedom associated with particular subsimplices. The keystep in there is the following characterization of the dual spaces $\mathcal{P}_r\Lambda^k(T)^*$ and $\mathcal{P}_r^-\Lambda^k(T)^*$ stated in [4, Theorem 5.5] and proved in [3, Sections 4.5 and 4.6]. See also [34].

Theorem 2.2.1. *Let r, k, and n be integers with $0 \le k \le n$ and $r > 0$, and let T be an n-simplex in $\mathbb{R}^n$.*

1. *To each $f \in \Delta(T)$, associate a space $W_r^k(T,f) \subset \mathcal{P}_r\Lambda^k(T)^*$:*

$$W_r^k(T,f) = \big\{\omega \mapsto \int_f \operatorname{tr}_{T,f}\omega \wedge \eta \mid \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f)\big\}.$$

Then $W_r^k(T,f) \cong \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f)$ and

$$\mathcal{P}_r\Lambda^k(T)^* = \bigoplus_{f\in\Delta(T)} W_r^k(T,f).$$

2. *To each $f \in \Delta(T)$, associate a space $W_r^{k,-}(T,f) \subset \mathcal{P}_r^-\Lambda^k(T)^*$:*

$$W_r^{k,-}(T,f) = \big\{\omega \mapsto \int_f \operatorname{tr}_{T,f}\omega \wedge \eta \mid \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f)\big\}.$$

Then $W_r^{k,-}(T,f) \cong \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f)$ and

$$\mathcal{P}_r^-\Lambda^k(T)^* = \bigoplus_{f\in\Delta(T)} W_r^{k,-}(T,f).$$

For $k = 0$ this is a standard result of $H^1(\Omega)$-conforming finite elements [80]: an element of $\mathcal{P}_r\Lambda^0(T)$ vanishes if it vanishes at the vertices, its moments of degree at most $r-2$ vanish on each edge, its moments of degree at most $r-3$ vanish on each 2-subsimplex and so on. Since the space $\mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f-k}(f)$ is defined for $1 \le r+k-\dim f$ and $0 \le \dim f - k \le \dim f$ we find that the dual space of $\mathcal{P}_r\Lambda^k(T)$ is the span of certain moments on all k- to $r+k-1$-subsimplices of T. And similar the dual space of $\mathcal{P}_r^-\Lambda^k(T)$ is the span of certain moments on all k- to $r+k-1$-subsimplices.

Theorem 2.2.1 shows that the dual spaces $\mathcal{P}_r\Lambda^k(T)^*$ and $\mathcal{P}_r^-\Lambda^k(T)^*$ are direct sums of certain spaces of functionals, whose definition is completely local, i.e. it depends only on specific subsimplices f. By requiring these functionals to be single valued for elements sharing the same subsimplex f we obtain global finite element spaces.

Definition 2.2.2. *The first family of finite element differential forms $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ is:*

$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) := \left\{\omega \in L^2\Lambda^k(\Omega),\ \begin{array}{l}\omega_{|T} \in \mathcal{P}_r^-\Lambda^k(T) \quad and \\ l(\omega_{|T_1}) = l(\omega_{|T_2})\,\forall l \in \mathcal{P}_r^-\Lambda^k(T_1)^* \cap \mathcal{P}_r^-\Lambda^k(T_2)^*\end{array}\right\}.$$

The second family of finite element differential forms $\mathcal{P}_r\Lambda^k(\mathcal{T})$ is:

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) := \left\{\omega \in L^2\Lambda^k(\Omega),\ \begin{array}{l}\omega_{|T} \in \mathcal{P}_r\Lambda^T(\mathcal{T}) \quad and \\ l(\omega_{|T_1}) = l(\omega_{|T_2})\,\forall l \in \mathcal{P}_r\Lambda^k(T_1)^* \cap \mathcal{P}_r\Lambda^k(T_2)^*\end{array}\right\}.$$

To conclude that the two spaces $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r\Lambda^k(\mathcal{T})$ are conforming, i.e. that $\mathcal{P}_r^-\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ and $\mathcal{P}_r\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$, we recall the following characterization of piecewise smooth differential forms due to Stokes' law (2.19) [3, Lemma 5.1]:

Lemma 2.2.3. *Let $\omega \in L^2\Lambda^k(\Omega)$ be piecewise smooth with respect to the triangulation $\mathcal{T}$. The following statements are equivalent:*

1. $\omega \in H\Lambda^k(\Omega)$,
2. $\mathrm{tr}_{\Omega,f}\,\omega$ *is single valued for all* $f \in \Delta_{n-1}(\mathcal{T})$,
3. $\mathrm{tr}_{\Omega,f}\,\omega$ *is single valued for all* $f \in \Delta_j(\mathcal{T}), k \leq j \leq n-1$.

As a corollary, we get the following Theorem stated and proved in [3, p. 59]:

Theorem 2.2.4.

$$\mathcal{P}_r\Lambda^k(\mathcal{T}) = \left\{\omega \in H\Lambda^k(\Omega),\ \omega_{|T} \in \mathcal{P}_r\Lambda^k(T)\right\},$$
$$\mathcal{P}_r^-\Lambda^k(\mathcal{T}) = \left\{\omega \in H\Lambda^k(\Omega),\ \omega_{|T} \in \mathcal{P}_r^-\Lambda^k(T)\right\}.$$

Further, since

$$\mathsf{d}\,\mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T}) \subset \mathsf{d}\,\mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}) \subset \mathcal{P}_r\Lambda^k(\mathcal{T}) \subset \mathcal{P}_{r+1}^-\Lambda^k$$

or

$$\left\{\begin{array}{c}\mathcal{P}_{r+1}\Lambda^{k-1}(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_{r+1}^-\Lambda^{k-1}(\mathcal{T})\end{array}\right\} \xrightarrow{\mathsf{d}} \left\{\begin{array}{c}\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T}) \\ \text{or} \\ \mathcal{P}_r\Lambda^k(\mathcal{T})\end{array}\right\},$$

we get several different polynomial subcomplexes of the de Rham complex. That polynomial subcomplex that is built from the first family

$$0 \longrightarrow \mathcal{P}_r^-\Lambda^0(\mathcal{T}) \xrightarrow{\mathsf{d}} \mathcal{P}_r^-\Lambda^1(\mathcal{T}) \xrightarrow{\mathsf{d}} \cdots \xrightarrow{\mathsf{d}} \mathcal{P}_r^-\Lambda^n(\mathcal{T}) \longrightarrow 0,$$

is called higher order Whitney complex, because it coincides for $r = 1$ with the one introduced by Whitney [87].

Finally we have to show, that our finite element differential forms spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ not only inherit the de Rham complex, but also approximate $H\Lambda^k(\Omega)$.

This means, for every $\omega \in H\Lambda^k(\Omega)$ we need to give approximations $\omega_h \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\omega_h \in \mathcal{P}_r^-\Lambda^k(\mathcal{T})$, such that the error, measured in a certain norm, tends to zero, when we refine the mesh $\mathcal{T}$ successively. As usually, we will define a global projection operator by means of local elementwise operators. The characterization of the dual spaces $\mathcal{P}_r\Lambda^k(T)^*$ and $\mathcal{P}_r^-\Lambda^k(T)^*$ in Theorem 2.2.1 shows that for $0 \leq k \leq n$, $r > 0$, $\omega \in \mathcal{P}_r\Lambda^k(T)$ is uniquely determined by the quantities

$$\int_f \operatorname{tr}_{T,f} \omega \wedge \eta, \quad \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f), \quad f \in \Delta T \tag{2.44}$$

and $\omega \in \mathcal{P}_r^-\Lambda^K(T)$ is uniquely determined by the quantities

$$\int_f \operatorname{tr}_{T,f} \omega \wedge \eta, \quad \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f), \quad f \in \Delta T. \tag{2.45}$$

Therefore we can define local projection operators $\Pi_r^k : \Lambda^k(T) \mapsto \mathcal{P}_r\Lambda^k(T)$ and $\Pi_r^{k,-} : \Lambda^k(T) \mapsto \mathcal{P}_r^-\Lambda^k(T)$ by requiring:

$$\int_f \operatorname{tr}_{T,f}(\omega - \Pi_r^k\omega) \wedge \eta = 0, \quad \eta \in \mathcal{P}^-_{r+k-\dim f}\Lambda^{\dim f - k}(f), \quad f \in \Delta T$$

and

$$\int_f \operatorname{tr}_{T,f}(\omega - \Pi_r^{k,-}\omega) \wedge \eta = 0, \quad \eta \in \mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f - k}(f), \quad f \in \Delta T.$$

This gives the global projection operators, called canonical projection operators, again denoted with Π_r^k and Π_r^{k-}:

$$(\Pi_r^k\omega)_{|T} = \Pi_r^k\omega_{|T} \quad \text{and} \quad (\Pi_r^{k,-}\omega)_{|T} = \Pi_r^{k,-}\omega_{|T}. \tag{2.46}$$

For these we can prove standard interpolation error estimates [3, Theorem 5.3].

Theorem 2.2.5. *Let $\mathcal{T}_h$ be a family of triangulations of Ω indexed by the discretization parameter*

$$h = \max_{T \in \mathcal{T}} \operatorname{diam} T.$$

We assume mesh regularity, i.e. there exists a constant $C_r > 0$ such that

$$|h|^n \leq C_r|T|, \quad \forall T \in \mathcal{T}_h.$$

Denote by Π_h the canonical projection for $\Lambda^k(\Omega)$ onto either $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}^-_{r+1}\Lambda^k(\mathcal{T})$. Let $1 \leq p \leq \infty$ and $\frac{n-k}{p} < r+1$. Then Π_h extends boundedly to $W^{s,p}\Lambda^k(\Omega)$, and there exists a constant C independent of h, such that

$$\|\omega - \Pi_h\omega\|_{L^2\Lambda^k(\Omega)} \leq Ch^{\min(s,r+1)} \|\omega\|_{W^{s,p}\Lambda^k(\Omega)}.$$

The larger the degree k of the form the less smoothness need to be assumed. As in the standard finite element theory we get best approximation results without such smoothness assumption by using Clement interpolation operators [21]. This yields the following approximation estimate (see [4, page 338] and [4, Theorem 5.8]).

Theorem 2.2.6. *Let $\mathcal{T}_h$ be a family of triangulations of Ω indexed by the discretization parameter*

$$h = \max_{T \in \mathcal{T}} \operatorname{diam} T.$$

We assume mesh regularity, i.e. there exists a constant $C_r > 0$ such that

$$|h|^n \le C_r |T|, \quad \forall T \in \mathcal{T}_h.$$

Assume the $\Lambda_h^k(\mathcal{T})$ is either $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$. Let $1 \le p \le \infty$ and $\frac{n-k}{p} < r+1$. Then there is a constant C independent of h, such that

$$\inf_{\omega \in \Lambda_h^k(\mathcal{T})} \|\omega - \omega_h\|_{L^2\Lambda^k(\Omega)} \le C h^{\min(s,r+1)} \|\omega\|_{W^{s,p}\Lambda^k(\Omega)}.$$

Since the Hodge operator is an isometry we obtain from Theorem 2.2.6 an approximation result for the approximations of $n-k$-forms in the finite element differential form spaces $\mathcal{P}_r\Lambda^{n-k}$ and $\mathcal{P}_r^-\Lambda^{n-k}$.

Corollary 2.2.7. *Let $\mathcal{T}_h$ be a family of triangulations of Ω indexed by the discretization parameter*

$$h = \max_{T \in \mathcal{T}} \operatorname{diam} T.$$

We assume mesh regularity, i.e. there exists a constant $C_r > 0$ such that

$$|h|^n \le C_r |T|, \quad \forall T \in \mathcal{T}_h.$$

Assume the $\Lambda_h^k(\mathcal{T})$ is either $\star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ or $\star\mathcal{P}_{r+1}^-\Lambda^{n-k}(\mathcal{T})$. Let $1 \le p \le \infty$ and $\frac{n-k}{p} < r+1$. Then there is a constant C independent of h, such that

$$\inf_{\omega_h \in \Lambda_h^k(\mathcal{T})} \|\omega - \omega_h\|_{L^2\Lambda^k(\Omega)} \le C h^{\min(s,r+1)} \|\omega\|_{W^{s,p}\Lambda^k(\Omega)}.$$

More explicit definitions of interpolation operators rest upon certain choices of basis functionals spanning the local degrees of freedom spaces $W_r^k(T,f)$ and $W_r^{k,-}(T,f)$. By Theorem 2.2.1 this is equivalent to fixing a basis of the local polynomial spaces $\mathcal{P}_{r+k-\dim f}^-\Lambda^{\dim f - k}(f)$ and $\mathcal{P}_{r+k-\dim f-1}\Lambda^{\dim f-k}(f)$. We can conclude that the definition of the finite element differential forms (2.2.2) allow to choose between different basis functions. Examples are hierarchical basis functions or problem adapted basis functions that give small condition numbers for the stiffness matrices.

Another important property of these canonical interpolation operators is the commutativity with exterior derivatives [3, Lemma 5.2]:

Lemma 2.2.8. *The following four diagrams commute:*

$$\begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{\mathsf{d}} & \Lambda^{k+1}(\Omega) \\ \Pi_r^k \downarrow & & \downarrow \Pi_{r-1}^{k+1} \\ \mathcal{P}_r\Lambda^k(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{\mathsf{d}} & \Lambda^{k+1}(\Omega) \\ \Pi_r^k \downarrow & & \downarrow \Pi_r^{k+1,-} \\ \mathcal{P}_r\Lambda^k(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}) \end{array}$$

$$\begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{\mathsf{d}} & \Lambda^{k+1}(\Omega) \\ \Pi_r^{k,-} \downarrow & & \downarrow \Pi_r^{k+1,-} \\ \mathcal{P}_r^-\Lambda^k(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}_r^-\Lambda^{k+1}(\mathcal{T}) \end{array} \qquad \begin{array}{ccc} \Lambda^k(\Omega) & \xrightarrow{\mathsf{d}} & \Lambda^{k+1}(\Omega) \\ \Pi_r^{k,-} \downarrow & & \downarrow \Pi_{r-1}^{k+1} \\ \mathcal{P}_r^-\Lambda^k(\mathcal{T}) & \xrightarrow{\mathsf{d}} & \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}) \end{array}$$

It is Theorem 2.2.5 and Lemma 2.2.8 that finally justify the name finite element differential form spaces for the spaces $\mathcal{P}_r\Lambda^k\mathcal{T}$ and $\mathcal{P}_r^-\Lambda^k\mathcal{T}$. Theorem 2.2.5 establishes standard finite element approximation estimates. A "corollary" of Lemma 2.2.8 says that the finite element differential forms spaces $\mathcal{P}_r\Lambda^k\mathcal{T}$ and $\mathcal{P}_r^-\Lambda^k\mathcal{T}$ are not only contained in the Sobolev spaces of differential forms $H\Lambda^k(\Omega)$, but also that the cohomology spaces of differential forms and finite element differential forms have the same dimensions [4, Section 5.5]. The proof is based on so-called smoothed projection operators that are defined for the spaces $H\Lambda^k(\Omega)$ and commute with the exterior derivative [4, Theorem 5.9]. With these one deduces the isomorphy of the cohomology spaces of differential form and finite element differential forms from the classical result for Whitney forms. We refer to [4, Section 5.5] for the detailed argumentation. Both $\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ deserve the name *conforming discrete differential forms*. One recognized advantage of this abstract treatment of finite element spaces is a unifying convergence analysis for general second order boundary value problems, including Poisson-type and Maxwell-type problems [35]. It is the notion of the Hodge Laplacian $\mathsf{d}\,\delta + \delta\,\mathsf{d}$, that unifies many common second order differential operators. Hence, studying source problems, eigenvalue problems and preconditioning for this Hodge Laplacian simultaneously shall cover many different problems. We refer to [3] for a detailed discussion. Many Hodge Laplacian problems in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been subject to intensive research before. It is thus not surprising that in the cases $\mathbb{R}^2$ and $\mathbb{R}^3$ the finite element differential forms $\mathcal{P}_r\Lambda^k$ and $\mathcal{P}_r^-\Lambda^k$ correspond to classical finite element spaces (see Tables 2.4 and 2.5).

Furthermore, there exist finite element differential forms spaces on hexahedral triangulations [36, page 276] and finite element differential forms spaces with non-homogeneous polynomial degree [36, page 273]. All these spaces can be considered as name finite element differential form spaces, since they feature both finite element approximation properties, like those in Theorem 2.2.5, and the de Rham cohomolgy (2.14) of smooth differential forms. We close this section with a discussion of lowest order finite element differential forms spaces $\mathcal{P}_1^-\Lambda^k$ also called Whitney forms, due to Whitney [87].

k	$\Lambda_h^k(\mathcal{T})$	Classical finite element space
0	$\mathcal{P}_r\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$ [19]
1	$\mathcal{P}_r\Lambda^1(\mathcal{T})$	Brezzi-Douglas-Marini $\boldsymbol{H}(\text{div}, \Omega)$ elements of degree $\leq r$ [15]
2	$\mathcal{P}_r\Lambda^2(\mathcal{T})$	discontinuous elements of degree $\leq r$
0	$\mathcal{P}_r^-\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$ [19]
1	$\mathcal{P}_r^-\Lambda^1(\mathcal{T})$	Raviart-Thomas $\boldsymbol{H}(\text{div}, \Omega)$ elements of degree $\leq r-1$ [73]
2	$\mathcal{P}_r^-\Lambda^2(\mathcal{T})$	discontinuous elements of degree $\leq r-1$

Table 2.4: Correspondence between finite element differential forms and classical finite element spaces for $\mathbb{R}^2$.

k	$\Lambda_h^k(\mathcal{T})$	Classical finite element space
0	$\mathcal{P}_r\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$ [19]
1	$\mathcal{P}_r\Lambda^1(\mathcal{T})$	Nédélec 2nd kind $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$ elements of degree $\leq r$ [67]
2	$\mathcal{P}_r\Lambda^2(\mathcal{T})$	Nédélec 2nd kind $\boldsymbol{H}(\text{div}, \Omega)$ elements of degree $\leq r$ [67]
3	$\mathcal{P}_r\Lambda^3(\mathcal{T})$	discontinuous elements of degree $\leq r$
0	$\mathcal{P}_r^-\Lambda^0(\mathcal{T})$	Lagrange elements of degree $\leq r$ [19]
1	$\mathcal{P}_r^-\Lambda^1(\mathcal{T})$	Nédélec 1st kind $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$ elements of degree $\leq r-1$ [66]
2	$\mathcal{P}_r^-\Lambda^2(\mathcal{T})$	Nédélec 1st kind $\boldsymbol{H}(\text{div}, \Omega)$ elements of degree $\leq r-1$ [66]
3	$\mathcal{P}_r^-\Lambda^3(\mathcal{T})$	discontinuous elements of degree $\leq r-1$

Table 2.5: Correspondence between finite element differential forms and classical finite element spaces for $\mathbb{R}^3$.

d-simplex f^d	basis forms
(f_i^0)	λ_i
(f_i^0, f_j^0)	$\lambda_i \,\mathbf{grad}\, \lambda_j - \lambda_j \,\mathbf{grad}\, \lambda_i$
(f_i^0, f_j^0, f_k^0)	$\lambda_i \,\mathbf{grad}\, \lambda_j \times \mathbf{grad}\, \lambda_l - \lambda_j \,\mathbf{grad}\, \lambda_i \times \mathbf{grad}\, \lambda_l + \lambda_l \,\mathbf{grad}\, \lambda_i \times \mathbf{grad}\, \lambda_j$
$(f_i^0, f_j^0, f_k^0, f_l^0)$	$\mathrm{vol}(f^3)^{-1}, \quad f^3 = (f_i^0, f_j^0, f_k^0, f_l^0)$

Table 2.6: Vector correspondences of basis forms of Whitney forms in $\mathbb{R}^3$, k-simplices are specified by their k vertices.

Whitney Forms

Recall that $\Delta_k(\mathcal{T})$ is the set of all k-simplices of $\mathcal{T}$. In words, $\Delta_0(\mathcal{T})$ is the set of vertices, $\Delta_1(\mathcal{T})$ the set of edges etc. Additionally, N_k denotes the cardinality of $\Delta_k\mathcal{T}$, i.e. N_0 is the number of vertices, N_1 the number of edges etc. We impose an arbitrary numbering on k-simplices of the mesh $\mathcal{T}$, i.e.

$$\Delta_k(\mathcal{T}) = (f_i^k)_{i=1}^{N_k}, \quad f_i^k \ k\text{-simplex}.$$

The dual space $\mathcal{P}_1^-\Lambda^k(\mathcal{T})^*$ is then spanned by the integral values on all k-simplices:

$$\mathcal{P}_1^-\Lambda^k(\mathcal{T})^* \cong (l_i^k)_{i=1}^{N_k}, \quad \text{with} \quad l_i^k(\omega) := \int_{f_i^k} \omega \quad f_i^k \in \Delta_k(\mathcal{T}). \tag{2.47}$$

The degrees of freedom l_i^0 of $\mathcal{P}_1^-\Lambda^0(\mathcal{T})$ are point evaluations on all vertices, the degrees of freedom l_i^1 of $\mathcal{P}_1^-\Lambda^1(\mathcal{T})$ are line integrals on all edges etc. The basis form $b_i^k \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ dual to the degrees of freedom l_i^k can be expressed in terms of the barycentric coordinate functions λ_i of n-simplices T and their gradients $\mathsf{d}\,\lambda_i$. If $I = (I_0, \dots I_k)$ denotes the index set of the vertices $f_{I_0}^0, \dots f_{I_k}^0$ of some k-simplex $f_i^k \in \Delta_k(\mathcal{T})$, then [4, Formula 4.3]:

$$b_{f_i^k} := b_i^k := \sum_{j=0}^{k} (-1)^j \lambda_{I_j} \bigwedge_{l=1, l\neq j}^{k} d\lambda_{I_l}. \tag{2.48}$$

The basis forms b_i^0, associated to vertices f_i^0, are the barycentric coordinate functions λ_i. For a basis form b_i^1, associated to an edge f_i^1 that is oriented from vertex $f_{I_1}^0$ to $f_{I_2}^0$ we get:

$$b_i^1 = \lambda_{I_1}\,\mathsf{d}\,\lambda_{I_2} - \lambda_{I_2}\,\mathsf{d}\,\lambda_{I_1}.$$

In table 2.6 we list the corresponding vector representations of basis forms in $\mathbb{R}^3$. With this it follows directly from Stokes law that the restriction of the discrete exterior derivate to Whitney forms

$$\mathsf{d} : \mathcal{P}_1^-\Lambda^k(\mathcal{T}) \mapsto \mathcal{P}_1^-\Lambda^{k+1}(\mathcal{T})$$

can be represented as incidence matrix $\mathbf{D}_k \in \mathbb{R}^{N_{k+1}\times N_k}$:

$$(\mathbf{D}_k)_{i,j} := \mathbf{D}^{f_j^k}_{f_i^{k+1}} := \int_{\partial f_i^{k+1}} b_j^k = \begin{cases} 0 & f_j^k \not\subset \partial f_i^{k+1} \\ 1 & f_j^k \subset \partial f_i^{k+1}, \text{induced orient. coincide} \\ -1 & f_j^k \subset \partial f_i^{k+1}, \text{induced orient. don't coincide} \end{cases} . \tag{2.49}$$

These incidence matrices can be used to find a recursive definition of the basis forms. For a k-simplex f^k and a $k-1$-simplex f^{k-1} with $f^{k-1} \subset \partial f^k$ let $\lambda^{f^k - f^{k-1}}$ denote the barycentric coordinate function associated to that vertex of f^k that is not in f^{k-1}. Then we have [12, Definition 23.1]:

$$\begin{aligned} b_{f_i^0} &= b_i^0 = \lambda_i, \\ b_{f_i^1} &= b_i^1 = \sum_{j=1}^{N_0} (\mathbf{D}_0)_{i,j}\, \mathsf{d}\, \lambda^{f_i^1 - f_j^0}\, \mathsf{d}\, b_j^0 = \sum_{f^0\in\Delta_0(\mathcal{T})} \mathbf{D}^{f^0}_{f_i^1} \lambda^{f_i^1 - f^0}\, \mathsf{d}\, b_{f^0}, \\ b_{f_i^2} &= b_i^2 = \sum_{j=1}^{N_1} (\mathbf{D}_1)_{i,j} \lambda^{f_i^2 - f_j^1}\, \mathsf{d}\, b_j^1 = \sum_{f^1\in\Delta_1(\mathcal{T})} \mathbf{D}^{f^1}_{f_i^2} \lambda^{f_i^2 - f^1}\, \mathsf{d}\, b_{f^1}, \\ b_{f_i^3} &= b_i^3 = \dots \end{aligned}$$

and in general

$$b_{f_i^k} = b_i^k = \sum_{j=1}^{N_{k-1}} (\mathbf{D}_k)_{i,j} \lambda^{f_i^k - f_j^{k-1}}\, \mathsf{d}\, b_j^{k-1} = \sum_{f^{k-1}\in\Delta_{k-1}(\mathcal{T})} \mathbf{D}^{f^{k-1}}_{f_i^k} \lambda^{f_i^k - f^{k-1}}\, \mathsf{d}\, b_{f^{k-1}}. \tag{2.50}$$

By the definition of the incidence matrices the summation over all $k-1$ simplices reduces to a summation over those $k-1$-simplices that are adjacent with k-simplex f_i^k, i.e (2.50) is a well determined formula. See Figure 2.1 for a sketch. In essence the lowest order differential forms $\mathcal{P}_1^-\Lambda^k(\mathcal{T})$ are represented by numbers associated to each k-simplex while exterior derivatives are represented as incidence matrices. This idea appears very frequently in the literature when it comes to discretization of problems formulated in terms of **grad**, **curl** or div-operators. Certain finite volume schemes [88], the mimetic finite differences [45,46], the cell method [54], the finite integration technique [20,79,86] and the discrete exterior calculus [24] are very closely related to Whitney forms, even though some of these methods are derived completely decoupled from the differential forms or the finite element framework.

2.2.2 Non-Conforming Discrete Differential Forms

Conforming finite element spaces for 0-forms exhibit global continuity. On the other hand there exist competitive Galerkin methods for scalar problems that are based on approximation spaces that do not enforce any kind of global continuity [2]. Such *Discontinuous Galerkin methods* have been successfully applied to various kinds of second order boundary value problems including source and eigenvalue problems for Maxwell [16,38,40].

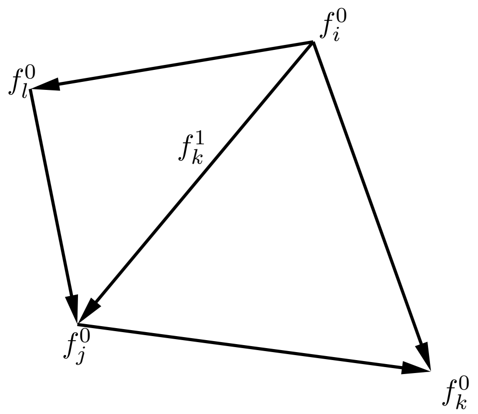

Figure 2.1: Sketch on recursive definition of Whitney basis forms: $b_{f_k^1} = \lambda_i \,\mathsf{d}\, \lambda_j - \lambda_j \,\mathsf{d}\, \lambda_i$ according to (2.48) or $b_{f_k^1} = b_{f_i^0} \,\mathsf{d}\, \lambda^{f_k^1 - f_i^0} - b_{f_j^0} \,\mathsf{d}\, \lambda^{f_k^1 - f_j^0}$ according to (2.50).

In light of these results we define non-conforming approximation spaces for differential forms.

Definition 2.2.9. *Let $r \geq 0$. The space of* non-conforming discrete differential forms *is*

$$\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}) := \{\omega \in L^2\Lambda^k(\Omega)\,,\, \omega_{|_T} \in \mathcal{P}_r\Lambda^k(T)\}.$$

In contrast to conforming discrete differential forms the case $r = 0$, i.e. piecewise constant approximation, is included here. As for the conforming spaces we have standard approximation estimates for non-conforming discrete differential forms as well:

Theorem 2.2.10. *Let $\mathcal{T}_h$ be a family of triangulations of Ω indexed by the discretization parameter*

$$h = \max_{T \in \mathcal{T}} \operatorname{diam} T.$$

We assume mesh regularity, i.e. there exists a constant $C_r > 0$ such that

$$|h|^n \leq C_r |T|, \quad \forall T \in \mathcal{T}_h. \tag{2.51}$$

Let $1 \leq p \leq \infty$ and $\frac{n-k}{p} < r + 1$. There is a constant C independent of h, such that

$$\inf_{\omega_h \in \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})} \|\omega - \Pi_h \omega\|_{L^2\Lambda^k(\Omega)} \leq C h^{\min(s, r+1)} \|\omega\|_{W^{s,p}\Lambda^k(\Omega)}\,.$$

Proof. The assertion follows directly from Theorem 2.2.5 and Corollary 2.2.7. ☐

Note that the elements of $\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ do not have a well-defined exterior derivative. Only its restriction to n-simplices is well-defined

$$(\mathsf{d}_{\mathcal{T}}\, \omega_h)_{|_T} := \mathsf{d}\, \omega_{h_{|_T}}. \tag{2.52}$$

Further, we get a commuting diagram for $r \geq 2$:

$$\begin{array}{ccc} \mathcal{P}_r\Lambda^k(\mathcal{T}) & \xrightarrow{\;\mathsf{d}\;} & \mathcal{P}_{r-1}\Lambda^{k+1}(\mathcal{T}) \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle i} \\ \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T}) & \xrightarrow{\;\mathsf{d}_{\mathcal{T}}\;} & \mathcal{P}_{r-1}^{\mathrm{d}}\Lambda^{k+1}(\mathcal{T}), \end{array}$$

where i denotes the inclusion map $\mathcal{P}_r\Lambda^k \mapsto \mathcal{P}_r^{\mathrm{d}}\Lambda^k$. Since $\mathcal{P}_r\Lambda^k(\mathcal{T})$ is subset of $\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ we can not use the cohomology groups of conforming discrete differential forms to characterize those of non-conforming discrete differential forms. Nevertheless we deduce $\mathsf{d}_{\mathcal{T}}\,\mathsf{d}_{\mathcal{T}} = 0$, which justifies the name *non-conforming discrete differential forms*. In Chapter 4, we define exterior derivatives of non-conforming discrete differential forms in the sense of distributions. Thereby the de Rham complex is lost but better approximation properties are attained.

2.2.3 Constraint Preserving Finite Volume Schemes

For the sake of completeness we attach here a discussion on so-called constraint preserving finite volume schemes. Such schemes are of great importance for the treatment of conservation laws.

We consider the following generic model problem for some time dependent k-form $\omega \in \Lambda^k(\Omega)$:

$$\partial_t\omega + \mathsf{d}\,g(\omega) = 0, \tag{2.53}$$

where g is a mapping $\Lambda^k(\Omega) \mapsto \Lambda^{k-1}(\Omega)$. For simplicity we consider here only the Cauchy problem and assume that $\omega(t)$ is compactly supported. In many applications, g is defined in a pointwise sense and for $k = n$ our model problem (2.53) is a standard conservation law, corresponding to

$$\partial_t\mathbf{u} + \operatorname{div}\mathbf{g}(\mathbf{u}) = 0.$$

We readily deduce that the evolution of the exterior derivative of the solution ω is constant:

$$\mathsf{d}\,\omega(t) = \mathsf{d}\,\omega(0) \quad \forall t. \tag{2.54}$$

Moreover, by the common assumption of compactly supported $\omega(t)$ and an application of the Leibniz rule (2.11), we get for any constant $c \in \mathcal{P}_0\Lambda^{n-k}(\mathbb{R}^n)$:

$$\int_{\mathbb{R}^n} \omega(t) \wedge c = \int_{\mathbb{R}^n} \omega(0) \wedge c \quad \forall t. \tag{2.55}$$

In $\mathbb{R}^3$ the property (2.54) corresponds either to the **curl** constraint ($k = 1$) or the div-constraint ($k = 2$). The property (2.55) corresponds to a preservation of total mass of each component of a vector representation of ω. Since this holds for any representation, i.e. choice of basis of $\mathbb{R}^3$, this is a global metric independent constraint.

Finding approximations to (2.53) that preserve both the constraints (2.55) and (2.54) has attracted considerable attention in the finite volume literature [23,28,44,59,62,84,85].

In classical finite volume schemes, both scalar and vector valued functions are represented by degrees of freedom associated to cell centers. While the preservation of total mass is the standard feature of finite volume schemes, the preservation of the constraint (2.54) requires sophisticated modifications of standard finite volume schemes. Finite volume schemes that preserve certain approximated values of the exterior derivative are called *constraint preserving finite volume schemes*.

Constraint Preserving Finite Volume Schemes on Cartesian Meshes

In order to get a rough idea on constraint preserving finite volume schemes the main principles of such schemes shall be sketched here. In contrast to earlier presentations, e.g. those in [59,62,84], we provide here a unifying framework, that allows for constraint preserving finite volumes schemes for k-forms in $\mathbb{R}^n$. An appropriate counterpart of the de Rham complex (2.13) is at the bottom of this unifying framework. To keep the presentation simple we will nevertheless stick to the case $n = 3$. But it will be clear that this framework extends straight forwardly to the general case.

We consider a uniform Cartesian mesh in $\mathbb{R}^3$ with mesh sizes Δx, Δy and Δz in x, y and z-directions respectively. It consists of cells $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}})\times[y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})\times[z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}})$, centered at mesh points $(x_i, y_j, z_k) = (i\Delta x, j\Delta y, k\Delta z), (i, j, k) \in \mathbb{Z}^3$. We introduce the averaging operators

$$\begin{aligned}
\mu_x a_{I,J,K} &:= \frac{a_{I+\frac{1}{2},J,K} + a_{I-\frac{1}{2},J,K}}{2},\\
\mu_y a_{I,J,K} &:= \frac{a_{I,J+\frac{1}{2},K} + a_{I,J-\frac{1}{2},K}}{2},\\
\mu_z a_{I,J,K} &:= \frac{a_{I,J,K+\frac{1}{2}} + a_{I,J,K-\frac{1}{2}}}{2}
\end{aligned} \tag{2.56}$$

and difference operators:

$$\begin{aligned}
\delta_x a_{I,J,K} &:= \frac{a_{I+\frac{1}{2},J,K} - a_{I-\frac{1}{2},J,K}}{\Delta x},\\
\delta_y a_{I,J,K} &:= \frac{a_{I,J+\frac{1}{2},K} - a_{I,J-\frac{1}{2},K}}{\Delta y},\\
\delta_z a_{I,J,K} &:= \frac{a_{I,J,K+\frac{1}{2}} - a_{I,J,K-\frac{1}{2}}}{\Delta z}.
\end{aligned} \tag{2.57}$$

With these we define the following approximation $\partial_{h,x}, \partial_{h,y}$ and $\partial_{h,z}$ to ∂_x, ∂_y and ∂_z:

$$\begin{aligned}
\partial_{h,x} &:= \delta_x \mu_y \mu_z,\\
\partial_{h,y} &:= \mu_x \delta_y \mu_z,\\
\partial_{h,z} &:= \mu_x \mu_y \delta_z.
\end{aligned}$$

Next, we need to introduce scalar and vectorial finite volume spaces with degrees of freedoms associated to cell centers (x_i, y_j, z_k) or vertices $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}, z_{k+\frac{1}{2}})$:

$$\begin{aligned}
S_c &:= \{u_{i,j,k} \in \mathbb{R},\ (i,j,k) \in \mathbb{Z}^3\},\\
S_v &:= \{u_{i,j,k} \in \mathbb{R},\ (i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}) \in \mathbb{Z}^3\},\\
V_c &:= \{\mathbf{u}_{i,j,k} = (u^1_{i,j,k}, u^2_{i,j,k}, u^3_{i,j,k}) \in \mathbb{R}^3,\ (i,j,k) \in \mathbb{Z}^3\},\\
V_v &:= \{\mathbf{u}_{i,j,k} = (u^1_{i,j,k}, u^2_{i,j,k}, u^3_{i,j,k}) \in \mathbb{R}^3,\ (i+\frac{1}{2}, j+\frac{1}{2}, k+\frac{1}{2}) \in \mathbb{Z}^3\}.
\end{aligned} \tag{2.58}$$

There are two important remarks on the previous definitions. First we see that

$$\begin{aligned}
\partial_{h,x}, \partial_{h,y}, \partial_{h,z} &: S_c \mapsto S_v,\\
\partial_{h,x}, \partial_{h,y}, \partial_{h,z} &: S_v \mapsto S_c
\end{aligned}$$

and second, we have a commuting property:

$$\begin{aligned}
\partial_{h,x}\partial_{h,y} &= \partial_{h,y}\partial_{h,x},\\
\partial_{h,z}\partial_{h,x} &= \partial_{h,x}\partial_{h,z},\\
\partial_{h,y}\partial_{h,z} &= \partial_{h,z}\partial_{h,y}.
\end{aligned}$$

Finally, we define the discrete counterparts of div, **curl** and **grad**:

$$\begin{aligned}
\operatorname{div}_h \mathbf{u}_{I,J,K} &:= \partial_{h,x} u^1_{I,J,K} + \partial_{h,y} u^2_{I,J,K} + \partial_{h,z} u^3_{I,J,K},\\
\mathbf{curl}_h\, \mathbf{u}_{I,J,K} &:= \begin{pmatrix} \partial_{h,y} u^3_{I,J,K} - \partial_{h,z} u^2_{I,J,K} \\ \partial_{h,z} u^1_{I,J,K} - \partial_{h,x} u^3_{I,J,K} \\ \partial_{h,x} u^2_{I,J,K} - \partial_{h,y} u^1_{I,J,K} \end{pmatrix},\\
\mathbf{grad}_h\, u_{I,J,K} &:= \begin{pmatrix} \partial_{h,x} u_{I,J,K} \\ \partial_{h,y} u_{I,J,K} \\ \partial_{h,z} u_{I,J,K} \end{pmatrix}.
\end{aligned} \tag{2.59}$$

With these we get the two cochain complexes:

$$S_c \overset{\mathbf{grad}_h}{\longrightarrow} V_v \overset{\mathbf{curl}_h}{\longrightarrow} V_c \overset{\operatorname{div}_h}{\longrightarrow} S_v \tag{2.60}$$

and

$$S_v \overset{\mathbf{grad}_h}{\longrightarrow} V_c \overset{\mathbf{curl}_h}{\longrightarrow} V_v \overset{\operatorname{div}_h}{\longrightarrow} S_c. \tag{2.61}$$

Analogue sequences can be found in $\mathbb{R}^2$. From the cochain complexes (2.60) and (2.61) we get the appropriate finite volume spaces and definitions of approximative exterior derivatives for the discretization of our generic model problem (2.53). If we approximate, e.g. in the case $k = 1$, ω by degrees of freedom associated to vertices (cell centers), we need to represent the discrete counterpart g_h of g by degrees of freedoms on the cell centers (vertices). Then, the quantity $\mathbf{curl}_h\, \omega_h$ is automatically preserved during the evolution. Note, that a proper definition of the discrete counterpart of g builds on a proper choice of so-called numerical flux functions [53], since the approximation of ω

has no well defined values at vertices (cell centers). Because we concentrate here on the preservation of constraints, we do not discuss numerical flux functions in detail and refer to [55–58] for an extensive study of numerical flux functions within the presented framework.

Averaging and difference operators (2.56) and (2.57) have been introduced in [62] for **curl**-preserving finite volume methods. In [59] this idea is extended to div-preserving schemes. Similar ideas appeared elsewhere [84]. A comprehensive treatment of constraint preserving finite volume schemes with average and difference operators seems to be limited to tensor-product meshes.

Discrete Differential Forms and Preservation of Total Mass

To overcome these limitations we move on to the discretization of our model problem (2.53) in terms of the Whitney forms introduced in Section 2.2.1. We stay here with the Whitney forms introduced for simplical triangulations. However, the results hold for any other triangulation, e.g. quadrilateral meshes for which one can define Whitney forms.

We consider a simplicial triangulation $\mathcal{T}$ and approximate ω and g with Whitney k- and $k-1$-forms $\omega_h \in \mathcal{P}_1^-\Lambda^k(\mathcal{T})$ and $g_h \in \mathcal{P}_1^-\Lambda^{k-1}(\mathcal{T})$. ω_h is non-smooth on $k-1$-simplices. Thus, to define g_h one has to adapt the idea of numerical fluxes [53] of finite volume schemes: g_h is an expansion in basis forms b_i^{k-1}:

$$g_h = \sum_{i=1}^{N_{k-1}} g_i b_i^{k-1}$$

with coefficients $g_i \in \mathbb{R}$, associated to $k-1$-simplices f_i^{k-1}. Ideally we would take g_i to be $\int_{f_i^{k-1}} g(\omega_h)$. But since $\int_{f_i^{k-1}} g(\omega_h)$ is not well-defined, one defines the coefficients g_i as functions of the values $\int_{f_i^{k-1}} g(\omega_{h_{|f_j^n}})$ instead:

$$g_i = g_i\left(\int_{f_i^{k-1}} g(\omega_{h_{|f_1^n}}), \ldots \int_{f_i^{k-1}} g(\omega_{h_{|f_{N_n}^n}})\right),$$

which indeed are well defined. For pointwise defined g the value $\int_{f_i^{k-1}} g(\omega_{h_{|f_j^n}})$ is non-zero only if n-simplex f_j^n is adjacent with f_i^{k-1}. For $k = n$ this gives the standard finite volume scheme with numerical flux functions g_i associates to $n-1$ simplices f_j^{n-1}. A clever choice of flux functions g_i for given g is subject to intensive research and shall not discussed in detail here.

Finally we end up with the discrete problem for $\omega_h \in \mathcal{P}_1^-\Lambda^k(\mathcal{T})$:

$$\partial_t\omega_h + \mathsf{d}\, g_h(\omega_h) = 0. \tag{2.62}$$

The solution ω_h to the discrete problem (2.62) has the property $\mathsf{d}\,\omega_h(t) = \mathsf{d}\,\omega_h(0)$ since $\mathcal{P}_1^-\Lambda^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ and $\mathsf{d}\,\mathsf{d} = 0$. To show that also the total mass is preserved we prove the following Lemma.

Lemma 2.2.11. *We consider simplicial triangulations $\mathcal{T}$ of $\Omega \subset \mathbb{R}^n$. T denotes n-simplices, s is a fixed $k-1$-simplex with $k \leq n$ and $s \not\subset \partial\Omega$. Let b_s denote the Whitney basis-form* (2.48) *associated to s, then for all $c \in \mathcal{P}_0\Lambda^k(\mathbb{R}^n)$*

$$\sum_T \int_T \mathsf{d}\, b_s \wedge c = 0.$$

Proof. Let $\Omega_s = \overline{\cup_{T:s\in\Delta_{k-1}(T)} T}$ denote the union of all n-simplices that share the $k-1$-simplex s. Then we deduce:

$$\begin{aligned}\sum_T \int_T \mathsf{d}\, b_s \wedge c &\overset{1.}{=} \sum_{T\,:\,s\in\Delta_{k-1}(T)} \int_T \mathsf{d}\, b_s \wedge c \\ &\overset{2.}{=} \sum_{T\,:\,s\in\Delta_{k-1}(T)} \int_{\partial T} \operatorname{tr}_{\partial T}(b_s \wedge c) \\ &\overset{3.}{=} 0.\end{aligned}$$

See Figure 2.2 for an illustration of the three steps.

1. The support of $\mathsf{d}\, b_s$ consists of all n-simplices T with $s \in \Delta_{k-1}(T)$.
2. Leibniz rule (2.11) on each n-simplex T and $\mathsf{d}\, c = 0$.
3. The sum over all n-simplices T can be written as a sum over all $(n-1)$-simplices f with $f \subset \Omega_s$.
   - $f \subset \partial\Omega_s$: For those facets f with $f \subset \partial\Omega_s$ we have $\operatorname{tr}_f b_s = 0$, since for Whitney basis forms $\operatorname{tr}_f b_s = 0$ for all f that do not contain s. This follows because Ω_s is the support of b_s and tr_f is single valued for Whitney $k-1$-forms, $k \leq n$.
   - $f \subset \Omega, f \not\subset \partial\Omega_s$: The integrals over facets f with $f \subset \Omega_s$, $f \not\subset \partial\Omega$ appear twice with different signs, since the induced orientations are different for the two adjacent elements. But since $\operatorname{tr}_f(b_s \wedge c) = \operatorname{tr}_f b_s \wedge \operatorname{tr}_f c$ is single valued by the continuity of $\operatorname{tr}_f b_s$ and c these integrals vanish as well.

□

Remark 2.2.12. *By the geometrical decomposition of the dual space of the space of local high order finite element differential forms [3, page 53–54] we have more generally:*

$$\sum_T \int_T \mathsf{d}\, b_s \wedge c = 0,$$

for all local basis forms $b_s \in \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $b_s \in \mathcal{P}_r^-\Lambda^k(\mathcal{T})$ associated to simplex s and $c \in \mathcal{P}_0^{\mathrm{d}}\Lambda^{n-k}(\mathbb{R}^n)$.

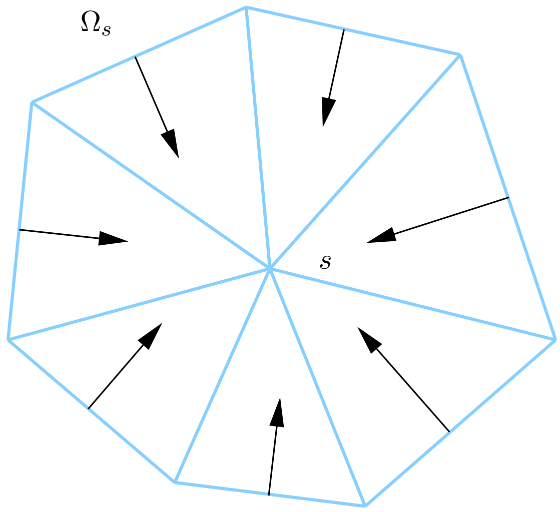

Figure 2.2: Illustration for the proof of Lemma 2.2.11. In this situation, $k = 1$ and $n = 2$, the $k-1$-simplex s is a vertex and b_s is the barycentric coordinate function associated to vertex s. The gradients in the surrounding elements scaled by the volumes of the elements sum up to zero. The gradients are depicted with black arrows.

As a corollary we get mass preservation for the Whitney form discretization (2.62). We only need to show

$$\sum_T \int_T \mathsf{d}\, g_h(\omega_h) \wedge c = 0,$$

for arbitrary constant $n-k$ forms $c \in \mathcal{P}_0\Lambda^{n-k}(\mathbb{R}^n)$. By the definition of g_h we have the expansion $g_h(\omega_h) = \sum_{s\in\Delta_{k-1}(\mathcal{T})} a_s b_s$ with real valued coefficients $a_s = a_s(\omega_h, g)$. Then we see immediately

$$\sum_T \int_T \mathsf{d}\, g_h(\omega_h) \wedge c = \sum_T \int_T \sum_s a_s \,\mathsf{d}\, b_s \wedge c = \sum_s a_s \sum_T \int_T \mathsf{d}\, b_s \wedge c = 0.$$

Since only a Cauchy problem has been considered we do not encounter the case $s \in \partial\Omega$. For boundary value problems we will obtain preservation of mass modulo an in- and outflow flux of mass across the boundary.

Remark 2.2.13. *The proof of Lemma* (2.2.11) *uses canonical properties of lowest order Whitney forms. Therefore the assertion on automatic mass preservation of the discretization of* (2.53) *with Whitney forms holds also for other than simplicial triangulations.*

In light of these results, a discretization of (2.53) in terms of conforming differential forms seems to be the method of choice when it comes to constraint preservation.

Obviously, a big drawback is the non-standard degrees of freedom, that are difficult to implement in usual finite volume codes.

Constraint Preserving Finite Volume Schemes and Discrete Differential Forms

Now, we will explain how to deduce finite volume schemes from our Whitney forms discretization (2.62), that allow for a notion of exterior derivatives. This gives finite volume schemes preserving an approximation of the constraint (2.54).

In light of remark 2.2.13, here and in the subsequent discussion, T stands for finite volume cells of the triangulation $\mathcal{T}$. On simplicial meshes cells are the n-simplices. Our discrete differential forms spaces $\mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ are the counterparts to the cell centered finite volume spaces introduced in (2.58).

Let $A_h : \Lambda^k(\Omega) \mapsto \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ denote the standard finite volume averaging operator that assigns a piecewise discontinuous differential form $\widetilde{\omega}_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ to each differential form ω such that $\int_T \widetilde{\omega}_h := \int_T A_h\omega = \int_T \omega$, for all cells T. Applying the averaging operator A_h to both sides of equation (2.62), yields the following scheme for $\omega_h \in \mathcal{P}_1^-\Lambda^k(\mathcal{T})$:

$$\partial_t\widetilde{\omega}_h + A_h \,\mathsf{d}\, g_h(\omega_h) = 0.$$

If we assume further, that the numerical flux $g_h \in \mathcal{P}_1^-\Lambda^{k-1}(\mathcal{T})$ is not a function of $\omega_h \in \mathcal{P}_1^-\Lambda^{k-1}(\mathcal{T})$ but a function of $\widetilde{\omega}_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$, i.e. $g_h(\omega_h) = g_h(\widetilde{\omega}_h)$, we get a finite volume scheme for $\widetilde{\omega}_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ formulated entirely in terms of cell centered unknowns:

$$\partial_t\widetilde{\omega}_h + A_h \,\mathsf{d}\, g_h(\widetilde{\omega}_h) = 0. \tag{2.63}$$

Clearly, the averaging procedure does not destroy the preservation of constraint (2.55): the scheme (2.63) preserves the total mass of the solution $\widetilde{\omega}_h$. Moreover, we can find approximations d_h of exterior derivatives d such that even $\mathsf{d}_h\,\widetilde{\omega}_h(t)$ is constant: We call an operator d_h defined on $\mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ *approximative exterior derivative*, if there exists a linear operator C_h defined on $\mathcal{P}_1^-\Lambda^{k+1}(\mathcal{T})$ such that:

$$\mathsf{d}_h\, A_h\omega_h = C_h\,\mathsf{d}\,\omega_h, \quad \forall\omega_h \in \mathcal{P}_1^-\Lambda^k(\mathcal{T}). \tag{2.64}$$

For $\widetilde{\omega}_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ solving (2.63) we deduce that $\mathsf{d}_h\,\widetilde{\omega}_h(t)$ is constant in time, since

$$\mathsf{d}_h\, A_h\,\mathsf{d}\, g_h(\widetilde{\omega}_h) = C_h\,\mathsf{d}\,\mathsf{d}\, g_h(\widetilde{\omega}_h) = 0. \tag{2.65}$$

If there exists a right inverse $A_h^+ : \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T}) \mapsto \mathcal{P}_0^-\Lambda^k(\mathcal{T})$ of A_h, i.e. $A_hA_h^+ = \mathsf{id}$, then we have for $\widetilde{\omega}_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k$

$$\mathsf{d}_h\,\widetilde{\omega}_h = C_h\,\mathsf{d}\,A_h^+\widetilde{\omega}_h. \tag{2.66}$$

The right inverse A_h^+ permits the identity $A_hA_h^+A_h = A_h$, hence A_h^+ is a mass preserving reconstruction operator:

$$\int_T \widetilde{\omega}_h = \int_T A_h\widetilde{\omega}_h = \int_T A_hA_h^+A_h\widetilde{\omega}_h = \int_T A_h^+\widetilde{\omega}_h.$$

From that perspective the representation (2.66) reads as: *The approximative exterior derivative d_h of $\widetilde{\omega}_h$ is a linear operator of the well defined exterior derivative of a reconstruction of $\widetilde{\omega}_h$ in the space $\mathcal{P}_1^- \Lambda^k(\mathcal{T})$.* How such approximative exterior derivatives can be actually constructed is indicated by (2.66). Provided that a right inverse of A_h^+ is found, we could choose any C_h, define d_h by (2.66) and try to show that (2.64) holds true. Note that this procedure simplifies a lot, if we restrict the definition of d_h to discrete forms on submeshes. Since $\mathsf{d}\,\omega_h$ is piecewise constant, a submesh should contain at least two elements.

Let us illustrate this last idea for structured Cartesian and unstructured triangular meshes.

1. **Discrete divergence on Cartesian meshes**
   Assume a Cartesian mesh in $\mathbb{R}^2$. The submesh for which we define an approximative exterior derivative consists of the four cells sharing the same vertex. The submesh and the numbering schemes are depicted in figure 2.3. The degrees of

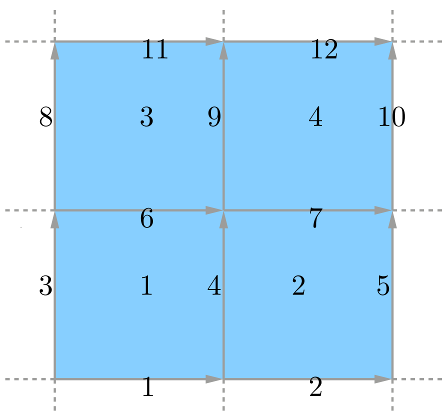

Figure 2.3: Numbering of edges and cells.

freedom of discrete differential 2-forms are associated to the cells, those of 1-forms to the edges. The matrix representation $\mathbf{D}_2$ of div reads:

$$\mathbf{D}_2 = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & -1 \end{pmatrix}.$$

The degrees of freedom of averaged 1-forms are associated to the cells as well. In here we first list all the second components sorted according to the corresponding square number and then the first components. The matrix $\mathbf{A}$ representing the averaging A_h is:

$$\mathbf{A} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

From this, one computes a right inverse:

$$\mathbf{A}^{+} = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1} = \frac{1}{3}\begin{pmatrix} -4 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 4 & 0 & 0 \\ -2 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 4 \\ 2 & 0 & -4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The first column says that a discontinuous function that has vanishing first component everywhere and non-vanishing second component only on the first cell is reconstructed as discrete differential form that has non-vanishing first components on the first and third cell. Nevertheless the averages of the reconstruction are the same. Next, we compute the matrix representation of $\operatorname{div} A_h^+$:

$$\mathbf{D}_2\mathbf{A}^{+} = \frac{1}{3}\begin{pmatrix} -2 & 0 & 4 & 0 & -2 & 4 & 0 & 0 \\ 0 & -2 & 0 & 4 & -4 & 2 & 0 & 0 \\ -4 & 0 & 2 & 0 & 0 & 0 & -2 & 4 \\ 0 & -4 & 0 & 2 & 0 & 0 & -4 & 2 \end{pmatrix}.$$

Recall that the ith line corresponds to the divergence on the ith square. The divergence on the 1st line for example is a finite difference stencil using the second components on cells 1 and 3 and the first components on 1 and 2. It is now very natural to assign the average of the four divergence values to the vertex that is shared by all four cells, i.e. we propose a C_h with the following matrix representation:

$$\mathbf{C} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}.$$

Next, we define the matrix representation of d_h:

$$\mathbf{D}_h = \mathbf{C}\mathbf{D}_2\mathbf{A}^+ = \frac{1}{2}\begin{pmatrix} -1 & -1 & 1 & 1 & -1 & 1 & -1 & 1 \end{pmatrix}$$

and check that indeed

$$\mathbf{D}_h\mathbf{A} = \mathbf{C}\mathbf{D}_2.$$

We want to emphasize, that this is exactly the same stencil we would get in terms of averaging and difference operators in (2.59). Such calculations can be done in principle for any kind of mesh, once we have finite element like discrete differential forms on such meshes.

In the general case of unstructured grids the explicit computation of the right inverse A_h^+ does not seem to be very satisfactory albeit realizable.

2. **Discrete divergence on triangular meshes**
Let T, f and e denote the oriented n-, $n-1$-, and $n-2$-simplices (cells, faces, edges) of a simplicial mesh. In light of the previous example we would like to define approximative exterior derivatives d_h that assign values to vertices on the basis of the discrete form values on surrounding cells. For a fixed vertex v we therefore consider C_h to be the average value over all cells that share v, i.e.

$$C_h \text{ div } \omega_h := \frac{\sum_{T\,:\,v\in\Delta(T)} \int_T \operatorname{div}\omega_h}{\sum_{T\,:\,v\in\Delta(T)} |T|}, \quad \omega_h \in \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T}).$$

Then Gauss law and normal continuity of $\omega_h \in \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T})$ give

$$C_h \text{ div } \omega_h = \frac{\sum_{T\,:\,v\in\Delta(T)} \mathbf{D}_T^{f_{t,v}} \int_{f_{T,v}} \omega_{h|T}}{\sum_{T\,:\,v\in\Delta(T)} |T|}, \tag{2.67}$$

where $f_{T,v}$ is that face of T that is opposite to vertex v and $\mathbf{D}_T^{f_{T,v}}$ the incidence matrix of cells and faces (2.49). $|T|$ is the volume of n-simplex T. We observe that the right hand side of (2.67) is also well-defined for $\omega_h \in \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$. However, because of (2.48) and the midpoint quadrature rule

$$\int_{f_{\widetilde{T},v}} \left(A_h b_{f_{T,v}}\right)_{|\widetilde{T}} = \begin{cases} \frac{n}{n+1} & T = \widetilde{T} \\ 0 & T \neq \widetilde{T} \end{cases}$$

this suggests to define

$$\operatorname{div}_h \widetilde{\omega}_h := \frac{n+1}{n\sum_{T\,:\,v\in\Delta(T)} |T|} \sum_{T\,:v\,\in\Delta(T)} \mathbf{D}_T^{f_{T,v}} \int_{f_{T,v}} \widetilde{\omega}_{h|T}.$$

So far, div_h is just a candidate definition for the discrete divergence. It remains to prove

$$\operatorname{div}_h A_h\omega_h = C_h \text{ div } \omega_h, \quad \forall \omega_h \in \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T}).$$

The crucial step is the following Lemma.

Lemma 2.2.14. *For a fixed vertex v and $n-1$-form $\omega_h \in \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T})$:*

$$\sum_{T\,:\,v\in\Delta(T)} \int_T \operatorname{div}\,\omega_h = \frac{n+1}{n} \sum_{T\,:\,v\in\Delta(T)} \mathbf{D}_T^{f_{T,v}} \int_{f_{T,v}} A_h \omega_{h|T}$$

Proof. Let x_T denote the barycenter of a n-simplex T, then $A_h\omega_{h|T} = \omega_h(x_T)\chi_T$, with χ_T the characteristic function of T. The discrete $n-1$-form $\omega_h \in \mathcal{P}_1^-\Lambda^{n-1}(\mathcal{T})$ is an expansion in basis forms b_f associated to $n-1$-simplices f with coefficients $(\omega_h)_f$:

$$\omega_h = \sum_f (\omega_h)_f b_f.$$

For barycentric coordinate functions λ^{f-e} associated to the vertex in f that is opposite to e we can write

$$b_f = \sum_e \mathbf{D}_f^e \lambda^{f-e}\,\mathsf{d}\,b_e,$$

with incidence matrix $\mathbf{D}_f^e$ of edges and faces and basis forms b_e associated to edges e (2.50). Then we compute:

$$\begin{aligned}
\int_{f_{T,v}} \chi_T \omega_h(x_T) &= \sum_{f\in\Delta(T)} (\omega_h)_f \int_{f_{T,v}} \chi_T b_f(x_T) \\
&= (\omega_h)_{f_{T,v}} \frac{n}{n+1} + \sum_{f\in\Delta(T), f\neq f_{T,v}} (\omega_h)_f \int_{f_{T,v}} \chi_T b_f(x_T) \\
&= (\omega_h)_{f_{T,v}} \frac{n}{n+1} + \sum_{f\in\Delta(T), f\neq f_{T,v}} (\omega_h)_f \sum_e \int_{f_{T,v}} \frac{1}{n+1} \mathbf{D}_f^e\,\mathsf{d}\,b_e \\
&= (\omega_h)_{f_{T,v}} \frac{n}{n+1} + \sum_{f\in\Delta(T), f\neq f_{T,v}} (\omega_h)_f \sum_e \frac{1}{n+1} \mathbf{D}_{f_{T,v}}^e \mathbf{D}_f^e.
\end{aligned}$$

For each $n-1$-simplex f in the last sum exists only one $n-2$-simplex e such that $\mathbf{D}_{f_{T,v}}^e \mathbf{D}_f^e \neq 0$. Further, if we multiply this equality with $\mathbf{D}_T^{f_{T,v}}$ and sum over all n-simplices T sharing vertex v, there is exactly one other n-simplex $\widetilde{T}$ that shares both f and e, i.e. $\mathbf{D}_{f_{\widetilde{T},v}}^e \mathbf{D}_f^e \neq 0$. But for these we deduce (see Figure 2.4).

$$\begin{aligned}
\left(\mathbf{D}_T^{f_{T,v}} \mathbf{D}_{f_{T,v}}^e + \mathbf{D}_{\widetilde{T}}^{f_{\widetilde{T},v}} \mathbf{D}_{f_{\widetilde{T},v}}^e\right) &= \left(\mathbf{D}_T^{f_{T,v}} \mathbf{D}_{f_{T,v}}^e + \mathbf{D}_{\widetilde{T}}^{f_{\widetilde{T},v}} \mathbf{D}_{f_{\widetilde{T},v}}^e\right) + \left(\mathbf{D}_T^f + \mathbf{D}_{\widetilde{T}}^f\right) \mathbf{D}_f^e \\
&= \left(\mathbf{D}_T^{f_{T,v}} \mathbf{D}_{f_{T,v}}^e + \mathbf{D}_T^f \mathbf{D}_f^e\right) + \left(\mathbf{D}_{\widetilde{T}}^{f_{\widetilde{T},v}} \mathbf{D}_{f_{\widetilde{T},v}}^e + \mathbf{D}_{\widetilde{T}}^f \mathbf{D}_f^e\right) \\
&= \sum_f \mathbf{D}_T^f \mathbf{D}_f^e + \sum_f \mathbf{D}_{\widetilde{T}}^f \mathbf{D}_f^e = 0,
\end{aligned}$$

where we used first that positive orientation of n-simplices T implies $\left(\mathbf{D}_T^f + \mathbf{D}_{\widetilde{T}}^f\right) = 0$ and second that $\mathsf{d} \circ \mathsf{d} = 0$ and that any a $n-2$ subsimplex e of an n-simplex T is subsimplex of exactly two $n-1$ subsimlices of T. That gives:

$$\sum_{T\,:\,v \in \Delta(T)} \mathbf{D}_T^{f_{T,v}} \int_{f_{T,v}} \chi_T \omega_h(x_T) = \frac{n}{n+1} \sum_{T\,:\,v \in \Delta(T)} \mathbf{D}_T^{f_{T,v}} (\omega_h)_{f_{T,v}}.$$

Hence we can conclude:

$$\begin{aligned}\sum_{T\,:\,v \in \Delta(T)} \int_T \operatorname{div}\, \omega_h &= \sum_{T\,:\,v \in \Delta(T)} \mathbf{D}_T^{f_{T,v}} (\omega_h)_{f_{T,v}} \\ &= \frac{n+1}{n} \sum_{T\,:\,v \in \Delta(T)} \mathbf{D}_T^{f_{T,v}} \int_{f_{T,v}} A_h \omega_h.\end{aligned}$$

□

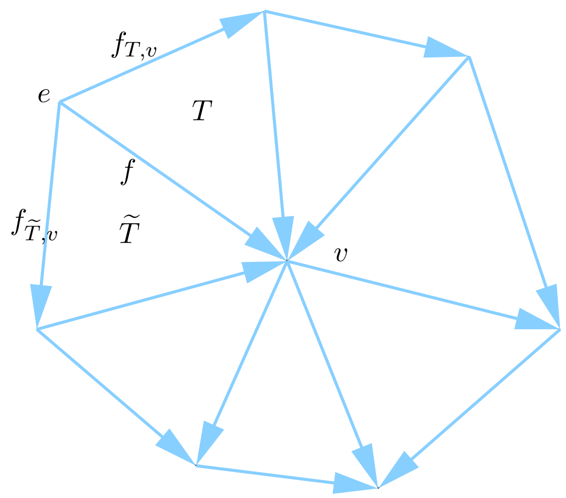

Figure 2.4: Illustration for the proof of Lemma 2.2.14.

3 Generalized Advection-Diffusion Problem

3.1 Introduction

In this chapter we introduce the generalized advection-diffusion problem for differential forms and show that the eddy current model in moving conductors can be formulated within this framework. We also illustrate the main challenges on approximation of advection-diffusion problems for the example of the well studied scalar advection-diffusion problems. In light of our motivating example of the eddy current problem with high magnetic Reynolds number we focus on the case of small diffusion, or equivalently large advection. We conclude that viable numerical methods for such singular perturbed problems build on practical numerical methods for the limiting case. We derive adequate boundary conditions and show well-posedness in appropriate spaces. The following chapters will deal with different discretization strategies of such advection problems.

Recall the two classical linear transient 2nd-order advection-diffusion problems for an unknown scalar function $u = u(x,t)$ on a bounded domain $\Omega \subset \mathbb{R}^n$:

$$\begin{aligned} \partial_t u - \varepsilon \operatorname{div} \mathbf{grad}\, u + \boldsymbol{\beta} \cdot \mathbf{grad}\, u &= f && \text{in } \Omega\,, \\ u &= g_D && \text{on } \Gamma_D\,, \\ \mathbf{n}_\Omega \cdot \mathbf{grad}\, u &= g_N && \text{on } \Gamma_N\,, \\ u(\cdot,0) &= u_0 && \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \partial_t u - \varepsilon \operatorname{div} \mathbf{grad}\, u + \operatorname{div}(u\,\boldsymbol{\beta}) &= f && \text{in } \Omega\,, \\ u &= g_D && \text{on } \Gamma_D\,, \\ \mathbf{n}_\Omega \cdot \mathbf{grad}\, u &= g_N && \text{on } \Gamma_N\,, \\ u(\cdot,0) &= u_0\,. && \end{aligned} \tag{3.2}$$

The non-negative parameter $\varepsilon \in \mathbb{R}$ will be called diffusion constant, $\boldsymbol{\beta} : \overline{\Omega} \mapsto \mathbb{R}^n$ stands for a given smooth vector field and $f : \Omega \mapsto \mathbb{R}$ is a given source function. For $\operatorname{div} \boldsymbol{\beta} = 0$ problems (3.1) and (3.2) coincide. The boundary splits into two disjoint parts $\Gamma_N \cup \Gamma_D = \partial\Omega$, with $\Gamma_N \cap \Gamma_D = \{\,\}$. g_D and g_N are the boundary data on Γ_D and Γ_N. Note that the advection operators $\boldsymbol{\beta} \cdot \mathbf{grad}\, u$ in (3.1) and $\operatorname{div}(u\,\boldsymbol{\beta})$ in (3.2) are the vector representation of Lie derivatives of 0-forms and n-forms (Table 2.3).

Solving the advection-diffusion problems numerically is usually challenging in the case of dominant advection, because we encounter a singular perturbation. In the limit of vanishing diffusion the problem type changes from parabolic to hyperbolic and the standard methods for parabolic problems usually fail. We refer to [76] and the many references cited therein for an overview of numerical methods for the stationary singularly perturbed advection-diffusion problems.

It is the notion of the *Lie derivative* (2.24) that permits generalizations of scalar advection operators to differential forms. The generalized advection-diffusion problems for time dependent differential forms are:

$$\begin{aligned} \partial_t\omega(t) + \varepsilon\,\delta\,\mathsf{d}\,\omega(t) + \gamma_1\,\mathsf{d}\,\delta\,\omega(t) + \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega(t) + \gamma_2\,\mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega(t) &= \varphi(t) && \text{in } \Omega \subset \mathbb{R}^n, \\ \operatorname{tr}\omega &= \psi_D && \text{on } \Gamma_D\,, \\ \operatorname{tr}(\star\,\mathsf{d}\,\omega) &= \psi_N && \text{on } \Gamma_N\,, \\ \omega(0) &= \omega_0 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \partial_t\omega(t) + \varepsilon\,\delta\,\mathsf{d}\,\omega(t) + \gamma_1\,\mathsf{d}\,\delta\,\omega(t) - \delta\,\mathsf{j}_{\boldsymbol{\beta}}\,\omega(t) - \gamma_2\,\mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\omega(t) &= \varphi(t) && \text{in } \Omega \subset \mathbb{R}^n, \\ \operatorname{tr}\omega &= \psi_D && \text{on } \Gamma_D\,, \\ \operatorname{tr}(\star\,\mathsf{d}\,\omega) &= \psi_N && \text{on } \Gamma_N\,, \\ \omega(0) &= \omega_0\,. \end{aligned} \tag{3.4}$$

These are equations for an unknown time dependent k-form $\omega(t) \in \Lambda^k(\Omega)$, $0 \le k \le n$, on the domain $\Omega \subset \mathbb{R}^n$. d and $\mathsf{i}_{\boldsymbol{\beta}}$ are the exterior derivative (2.9) and contraction operator (2.23). δ and $\mathsf{j}_{\boldsymbol{\beta}}$ are there formal adjoints (2.31) and (2.32) and $\star$ is the Hodge operator (2.3). The source term φ is a $n-k$-form. The boundary data ψ_D and ψ_N are in $\Lambda^k(\Gamma_D)$ and $\Lambda^{n-k-1}(\Gamma_N)$. γ_1 and γ_2 are non-negative scalar parameters. Depending on the choice of γ_1 and γ_2 well-posedness will require additional boundary conditions.

Equations (3.3) and (3.4) are identical if $\gamma_2 = 1$ and $\mathsf{L}_{\boldsymbol{\beta}}\,\omega + \mathcal{L}_{\boldsymbol{\beta}}\,\omega = 0$, compare (2.33). In Tables 2.2 and 2.3 we listed the operators corresponding to $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ in $\mathbb{R}^3$ and $\mathbb{R}^2$. Clearly, since $\delta\,\omega = 0$ for $\omega \in \Lambda^0(\Omega)$ problems (3.1) and (3.2) are generalizations of problems (3.3) and (3.4). Before we proceed in discussing well-posedness of certain variational formulations of problems (3.3) and (3.4) we show that the *magnetoquasistatic electrodynamic equations in moving conductors* can be formulated as (3.3) and (3.4).

3.2 Magnetoquasistatic Electrodynamic Equations in Moving Conductors

We consider $\Omega \subset \mathbb{R}^3$ and Maxwell's system in the magnetoquasistatic approximation. This reduced model, also called eddy current model, is a system of equations for the magnetic field $h \in \Lambda^1(\Omega)$, the electric field $e \in \Lambda^1(\Omega)$, the magnetic induction $b \in \Lambda^2(\Omega)$, the current density $j \in \Lambda^2(\Omega)$ and imposed current density $f \in \Lambda^2(\Omega)$:

$$\mathsf{d}\,e = -\partial_t b \qquad \mathbf{curl}\,\mathbf{E} = -\partial_t\mathbf{B} \qquad \text{in } \Omega\,, \tag{3.5a}$$

$$\mathsf{d}\,h = j + f \qquad \mathbf{curl}\,\mathbf{H} = \mathbf{J} + \mathbf{F} \qquad \text{in } \Omega\,, \tag{3.5b}$$

$$j = \star_\sigma(e - \mathsf{i}_{\boldsymbol{\beta}}\,b) \qquad \mathbf{J} = \sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \qquad \text{in } \Omega\,, \tag{3.5c}$$

$$\star_\mu h = b \qquad \mu\mathbf{H} = \mathbf{B} \qquad \text{in } \Omega\,. \tag{3.5d}$$

Commonly the boundary $\partial\Omega$ splits into two disjoint parts $\Gamma_e \cup \Gamma_h = \partial\Omega$, $\Gamma_e \cap \Gamma_h = \{\}$ and one imposes the boundary conditions

$$\begin{aligned} \operatorname{tr} e &= g_e \quad \mathbf{n}\times\mathbf{E} = \mathbf{G}_e \quad \text{on } \Gamma_e,\\ \operatorname{tr} h &= g_h \quad \mathbf{n}\times\mathbf{H} = \mathbf{G}_h \quad \text{on } \Gamma_h. \end{aligned} \tag{3.6}$$

By (3.5a)-(3.5d) these conditions imply

$$\begin{aligned} \operatorname{tr}(\star_{\sigma^{-1}} \mathsf{d}\, h) + \operatorname{tr}(\mathsf{i}_{\boldsymbol\beta}\, b) &= g_e + \operatorname{tr}(\star_{\sigma^{-1}} f) \quad \text{on } \Gamma_e,\\ \operatorname{tr}(\star_{\mu^{-1}} \mathsf{d}\, e) &= -\partial_t g_h \quad \text{on } \Gamma_h. \end{aligned}$$

For simplicity we assume in the sequel that the Hodge operator $\star_\sigma$ and $\star_\mu$ encodes linear scalar material laws, i.e.

$$\int_\Omega e \wedge \star_\sigma e' = \left(e, \sigma\, e'\right)_\Omega$$

for a uniformly positive $\sigma : \Omega \mapsto \mathbb{R}$. Eliminating in system (3.5a)-(3.5d) all fields except h yields:

$$\begin{aligned} \partial_t \mu h + \delta\, \sigma^{-1} \mathsf{d}\, h - \delta\, \mathsf{j}_{\boldsymbol\beta}\, \mu h &= \delta\, \sigma^{-1} f \quad &&\text{in } \Omega,\\ \operatorname{tr}(\sigma^{-1} \star \mathsf{d}\, h) - \operatorname{tr}(\star \mathsf{j}_{\boldsymbol\beta}\, \mu h) &= g_e + \operatorname{tr} \sigma^{-1} \star f \quad &&\text{on } \Gamma_e,\\ \operatorname{tr} h &= g_h \quad &&\text{on } \Gamma_h, \end{aligned} \tag{3.7}$$

where we used that $\mathsf{i}_{\boldsymbol\beta}\, b = \mathsf{i}_{\boldsymbol\beta} \star \mu h = - \star \mathsf{j}_{\boldsymbol\beta}\, \mu h$ by (2.32). Solenoidal initial magnetic induction $\mathsf{d}\, b(0) = 0$ implies $\mathsf{d}\, b(t) = 0$ by (3.5a). Hence a solution of the system (3.5a)-(3.5d) with solenoidal initial magnetic induction $\mathsf{d}\, b(0) = 0$ is also a solution to the problem:

$$\begin{aligned} \partial_t \mu h + \delta\, \sigma^{-1} \mathsf{d}\, h + \gamma_1\, \mathsf{d}\, \sigma^{-1}\, \delta\, h - \delta\, \mathsf{j}_{\boldsymbol\beta}\, \mu h - \gamma_2\, \mathsf{j}_{\boldsymbol\beta}\, \delta\, \mu h &= \delta\, \sigma^{-1} f \quad &&\text{in } \Omega,\\ \operatorname{tr}(\sigma^{-1} \star \mathsf{d}\, h) - \operatorname{tr}(\star \mathsf{j}_{\boldsymbol\beta}\, \mu h) &= g_e + \operatorname{tr} \sigma^{-1} \star f \quad &&\text{on } \Gamma_e,\\ \operatorname{tr} h &= g_h \quad &&\text{on } \Gamma_h, \end{aligned} \tag{3.8}$$

which is a problem of type (3.4) for the 1-form h.

On the other hand we could introduce a vector potential $a \in \Lambda^1(\Omega)$ and scalar potential $\phi \in \Lambda^0(\Omega)$ with $b = \mathsf{d}\, a$ and $e = -\partial_t a - \mathsf{d}\, \phi$. Eliminating all fields except a and ϕ yields:

$$\begin{aligned} \partial_t a + \sigma^{-1}\, \delta\, \mu^{-1}\, \mathsf{d}\, a + \mathsf{i}_{\boldsymbol\beta}\, \mathsf{d}\, a + \mathsf{d}\, \phi &= \sigma^{-1} \star f \quad &&\text{in } \Omega,\\ -\operatorname{tr}(\partial_t a + \mathsf{d}\, \phi) &= g_e \quad &&\text{on } \Gamma_e,\\ \operatorname{tr}(\star \mu^{-1}\, \mathsf{d}\, a) &= g_h \quad &&\text{on } \Gamma_h. \end{aligned} \tag{3.9}$$

But since for arbitrary $\lambda \in \Lambda^0(\Omega)$ also $a' = a + \mathsf{d}\, \lambda$ and $\phi' = \phi - \partial_t \lambda$ yield $\mathsf{d}\, a' = b$ and $-\partial_t a' - \mathsf{d}\, \phi' = e$ we could choose here $\phi = \gamma_1 \mu^{-1}\, \delta\, a + \gamma_2\, \mathsf{i}_{\boldsymbol\beta}\, a$: if $\phi \neq \gamma_1 \mu^{-1}\, \delta\, a + \gamma_2\, \mathsf{i}_{\boldsymbol\beta}\, a$ we could find λ solving the scalar advection-diffusion problem:

$$\partial_t \lambda + \gamma_1 \mu^{-1}\, \delta\, \mathsf{d}\, \lambda + \gamma_2\, \mathsf{i}_{\boldsymbol\beta}\, \mathsf{d}\, \lambda = \phi - \gamma_1 \mu^{-1}\, \delta\, a - \gamma_2\, \mathsf{i}_{\boldsymbol\beta}\, a \tag{3.10}$$

and get $\phi' = \gamma_1 \mu^{-1}\,\delta\, a' + \gamma_2\, \mathsf{i}_{\boldsymbol{\beta}}\, a'$. Hence gauging allows for a problem formulation of type (3.3):

$$\begin{aligned} \partial_t a + \sigma^{-1}\,\delta\,\mu^{-1}\,\mathsf{d}\,a + \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,a + \gamma_1\,\mathsf{d}\,\mu^{-1}\,\delta\,a + \gamma_2\,\mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,a &= \sigma^{-1} f \quad \text{in } \Omega, \\ -\operatorname{tr}(\partial_t a + \gamma_1\,\mathsf{d}\,\mu^{-1}\,\delta\,a + \gamma_2\,\mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,a) &= g_e \quad \text{on } \Gamma_e, \\ \operatorname{tr}(\star\mu^{-1}\,\mathsf{d}\,a) &= g_h \quad \text{on } \Gamma_h. \end{aligned} \tag{3.11}$$

Once again we would like to stress that different choices for γ_1 and γ_2 in (3.8) and (3.11) will require different boundary conditions in order to establish well-posedness in certain Sobolev-spaces. Actually the boundary conditions (3.6) stated here imply certain additional conditions. The trace tr commutes with exterior derivative d and we deduce:

$$\begin{aligned} \operatorname{tr}(\mathsf{d}\,e) &= \mathsf{d}\,g_e \quad \text{on } \Gamma_e, \\ -\partial_t \operatorname{tr}(b) &= \mathsf{d}\,g_e \quad \text{on } \Gamma_e, \\ \operatorname{tr}(\mathsf{d}\,h) &= \mathsf{d}\,g_h \quad \text{on } \Gamma_h, \\ \operatorname{tr}(\star_\sigma e - \star_\sigma \mathsf{i}_{\boldsymbol{\beta}}\,b + f) &= \mathsf{d}\,g_h \quad \text{on } \Gamma_h. \end{aligned} \tag{3.12}$$

We will see that in some cases some of these conditions need to be imposed explicitly.

3.2.1 Perfect Conductor Limit

In the limit $\sigma \to \infty$ both the formulation (3.8) for the magnetic field h with $\gamma_2 = 1$ and the formulation (3.11) for the vector potential a with $\gamma_1 = 0$ and $\gamma_2 = 1$ reduce to first order transport problems:

$$\begin{aligned} \partial_t \star_\mu h + \mathsf{L}_{\boldsymbol{\beta}} \star_\mu h &= 0 \quad \text{in } \Omega, \\ \operatorname{tr}(\mathsf{i}_{\boldsymbol{\beta}} \star_\mu h) &= g_e \quad \text{on } \Gamma_e, \\ \operatorname{tr} h &= g_h \quad \text{on } \Gamma_h, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \partial_t a + \mathsf{L}_{\boldsymbol{\beta}}\,a &= 0 \quad \text{in } \Omega, \\ -\operatorname{tr}(\partial_t a + \mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,a) &= g_e \quad \text{on } \Gamma_e, \\ \operatorname{tr}(\star_{\mu^{-1}}\,\mathsf{d}\,a) &= g_h \quad \text{on } \Gamma_h. \end{aligned} \tag{3.14}$$

The Lie derivative $\mathsf{L}_{\boldsymbol{\beta}}$ (2.28) is the derivative of the pullback $X_t^*\omega(t_0)$ with respect to t at $t = 0$, where X_t is the flow induced by $\boldsymbol{\beta}$. Hence, the formal solution to a transport problem on entire $\mathbb{R}^n$, $\omega \in \Lambda^k(\mathbb{R}^n)$:

$$\partial_t \omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega(t) = 0 \quad \text{in } \mathbb{R}^n \tag{3.15}$$

is given as:

$$\omega(t) = X_{-t}^*\omega(0). \tag{3.16}$$

In the case of a bounded domain $\Omega \subset \mathbb{R}^n$, the solution also depends on the boundary values:

$$(\omega(t))_x = \begin{cases} \left(X_{-t}^*\omega(0)\right)_x, & X_{\tau - t}(x) \notin \partial\Omega\,\forall\,\tau \in [0, t]; \\ \left(X_{t(x)-t}^*\omega(t(x))\right)_x, & X_{t(x)-t}(x) \in \partial\Omega. \end{cases} \tag{3.17}$$

If a trajectory through a point x hits the boundary at some time $0 < t(x) \leq t$, the solution ω at x depends on the values on the boundary. Since tr and pullback X^*_{-t} do not commute, we see that the well-posedness of transport problems cannot be guaranteed by prescribing only the traces: traces determine only the tangential components forms.

3.3 Functional Analytic Framework for Variational Problems

In this section we will give a short review of the functional analytic framework, which is used to establish existence and uniqueness of solutions to variational formulations of boundary value problems. The main reference is the textbook [27].

3.3.1 Stationary Problems

Let W and V be normed vector spaces with norms $\|\cdot\|_W$ and $\|\cdot\|_V$. Further $\mathsf{a} : W \times V \mapsto \mathbb{R}$ is a bounded sesquilinear form, i.e. it satisfies the continuity estimate:

$$|\mathsf{a}(u, v)| \leq C \|u\|_W \|v\|_V \quad \forall u \in W, v \in V,$$

and f is a continuous linear form on V, i.e. $f \in V'$. The following Theorem [27, Theorem 2.6] establishes well-posedness of the variational problem: Find $u \in W$ such that

$$\mathsf{a}(u, v) = f(v), \quad \forall v \in V. \tag{3.18}$$

Theorem 3.3.1 (Banach-Nečas-Babuška). *Let W be a Banach space and let L be a reflexive Banach space. L' is the dual space of L. Let $\mathsf{a} : W \times L \mapsto \mathbb{R}$ be a bounded bilinear form and let $f \in L'$. Then problem (3.18) is well-posed if and only if:*

$$\exists \alpha > 0, \inf_{u \in W} \sup_{v \in L} \frac{\mathsf{a}(u, v)}{\|u\|_W \|v\|_L} \geq \alpha,$$
$$(\forall u \in W, \mathsf{a}(u, v) = 0) \Rightarrow (v = 0).$$

This Theorem is basically a rephrasing of the *open range Theorem* and *open mapping Theorem* from functional analysis. For the special case $W = L$ the Theorem is also known as the Lax-Milgram-Lemma [27, Lemma 2.2].

Lemma 3.3.2 (Lax-Milgram). *Let V in (3.18) be a Hilbert space, let $\mathsf{a} : V \times V \mapsto \mathbb{R}$ be a bounded bilinear form and let $f \in V'$. Assume that the bilinear form a is coercive, i.e. there exists $\alpha > 0$ such that*

$$\mathsf{a}(v, v) \geq \alpha \|v\|_V^2, \quad \forall v \in V. \tag{3.19}$$

Then, the problem (3.18) is well-posed with a priori estimate:

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}$$

3.3.2 Non-Stationary Problems

The well-posedness results for time dependent problems distinguish again between a symmetric and a non-symmetric setting. If V denotes a Banach space we define $C^k([0,T];V)$ to be the set of V-valued functions $u(t) \in V$ that are k-times continuously differentiable with respect to t. Similarly we define the space $L^2(]0,T[;V)$ to be the set of V-valued functions $u(t) \in V$, whose norm in V is in $L^2(]0,T[)$. $d_t u$ denotes the distributional time-derivative of $u \in L^2(]0,T[;V)$.

We start with the non-symmetric setting. Let L be a separable Hilbert space with inner product $(\cdot,\cdot)_L$. Let $A : D(A) \subset L \mapsto L$ be a linear, maximal and monotone operator, i.e.

$$\forall f \in L, \exists v \in D(A), v + Av = f \tag{3.20}$$

and

$$\forall v \in D(A),\ (Av, v)_L \geq 0. \tag{3.21}$$

It can be shown that in this case the space $W = D(A)$, equipped with the scalar product $(u,v)_L + (Au, Av)_L$ is a Hilbert space. We define a bilinear form a as $\mathsf{a}(u,v) = (Au, v)_L$ for all $u \in W$ and $v \in L$ and consider the following model problem:

For $f \in C^1([0,T];L)$ and $u_0 \in W$, find $u \in C^1([0,T];L) \cap C^0([0,T];W)$ such that

$$\begin{aligned} (d_t u, v)_L + \mathsf{a}(u,v) &= (f,v)_L, \quad \forall v \in L, \forall t > 0, \\ (u(0), v)_L &= (u_0, v)_L, \quad \forall v \in L. \end{aligned} \tag{3.22}$$

The Hille-Yosida Theorem [27, Theorem 6.52] gives existence and uniqueness:

Theorem 3.3.3 (Hille-Yosida). *Let L be a separable Hilbert space with inner product $(\cdot,\cdot)_L$. Let $A : D(A) \subset L \mapsto L$ be a linear, maximal and monotone operator and $\mathsf{a}(u,v) = (Au,v)_L$ for all $u \in D(A)$ and $v \in L$. For all $f \in C^1([0,T];L)$ and $u_0 \in D(A)$ the problem (3.22) has a unique solution.*

In the symmetric case we assume $\mathsf{a}(\cdot,\cdot)$ to be a bilinear form on $V \times V$, where V is a Hilbert space such that $V \subset L \equiv L' \subset V'$. Hence the duality pairing $\langle \cdot,\cdot \rangle_{V',V}$ can be viewed as an extension of the scalar product on L. We consider the following problem:

For $f \in L^2(]0,T[;V')$ and $u_0 \in L$, find $u \in L^2(]0,T[;V)$ with $d_t u \in L^2(]0,T[;V')$ such that

$$\begin{aligned} \langle d_t u, v \rangle_{V',V} + \mathsf{a}(u,v) &= \langle f, v \rangle_{V',V}, \quad \forall v \in V, \forall t > 0, \\ u(0) &= u_0. \end{aligned} \tag{3.23}$$

Existence and uniqueness are due to a result of J.L. Lions [27, Theorem 6.6].

Theorem 3.3.4 (J.L. Lions). *Let $V \subset L$ be two Hilbert spaces, V dense in L, with norms $\|\cdot\|_V$ and $\|\cdot\|_L$. Let $\mathsf{a} : V \times V \mapsto \mathbb{R}$ be a bounded bilinear form. Assume that there exists $\alpha > 0$ and $\gamma > 0$ such that*

$$\mathsf{a}(u,u) + \gamma \|u\|_L^2 \geq \alpha \|u\|_V^2, \quad \forall u \in V.$$

Let $f \in L^2(]0,T[;V')$ and $u_0 \in L$, then problem (3.23) has a unique solution.

3.4 Variational Formulations

Now, we state variational formulations of the generalized advection problems (3.3) and (3.4). We will differentiate between

- $\varepsilon > 0$ and $\varepsilon = 0$;
- $\gamma_1 = 0$, $\gamma_2 = 0$ and $\gamma_1 \neq 0$, $\gamma_2 \neq 0$.

The key formulas in the derivation of the variational formulations are the various integration-by-parts formulas (2.34)-(2.37). We will not discuss in detail admissible sets of boundary conditions but accentuate the stability properties of different formulations with respect to ε. The main result will be the observation that in the limit $\varepsilon = 0$ well-posedness requires a different set of boundary conditions than the case $\varepsilon > 0$. We will see that as in the scalar case one only has to impose boundary conditions in the inflow part of the boundary. Further, and this is a distinctive property of the non-scalar case, it is not enough to prescribe the standard Dirichlet data, i.e. the trace of the unknown form. This is in perfect agreement with the solution formula (3.17): the traces fix only tangential components. For advection problems with velocity fields, that have vanishing normal component everywhere on the boundary, we do not face this special feature.

3.4.1 Well-Posedness: Case $\varepsilon > 0$, $\gamma_1 = 0$, $\gamma_2 = 0$

We consider the Dirichlet problems (3.3) and (3.4) with $\Gamma_D = \partial\Omega$, $\varepsilon > 0$, $\gamma_1 = 0$ and $\gamma_2 = 0$. We set $V = H\Lambda^k(\Omega, \psi_D)$ and $V_0 = H\Lambda^k(\Omega, 0)$ and assume that ψ_D can be extended to $H\Lambda^k(\Omega)$. A variational formulation for a problem of type (3.3) is:

For $\varphi \in L^2(]0,T[; V_0')$ and $\omega_0 \in V$, find $\omega \in L^2(]0,T[; V)$ with $d_t\omega \in L^2(]0,T[; V_0')$ such that for all $\eta \in V_0$:

$$\begin{aligned} \langle d_t\omega, \eta\rangle_{V',V} + \varepsilon\,(\mathsf{d}\,\omega, \mathsf{d}\,\eta)_\Omega + (\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega, \eta)_\Omega &= \langle \varphi, \eta\rangle_{V',V}, \\ \omega(0) &= \omega_0. \end{aligned} \tag{3.24}$$

This corresponds to the initial value problem: Find ω such that:

$$\partial_t\omega + \delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega = \varphi \text{ in } \Omega, \quad \operatorname{tr}\omega = \psi_D \text{ on } \partial\Omega, \quad \omega(0) = \omega_0.$$

Similar a variational formulation for a problem of type (3.4) with is $\Gamma_D = \partial\Omega$, $\varepsilon > 0$, $\gamma_1 = 0$ and $\gamma_2 = 0$:

For $\varphi \in L^2(]0,T[; V_0')$ and $\omega_0 \in V$, find $\omega \in L^2(]0,T[; V)$ with $d_t\omega \in L^2(]0,T[; V_0')$ such that for all $\eta \in V_0$:

$$\begin{aligned} \langle d_t\omega, \eta\rangle_{V',V} + \varepsilon\,(\mathsf{d}\,\omega, \mathsf{d}\,\eta)_\Omega - (\omega, \mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\eta)_\Omega &= \langle \varphi, \eta\rangle_{V',V}, \\ \omega(0) &= \omega_0. \end{aligned} \tag{3.25}$$

This corresponds to the initial value problem: Find ω such that:

$$\partial_t\omega + \delta\,\varepsilon\,\mathsf{d}\,\omega - \delta\,\mathsf{j}_{\boldsymbol\beta}\,\omega = \varphi \text{ in } \Omega, \quad \operatorname{tr}\omega = \psi_D \text{ on } \partial\Omega, \quad \omega(0) = \omega_0.$$

Lemma 3.4.1 proves coercivity for the bilinear form in (3.24) and (3.25).

Lemma 3.4.1. *If $\varepsilon > 0$, there are $\gamma > 0$ and $\kappa(\varepsilon) > 0$ such that for all $\omega \in V_0$*

$$\varepsilon \left(\mathsf{d}\,\omega, \mathsf{d}\,\omega\right)_\Omega + \left(\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega, \omega\right)_\Omega + \gamma \left(\omega, \omega\right)_\Omega \geq \kappa \|\omega\|^2_{H\Lambda^k(\Omega)}.$$

Proof. The proof is based on a generalization of the proof of a Lemma in [30, Lemma 2.1]. We set $\gamma = \varepsilon^{-1} \|\boldsymbol{\beta}\|^2_{\boldsymbol{W}^{0,\infty}(\Omega)}$ and $\kappa = \frac{1}{2}\min(\varepsilon, \gamma)$. Then for all $\omega \in V_0$

$$\begin{aligned}
&\varepsilon \left(\mathsf{d}\,\omega, \mathsf{d}\,\omega\right)_\Omega + \left(\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega, \omega\right)_\Omega + \gamma \left(\omega, \omega\right)_\Omega \\
&\qquad \geq \varepsilon \|\mathsf{d}\,\omega\|^2_{L^2\Lambda^{k+1}(\Omega)} - \sqrt{\gamma\varepsilon}\, \|\omega\|_{L^2\Lambda^k(\Omega)} \|\mathsf{d}\,\omega\|_{L^2\Lambda^{k+1}(\Omega)} + \gamma \|\omega\|^2_{L^2\Lambda^k(\Omega)} \\
&\qquad \geq \frac{\varepsilon}{2} \|\mathsf{d}\,\omega\|^2_{L^2\Lambda^{k+1}(\Omega)} + \frac{\gamma}{2} \|\omega\|^2_{L^2\Lambda^k(\Omega)} \\
&\qquad \geq \kappa \|\omega\|^2_{H\Lambda^k(\Omega)}
\end{aligned} .$$

□

Theorem 3.4.2. *Problems* (3.24) *and* (3.25) *are well-posed if $\varepsilon > 0$.*

Proof. This follows directly from Lemma 3.4.1 and Theorem 3.3.4. □

Remark 3.4.3. *Note that the result of Theorem 3.4.2 is particularly weak when ε is small. We encounter $\gamma \to \infty$ and $\kappa \to 0$ for $\varepsilon \to 0$.*

3.4.2 Well-Posedness: Case $\varepsilon > 0$, $\gamma_1 = \varepsilon$, $\gamma_2 = 1$

We consider the advection-diffusion problems (3.3) and (3.4) with $\Gamma_D = \partial\Omega$, $\varepsilon > 0$, $\gamma_1 = \varepsilon$, $\gamma_2 = 1$ and assume that ψ_D can be extended to a function in $H\Lambda^k(\Omega)$. We set $V = H\Lambda^k(\Omega, \psi_D) \cap H^*\Lambda^k(\Omega)$ and $V_0 = H\Lambda^k(\Omega, 0) \cap H^*\Lambda^k(\Omega)$. The variational formulation of an advection-diffusion problem of type (3.3) is:

For $\varphi \in L^2(]0, T[; V_0')$ and $\omega_0 \in V$, find $\omega \in L^2(]0, T[; V)$ with $d_t\omega \in L^2(]0, T[; V_0')$ such that for all $\eta \in V_0$:

$$\begin{aligned}
\langle d_t\omega, \eta\rangle_{V',V} + \varepsilon \left(\mathsf{d}\,\omega, \mathsf{d}\,\eta\right)_\Omega + \varepsilon \left(\delta\,\omega, \delta\,\eta\right)_\Omega + \left(\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega, \eta\right)_\Omega + \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\eta\right)_\Omega &= \langle \phi, \eta\rangle_{V',V}, \\
\omega(0) &= \omega_0.
\end{aligned} \tag{3.26}$$

This formulation corresponds to the initial boundary value problem:

$$\begin{aligned}
\partial_t\omega + \delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{d}\,\varepsilon\,\delta\,\omega + \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega + \mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega, \\
\operatorname{tr}\omega &= \psi_D, && \text{on } \partial\Omega, \\
\operatorname{tr}(\varepsilon\,\delta\,\omega - \mathsf{i}_{\boldsymbol{\beta}}\,\omega) &= 0, && \text{on } \partial\Omega, \\
\omega(0) &= \omega_0.
\end{aligned}$$

Similar, the variational formulation of problem of type (3.4) for the setting $\Gamma_D = \partial\Omega$, $\varepsilon > 0$, $\gamma_1 = \varepsilon$ and $\gamma_2 = 1$ is:

For $\varphi \in L^2\left(]0,T[;V\right)$ and $\omega_0 \in V$, find $\omega \in L^2\left(]0,T[;V\right)$ with $d_t\omega \in L^2\left(]0,T[;V_0'\right)$ such that for all $\eta \in V_0$:

$$\begin{aligned}\langle d_t\omega,\eta\rangle_{V',V} + \varepsilon\,(\mathsf{d}\,\omega,\mathsf{d}\,\eta)_\Omega + \varepsilon\,(\delta\,\omega,\delta\,\eta)_\Omega - \;(\omega,\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\eta)_\Omega - \;\left(\mathsf{j}_{\boldsymbol\beta}\,\delta\,\omega,\eta\right)_\Omega &= \langle\varphi,\eta\rangle_{V',V},\\ \omega(0) &= \omega_0.\end{aligned} \tag{3.27}$$

This is a variational formulation of the initial boundary value problem:

$$\begin{aligned}\partial_t\omega + \delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{d}\,\varepsilon\,\delta\,\omega - \mathsf{j}_{\boldsymbol\beta}\,\delta\,\omega - \delta\,\mathsf{j}_{\boldsymbol\beta}\,\omega &= \varphi, &&\text{in } \Omega,\\ \operatorname{tr}\omega &= \psi_D, &&\text{on } \partial\Omega,\\ \operatorname{tr}\delta\,\omega &= 0, &&\text{on } \partial\Omega,\\ \omega(0) &= \omega_0. &&\end{aligned}$$

Lemma 3.4.4 proves coercivity for the bilinear forms in (3.26) and (3.27).

Lemma 3.4.4. *If $\varepsilon > 0$, there are $\gamma > 0$ and $\kappa(\varepsilon) > 0$ such that for all $\omega \in V_0$*

$$\begin{aligned}\varepsilon\,(\mathsf{d}\,\omega,\mathsf{d}\,\omega)_\Omega + \varepsilon\,(\delta\,\omega,\delta\,\omega)_\Omega + \;(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\omega)_\Omega + \;\left(\omega,\mathsf{j}_{\boldsymbol\beta}\,\delta\,\omega\right)_\Omega + \gamma\,(\omega,\omega)_\Omega&\\ \geq \kappa\left(\|\omega\|^2_{H\Lambda^k(\Omega)} + \|\omega\|^2_{H^*\Lambda^k(\Omega)}\right)&.\end{aligned}$$

Proof. The proof is based on a generalization of the proof of a Lemma in [30, Lemma 2.1]. We set $\gamma = \varepsilon^{-1}\,\|\boldsymbol\beta\|^2_{\boldsymbol{W}^{0,\infty}(\Omega)}$ and $\kappa = \frac{1}{2}\min(\varepsilon,\gamma)$. Then for all $\omega \in V_0$

$$(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\omega)_\Omega \leq \sqrt{\gamma\varepsilon}\,\|\omega\|_{L^2\Lambda^k(\Omega)}\,\|\mathsf{d}\,\omega\|_{L^2\Lambda^{k+1}(\Omega)}$$

and

$$\left(\mathsf{j}_{\boldsymbol\beta}\,\delta\,\omega,\omega\right)_\Omega \leq \sqrt{\gamma\varepsilon}\,\|\omega\|_{L^2\Lambda^k(\Omega)}\,\|\delta\,\omega\|_{L^2\Lambda^{k-1}(\Omega)},$$

hence

$$\begin{aligned}\varepsilon\,\|\mathsf{d}\,\omega\|^2_{L^2\Lambda^{k+1}(\Omega)} + \varepsilon\,\|\delta\,\omega\|^2_{L^2\Lambda^{k-1}(\Omega)} + \;(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\omega)_\Omega - \;\left(\omega,\mathsf{j}_{\boldsymbol\beta}\,\delta\,\omega\right)_\Omega + \gamma\,(\omega,\omega)_\Omega&\\ \geq \frac{\varepsilon}{2}\,\|\mathsf{d}\,\omega\|^2_{L^2\Lambda^{k+1}(\Omega)} + \frac{\varepsilon}{2}\,\|\delta\,\omega\|^2_{L^2\Lambda^{k-1}(\Omega)} + \gamma\,\|\omega\|^2_{L^2\Lambda^k(\Omega)}&\\ \geq \kappa\left(\|\omega\|^2_{H\Lambda^k(\Omega)} + \|\omega\|^2_{H^*\Lambda^k(\Omega)}\right)&.\end{aligned}$$

□

Theorem 3.4.5. *Problems* (3.26) *and* (3.27) *are well-posed if $\varepsilon > 0$.*

Proof. This follows directly from Lemma 3.4.4 and Theorem 3.3.4. □

Remark 3.4.6. *Note that the assertion of Theorem 3.4.5 is particularly weak when ε is small. We encounter $\gamma \to \infty$ and $\kappa \to 0$ for $\varepsilon \to 0$.*

3.4.3 Well-Posedness: Case $\varepsilon = 0$, $\gamma_1 = 0$, $\gamma_2 = 1$

In this case the boundary $\partial\Omega$ splits into inflow part Γ_{in} with $\boldsymbol{\beta}_{|\partial\Omega} \cdot \mathbf{n}_\Omega < 0$, outflow part Γ_{out} with $\boldsymbol{\beta}_{|\partial\Omega} \cdot \mathbf{n}_\Omega > 0$ and characteristic part Γ_0 with $\boldsymbol{\beta}_{|\partial\Omega} \cdot \mathbf{n}_\Omega = 0$. We consider advection problems (3.3) and (3.4) with $\varepsilon = 0$, $\gamma_1 = 0$, $\gamma_2 = 1$ and impose the boundary conditions $\operatorname{tr}\omega = \operatorname{tr}\psi_D, \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\psi_D$ on the inflow part Γ_{in} with $\psi_D \in \Lambda^k(\mathbb{R}^n \setminus \Omega)$. The two bilinear forms $(\cdot,\cdot)_{\Gamma_{\text{in}},\boldsymbol{\beta}}$ and $(\cdot,\cdot)_{\Gamma_{\text{out}},\boldsymbol{\beta}}$ introduced in (2.35) are negative and positive definite since $(\boldsymbol{\beta} \cdot \mathbf{n}_\Omega)_{|\Gamma_{\text{in}}} < 0$ and $(\boldsymbol{\beta} \cdot \mathbf{n}_\Omega)_{|\Gamma_{\text{out}}} > 0$. This gives rise to two semi-norms $|\omega|^2_{\Gamma_{\text{in}},-\boldsymbol{\beta}} := \frac{1}{2}(\omega,\omega)_{\Gamma_{\text{in}},-\boldsymbol{\beta}}$ and $|\omega|^2_{\Gamma_{\text{out}},\boldsymbol{\beta}} := \frac{1}{2}(\omega,\omega)_{\Gamma_{\text{out}},\boldsymbol{\beta}}$. We set $W = \{\omega \in L^2\Lambda^k(\Omega), \mathsf{L}_{\boldsymbol{\beta}}\,\omega \in L^2\Lambda^k(\Omega), |\omega|_{\Gamma_{\text{in}},-\boldsymbol{\beta}} < \infty\}$ and $L = L^2\Lambda^k(\Omega)$ and equip W with the norm

$$\|\omega\|^2_W := \|\omega\|^2_{L^2\Lambda^k(\Omega)} + \|\mathsf{L}_{\boldsymbol{\beta}}\,\omega\|^2_{L^2\Lambda^k(\Omega)}.$$

Without loss of generality we can assume $\psi_D = 0$. We set $V = \{\omega \in W, \operatorname{tr}\omega = 0, \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega = 0\}$. The variational formulation of an advection problem of type (3.3) is:

For $\varphi \in C^1([0,T];L)$ and $\omega_0 \in V$, find $\omega \in C^1([0,T];L) \cap C^0([0,T];V)$ such that for all $\eta \in L$:

$$\begin{aligned} (d_t\omega,\eta)_\Omega + (\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega,\eta)_\Omega + (\mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega,\eta)_\Omega &= (\varphi,\eta)_\Omega\,, \\ (\omega(0),\eta)_\Omega &= (\omega_0,\eta)_\Omega\,. \end{aligned} \tag{3.28}$$

This is a variational formulation of the initial boundary value problem:

$$\begin{aligned} \partial_t\omega + \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega + \mathsf{d}\,\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega, \\ \operatorname{tr}\omega &= 0, && \text{on } \Gamma_{\text{in}}, \\ \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= 0, && \text{on } \Gamma_{\text{in}}, \\ \omega(0) &= \omega_0. \end{aligned}$$

Since we can always introduce a rescaling $\omega' = e^{-\alpha t}\omega$ in (3.28) the following Lemma establishes the crucial step for proving well-posedness.

Lemma 3.4.7. *Assume that there exists $\alpha \in \Lambda^0(\Omega)$, $\alpha > 0$, such that*

$$\alpha\,\omega \wedge \star\omega + \frac{1}{2}(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\,\omega \wedge \star\omega \geq \omega \wedge \star\omega, \quad \forall\omega \in \Lambda^k(\Omega)\,. \tag{3.29}$$

The operator $A : V \subset L \mapsto L$, defined as

$$(A\omega,\eta)_\Omega := (\alpha\omega,\eta)_\Omega + (\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_\Omega\,, \quad \forall\eta \in L$$

is a maximal monotone operator.

Proof. The operator A is monotone and it only remains to prove maximality (3.20), which is equivalent to existence and uniqueness of solutions of the following variational formulation:

For $f \in L$ find $\omega \in V$ such that

$$((\alpha + 1)\,\omega,\eta)_\Omega + (\mathsf{L}_{\boldsymbol{\beta}}\,\omega,\eta)_\Omega = (f,\eta)_\Omega\,, \quad \forall\eta \in L.$$

To prove existence and uniqueness we verify the assumptions of Theorem 3.3.1:

- The bilinear form

$$\mathsf{a}(\omega,\eta) := ((\alpha+1)\,\omega,\eta)_\Omega + (\mathsf{L}_{\boldsymbol\beta}\,\omega,\eta)_\Omega$$

is continuous on $V \times L$.

- We continue in proving the inf-sup-condition in Theorem 3.3.1. First the positivity conditions and the Leibniz rule for Lie derivatives (2.36) imply L^2-stability:

$$\begin{aligned}\mathsf{a}(\omega,\omega) &= ((\alpha+1)\omega,\omega)_\Omega + (\mathsf{L}_{\boldsymbol\beta}\,\omega,\omega)_\Omega \\ &= ((\alpha+1)\omega,\omega)_\Omega + \frac{1}{2}\left((\mathsf{L}_{\boldsymbol\beta}+\mathcal{L}_{\boldsymbol\beta})\omega,\omega\right)_\Omega + \frac{1}{2}\left((\omega,\omega)_{\Gamma_{\mathrm{out}},\boldsymbol\beta} + (\omega,\omega)_{\Gamma_{\mathrm{in}},\boldsymbol\beta}\right) \\ &\geq 2\,\|\omega\|_L^2 .\end{aligned}$$

The last inequality follows from $(\boldsymbol\beta\cdot\mathbf{n}_\Omega)_{|\Gamma_{\mathrm{out}}} > 0$ and the imposed boundary conditions, since

$$(\omega,\omega)_{\Gamma_{\mathrm{in}},\boldsymbol\beta} = \int_{\Gamma_{\mathrm{in}}} \operatorname{tr} \mathsf{i}_{\boldsymbol\beta}\,\omega \wedge \operatorname{tr}\star\omega + (-1)^k \int_{\Gamma_{\mathrm{in}}} \operatorname{tr}\omega\wedge\operatorname{tr}\mathsf{i}_{\boldsymbol\beta}\star\omega, \tag{3.30}$$

by (2.35), (2.32) and (2.21). The L^2-stability implies

$$\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L} \geq \frac{\mathsf{a}(\omega,\omega)}{\|\omega\|_L} \geq 2\,\|\omega\|_L .$$

We set $\alpha_1 = \|\alpha+1\|_{L^\infty(\Omega)}$ and deduce

$$\begin{aligned}\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L} &= \sup_{\eta\in L}\frac{((\alpha+1)\omega,\eta)_\Omega + (\mathsf{L}_{\boldsymbol\beta}\,\omega,\eta)_\Omega}{\|\eta\|_L} \\ &\geq \sup_{\eta\in L}\frac{(\mathsf{L}_{\boldsymbol\beta}\,\omega,\eta)_\Omega}{\|\eta\|_L} - \sup_{\eta\in L}\frac{((\alpha+1)\omega,\eta)_\Omega}{\|\eta\|_L} \\ &\geq \sup_{\eta\in L}\frac{(\mathsf{L}_{\boldsymbol\beta}\,\omega,\eta)_\Omega}{\|\eta\|_L} - \alpha_1\,\|\omega\|_L \\ &\geq \|\mathsf{L}_{\boldsymbol\beta}\,\omega\|_{L^2\Lambda^k(\Omega)} - \frac{\alpha_1}{2}\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L}.\end{aligned}$$

This yields

$$\left(\left(1+\frac{\alpha_1}{2}\right)^2 + \frac{1}{4}\right)\left(\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\eta\|_L}\right)^2 \geq \|\mathsf{L}_{\boldsymbol\beta}\,\omega\|^2_{L^2\Lambda^k(\Omega)} + \|\omega\|_L^2 ,$$

i.e. the inf-sup-inequality

$$\inf_{\omega\in V}\sup_{\eta\in L}\frac{\mathsf{a}(\omega,\eta)}{\|\omega\|_W\,\|\eta\|_L} \geq 2\left(1+(2+\alpha_1)^2\right)^{-\frac{1}{2}} .$$

- Next we establish the injectivity condition in Theorem 3.3.1. Let $\eta \in L$ such that $\mathsf{a}(\omega, \eta) = 0$ for all $\omega \in V$. A density argument gives $(\alpha+1)\eta + \mathcal{L}_{\boldsymbol{\beta}}\eta = 0$, which implies in particular $\eta \in W$. Testing with $\omega \in \Lambda^k(\Omega) \cap V$ we find $\operatorname{tr}\star\eta = 0$ and $\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\star\eta = 0$ at Γ_{out} and deduce

$$\begin{aligned} 0 =& \ (\alpha\eta, \eta)_\Omega + (\eta, \mathcal{L}_{\boldsymbol{\beta}}\eta)_\Omega \\ =& \ (\alpha\eta, \eta)_\Omega + \frac{1}{2}\left((\eta, \mathcal{L}_{\boldsymbol{\beta}}\eta)_\Omega + (\eta, \mathsf{L}_{\boldsymbol{\beta}}\eta)_\Omega\right) - \frac{1}{2}\left((\eta,\eta)_{\Gamma_{\text{in}},\boldsymbol{\beta}} + (\eta,\eta)_{\Gamma_{\text{out}},\boldsymbol{\beta}}\right) \\ \geq& \ \|\eta\|^2_{L^2\Lambda^k(\Omega)}, \end{aligned}$$

  i.e. $\eta = 0$.

□

Remark 3.4.8. *The identity* (3.30) *shows that we could impose the boundary conditions differently. For the following four conditions we can show well-posedness:*

1. $\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\psi_D$ *and* $\operatorname{tr}\omega = \operatorname{tr}\psi_D$ *on* Γ_{in}*;*
2. $\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\psi_D$ *and* $\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\star\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\star\psi_D$ *on* Γ_{in}*;*
3. $\operatorname{tr}\star\omega = \operatorname{tr}\star\psi_D$ *and* $\operatorname{tr}\omega = \operatorname{tr}\psi_D$ *on* Γ_{in}*;*
4. $\operatorname{tr}\star\omega = \operatorname{tr}\star\psi_D$ *and* $\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\star\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\star\psi_D$ *on* Γ_{in}*;*

Theorem 3.4.9. *Problem* (3.28) *is well-posed.*

Proof. In introducing $\nu = \exp(-\alpha t)\omega$ we can always write (3.28) in terms of the operator $\alpha + \mathsf{L}_{\boldsymbol{\beta}}$ instead of $\mathsf{L}_{\boldsymbol{\beta}}$, hence well-posedness follows from Lemma 3.4.7 and Theorem 3.3.3. □

Next we consider the advection problem of type (3.3) for the adjoint Lie derivative $\mathcal{L}_{\boldsymbol{\beta}}$. We impose the boundary conditions $\operatorname{tr}\omega = \operatorname{tr}\psi_D, \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\omega = \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\psi_D$ on the inflow part Γ_{in} with $\psi_D \in \Lambda^k(\mathbb{R}^n \setminus \Omega)$. We set $W = \{\omega \in L^2\Lambda^k(\Omega), \mathcal{L}_{\boldsymbol{\beta}}\omega \in L^2\Lambda^k(\Omega), |\omega|_{\Gamma_{\text{in}},-\boldsymbol{\beta}} < \infty\}$ and $L = L^2\Lambda^k(\Omega)$ and equip W with norm

$$\|\omega\|^2_W := \|\omega\|^2_{L^2\Lambda^k(\Omega)} + \|\mathcal{L}_{\boldsymbol{\beta}}\omega\|^2_{L^2\Lambda^k(\Omega)}.$$

We assume again homogeneous boundary conditions and set again $V = \{\omega \in W, \operatorname{tr}\omega = 0, \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\omega = 0\}$. The variational formulation of (3.4) is:

For $\varphi \in C^1([0,T]; L)$ and $\omega_0 \in V$ find $\omega \in C^1([0,T]; L) \cap C^0([0,T]; V)$ such that for all $\eta \in L$:

$$\begin{aligned} (d_t\omega, \eta)_\Omega - \left(\delta\,\mathsf{j}_{\boldsymbol{\beta}}\omega, \eta\right)_\Omega - \left(\mathsf{j}_{\boldsymbol{\beta}}\delta\,\omega, \eta\right)_\Omega &= (\varphi, \eta)_\Omega, \\ (\omega(0), \eta)_\Omega &= (\omega_0, \eta)_\Omega. \end{aligned} \tag{3.31}$$

This is a variational formulation of the initial boundary value problem:

$$\begin{aligned} \partial_t \omega - \mathsf{j}_{\boldsymbol{\beta}} \delta\, \omega - \delta\, \mathsf{j}_{\boldsymbol{\beta}}\, \omega &= \varphi, && \text{in } \Omega, \\ \operatorname{tr} \omega &= 0, && \text{on } \Gamma_{\text{in}}, \\ \operatorname{tr} \mathsf{j}_{\boldsymbol{\beta}}\, \omega &= 0, && \text{on } \Gamma_{\text{in}}, \\ \omega(0) &= \omega_0. \end{aligned}$$

Analogue to Lemma 3.4.7 we get the following result on existence and uniqueness for the stationary problem.

Lemma 3.4.10. *If there exists $\alpha \in \Lambda^0(\Omega)$, $\alpha > 0$, such that*

$$\alpha\, \omega \wedge \star\omega - \frac{1}{2} (\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\, \omega \wedge \star\omega \geq \omega \wedge \star\omega, \quad \forall \omega \in \Lambda^k(\Omega).$$

The operator $A : V \subset L \mapsto L$, defined as

$$(A\omega, \eta)_\Omega := (\alpha\omega, \eta)_\Omega - (\mathcal{L}_{\boldsymbol{\beta}}\, \omega, \eta)_\Omega, \quad \forall \eta \in L$$

is a maximal monotone operator.

Theorem 3.4.11. *Problem* (3.31) *is well-posed.*

Proof. Follows from Lemma 3.4.10 and Theorem 3.3.3. □

Remark 3.4.12. *The positivity conditions in Lemmas 3.4.7 and 3.4.10 are conditions on $\boldsymbol{\beta}$ and partial derivatives of $\boldsymbol{\beta}$, since $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}} = C(\boldsymbol{\beta}, D\boldsymbol{\beta})\mathsf{id}$ by* (2.40)*. Table 2.2 summarizes explicit values of $C(\boldsymbol{\beta}, D\boldsymbol{\beta})$ for the different forms in $\mathbb{R}^3$.*

Remark 3.4.13. *If $\boldsymbol{\beta}$ has vanishing normal components everywhere on the boundary we do not need to impose boundary conditions.*

3.4.4 Discussion

In light of the results of Theorems 3.4.9 and 3.4.11 we change in the variational formulations (3.26) and (3.24) the boundary conditions. Instead of $V = H\Lambda^k(\Omega, \psi_D) \cap H^*\Lambda^k(\Omega)$ and $V_0 = H\Lambda^k(\Omega, 0) \cap H^*\Lambda^k(\Omega)$ we set $V = H\Lambda^k(\Omega, \psi_D) \cap H^*\Lambda^k(\Omega, 0)$ and $V_0 = H\Lambda^k(\Omega, 0) \cap H^*\Lambda^k(\Omega, 0)$. Then the variational formulation (3.26) for advection-diffusion problems of type (3.3) corresponds to the initial boundary value problem:

$$\begin{aligned} \partial_t \omega + \delta\, \varepsilon\, \mathsf{d}\, \omega + \mathsf{d}\, \varepsilon\, \delta\, \omega + \mathsf{i}_{\boldsymbol{\beta}}\, \mathsf{d}\, \omega + \mathsf{d}\, \mathsf{i}_{\boldsymbol{\beta}}\, \omega &= \varphi, && \text{in } \Omega, \\ \operatorname{tr} \omega &= \psi_D, && \text{on } \partial\Omega, \\ \operatorname{tr}(\star\omega) &= 0, && \text{on } \partial\Omega, \\ \omega(0) &= \omega_0. \end{aligned}$$

The variational formulation (3.27) for advection-diffusion problems of type (3.4) corresponds to the initial boundary value problem

$$\begin{aligned} \partial_t\omega + \delta\,\varepsilon\,\mathsf{d}\,\omega + \mathsf{d}\,\varepsilon\,\delta\,\omega - \mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\omega - \delta\,\mathsf{j}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega, \\ \operatorname{tr}\omega &= \psi_D, && \text{on } \partial\Omega, \\ \operatorname{tr}\star\omega &= 0, && \text{on } \partial\Omega, \\ \omega(0) &= \omega_0. \end{aligned}$$

Under the positivity assumption of Lemmas 3.4.7 and 3.4.10 we can prove uniform L^2-stability for the bilinear forms in the variational formulations (3.26) and (3.27).

Lemma 3.4.14. *If there exists $\alpha \in \Lambda^0(\Omega)$, $\alpha > 0$, such that*

$$\alpha\,\omega\wedge\star\omega + \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega\wedge\star\omega \geq \omega\wedge\star\omega, \quad \forall\omega\in\Lambda^k(\Omega),$$

then for $\omega \in H\Lambda^k(\Omega, 0) \cap H^*\Lambda^k(\Omega, 0)$

$$\begin{aligned} \varepsilon\left(\mathsf{d}\,\omega, \mathsf{d}\,\omega\right)_\Omega + \varepsilon\left(\delta\,\omega, \delta\,\omega\right)_\Omega + \left(\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega, \omega\right)_\Omega + \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\omega\right)_\Omega + \alpha\left(\omega,\omega\right)_\Omega \\ \geq \varepsilon\left\|\mathsf{d}\,\omega\right\|^2_{L^2\Lambda^k(\Omega)} + \varepsilon\left\|\delta\,\omega\right\|^2_{L^2\Lambda^k(\Omega)} + \left\|\omega\right\|^2_{L^2\Lambda^k(\Omega)}. \end{aligned}$$

If there exists $\alpha \in \Lambda^0(\Omega)$, $\alpha > 0$, such that

$$\alpha\,\omega\wedge\star\omega - \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega\wedge\star\omega \geq \omega\wedge\star\omega, \quad \forall\omega\in\Lambda^k(\Omega),$$

then for $\omega \in H\Lambda^k(\Omega, 0) \cap H^*\Lambda^k(\Omega, 0)$

$$\begin{aligned} \varepsilon\left(\mathsf{d}\,\omega, \mathsf{d}\,\omega\right)_\Omega + \varepsilon\left(\delta\,\omega, \delta\,\omega\right)_\Omega - \left(\mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\omega, \omega\right)_\Omega - \left(\omega, \mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega\right)_\Omega + \alpha\left(\omega,\omega\right)_\Omega \\ \geq \varepsilon\left\|\mathsf{d}\,\omega\right\|^2_{L^2\Lambda^k(\Omega)} + \varepsilon\left\|\delta\,\omega\right\|^2_{L^2\Lambda^k(\Omega)} + \left\|\omega\right\|^2_{L^2\Lambda^k(\Omega)}. \end{aligned}$$

Proof. The proof follows by (2.34), (2.36) and (2.37) from

$$\left(\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega, \omega\right)_\Omega + \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\omega\right)_\Omega = \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\,\omega, \omega\right) - \frac{1}{2}\int_{\partial\Omega}\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega\wedge\star\omega + \frac{1}{2}\int_{\partial\Omega}\operatorname{tr}\omega\wedge\star\mathsf{j}_{\boldsymbol{\beta}}\,\omega$$

. □

For $k = 0$ we find from Table 2.2 that $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ corresponds to $-\operatorname{div}\boldsymbol{\beta}$, hence we have found a generalization of the scalar advection-diffusion problem. It is somehow irritating that we need to impose boundary conditions on both $\operatorname{tr}\omega$ and $\operatorname{tr}\star\omega$ which is different from the pure diffusion problem. There well-posedness can be guaranteed by imposing conditions on either of these two.

It well is well known that Galerkin discretizations for variational problems in the spaces $H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega)$ need to be formulated with caution. At first glance the simplest choice for approximation spaces would be some $H^1\Lambda^k(\Omega)$-conforming space. But there are situations [22], e.g. Ω non-convex, where the complement of $H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \cap H^1\Lambda^k(\Omega)$ in $H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega)$ is non-trivial. In that case there are ω in $H\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega)$ that can not be approximated by $H^1\Lambda^k(\Omega)$-conforming approximation spaces and we can not guarantee that the solution of a Galerkin scheme based on such spaces will converge to the right solution.

In the following chapters we will therefore present Galerkin discretizations with either $H\Lambda^k(\Omega)$-conforming approximation spaces, $H^*\Lambda^k(\Omega)$-conforming approximations spaces or $L^2\Lambda^k(\Omega)$-conforming approximation spaces.

3.5 Numerical Methods for the Scalar Problem

To motivate the subsequent chapters we recall here a few important issues for solving non-stationary advection-diffusion methods and give short references to the main methods.

The numerical methods for the scalar advection-diffusion problems (3.1) and (3.2) have to deal with two main difficulties when the diffusion parameter ε is small:

1. The analytical solution of the stationary advection-diffusion problem can have very steep layers along characteristic lines or at the outflow boundary. In such cases the standard numerical methods for elliptic problems usually suffer from so-called spurious oscillations: the numerical solution is highly oscillatory on the entire domain. Only very fine discretizations, meaning very expensive discretizations, can yield reasonable numerical solutions. If one uses the methods of lines approach with implicit numerical integrators for the non-stationary problem, it will be crucial to have cheap and stable methods for the stationary problem.

2. Explicit time-stepping schemes on the other hand, that would circumvent the successive solution of stationary advection-diffusion problems, face serious stability issues when the diffusion parameter tends to zero. Is is for example well known that the explicit Euler method and spatial discretization with central finite differences will be unconditionally unstable for the limit problem.

This first issue is addressed in designing numerical methods that work even in the limit case of advection problems. Here the methods of choice are the Discontinuous Galerkin methods [42,51,74], Galerkin/Least-Squares methods [43] or subgrid viscosity techniques [27]. We also refer to the monograph [76] for a detailed discussion of such methods. Compared to standard Galerkin methods the non-standard methods permit stability and error estimates in certain mesh dependent norms $\|\cdot\|_h$ that are stronger than the L^2-norm, even for $\varepsilon = 0$. Although numerical experiments confirm the superior quality of solutions of non-standard methods the usual error estimates of type:

$$\|\omega - \omega_h\|_h \leq Ch^s$$

do not justify this rigorously. As for the standard Galerkin methods the constant C depends here on higher order derivatives of the solution ω. In the case of layers this constant can be very large. The ultimate justification of non-standard methods are localized error estimates that bound the error on subdomains by constants that depend only on slightly larger subdomains. This proves that steep layers of the analytical solution do have very little influence on the approximation quality in regions away from these layers. We refer to [31] for such estimates for Discontinuous Galerkin methods and to [65] and [76, Theorem 3.41] for a Galerkin/Least-Squares method.

Another issue that is frequently neglected are the different boundary conditions for the limit problem. While for $\varepsilon > 0$ well-posedness requires boundary conditions on the entire boundary, the limiting advection problem needs boundary conditions only on the inflow part of the boundary. It is shown in [7] that an appropriate treatment of boundary conditions can improve the quality of the numerical solutions of non-standard schemes. Here, the key technique is to enforce the boundary conditions weakly such that in the limit $\varepsilon = 0$ only those on the inflow part remain. This basic idea goes goes back to [68] and is related a Lagrange multiplier technique for enforcing boundary conditions [82]. In Discontinuous Galerkin methods for advection-diffusion problems such a treatment of boundary conditions appears naturally in the derivation [42].

A recent result [78] shows even that Galerkin methods that are non-standard only in the treatment of the boundary conditions yield numerical solution that do not suffer from spurious oscillations. To illustrate this we consider the following Galerkin formulation/ of the stationary advection-diffusion problem: Find $u_h \in V_h \subset H^1\Lambda^0(\Omega)$, where V_h is finite dimensional, such that

$$\varepsilon a_1(u_h, v) + a_2(u_h, v) = (f, v)_\Omega + \varepsilon l_1(v) + l_2(v), \quad \forall v \in V_h, \tag{3.32}$$

where

$$\begin{aligned} a_1(u,v) &:= (\mathbf{grad}\, u, \mathbf{grad}\, v)_\Omega - (\mathbf{n}_\Omega \cdot \mathbf{grad}\, u, v)_{\partial\Omega} - (u, \mathbf{n}_\Omega \cdot \mathbf{grad}\, v)_{\partial\Omega} + (\alpha u, v)_{\partial\Omega}, \\ a_2(u,v) &= (\boldsymbol{\beta} \cdot \mathbf{grad}\, u, v)_\Omega - (\boldsymbol{\beta} \cdot \mathbf{n}_\Omega u, v)_{\Gamma_{\text{in}}}, \\ l_1(v) &= (\alpha g, v)_{\partial\Omega}, \\ l_2(v) &= -(\boldsymbol{\beta} \cdot \mathbf{n}_\Omega g, v)_{\Gamma_{\text{in}}}. \end{aligned}$$

The stabilization parameter α is anti-proportional to the local mesh size h. For $\varepsilon > 0$ the variational formulation (3.32) is a discretization of the boundary value problem:

$$-\varepsilon \operatorname{div} \mathbf{grad}\, u + \boldsymbol{\beta} \cdot \mathbf{grad}\, u = f, \quad u = g \text{ on } \partial\Omega. \tag{3.33}$$

In the limiting case $\varepsilon \to 0$ we get a variational formulation for the advection problem with weakly enforced Dirichlet boundary conditions on the inflow part Γ_{in} of the boundary. Then (3.32) is a discretization of the boundary value problem:

$$\boldsymbol{\beta} \cdot \mathbf{grad}\, u = f, \quad u = g \text{ on } \Gamma_{\text{in}}. \tag{3.34}$$

For the solutions of (3.32) we can prove the following local estimates [78, Theorem 5.1].

Theorem 3.5.1 (Schieweck). *Let $V_h \subset H^1\Lambda^0(\Omega)$ be the standard Lagrangian finite element space with local polynomial degree $r > 0$ on some uniform mesh with mesh size h and u_h the solution of the discrete problem* (3.32)*. Assume* $\operatorname{div}\boldsymbol{\beta} > 0$*, $\widetilde{u} \in H^{r+1}\Lambda^0(\Omega)$ for the solution $\widetilde{u}$ of* (3.34) *and $u \in H^2(\Omega)$ for the solution u of* (3.33)*. Furthermore,* *let $\Omega_0 \subset \Omega$ be a subdomain excluding all boundary layers in the sense that there exists a constant C independent of ε such that:*

$$\varepsilon^{-\frac{1}{2}}|u-\widetilde{u}|_{H^1\Lambda^0(\Omega_0)} + \|u-\widetilde{u}\|_{L^2\Lambda^0(\Omega_0)} \leq C\varepsilon.$$

Then for $\varepsilon \leq h$ and $m := 2 - \frac{n}{2}$, it holds

$$\|u-u_h\|_{L^2\Lambda^0(\Omega_0)} \leq C\left(\varepsilon h^{-m} + h^r\right)\|\widetilde{u}\|_{H^{r+1}\Lambda^0(\Omega)}. \tag{3.35}$$

The original results in [78] apply also to non-uniform meshes. The convergence estimate (3.35) states that in the case when h is sufficiently larger than ε the error in a domain excluding the boundary layer is independent of ε and converges with rate r.

Let us illustrate this result for the data $\Omega = [0,1]$, $f(x) = \sin(11\pi x)$, $g(0) = 0$, $g(1) = 0$, $\boldsymbol{\beta} = 1$ and piecewise linear approximation $r = 1$. For small ε the analytical solution has a steep boundary layer at $x = 1$. The standard method, i.e. formulation (3.32) with strongly imposed boundary conditions will yield numerical solutions with large oscillations on the entire domain. For moderate diffusion, i.e. $\varepsilon = 1$ the L^2-error of the numerical solution with weakly imposed boundary conditions does not differ much from the error of the solution with strongly imposed boundary conditions (Figure 3.1). This changes dramatically when we decrease ε to 10^{-6} (Figure 3.2) and 10^{-12} (Figure

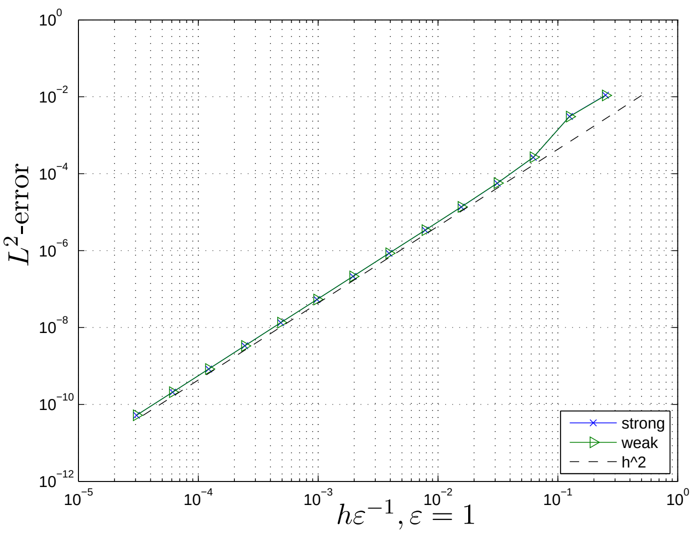

Figure 3.1: L^2-error on $[0, 0.889]$ of the numerical solutions with weakly imposed and strongly imposed boundary conditions, $\varepsilon = 1$.

3.3). Figures 3.2 and 3.3 illustrate the L^2-error for solutions of the different methods on the interval $[0, 0.889]$ excluding the boundary layer. For both methods we see convergence of second order for h large enough, but the error of the solution with weakly imposed boundary conditions is orders of magnitude smaller. The numerical solution does not suffer form spurious oscillations. Figure 3.4 shows the typical numerical solutions for

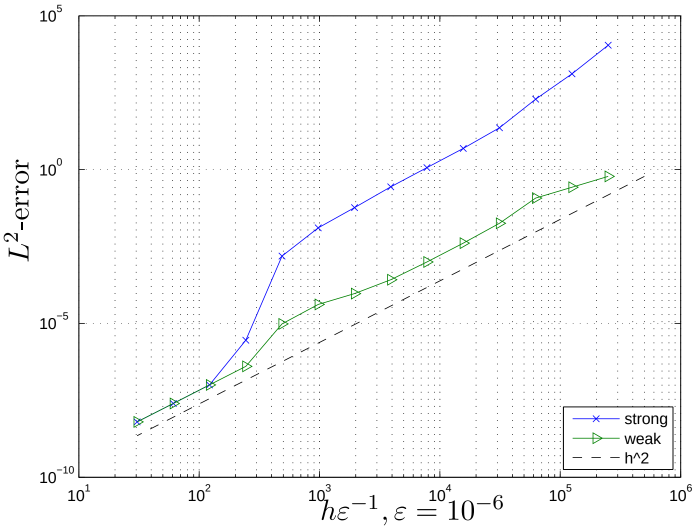

Figure 3.2: L^2-error on $[0, 0.889]$ of the numerical solutions with weakly imposed and strongly imposed boundary conditions, $\varepsilon = 10^{-6}$.

$\varepsilon = 10^{-6}$ on three different meshes. When the mesh size h is larger than ε the solution with strongly imposed boundary conditions is highly oscillatory, while the solution with weakly imposed boundary conditions yields quite accurate approximations.

The difference between the two methods is the ε-dependency of the error for fixed h. In case $h > \varepsilon$ the error of the method with strongly imposed boundary conditions increases dramatically when ε decreases (Figure 3.5).

In conclusion we find that the method of lines with implicit numerical integrators yields valuable methods for the non-stationary problem if the spatial discretization accounts for a proper treatment of the limit problem.

An entirely different approach that circumvents both the stability constraint of explicit numerical integrators and the difficult treatment of stationary advection-diffusion problems are the so-called semi-Lagrangian methods. Semi-Lagrangian methods combine the partial time derivative and the advection operator as one operator. For the limiting problem, the advection problem, the Lagrangian methods are known to be unconditionally stable. Further, in the advection-diffusion case we only need to solve stationary parabolic problems, that do not cause the difficulties encountered for stationary advection-diffusion problems.

In view of the rationale that good numerical methods for singularly perturbed advection-diffusion problems should provide also good solutions for the limit problem we will focus in the following chapters on stationary and non-stationary advection problems.

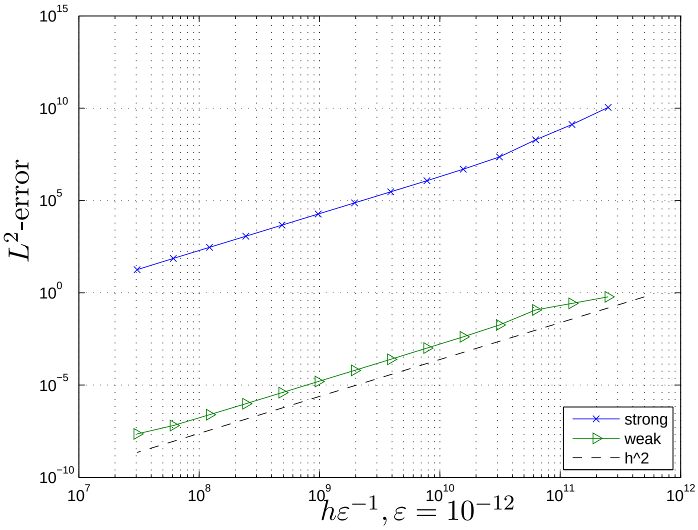

Figure 3.3: L^2-error on $[0, 0.889]$ of the numerical solutions with weakly imposed and strongly imposed boundary conditions, $\varepsilon = 10^{-6}$.

In Chapter 4 we present stabilized methods for the generalized stationary advection-diffusion problem for differential forms. In the next chapter, Chapter 5, we use these stabilized methods to introduce Eulerian methods for the non-stationary problem and present the semi-Lagrangian methods, as an alternative method.

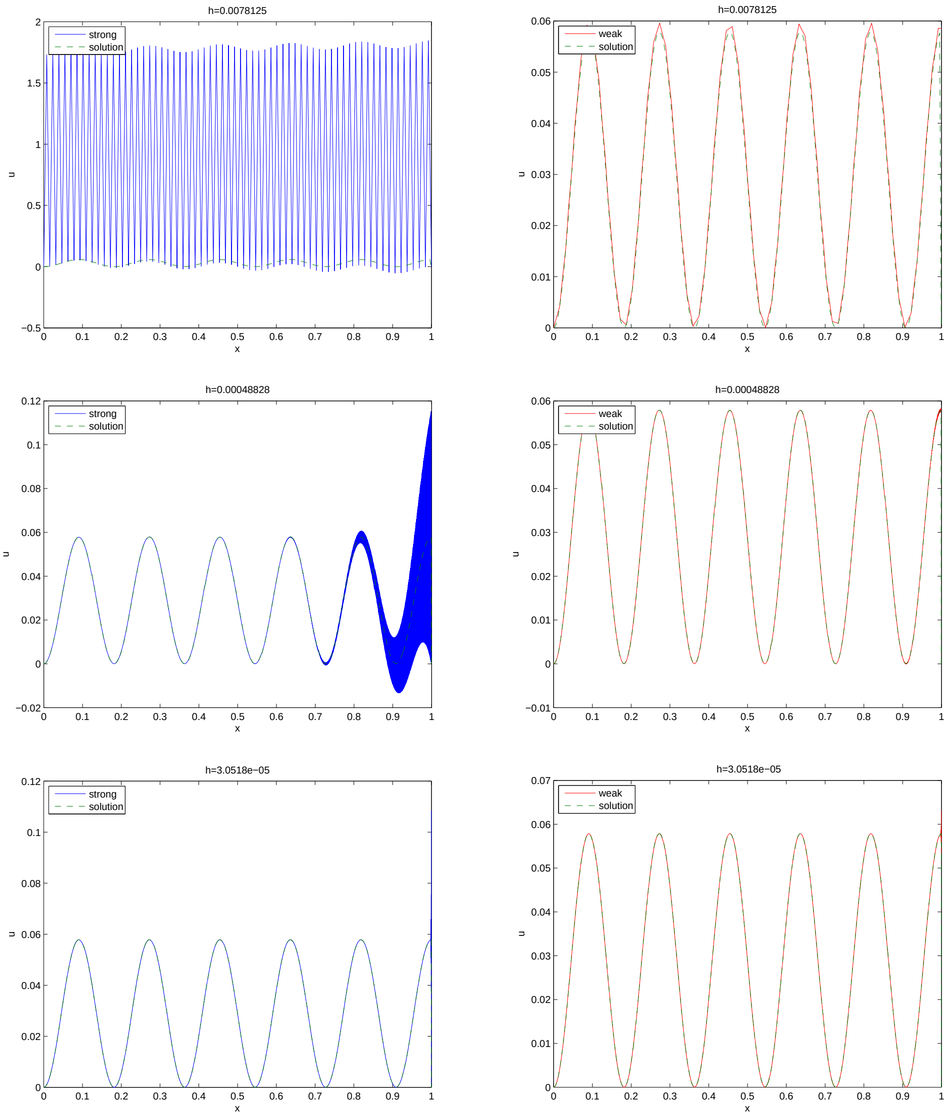

Figure 3.4: Numerical solutions with strongly (left) and weakly (right) imposed boundary conditions on three different meshes with 1.) $h = 7.8125\,10^{-3}$ (top), 2.) $h = 4.8828\,10^{-4}$ and 3.) $h = 3.0518\,10^{-5}$ (bottom) for problem (3.33) with right-hand side $f = \sin(11\pi x)$, velocity $\boldsymbol{\beta} = 1$ and boundary data $u(0) = u(1) = 0$. The analytical solution has a steep layer in the vicinity of $x = 1$. The solution with strongly imposed boundary conditions is highly oscillatory for larger mesh size h. The solution with weakly imposed boundary conditions yields good solutions for small and large mesh sizes.

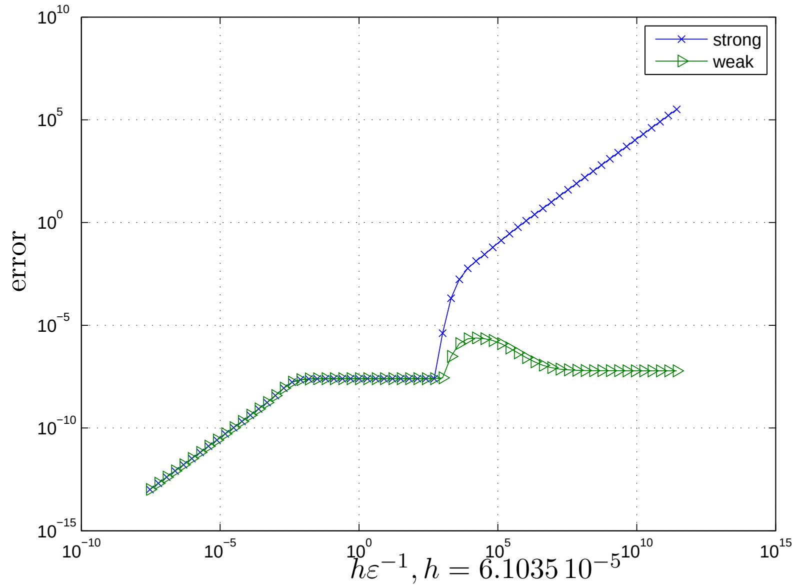

Figure 3.5: L^2-error on $[0, 0.889]$ of the numerical solutions with weakly imposed and strongly imposed boundary conditions, $h = 6.103510^{-5}$.

4 Stationary Advection Problem

The method of lines is a popular method to solve non-stationary problems. If we use such methods with implicit time integrators for the generalized advection-diffusion problem of type (3.3) or (3.3), we have to solve iteratively stationary advection-diffusion problems. Since we focus here on advection-diffusion problems with dominating advection, we have to solve iteratively singularly perturbed stationary advection-diffusion problems. In the scalar case this is known to be a difficult task as outlined in Section 3.5.

Good numerical methods for the scalar stationary advection-diffusion problem, like the SUPG/SDFEM-method [43] or discontinuous Galerkin method with upwind fluxes [42, 51, 74], yield admissible solutions even for the limiting problem, the advection problem. Further, since discretization of generalized diffusion problems for differential forms is well established nowadays [3, 36] we can focus here as well on numerical methods for stationary advection problems.

In this chapter we present two different stabilized Galerkin methods for the stationary advection problems

$$\begin{aligned} \alpha\omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega,\\ \operatorname{tr}\omega &= \operatorname{tr}\psi_D, && \text{on } \Gamma_{\text{in}},\\ \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\psi_D, && \text{on } \Gamma_{\text{in}} \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} \alpha\widetilde{\omega} - \mathcal{L}_{\boldsymbol{\beta}}\,\widetilde{\omega} &= \widetilde{\varphi}, && \text{in } \Omega,\\ \operatorname{tr}\widetilde{\omega} &= \operatorname{tr}\widetilde{\psi}_D, && \text{on } \Gamma_{\text{in}},\\ \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\widetilde{\omega} &= \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\widetilde{\psi}_D, && \text{on } \Gamma_{\text{in}} \end{aligned} \tag{4.2}$$

with data $\varphi, \widetilde{\varphi} \in L^2\Lambda^k(\Omega)$ and $\psi_D, \widetilde{\psi}_D \in \Lambda^k(\mathbb{R}^n \setminus \Omega)$. $\alpha \in \Lambda^0(\Omega)$ is a given scalar parameter and $\beta : \Omega \mapsto \mathbb{R}^n$ is a given Lipschitz continuous velocity field. The advection problem (4.1) is the limiting problem of an advection-diffusion problem of type (3.3). Problem (4.2) is the limiting problem of a generalized advection-diffusion problem of type (3.4).

We have shown in Lemma 3.4.7 that (4.1) is well-posed for $\omega \in W$, with $W = \{\omega \in L^2\Lambda^k(\Omega), \mathsf{L}_{\boldsymbol{\beta}}\,\omega \in L^2\Lambda^k(\Omega), |\omega|_{\Gamma_{\text{in}},-\boldsymbol{\beta}} < \infty\}$ under the assumption that there exist $\alpha_0 \in \mathbb{R}$, $\alpha_0 > 0$ with

$$\alpha\omega \wedge \star\omega + \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega \wedge \star\omega \geq \alpha_0\omega \wedge \star\omega, \quad \forall\omega \in \Lambda^k(\Omega). \tag{4.3}$$

Similar Lemma 3.4.10 proves well-posedness of (4.1) for $\widetilde{\omega} \in \widetilde{W} = \{\omega \in L^2\Lambda^k(\Omega), \mathcal{L}_{\boldsymbol{\beta}}\,\omega \in L^2\Lambda^k(\Omega), |\omega|_{\Gamma_{\text{in}},-\boldsymbol{\beta}} < \infty\}$ if there exist $\alpha_0 \in \mathbb{R}$, $\alpha_0 > 0$ with

$$\alpha\omega \wedge \star\omega - \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega \wedge \star\omega \geq \alpha_0\omega \wedge \star\omega, \quad \forall\omega \in \Lambda^k(\Omega). \tag{4.4}$$

By (2.39) and Proposition 2.1.3 we know that $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ is a self-adjoint operator on $L^2\Lambda^k(\Omega)$, i.e. (4.3) and (4.4) are positivity assumptions on the self-adjoint operators $\alpha\mathsf{id} + \frac{1}{2}(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})$ and $\alpha\mathsf{id} - \frac{1}{2}(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})$. In particular the assumption (4.3) and (4.4) are assumption on the problem parameters α and $\boldsymbol{\beta}$. We refer to the Tables 2.2 and 2.3 for concrete representation of $\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}$ as non-differential operator for forms in $\mathbb{R}^2$ and $\mathbb{R}^3$. For the cases $k = 0$ and $k = n$ we recover the standard positivity assumptions $\alpha - \frac{1}{2}\operatorname{div}\boldsymbol{\beta} \geq \alpha_0$ and $\alpha + \frac{1}{2}\operatorname{div}\boldsymbol{\beta} \geq \alpha_0$. In introducing in (3.3) and (3.4) a scaling $\omega' = e^{\alpha'}\omega$ with appropriate $\alpha' \in \mathbb{R}$ we can always take the positivity assumptions (4.3) and (4.4) for granted.

Note that problems (4.1) and (4.2) are two different advection problems for k-forms. Replacing in problem (4.2) the k-form $\widetilde{\omega}$ with the k-form $\star\omega$, where ω is a $n-k$ form we obtain a problem of type (4.1) for ω. This means that it is enough to study problem (4.1), if we consider both discrete differential k- and $n-k$ forms as approximation spaces for the advection problem (4.1) for k-forms.

We will first introduce a family of stabilized Galerkin methods for advection problems of k-forms that include the Discontinuous Galerkin methods for scalar advection problems. The second family of methods, so-called characteristic methods, presented thereafter is inspired by semi-Lagrangian time-stepping schemes. Here the "time"-parameter τ appears as an artificial, user-defined parameter. We show that our stabilized Galerkin methods are the limit of the characteristic methods when $\tau \to 0$.

4.1 Stabilized Galerkin Methods

The derivation of the method for k-forms corresponds to the derivation of the Discontinuous Galerkin method for scalar advection in [13]. Accordingly we get stability and consistency of the method in the general case. The convergence proofs we give afterwards will cover only the different cases in $\mathbb{R}^3$. The proofs involve certain technicalities that can not be expressed with the limited notations from exterior calculus introduced here. Since $\mathbb{R}^3$ is our main interest this is no severe limitation. Apparently, similar results for the lower-dimensional cases $\mathbb{R}^2$ and $\mathbb{R}^1$ follow by identical arguments.

4.1.1 Derivation of the Method

Let $\mathcal{T}$ be a simplicial triangulation of Ω. We will call the oriented n-simplices $T \in \Delta_n(\mathcal{T})$ and $n-1$-simplices $f \in \Delta_{n-1}(\mathcal{T})$ elements and facets of the mesh $\mathcal{T}$. An oriented facet f has a distinguished normal $\mathbf{n}_f$. If a facet f is contained in the boundary of some element T then either $\mathbf{n}_f = \mathbf{n}_{\Omega|_f}$ or $\mathbf{n}_f = -\mathbf{n}_{\Omega|_f}$. Then ω^+ and ω^- denote the two different restrictions of $\omega \in \Lambda^k(\Omega)$ to f, e.g. $\omega^+ := \omega_{|_{T^+}}$ where element T^+ has outward normal $\mathbf{n}_f$. With these restriction we define also the jump $[\omega]_f = \omega^+ - \omega^-$ and the average $\{\omega\}_f = \frac{1}{2}(\omega^+ + \omega^-)$. For $f \subset \partial\Omega$ we assume f to be oriented such that $\mathbf{n}_f$ points outwards. Let $\mathcal{F}^\circ$ and $\mathcal{F}^\partial$ be the set of interior and boundary facets. $\mathcal{F}^\partial_-, \mathcal{F}^\partial_+ \subset \mathcal{F}^\partial$ is the set of facets on the inflow and outflow boundary.

Further let $\Lambda_h^k(\mathcal{T})$ denote some piecewise polynomial approximation space for differential k-forms. Here $\Lambda_h^k(\mathcal{T})$ could be either one of the conforming approximation spaces $\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ but also the non-conforming space $\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ or even $\star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ or $\star\mathcal{P}_r^-\Lambda^{n-k}(\mathcal{T})$.

We fix some element T, test (4.1) with $\star\eta$, $\eta \in \Lambda_h^k(\mathcal{T})$, integrate the product over T and apply the Leibniz rule for Lie derivatives (2.36):

$$(\alpha\omega,\eta)_T + (\omega,\mathcal{L}_{\boldsymbol{\beta}}\eta)_T + (\omega,\eta)_{\partial T,\boldsymbol{\beta}} = (\varphi,\eta)_T\,.$$

Summing this equation over all elements yields:

$$(\alpha\omega,\eta)_\Omega + \sum_T (\omega,\mathcal{L}_{\boldsymbol{\beta}}\eta)_T + \sum_T (\omega,\eta)_{\partial T,\boldsymbol{\beta}} = (\varphi,\eta)_\Omega\,,$$

or, if we write the sum over boundaries of elements as sum over facets:

$$(\alpha\omega,\eta)_\Omega + \sum_T (\omega,\mathcal{L}_{\boldsymbol{\beta}}\eta)_T + \sum_{f\in\mathcal{F}^\circ} \left(\omega^+,\eta^+\right)_{f,\boldsymbol{\beta}} - \left(\omega^-,\eta^-\right)_{f,\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial} (\omega,\eta)_{f,\boldsymbol{\beta}} = (\varphi,\eta)_\Omega\,.$$

The identity

$$\left(\omega^+,\eta^+\right)_{f,\boldsymbol{\beta}} - \left(\omega^-,\eta^-\right)_{f,\boldsymbol{\beta}} = \Big([\omega]_f,\{\eta\}_f\Big)_{f,\boldsymbol{\beta}} + \Big(\{\omega\}_f,[\eta]_f\Big)_{f,\boldsymbol{\beta}} \tag{4.5}$$

shows

$$\left(\omega^+,\eta^+\right)_{f,\boldsymbol{\beta}} - \left(\omega^-,\eta^-\right)_{f,\boldsymbol{\beta}} = \Big(\{\omega\}_f,[\eta]_f\Big)_{f,\boldsymbol{\beta}}$$

for solutions $\omega \in W$ of the advection problem (4.1), since ω is non-smooth only across characteristic faces, i.e. those faces f with $\mathbf{n}_f\cdot\boldsymbol{\beta} = 0$. But for f with $\mathbf{n}_f\cdot\boldsymbol{\beta} = 0$ we have $(\cdot,\cdot)_{f,\boldsymbol{\beta}} = 0$, anyway. We are now in the position to define a stabilized Galerkin scheme for the advection problem (4.1):

Find $\omega_h \in \Lambda_h^k(\mathcal{T})$ such that:

$$\mathsf{a}(\omega_h,\eta) = \mathsf{l}(\eta)\,,\quad \forall\eta\in\Lambda_h^k(\mathcal{T})\,, \tag{4.6}$$

with

$$\mathsf{l}(\eta) := (\varphi,\eta)_\Omega - \sum_{f\in\mathcal{F}_-^\partial} (\psi_D,\eta)_{f,\boldsymbol{\beta}} \tag{4.7}$$

and

$$\begin{aligned}\mathsf{a}(\omega,\eta) := {} & (\alpha\omega,\eta)_\Omega + \sum_T (\omega,\mathcal{L}_{\boldsymbol{\beta}}\eta)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} (\omega,\eta)_{f,\boldsymbol{\beta}} \\ & + \sum_{f\in\mathcal{F}^\circ} \Big(\{\omega\}_f,[\eta]_f\Big)_{f,\boldsymbol{\beta}} + \Big(c_f\,[\omega]_f,[\eta]_f\Big)_{f,\boldsymbol{\beta}}\,,\end{aligned} \tag{4.8}$$

where c_f is a stabilization parameter that need to be specified later. Since the stabilization terms $\Big(c_f\,[\omega]_f,[\eta]_f\Big)_{f,\boldsymbol{\beta}}$ vanish for $\omega \in W$ solution to (4.1), the derivation proves consistency.

Lemma 4.1.1. *The variational formulation* (4.6) *is consistent with problem* (4.1).

Remark 4.1.2. *The choice* $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$ *yields a scheme with so-called upwind fluxes:*

$$\left(\{\omega\}_f, [\eta]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\omega]_f, [\eta]_f\right)_{f,\boldsymbol{\beta}} = \left(\frac{1}{2}\left(1+\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}\right)\omega^+ + \frac{1}{2}\left(1-\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}\right)\omega^-, [\eta]_f\right)_{f,\boldsymbol{\beta}}.$$

When we want to implement our variational formulation we realize that the evaluation of the terms $(\omega, \mathcal{L}_{\boldsymbol{\beta}}\,\eta)_T$ requires knowledge of first order derivatives of $\boldsymbol{\beta}$ due to $\mathcal{L}_{\boldsymbol{\beta}} = \delta\,\mathsf{j}_{\boldsymbol{\beta}} + \mathsf{j}_{\boldsymbol{\beta}}\,\delta$. Therefore, the representation of $\mathsf{a}(\omega,\eta)$ in the following proposition is much more convenient for implementation. See also the Appendix 4.1.5 for a representation of $\mathsf{a}(\omega,\eta)$ in vector proxies in $\mathbb{R}^3$.

Proposition 4.1.3. *The following equality holds for all* $\omega,\eta \in \Lambda_h^k(\mathcal{T})$*:*

$$\begin{aligned}\mathsf{a}(\omega,\eta) = (\alpha\omega,\eta)_\Omega &+ \sum_T (\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega,\eta)_T + \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\delta\,\eta\right)_T \\ &+ \sum_{f\in\mathcal{F}^\circ} \left(\mathsf{i}_{\boldsymbol{\beta}}\,\{\omega\}_f, [\eta]_f\right)_{f,\mathrm{tr}} - \left([\omega]_f, \mathsf{j}_{\boldsymbol{\beta}}\,\{\eta\}_f\right)_{f,\mathrm{tr}} + \left(c_f\,[\omega]_f, [\eta]_f\right)_{f,\boldsymbol{\beta}} \\ &+ \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\mathsf{i}_{\boldsymbol{\beta}}\,\omega,\eta)_{f,\mathrm{tr}} - \sum_{f\in\mathcal{F}^\partial_-} \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{f,\mathrm{tr}}.\end{aligned}$$

Proof. The proof follows from the Leibniz rules for exterior derivatives (2.11) and contractions (2.25) and the identity $\omega^+\wedge\eta^+ - \omega^-\wedge\eta^- = [\omega]_f \wedge \{\eta\}_f + \{\omega\}_f\,[\eta]_f$:

$$\begin{aligned}\sum_T \left(\omega, \delta\,\mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_T &\overset{(2.34)}{=} \sum_T \left(\mathsf{d}\,\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_T - \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{\partial T,\mathrm{tr}} \\ &\overset{(2.37)}{=} \sum_T (\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega,\eta)_T - \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{\partial T,\mathrm{tr}} \\ &= \sum_T (\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega,\eta)_T - \sum_{f\in\mathcal{F}^\partial} \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{f,\mathrm{tr}} \\ &\quad - \sum_{f\in\mathcal{F}^\circ} \left(\omega^+, \mathsf{j}_{\boldsymbol{\beta}}\,\eta^+\right)_{f,\mathrm{tr}} - \left(\omega^-, \mathsf{j}_{\boldsymbol{\beta}}\,\eta^-\right)_{f,\mathrm{tr}} \\ &= \sum_T (\mathsf{i}_{\boldsymbol{\beta}}\,\mathsf{d}\,\omega,\eta)_T - \sum_{f\in\mathcal{F}^\partial} \left(\omega, \mathsf{j}_{\boldsymbol{\beta}}\,\eta\right)_{f,\mathrm{tr}} \\ &\quad - \sum_{f\in\mathcal{F}^\circ} \left([\omega]_f, \mathsf{j}_{\boldsymbol{\beta}}\,\{\eta\}_f\right)_{f,\mathrm{tr}} + \left(\{\omega\}_f, \mathsf{j}_{\boldsymbol{\beta}}\,[\eta]_f\right)_{f,\mathrm{tr}}.\end{aligned}$$

Then we get

$$\begin{aligned}\mathsf{a}(\omega,\eta) = (\alpha\omega,\eta)_\Omega &+ \sum_T \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\delta\,\eta\right)_T + \left(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\eta\right)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\omega,\eta)_{f,\boldsymbol\beta} \\ &+ \sum_{f\in\mathcal{F}^\circ} \left(\{\omega\}_f, [\eta]_f\right)_{f,\boldsymbol\beta} + \left(c_f\,[\omega]_f, [\eta]_f\right)_{f,\boldsymbol\beta} \\ &- \sum_{f\in\mathcal{F}^\partial} \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\eta\right)_{f,\mathrm{tr}} - \sum_{f\in\mathcal{F}^\circ} \left([\omega]_f, \mathsf{j}_{\boldsymbol\beta}\,\{\eta\}_f\right)_{f,\mathrm{tr}} + \left(\{\omega\}_f, \mathsf{j}_{\boldsymbol\beta}\,[\eta]_f\right)_{f,\mathrm{tr}}\end{aligned}$$

and the assertion follows from (2.35). □

If we use conforming discrete differential forms spaces as approximation spaces the bilinear form in proposition 4.1.3 simplifies further.

Proposition 4.1.4. *For $\omega,\eta\in\mathcal{P}_r\Lambda^k(\mathcal{T})$ and $\omega,\eta\in\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ it holds:*

$$\begin{aligned}\mathsf{a}(\omega,\eta) = (\alpha\omega,\eta)_\Omega &+ \sum_T \left(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\eta\right)_T + \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\delta\,\eta\right)_T \\ &+ \sum_{f\in\mathcal{F}^\circ} \left(\mathsf{i}_{\boldsymbol\beta}\,\{\omega\}_f, [\eta]_f\right)_{f,\mathrm{tr}} + \left(c_f\,\mathsf{i}_{\boldsymbol\beta}\,[\omega]_f, [\eta]_f\right)_{f,\mathrm{tr}} \\ &+ \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \left(\mathsf{i}_{\boldsymbol\beta}\,\omega,\eta\right)_{f,\mathrm{tr}} - \sum_{f\in\mathcal{F}^\partial_-} \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\eta\right)_{f,\mathrm{tr}}.\end{aligned}$$

For $\omega,\eta\in\star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ and $\omega,\eta\in\star\mathcal{P}_r^-\Lambda^{n-k}(\mathcal{T})$ it holds:

$$\begin{aligned}\mathsf{a}(\omega,\eta) = (\alpha\omega,\eta)_\Omega &+ \sum_T \left(\mathsf{i}_{\boldsymbol\beta}\,\mathsf{d}\,\omega,\eta\right)_T + \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\delta\,\eta\right)_T \\ &- \sum_{f\in\mathcal{F}^\circ} \left([\omega]_f, \mathsf{j}_{\boldsymbol\beta}\,\{\eta\}_f\right)_{f,\mathrm{tr}} - \left(c_f\,[\omega]_f, \mathsf{j}_{\boldsymbol\beta}\,[\eta]_f\right)_{f,\mathrm{tr}} \\ &+ \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \left(\mathsf{i}_{\boldsymbol\beta}\,\omega,\eta\right)_{f,\mathrm{tr}} - \sum_{f\in\mathcal{F}^\partial_-} \left(\omega, \mathsf{j}_{\boldsymbol\beta}\,\eta\right)_{f,\mathrm{tr}}.\end{aligned}$$

Proof. Recall that $(\omega,\eta)_{f,\mathrm{tr}} = \int_f \mathrm{tr}(\omega)\wedge\mathrm{tr}(\star\eta)$, $\mathrm{tr}([\omega]_f) = 0$ for $\omega\in\mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\omega\in\mathcal{P}_r^-\Lambda^k(\mathcal{T})$ and $\mathrm{tr}(\star\,[\omega]_f) = 0$ for $\omega\in\star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ or $\omega\in\star\mathcal{P}_r^-\Lambda^{n-k}(\mathcal{T})$. □

We proceed by proving stability in the mesh dependent norm:

$$\|\omega\|_h^2 := \|\omega\|^2_{L^2\Lambda^k(\Omega)} + \sum_{f\in\mathcal{F}^\circ} \left\|[\omega]_f\right\|^2_{f,c_f\boldsymbol\beta} + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \|\omega\|^2_{f,\frac{1}{2}\boldsymbol\beta} + \sum_{f\in\mathcal{F}^\partial_-} \|\omega\|^2_{f,-\frac{1}{2}\boldsymbol\beta}, \quad (4.9)$$

with the obvious definition $\|\cdot\|^2_{f,\boldsymbol\beta} := (\omega,\omega)_{f,\boldsymbol\beta}$. $\|\cdot\|_h$ is a norm for any choice c_f with $c_f\boldsymbol\beta\cdot\mathbf{n}_f\geq 0$, because then $(c_f\omega,\omega)_{f,\boldsymbol\beta}$ is non-negative according to remark 2.1.2.

Lemma 4.1.5. *Assume the parameter c_f satisfies for all faces f the non-negativity condition*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f \geq 0$$

and (4.3)*, i.e. that α and β are such that there is a positive constant α_0 with*

$$\alpha\omega \wedge \star\omega + \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega \wedge \star\omega \geq \alpha_0 \omega \wedge \star\omega, \quad \forall \omega \in \Lambda^k(\Omega).$$

Then we have for all $\omega \in \Lambda^k_h(\mathcal{T})$:

$$\mathsf{a}(\omega, \omega) \geq \min(\alpha_0, 1) \|\omega\|_h^2.$$

Proof. This follows from the Leibniz rule (2.36), starting from (4.8):

$$\begin{aligned}
\mathsf{a}(\omega,\omega) = &(\alpha\omega,\omega)_\Omega + \sum_T (\omega, \mathcal{L}_{\boldsymbol{\beta}}\omega)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\omega,\omega)_{f,\boldsymbol{\beta}} \\
&+ \sum_{f\in\mathcal{F}^\circ} \left(\{\omega\}_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f [\omega]_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} \\
= &(\alpha\omega,\omega)_\Omega + \sum_T \left(\omega, \frac{1}{2}(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega\right)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\omega,\omega)_{f,\boldsymbol{\beta}} \\
&+ \sum_{f\in\mathcal{F}^\circ} \left(\{\omega\}_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f [\omega]_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} \\
&- \frac{1}{2}\sum_{f\in\mathcal{F}^\circ} \left(\{\omega\}_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} + \left([\omega]_f, \{\omega\}_f\right)_{f,\boldsymbol{\beta}} - \frac{1}{2}\sum_{f\in\mathcal{F}^\partial} (\omega,\omega)_{f,\boldsymbol{\beta}} \\
= &\sum_T \left(\omega, \alpha + \frac{1}{2}(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}})\omega\right)_T + \sum_{f\in\mathcal{F}^\circ} \left(c_f [\omega]_f, [\omega]_f\right)_{f,\boldsymbol{\beta}} \\
&+ \frac{1}{2}\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\omega,\omega)_{f,\boldsymbol{\beta}} - \frac{1}{2}\sum_{f\in\mathcal{F}^\partial_-} (\omega,\omega)_{f,\boldsymbol{\beta}} \\
\geq &\min(\alpha_0, 1)\|\omega\|_h^2,
\end{aligned}$$

since $(\omega,\omega)_{f,\boldsymbol{\beta}} \geq 0$ for $f \in \mathcal{F}^\partial \setminus \mathcal{F}^\partial_-$. □

Remark 4.1.6. *If we have in Lemma 4.1.5 that $c_f = 0$ then*

$$\mathsf{a}(\omega, \omega) \geq \min(\alpha_0, 1) \|\omega\|^2_{L^2\Lambda^k(\Omega)}.$$

Remark 4.1.7. *All assertions on convergence of Galerkin schemes, e.g. Theorems 4.1.8-4.1.16 will be based on Lemma 4.1.5. That means they will assume* (4.3)*. For concrete problems with solution ω and Galerkin solution ω_h it would be enough to assume L^2-stability of $\mathsf{a}(\bar{\omega}_h - \omega_h, \bar{\omega}_h - \omega_h)$, where $\bar{\omega}_h$ is the L^2-orthogonal projection of ω.*

4.1.2 Convergence: Non-Conforming Discrete Differential Forms in $\mathbb{R}^n$

Now, we prove convergence of our formulation (4.6) for the non-conforming discrete differential forms space $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$.

Theorem 4.1.8. *Assume that* (4.3) *holds for α and β in* (4.1) *and that c_f in* (4.6) *is such that*

$$c_f\boldsymbol{\beta}\cdot\mathbf{n}_f > 0 \quad \forall f \in \mathcal{F}^\circ.$$

Let $\omega \in W$ and $\omega_h \in \Lambda_h^k(\mathcal{T})$ be the solutions to the advection problem (4.1) *and its variational formulation* (4.6)*. If additionally $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ and $\omega \in H^{r+1}\Lambda^k(\Omega)$, $r \geq 0$, we get with $C > 0$ independent of mesh size $h := \max_T(h_T)$*

$$\|\omega - \omega_h\|_h \leq Ch^{r+\frac{1}{2}}\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}.$$

Proof. Let $\bar{\omega}_h$ denote the L^2-projection of ω on $\mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$. Then stability (Lemma 4.1.5) and consistency (Lemma 4.1.1) show

$$\min(\alpha_0, 1)\,\|\omega_h - \bar{\omega}_h\|_h^2 \leq \mathsf{a}\,(\omega_h - \bar{\omega}_h, \omega_h - \bar{\omega}_h) = \mathsf{a}\,(\eta, \gamma_h)$$

with $\eta := \omega - \bar{\omega}_h$ and $\gamma_h = \omega_h - \bar{\omega}_h$. Let $\boldsymbol{\beta}_h$ denote the L^2-projection of $\boldsymbol{\beta}$ onto $\left(\mathcal{P}_0^{\mathrm{d}}\Lambda^0(\mathcal{T})\right)^n$, then $\mathcal{L}_{\boldsymbol{\beta}_h}\gamma_h \in \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$, i.e. $\left(\eta, \mathcal{L}_{\boldsymbol{\beta}_h}\gamma_h\right)_T = 0$, and

$$\begin{aligned}\mathsf{a}\,(\eta, \gamma_h) = (\alpha\eta, \gamma_h)_\Omega + \sum_T \left(\eta, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\gamma_h\right)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} (\eta, \gamma_h)_{f,\boldsymbol{\beta}} \\ + \sum_{f\in\mathcal{F}^\circ} \left(\{\eta\}_f, [\gamma_h]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\eta]_f, [\gamma_h]_f\right)_{f,\boldsymbol{\beta}}.\end{aligned} \tag{4.10}$$

The pairing $\left([\eta]_f, [\gamma_h]_f\right)_{f,c_f\boldsymbol{\beta}} = \left(c_f\,[\eta]_f, [\gamma_h]_f\right)_{f,\boldsymbol{\beta}}$ is a semi-definite bilinear form by the assumption $c_f\boldsymbol{\beta}\cdot\mathbf{n}_f \geq 0$. Hence Cauchy-Schwarz inequalities yield:

$$\begin{aligned}(\eta, \gamma_h)_{f,\boldsymbol{\beta}} &\leq \|\eta\|_{f,\boldsymbol{\beta}}\,\|\gamma_h\|_{f,\boldsymbol{\beta}}, && \text{for } f \in \mathcal{F}^\partial\setminus\mathcal{F}^\partial_-,\\ \left(c_f^{-1}\{\eta\}_f + [\eta]_f, [\gamma_h]_f\right)_{f,c_f\boldsymbol{\beta}} &\leq \left\|c_f^{-1}\{\eta\}_f + [\eta]_f\right\|_{f,c_f\boldsymbol{\beta}}\left\|[\gamma_h]_f\right\|_{f,c_f\boldsymbol{\beta}}, && \text{for } f \in \mathcal{F}^\circ\\ \left(\eta, (\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\gamma_h\right)_T &\leq \|\eta\|_{L^2\Lambda^k(T)}\left\|(\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\gamma_h\right\|_{L^2\Lambda^k(T)},\\ (\alpha\eta, \gamma_h)_\Omega &\leq \|\alpha\|_{W^{0,\infty}\Lambda^0(\Omega)}\,\|\eta\|_{L^2\Lambda^k(\Omega)}\,\|\gamma_h\|_{L^2\Lambda^k(\Omega)}.\end{aligned}$$

Analogous to the scalar case, we find

- the multiplicative trace inequality [2]

$$\|\eta\|_{f,c_f\boldsymbol{\beta}}^2 \leq C(h_f^{-1}\,\|\eta\|_{L^2\Lambda^k(T)}^2 + h_f|\eta|_{H^1\Lambda^k(T)}^2),$$

  with diameter h_f of face f and $C > 0$ depending on the minimum angle of T and $\boldsymbol{\beta}$,

- the estimate

$$\left\|(\mathcal{L}_{\boldsymbol{\beta}} - \mathcal{L}_{\boldsymbol{\beta}_h})\gamma_h\right\|_{L^2\Lambda^k(T)} \\ \leq |\boldsymbol{\beta} - \boldsymbol{\beta}_h|_{\boldsymbol{W}^{1,\infty}(T)} \|\gamma_h\|_{L^2\Lambda^k(T)} + \|\boldsymbol{\beta} - \boldsymbol{\beta}_h\|_{\boldsymbol{L}^\infty(T)} |\gamma_h|_{H^1\Lambda^k(T)},$$

- the inverse inequality,

$$|\gamma_h|_{H^1\Lambda^k(T)} \leq C h_T^{-1} \|\gamma_h\|_{L^2\Lambda^k(T)},$$

  with element diameter h_T and $C > 0$ independent of h_T.

These results follow from those for the scalar case by an argument for vector proxies.
In conclusion we find with $C > 0$ independent of h

$$\|\omega_h - \bar{\omega}_h\|_h^2 \leq C \max_T \left(h^{-\frac{1}{2}} \|\omega - \bar{\omega}_h\|_{L^2\Lambda^k(T)} + h^{\frac{1}{2}} \|\omega - \bar{\omega}_h\|_{H^1\Lambda^k(T)}\right) \|\omega_h - \bar{\omega}_h\|_h .$$

Then triangle inequality and the approximation estimates in Theorem 2.2.10 yield the assertion. □

For the particular choice $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k$ the formulation (4.6) can be seen as finite volume scheme. In the case $k = 2$ and $n = 3$ we discover an upwind finite volume scheme that reduces to the one in [29] for Cartesian grids.

Remark 4.1.9. *Consider the variational formulation* (4.6) *with* $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$, $n = 3$, $k = 2$ *and* $\Lambda_h^2(\mathcal{T}) = \mathcal{P}_0^{\mathrm{d}}\Lambda^2(\mathcal{T})$. *Let* $\mathbf{B}$ *denote the vector correspondence to* $\omega \in \mathcal{P}_0^{\mathrm{d}}\Lambda^2(\mathcal{T})$. $\mathbf{B}$ *has three components* $\mathbf{B}^i$ *that are piecewise constant on the elements* T. *A simple calculation shows that* $\mathcal{L}_{\boldsymbol{\beta}}\omega$ *corresponds to* $-D\boldsymbol{\beta}^T\mathbf{B}$. *We introduce the coefficients*

$$\beta_{f,T}^- := \min\left(0, \int_{\partial T} \boldsymbol{\beta}_{|_f} \cdot \mathbf{n}_T \,\mathrm{d}S\right)$$
$$\beta_{f,T}^+ := \max\left(0, \int_{T} \boldsymbol{\beta}_{|_f} \cdot \mathbf{n}_T \,\mathrm{d}S\right)$$

and find that the variational formulation (4.6) *is*

$$\alpha\mathbf{B}_{|_T}^i - (D\boldsymbol{\beta}\mathbf{B})_{|_T}^i + \frac{1}{|T|}\sum_{f\in\mathcal{F}\setminus\mathcal{F}_-^\partial} \left(\beta_{f,T}^+\mathbf{B}_{|_T}^i - \beta_{f,T}^-\mathbf{B}_{|_{T^-}}^i\right) = \mathbf{F}_{|_T}^i, \quad \forall i \text{ and } \forall T,$$

where $T^- \neq T$ *is the other adjacent element of* f_T *if* $f_T \not\subset \partial\Omega$ *and* $\mathbf{B}_{|_{T^-}}^i = 0$ *otherwise. Note that the coefficients* $\beta_{f,T}^+$ *and* $\beta_{f,T}^-$ *select the upwind values of* $\mathbf{B}$ *and follow from our particular choice of* c_f *(see Remark 4.1.2). Lemma 4.1.5 and Theorem 4.1.8 prove stability and convergence of such finite volume schemes.*

For the non-stabilized scheme, i.e. $c_f = 0$ in (4.6), we get a sub-optimal convergence estimate, since we have to use another inverse inequality to bound the facet integrals $\|\gamma_h\|_{f,\boldsymbol{\beta}}$ by L^2-norms on elements [13, p. 1902].

Remark 4.1.10. *Assume that* (4.3) *holds for* α *and* β *in* (4.1) *and that* $c_f = 0$ *in* (4.6). *Let* $\omega \in W$ *and* $\omega_h \in \Lambda_h^k(\mathcal{T})$ *be the solutions to the advection problem* (4.1) *and its variational formulation* (4.6). *If additionally* $\omega \in H^{r+1}\Lambda^k(\Omega)$ *and* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$, $r \geq 0$ *for we get with* $C > 0$ *independent of mesh size* h

$$\|\omega - \omega_h\|_{L^2\Lambda^k(\Omega)} \leq Ch^r \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}.$$

In particular we have no convergence for $r = 0$*: If we work with piecewise constant approximation spaces* $\mathcal{P}_0^{\mathrm{d}}\Lambda^k$ *we can not prove that the numerical solution of the non-stabilized scheme converges.*

The crucial step (4.10) in the proof of Theorem 4.1.8 is based on the property $\mathcal{L}_{\boldsymbol{\beta}_h}\gamma_h \in \Lambda_h^k(\mathcal{T})$ for $\gamma_h \in \Lambda_h^k(\mathcal{T})$ and piecewise constant velocity fields $\boldsymbol{\beta}_h \in \left(\mathcal{P}_0^{\mathrm{d}}\Lambda^0(\mathcal{T})\right)^n$. For approximation spaces $\Lambda_h^k(\mathcal{T})$ with global continuity properties this will not follow straight forwardly. At first, we can only give a suboptimal estimate.

Theorem 4.1.11. *Assume that* (4.3) *holds for* α *and* β *in* (4.1) *and that* c_f *in* (4.6) *is such that*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f \geq 0, \quad \forall f \in \mathcal{F}^\circ.$$

Let $\omega \in H^{r+1}\Lambda^k(\Omega)$ *and* $\omega_h \in \Lambda_h^k(\mathcal{T})$ *be the solutions to the advection problem* (4.1) *and its variational formulation* (4.6). *If additionally either* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$, $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$, $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ *or* $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^{n-k}(\mathcal{T})$ *we get with* $C > 0$ *independent of the mesh size* $h := \max_T(h_T)$

$$\|\omega - \omega_h\|_h \leq Ch^r \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}.$$

Proof. The proof is similar to the proof of Theorem (4.1.8), but without step (4.10). This means that we need an additional inverse estimate that causes the non-optimal rate. □

This result is very unsatisfactory, because it does not prove convergence for lowest order conforming discrete differential forms. In the next section we show therefore how to establish optimal convergence estimates for conforming discrete differential forms spaces in $\mathbb{R}^3$. As explained earlier, the proofs involve certain technicalities that can not be expressed with the limited notations from exterior calculus introduced here.

4.1.3 Convergence: Conforming Discrete Differential Forms in $\mathbb{R}^3$

Notation: We use here boldface letters to denote the vector proxies of 1-form and 2-forms. Non-boldface letters stand for 0-forms or 3-forms. While we keep the introduced notion for the various spaces, e.g. $H\Lambda^1(\Omega)$ instead of $\boldsymbol{H}(\mathbf{curl}, \Omega)$, we use now $\|\cdot\|_{L^2(\Omega)}$ ($\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$) and $\|\cdot\|_{H^r(\Omega)}$ ($\|\cdot\|_{\boldsymbol{H}^r(\Omega)}$) to denote L^2 and Sobolev norms of scalar (vectorial) functions. Besides the obvious correspondences to L^2 and Sobolev norm for differential forms we have e.g. that $\left\|\sqrt{c_f \boldsymbol{\beta}\cdot\mathbf{n}_f}\cdot\right\|_{L^2(f)}$ and $\left\|\sqrt{c_f \boldsymbol{\beta}\cdot\mathbf{n}_f}\cdot\right\|_{\boldsymbol{L}^2(f)}$ corresponds to $\|\cdot\|_{f,c_f\boldsymbol{\beta}}$, compare (2.35).

The convergence results for the advection problems for 3-form and 0-forms are classical results in numerical analysis. The approximation spaces are either piecewise polynomial globally discontinuous or piecewise polynomial globally continuous approximation spaces and the convergence proofs are standard theory of either discontinuous or continuous Galerkin methods (see [27, page 265] and [13]). To adapt to the proof of Theorem 4.1.8 we need to introduce so-called *averaging interpolations operators* mapping non-conforming discrete differential forms to conforming discrete differential forms. Such interpolation operators have been used previously in the analysis of Discontinuous Galerkin methods ([41, Appendix], [39, Appendix] and [49]). Here, we need averaging interpolation operators for discrete 0-forms, 1-forms and 2-forms. As a result we get the following approximation results on conforming approximation for discontinuous scalar and vectorial functions.

Proposition 4.1.12. *Let* $\mathbf{u} \in \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T}) = \star\mathcal{P}_r^{\mathrm{d}}\Lambda^2(\mathcal{T})$ *and* $u \in \mathcal{P}_r^{\mathrm{d}}\Lambda^0(\mathcal{T}) = \star\mathcal{P}_r^{\mathrm{d}}\Lambda^3(\mathcal{T}) = \star\mathcal{P}_r\Lambda^3(\mathcal{T})$. *Then there exist* $\mathbf{u}^{c,1} \in \mathcal{P}_r\Lambda^1(\mathcal{T}) \subset \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$, $\mathbf{u}^{c,2} \in \mathcal{P}_r\Lambda^2(\mathcal{T}) \subset \mathcal{P}_{r+1}^-\Lambda^2(\mathcal{T})$ *and* $u^{c,0} \in \mathcal{P}_r\Lambda^0(\mathcal{T}) \subset \mathcal{P}_{r+1}^-\Lambda^0(\mathcal{T})$ *such that*

$$\left\|\mathbf{u}-\mathbf{u}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \le C_1\left(\sum_{f\in\mathcal{F}^\circ} h_f \int_f \left|[\mathbf{u}]_f \times \mathbf{n}_f\right|^2 \,\mathrm{d}S + \sum_{f\in\mathcal{F}^\partial} h_f \int_f \left|\mathbf{u}\times\mathbf{n}_f\right|^2 \,\mathrm{d}S\right), \quad (4.11)$$

$$\left\|\mathbf{u}-\mathbf{u}^{c,2}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \le C_2\left(\sum_{f\in\mathcal{F}^\circ} h_f \int_f \left|[\mathbf{u}]_f \cdot \mathbf{n}_f\right|^2 \,\mathrm{d}S + \sum_{f\in\mathcal{F}^\partial} h_f \int_f \left|\mathbf{u}\cdot\mathbf{n}_f\right|^2 \,\mathrm{d}S\right), \quad (4.12)$$

and

$$\left\|u-u^{c,0}\right\|_{L^2(\Omega)}^2 \le C_3\left(\sum_{f\in\mathcal{F}^\circ} h_f \int_f \left|[u]_f\right|^2 \,\mathrm{d}S + \sum_{f\in\mathcal{F}^\partial} h_f \int_f \left|u\right|^2 \,\mathrm{d}S\right), \quad (4.13)$$

where h_f *is the diameter of facet* f *and* C_1, C_2 *and* C_3 *are independent of the mesh size.*

Proof. The proof of (4.11) can be found in [41, Proposition 4.5]. We give here only a sketch of the proof. By Theorem 2.2.1 the dual space of polynomial differential forms $\mathcal{P}_r\Lambda^1(T)$ on elements T is the span of degrees of freedoms l_T associated to T, degrees of freedom l_f associated to faces $f \in \Delta_2(T)$ and degrees of freedom l_e associated to edges $e \in \Delta_1(T)$. Since a $\mathbf{v} \in \mathcal{P}_r\Lambda^1(\mathcal{T})$ satisfies for any face f with $f \in \Delta_2(T_1)$ and $f \in \Delta_2(T_2)$ $l_f(\mathbf{v}_{|T_1}) = l_f(\mathbf{v}_{|T_2})$ and for any edge e with $e \in \Delta_1(T_1)$ and $e \in \Delta_1(T_2)$

$l_e(\mathbf{v}_{|T_1}) = l_e(\mathbf{v}_{|T_2})$, we can define $\mathbf{u}^{c,1} \in \mathcal{P}_r\Lambda^1(\mathcal{T})$ to be that function with

$$\begin{aligned} l_T(\mathbf{u}^{c,1}) &= l_T(\mathbf{u}), && \forall l_T \in \left(\mathcal{P}_r\Lambda^1(\mathcal{T})\right)^*, \\ l_f(\mathbf{u}^{c,1}) &= \frac{\sum_{T\colon f\in\Delta_2(T)} l_f(\mathbf{u}_{|T})}{\sum_{T\colon f\in\Delta_2(T)} 1}, && \forall l_f \in \left(\mathcal{P}_r\Lambda^1(\mathcal{T})\right)^*, \\ l_e(\mathbf{u}^{c,1}) &= \frac{\sum_{T\colon e\in\Delta_2(T)} l_e(\mathbf{u}_{|T})}{\sum_{T\colon e\in\Delta_2(T)} 1}, && \forall l_e \in \left(\mathcal{P}_r\Lambda^1(\mathcal{T})\right)^*. \end{aligned} \tag{4.14}$$

The averages of the local degrees of freedom of $\mathbf{u}$ define the conforming approximation $\mathbf{u}^{c,1}$. Norm equivalence on finite dimensional spaces and a technical scaling argument proves the assertion. The proof of (4.12) and (4.13) follows similarly. □

1-Forms in $\mathbb{R}^3$

The advection problem (4.1) for 1-forms corresponds to

$$\begin{aligned} \alpha\mathbf{u} + \mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta} + \mathbf{grad}\,(\mathbf{u}\cdot\boldsymbol{\beta}) = \mathbf{f}, &\quad \text{in } \Omega, \\ \mathbf{u}_{|\Gamma_{\text{in}}} = \mathbf{g}, &\quad \text{in } \Gamma_{\text{in}}. \end{aligned} \tag{4.15}$$

For convenience we rewrite also the discrete variational formulation (4.6) in vector proxy notation: Find $\mathbf{u} \in \Lambda_h^1(\mathcal{T})$, such that:

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \mathsf{l}(\mathbf{v}), \quad \forall \mathbf{v} \in \Lambda_h^1(\mathcal{T}), \tag{4.16}$$

with

$$\mathsf{l}(\mathbf{v}) := (\mathbf{f},\mathbf{v})_\Omega - \sum_{f\in\mathcal{F}_-^\partial} (\mathbf{g},\mathbf{v})_{f,\boldsymbol{\beta}} \tag{4.17}$$

and

$$\begin{aligned} \mathsf{a}(\mathbf{u},\mathbf{v}) :=& (\alpha\mathbf{u},\mathbf{v})_\Omega - \sum_T (\mathbf{u}, \mathbf{curl}\,(\mathbf{v}\times\boldsymbol{\beta}) + \boldsymbol{\beta}\,\mathrm{div}\,\mathbf{v})_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} (\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}} \\ &+ \sum_{f\in\mathcal{F}^\circ} \left(\{\mathbf{u}\}_f, [\mathbf{v}]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f\,[\mathbf{v}]_f, [\mathbf{u}]_f\right)_{f,\boldsymbol{\beta}}. \end{aligned} \tag{4.18}$$

The stability assumption (4.3) of Lemma 4.1.5 corresponds by Table 2.2 to positive definiteness of $(2\alpha - 2\alpha_0 - \mathrm{div}\,\boldsymbol{\beta})\mathsf{id} + D\boldsymbol{\beta} + D\boldsymbol{\beta}^T$ and we have stability

$$\mathsf{a}(\mathbf{u},\mathbf{u}) \geq \min(\alpha_0, 1)\,\|\mathbf{u}\|_h^2, \quad \forall \mathbf{u} \in \Lambda_h^1(\mathcal{T}).$$

for a choice c_f, such that $c_f\boldsymbol{\beta}\cdot\mathbf{n}_f > 0$. We get the following convergence result for $\boldsymbol{H}(\mathbf{curl},\Omega)$-conforming approximation spaces $\mathcal{P}_r\Lambda^1(\mathcal{T})$ or $\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$.

Theorem 4.1.13. *Assume that* (4.3) *holds for* α *and* β *in* (4.15)*, i.e. there exists* $\alpha_0 > 0$ *such that* $(2\alpha - 2\alpha_0 + \mathrm{div}\,\boldsymbol{\beta})\mathsf{id} - D\boldsymbol{\beta} - D\boldsymbol{\beta}^T$ *is positive definite and assume that* c_f *in* (4.18) *is such that*

$$c_f\boldsymbol{\beta}\cdot\mathbf{n}_f \geq 0.$$

Let $\mathbf{u} \in W$ *and* $\mathbf{u}_h \in \Lambda^1_h(\mathcal{T})$ *be the solutions of* (4.15) *and* (4.18). *Then if* $\mathbf{u} \in H^{r+1}\Lambda^1(\Omega)$ *and* $\Lambda^1_h(\mathcal{T}) = \mathcal{P}_r\Lambda^1(\mathcal{T})$ *or* $\Lambda^1_h(\mathcal{T}) = \mathcal{P}^-_{r+1}\Lambda^1(\mathcal{T})$ *we get with* $C > 0$ *independent of the mesh size* $h = \max_T(h_T)$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}.$$

Proof. Let $\bar{\mathbf{u}}_h$ denote the L^2-projection of $\mathbf{u}$ onto the $\Lambda^1_h(\mathcal{T})$ and define $\boldsymbol{\eta} := \mathbf{u} - \bar{\mathbf{u}}_h$ and $\boldsymbol{\gamma}_h := \mathbf{u}_h - \bar{\mathbf{u}}_h$. At first we recall that by Theorem 2.2.6

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \leq C h^{r+1} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}.$$

Then by stability, consistency and $\boldsymbol{\gamma}_h \in \Lambda^1_h(\mathcal{T})$:

$$\min(\alpha_0, 1)\, \|\mathbf{u} - \mathbf{u}_h\|_h^2 \leq \mathsf{a}\,(\boldsymbol{\eta}, \boldsymbol{\gamma}_h)\,.$$

Let $\boldsymbol{\beta}_h$ be the L^2-projection of $\boldsymbol{\beta}$ onto $\left(\mathcal{P}^{\mathrm{d}}_0\Lambda^0(\mathcal{T})\right)^n$. As in the proof of Theorem 4.1.8 we add and subtract the Lie-derivative with respect to the projected velocity field $\boldsymbol{\beta}_h$. But since this time $\sum_T \left(\boldsymbol{\eta}, \mathcal{L}_{\boldsymbol{\beta}_h}\boldsymbol{\gamma}_h\right)_T \neq 0$ we need to prove additionally

$$\left|\sum_T \left(\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) + \boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h\right)_T\right| \leq C h^{-\frac{1}{2}} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_h\,.$$

Since we have by Table 2.3 for piecewise constant $\boldsymbol{\beta}_h$ the local identity $\mathsf{L}_{\boldsymbol{\beta}_h} = -\,\mathcal{L}_{\boldsymbol{\beta}_h}$ this is implied by

$$\left|\sum_T \left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h\right)_T\right| \leq C h^{-\frac{1}{2}} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_h \tag{4.19}$$

and

$$\left|\sum_T \left(\boldsymbol{\eta}, \mathbf{grad}\,(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h)\right)_T\right| \leq C h^{-\frac{1}{2}} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_h\,. \tag{4.20}$$

We use the approximation results of Proposition 4.1.12 to prove the two assertions (4.19) and (4.20). Let $\mathbf{w}^{c,1} \in \mathcal{P}_r\Lambda^1(\mathcal{T})$ and $w^{c,0} \in \mathcal{P}_{r+1}\Lambda^0(\mathcal{T})$ be the conforming approximations of $\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h \in \mathcal{P}^{\mathrm{d}}_r\Lambda^1(\mathcal{T})$ and $\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h \in \mathcal{P}^{\mathrm{d}}_{r+1}\Lambda^0(\mathcal{T})$. Since $\boldsymbol{\eta} = \mathbf{u} - \bar{\mathbf{u}}_h$ and both $\mathbf{w}^{c,1} \in \mathcal{P}_r\Lambda^1(\mathcal{T})$ and $\mathbf{grad}\,w^{c,0} \in \mathcal{P}_r\Lambda^1(\mathcal{T})$ we find

$$\begin{aligned}\left|\sum_T \left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h\right)_T\right| &= \left|\sum_T \left(\boldsymbol{\eta}, \boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right)_T\right| \\ &\leq \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}\end{aligned}$$

and

$$\begin{aligned}\left|\sum_T \left(\boldsymbol{\eta}, \mathbf{grad}\,(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h)\right)_T\right| &= \left|\sum_T \left(\boldsymbol{\eta}, \mathbf{grad}\left(\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right)\right)_T\right| \\ &\leq C_0 h^{-1} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|_{L^2(\Omega)}\end{aligned}.$$

The approximation results (4.11) and (4.13) give

$$\left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|^2_{\boldsymbol{L}^2(\Omega)} \leq C_1 h \left(\sum_{f\in\mathcal{F}^\circ} \left\| [\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f \right\|^2_{\boldsymbol{L}^2(f)} + \sum_{f\in\mathcal{F}^\partial} \left\| (\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h) \times \mathbf{n}_f \right\|^2_{\boldsymbol{L}^2(f)} \right)$$

and

$$\left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|^2_{L^2(\Omega)} \leq C_2 h \left(\sum_{f\in\mathcal{F}^\circ} \left\| [\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h]_f \right\|^2_{L^2(f)} + \sum_{f\in\mathcal{F}^\partial} \left\| (\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h) \right\|^2_{L^2(f)} \right).$$

Further we have by inverse inequalities, approximation properties of $\boldsymbol{\beta}_h$ and normal continuity of $\mathbf{curl}\,\boldsymbol{\gamma}_h \in H\Lambda^2(\Omega)$:

$$\begin{aligned}
&\left\| [\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} \\
&\quad\leq \left\| [(\boldsymbol{\beta}_h - \boldsymbol{\beta}) \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} + \left\| [\boldsymbol{\beta} \times \mathbf{curl}\,\boldsymbol{\gamma}_h]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)} \\
&\quad\leq C_3 h^{\frac{1}{2}} \left\| \mathbf{curl}\,\boldsymbol{\gamma}_h \right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + \left\| [\mathbf{curl}\,\boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \boldsymbol{\beta} - \boldsymbol{\beta}\cdot\mathbf{n}_f \left[\mathbf{curl}\,\boldsymbol{\gamma}_h\right]_f \right\|_{\boldsymbol{L}^2(f)} \\
&\quad\leq C_3 h^{-\frac{1}{2}} \left\| \boldsymbol{\gamma}_h \right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + C_4 \left\| \boldsymbol{\beta}\cdot\mathbf{n}_f \left[\mathbf{curl}\,\boldsymbol{\gamma}_h\right]_f \right\|_{\boldsymbol{L}^2(f)} \\
&\quad\leq C_3 h^{-\frac{1}{2}} \left\| \boldsymbol{\gamma}_h \right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + C_4 h^{-1} \left\| \boldsymbol{\beta}\cdot\mathbf{n}_f \left[\boldsymbol{\gamma}_h\right]_f \right\|_{\boldsymbol{L}^2(f)}
\end{aligned}$$

and similar by tangential continuity of $\boldsymbol{\gamma}_h \in H\Lambda^1(\Omega)$:

$$\begin{aligned}
\left\| [\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h]_f \right\|_{L^2(f)} &\leq \left\| [(\boldsymbol{\beta}_h - \boldsymbol{\beta}) \cdot \boldsymbol{\gamma}_h]_f \right\|_{L^2(f)} + \left\| [\boldsymbol{\beta} \cdot \boldsymbol{\gamma}_h]_f \right\|_{L^2(f)} \\
&\leq C_5 h^{\frac{1}{2}} \left\| \boldsymbol{\gamma}_h \right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + \left\| \boldsymbol{\beta}\cdot\mathbf{n}_f \left[\boldsymbol{\gamma}_h\right]_f \right\|_{\boldsymbol{L}^2(f)},
\end{aligned}$$

with constants C_3, C_4 and C_5 independent of h, and T_1 and T_2 those elements that share f. Hence we have proved

$$\left\|\boldsymbol{\beta}_h \times \mathbf{curl}\,\boldsymbol{\gamma}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)} \leq C_6 h^{-\frac{1}{2}} \left\| \boldsymbol{\gamma}_h \right\|_h$$

and

$$\left\|\boldsymbol{\beta}_h \cdot \boldsymbol{\gamma}_h - w^{c,0}\right\|_{L^2(\Omega)} \leq C_7 h^{\frac{1}{2}} \left\| \boldsymbol{\gamma}_h \right\|_h,$$

which yields estimates (4.19) and (4.20). □

A similar convergence result is obtained for $\boldsymbol{H}(\mathrm{div},\Omega)$-conforming approximation spaces $\mathcal{P}_r\Lambda^2(\mathcal{T})$ or $\mathcal{P}_r^-\Lambda^2(\mathcal{T})$.

Theorem 4.1.14. *Assume that* (4.3) *holds for* α *and* β *in* (4.15)*, i.e. there exists* $\alpha_0 > 0$ *such that* $(2\alpha - 2\alpha_0 - \operatorname{div}\boldsymbol{\beta})\mathsf{id} + D\boldsymbol{\beta} + D\boldsymbol{\beta}^T$ *is positive definite and that* c_f *in* (4.18) *is such that*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f > 0.$$

Let $\mathbf{u} \in W$ *and* $\mathbf{u}_h \in \Lambda_h^1(\mathcal{T})$ *be the solutions of* (4.15) *and* (4.18)*. Then if* $\mathbf{u} \in H^{r+1}\Lambda^1(\Omega)$ *and* $\Lambda_h^1(\mathcal{T}) = \star\mathcal{P}_r\Lambda^2(\mathcal{T})$ *or* $\Lambda_h^1(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^2(\mathcal{T})$ *for we get with* $C > 0$ *independent of* $h := \max_T(h_T)$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}.$$

Proof. As in the proof of Theorem 4.1.13 the crucial step is a proof of the estimate

$$\left| \sum_T (\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) + \boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h)_T \right| \leq C h^{-\frac{1}{2}} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \|\boldsymbol{\gamma}_h\|_h . \tag{4.21}$$

This time $\boldsymbol{\eta} := \mathbf{u} - \bar{\mathbf{u}}_h$, $\bar{\mathbf{u}}_h \in \Lambda_h^1(\mathcal{T}) \subset \star H\Lambda^2(\Omega)$, is the L^2-projection error for $\boldsymbol{H}(\operatorname{div}, \Omega)$-conforming approximation spaces and we can work directly with the adjoint Lie-derivative $\mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) + \boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h$. We recall that by Corollary 2.2.7

$$\|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \leq C h^{r+1} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}.$$

Let $\mathbf{w}^{c,1} \in \mathcal{P}_{r+1}\Lambda^1(\mathcal{T})$ and $\mathbf{w}^{c,2} \in \mathcal{P}_r\Lambda^2(\mathcal{T})$ be the conforming approximations of $\boldsymbol{\beta}_h \times \boldsymbol{\gamma}_h \in \mathcal{P}_{r+1}^{\mathrm{d}}\Lambda^1(\mathcal{T})$ and $\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h \in \mathcal{P}_r^{\mathrm{d}}\Lambda^2(\mathcal{T})$. Since $\boldsymbol{\eta} = \mathbf{u} - \bar{\mathbf{u}}_h$ and both $\mathbf{curl}\,\mathbf{w}^{c,1} \in \mathcal{P}_r\Lambda^2(\mathcal{T})$ and $\mathbf{w}^{c,2} \in \mathcal{P}_r\Lambda^2(\mathcal{T})$ we find

$$\left| \sum_T (\boldsymbol{\eta}, \mathbf{curl}(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h))_T \right| \leq C_0 h^{-1} \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}$$

and

$$\left| \sum_T (\boldsymbol{\eta}, \boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h)_T \right| \leq \|\boldsymbol{\eta}\|_{\boldsymbol{L}^2(\Omega)} \left\|\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h - \mathbf{w}^{c,2}\right\|_{\boldsymbol{L}^2(\Omega)} .$$

The approximation results (4.11) and (4.12) give

$$\begin{aligned}
&\left\|\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h - \mathbf{w}^{c,1}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \\
&\qquad \leq C_1 h \left(\sum_{f \in \mathcal{F}^\circ} \left\| [\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h]_f \times \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)}^2 + \sum_{f \in \mathcal{F}^\partial} \|(\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h) \times \mathbf{n}_f\|_{\boldsymbol{L}^2(f)}^2 \right)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h - \mathbf{w}^{c,2}\right\|_{\boldsymbol{L}^2(\Omega)}^2 \leq \\
&\qquad C_2 h \left(\sum_{f \in \mathcal{F}^\circ} \left\| [\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h]_f \cdot \mathbf{n}_f \right\|_{\boldsymbol{L}^2(f)}^2 + \sum_{f \in \mathcal{F}^\partial} \|(\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h) \cdot \mathbf{n}_f\|_{\boldsymbol{L}^2(f)}^2 \right).
\end{aligned}$$

Inverse inequalities, approximation properties of $\boldsymbol{\beta}_h$, normal continuity of $\boldsymbol{\gamma}_h$ and tangential continuity yield:

$$\begin{aligned}\left\|[\boldsymbol{\gamma}_h \times \boldsymbol{\beta}_h]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} &\leq \left\|[\boldsymbol{\gamma}_h \times (\boldsymbol{\beta}_h - \boldsymbol{\beta})]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} + \left\|[\boldsymbol{\gamma}_h \times \boldsymbol{\beta}]_f \times \mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} \\ &\leq C_3 h^{\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + \left\|\boldsymbol{\beta}\cdot\mathbf{n}_f\,[\boldsymbol{\gamma}_h]_f - [\boldsymbol{\gamma}_h]_f\cdot\mathbf{n}_f\boldsymbol{\beta}\right\|_{\boldsymbol{L}^2(f)} \\ &\leq C_3 h^{\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + \left\|\boldsymbol{\beta}\cdot\mathbf{n}_f\,[\boldsymbol{\gamma}_h]_f\right\|_{\boldsymbol{L}^2(f)}\end{aligned}$$

and

$$\begin{aligned}\left\|[\boldsymbol{\beta}_h \operatorname{div}\boldsymbol{\gamma}_h]_f\cdot\mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} &\leq \left\|[(\boldsymbol{\beta}_h - \boldsymbol{\beta})\operatorname{div}\boldsymbol{\gamma}_h]_f\cdot\mathbf{n}_f\right\|_{\boldsymbol{L}^2(f)} + \left\|\boldsymbol{\beta}\cdot\mathbf{n}_f\,[\operatorname{div}\boldsymbol{\gamma}_h]_f\right\|_{\boldsymbol{L}^2(f)} \\ &\leq C_4 h^{-\frac{1}{2}} \left\|\boldsymbol{\gamma}_h\right\|_{\boldsymbol{L}^2(T_1\cup T_2)} + C_5 h^{-1}\left\|\boldsymbol{\beta}\cdot\mathbf{n}_f\,[\boldsymbol{\gamma}_h]_f\right\|_{\boldsymbol{L}^2(f)},\end{aligned}$$

with constants C_3, C_4 and C_5 independent of h, and T_1 and T_2 those elements that share f. Hence we deduce (4.21). □

2-**Forms in** $\mathbb{R}^3$

The advection problem (4.1) for 2-forms corresponds to

$$\begin{aligned}\alpha\mathbf{u} + \mathbf{curl}\,(\mathbf{u}\times\boldsymbol{\beta}) + \boldsymbol{\beta}\operatorname{div}\mathbf{u} = \mathbf{f}, &\quad \text{in } \Omega, \\ \mathbf{u}_{|\Gamma_{\text{in}}} = \mathbf{g}, &\quad \text{in } \Gamma_{\text{in}}.\end{aligned} \tag{4.22}$$

The discrete variational formulation (4.6) in vector proxy notation is: Find $\mathbf{u}\in\Lambda_h^2(\mathcal{T})$ such that:

$$\mathsf{a}(\mathbf{u},\mathbf{v}) = \mathsf{l}(\mathbf{v}), \quad \forall \mathbf{v}\in\Lambda_h^2(\mathcal{T}), \tag{4.23}$$

with

$$\mathsf{l}(\mathbf{v}) := (\mathbf{f},\mathbf{v})_\Omega - \sum_{f\in\mathcal{F}_-^\partial} (\mathbf{g},\mathbf{v})_{f,\boldsymbol{\beta}} \tag{4.24}$$

and

$$\begin{aligned}\mathsf{a}(\mathbf{u},\mathbf{v}) := (\alpha\mathbf{u},\mathbf{v})_\Omega - \sum_T (\mathbf{u}, \mathbf{curl}\,\mathbf{v}\times\boldsymbol{\beta} + \mathbf{grad}\,(\boldsymbol{\beta}\cdot\mathbf{v}))_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} (\mathbf{u},\mathbf{v})_{f,\boldsymbol{\beta}} \\ + \sum_{f\in\mathcal{F}^\circ} \Big(\{\mathbf{u}\}_f, [\mathbf{v}]_f\Big)_{f,\boldsymbol{\beta}} + \Big(c_f\,[\mathbf{v}]_f, [\mathbf{u}]_f\Big)_{f,\boldsymbol{\beta}}.\end{aligned} \tag{4.25}$$

The stability assumption of Lemma (4.4) corresponds by Table 2.2 to positive definiteness of $(2\alpha - 2\alpha_0 + \operatorname{div}\boldsymbol{\beta})\mathsf{id} - D\boldsymbol{\beta} - D\boldsymbol{\beta}^T$ and we have stability

$$\mathsf{a}(\mathbf{u},\mathbf{u}) \geq C\left\|\mathbf{u}\right\|_h^2, \quad \forall\mathbf{u}\in\Lambda_h^2(\mathcal{T})$$

for a choice c_f, such that $c_f\boldsymbol{\beta}\cdot\mathbf{n}_f > 0$.

As for the advection problem for 1-forms we can prove convergence for both $\boldsymbol{H}(\mathbf{curl},\Omega)$ and $\boldsymbol{H}(\operatorname{div},\Omega)$-conforming approximations. We omit the proofs, because they build on exactly the same arguments as the proofs of Theorems 4.1.13 and 4.1.14.

Theorem 4.1.15. *Assume that* (4.3) *holds for* α *and* β *in* (4.22)*, i.e. there exists* $\alpha_0 > 0$ *such that* $(2\alpha - 2\alpha_0 + \operatorname{div}\boldsymbol{\beta})\mathsf{id} - D\boldsymbol{\beta} - D\boldsymbol{\beta}^T$ *is positive definite and that* c_f *in* (4.25) *is such that*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f > 0.$$

Let $\mathbf{u} \in W$ *and* $\mathbf{u}_h \in \Lambda_h^2(\mathcal{T})$ *be the solutions of* (4.22) *and* (4.25)*. Then if* $\mathbf{u} \in H^{r+1}\Lambda^2(\Omega)$ *and* $\Lambda_h^2(\mathcal{T}) = \mathcal{P}_r\Lambda^2(\mathcal{T})$ *or* $\Lambda_h^2(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^2(\mathcal{T})$ *for we get with* $C > 0$ *independent of the mesh size* $h := \max_T(h_T)$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}. \tag{4.26}$$

Theorem 4.1.16. *Assume that for* α *and* β *in* (4.22) *there exists* $\alpha_0 > 0$ *such that* $(\alpha - \alpha_0 - \frac{1}{2}\operatorname{div}\boldsymbol{\beta})\mathsf{id} + D\boldsymbol{\beta} + D\boldsymbol{\beta}^T$ *positive definite and that* c_f *in* (4.25) *is such that*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f > 0.$$

Let $\mathbf{u} \in W$ *and* $\mathbf{u}_h \in \Lambda_h^2(\mathcal{T})$ *be the solutions of* (4.22) *and* (4.25)*. Then if* $\mathbf{u} \in H^{r+1}\Lambda^2(\Omega)$ *and* $\Lambda_h^2(\mathcal{T}) = \star\mathcal{P}_r\Lambda^1(\mathcal{T})$ *or* $\Lambda_h^2(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$ *for we get with* $C > 0$ *independent of the mesh size* $h := \max_T(h_T)$

$$\|\mathbf{u} - \mathbf{u}_h\|_h \leq C h^{r+\frac{1}{2}} \|\mathbf{u}\|_{\boldsymbol{H}^{r+1}(\Omega)}. \tag{4.27}$$

4.1.4 Numerical Experiments

In this section we set $\Omega \subset \mathbb{R}^2$ and look at the advection problem

$$\begin{aligned} \alpha\omega - \mathcal{L}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega \subset \mathbb{R}^2, \\ \omega_{|\Gamma_{\text{in}}} &= \psi_D, && \text{on } \Gamma_{\text{in}}, \end{aligned} \tag{4.28}$$

for 1-forms $\omega \in \Lambda^1(\Omega)$. From Table 2.3 we find that this corresponds to

$$\begin{aligned} \alpha\mathbf{u} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) - \mathbf{R}\operatorname{div}(\mathbf{R}\mathbf{u})\boldsymbol{\beta} &= \mathbf{f}, && \text{in } \Omega, \\ \mathbf{u}_{|\Gamma_{\text{in}}} &= \mathbf{g}, && \text{on } \Gamma_{\text{in}} \end{aligned} \tag{4.29}$$

with $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\mathbf{u}$ doubling for ω. We consider approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ with no global continuity, approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^-\Lambda^1(\mathcal{T})$ with global normal continuity and approximation spaces

$$\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_r^-\Lambda^1(\mathcal{T})\right)^{\perp} := \mathbf{R}\mathcal{P}_r^-\Lambda^1(\mathcal{T})$$

with tangential global continuity. The case $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^-\Lambda^1(\mathcal{T})$ corresponds to that setting, where we use conforming discrete $n-k$-forms to approximate Lie derivatives of k-forms.

Experiment 1: Smooth Data

We set $\Omega = [0,1]^2$, $\alpha = 2$ and take:

$$\boldsymbol{\beta} = \begin{pmatrix} 0.66(1-x^2)(1-y^2) \\ 0.2 + \sin(\pi x)\cos(\pi x) \end{pmatrix},$$

and chose $\mathbf{f}$ and $\mathbf{g}$ such that

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x)(1-y^2) \\ (1-x^2)(1-y^2) \end{pmatrix}$$

is the solution.

With this data we compute

$$\begin{aligned} &D\boldsymbol{\beta} + (D\boldsymbol{\beta})^T - (\operatorname{div}\boldsymbol{\beta} - \alpha)(id) \\ &\quad = \begin{pmatrix} 4 - \frac{33}{25}x(1-y^2) + \pi\sin(\pi x)\sin(\pi y) & -\frac{33}{25}(1-x^2)y + \pi\cos(\pi x)\cos(\pi y) \\ -\frac{33}{25}(1-x^2)y + \pi\cos(\pi x)\cos(\pi y) & 4 + \frac{33}{25}x(1-y^2) - \pi\sin(\pi x)\sin(\pi y), \end{pmatrix} \end{aligned}$$

i.e. the stability assumption (4.4) requires this matrix to be positive definite. The sketch of the values of the eigenvalues on Ω in Figure 4.1 shows that for our choice of parameters the assumption holds true.

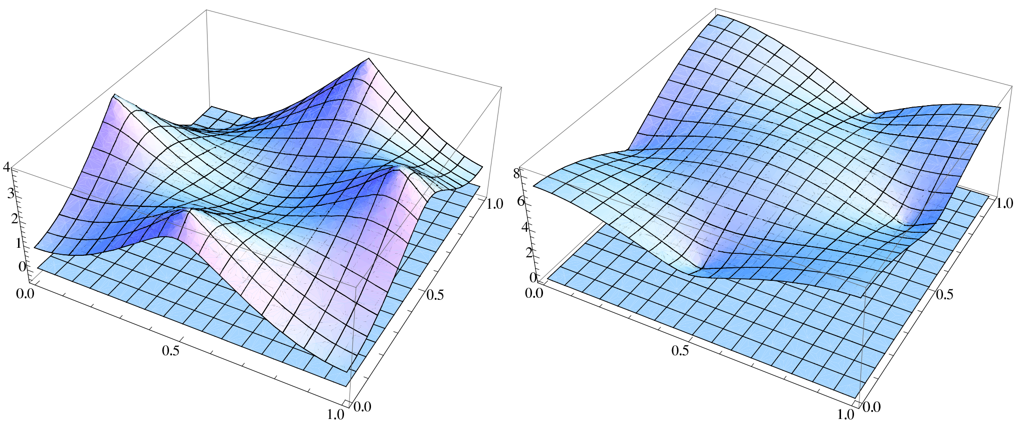

Figure 4.1: Sketch of two eigenvalues of the matrix $D\boldsymbol{\beta} + (D\boldsymbol{\beta})^T - (\operatorname{div}\boldsymbol{\beta} - \alpha)(id)$. The blue plane is the (x,y)-plane. In the entire domain Ω both eigenvalues are positive.

We first determine numerical convergence rates for stabilized schemes with stabilization $c_f \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. Figures 4.2, 4.3 and 4.4 show the error in the semi-norm

$$|\omega|_h^2 := \sum_{f\in\mathcal{F}^\circ} \left\| [\omega]_f \right\|^2_{f,c_f\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \|\omega\|^2_{f,\frac{1}{2}\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\partial_-} \|\omega\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}.$$

The rates of convergence confirm the theoretical results of Theorems 4.1.8, 4.1.13 and 4.1.14. Figures 4.5, 4.6 and 4.7 show the error in the L^2-norm. The rates of convergence improve by $\frac{1}{2}$ compared to the theoretical results. This phenomenon has also been observed for stabilized Galerkin methods for scalar advection. Only on certain very special meshes one could find there that the theoretical results are also sharp for the L^2-norm [70, 89].

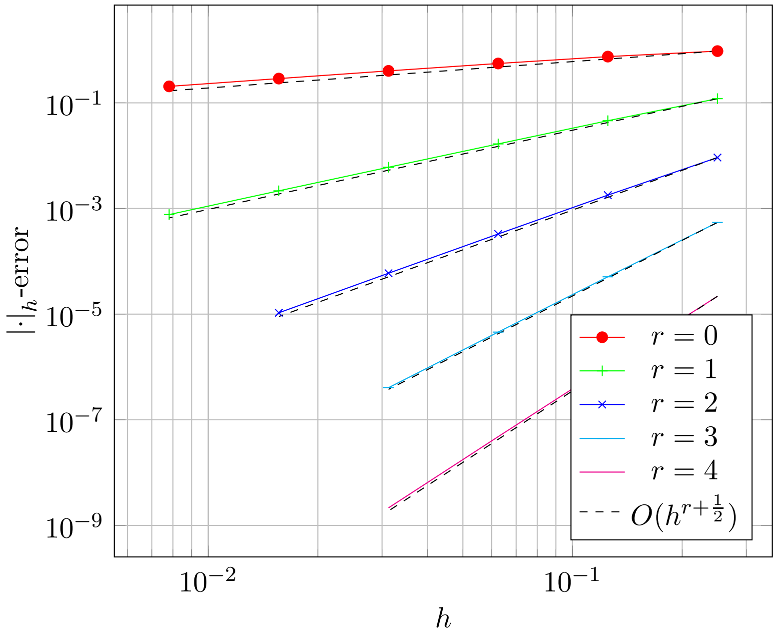

Figure 4.2: Experiment 1: Discontinuous approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ and stabilization. The results coincide with the assertions of Theorem 4.1.8.

Second we determine numerical convergence rates for the non-stabilized schemes, i.e. $c_f = 0$. Figures 4.8, 4.9 and 4.10 show the error in the L^2-norm. For $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ and r odd the experiments confirm the theoretical results in Remark 4.1.10, while for r even we observe even higher rates (see 4.8). This is also known from the scalar problem. For $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})$ and $\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})\right)^\perp$ the results coincide with Theorem 4.1.11 (see Figures 4.9 and 4.10).

Experiment 2: Non-Smooth Data

We set in problem (4.29) $\Omega = [-1, 1]^2$, $\alpha = 0$

$$\boldsymbol{\beta} = \begin{pmatrix} 4(4+y) \\ 4+x \end{pmatrix}, \tag{4.30}$$

$\mathbf{f} = 0$ and

$$\mathbf{g} = \begin{pmatrix} 1 + \sin(0.5\pi x)\sin(0.5\pi y) \\ -0.5 + \cos(0.5\pi x)\cos(0.5\pi y) \end{pmatrix}. \tag{4.31}$$

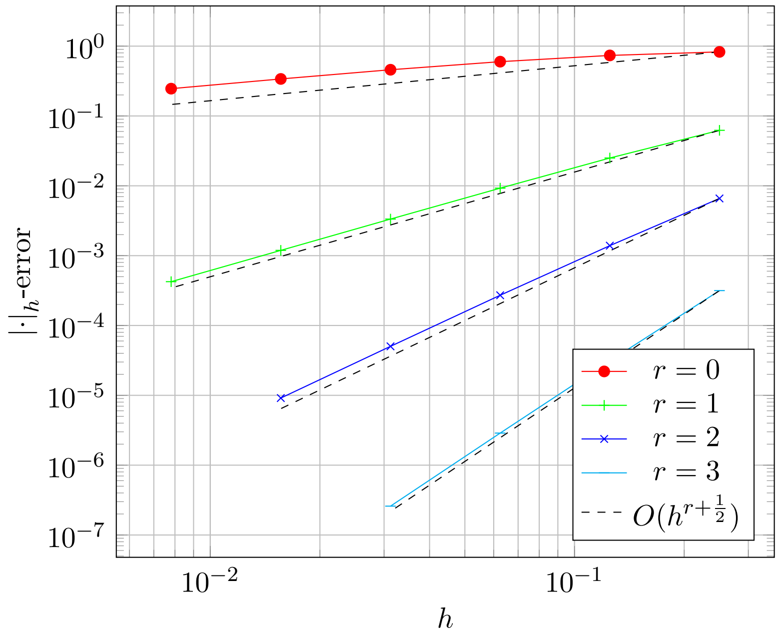

Figure 4.3: Experiment 1: Conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ and stabilization. The results coincide with the assertions of Theorem 4.1.15.

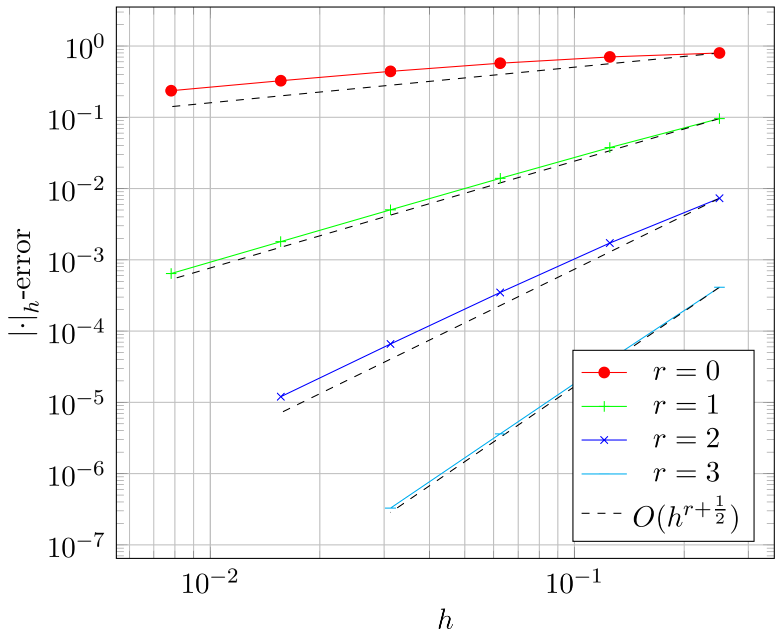

Figure 4.4: Experiment 1: Non-conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})\right)^{\perp}$ and stabilization. The resultes coincide with the assertions of Theorem 4.1.16.

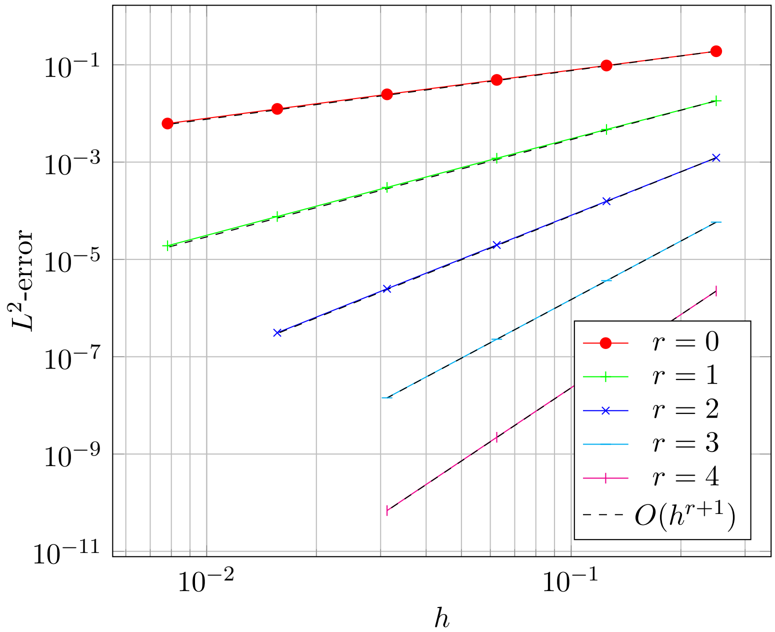

Figure 4.5: Experiment 1: Discontinuous approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ and stabilization. As for the scalar problems we observe on ad-hoc meshes super-convergence for the L^2-error compared to the theoretical results.

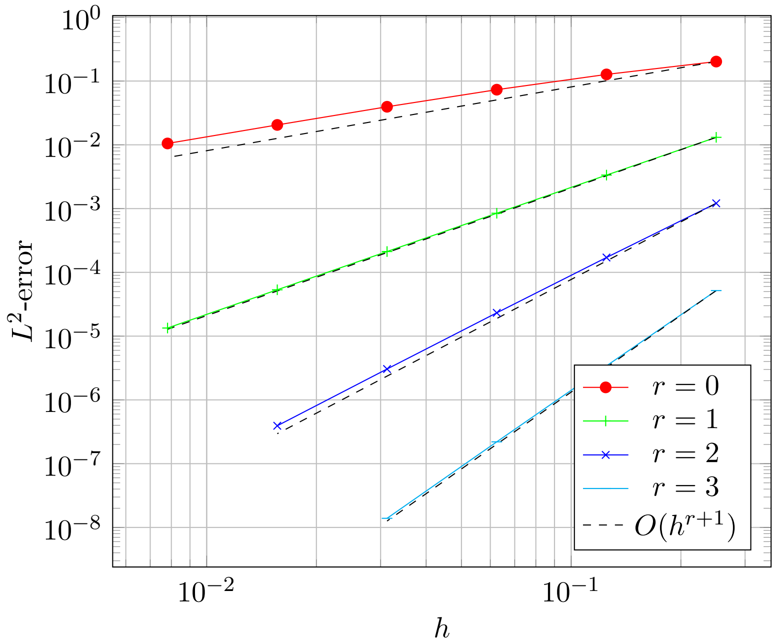

Figure 4.6: Experiment 1: Conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ and stabilization. As for the scalar problems we observe on ad-hoc meshes super-convergence for the L^2-error compared to the theoretical results.

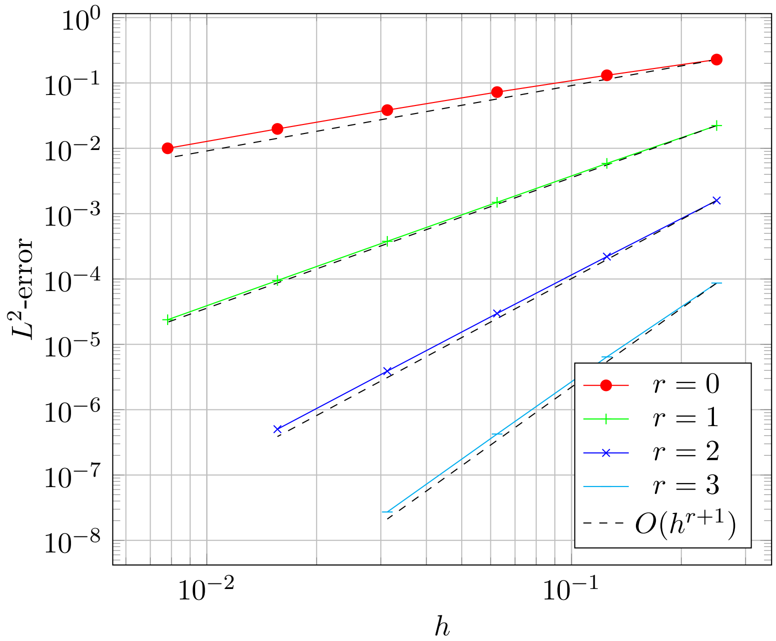

Figure 4.7: Experiment 1: Non-conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})\right)^{\perp}$ and stabilization. As for the scalar problems we observe on ad-hoc meshes super-convergence for the L^2-error compared to the theoretical results.

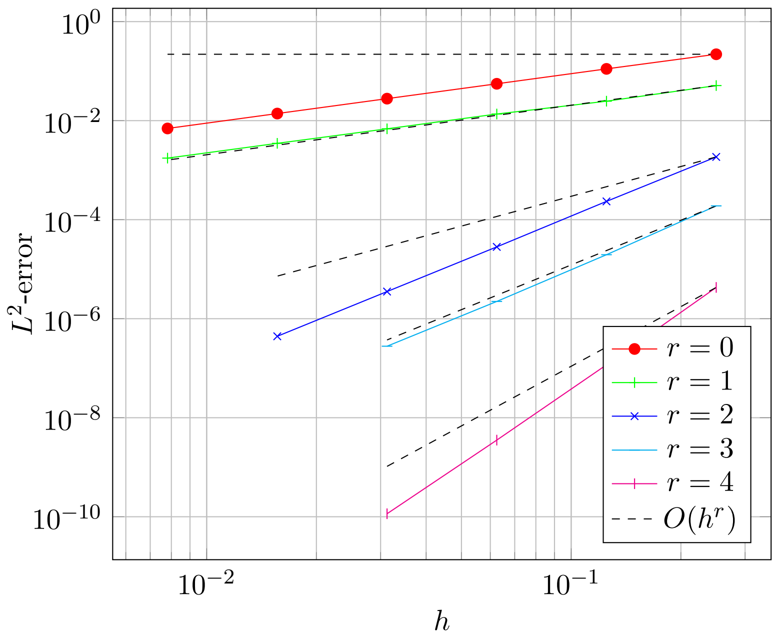

Figure 4.8: Experiment 1: Discontinuous approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ and no stabilization. For r odd the results coincide with the assertions of Remark 4.1.10.

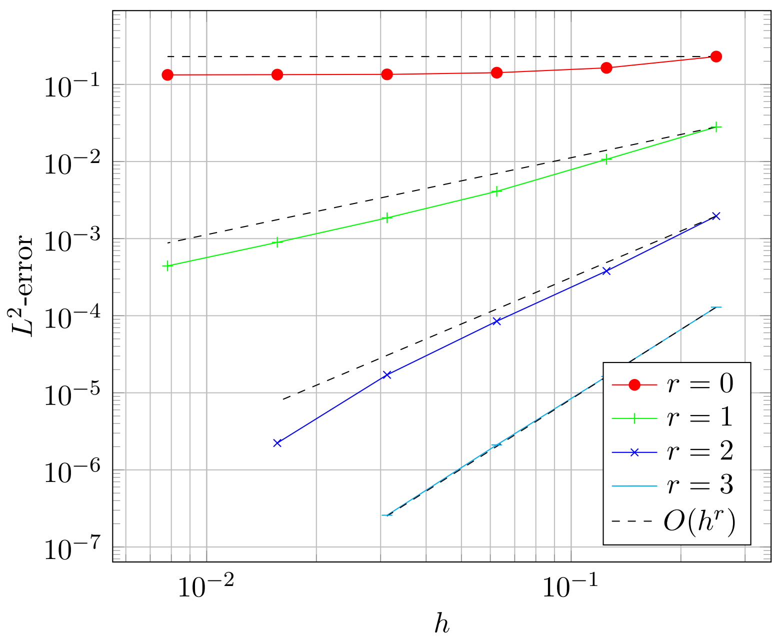

Figure 4.9: Experiment 1: Conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ and no stabilization. The results coincide with the assertions of Theorem 4.1.11.

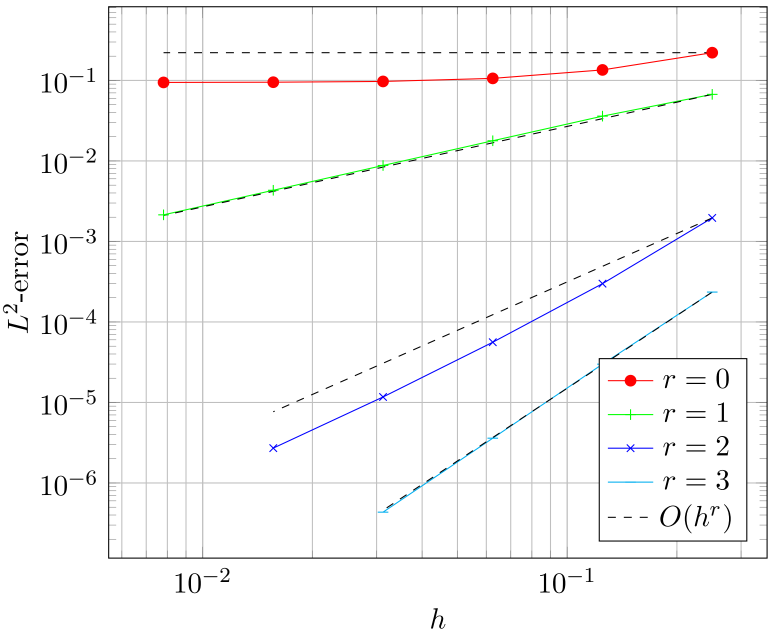

Figure 4.10: Experiment 1: Non-conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_{r+1}^-\Lambda^1(\mathcal{T})\right)^\perp$ and no stabilization. The results coincide with the assertions of Theorem 4.1.11.

Since $\boldsymbol{\beta}$ is linear we can derive a closed form expression of the solution. This time the stability assertion (4.4) is violated. Since we observe convergence for our Galerkin schemes we can be confident to have the situation outlined in Remark 4.1.7. Figure 4.11, 4.12 and 4.13 show the numerical convergence rates for stabilized schemes, where $c_f \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$. Since the analytic solution is in this case non-smooth along the trajectory of the vertex $(-1, 1)$ we observe reduced convergence rates. Figure 4.14 visualizes a characteristic error distribution.

4.1.5 Appendix

For convenience we write here explicitly the representation of the bilinear form $\mathsf{a}(\cdot,\cdot)$ for forms in $\mathbb{R}^3$ stated on proposition 4.1.3. For these cases an implementation does not require partial derivatives of $\boldsymbol{\beta}$.

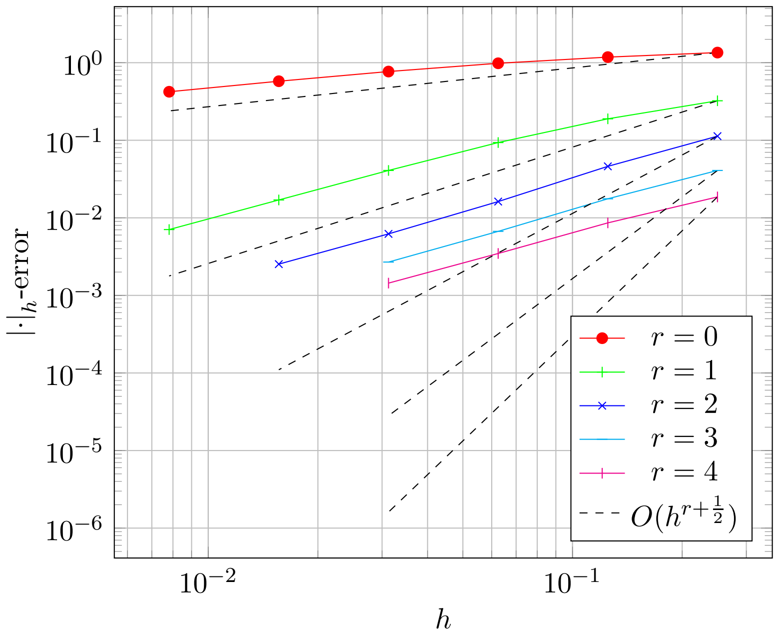

Figure 4.11: Experiment 2: Discontinuous approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$ with stabilization.

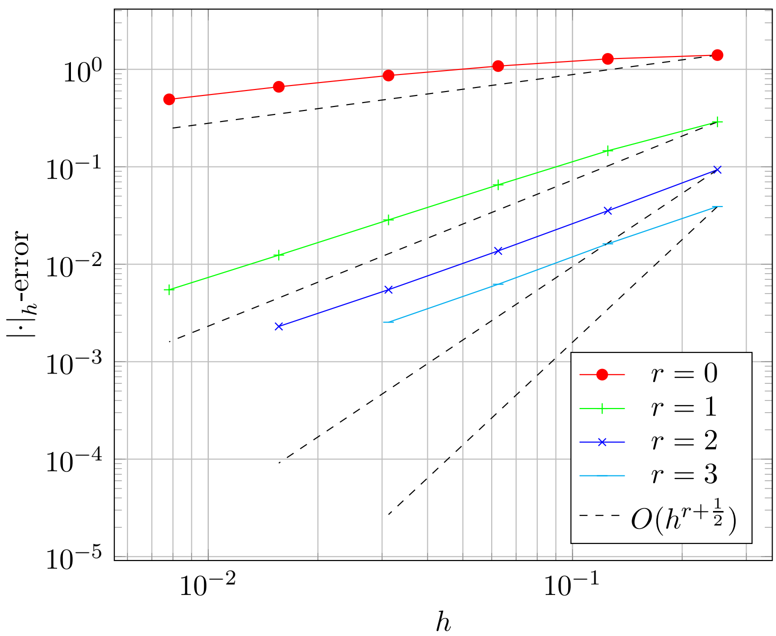

Figure 4.12: Experiment 2: Conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})$ with stabilization.

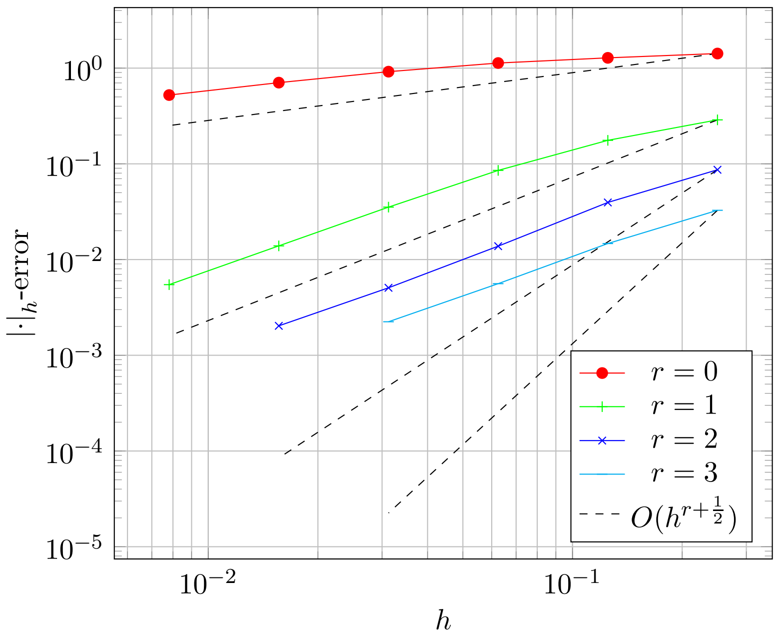

Figure 4.13: Experiment 2: Non-conforming approximation spaces $\Lambda_h^1(\mathcal{T}) = \left(\mathcal{P}_{r+1}^{-}\Lambda^1(\mathcal{T})\right)^{\perp}$ with stabilization.

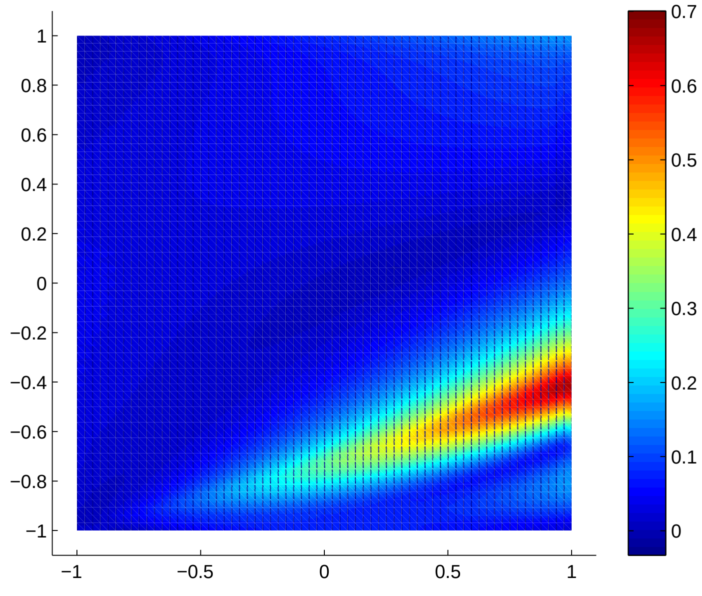

Figure 4.14: Experiment 2: Characteristic distribution of the error.

- $k = 0$:

$$
\begin{aligned}
\mathsf{a}\left(u,v\right) &= \int_\Omega \alpha uv \,\mathrm{d}\boldsymbol{x} + \sum_T \int_T \boldsymbol{\beta}\cdot\mathbf{grad}\, uv\,\mathrm{d}\boldsymbol{x} \\
&\quad - \sum_{f\in\mathcal{F}^\circ} \int_f \boldsymbol{\beta}\cdot\mathbf{n}_f \left[u\right]_f \{v\}_f \,\mathrm{d}S - \int_f \boldsymbol{\beta}\cdot\mathbf{n}_f c_f \left[u\right]_f [v]_f \,\mathrm{d}S \\
&\quad - \sum_{f\in\mathcal{F}^\partial_-} \int_f \boldsymbol{\beta}\cdot\mathbf{n}_f uv\,\mathrm{d}S, \\
\mathsf{l}\left(v\right) &= \int_\Omega fv\,\mathrm{d}\boldsymbol{x} - \sum_{f\in\mathcal{F}^\partial_-} \int_f \boldsymbol{\beta}\cdot\mathbf{n}_f gv\,\mathrm{d}S
\end{aligned}
$$

- $k = 1$:

$$
\begin{aligned}
\mathsf{a}\left(\mathbf{u},\mathbf{v}\right) &= \int_\Omega \alpha\mathbf{u}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + \sum_T \int_T \left(\mathbf{curl}\,\mathbf{u}\times\boldsymbol{\beta}\right)\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x} - \int_T \mathbf{u}\cdot\boldsymbol{\beta}\mathrm{div}\,\mathbf{v}\,\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{f\in\mathcal{F}^\circ} \int_f \boldsymbol{\beta}\cdot\{\mathbf{u}\}_f[\mathbf{v}]_f\cdot\mathbf{n}_f\,\mathrm{d}S - \int_f \Big([\mathbf{u}]_f\times\mathbf{n}_f\Big)\cdot\Big(\{\mathbf{v}\}_f\times\boldsymbol{\beta}\Big)\,\mathrm{d}S \\
&\quad + \sum_{f\in\mathcal{F}^\circ} \int_f c_f\boldsymbol{\beta}\cdot\mathbf{n}_f\left[\mathbf{u}\right]_f\cdot\left[\mathbf{v}\right]_f\mathrm{d}S \\
&\quad + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \boldsymbol{\beta}\cdot\mathbf{u}\mathbf{v}\cdot\mathbf{n}_f\,\mathrm{d}S - \sum_{f\in\mathcal{F}^\partial_-} \int_f \left(\mathbf{u}\times\mathbf{n}_f\right)\cdot\left(\mathbf{v}\times\boldsymbol{\beta}\right)\,\mathrm{d}S, \\
\mathsf{l}\left(\mathbf{v}\right) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x} - \sum_{f\in\mathcal{F}^\partial_-} \int_f \left(\mathbf{g}\times\mathbf{n}_f\right)\cdot\left(\mathbf{v}\times\boldsymbol{\beta}\right)\,\mathrm{d}S
\end{aligned}
$$

- $k = 2$:

$$
\begin{aligned}
\mathsf{a}\left(\mathbf{u},\mathbf{v}\right) &= \int_\Omega \alpha\mathbf{u}\mathbf{v}\,\mathrm{d}\boldsymbol{x} + \sum_T \int_T \boldsymbol{\beta}\cdot\mathbf{v}\,\mathrm{div}\,\mathbf{u}\,\mathrm{d}\boldsymbol{x} - \int_T \mathbf{u}\cdot\left(\mathbf{curl}\,\mathbf{v}\times\boldsymbol{\beta}\right)\mathrm{d}\boldsymbol{x} \\
&\quad + \sum_{f\in\mathcal{F}^\circ} \int_f \Big(\{\mathbf{u}\}_f\times\boldsymbol{\beta}\Big)\cdot\Big([\mathbf{v}]_f\times\mathbf{n}_f\Big)\,\mathrm{d}S - \int_f [\mathbf{u}]_f\cdot\mathbf{n}_f\,\{\mathbf{v}\}_f\cdot\boldsymbol{\beta}\,\mathrm{d}S \\
&\quad + \sum_{f\in\mathcal{F}^\circ} \int_f c_f\boldsymbol{\beta}\cdot\mathbf{n}_f\left[\mathbf{u}\right]_f\cdot\left[\mathbf{v}\right]_f\mathrm{d}S \\
&\quad + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-} \int_f \left(\mathbf{u}\times\boldsymbol{\beta}\right)\cdot\left(\mathbf{v}\times\mathbf{n}_f\right)\,\mathrm{d}S - \sum_{f\in\mathcal{F}^\partial_-} \int_f \mathbf{u}\cdot\mathbf{n}_f\mathbf{v}\cdot\boldsymbol{\beta}\,\mathrm{d}S, \\
\mathsf{l}\left(\mathbf{v}\right) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x} - \sum_{f\in\mathcal{F}^\partial_-} \int_f \mathbf{g}\cdot\mathbf{n}_f\mathbf{v}\cdot\boldsymbol{\beta}\,\mathrm{d}S
\end{aligned}
$$

- $k = 3$:

$$\mathsf{a}\left(u,v\right) = \int_{\Omega} \alpha u v \,\mathrm{d}\boldsymbol{x} - \sum_{T} \int_{t} u\boldsymbol{\beta} \cdot \mathbf{grad}\, v \,\mathrm{d}\boldsymbol{x} + \sum_{f\in\mathcal{F}^{\circ}} \int_{T} \boldsymbol{\beta}\cdot\mathbf{n}_f \left\{u\right\}_f [v]_f \,\mathrm{d}S + \int_{T} c_f \boldsymbol{\beta}\cdot\mathbf{n}_f \left[u\right]_f [v]_f \,\mathrm{d}S + \sum_{f\in\mathcal{F}^{\partial}\setminus\mathcal{F}^{\partial}_{-}} \int \boldsymbol{\beta}\cdot\mathbf{n}_f u v \,\mathrm{d}S,$$

$$\mathsf{l}\left(v\right) = \int_{\Omega} f v \,\mathrm{d}\boldsymbol{x} - \sum_{f\in\mathcal{F}^{\partial}_{-}} \int_{f} \boldsymbol{\beta}\cdot\mathbf{n}_f g v \,\mathrm{d}S$$

4.2 Characteristic Methods

Recall the identity (2.28)

$$\mathsf{L}_{\boldsymbol{\beta}}\,\omega = \frac{\partial_t X_t^* \omega}{\partial t}|_{t=0}$$

for the pullback X_t^* induced by the flow of the velocity field $\boldsymbol{\beta}$, i.e.

$$\frac{\partial X_t(x)}{\partial t} = \boldsymbol{\beta}(X_t(x)), \quad X_0(x) = x. \tag{4.32}$$

We will use this identity to introduce so-called characteristic methods for the advection problem of differential forms. Characteristic methods for scalar stationary advection problems have been introduced in [8], convergence for scalar advection problems in $\mathbb{R}^2$ was proven in [5]. Although we do prove convergence for characteristic methods for k-forms in $\mathbb{R}^n$ we do not consider these characteristic methods as practical methods for the stationary advection problem (4.1). But, we will show that the characteristic methods are closely related to our stabilized Galerkin methods, which will be an important property for the analysis of non-stationary problems. For the scalar problems such close relationship has been recognized earlier [5].

Let X_t be the flow of the velocity field $\boldsymbol{\beta}$ and τ a fixed parameter. Here we assume that $\boldsymbol{\beta}$ is defined on an open neighbourhood of Ω. The flow $X_{-\tau}$ induces the decomposition $\Omega = \Omega_{\mathrm{in}} \cup \Omega_0$, with $X_{-\tau}(\Omega_{\mathrm{in}}) \cap \Omega = \{\}$ and $X_{-\tau}(\Omega_0) \subset \Omega$. Further we have $X_\tau(\Omega) = \Omega_0 \cup \Omega_{\mathrm{out}}$ with $\Omega_{\mathrm{out}} = X_\tau(\Omega) \setminus \Omega_0$ (see Figure 4.15).

Here again $\Lambda_h^k(\mathcal{T})$ denotes some piecewise polynomial approximation space on the triangulation $\mathcal{T}$ for k-forms in Ω.

We then define the characteristic Galerkin scheme for the advection problem (4.1): Find $\omega_h \in \Lambda_h^k(\mathcal{T})$ such that:

$$\mathsf{a}_\tau\left(\omega_h, \eta\right) = \mathsf{l}_\tau\left(\eta\right), \quad \forall \eta \in \Lambda_h^k\left(\mathcal{T}\right), \tag{4.33}$$

with

$$\mathsf{a}_\tau\left(\omega,\eta\right) := \left(\alpha\omega,\eta\right)_\Omega + \frac{1}{\tau}\left(\omega,\eta\right)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega,\eta\right)_{\Omega_0}, \quad \omega \in L^2\Lambda^k\left(\Omega\right), \tag{4.34}$$

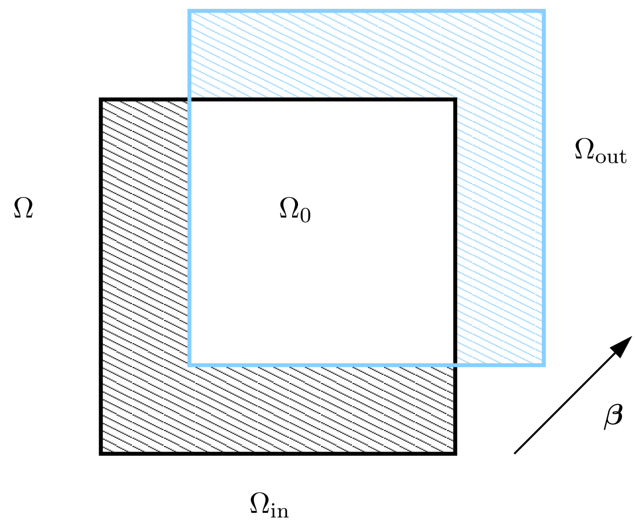

Figure 4.15: Illustration for definition of the domains Ω_0, Ω_{in} and Ω_{out}: the black lines and the light blue lines bound Ω and $X_\tau(\Omega)$, respectively. The black shaded area is Ω_{in} and the light blue shaded area is Ω_{out}.

and

$$\mathsf{l}_\tau(\eta) := (\phi, \eta)_\Omega + \frac{1}{\tau}\left(\widetilde{\psi}_D, \eta\right)_{\Omega_{\text{in}}}, \tag{4.35}$$

where $\widetilde{\psi}_D(x)$ is an extension of ψ_D onto Ω_{in}, that is constant along the characteristic lines of $\boldsymbol{\beta}$. More precisely, if we define for $x \in \Omega_{\text{in}}$ the time $t(x)$ such that $X_{t(x)-\tau}(x) \in \Gamma_{\text{in}}$ we set

$$\widetilde{\psi}_D(x) = \left(X^*_{t(x)-\tau}\psi_D\right)(x).$$

To evaluate the bilinear form $\mathsf{a}_\tau(\omega, \eta)$ one needs to solve the ordinary differential equation (4.32) to determine X_τ. We postpone the details of an implementation to the Chapter 5 on Lagrangian methods.

Convergence

As for the stabilized Galerkin methods we prove convergence in a mesh dependent norm, but here the natural norm depends also on τ. We define

$$\begin{aligned} \|\omega\|^2_{h,\tau} := \|\omega\|^2_{L^2\Lambda^k(\Omega)} + \frac{1}{2\tau}\left\|\omega - X^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega_0)} \\ + \frac{1}{2\tau}\|\omega\|^2_{L^2\Lambda^k(\Omega_{\text{in}})} + \frac{1}{2\tau}\left\|X^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega_{\text{out}})} \end{aligned} \tag{4.36}$$

and prove stability of the formulation (4.33) in this norm.

Lemma 4.2.1. *Assume that α and β in (4.1) are such that there is a positive constant α_0 with*

$$\alpha\omega \wedge \star\omega + \frac{1}{2\tau}\left(\omega \wedge \star\omega - \omega \wedge X_\tau^* \star X_{-\tau}^*\omega\right) \geq \alpha_0\omega \wedge \star\omega, \quad \forall\omega \in \Lambda^k(\Omega). \tag{4.37}$$

Then we have for all $\omega \in \Lambda_h^k(\mathcal{T})$:

$$\mathsf{a}_\tau(\omega,\omega) \geq \min(\alpha_0, 1)\,\|\omega\|_{h,\tau}^2.$$

Proof.

$$\begin{aligned}
\mathsf{a}_\tau(\omega,\omega) &= (\alpha\omega,\omega)_\Omega + \frac{1}{\tau}(\omega,\omega)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega,\omega\right)_{\Omega_0} \\
&= (\alpha\omega,\omega)_\Omega + \frac{1}{2\tau}(\omega,\omega)_{\Omega_0} - \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0} \\
&\quad + \frac{1}{2\tau}\left(\omega - X_{-\tau}^*\omega, \omega - X_{-\tau}^*\omega\right)_{\Omega_0} + \frac{1}{\tau}(\omega,\omega)_{\Omega_{\text{in}}} \\
&= (\alpha\omega,\omega)_\Omega + \frac{1}{2\tau}(\omega,\omega)_\Omega - \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0\cup\Omega_{\text{out}}} \\
&\quad + \frac{1}{2\tau}\left(\omega - X_{-\tau}^*\omega, \omega - X_{-\tau}^*\omega\right)_{\Omega_0} + \frac{1}{2\tau}(\omega,\omega)_{\Omega_{\text{in}}} + \frac{1}{2\tau}\left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_{\text{out}}} \\
&\geq \min(\alpha_0,1)\,\|\omega\|_{h,\tau}^2,
\end{aligned}$$

where the last estimate follows from the positivity assumption (4.37) by the following identity:

$$(\omega,\omega)_\Omega - \left(X_{-\tau}^*\omega, X_{-\tau}^*\omega\right)_{\Omega_0\cup\Omega_{\text{out}}} = \int_\Omega \omega \wedge \star\omega - \int_\Omega \omega \wedge X_\tau^* \star X_{-\tau}^*\omega. \tag{4.38}$$

□

The identity

$$\begin{aligned}
\omega \wedge \star\omega - \omega \wedge X_\tau^* \star X_{-\tau}^*\omega &= \omega \wedge (\star\omega - X_\tau^* \star \omega) + \omega \wedge \left(X_\tau^* \star \omega - X_\tau^* \star X_{-\tau}^*\omega\right) \\
&= \tau\omega \wedge \star\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega + O(\tau^2),
\end{aligned} \tag{4.39}$$

$\omega \in \Lambda^k(\Omega)$, shows that the stability assumption (4.37) in Lemma 4.2.1 is very similar to the stability assumption of the stabilized Galerkin methods in Lemma 4.1.5:

$$\alpha\omega \wedge \star\omega + \frac{1}{2}\left(\mathsf{L}_{\boldsymbol{\beta}} + \mathcal{L}_{\boldsymbol{\beta}}\right)\omega \wedge \star\omega \geq \alpha_0\omega \wedge \star\omega, \quad \forall\omega \in \Lambda^k(\Omega).$$

The next Lemma gives a continuity estimate for $\mathsf{a}_\tau(\omega,\eta)$.

Lemma 4.2.2. *For $\omega \in L^2\Lambda^k(\Omega)$ and $\eta \in \Lambda_h^k(\mathcal{T})$, τ sufficiently small we have*

$$\mathsf{a}_\tau(\omega,\eta) \leq \frac{C}{\sqrt{\tau}}\|\omega\|_{L^2\Lambda^k(\Omega)}\|\eta\|_{h,\tau}$$

with $C = C(\boldsymbol{\beta}) \geq 0$ independent of τ and mesh size h.

Proof. First we rewrite a_τ:

$$\begin{aligned}\mathsf{a}_\tau(\omega,\eta) &= (\alpha\omega,\eta)_\Omega + \frac{1}{\tau}(\omega,\eta)_\Omega + \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta - \eta\right)_{\Omega_0} \\ &\quad - \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{\Omega_0} \\ &= (\alpha\omega,\eta)_\Omega + \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta - \eta\right)_{\Omega_0} + \frac{1}{\tau}(\omega,\eta)_\Omega \\ &\quad - \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{\Omega_0\cup\Omega_{\mathrm{out}}} + \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{\Omega_{\mathrm{out}}}\end{aligned}$$

and estimate then the individual terms in the last sum:

$$\begin{aligned}|(\alpha\omega,\eta)_\Omega| &\le \|\alpha\|_{L^\infty\Lambda^0(\Omega)}\|\omega\|_{L^2\Lambda^k(\Omega)}\|\eta\|_{L^2\Lambda^k(\Omega)}, \\ \left|\frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta-\eta\right)_{\Omega_0}\right| &\le \sqrt{\frac{1+C\tau}{\tau}}\|\omega\|_{L^2\Lambda^k(\Omega)}\frac{1}{\sqrt{\tau}}\left\|\eta - X^*_{-\tau}\eta\right\|_{L^2\Lambda^k(\Omega_0)}, \\ \left|\frac{1}{\tau}(\omega,\eta)_\Omega - \frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{\Omega_0\cup\Omega_{\mathrm{out}}}\right| &\le C(\boldsymbol{\beta})\|\omega\|_{L^2\Lambda^k(\Omega)}\|\eta\|_{L^2\Lambda^k(\Omega)}, \\ \left|\frac{1}{\tau}\left(X^*_{-\tau}\omega, X^*_{-\tau}\eta\right)_{\Omega_{\mathrm{out}}}\right| &\le \sqrt{\frac{1+C\tau}{\tau}}\|\omega\|_{L^2\Lambda^k(\Omega)}\frac{1}{\sqrt{\tau}}\left\|X^*_{-\tau}\eta\right\|_{L^2\Lambda^k(\Omega_{\mathrm{out}})}.\end{aligned}$$

The third estimate is based on the identities (4.38), (4.39) and the bound (2.40). The second and fourth estimate use boundedness of the pullback for sufficiently small τ (see Proposition 4.2.3):

$$\left\|X^*_{-\tau}\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})} \le \sqrt{1+C\tau}\,\|\omega\|_{L^2\Lambda^k(\Omega)}.$$

For sufficiently small τ we have both $\|\alpha\|_{L^\infty(\Omega)} \le \tau^{-\frac{1}{2}}$ and $C(\boldsymbol{\beta}) \le \tau^{-\frac{1}{2}}$ and we deduce the assertion. ∎

Proposition 4.2.3. *For sufficiently small τ there exists $C = C(\boldsymbol{\beta}) \ge 0$ independent of τ such that*

$$\left\|X^*_{-\tau}\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})} \le \sqrt{1+C\tau}\,\|\omega\|_{L^2\Lambda^k(\Omega)}.$$

Proof. For $n = 2$ and $n = 3$ the assertion follows immediately from the explicit representation of the pullbacks in Tables 2.2 and 2.3. We find e.g. for differential forms ω in $\mathbb{R}^3$ with vector correspondences u or $\mathbf{u}$:

$$\begin{aligned}k=0: &\quad (X^*_\tau * X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))u(x), \\ k=1: &\quad (X^*_\tau * X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))DX_\tau^{-1}(x)DX_\tau^{-T}(x)\mathbf{u}(x), \\ k=2: &\quad (X^*_\tau * X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))^{-1}DX_\tau^{T}(x)DX_\tau(x)\mathbf{u}(x), \\ k=3: &\quad (X^*_\tau * X^*_{-\tau}\omega)(x) \sim \det(DX_\tau(x))^{-1}u(x).\end{aligned} \tag{4.40}$$

and compute

$$\begin{aligned}\left\|X_{-\tau}^{*}\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})}^2 &= \left(X_{-\tau}^{*}\omega, X_{-\tau}^{*}\omega\right)_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})}\\ &= \int_\Omega \omega\wedge X_\tau^*\star X_{-\tau}^*\omega\\ &\le (1+C(\boldsymbol{\beta})\tau)\int_\Omega \omega\wedge\star\omega,\end{aligned}$$

since $DX_\tau = \mathsf{id} + \tau D\boldsymbol{\beta} + O(\tau^2)$.

The proof for the general case uses the notion of k-minors introduced in (2.17). By density of $\Lambda^k(\Omega)$ in $L^2\Lambda^k(\Omega)$ it is enough to consider $\omega\in\Lambda^k(\Omega)$. Recall that vol is the volume form on Ω and that we have by (2.8)

$$\left\|X_{-\tau}^{*}\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})}^2 = \int_{\Omega_0\cup\Omega_{\mathrm{out}}} \left(\left(X_{-\tau}^*\omega\right)_x, \left(X_{-\tau}^*\omega\right)_x\right)\mathrm{vol}\,.$$

Then by (2.17) for ω and vol we find:

$$\begin{aligned}\left\|X_{-\tau}^{*}\omega\right\|_{L^2\Lambda^k(\Omega_0\cup\Omega_{\mathrm{out}})}^2 &= \int_{\Omega_0\cup\Omega_{\mathrm{out}}} \left(\mathbf{M}_k(X_{-\tau,x})\omega_{X_{-\tau}(x)}, \mathbf{M}_k(X_{-\tau,x})\omega_{X_{-\tau}(x)}\right)\mathrm{vol},\\ &\le \sup_{x\in\Omega}\rho\left(\mathbf{M}_k(X_{-\tau,x})\right)^2\int_{\Omega_0\cup\Omega_{\mathrm{out}}}\left(\omega_{X_{-\tau}(x)},\omega_{X_{-\tau}(x)}\right)\mathrm{vol},\\ &\le \sup_{x\in\Omega}\left(\rho\left(\mathbf{M}_k(X_{-\tau,x})\right)^2\rho\left(\mathbf{M}_n(X_{-\tau,x})\right)\right)\int_\Omega(\omega_x,\omega_x)\,\mathrm{vol},\end{aligned}$$

where ρ denotes the spectral radius. The assertion follows, since we have $\mathbf{M}_k(X_{-\tau,x}) = \mathsf{id} + \tau\mathbf{M}_k'(X_{0,x})$, with $\mathbf{M}_k'(X_{0,x}) = \mathbf{M}_k'(D\boldsymbol{\beta}(x))$ defined in the proof of Proposition 2.1.3. □

This time we do not have consistency of the methods. But we can control the consistency error and prove convergence.

Theorem 4.2.4. *Assume that α, β and τ in* (4.1) *are such that there is a positive constant α_0 with*

$$\alpha\omega\wedge\star\omega + \frac{1}{2\tau}\left(\omega\wedge\star\omega - \omega\wedge X_\tau^*\star X_{-\tau}^*\omega\right) \ge \alpha_0\omega\wedge\star\omega,\quad \forall\omega\in\Lambda^k(\Omega)\,. \tag{4.41}$$

Let $\omega\in H^{r+1}\Lambda^k(\Omega)$ and $\omega_h\in\Lambda_h^k(\mathcal{T})$ be the solutions to the advection problem (4.1) *and its variational formulation* (4.33). *If additionally $\Lambda_h^k(\mathcal{T})$ permits the approximation property*

$$\inf_{\eta\in\Lambda_h^k(\mathcal{T})}\|\omega-\eta\|_{L^2\Lambda^k(\Omega)} \le Ch^{r+1}\|\omega\|_{H^{r+1}\Lambda^k(\Omega)},$$

we get with $C>0$ independent of mesh size $h := \max_T(h_T)$ and τ:

$$\|\omega-\omega_h\|_{h,\tau} \le C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}}\right)\|\omega\|_{H^{r+1}\Lambda^k(\Omega)}.$$

Proof. Let $\bar{\omega}_h$ denote the L^2-projection of ω onto $\Lambda_h^k(\mathcal{T})$, then:

$$\|\omega - \omega_h\|_{h,\tau}^2 \le \|\omega - \bar{\omega}_h\|_{h,\tau}^2 + \|\bar{\omega}_h - \omega_h\|_{h,\tau}^2 .$$

The stability estimate in Lemma 4.2.1 yields

$$\min(\alpha_0, 1)\, \|\bar{\omega}_h - \omega_h\|_{h,\tau}^2 \le \mathsf{a}_\tau\left(\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h\right) + \mathsf{a}_\tau\left(\omega - \omega_h, \bar{\omega}_h - \omega_h\right).$$

We find for the consistency error $\mathsf{a}_\tau(\omega - \omega_h, \eta_h)$, $\eta_h \in \Lambda_h^k(\mathcal{T})$ by the definition of $\mathsf{a}_\tau(\cdot,\cdot)$, $\mathsf{l}_\tau(\cdot)$ and $\alpha\omega + \mathsf{L}_{\boldsymbol{\beta}}\omega = \varphi$

$$\begin{aligned} \left|\mathsf{a}_\tau(\omega - \omega_h, \eta_h)\right| &= \left|\mathsf{a}_\tau(\omega, \eta_h) - \mathsf{l}_\tau(\eta_h)\right| \\ &= \left| \frac{1}{\tau}(\omega, \eta_h)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega, \eta_h\right)_{\Omega_0} - \frac{1}{\tau}\left(\widetilde{\psi}_D, \eta_h\right)_{\Omega_{\mathrm{in}}} - (\mathsf{L}_{\boldsymbol{\beta}}\omega, \eta_h)_\Omega \right| \\ &= \left| \left(\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\omega, \eta_h\right)_{\Omega_0} + \left(\frac{1}{\tau}\left(\omega - \widetilde{\psi}_D\right) - \mathsf{L}_{\boldsymbol{\beta}}\omega, \eta_h\right)_{\Omega_{\mathrm{in}}} \right| . \end{aligned}$$

A bound for the first term in the last inequality follows from Taylor expansion

$$\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\omega = \frac{1}{\tau}\int_0^\tau (-s) \frac{\partial^2 X_t^*\omega}{\partial t^2}_{|t=s} \mathrm{d}s$$

and we find

$$\left| \left(\frac{1}{\tau}\left(\omega - X_{-\tau}^*\omega\right) - \mathsf{L}_{\boldsymbol{\beta}}\omega, \eta_h\right)_{\Omega_0} \right| \le C\tau \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{2,\infty}(\Omega)} \|\omega\|_{H^2\Lambda^k(\Omega)} \|\eta_h\|_{L^2\Lambda^k(\Omega)}$$

with C independent of h. Recall that $\widetilde{\psi}_D(x) = \left(X_{t(x)-\tau}^*\psi_D\right)(x)$ with $X_{t(x)-\tau}(x) \in \Gamma_{\mathrm{in}}$ and $\psi_D = \omega$ on Γ_{in}. Whence Taylor expansion for the second term yields

$$\begin{aligned} \left| \left(\frac{1}{\tau}\left(\omega - \widetilde{\psi}_D\right) - \mathsf{L}_{\boldsymbol{\beta}}\omega, \eta_h\right)_{\Omega_{\mathrm{in}}} \right| &\le C \|\mathsf{L}_{\boldsymbol{\beta}}\omega\|_{L^2\Lambda^k(\Omega_{\mathrm{in}})} \|\eta_h\|_{L^2\Lambda^k(\Omega_{\mathrm{in}})} \\ &\le C\tau^{\frac{1}{2}} \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{1,\infty}(\Omega)} \|\omega\|_{H^1\Lambda^k(\Omega)} \|\eta_h\|_{h,\tau} , \end{aligned}$$

since $(2\tau)^{-\frac{1}{2}} \|\eta_h\|_{L^2\Lambda^k(\Omega_{\mathrm{in}})} \le \|\eta_h\|_{h,\tau}$. That means that we have the following bound for the consistency error:

$$\left|\mathsf{a}_\tau(\omega - \omega_h, \bar{\omega}_h - \omega_h)\right| \le C\tau^{\frac{1}{2}} \|\boldsymbol{\beta}\|_{\boldsymbol{W}^{2,\infty}(\Omega)} \|\omega\|_{H^2\Lambda^k(\Omega)} \|\bar{\omega}_h - \omega_h\|_{h,\tau} .$$

The continuity estimate in Lemma 4.2.2 gives:

$$\mathsf{a}_\tau(\bar{\omega}_h - \omega, \bar{\omega}_h - \omega_h) \le C\tau^{-\frac{1}{2}} \|\bar{\omega}_h - \omega\|_{L^2\Lambda^k(\Omega)} \|\bar{\omega}_h - \omega_h\|_{h,\tau} .$$

In summary we find

$$\min(\alpha_0, 1)\, \|\bar{\omega}_h - \omega_h\|_{h,\tau} \le C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}}\right) \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

and

$$\|\bar{\omega}_h - \omega\|_{h,\tau} \le Ch^{r+1}\tau^{-\frac{1}{2}} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)},$$

which yields the assertion. □

The proof of the Theorem 4.2.4 assumes that the bilinear form $\mathsf{a}_\tau(\omega, \eta)$ can be evaluated exactly for discrete forms $\omega, \eta \in \Lambda_h^k(\mathcal{T})$. Real implementations will barely match this assumption. The standard technique to evaluate bilinear forms for piecewise smooth finite element functions is the use of local, i.e. elementwise, quadrature rules of sufficient accuracy. This simple approach will fail for our characteristic methods due to the term $\left(X_{-\tau}^*\omega, \eta\right)_{\Omega_0}$. While η is piecewise smooth on the mesh $\mathcal{T}$, the pullback $X_{-\tau}^*\omega$ of ω is picewise smooth on $X_\tau(\mathcal{T})$. $X_\tau(\mathcal{T})$ is the image mesh of mesh $\mathcal{T}$ (see Figure 4.16). A sound approximation of $\left(X_{-\tau}^*\omega, \eta\right)_{\Omega_0}$ must split integration over Ω_0 in a sum of integrals over intersections of elements of $\mathcal{T}$ with elements of $X_\tau(\mathcal{T})$. This might be very expensive.

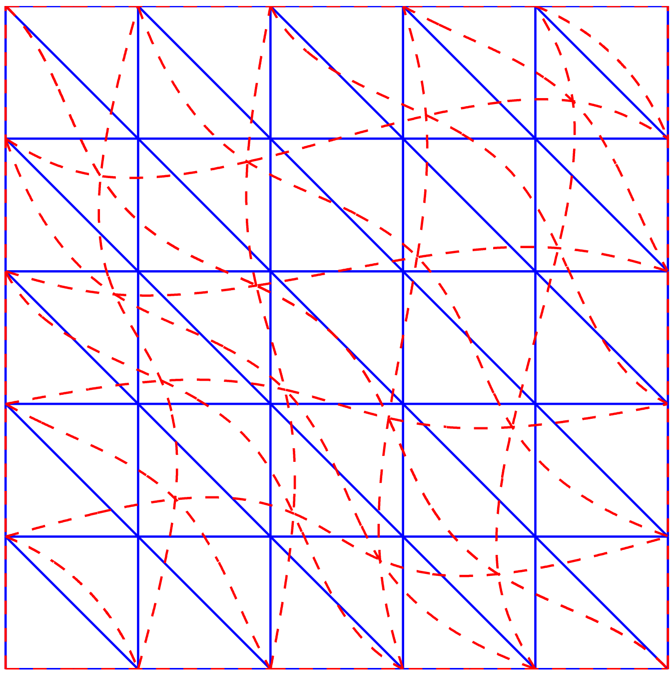

Figure 4.16: A mesh $\mathcal{T}$ (blue, solid lines) on $\Omega = [0,1]^2$ and its image mesh $X_\tau(\mathcal{T})$ under the flow induced by $\boldsymbol{\beta} = \left(\frac{1}{16}\sin(2\pi x)\sin(2\pi y), \frac{1}{16}\sin(2\pi x)\sin(2\pi y)\right)$.

Characteristic Methods and Stabilized Methods

As mentioned earlier there is a very close relationship between these characteristic methods and the stabilized Galerkin methods. The stabilized Galerkin methods are the limit case of characteristic methods when the parameter τ tends to zero.

Theorem 4.2.5. *Let $\Lambda_h^k(\mathcal{T})$ be a piecewise polynomial approximation space of k-forms. Let $\mathsf{a}(\omega, \eta)$ and $\mathsf{a}_\tau(\omega, \eta)$ be the two bilinear forms defined in* (4.8) *and* (4.34)*. Further* *assume that* $c_f = \frac{1}{2}\frac{\mathbf{n}_f\cdot\boldsymbol{\beta}}{|\mathbf{n}_f\cdot\boldsymbol{\beta}|}$. *The limit* $\lim_{\tau\to 0}\mathsf{a}_\tau(\omega,\eta)$ *exists and*

$$\lim_{\tau\to 0}\mathsf{a}_\tau(\omega,\eta) = \mathsf{a}(\omega,\eta), \quad \omega,\eta\in\Lambda_h^k(\mathcal{T}).$$

Proof. By Leibniz rule (2.36) we find

$$\begin{aligned}\mathsf{a}(\omega,\eta) = (\alpha\omega,\eta)_\Omega + \sum_T (\mathsf{L}_{\boldsymbol{\beta}}\omega,\eta)_T - \sum_{f\in\mathcal{F}_-^\partial}(\omega,\eta)_{f,\boldsymbol{\beta}} \\ + \sum_{f\in\mathcal{F}^\circ}\Big([\omega]_f, c_f[\eta]_f - \{\eta\}_f\Big)_{f,\boldsymbol{\beta}}.\end{aligned} \tag{4.42}$$

We first prove

$$\lim_{\tau\to 0}\frac{1}{\tau}(\omega - X_\tau^*\omega,\eta)_{\Omega_0} = \sum_T(\mathsf{L}_{\boldsymbol{\beta}}\omega,\eta)_T + \sum_{f\in\mathcal{F}^\circ}\Big([\omega]_f, c_f[\eta]_f - \{\eta\}_f\Big)_{f,\boldsymbol{\beta}}, \tag{4.43}$$

with $c_f = \frac{1}{2}\frac{\mathbf{n}_f\cdot\boldsymbol{\beta}}{|\mathbf{n}_f\cdot\boldsymbol{\beta}|}$. T and f denotes n-simplices and $n-1$-simplices of the mesh. ω and η are picewise smooth on the mesh $\mathcal{T}$ and $X_{-\tau}^*\omega$ is piecewise smooth on the mesh $X_\tau(\mathcal{T})$. For any two elements $T, T'\in\Delta_n(\mathcal{T})$ we define the patches

$$P_0(T,T') := T\cap X_\tau(T')\cap\Omega_0.$$

Then we split the integration over Ω_0 in a sum of intgrals over the patches:

$$\frac{1}{\tau}\big(\omega - X_{-\tau}^*\omega,\eta\big)_{\Omega_0} = \sum_{P_0(T,T')}\frac{1}{\tau}\big(\omega - X_{-\tau}^*\omega,\eta\big)_{P_0(T,T')}.$$

Each of the integrals in the last sum has smooth integrands. We distinguish now three different cases (see Figure 4.17):

- $T\neq T'$ and $\Delta_{n-1}(T)\cup\Delta_{n-1}(T') = \{\}$, i.e. the elements T and T' do not share any facet f: Since $|P_0(T,T')| = O(\tau^2)$ we find by the Lebesgue's dominated convergence theorem:
$$\lim_{\tau\to 0}\frac{1}{\tau}\big(\omega - X_{-\tau}^*\omega,\eta\big)_{P_0(T,T')} = 0.$$
- $T\neq T'$ and $\Delta_{n-1}(T)\cup\Delta_{n-1}(T') = \{f\}$: Here $|P_0(T,T')| = |\operatorname{extr}(f,X_\tau)| + O(\tau^2)$ and the orientations of $P_0(T,T')$ and $\operatorname{extr}(f,X_\tau)$ coincide if $\boldsymbol{\beta}\cdot\mathbf{n}_f > 0$. We use the coefficent c_f to reflect the change of orientation. Let $\widetilde{c}_f$ denote the piecewise constant extension of c_f to $\operatorname{extr}(f,X_\tau)$ (see Figure 4.18), then we find:

$$\begin{aligned}\lim_{\tau\to 0}\frac{1}{\tau}\big(\omega - X_{-\tau}^*\omega,\eta\big)_{P_0(T,T')} &= \lim_{\tau\to 0}\frac{2}{\tau}\big(\omega - X_{-\tau}^*\omega,\widetilde{c}_f\eta\big)_{\operatorname{extr}(f,X_\tau)} \\ &= \lim_{\tau\to 0}\frac{2}{\tau}\left(\int_{\operatorname{extr}(f,X_\tau)}\widetilde{c}_f\omega\wedge\star\eta - \int_{\operatorname{extr}(f,X_\tau)}\widetilde{c}_f X_{-\tau}^*\omega\wedge\star\eta\right).\end{aligned}$$

To determine the limit values of the last two terms we recall that ω and η are discontinuous across facet f. But because the extrusion $\mathrm{extr}(f, X_\tau)$ is an extrusion of f in the direction of the flow, the limit selects the values from the downwind side for ω and η and the values from the upwind side for $X_{-\tau}\omega$ (see Figure 4.17). According to Remark 4.1.2 the downwind and upwind values of ω are given as $\{\omega\}_f - c_f [\omega]_f$ and $\{\omega\}_f + c_f [\omega]_f$. We get by (2.30)

$$\begin{aligned}\lim_{\tau\to 0}\frac{1}{\tau}\left(\omega - X^*_{-\tau}\omega, \eta\right)_{P_0(T,T')} =& 2\int_f c_f\, \mathsf{i}_{\boldsymbol{\beta}}\left(\left(\{\omega\}_f - c_f[\omega]_f\right)\wedge\star\left(\{\eta\}_f - c_f[\eta]_f\right)\right)\\ &- 2\int_f c_f\, \mathsf{i}_{\boldsymbol{\beta}}\left(\left(\{\omega\}_f + c_f[\omega]_f\right)\wedge\star\left(\{\eta\}_f - c_f[\eta]_f\right)\right)\\ =& \left([\omega]_f, c_f[\eta]_f - \{\eta\}_f\right)_{f,\boldsymbol{\beta}}.\end{aligned}$$

Note that for any f the existence of T, T' with $\Delta_{n-1}(T)\cup\Delta_{n-1}(T') = \{f\}$ and $f \in P_0(T,T')$ imply $f\in\mathcal{F}^\circ$.

- $T = T'$: Since $|P_0(T,T)| = O(\tau^0)$ we find by the Lebesgue's dominated convergence theorem:

$$\lim_{\tau\to 0}\frac{1}{\tau}\left(\omega - X^*_{-\tau}\omega, \eta\right)_{P_0(T,T)} = \left(\mathsf{L}_{\boldsymbol{\beta}}\,\omega, \eta\right)_T.$$

This proves (4.43). Similarly we deduce

$$\lim_{\tau\to 0}\,(\omega,\eta)_{\Omega_{\mathrm{in}}} = -\sum_{f\in\mathcal{F}^\partial_-}(\omega,\eta)_{f,\boldsymbol{\beta}}.$$

and get the assertion. □

A corollary of Theorem 4.2.5 says that the norm $\|\cdot\|_h$ is the limit of the norm $\|\cdot\|_{h,\tau}$.

Lemma 4.2.6. *Let $\Lambda^k_h(\mathcal{T})$ be a piecewise polynomial approximation space of k-forms and assume $c_f = \frac{1}{2}\frac{\mathbf{n}_f\cdot\boldsymbol{\beta}}{|\mathbf{n}_f\cdot\boldsymbol{\beta}|}$ in (4.9) then*

$$\lim_{\tau\to 0}\|\omega\|_{h,\tau} = \|\omega\|_h, \quad \forall\omega\in\Lambda^k_h(\mathcal{T}).$$

Proof. The same arguments as in the proof of Theorem 4.2.5 show:

$$\lim_{\tau\to 0}\frac{1}{2\tau}\left\|\omega - X^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega_0)} = \sum_{f\in\mathcal{F}^\circ}\left\|[\omega]_f\right\|^2_{f,c_f\boldsymbol{\beta}},$$

$$\lim_{\tau\to 0}\frac{1}{2\tau}\|\omega\|^2_{L^2\Lambda^k(\Omega_{\mathrm{out}})} = \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}^\partial_-}\|\omega\|^2_{f,\frac{1}{2}\boldsymbol{\beta}},$$

and

$$\lim_{\tau\to 0}\frac{1}{2\tau}\|\omega\|^2_{L^2\Lambda^k(\Omega_{\mathrm{in}})} = \sum_{f\in\mathcal{F}^\partial_-}\|\omega\|^2_{f,-\frac{1}{2}\boldsymbol{\beta}}.$$

□

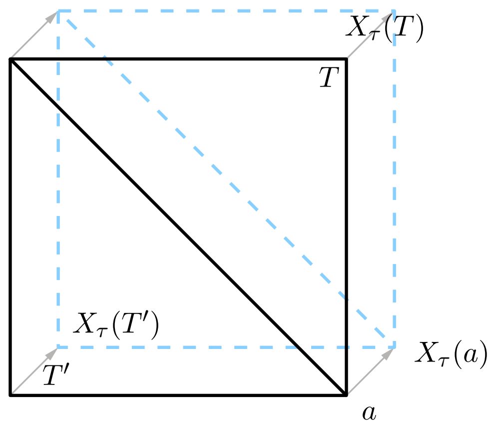

Figure 4.17: Illustration for proof of Theorem 4.2.5. Discrete k-forms $\omega \in \Lambda_h^k(\mathcal{T})$, $k > 0$ are discontinuous across edges of the triangulation $\mathcal{T}$ (black, solid lines). Their pullbacks $X_{-\tau}^*\omega$ are discontinuous across edges of the triangulation $X_\tau(\mathcal{T})$ (blue, dashed lines), the image of $\mathcal{T}$.

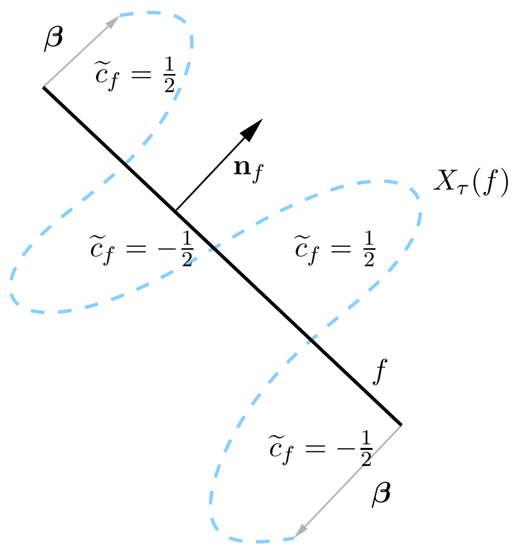

Figure 4.18: Illustration for the definition of $\widetilde{c}_f$ in the proof of Theorem 4.2.5.

5 Non-Stationary Advection Problem

In this chapter we will present numerical methods for the non-stationary advection problems

$$
\begin{aligned}
\partial_t\omega + \alpha\omega + \mathsf{L}_{\boldsymbol{\beta}}\,\omega &= \varphi, && \text{in } \Omega,\\
\operatorname{tr}\omega &= \operatorname{tr}\psi_D, && \text{on } \Gamma_{\text{in}},\\
\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\omega &= \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\psi_D, && \text{on } \Gamma_{\text{in}},\\
\omega(0) &= \omega_0, && \text{in } \Omega,
\end{aligned}
\tag{5.1}
$$

and

$$
\begin{aligned}
\partial_t\widetilde{\omega} + \alpha\widetilde{\omega} - \mathcal{L}_{\boldsymbol{\beta}}\,\widetilde{\omega} &= \widetilde{\varphi}, && \text{in } \Omega,\\
\operatorname{tr}\widetilde{\omega} &= \operatorname{tr}\widetilde{\psi}_D, && \text{on } \Gamma_{\text{in}},\\
\operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\widetilde{\omega} &= \operatorname{tr}\mathsf{i}_{\boldsymbol{\beta}}\,\widetilde{\psi}_D, && \text{on } \Gamma_{\text{in}},\\
\widetilde{\omega}(0) &= \omega_0, && \text{in } \Omega
\end{aligned}
\tag{5.2}
$$

with data $\varphi, \widetilde{\varphi} \in L^2\Lambda^k(\Omega)$ and $\psi_D, \widetilde{\psi}_D \in \Lambda^k(\mathbb{R}^n \setminus \Omega)$. $\alpha \in \Lambda^0(\Omega)$ is a given scalar parameter and $\beta : \Omega \mapsto \mathbb{R}^n$ is a given Lipschitz continuous velocity field.

Again we consider (5.1) and (5.2) as the limiting problems of advection-diffusion problems of type (3.3) and (3.4). We have shown in Theorems 3.4.9 and 3.4.11 that (5.1) and (5.2) are well-posed in appropriate spaces.

As for the stationary problem in Chapter 4 it is enough to study the problem (5.1). We will present two different families of timestepping methods for tackling the advection problem (5.1). These are the Eulerian methods and the Lagrangian methods. The former are based on spatial discretization to which some numerical timestepping procedure is applied. In our case we use the stabilized Galerkin methods from Chapter 4 for the spatial discretization and explicit and implicit Euler timestepping methods as timestepping procedures. Lagrangian methods dispense with a fixed mesh and approximately track the flow induced by the velocity field $\boldsymbol{\beta}$.

5.1 Eulerian Methods

Eulerian methods for the non-stationary advection problem (5.1) build on semi-discretization in space. The resulting system of ordinary differential equations is solved with standard numerical integrators. Here, we will use the stabilized Galerkin discretization from Chapter 4 for the semi-discretization in space and simple lowest order one step Euler methods as numerical integrator. Since the main objective is a comparison with the semi-Lagrangian methods from the next section we do not provide an analysis for higher order numerical integrators, even though the formulations are straight forward.

In the following $\Lambda_h^k(\mathcal{T})$ denotes again some piecewise polynomial approximation space of differential k-forms on a triangulation $\mathcal{T}$ of Ω, i.e. $\Lambda_h^k(\mathcal{T}) \subset L^2\Lambda^k(\Omega)$, $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ or $\Lambda_h^k(\mathcal{T}) \subset \star H\Lambda^{n-k}(\Omega)$.

The semi-discrete variational formulation of (5.1) is: Find $\omega_h \in C^1\left([0,T];\Lambda_h^k(\mathcal{T})\right)$ such that

$$\begin{aligned} (\partial_t\omega_h,\eta)_\Omega + \mathsf{a}(\omega_h,\eta) &= \mathsf{l}(\eta), \quad \forall t \in [0,T], \quad \forall \eta \in \Lambda_h^k(\mathcal{T}) \\ \omega_h(0) &= \omega_0, \end{aligned} \tag{5.3}$$

where $\mathsf{a}(\omega_h,\eta)$ and $\mathsf{l}(\eta)$ are defined in (4.7) and (4.8).

The stability estimate for $\mathsf{a}(\omega_h,\eta)$ in Lemma 4.1.5 and convergence estimates as in Theorems 4.1.8 and 4.1.13-4.1.16 show that the solutions $\omega_h(t)$ of (5.3) are accurate approximations to $\omega(t)$:

Theorem 5.1.1. *Let ω and ω_h be the solution to* (5.1) *and* (5.3)*. Assume that* (4.3) *holds for α and β in* (5.1)*. Assume in the definition* (4.7) *of $\mathsf{a}(\cdot,\cdot)$ that the parameter c_f satisfies for all faces f the positivity condition*

$$c_f \boldsymbol{\beta}\cdot\mathbf{n}_f > 0.$$

Further assume that the Ritz-Galerkin projection $P_h\omega(t) \in \Lambda_h^k(\mathcal{T})$ with

$$\mathsf{a}(P_h\omega(t),\eta) := \mathsf{a}(\omega(t),\eta), \quad \forall \eta \in \Lambda_h^k(\mathcal{T})$$

fulfills the estimate

$$\|\omega(t) - P_h\omega(t)\|_{L^2\Lambda^k(\Omega)} \leq C_c h^{r+\frac{1}{2}} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)} \tag{5.4}$$

for $\omega \in H^{r+1}\Lambda^k(\Omega)$ and $C_c > 0$ independent of h. Then we have

$$\max_{t\in[0,T]} \|\omega(t) - \omega_h(t)\|_{L^2\Lambda^k(\Omega)} \leq \|\omega(0) - \omega_h(0)\|_{L^2\Lambda^k(\Omega)}\, e^{-\min(\alpha_0,1)\frac{T}{2}} + h^{r+\frac{1}{2}} c(\omega),$$

where $c(\omega) = C_c \max_{t\in[0,T]}\left(\frac{1}{\min(\alpha_0,1)}\|\partial_t\omega(t)\|_{H^{r+1}\Lambda^k(\Omega)} + \|\omega(t)\|_{H^{r+1}\Lambda^k(\Omega)}\right)$.

Proof. We set $\gamma_h(t) := P_h\omega(t) - \omega_h(t)$ and find

$$(\partial_t\gamma_h(t),\eta)_\Omega + \mathsf{a}(\gamma_h(t),\eta) = (\partial_t P_h\omega(t) - \partial_t\omega(t),\eta)_\Omega,$$

because $\mathsf{a}(P_h\omega(t) - \omega(t),\eta) = 0$. Let $\bar{\alpha}_0 = \min(\alpha_0,1)$ for the constant α_0 in (4.3). Setting $\eta = \gamma_h$ the stability in Lemma 4.1.5 and Young's inequality yield:

$$\begin{aligned} &\partial_t \frac{1}{2}\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} + \bar{\alpha}_0\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} \\ &\qquad\leq \frac{\bar{\alpha}_0}{2}\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} + \frac{1}{2\bar{\alpha}_0}\|\partial_t P_h\omega(t) - \partial_t\omega_h(t)\|^2_{L^2\Lambda^k(\Omega)}, \end{aligned}$$

or

$$\partial_t\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} + \bar{\alpha}_0\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} \leq \frac{1}{\bar{\alpha}_0}\|\partial_t P_h\omega(t) - \partial_t\omega_h(t)\|^2_{L^2\Lambda^k(\Omega)}.$$

The Gronwall's Lemma [27, Lemma 6.9] gives then:

$$\|\gamma_h(t)\|^2_{L^2\Lambda^k(\Omega)} \leq \|\omega_h(0)\|^2_{L^2\Lambda^k(\Omega)}\, e^{-\bar{\alpha}_0 t} + C_c^2 \frac{1-e^{-\bar{\alpha}_0 t}}{\bar{\alpha}_0^2} h^{2r+1} \max_{t\in[0,T]} \|\partial_t \omega(t)\|^2_{H^{r+1}\Lambda^k(\Omega)}$$

and

$$\|\omega(t)-\omega_h(t)\|_{L^2\Lambda^k(\Omega)} \leq \|\omega_h(0)\|_{L^2\Lambda^k(\Omega)}\, e^{-\bar{\alpha}_0 \frac{t}{2}} + C_c h^{r+\frac{1}{2}} \max_{t\in[0,T]} \left(\frac{1}{\bar{\alpha}_0}\|\partial_t\omega(t)\|_{H^{r+1}\Lambda^k(\Omega)} + \|\omega(t)\|_{H^{r+1}\Lambda^k(\Omega)}\right).$$

□

Remark 5.1.2. *In Chapter 4 we have proved* (5.4) *for the following cases:*

- $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ *(Theorem 4.1.8),*
- $n=3,\ k=1$ *and* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k$ *or* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k$, *(Theorem 4.1.13),*
- $n=3,\ k=1$ *and* $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_r\Lambda^{n-k}$ *or* $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^{n-k}$, *(Theorem 4.1.14),*
- $n=3,\ k=2$ *and* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k$ *or* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k$, *(Theorem 4.1.15),*
- $n=3,\ k=2$ *and* $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_r\Lambda^{n-k}$ *or* $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^{n-k}$, *(Theorem 4.1.16).*

Given a positive number N, we set $\tau = \frac{T}{N}$, $t^n = \tau n$ for $0 \leq n \leq N$ and consider a partitioning of the time interval in the form $[0,T] = \bigcup_{n=0}^{N-1}[t^n, t^{n+1}]$. We introduce $\mathsf{I}^{\mathsf{n}}(\eta) := (\varphi(t^n),\eta)_\Omega - \sum_{f\in\mathcal{F}_-^\partial} (\psi_D(t^n),\eta)_{f,\boldsymbol{\beta}}$. and consider now two different time stepping schemes for constructing sequences $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, that approximate $(\omega(t^n))_{n=0}^N$.

- Explicit Euler time stepping scheme:
  Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned} \left(\omega_h^0,\eta\right)_\Omega &= (\omega(0),\eta)_\Omega\,, \\ \frac{1}{\tau}\left(\omega_h^{n+1}-\omega_h^n,\eta\right)_\Omega + \mathsf{a}\left(\omega_h^n,\eta\right) &= \mathsf{I}^{\mathsf{n+1}}(\eta)\,. \end{aligned} \tag{5.5}$$

- Implicit Euler time stepping scheme:
  Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned} \left(\omega_h^0,\eta\right)_\Omega &= (\omega(0),\eta)_\Omega\,, \\ \frac{1}{\tau}\left(\omega_h^{n+1}-\omega_h^n,\eta\right)_\Omega + \mathsf{a}\left(\omega_h^{n+1},\eta\right) &= \mathsf{I}^{\mathsf{n+1}}(\eta)\,. \end{aligned} \tag{5.6}$$

Note that in any case the two schemes (5.6) and (5.5) treat the right hand sides implicitly. Similar to the analysis of parabolic problems we can prove convergence of these schemes. We give here only the proof for the explicit scheme, the proof for the implicit scheme follows analogue.

Theorem 5.1.3. *Let $(\omega(t^n))_{n=0}^N$, $(\omega_h^n)_{n=0}^N$ be the solution to (5.1) and (5.5). Assume that (4.3) holds for α and β in (5.1). Assume in the definition (4.7) of $\mathsf{a}(\cdot,\cdot)$ that the parameter c_f satisfies for all faces f the positivity condition*

$$c_f\boldsymbol{\beta}\cdot\mathbf{n}_f > 0.$$

Further assume that the Ritz-Galerkin projection $P_h\omega(t) \in \Lambda_h^k(\mathcal{T})$ with

$$\mathsf{a}\left(P_h\omega(t),\eta\right) := \mathsf{a}\left(\omega(t),\eta\right), \quad \forall\eta \in \Lambda_h^k(\mathcal{T})$$

fulfills the estimate

$$\|\omega(t) - P_h\omega(t)\|_{L^2\Lambda^k(\Omega)} \le C_c h^{r+\frac{1}{2}} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

for $\omega \in H^{r+1}\Lambda^k(\Omega)$ and $C_c > 0$ independent of h. Let $C(h)$ be a positive function of the mesh size h such that

$$\mathsf{a}\left(\omega,\eta\right) \le C(h)\,\|\omega\|_{L^2\Lambda^k(\Omega)}\,\|\eta\|_h\,, \quad \forall\omega,\eta \in \Lambda_h^k(\mathcal{T})\,. \tag{5.7}$$

Then if $\tau \le \kappa\min(\alpha_0,1)C(h)^{-2}, 0 \le \kappa < 1$, we have

$$\max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \le C\left(h^{r+\frac{1}{2}} + \tau\right),$$

with $C > 0$ independent of τ and h.

Proof. The proof is similar to the proof of Theorem 7.1.15 in [26]. Let $\bar{\omega}_h^n := P_h\omega(t^n)$, then

$$\frac{1}{\tau}\left(\bar{\omega}_h^{n+1} - \bar{\omega}_h^n,\eta\right)_\Omega + \mathsf{a}\left(\bar{\omega}_h^n,\eta\right) = \mathsf{l}^{n+1}(\eta) + \left(R^{n+1},\eta\right)_\Omega$$

with

$$\left(R^{n+1},\eta\right)_\Omega = \left(\frac{1}{\tau}\left(\bar{\omega}_h^{n+1} - \bar{\omega}_h^n\right) - \partial_t\omega(t^n),\eta\right)_\Omega + \mathsf{l}^n(\eta) - \mathsf{l}^{n+1}(\eta)\,.$$

We define $\gamma_h^n := \bar{\omega}_h^n - \omega_h^n$ and find:

$$\frac{1}{\tau}\left(\gamma_h^{n+1} - \gamma_h^n,\eta\right)_\Omega + \mathsf{a}\left(\gamma_h^n,\eta\right) = \left(R^{n+1},\eta\right)_\Omega.$$

We take $\eta = 2\tau\gamma_h^{n+1}$ and use $2p(p-q) = p^2 + (p-q)^2 - q^2$:

$$\begin{aligned}\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2 + \left\|\gamma_h^{n+1} - \gamma_h^n\right\|_{L^2\Lambda^k(\Omega)}^2 + 2\tau\mathsf{a}\left(\gamma_h^n,\gamma_h^{n+1}\right)& \\ \le \|\gamma_h^n\|_{L^2\Lambda^k(\Omega)}^2 + 2\tau\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}& \\ \le \|\gamma_h^n\|_{L^2\Lambda^k(\Omega)}^2 + \frac{\tau}{\bar{\alpha}_0}\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2 + \bar{\alpha}_0\tau\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)}^2&,\end{aligned}$$

where $\bar{\alpha}_0 = \min(\alpha_0, 1)$ with α_0 the constant in (4.3). Continuity and stability of $\mathsf{a}(\cdot,\cdot)$ and Young's inequality give for $\kappa > 0$:

$$\begin{aligned} \mathsf{a}\left(\gamma_h^n, \gamma_h^{n+1}\right) &= \mathsf{a}\left(\gamma_h^{n+1}, \gamma_h^{n+1}\right) + \mathsf{a}\left(\gamma_h^n - \gamma_h^{n+1}, \gamma_h^{n+1}\right) \\ &\geq \bar{\alpha}_0 \left\|\gamma_h^{n+1}\right\|_h^2 - C(h) \left\|\gamma_h^n - \gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)} \left\|\gamma_h^{n+1}\right\|_h \\ &\geq \bar{\alpha}_0 \left\|\gamma_h^{n+1}\right\|_h^2 - \frac{C(h)^2}{2\kappa\bar{\alpha}_0} \left\|\gamma_h^n - \gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} - \frac{\kappa\bar{\alpha}_0}{2} \left\|\gamma_h^{n+1}\right\|_h^2 . \end{aligned}$$

Combining the last two results we get:

$$\begin{aligned} \left\|\gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} + \left(1 - \frac{C(h)^2\tau}{\kappa\bar{\alpha}_0}\right) \left\|\gamma_h^{n+1} - \gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + \tau\bar{\alpha}_0(1-\kappa) \left\|\gamma_h^{n+1}\right\|_h^2 \\ \leq \left\|\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + \frac{\tau}{\bar{\alpha}_0} \left\|R^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} , \end{aligned}$$

and in particular for $\tau \leq \frac{\kappa\bar{\alpha}_0}{C(h)^2}$:

$$(1 + \tau\bar{\alpha}_0(1-\kappa)) \left\|\gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} \leq \left\|\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + \frac{\tau}{\bar{\alpha}_0} \left\|R^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} .$$

A discrete Gronwall's Lemma [26, Lemma 7.1.12] yields for this recursion and $\kappa > 0$

$$\left\|\gamma_h^n\right\|_{L^2\Lambda^k(\Omega)} \leq e^{-\bar{\alpha}_0(1-\kappa)t^n} \left\|\gamma_h^0\right\|_{L^2\Lambda^k(\Omega)} + \frac{1}{\bar{\alpha}_0\sqrt{1-\kappa}} \max_{0\leq i\leq N} \left\|R^i\right\|_{L^2\Lambda^k(\Omega)} .$$

From the definitions of R^n and $\mathsf{l}^n(\eta)$ we infer

$$\begin{aligned} \left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)} \leq & C_c \max_{t\in[0,T]} \left(h^{r+\frac{1}{2}} \left\|\partial_t\omega(t)\right\|_{L^2\Lambda^k(\Omega)} + \tau \left\|\partial_t^2\omega(t)\right\|_{L^2\Lambda^k(\Omega)}\right) \\ & + C_1\tau \max_{t\in[0,T]} \left(\left\|\partial_t\varphi(t)\right\|_{L^2\Lambda^k(\Omega)} + \left\|\partial_t\psi_D(t)\right\|_{L^2\Lambda^k(\partial\Omega)}\right) . \end{aligned}$$

This proves the assertion. □

It remains to specify $C(h)$ in (5.7).

Remark 5.1.4. *From the definition* (4.8) *of* $\mathsf{a}(\omega,\eta)$, $\omega,\eta \in \Lambda_h^k(\mathcal{T})$ *we deduce by inverse inequalities:*

$$\begin{aligned} \mathsf{a}(\omega,\eta) = & (\bar{\alpha}\omega,\eta)_\Omega + \sum_T (\omega, \mathcal{L}_{\boldsymbol{\beta}}\eta)_T + \sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} (\omega,\eta)_{f,\boldsymbol{\beta}} \\ & + \sum_{f\in\mathcal{F}^\circ} \left(\{\omega\}_f, [\eta]_f\right)_{f,\boldsymbol{\beta}} + \left(c_f [\omega]_f, [\eta]_f\right)_{f,\boldsymbol{\beta}} \\ \leq & C \left\|\omega\right\|_{L^2\Lambda^k(\Omega)} \left(\left\|\eta\right\|_{L^2\Lambda^k(\Omega)} + h^{-1} \left\|\eta\right\|_{L^2\Lambda^k(\Omega)}\right) \\ & + Ch^{-\frac{1}{2}} \left\|\omega\right\|_{L^2\Lambda^k(\Omega)} \left(\sum_{f\in\mathcal{F}^\partial\setminus\mathcal{F}_-^\partial} \left\|\omega\right\|_{f,c_f\boldsymbol{\beta}} + \sum_{f\in\mathcal{F}^\circ} \left\|[\omega]_f\right\|_{f,c_f\boldsymbol{\beta}}\right) \\ \leq & Ch^{-1} \left\|\omega\right\|_{L^2\Lambda^k(\Omega)} \left\|\eta\right\|_h . \end{aligned}$$

If we have $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ *then*

$$(\omega, \mathcal{L}_{\boldsymbol{\beta}}\eta)_T \leq C(D\boldsymbol{\beta}) \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta\|_{L^2\Lambda^k(\Omega)},$$

hence

$$\mathsf{a}(\omega, \eta) \leq Ch^{-\frac{1}{2}} \|\omega\|_{L^2\Lambda^k(\Omega)} \|\eta\|_h.$$

By remark 5.1.4 we have in the general case $C(h) = Ch^{-1}$ in (5.7) which yields the timestep constraint $\tau = O(h^2)$. Only for piecewise constant approximation space $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_0^{\mathrm{d}}\Lambda^k(\mathcal{T})$ we find $C(h) = Ch^{-\frac{1}{2}}$, i.e. $\tau = O(h)$. This is the standard timestep constraint for upwind finite volume or finite difference schemes for scalar advection problems. That result means that for explicit Euler timestepping and for lowest order approximation spaces $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_1^-\Lambda^k$ or $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_1^-\Lambda^{n-k}$, $k > 0$, we have to impose the strict timestep constraint $\tau = O(h^2)$.

For the implicit scheme (5.6), on the other hand, we have stability and convergence for timestep sizes τ independent of the mesh size h.

Theorem 5.1.5. *Let* $(\omega(t^n))_{n=0}^N$, $(\omega_h^n)_{n=0}^N$ *be the solution to* (5.1) *and* (5.6). *Assume in the definition* (4.7) *of* $\mathsf{a}(\cdot,\cdot)$ *that the parameter* c_f *satisfies for all faces* f *the positivity condition*

$$c_f \boldsymbol{\beta} \cdot \mathbf{n}_f > 0.$$

Further assume that the Ritz-Galerkin projection $P_h\omega(t) \in \Lambda_h^k(\mathcal{T})$ *with*

$$\mathsf{a}(P_h\omega(t), \eta) := \mathsf{a}(\omega(t), \eta), \quad \forall \eta \in \Lambda_h^k(\mathcal{T})$$

fulfills the estimate

$$\|\omega(t) - P_h\omega(t)\|_{L^2\Lambda^k(\Omega)} \leq C_c h^{r+\frac{1}{2}} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

for $\omega \in H^{r+1}\Lambda^k(\Omega)$ *and* $C_c > 0$ *independent of* h. *Then we have*

$$\max_{0 \leq n \leq N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq C\left(h^{r+\frac{1}{2}} + \tau\right),$$

with $C > 0$ *independent of* τ *and* h.

5.1.1 Numerical Experiments

In this section we set $\Omega = [-1,1]^2 \subset \mathbb{R}^2$ and look at the non-stationary advection problem for a vector $\mathbf{u}$

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{grad}(\boldsymbol{\beta}\cdot\mathbf{u}) - \mathbf{R}\,\mathrm{div}(\mathbf{R}\mathbf{u})\boldsymbol{\beta} &= \mathbf{f}, && \text{in } \Omega, \\ \mathbf{u}_{|\Gamma_{\mathrm{in}}} &= \mathbf{g}, && \text{on } \Gamma_{\mathrm{in}} \\ \mathbf{u}(0) &= \mathbf{u}_0, && \text{in } \Omega \end{aligned} \tag{5.8}$$

with $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We consider approximation spaces $\Lambda^1_h(\mathcal{T}) = \left(\mathcal{P}_1^- \Lambda^1(\mathcal{T})\right)^\perp := \mathbf{R}\mathcal{P}_1^- \Lambda^1(\mathcal{T})$ that are globally tangential continuous. The velocity field

$$\boldsymbol{\beta} = \begin{pmatrix} 0.66(1-x^2)(1-y^2) \\ \sin(\pi x)\sin(\pi y) \end{pmatrix}$$

has vanishing normal components on the boundary, i.e. we have no inflow boundary $\Gamma_{in} = \{\}$. The data $\mathbf{f}$ and $\mathbf{u}_0$ is such that

$$\mathbf{u} = \cos(2\pi t) \begin{pmatrix} \sin(\pi x)\sin(\pi y) \\ (1-x^2)(1-y^2) \end{pmatrix}$$

is the solution. For a timestep $\tau = 0.8h^2$ we observe convergence for both the explicit and the implicit schemes (see Figure 5.1). For a timestep $\tau = 0.8h$ we do observe convergence only for the implicit scheme (see Figure 5.2). In contrast to instability phenomena with parabolic problems, the instability appears only after sufficient refinements. For a timestep $\tau = 0.2h$ we do not reach this limit (see Figure 5.3), but we can not guarantee that for even smaller meshes the error will not explode.

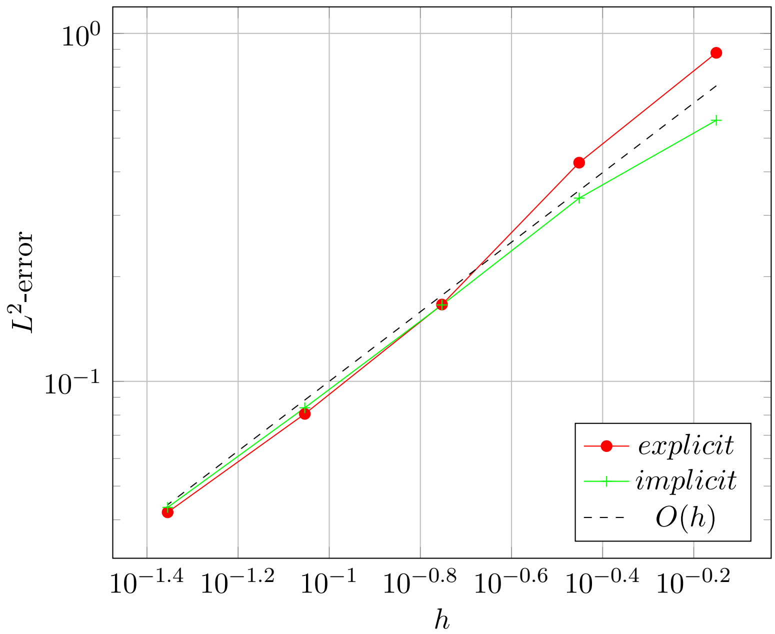

Figure 5.1: Experiment: Convergence rate of the L^2-error at $t = 0.4$ for simulations on interval $[0, 0.4]$ for timestep size $\tau = 0.8h^2$.

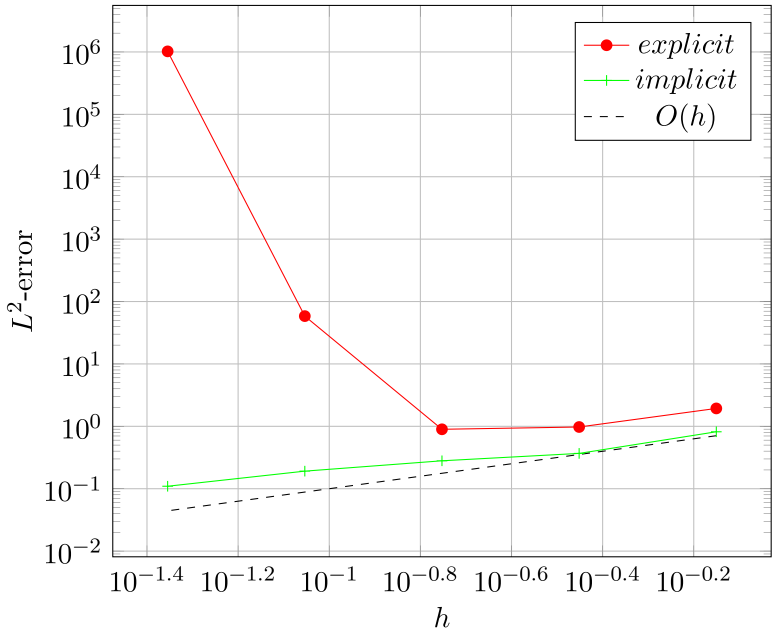

Figure 5.2: Experiment: Convergence rate of the L^2-error at $t = 0.4$ for simulations on interval $[0, 0.4]$ for timestep size $\tau = 0.8h$.

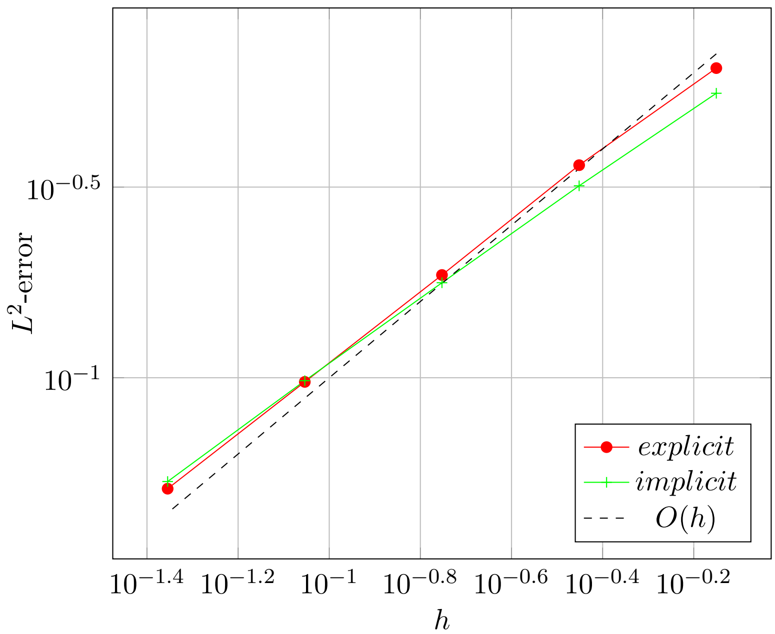

Figure 5.3: Experiment: Convergence rate of the L^2-error at $t = 0.4$ for simulations on interval $[0, 0.4]$ for timestep size $\tau = 0.2h$.

5.2 Semi-Lagrangian Formulations

Throughout this section we will assume for simplicity that the velocity $\boldsymbol{\beta}$ has vanishing normal components on the boundary $\partial\Omega$ of the domain. The formal solution of our advection problem (5.1) with $\alpha = 0$ is

$$\omega(t) = X_{-t}^{*}\omega_0 + \int_0^t X_{\tau-t}^{*}\varphi(\tau)\mathrm{d}\tau.$$

Semi-Lagrangian methods are based on such a representation of the solution. Let $P_h : L^2\Lambda^k(\Omega) \mapsto \Lambda_h^k(\mathcal{T})$ denote an abstract projection operator, where $\Lambda_h^k(\mathcal{T})$ denotes again some piecewise polynomial approximation space of differential k-forms on a triangulation $\mathcal{T}$ of Ω, i.e. $\Lambda_h^k(\mathcal{T}) \subset L^2\Lambda^k(\Omega)$, $\Lambda_h^k(\mathcal{T}) \subset H\Lambda^k(\Omega)$ or $\Lambda_h^k(\mathcal{T}) \subset \star H\Lambda^{n-k}(\Omega)$.

We consider again a partitioning of the time interval of the form $[0,T] = \bigcup_{n=0}^{N-1}[t^n, t^{n+1}]$ with $t^n = \tau n$ and $\tau = \frac{T}{N}$. The semi-Lagrangian timestepping scheme for constructing sequences $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, that approximate $(\omega(t^n))_{n=0}^N$ is:

- abstract semi-Lagrangian timestepping:
  Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that:

$$\begin{aligned}\omega_h^0 &= P_h\omega(0)\\ \omega_h^{n+1} &= P_h X_{-\tau}^{*}\omega_h^n + \int_{t^n}^{t^{n+1}} P_h X_{\tau-t^{n+1}}^{*}\varphi(\tau)\,\mathrm{d}\tau.\end{aligned} \tag{5.9}$$

For the scalar advection problem such methods have been formulated and analysed in [25, 61, 63, 71, 72, 83]. For L^2-continuous projections we can prove convergence of our abstract semi-Lagrangian scheme for k-forms in $\mathbb{R}^n$.

Theorem 5.2.1. *Let $\Lambda_h^k(\mathcal{T})$ be a piecewise polynomial space of discrete differential k-forms such that for P_h in (5.9) and for $r \geq 0$:*

$$\|\omega - P_h\omega\|_{L^2\Lambda^k(\Omega)} \leq C_c h^{r+1}\|\omega\|_{H^{r+1}\Lambda^k(\Omega)} \tag{5.10}$$

with C_c independent of h. Let ω and ω_h be the solutions to (5.1) and (5.9). Additionally we assume

$$\|P_h\omega\|_{L^2\Lambda^k(\Omega)} \leq \|\omega\|_{L^2\Lambda^k(\Omega)}. \tag{5.11}$$

Then we have

$$\max_{0\leq n\leq N}\|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq Ch^{r+1}(\tau^{-1}+1)\max_{0\leq n\leq N}\|\omega(t^n)\|_{H^{r+1}\Lambda^k(\Omega)}, \tag{5.12}$$

with $C > 0$ independent of h and τ.

Proof. To bound the error $\|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)}$ we add and subtract the projection $P_h\omega(t^n)$, use Cauchy-Schwarz inequality, formulas (5.1) and (5.9) and the assumption

(5.11):

$$\begin{aligned}\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)} &\leq \|\omega(t^n)-P_h\omega(t^n)\|_{L^2\Lambda^k(\Omega)} + \|P_h\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)} \\ &\leq \|\omega(t^n)-P_h\omega(t^n)\|_{L^2\Lambda^k(\Omega)} + \left\|X_{-\tau}^*\omega(t^{n-1})-X_{-\tau}^*\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)} \\ &\leq \|\omega(t^n)-P_h\omega(t^n)\|_{L^2\Lambda^k(\Omega)} + C_e\left\|\omega(t^{n-1})-\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)}.\end{aligned}$$

In the last inequality we have $C_e=\sqrt{1+C\tau}$ according to Proposition 4.2.3. Then a discrete Gronwall-like inequality (see Appendix 5.2.4) and the approximation assumption (5.10) yield:

$$\begin{aligned}\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq & \frac{e^{C\tau n}-1}{C\tau}\max_{1\leq j\leq n}\left\|\omega(t^j)-P_h\omega(t^j)\right\|_{L^2\Lambda^k(\Omega)} \\ &+ e^{C\tau n}\left\|\omega(t^0)-\omega_h^0\right\|_{L^2\Lambda^k(\Omega)} \\ \leq & C_c\frac{e^{C\tau n}-1}{C\tau}h^{r+1}\max_{1\leq j\leq n}\left\|\omega(t^j)\right\|_{H^{r+1}\Lambda^k(\Omega)} \\ &+ C_c e^{C\tau n}h^{r+1}\left\|\omega(t^0)\right\|_{L^2\Lambda^k(\Omega)}\end{aligned}$$

and the assertion follows by (5.10). □

Remark 5.2.2. *For $\tau=O(h)$ the estimate in Theorem 5.2.1 is suboptimal in comparison with the approximation assumption* (5.10). *In particular for $r=0$, e.g. the cases $\Lambda_h^k(\mathcal{T})=\mathcal{P}_1^-\Lambda^k$, $\Lambda_h^k(\mathcal{T})=\mathcal{P}_0^{\mathrm{d}}\Lambda^k$ and $\Lambda_h^k(\mathcal{T})=\star\mathcal{P}_1^-\Lambda^{n-k}$, we can not prove convergence. This phenomenon is also observed for semi-Lagrangian methods of scalar transport problems. Up to our knowledge there exists no proof of convergence for the case $r=0$, except for certain simplified problems in $\mathbb{R}^1$ and $\mathbb{R}^2$ with constant velocity [63].*

We present now two different concrete choices for the projection operator P_h. The L^2-orthogonal projection gives rise to the so-called *semi-Lagrangian Galerkin method*. The canonical approximation operators associated with the finite element like approximation spaces $\Lambda_h^k(\mathcal{T})$ yield *semi-Lagrangian interpolation methods*. While the L^2-projection is L^2-continuous most practical interpolation operators lack this property. Hence Theorem 5.2.1 will not give convergence for such semi-Lagrangian interpolation schemes.

5.2.1 Semi-Lagrangian Galerkin Methods

Let Π_h denote the L^2-projection operator onto $\Lambda_h^k(\mathcal{T})$:

$$(\Pi_h\omega,\eta)_\Omega := (\omega,\eta)_\Omega, \quad \forall\eta\in\Lambda_h^k(\mathcal{T}).$$

An actual implementation of (5.9) will require more approximations than simply specifying $P_h=\Pi_h$. This needs to be done very carefully [61] to preserve the nice stability properties of semi-Lagrangian schemes established in Theorem 5.2.1.

(i) Approximate flow map. First we would like to introduce approximations $\bar{X}_\tau$ of the flow map X_τ that depend on both Ω_h and the timestep τ. Since we have $\Omega\subset\mathbb{R}^n$ we can consider the difference $X_\tau(x)-\bar{X}_\tau(x)$ and require consistency in the following sense:

- $\bar{X}_\tau : \Omega \mapsto \Omega$ is $\mathcal{T}$-piecewise smooth,
- there are $l_1, l_2 \geq 1$ such that for $h \to 0$ and $\tau \to 0$

$$\|X_\tau - \bar{X}_\tau\|_{\boldsymbol{W}^{0,\infty}(\Omega)} \leq O(h^{l_1+1}\tau + \tau^{l_2}) \quad \text{and} \quad \|X_\tau - \bar{X}_\tau\|_{\boldsymbol{W}^{1,\infty}(\Omega)} \leq O(h^{l_1}\tau + \tau^{l_2}). \tag{5.13}$$

A simple construction of approximate flow maps relies on the nodal basis functions λ_i spanning the space $\mathcal{P}_{l_1}\Lambda^0(\mathcal{T})$ of continuous piecewise polynomial Lagrangian finite element functions of degree l_1. The degrees of freedom associated to these basis functions are point evaluations at particular nodal points $\mathbf{a}_i$ defined by affine coordinates inside the simplices of the mesh. Then we define

$$\bar{X}_\tau(\mathbf{x}) := \sum_i \bar{X}_{\tau,i}\lambda_i(\mathbf{x}), \tag{5.14}$$

where the coefficients $\bar{X}_{\tau,i}$ are approximations to the trajectories $X_\tau(\mathbf{a}_i)$ of the degrees of freedoms $\mathbf{a}_i$ with

$$\|X_\tau(\mathbf{a}_i) - \bar{X}_{\tau,i}\| \leq O(\tau^{l_1}) \quad \text{for} \quad \tau \to 0\,, \tag{5.15}$$

see figure 5.4 for an illustration. This approximation is consistent by construction. The

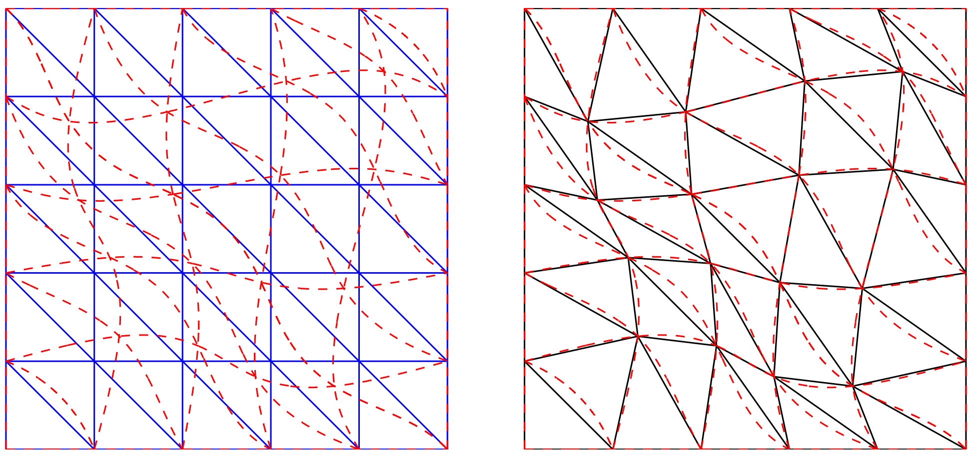

Figure 5.4: Illustration of the approximation of the trajectories X_τ. Left: The fixed mesh $\mathcal{T}$ (blue solid lines) and its image $X_\tau(\mathcal{T})$ under the exact flow. In the general case $X_\tau(\mathcal{T})$ consists of non-polynomial curved polygons. Right: A low order consistent approximation $\bar{X}_\tau(\mathcal{T})$ (black solid lines) of $X_\tau(\mathcal{T})$. Here we used linear Lagrangian elements and exact trajectories for the vertices, hence $\bar{X}_\tau(\mathcal{T})$ has again straight edges and the vertices of $X_\tau(\mathcal{T})$ and $\bar{X}_\tau(\mathcal{T})$ coincide.

errors $\left\|X_\tau - \bar{X}_\tau\right\|_{\boldsymbol{W}^{0,\infty}(\Omega)}$ and $\left\|X_\tau - \bar{X}_\tau\right\|_{\boldsymbol{W}^{1,\infty}(\Omega)}$ split into an error due to approximations of the trajectories of the degrees of freedom, which is assumed to be of order $O(\tau^{l_2})$, and an error due to interpolation in Lagrangian finite element functions. The bound on the interpolation error follows from standard interpolation estimates for Lagrangian finite elements [18]. If Π_h denotes the interpolation operator onto $\mathcal{P}_{l_1}\Lambda^0(\mathcal{T})$ we have for $s = 0, 1$:

$$\begin{aligned}\|X_\tau - \Pi_h X_\tau\|_{\boldsymbol{W}^{s,\infty}(\Omega)} &= \|X_\tau - \mathrm{id} - \Pi_h(X_\tau - \mathrm{id})\|_{\boldsymbol{W}^{s,\infty}(\Omega)} \\ &\leq Ch^{l_1+1-s} \left|X_\tau - \mathrm{id}\right|_{\boldsymbol{W}^{l_1+1,\infty}(\Omega)} \\ &\leq Ch^{l_1+1-s}\tau \left|\boldsymbol{\beta}\right|_{\boldsymbol{W}^{l_1+1,\infty}(\Omega)} ,\end{aligned}$$

where the last inequality follows from (2.27).

(ii) Approximation of source. We have to approximate the time integration of the right-hand side in (5.9). Since φ does not depend on ω it is reasonable to choose some quadrature method for the approximation $Q(\varphi, t, t+\tau) \approx \int_t^{t+\tau} \varphi(s)ds$ which satisfies

$$\left|\int_t^{t+\tau} \varphi(s)ds - Q(\varphi, t, \tau)\right| \leq C\tau^m \max_{t\leq s\leq t+\tau} \left|\frac{d^m}{dt^m}\varphi(s)\right|, \quad m \geq 1 . \tag{5.16}$$

Now we are in a position to formulate fully discrete semi-Lagrangian timestepping schemes.

- Semi-Lagrangian Galerkin method
  Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned}\left(\omega_h^0, \eta\right)_\Omega &= (\omega_0, \eta)_\Omega , \\ \left(\omega_h^{n+1}, \eta\right)_\Omega &= \left(\bar{X}_{-\tau}^*\omega_h^n, \eta\right)_\Omega + \left(Q(\bar{X}_{s-t^{n+1}}^*\varphi(s), t^n, t^{n+1}), \eta\right)_\Omega .\end{aligned} \tag{5.17}$$

For $k = 0$ and continuous piecewise linear approximation spaces this is exactly the scheme in [72].

Now the pullbacks are represented as piecewise polynomials. The right-hand side in (5.17) can be computed exactly, by determining the intersection of all elements T of the mesh $\mathcal{T}$ with all elements $\bar{X}_\tau(T')$ of the transported mesh $\bar{X}_\tau(\mathcal{T})$ (see Figure 5.5 for illustration). At a first glance this seems to be very expensive. Nevertheless we think that at least for the case of low order approximations such schemes are competitive methods, since semi-Lagrangian schemes enjoy unconditional stability. Moreover, for discontinuous finite element approximation spaces, i.e. the natural approximation spaces for k-forms, $k > 0$, there is hardly any other choice.

Remark 5.2.3. *Inspired from standard finite element techniques one could be tempted to split the inner product* $\left(\bar{X}_{-\tau}^*\omega_h, \eta\right)_\Omega$ *in a sum of integrals over elements of* $\mathcal{T}$ *and apply some quadrature rule there. We will call this scheme the* **quadrature-based scheme***:*
Find $\omega_h^n \in \Lambda_h^k(\mathcal{T})$ *such that for all* $\eta \in \Lambda_h^k(\mathcal{T})$*:*

$$\begin{aligned}\left(\omega_h^0, \eta\right)_\Omega &= (\omega_0, \eta)_\Omega , \\ \left(\omega_h^{n+1}, \eta\right)_{\Omega,h} &= \left(\bar{X}_{-\tau}^*\omega_h^n, \eta\right)_{\Omega,h} + \left(Q(\bar{X}_{s-t^{n+1}}\tilde{\varphi}(s), t^n, t^{n+1}), \eta\right)_{\Omega,h} ,\end{aligned} \tag{5.18}$$

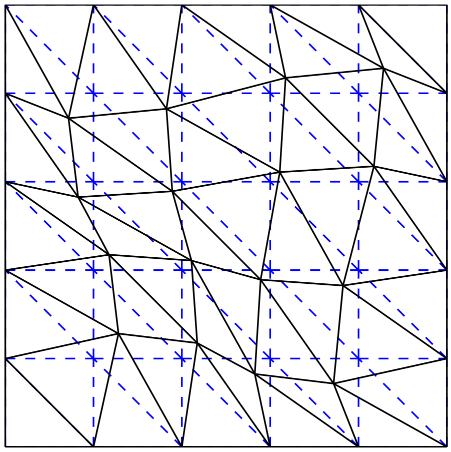

Figure 5.5: The inner products on the right-hand side of the Galerkin projection scheme (5.17) pair finite element functions defined on two different meshes, namely the fixed mesh (blue dashed lines) and the approximated transported mesh. Since in both meshes the facets are polynomial, we can algorithmically determine a partitioning of Ω such that all appearing finite element functions are smooth on each part. The finite element functions and the pullbacks are polynomials, hence the inner products can computed exactly.

with

$$(\omega, \eta)_{\Omega,h} = \sum_T \sum_i w_{i,T} \omega(x_{i,T}) \wedge \star\eta(x_{i,T}) \tag{5.19}$$

for suitable quadrature points $(x_{i,T})_i$ *and quadrature weights* $(w_{i,T})_i$. *Compared to the projection scheme this reduces the computational cost, since only the flows for the quadrature points need to be computed. Nevertheless this scheme is questionable since we apply quadrature on domains with discontinuous integrands. Our numerical experiments support these doubts.*

In analogy to Theorem 5.2.1 we can prove convergence for the solutions of the Galerkin projection scheme (5.17). A crucial step is the following Proposition:

Proposition 5.2.4. *Let* $\bar{X}_\tau$ *be a consistent approximation to* X_τ *according to* (5.13). *Then*

$$\left\| X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega \right\|_{L^2\Lambda^k(\Omega)} \le C_l(h^{l_1}\tau + \tau^{l_2}) \|\omega\|_{H^1\Lambda^k(\Omega)}, \tag{5.20}$$

with C_l *independent of* τ *and* h.

Proof. The proof follows the lines of the proof of Proposition 4.2.3. By the definition (2.8) we have

$$\left\| X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega \right\|^2_{L^2\Lambda^k(\Omega)} = \int_\Omega \left(\left(X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega \right)_x, \left(X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega \right)_x \right) \mathrm{vol},$$

where $(\cdot,\cdot)$ is the inner product on alternating forms defined in (2.2). In what follows σ and σ' are increasing sequences $\{1,\dots,p\} \mapsto \{1,\dots,n\}$ and $\mathbf{e}_1,\dots,\mathbf{e}_n$ is a basis of $\mathbb{R}^n$. For fixed $x \in \Omega$ and τ we introduce the abbreviations $\omega_x(\mathbf{e}_\sigma) := \omega_x\left(\mathbf{e}_{\sigma(1)},\dots,\mathbf{e}_{\sigma(p)}\right)$, $\omega_X := \omega_{X_{-\tau}(x)}$ and $X_x := X_{-\tau}(x)$. Then we find

$$\left(X^*_{-\tau}\omega\right)_x(\mathbf{e}_\sigma) = \mathbf{M}_k(X_x)\omega_X(\mathbf{e}_\sigma) \quad \text{and} \quad \left(\bar{X}^*_{-\tau}\omega\right)_x(\mathbf{e}_\sigma) = \mathbf{M}_k(\bar{X}_x)\omega_{\bar{X}}(\mathbf{e}_\sigma)$$

where $\mathbf{M}_k(\cdot)$ is the operator introduced in (2.17). Together this yields

$$\begin{aligned}\left(X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega\right)_x(\mathbf{e}_\sigma) = &\left(\mathbf{M}_k(X_x)\omega_X - \mathbf{M}_k(\bar{X}_x)\omega_X\right)(\mathbf{e}_\sigma)\\ &+ \left(\mathbf{M}_k(\bar{X}_x)\omega_X - \mathbf{M}_k(\bar{X}_x)\omega_{\bar{X}}\right)(\mathbf{e}_\sigma).\end{aligned}$$

For each σ' we have that $\omega_X(\mathbf{e}_{\sigma'})$ is a function of X, i.e. for smooth differential forms Taylor expansion yields

$$\omega_X(\mathbf{e}_{\sigma'}) = \omega_{\bar{X}}(\mathbf{e}_{\sigma'}) + (X-\bar{X})\partial_x\omega_x(\mathbf{e}_{\sigma'})_{|x=X+s(X-\bar{X})}$$

for some s with $0 \le s \le 1$. We find

$$\left\|X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega)} \le a_1 + a_2,$$

with

$$a_1 = \sup_x \rho\left(\mathbf{M}_k\left(X_x\right) - \mathbf{M}_k\left(\bar{X}_x\right)\right)^2 \sup_x \left|\mathbf{M}_n\left(X_{-\tau,x}\right)\right| \|\omega\|^2_{L^2\Lambda^k(\Omega)}$$

and

$$a_2 = \sup_x \rho\left(\mathbf{M}_k\left(\bar{X}_x\right)\right)^2 \sup_x \left|\mathbf{M}_n(X_{-\tau,x})\right| \left\|X-\bar{X}\right\|_{\boldsymbol{W}^{0,\infty}(\Omega)} |\omega|^2_{H^1\Lambda^k(\Omega)},$$

where $\rho(\cdot)$ denotes the spectral radius. We get the bound

$$\begin{aligned}&\left\|X^*_{-\tau}\omega - \bar{X}^*_{-\tau}\omega\right\|^2_{L^2\Lambda^k(\Omega)} \le\\ &\qquad C\left|X_{-\tau} - \bar{X}_{-\tau}\right|^2_{\boldsymbol{W}^{1,\infty}(\Omega)} \|\omega\|^2_{L^2\Lambda^k(\Omega)} + C\left\|X_{-\tau} - \bar{X}_{-\tau}\right\|^2_{\boldsymbol{W}^{0,\infty}(\Omega)} |\omega|^2_{H^1\Lambda^k(\Omega)}\end{aligned}$$

and the assertion follows by (5.13). □

Theorem 5.2.5. *Let $\omega(t)$ and $(\omega_h^n)_{n=0}^N$ be the solution of* (5.1) *and* (5.17). *Further assume, that the approximation $\bar{X}_\tau$ is consistent with X_τ according to* (5.13). *The approximation of the time integration of the right-hand side in* (5.17) *is assumed to be of order m according to* (5.16) *Then for h and τ sufficiently small*

$$\begin{aligned}\max_{0\le n\le N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \le &C\left(\left(1+\frac{1}{\tau}\right)h^{r+1} + h^{l_1} + \tau^{l_2-1}\right)\max_{0\le n\le N}\|\omega(t^n)\|_{H^{r+1}\Lambda^k(\Omega)}\\ &+ C\left(\tau^{m-1} + \tau h^{l_1} + \tau^{l_2}\right)C(\varphi).\end{aligned} \tag{5.21}$$

Proof. The proof is similar to the proof of Theorem 5.2.1. The additional approximations give additional consistency errors in the recursion for the error $\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)}$:

$$\begin{aligned}\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq& \|\omega(t^n)-\Pi_h\omega(t^n)\|_{L^2\Lambda^k(\Omega)}+\|\Pi_h\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)}\\ \leq& \|\omega(t^n)-\Pi_h\omega(t^n)\|_{L^2\Lambda^k(\Omega)}\\ &+\left\|\Pi_h X_{-\tau}^*\omega(t^{n-1})-\Pi_h\bar{X}_{-\tau}^*\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)}\\ &+\left\|\Pi_h\int_{t^{n-1}}^{t^n}X_{s-t^n}^*\varphi(s)\mathrm{d}s-\Pi_h Q(\bar{X}_{s-t^n}^*\varphi(s),t^{n-1},t^n)\right\|_{L^2\Lambda^k(\Omega)}\\ =&E_1+E_2+E_3\end{aligned}$$

For the second term in the last line we find by L^2-stability, Proposition 4.2.3 and Proposition 5.2.4 and $l\geq 1$:

$$\begin{aligned}E_2\leq&\left\|X_{-\tau}^*\omega(t^{n-1})-\bar{X}_{-\tau}^*\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)}\\ \leq&\left\|\bar{X}_{-\tau}^*\omega(t^{n-1})-\bar{X}_{-\tau}^*\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)}+\left\|\left(X_{-\tau}^*-\bar{X}_{-\tau}^*\right)\omega(t^{n-1})\right\|_{L^2\Lambda^k(\Omega)}\\ \leq&(1+C\tau)\left\|\omega(t^{n-1})-\omega_h^{n-1}\right\|_{L^2\Lambda^k(\Omega)}+C_l(h^{l_1}\tau+\tau^{l_2})\left\|\omega(t^{n-1})\right\|_{H^1\Lambda^k(\Omega)}.\end{aligned}$$

For the term E_3 we get by Proposition 5.2.4:

$$\begin{aligned}E_3\leq&\left\|\int_{t_{n-1}}^{t_n}X_{s-t^n}^*\tilde{\varphi}(s)\mathrm{d}s-Q(\bar{X}_{s-t^n}^*\tilde{\varphi}(s),t^{n-1},t^n)\right\|_{L^2\Lambda^k(\Omega)}\\ \leq&\left\|\int_{t^{n-1}}^{t^n}\bar{X}_{s-t^n}^*\tilde{\varphi}(s)\mathrm{d}s-Q(\bar{X}_{s-t^n}^*\tilde{\varphi}(s),t^{n-1},t^n)\right\|_{L^2\Lambda^k(\Omega)}\\ &+\left\|\int_{t^{n-1}}^{t^n}\bar{X}_{s-t^n}^*\tilde{\varphi}(s)\mathrm{d}s-\int_{t^{n-1}}^{t^n}X_{s-t^n}^*\tilde{\varphi}(s)\mathrm{d}s\right\|_{L^2\Lambda^k(\Omega)}\\ \leq&(\tau^m+\tau^2h^{l_1}+\tau^{l_2+1})C(\varphi).\end{aligned}$$

We can use a discrete Gronwall-like inequality (see Appendix 5.2.4):

$$\begin{aligned}&\|\omega(t^n)-\omega_h^n\|_{L^2\Lambda^k(\Omega)}\leq\\ &\quad\frac{e^{(C+C_l)\tau n}-1}{(C+C_l)\tau}\max_{1\leq j\leq n}\left(\left\|\omega(t^j)-\Pi_h\omega(t^j)\right\|_{L^2\Lambda^k(\Omega)}+C_l(h^{l_1}\tau+\tau^{l_2})\left\|\omega(t^{j-1})\right\|_{H^1\Lambda^k(\Omega)}\right)\\ &\quad+\frac{e^{(C+C_l)\tau n}-1}{(C+C_l)\tau}(\tau^m+\tau^2h^{l_1}+\tau^{l_2+1})C(\varphi)+e^{(C+C_l)\tau n-1}\left\|\omega(t^0)-\omega_h^0\right\|_{L^2\Lambda^k(\Omega)}\end{aligned}$$

and the assertion follows. □

Although we have now a proof of convergence that reflects explicitly the various discretization parameters, still, we do not have convergence for lowest order polynomial approximations spaces. In the next section we will give a proof of convergence of such schemes. This proof is very much inspired by the convergence proof for the explicit Eulerian scheme.

Convergence of Lowest Order Method

Since we focus on low order approximation spaces we use now lowest order collocation methods and consider the following semi-Lagrangian scheme for problem (5.1):

- Low order semi-Lagrangian Galerkin time stepping scheme:
  Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned} \left(\omega_h^0, \eta\right)_\Omega &= (\omega_0, \eta)_\Omega\,, \\ \left(\omega_h^{n+1}, \eta\right)_\Omega &= \left(X_{-\tau}^* \omega_h^n, \eta\right)_\Omega + \tau\left(\varphi^{n+1}, \eta\right)_\Omega\,. \end{aligned} \tag{5.22}$$

This is basically the semi-Lagrange Galerkin method in (5.17) where we assume here that $\left(X_{-\tau}^* \omega_h^n, \eta\right)_\Omega$ can be evaluated exactly and choose a low order quadrature (5.16) for the evaluation of the right hand side. These simplifications are made to accentuate the main arguments. Recall that we assumed vanishing normal components of $\boldsymbol{\beta}$ at the boundary $\partial\Omega$. Hence we can rewrite the semi-Lagrangian scheme (5.22) in terms of the bilinear form

$$\mathsf{a}_\tau(\omega, \eta) = (\alpha\omega, \eta)_\Omega + \frac{1}{\tau}(\omega, \eta)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^* \omega, \eta\right)_\Omega$$

introduced in (4.34) for the characteristic methods, where in this case $\alpha = 0$. We find an equivalent formulation that resembles the explicit Eulerian schemes (5.5):

- Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned} \left(\omega_h^0, \eta\right)_\Omega &= (\omega_0, \eta)_\Omega\,, \\ \frac{1}{\tau}\left(\omega_h^{n+1} - \omega_h^n, \eta\right)_\Omega + \mathsf{a}_\tau(\omega_h^n, \eta) &= \left(\varphi^{n+1}, \eta\right)_\Omega\,. \end{aligned} \tag{5.23}$$

In light of Theorems 4.2.5 and 5.1.3 it is very likely that this scheme converges at least for sufficiently small timesteps τ also for lowest order spatial approximations. Moreover, we can even prove convergence for sufficiently small timesteps under the assumption that $\mathsf{a}_\tau(\cdot,\cdot)$ allows for a Ritz-Galerkin projector.

Theorem 5.2.6. *Let $(\omega(t^n))_{n=0}^N$, $(\omega_h^n)_{n=0}^N$ be the solution to* (5.1) *and* (5.23)*. Assume that β is such that there is a constant $\alpha_0 > 0$ with*

$$\frac{1}{2\tau}\left(\omega \wedge \star\omega - \omega \wedge X_\tau^* \star X_{-\tau}^* \omega\right) \geq \alpha_0 \omega \wedge \star\omega, \quad \forall \omega \in \Lambda^k(\Omega)\,. \tag{5.24}$$

If additionally $\Lambda_h^k(\mathcal{T})$ permits the approximation property:

$$\inf_{\eta \in \Lambda_h^k(\mathcal{T})} \|\omega - \eta\|_{L^2\Lambda^k(\Omega)} \leq C h^{r+1} \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

we get for sufficiently small τ

$$\max_{0 \leq n \leq N} \|\omega(t^n) - \omega_h^n\|_{L^2\Lambda^k(\Omega)} \leq C\left(h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}} + \tau\right),$$

with $C > 0$ independent of τ and h.

Proof. By Theorem 4.2.4 we have a Ritz-Galerkin projection $P_h\omega(t) \in \Lambda_h^k(\mathcal{T})$ with

$$\mathsf{a}_\tau(P_h\omega(t), \eta) := \mathsf{a}(\omega(t), \eta), \quad \forall \eta \in \Lambda_h^k(\mathcal{T})$$

that fulfills the estimate

$$\|\omega(t) - P_h\omega(t)\|_{L^2\Lambda^k(\Omega)} \leq C_1 \left(h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}}\right) \|\omega\|_{H^{r+1}\Lambda^k(\Omega)}$$

for $\omega \in H^{r+1}\Lambda^k(\Omega)$ and $C_1 > 0$ independent of h.

Let $\bar{\omega}_h^n := P_h\omega(t^n)$, then

$$\frac{1}{\tau}\left(\bar{\omega}_h^{n+1} - \bar{\omega}_h^n, \eta\right)_\Omega + \mathsf{a}_\tau(\bar{\omega}_h^n, \eta) = \left(\varphi(t^{n+1}), \eta\right)_\Omega + \left(R^{n+1}, \eta\right)_\Omega$$

with

$$\left(R^{n+1}, \eta\right)_\Omega = \left(\frac{1}{\tau}(\bar{\omega}_h^{n+1} - \bar{\omega}_h^n) - \partial_t\omega(t^n), \eta\right)_\Omega + \left(\varphi(t^n), \eta\right)_\Omega - \left(\varphi(t^{n+1}), \eta\right)_\Omega .$$

We define $\gamma_h^n := \bar{\omega}_h^n - \omega_h^n$ and find:

$$\frac{1}{\tau}\left(\gamma_h^{n+1} - \gamma_h^n, \eta\right)_\Omega + \mathsf{a}_\tau(\gamma_h^n, \eta) = \left(R^{n+1}, \eta\right)_\Omega ,$$

or, equivalently

$$\frac{1}{\tau}\left(\gamma_h^{n+1} - X_{-\tau}^*\gamma_h^n, \eta\right)_\Omega = \left(R^{n+1}, \eta\right)_\Omega .$$

We take $\eta = 2\tau\gamma_h^{n+1}$ and use $2p(p-q) = p^2 + (p-q)^2 - q^2$:

$$\begin{aligned}
&\left\|\gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} + \left\|\gamma_h^{n+1} - X_{-\tau}^*\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} \\
&\qquad \leq \left\|X_{-\tau}^*\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + 2\tau\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)}\left\|\gamma_h^{n+1}\right\|_{L^2\Lambda^k(\Omega)} \\
&\qquad \leq \left\|X_{-\tau}^*\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + \frac{\tau}{\kappa}\left\|R^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} + \kappa\tau\left\|\gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} ,
\end{aligned}$$

for $\kappa > 0$. By Proposition 4.2.3 we deduce

$$\left\|\gamma_h^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} \leq \frac{1 + C_2\tau}{1 - \kappa\tau}\left\|\gamma_h^n\right\|^2_{L^2\Lambda^k(\Omega)} + \frac{\tau}{(1-\kappa\tau)\kappa}\left\|R^{n+1}\right\|^2_{L^2\Lambda^k(\Omega)} .$$

From the definition of R^n we infer

$$\begin{aligned}
&\left\|R^{n+1}\right\|_{L^2\Lambda^k(\Omega)} \leq \\
&C_3 \max_{t\in[0,T]}\left((h^{r+1}\tau^{-\frac{1}{2}} + \tau^{\frac{1}{2}})\left\|\partial_t\omega(t)\right\|_{L^2\Lambda^k(\Omega)} + \tau\left\|\partial_t^2\omega(t)\right\|_{L^2\Lambda^k(\Omega)} + \tau\left\|\partial_t\varphi(t)\right\|_{L^2\Lambda^k(\Omega)}\right)
\end{aligned}$$

hence the assertion follows from the discrete Gronwall-like inequality (see Appendix 5.2.4).

□

Remark 5.2.7. *Theorem 5.2.6 gives convergence for $\tau = O(h)$ and lowest order spatial approximation spaces. The assumption (5.24) here is stricter than the assumption (4.37) for the characteristic methods. While the assumption (4.37) can always be taken for granted due to rescaling $\omega' = e^{\alpha'}\omega$ the assumption (5.24) is an explicit assumption on the velocity field $\boldsymbol{\beta}$. However, a proof of convergence for the semi-Lagrangian scheme for the rescaled variables follows the same lines as the proof of Theorem 5.2.6, and we can in particuar establish convergence for $\tau = O(h)$ and lowest order approximation spaces.*

Remark 5.2.8. *For non-vanishing normal components of $\boldsymbol{\beta}$ at the boundary we can prove the same results as in Theorem 5.2.6 without any additional technicalities for the following formulation:*

- *Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:*

$$\left(\omega_h^0, \eta\right)_\Omega = (\omega_0, \eta)_\Omega\,,$$
$$\frac{1}{\tau}\left(\omega_h^{n+1}, \eta\right)_\Omega - \frac{1}{\tau}\left(X_{-\tau}^*\omega_h^n, \eta\right)_{\Omega_0} = \left(\varphi^{n+1}, \eta\right)_\Omega - \left(\psi_D^{n+1}, \eta\right)_{\Omega_{\mathrm{in}}}.$$

5.2.2 Semi-Lagrangian Interpolation Methods

Without doubt, the semi-Lagrangian Galerkin methods have apparent advantages compared to the Eulerian methods. They do not have the strict timestep constraints of explicit Eulerian methods, while the algebraic systems remain positive definite. For very large problems the positive definiteness is important to speedup the simulation time with appropriate linear solvers. The disadvantage of semi-Lagrangian methods is the need to evaluate inner products of finite element functions that are defined on two different meshes.

We present here another family of semi-Lagrangian methods that decimates this disadvantage for low order approximation spaces. They build on the interpolation operators (2.46) that are defined via the degrees of freedom of the differential form finite element spaces $\Lambda_h^k(\mathcal{T})$. In case of the non-conforming discrete differential form spaces $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r^{\mathrm{d}}\Lambda^k(\mathcal{T})$ the degrees of freedom could be simple point evaluations. Such schemes would strongly resemble the quadrature-based Galerkin schemes in Remark 5.2.3. But since the quadrature-based schemes are dubious we will exclude the spaces $\mathcal{P}_r^{\mathrm{d}}\Lambda^k$ in the following discussion. Second we will treat the two problems (5.1) and (5.2) separately. If Π_h is a canonical projection operator (2.46) for $\mathcal{P}_{r+1}^-\Lambda^k$ or $\mathcal{P}_r\Lambda^k$ based on the canonical degrees of freedom of type (2.45) and (2.44), then $\Pi_h X_{-\tau}^* \star \omega_h$ with ω_h in $\mathcal{P}_{r+1}^-\Lambda^k$ or $\mathcal{P}_r\Lambda^k$ is not well-defined for $k \leq \frac{n}{2}$: the degrees of freedom contain the functionals $\omega \mapsto \int_f \omega$, with f k-subsimplices of the mesh, that are not defined for $n-k$-forms like $X_{-\tau}^* \star \omega_h$. Rephrasing this for the case $\mathbb{R}^3$: *line integrals are not well defined for 2-forms.*

In what follows we assume again that the velocity has vanishing normal components on the boundary of Ω. Let $\bar{X}_\tau$ be an approximation to the flow X_τ that is consistent according to (5.13) and $Q(\varphi, t, s)$ a quadrature method as in (5.16). Let now Π_h denote a canonical approximation operator (2.46) that is based on the degrees of freedom of the approximation space $\Lambda_h^k(\mathcal{T})$. The degrees of freedoms are certain moments associated to subsimplices of the mesh. We define then a fully discrete interpolation based semi-Lagrangian method.

- semi-Lagrangian interpolation method for problem (5.1)
  Set $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k$. Find $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\eta \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned}\left(\omega_h^0, \eta\right)_\Omega &= \left(\Pi_h\omega_0, \eta\right)_\Omega \\ \left(\omega_h^{n+1}, \eta\right)_\Omega &= \left(\Pi_h \bar{X}_{-\tau}^*\omega_h^n, \eta\right)_\Omega + \left(Q(\Pi_h \bar{X}_{s-t^{n+1}}^*\varphi(s), t^n, t^{n+1}), \eta\right)_\Omega .\end{aligned} \tag{5.25}$$

- semi-Lagrangian interpolation method for problem (5.2)
  Set $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k$. Find $(\widetilde{\omega}_h^n)_{n=0}^N$, $\widetilde{\omega}_h^n \in \Lambda_h^k(\mathcal{T})$, such that for all $\widetilde{\eta} \in \Lambda_h^k(\mathcal{T})$:

$$\begin{aligned}\left(\widetilde{\omega}_h^0, \widetilde{\eta}\right)_\Omega &= \left(\Pi_h\widetilde{\omega}_0, \widetilde{\eta}\right)_\Omega \\ \left(\widetilde{\omega}_h^{n+1}, \widetilde{\eta}\right)_\Omega &= \left(\bar{X}_\tau^*\Pi_h\bar{X}_{-\tau}^*\widetilde{\omega}_h^n, \bar{X}_\tau^*\widetilde{\eta}\right)_\Omega \\ &\quad + Q\left(\left(\bar{X}_{t^{n+1}-s}^*\Pi_h\bar{X}_{s-t^{n+1}}^*\widetilde{\varphi}(s), \bar{X}_{t^{n+1}-s}^*\widetilde{\eta}\right)_\Omega, t^n, t^{n+1}\right).\end{aligned} \tag{5.26}$$

To motivate the definition of the semi-Lagrangian scheme (5.26) for problem (5.2) we prove a consistency result in the following proposition.

Proposition 5.2.9. *For smooth $\omega \in \Lambda^k(\Omega)$ and piecewise smooth $\eta \in \Lambda_h^k(\mathcal{T})$ we have*

$$\left|\frac{1}{\tau}\left(\left(\Pi_h\omega, \eta\right)_\Omega - \left(X_\tau^*\Pi_h X_{-\tau}^*\omega, X_\tau^*\eta\right)_\Omega\right) + \left(\mathcal{L}_{\boldsymbol{\beta}}\omega, \eta\right)_\Omega\right| \to 0 \quad \textit{for} \quad h, \tau \to 0.$$

Proof. Due to $X_{-\tau}(\Omega) = \Omega$ we compute

$$\begin{aligned}\left(\Pi_h\omega, \eta\right)_\Omega - \left(X_\tau^*\Pi_h X_{-\tau}^*\omega, X_\tau^*\eta\right)_\Omega &= \int_\Omega \eta \wedge \star\Pi_h\omega - \int_\Omega X_\tau^*\eta \wedge \star X_\tau^*\Pi_h X_{-\tau}^*\omega \\ &= \int_\Omega \eta \wedge \star\Pi_h\omega - \int_\Omega X_\tau^*\eta \wedge \star\Pi_h\omega \\ &\quad + \int_\Omega X_\tau^*\eta \wedge \star\Pi_h\omega - \int_\Omega X_\tau^*\eta \wedge \star\Pi_h X_{-\tau}^*\omega \\ &\quad + \int_\Omega X_\tau^*\eta \wedge \star\Pi_h X_{-\tau}^*\omega - \int_\Omega X_\tau^*\eta \wedge \star X_\tau^*\Pi_h X_{-\tau}^*\omega \\ &= A_1 + A_2 + A_3\end{aligned}$$

As in the proof of Theorem 4.2.5 we find with T and f denoting n- and $n-1$ simplices and $c_f = \frac{1}{2}\frac{\boldsymbol{\beta}\cdot\, \mathbf{n}_f}{|\boldsymbol{\beta}\cdot\mathbf{n}_f|}$,

$$\lim_{\tau\to 0}\frac{1}{\tau}A_1 = -\sum_T \left(\mathsf{L}_{\boldsymbol{\beta}}\eta, \Pi_h\omega\right)_T - \sum_f \left([\eta]_f, c_f\,[\Pi_h\omega]_f - \{\Pi_h\omega\}_f\right)_{f,\boldsymbol{\beta}},$$

or using Leibniz rule (2.36) and formula (4.5) we get

$$\lim_{\tau\to 0}\frac{1}{\tau}A_1 = -\sum_T (\eta, \mathcal{L}_{\boldsymbol\beta}\Pi_h\omega)_T - \sum_f \Big(c_f\,[\eta]_f + \{\eta\}_f, [\Pi_h\omega]_f\Big)_{f,\boldsymbol\beta}.$$

By introducing $\widetilde{\eta} := X_\tau^*\eta$ and $\widetilde{\omega} := X_\tau^*\Pi_h X_{-\tau}^*\omega$ we find for A_3 by the same arguments as in the proof of Theorem 4.2.5:

$$\lim_{\tau\to 0}\frac{1}{\tau}A_3 = -\sum_T (\mathsf{L}_{\boldsymbol\beta}\Pi_h\omega, \eta)_T - \sum_f \Big([\Pi_h\omega]_f, c_f\,[\eta]_f - \{\eta\}_f\Big)_{f,\boldsymbol\beta}.$$

Since ω is smooth we find also:

$$\lim_{\tau\to 0}\frac{1}{\tau}A_2 = \sum_T (\Pi_h\,\mathsf{L}_{\boldsymbol\beta}\,\omega, \eta)_T$$

Collecting the results for A_1, A_2 and A_3 we get:

$$\lim_{\tau\to 0}\left|\frac{1}{\tau}\left((\Pi_h\omega,\eta)_\Omega - \left(X_\tau^*\Pi_h X_{-\tau}^*\omega, X_\tau^*\eta\right)_\Omega\right) + (\mathcal{L}_{\boldsymbol\beta}\,\omega,\eta)_\Omega\right| =$$
$$\left|(\Pi_h\,\mathsf{L}_{\boldsymbol\beta}\,\omega - \mathsf{L}_{\boldsymbol\beta}\,\Pi_h\omega, \eta)_\Omega + (\mathcal{L}_{\boldsymbol\beta}\,\omega - \mathcal{L}_{\boldsymbol\beta}\,\Pi_h\omega,\eta)_\Omega - 2\sum_f \Big([\Pi_h\omega]_f, c_f\,[\eta]_f\Big)_{f,\boldsymbol\beta}\right|$$

and the assertion follows, since for smooth ω all $\Pi_h\,\mathsf{L}_{\boldsymbol\beta}\,\omega - \mathsf{L}_{\boldsymbol\beta}\,\Pi_h\omega \to 0$, $\mathcal{L}_{\boldsymbol\beta}\,\omega - \mathcal{L}_{\boldsymbol\beta}\,\Pi_h\omega \to 0$ and $[\omega]_f \to 0$ when $h \to 0$. □

Remark 5.2.10. *At first glance there seems to be a simpler way to define semi-Lagrangian interpolation schemes for problem* (5.2)*:*

- *wrong semi-Lagrangian interpolation method for problem* (5.2)
  Set $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$ *or* $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k$. *Find* $(\omega_h^n)_{n=0}^N$, $\omega_h^n \in \Lambda_h^k(\mathcal{T})$, *such that for all* $\eta \in \Lambda_h^k(\mathcal{T})$*:*

$$\begin{aligned}\left(\omega_h^0,\eta\right)_\Omega &= (\Pi_h\omega_0,\eta)_\Omega \\ \left(\omega_h^{n+1},\eta\right)_\Omega &= \left(\omega_h^n, \Pi_h\bar{X}_\tau^*\eta\right)_\Omega + Q\left(\left(\varphi(s), \Pi_h\bar{X}_{t_{i+1}-s}^*\eta\right)_\Omega, t_i, t_{i+1}\right).\end{aligned} \tag{5.27}$$

Note that the consistency result of proposition 5.2.9 will not apply for this scheme since $X_\tau^*\eta$ *is non-smooth in any case.*

As we mentioned earlier we can not prove convergence of such interpolation schemes, since the interpolation operators lack of L^2-stability. We think that it might be possible to give convergence results analogue to the proof of Theorem 5.2.6. The difficult part here is the analysis of the stationary problem that would establish the appropriate Ritz-Galerkin projections.

Nevertheless it is worth to consider such semi-Lagrangian interpolation-schemes. This is mainly for two reason:

- In accurate semi-Lagrangian Galerkin schemes we have to determine in each iteration all intersections of n-simplices of the fixed mesh with n-simplices of the transported mesh. The canonical interpolation operators (2.46) are based on moments of subsimplices. Hence in accurate semi-Lagrangian interpolation schemes we only need to find intersections of n-simplices of the fixed mesh with l-simplices of the transported mesh, where by Theorem 2.2.1 $k \leq l \leq r+k-1$ for $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_r\Lambda^k(\mathcal{T})$ or $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_r\Lambda^{n-k}(\mathcal{T})$ and $k \leq l \leq r+k$ for $\Lambda_h^k(\mathcal{T}) = \mathcal{P}_{r+1}^-\Lambda^k(\mathcal{T})$ or $\Lambda_h^k(\mathcal{T}) = \star\mathcal{P}_{r+1}^-\Lambda^{n-k}(\mathcal{T})$. For low order approximation spaces the interpolation schemes are cheaper.

- We know that for $\varphi = 0$ in (5.1) the exterior derivative of the initial data is preserved during the evolution. Since exterior derivative commutes with a canonical approximation operator (Lemma 2.2.8) the same holds true for the semi-Lagrangian interpolation scheme (5.25).

There is a remarkable difference between the two methods (5.25) and (5.26). While the first method is basically a non-variational method, the second scheme (5.26) is again a Galerkin method. But still the use of a canonical approximation operator simplifies the algorithmic treatment compared to the semi-Lagrangian Galerkin methods (5.17) introduced previously.

We close the presentation of semi-Lagrangian interpolation schemes with algorithmic details on an efficient implementation for lowest order conforming discrete 0- and 1-forms.

Nodal interpolation of transported Whitney 0-forms

To determine the interpolation $\Pi_h\bar{X}_\tau^*\omega_h \in \mathcal{P}_1\Lambda^0(\mathcal{T})$ for a transported conforming discrete 0-form $\bar{X}_\tau^*\omega_h, \omega_h \in \mathcal{P}_1\Lambda^0(\mathcal{T})$ it is by linearity enough to consider the basis functions λ_i defined in (2.48). The canonical degrees of freedom of Whitney 0-forms are point evaluations at the vertices $f_i^0 \in \Delta_0(\mathcal{T})$ of the triangulation. Hence we can represent the mapping $\Pi_h\bar{X}_\tau^* : \mathcal{P}_1\Lambda^0(\mathcal{T}) \mapsto \mathcal{P}_1\Lambda^0(\mathcal{T})$ by a matrix operator $\mathbf{P}_\tau^0$ with entries

$$\left(\mathbf{P}_\tau^0\right)_{ij} := \lambda_j(\bar{X}_\tau(f_i^0)) \tag{5.28}$$

that maps the expansions coefficients of ω_h to the coefficients of $\Pi_h\bar{X}_\tau^*\omega_h$. This means that in each time step we need the points $\bar{X}_\tau(f_i^0) \in \Omega$ that are determined by the approximation of the flow. When we use in (5.14) linear Lagrangian elements it is enough to solve (2.27) approximatively for the vertices f_i^0. We not only need to find the position but also the location within the mesh. To find the element, in which $\bar{X}_\tau(f_i^0)$ is located we trace the path of the trajectory from one element to the next. Based on this data the matrix entries (5.28) can be assembled element by element (see Figure 5.6).

Nodal interpolation of transported Whitney 1-forms

The canonical degrees of freedom of lowest order conforming discrete 1-forms are line integrals over all oriented 1-subsimplices of the triangulation. Hence, the interpolation of a transported discrete 1-form $\bar{X}_\tau^*\omega_h, \omega_h \in \mathcal{P}_1^-\Lambda^1(\mathcal{T})$ is determined through the

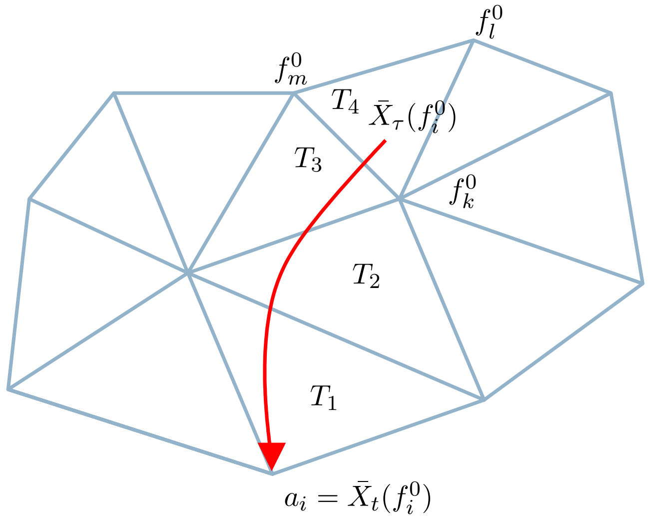

Figure 5.6: To determine the location of $\bar{X}_\tau(f_i^0)$ we move along the trajectory $\bar{X}_\cdot(f_i^0)$ starting from f_i^0 and identify the crossed elements T_1, T_2, T_3 and T_4. In this case $\left(\mathbf{P}_\tau^0\right)_{ik}$, $\left(\mathbf{P}_\tau^0\right)_{il}$ and $\left(\mathbf{P}_\tau^0\right)_{im}$ are the only non-zero entries in the i–th row of $\mathbf{P}_\tau^0$.

interpolation of transported basis forms $\bar{X}_\tau^* b_{f_j^1}$, where $b_{f_j^1}$ are the basis forms associated to edges $f_j^1 \in \Delta_1(\mathcal{T})$ and defined in (2.48). By the interpolation condition $\int_{f_i^1}(\Pi \bar{X}_\tau^* \omega_h - \bar{X}_\tau^* \omega_h) = 0, \forall f_i^1 \in \Delta_1(\mathcal{T})$ we find for $\Pi_h \bar{X}_\tau^* : \mathcal{P}_1^- \Lambda^1(\mathcal{T}) \mapsto \mathcal{P}_1^- \Lambda^1(\mathcal{T})$ a matrix representation $\mathbf{P}_\tau^1$, mapping the expansion coefficients of ω_h to those of $\Pi_h \bar{X}_\tau^* \omega_h$. The matrix entries

$$\left(\mathbf{P}_\tau^1\right)_{ij} := \int_{f_i^1} \bar{X}_\tau^* b_{f_j^1} = \int_{\bar{X}_\tau(f_i^1)} b_{f_j^1} \tag{5.29}$$

are path integrals of basis forms $b_{f_j^1}$ associated to 1-subsimplices $f_j^1 \in \Delta_1(\mathcal{T})$ along the transported 1-subsimplex $\bar{X}_\tau(f_i^1)$ (see Figure 5.7). When we use linear Lagrangian elements to define the approximative flow $\bar{X}_\tau$ in (5.14) the transported 1-simplex $\bar{X}_\tau(f_i^1)$ is again a straight line. To determine the entries of the i-th row, where $f_i^1 \in \Delta_1(\mathcal{T})$ is oriented from vertex f_1^0 to vertex f_2^0, we trace the path from $X_\tau(f_1^0)$ to $X_\tau(f_2^0)$ and calculate for each crossed element the line integrals for the attached basis functions. If e.g. the line crosses an element from point a to point b and if this element contains an edge f_j^1, that is oriented from vertex f_l^0 to f_m^0, (see Figure 5.8), then the element contribution to $\left(\mathbf{P}_\tau^1\right)_{ij}$ is:

$$\int_{X_\tau(f_i^1)\cap T} b_{f_j^1} = \lambda_{f_l^0}(a)\lambda_{f_m^0}(b) - \lambda_{f_m^0}(a)\lambda_{f_l^0}(b).$$

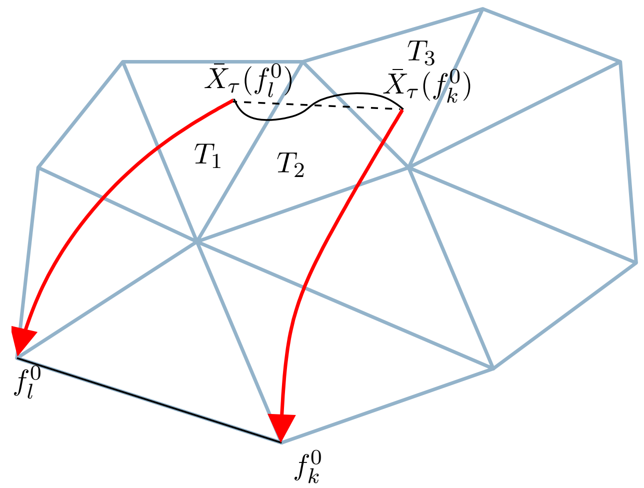

Figure 5.7: The transported 1-subsimplex $X_\tau^*(f_i^1)$ (black curved line) is approximated by a straight line $\bar{X}_\tau^*(f_i^1)$ (black dashed line). In the case depicted here all basis function associated with 1-subsimplices f_j^1 of elements T_1, T_2 and T_3 yield a nonzero entry $\left(\mathbf{P}_\tau^1\right)_{i,j}$.

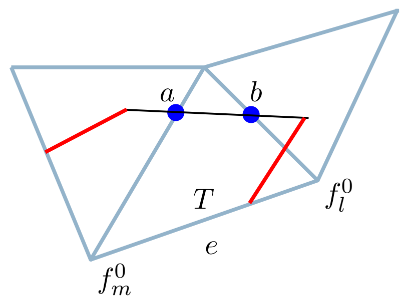

Figure 5.8: The line from a to b is the intersection of the approximation of the transported edge with element T.

5.2.3 Numerical Experiments

In this section we take $\Omega \subset \mathbb{R}^2$ and look at the advection problem for $\mathbf{u}$

$$\begin{aligned} \partial_t \mathbf{u} + \mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{u}) - \mathbf{R}\,\mathrm{div}(\mathbf{R}\mathbf{u})\boldsymbol{\beta} &= \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u}(0) &= \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{5.30}$$

with $\mathbf{R} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We approximate $\mathbf{u}$ by lowest order discrete 1-forms $\mathbf{u}_h \in \left(\mathcal{P}_1^- \Lambda^1(\mathcal{T})\right)^{\perp} = \mathbf{R}\mathcal{P}_1^- \Lambda^1(\mathcal{T})$ on a triangular mesh $\mathcal{T}$. In the following we will study the performance of the *semi-Lagrangian Galerkin* schemes 5.17 and the *semi-Lagrangian interpolation* schemes 5.25. The discrete space $\left(\mathcal{P}_1^- \Lambda^1(\mathcal{T})\right)^{\perp}$ consists of tangentially continuous, piecewise polynomial functions, with piecewise constant exterior derivatives. The basis functions are associated to the edges of the mesh and the degrees of freedom are line integrals on edges. We further use in (5.14) continuous piecewise linear Lagrangian finite elements to approximate the flow function X_τ. To this end we use, if not stated differently, explicit Euler timesteps to determine the flow of the vertices. Thus, the transported mesh $\bar{X}_\tau(\mathcal{T})$ is again a mesh with straight edges. The collocation method in (5.16) for evaluating the right-hand sides uses the end point of the integration interval. In the following experiments we link the timestep size τ to the mesh size h by :

$$\tau = \gamma \frac{h}{\|\boldsymbol{\beta}\|}, \tag{5.31}$$

where γ is some constant. Mostly, we will take $\gamma < 1$, since solving an entire advection-diffusion problem with non-vanishing diffusion would dictate such a timestep restriction.

Example 1: Generic right-hand side

We consider $\Omega = [-1, 1]^2$ and choose in (5.30) the velocity

$$\boldsymbol{\beta} = (1 - x_1^2)(1 - x_2^2) \begin{pmatrix} 0.66 \\ 1 \end{pmatrix}.$$

That data $\mathbf{f}$ and $\mathbf{u}_0$ is chosen such

$$\mathbf{u} = \cos(2\pi t) \begin{pmatrix} \sin(\pi x_1)\sin(\pi x_2) \\ (1 - x_1^2)(1 - x_2^2) \end{pmatrix}.$$

is the solution. With this choice we have in (5.30) a non-zero right-hand side $\mathbf{f}$.

In Figures 5.9 we monitor the convergence for different γ values. We observe here convergence of order 1 in the L^2-norm for both the Galerkin and the interpolation scheme. In Theorem 5.2.6 we have proved convergence of order $\frac{1}{2}$ for the semi-Lagrangian Galerkin scheme. We believe that this discrepancy is due to super convergence effects. Although we could not prove convergence for the interpolation scheme, our experiments underpin the hypothesis that the interpolation schemes are only perturbations of the Galerkin schemes, hence have similar approximation properties.

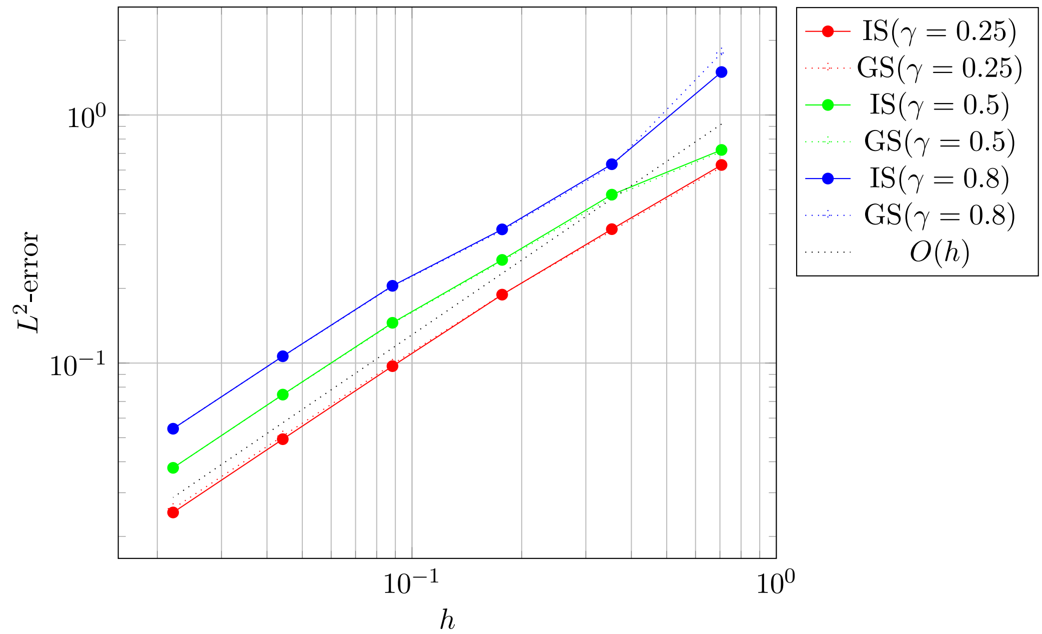

Figure 5.9: Example 1: Convergence rates of the L^2-error at $t = 0.4$ for the semi-Lagrangian interpolation scheme (IS) and the semi-Lagrangian Galerkin scheme (GS) on time interval $[0, 0.4]$ for $\gamma = 0.25$, $\gamma = 0.5$ and $\gamma = 0.8$. The convergence is of order $O(h)$, while we proved convergence of order $O(h^{\frac{1}{2}})$.

Example 2: A non-convergent fully-discrete scheme

The drawback of the Galerkin projection scheme is obviously the requirement to calculate the inner products $\left(\bar{X}^*_{-\tau}\omega_h, \eta\right)_\Omega$ exactly. A cheaper remedy, similar to standard finite element techniques, would be the quadrature-based scheme introduced in remark 5.2.3.

We consider the same data for problem (5.30) as in Example 1. Figure 5.10 shows the

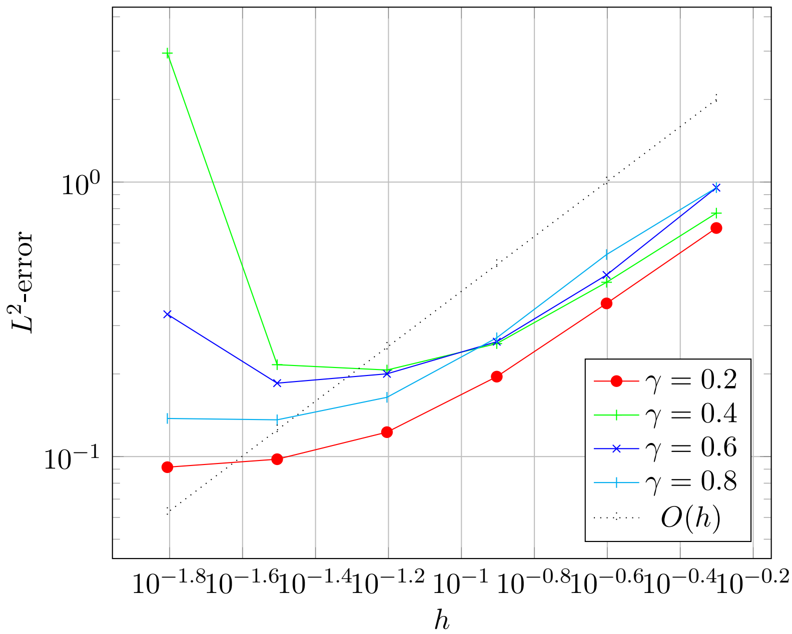

Figure 5.10: Example 2: Convergence rate of the L^2-error at $t = 0.4$ for a quadrature-based semi-Lagrangian scheme (5.18) on the time interval $[0, 0.4]$ with low order quadrature and $\gamma = 0.2$, $\gamma = 0.4$, $\gamma = 0.6$ and $\gamma = 0.8$.

convergence rate of a quadrature-based scheme build on the barycenters as quadrature points. Only for a few first refinements we see some sort of convergence, breaking down when we refine further. We observe the same phenomena if we use higher order quadrature rules to approximate the inner products. This result is as expected, since the quadrature-based scheme applies quadrature on domains with discontinuous integrands.

Example 3: Vanishing right-hand side and closed initial data

If we choose $\boldsymbol{\beta}$ in (5.30) such that $\operatorname{div} \boldsymbol{\beta} = 0$ and $\mathbf{u}(0) = \mathbf{R}\boldsymbol{\beta}$, then $\mathbf{u} = \mathbf{R}\boldsymbol{\beta}$ is a valid and a closed solution, i.e. $\operatorname{div} \mathbf{R}\mathbf{u} = 0$. Table 5.1 shows the values of the exterior derivatives of solutions of the interpolation scheme and the Galerkin scheme on a series of refined meshes. As expected, the interpolation schemes preserve the closedness of the initial data. The Galerkin scheme in contrast fails. Note that for the Galerkin projection schemes the error not even decreases if the mesh is refined.

meshwidth	interpolation	Galerkin
0.7995	1.110^{-15}	0.63
0.4257	2.710^{-15}	0.76
0.2120	5.910^{-15}	0.97
0.1077	1.310^{-15}	1.35

Table 5.1: Example 3: L^2-norm of the exterior derivative $\operatorname{div}\mathbf{Ru}(0.4)$ for the semi-Lagrangian interpolation scheme and the Galerkin scheme in time interval $[0, 0.4]$ on a series of refined meshes.

Example 4: Rotating hump problem 1

Here, we would like to study the behaviour of the interpolation scheme for the rotating hump problem. We consider problem (5.30) on a circular domain $\Omega := \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ with source term $\mathbf{f} = 0$, the velocity field:

$$\boldsymbol{\beta} = \begin{pmatrix} y \\ -x \end{pmatrix}$$

and initial data

$$\mathbf{u}_0(\mathbf{x}) = \begin{cases} \operatorname{grad} f(\mathbf{x}) & \sqrt{x_1^2 + (x_2 - 0.25)^2} \leq 0.5 \\ (0,0) & \sqrt{x_1^2 + (x_2 - 0.25)^2} > 0.5 \end{cases}.$$

with

$$f(\mathbf{x}) = \cos\left(\pi\sqrt{x_1^2 + (x_2 - 0.25)^2}\right)^4. \tag{5.32}$$

The exact solution is

$$\mathbf{u}(t, \mathbf{x}) = (\mathbf{R}(t))^{-1}\mathbf{u}_0(\mathbf{R}(t)\mathbf{x}), \quad \mathbf{R}(t) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}.$$

Figure 5.11 shows an example of a triangulation of Ω and a plot of the modulus of $\mathbf{u}(0)$. In order to study the impact of the approximation of the flow map, we use both (i) the explicit Euler method, and (ii) the explicit midpoint method in order to determine the positions of the vertices of the advected mesh. In Figure 5.12 we see that the global rate of convergence is not affected by these methods. Tables 5.2 and 5.3, list the L^2-errors of numerical solutions at $t = 2\pi$ for different mesh sizes h and timestep sizes τ. The numbers convey the need for balancing h and τ, with higher order integration of trajectories allowing larger timesteps. For fixed meshsize h we observe that the minimal error is not attained for the minimal timestep size, but for some medial values of τ. This observation is due to the negative power of τ in the estimate of Theorem 5.2.5. Second, when comparing the numbers of the two schemes, we see that the minimal error of the scheme with explicit midpoint method is attained for larger values of τ than for

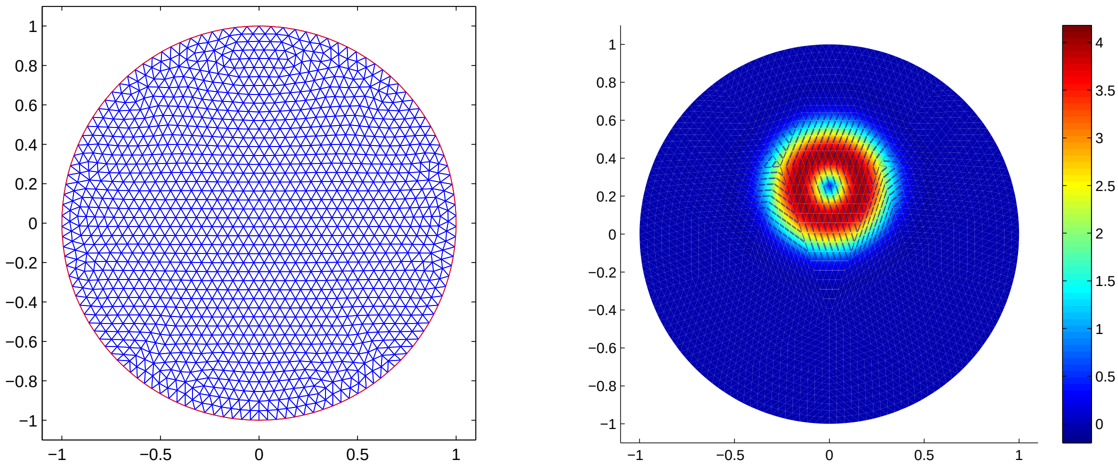

Figure 5.11: Example 4: A triangulation of the unit circle (left) and the plot of the modulus of the initial data (5.2.3).

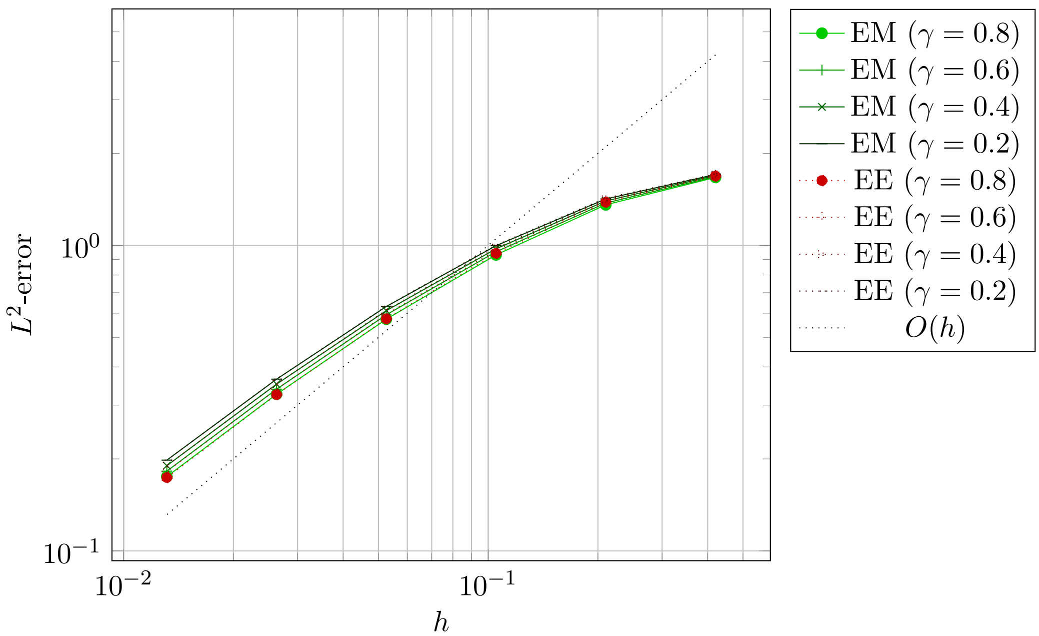

Figure 5.12: Example 4: Convergence rates of L^2-error at $t = 0.5\pi$ for the interpolation scheme with explicit midpoint rule (MM) and explicit Euler method (EE) on time interval $[0, 0.5\pi]$ for $\gamma = 0.2$, $\gamma = 0.4$, $\gamma = 0.6$ and $\gamma = 0.8$.

$\tau\backslash h$	0.420	0.210	0.105	0.052	0.026
1.5707	1.86	1.89	1.86	1.88	2.33
0.7853	1.86	1.88	1.88	2.01	2.36
0.3926	1.84	1.82	1.80	2.01	2.32
0.1963	1.85	1.79	1.52	1.51	1.79
0.0997	1.85	1.80	1.54	1.05	1.02
0.0498	1.85	1.81	1.59	1.18	0.63
0.0249	1.85	1.81	1.61	1.26	0.79
0.0124	1.85	1.81	1.62	1.30	0.88
0.0062	1.85	1.81	1.63	1.31	0.92

Table 5.2: Example 4, rotating hump: L^2-error at $t = 2\pi$ of the solution of the **interpolation scheme** (5.25) with explicit Euler method for different discretization parameters timestep τ (rows) and mesh size h (columns).

the scheme with explicit Euler method. This reflects the higher order approximation properties of the explicit midpoint method, that appear explicitly in the estimate of Theorem 5.2.5.

For our choice of data we find that the solution fulfills $\operatorname{div}\mathbf{Ru} = 0$ for all times, which we expect to hold also for the numerical solution produced by the interpolation scheme. Yet, Table 5.4 confirms this only for small times and fine meshes. We blame this on the approximate flow maps that will not map Ω exactly onto itself; backward trajectories may leave the domain and there may be edges, whose image under the flow will be at least partly outside the fixed mesh. In our implementation of the interpolant $\Pi_h \bar{X}^*_{-\tau}\omega_h$ we simply ignore the contribution of such edges, thus destroying the closedness property, see Figures 5.13. As long as ω_h has compact support away from $\partial\Omega$ this effect remains invisible. Yet, inevitable artificial diffusion will make $\operatorname{supp}\omega_h$ spread, reach $\partial\Omega$, and interpolation errors will pollute $\mathsf{d}\,\omega_h$, see Figures 5.14 and 5.15. Perversely, this happens earlier for the midpoint rule than the Euler method, because for the rotating flow the latter introduces a stronger drift towards the center, which partly offsets outward numerical diffusion.

Example 5: Rotating hump problem 2

We consider again the rotating hump problem, i.e. (5.30) on a circular domain $\Omega := \{(x,y) : x^2 + y^2 \leq 1\}$ with source term $\mathbf{f} = 0$, the velocity field:

$$\boldsymbol{\beta} = \begin{pmatrix} y \\ -x \end{pmatrix}.$$

$\tau \backslash h$	0.420	0.210	0.105	0.052	0.026
1.5707	1.85	1.86	1.91	2.12	2.47
0.7853	1.79	1.53	1.51	1.70	1.80
0.3926	1.83	1.56	0.99	0.55	0.50
0.1963	1.84	1.72	1.23	0.60	0.22
0.0997	1.84	1.77	1.49	0.85	0.33
0.0498	1.85	1.79	1.56	1.15	0.52
0.0249	1.85	1.80	1.59	1.24	0.79
0.0124	1.85	1.81	1.61	1.28	0.87
0.0062	1.85	1.81	1.62	1.31	0.92

Table 5.3: Example 4, rotating hump: L^2-error at $t = 2\pi$ of the solution of the **interpolation scheme** (5.25) with explicit midpoint method for different discretization parameters timestep τ (rows) and mesh size h (columns).

h	$\| \operatorname{div} \mathbf{Ru}_h(0.25\pi)\|_{0,1}$		$\| \operatorname{div} \mathbf{Ru}_h(0.5\pi)\|_{0,1}$		$\| \operatorname{div} \mathbf{Ru}_h(\pi)\|_{0,1}$	
	Euler	Midpoint	Euler	Midpoint	Euler	Midpoint
0.21	$1.5 \cdot 10^{-14}$	$1.7 \cdot 10^{-3}$	$1.5 \cdot 10^{-14}$	$4.6 \cdot 10^{-3}$	$6.2 \cdot 10^{-15}$	$1.1 \cdot 10^{-2}$
0.11	$4.1 \cdot 10^{-14}$	$7.3 \cdot 10^{-6}$	$3.1 \cdot 10^{-14}$	$4.6 \cdot 10^{-5}$	$3.1 \cdot 10^{-14}$	$5.3 \cdot 10^{-4}$
0.52	$8.2 \cdot 10^{-14}$	$5.9 \cdot 10^{-10}$	$9.2 \cdot 10^{-14}$	$4.5 \cdot 10^{-8}$	$9.1 \cdot 10^{-14}$	$2.1 \cdot 10^{-6}$
0.26	$1.9 \cdot 10^{-13}$	$1.9 \cdot 10^{-13}$	$2.1 \cdot 10^{-13}$	$1.8 \cdot 10^{-13}$	$2.1 \cdot 10^{-13}$	$1.2 \cdot 10^{-10}$

Table 5.4: Example 4, interpolation scheme, with $\gamma = 0.8$: The error $\| \operatorname{div} \mathbf{Ru}_h(0.25\pi)\|_{0,1}$, $\| \operatorname{div} \mathbf{Ru}_h(0.5\pi)\|_{0,1}$ and $\| \operatorname{div} \mathbf{Ru}_h(\pi)\|_{0,1}$ for solutions $\mathbf{u}_h$ of the interpolation scheme with explicit Euler and explicit midpoint methods for different mesh sizes h.

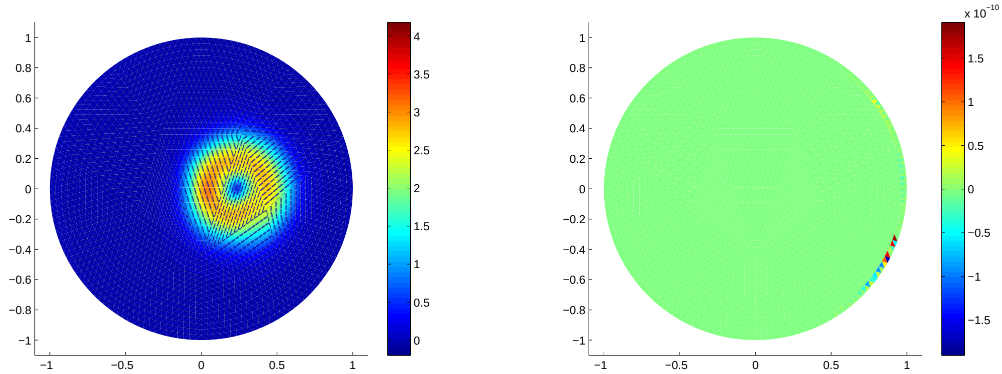

Figure 5.13: Example 4: Plot of the modulus of $\mathbf{u}_h(0.5\pi)$ (left) and $\operatorname{div}\mathbf{Ru}_h(0.5\pi)$ (right), with $\mathbf{u}_h$ obtained by the interpolation scheme with explicit midpoint rule on a mesh with mesh size $h = 0.0521$ and $\gamma = 0.8$.

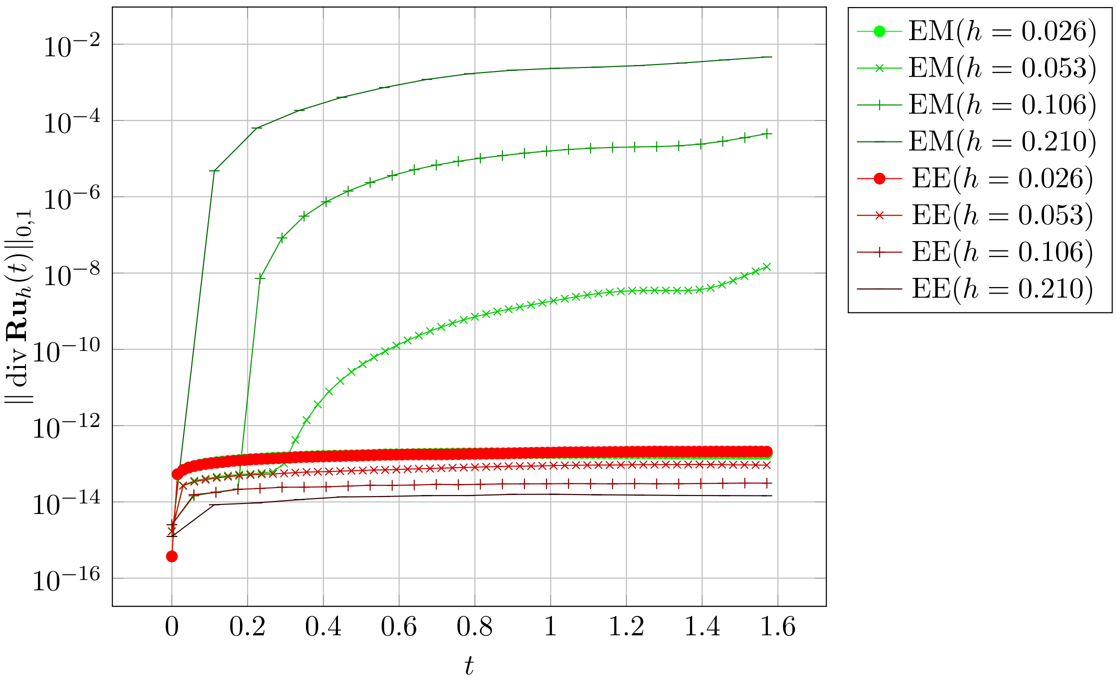

Figure 5.14: Example 4: Behavior of $\|\operatorname{div}\mathbf{Ru}_h\|_{0,1}$ as a function of t, with $\mathbf{u}_h$ produced by the interpolation scheme with explicit midpoint rule (EM) and explicit Euler (EE) on meshes with different mesh sizes for the time intervall $[0, 0.5\pi]$ and $\gamma = 0.8$.

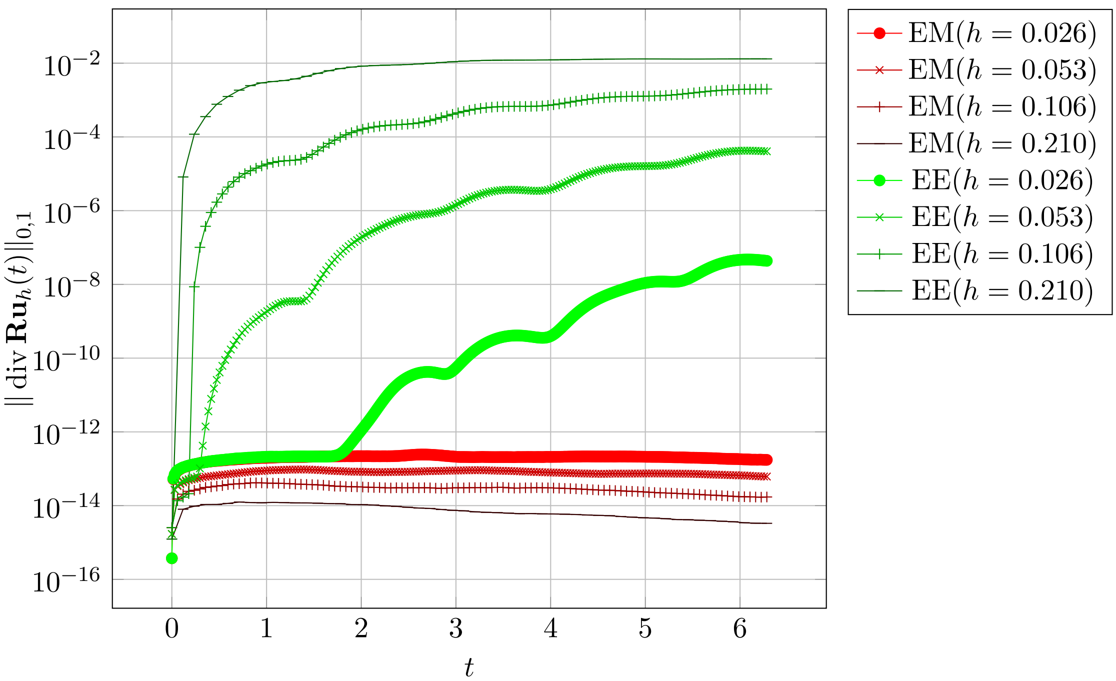

Figure 5.15: Example 4: Behavior of $\|\operatorname{div} \mathbf{R}\mathbf{u}_h\|_{0,1}$ as a function of t, with $\mathbf{u}_h$ produced by the interpolation scheme with explicit midpoint rule (EM) and explicit Euler (EE) on meshes with different mesh sizes for the time intervall $[0, 2\pi]$ and $\gamma = 0.8$.

Here we take very smooth initial data (see Figure 5.19)

$$\mathbf{u}_0 = e^{-\frac{(x-0.5)^2+y^2}{0.02}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{5.33}$$

Although semi-Lagrangian methods in principle permit very large timesteps, there are some subtle details to be considered. In each timestep there is a huge number of ordinary differential equations to be solved approximately up to some accuracy. We would like to stress here that the choice of the numerical integrator strongly affects the numerical solution. To examine the influence we use one timestep of

- explicit Euler method
- implicit Euler method
- implicit midpoint method

to determine the vertices of the transported mesh at $t = \frac{\pi}{4}$. For our specific choice of the velocity function $\boldsymbol{\beta}$, in all three cases the update can be computed explicitly. See Figures 5.16, 5.17 and 5.18 (left) for the corresponding transported meshes. It is well known [32, p. 12] that for our rotating problem

- the explicit Euler method gives an expanding solution;
- the implicit Euler method gives a collapsing solution;
- the midpoint rule gives a norm preserving solution.

We therefore encounter here that in comparison to the given circular domain the transported mesh covers

- a larger domain with the explicit Euler;
- a smaller domain with the implicit Euler;
- a domain of similar size with the midpoint rule.

These differences explain the quantitative differences of the solutions in Figures 5.16, 5.17 and 5.18. We could of course increase the number of timesteps to obtain more accurate solutions for the two Euler methods. Nevertheless we will encounter problems on the boundary of the domain. Even if the velocity $\boldsymbol{\beta}$ has vanishing normal components on the boundary on the domain, the approximate flow $\bar{X}_\tau$, used in any semi-Lagrangian methods doesn't need to map the domain Ω onto itself, i.e. in general $\bar{X}_{-\tau}(\Omega) \neq \Omega$.

Finally we would like to stress, that it is mainly the projection onto the fixed initial mesh, that introduces numerical diffusivity. Therefore, we consider

$$\mathbf{u}_0 = e^{-\frac{(x-0.5)^2+y^2}{0.08}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{5.34}$$

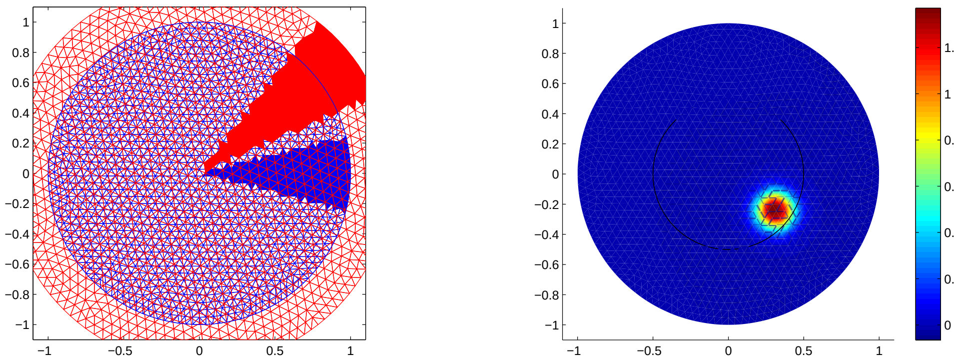

Figure 5.16: Example 5 with one explicit Euler step. Left: Mesh (blue) and transported mesh (red). The solid segments indicate the rotation. Right: Solution at $t = \frac{\pi}{4}$, for one Lagrangian time step. The black circle indicates the trajectory of the analytical solution.

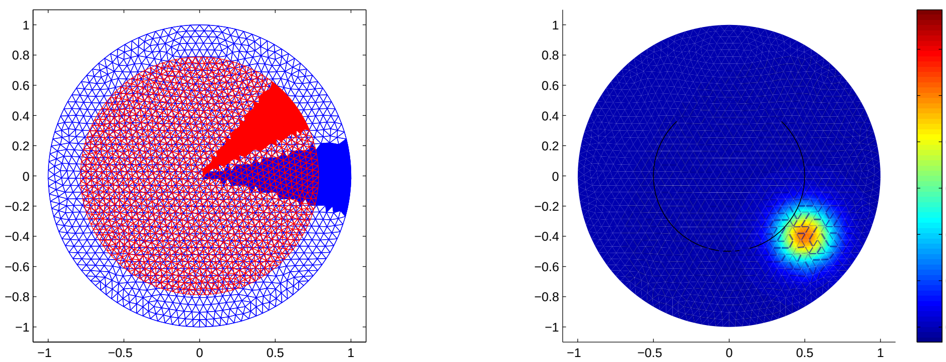

Figure 5.17: Example 5 with one implicit Euler step. Left: Mesh (blue) and transported mesh (red). The solid segments indicate the rotation. Right: Solution at $t = \frac{\pi}{4}$, for one Lagrangian time step. The black circle indicates the trajectory of the analytical solution.

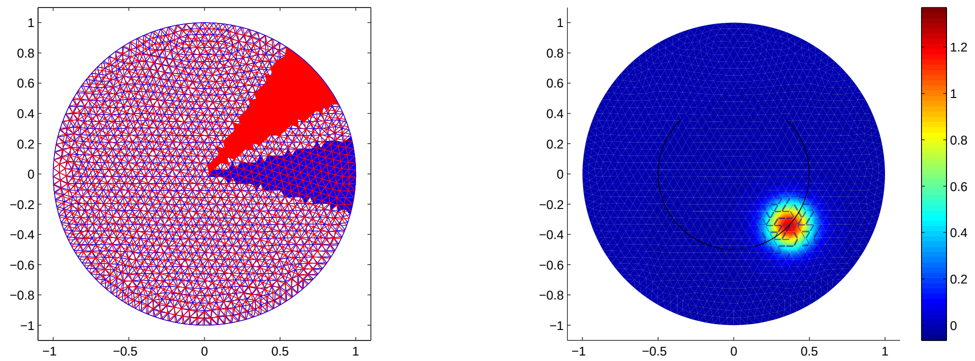

Figure 5.18: Example 5 with one step of the midpoint rule. Left: Mesh (blue) and transported mesh (red). The solid segments indicate the rotation. Right: Solution at $t = \frac{\pi}{4}$, for one Lagrangian time step. The black circle indicates the trajectory of the analytical solution.

as initial data (see Figure 5.19), take the implicit midpoint rule with very small local timesteps to compute the transported meshes and determine the solution at $t = \pi$ with 1, 4, 16 and 64 Lagrangian timesteps. Since each Lagrangian timestep maps the iterated solution back onto the initial mesh, it is the solution with just one Lagrangian timestep that suffers least from numerical diffusion (see figure 5.20).

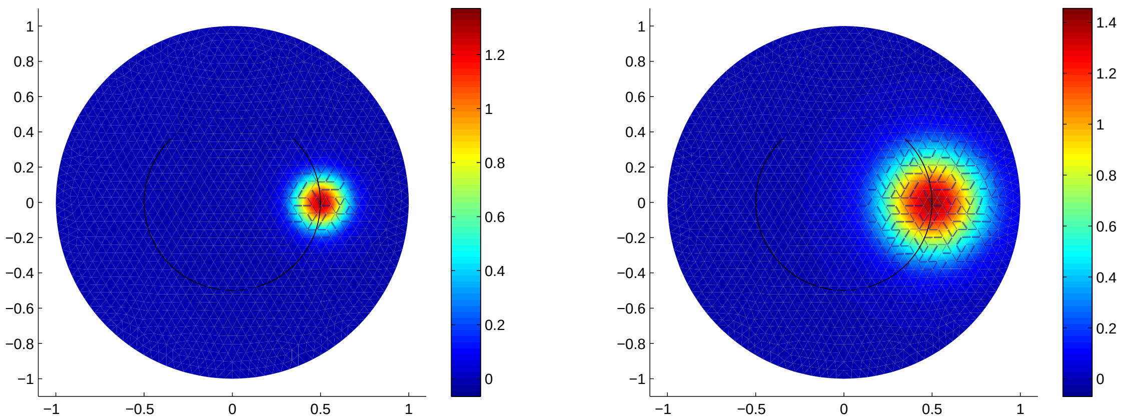

Figure 5.19: Example 5: Modulus of the initial solutions 5.33 (left) and (5.34).

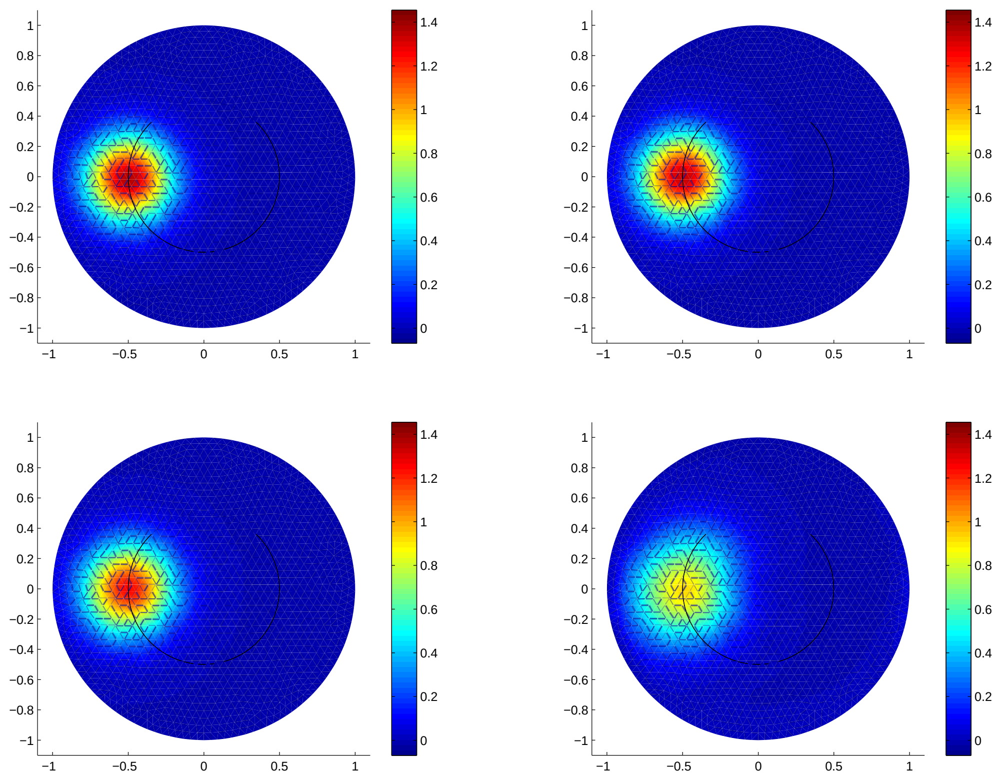

Figure 5.20: Example 5: Numerical solutions at $t = \pi$ for initial data as in figure 5.19, computed with 1.) upper left: 1 timestep ($\gamma \approx 68$), 2.) upper right: 4 timesteps ($\gamma \approx 17$), 3.) lower left: 16 timesteps ($\gamma \approx 4$) and 4.) lower right: 64 timesteps ($\gamma \approx 1$).

5.2.4 Appendix

A Discrete Gronwall Inequality

The recursion

$$\begin{aligned} b_0 &= a_0 \\ b_{i+1} &\leq a_{i+1} + (1 + C\tau)b_i, \quad C > 0 \end{aligned}$$

implies

$$b_N \leq \frac{e^{CN\tau} - 1}{C\tau} \max_{1 \leq i \leq N} a_i + e^{CN\tau} b_0.$$

Proof by induction:

$$\begin{aligned} b_{N+1} &\leq a_{N+1} + (1 + \tau C)b_N \\ &\leq a_{N+1} + (1 + \tau C)\left(\frac{e^{CN\tau} - 1}{C\tau} \max_{1 \leq i \leq N} a_i + e^{CN\tau} b_0\right) \\ &= a_{N+1} + (1 + \tau C)\left(\frac{e^{CN\tau} - 1}{C\tau} \max_{1 \leq i \leq N} a_i\right) + (1 + \tau C)e^{CN\tau} b_0 \\ &\leq \max_{1 \leq i \leq N+1} a_i \left(1 - \frac{1 + C\tau}{C\tau} + \frac{1 + C\tau}{C\tau} e^{CN\tau}\right) + (1 + C\tau)e^{CN\tau} b_0 \\ &\leq \frac{e^{C(N+1)\tau} - 1}{C\tau} \max_{1 \leq i \leq N+1} a_i + e^{C(N+1)\tau} b_0. \end{aligned}$$

6 Conclusions

6.1 Galerkin Methods for Magnetoquasistatic Equations in Moving Conductors

With the tools presented in the preceding chapters we are now in the position to state fully discrete Eulerian and Lagrangian timestepping schemes for the magnetoquasistatic equations in moving conductors (3.5a)-(3.5d).

For the sake of better readability, we stick here to an entirely vectorial notation. Electromagnetic fields are denoted by bold face capital letters, so that the eddy current model reads:

$$\mathbf{curl}\,\mathbf{E} = -\partial_t \mathbf{B} \qquad \text{in } \Omega, \tag{6.1a}$$
$$\mathbf{curl}\,\mathbf{H} = \mathbf{J} + \mathbf{F} \qquad \text{in } \Omega, \tag{6.1b}$$
$$\mathbf{J} = \sigma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) \qquad \text{in } \Omega, \tag{6.1c}$$
$$\mu\mathbf{H} = \mathbf{B} \qquad \text{in } \Omega. \tag{6.1d}$$

In Section 3.2 we have shown that this can be reformulated as a second order advection-diffusion problem for either the magnetic field $\mathbf{H}$ (3.8):

$$\partial_t \mu\mathbf{H} + \mathbf{curl}\,\sigma^{-1}\,\mathbf{curl}\,\mathbf{H} + \mathbf{grad}\,\sigma^{-1}\operatorname{div}\mathbf{H} + \boldsymbol{\beta}\operatorname{div}(\mu\mathbf{H}) + \mathbf{curl}(\mu\mathbf{H} \times \boldsymbol{\beta}) = \mathbf{curl}\,\sigma^{-1}\mathbf{F}$$

or for the vector potential $\mathbf{A}$ (3.11):

$$\partial_t \sigma\mathbf{A} + \mathbf{curl}\,\mu^{-1}\,\mathbf{curl}\,\mathbf{A} + \mathbf{grad}\,\mu^{-1}\operatorname{div}\mathbf{A} + \ \sigma\,\mathbf{curl}\,\mathbf{A} \times \boldsymbol{\beta} + \sigma\,\mathbf{grad}(\boldsymbol{\beta} \cdot \mathbf{A}) = \mathbf{F}.$$

We aim to find numerical solutions $\{\mathbf{H}_h^i\}_{i=0}^N$, $\mathbf{H}_h^i \in V_h$ or $\{\mathbf{A}_h^i\}_{i=0}^N$, $\mathbf{A}_h^i \in V_h$ approximating $\{\mathbf{H}(t_i)\}_{i=0}^N$ and $\{\mathbf{A}(t_i)\}_{i=0}^N$ at discrete time points $t_i = i\tau$ and $\tau = \frac{T}{N}$, where V_h is a finite dimensional approximation space. The most important candidates for V_h are the $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming spaces $\mathcal{P}_r\Lambda^1(\mathcal{T})$ and $\mathcal{P}_r^-\Lambda^1(\mathcal{T})$ and the $\boldsymbol{L}^2(\Omega)$-conforming spaces $\mathcal{P}_r^{\mathrm{d}}\Lambda^1(\mathcal{T})$. The discretization of the diffusion part $\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\operatorname{div}$ in such spaces is known and we refer to [60, Page 191] for $\boldsymbol{H}(\mathbf{curl}, \Omega)$-conforming spaces and to [69] for $\boldsymbol{L}^2(\Omega)$-conforming spaces.

Combining these discretizations of $\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\operatorname{div}$ with the explicit Eulerian schemes (5.5) or implicit Eulerian schemes (5.6) for the advection problem, one obtains Eulerian timestepping schemes for the complete advection-diffusion problem. Since we have proved convergence of the schemes (5.5) and (5.6), we get timestepping schemes for the advection-diffusion problem that are stable even for small diffusion coefficients. Moreover we can even choose whether we want to treat the diffusion part explicitly or implicitly. Since stability of explicit timestepping schemes for parabolic problems requires strong restrictions on the timestep size an implicit treatment appears preferable. Still it might be beneficial to consider a semi-implicit setting where only the advection part is treated explicitly. The resulting algebraic system is symmetric positive definite and there are fast solvers for such systems available. However, in this case the timestep size is limited by the stability constraint in Theorem 5.1.3. For the space $V_h = \mathcal{P}_0^- \Lambda^1(\mathcal{T})$ we could prove stability for $\tau = O(h)$. For the $\boldsymbol{H}\,(\mathbf{curl}, \Omega)$-conforming approximation spaces, in contrast, the result of Remark 5.1.4 and the numerical experiments in Section 5.1.1 indicate that the timestep choice $\tau = O(h)$ will not guarantee stability. Hence, in view of timestep constraints, the semi-implicit scheme is in general not advantageous compared to fully explicit timestepping schemes.

Another possibility to solve the complete advection-diffusion problems is to combine the discretization of $\mathbf{curl}\,\mathbf{curl} + \mathbf{grad}\,\mathrm{div}$ with one of the semi-Lagrangian timestepping schemes in Chapter 5.2. Semi-Lagrangian schemes have the advantage that they are unconditionally stable, while the algebraic systems that need to be solved in each timestep remain positive definite. A disadvantage is definitely the expensive evaluation of the right-hand side. Only for low-order approximation spaces we might use the semi-Lagrangian interpolation schemes (5.25) and (5.26), that are then less expensive than the semi-Lagrangian Galerkin schemes (5.17).

Finally it will strongly depend on the concrete problem setting whether an Eulerian timestepping scheme or a semi-Lagrangian scheme has an overall better performance.

6.2 Summary and Outlook

In Section 2.2.2 we presented a unifying framework to derive finite volume schemes from schemes for discrete differential forms. In doing so, constraints, that are preserved on the continuous level, can be consistently replaced in finite volume schemes by approximations. We think, that within this framework it is possible to define approximations of exterior derivatives for very general triangulations, such that (2.64) is fulfilled. This means one finds finite volumes spaces and discrete exterior derivatives that build a cochain complex similar to the complexes (2.60) and (2.61).

We believe that the convergence order $O(h^{r+\frac{1}{2}})$ given for the stabilized Galerkin methods in Theorems 4.1.8, 4.1.13, 4.1.14, 4.1.15 and 4.1.16 are sharp for estimates in L^2-norms. For the scalar case this has already been shown in [70, 75, 89]. Although the experiments in Section 4.1.4 show convergence rates of order $O(h^{r+1})$, we expect that on special meshes, meshes that are adapted to the velocity field, a convergence of order $O(h^{r+\frac{1}{2}})$ would be observed.

The analysis of the semi-Lagrangian interpolation schemes introduced in Section 5.2.2 seems to be much harder than the analysis of the semi-Lagrangian Galerkin schemes in 5.2.1. The standard analysis as in Theorems 5.2.1 and 5.2.5 fails, because of the lack of continuity of most interpolation operators in L^2. In Theorem 5.2.6 we have presented a new approach to analyse semi-Lagrangian methods and proved convergence even for lowest order approximations spaces. This approach is inspired by the analysis of the

Eulerian methods in Section 5.1 and uses so-called Ritz-Galerkin projectors defined via spatial discretizations of the stationary problem. For the case of the semi-Lagrangian Galerkin methods we used the characteristic method in Section 4.2 to define such Ritz-Galerkin methods. If it would be possible to define and analyse characteristic methods for the interpolation scheme this would allow a different kind of convergence analysis for semi-Lagrangian interpolation schemes.

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