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Minimal submanifolds from the abelian Higgs model

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Abstract Given a Hermitian line bundle $L \to M$ over a closed, oriented Riemannian manifold M, we study the asymptotic behavior, as $\epsilon \to 0$, of couples $(u_\epsilon, \nabla_\epsilon)$ critical for the rescalings

$$E_\epsilon(u, \nabla) = \int_M \left(|\nabla u|^2 + \epsilon^2 |F_\nabla|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2\right)$$

of the self-dual Yang–Mills–Higgs energy, where u is a section of L and ∇ is a Hermitian connection on L with curvature F_∇. Under the natural assumption $\limsup_{\epsilon \to 0} E_\epsilon(u_\epsilon, \nabla_\epsilon) < \infty$, we show that the energy measures converge subsequentially to (the weight measure μ of) a stationary integral $(n-2)$-varifold. Also, we show that the $(n-2)$-currents dual to the curvature forms converge subsequentially to $2\pi\Gamma$, for an integral $(n-2)$-cycle Γ with $|\Gamma| \le \mu$. Finally, we provide a variational construction of nontrivial critical points $(u_\epsilon, \nabla_\epsilon)$ on arbitrary line bundles, satisfying a uniform energy bound. As a byproduct, we obtain a PDE proof, in codimension two, of Almgren's existence result for (nontrivial) stationary integral $(n-2)$-varifolds in an arbitrary closed Riemannian manifold.

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1 Introduction

A level set approach for the variational construction of minimal hypersurfaces was born from the work of Modica–Mortola [30], Modica [29], and Sternberg [34]. Starting from a suggestion by De Giorgi [12], they highlighted a deep connection between minimizers $u_\epsilon : M \to \mathbb{R}$ of the Allen–Cahn functional

$$F_\epsilon(v) := \int_M \Big(\epsilon|dv|^2 + \frac{1}{4\epsilon}(1-v^2)^2\Big),$$

and two-sided minimal hypersurfaces in M, showing essentially that the functionals F_ϵ Γ-converge to ($\frac{4}{3}$ times) the perimeter functional on Caccioppoli sets. Several years later, Hutchinson and Tonegawa [19] initiated the asymptotic study of critical points v_ϵ of F_ϵ with bounded energy, without the energy-minimality assumption. They showed, in particular, that their energy measures concentrate along a stationary, integral $(n-1)$-varifold, given by the limit of the level sets $v_\epsilon^{-1}(0)$.

These developments, together with the deep regularity work by Tonegawa and Wickramasekera on stable solutions [38], opened the doors to a fruitful min–max approach to the construction of minimal hypersurfaces, providing a PDE alternative to the rather involved discretized min–max procedure implemented by Almgren and Pitts [5,31] in the setting of geometric measure theory. This promising min–max approach based on the Allen–Cahn functionals was recently developed by Guaraco and Gaspar–Guaraco [14,16], and has been used successfully to attack some profound questions concerning the structure of min–max minimal hypersurfaces—most notably in Chodosh and Mantoulidis's work on the multiplicity one conjecture [11].

The initial motivation for this paper is to find, in a similar vein, a natural way to construct minimal varieties of codimension two through PDE methods. Recently, other attempts in this direction have been made by Cheng [10] and the second-named author [33], based on the study of the Ginzburg–Landau functionals

$$F_\epsilon(v) := \frac{1}{|\log\epsilon|}\int_M \Big(|dv|^2 + \frac{1}{4\epsilon^2}(1-|v|^2)^2\Big)$$

on complex-valued maps $v : M \to \mathbb{C}$. While the Ginzburg–Landau approach can be employed successfully to produce nontrivial stationary *rectifiable* $(n-2)$-varifolds (building on the analysis of [8,28], and others), and leads to existence results of independent interest for solutions of the Ginzburg–Landau equations, it is not yet known whether the varifolds produced in this way are *integral*, nor is it known whether the full energies $F_\epsilon(v_\epsilon)$ of the min–max critical points converge to the mass of the limiting minimal variety in the case $b_1(M) \neq 0$.

While it is possible that these and other technical difficulties may be overcome with sufficient effort—and establishing integrality in particular remains a fascinating open problem—they point to the deeper fact that the Ginzburg–Landau functionals, though intimately related to the $(n-2)$-area, do *not* provide a straightforward regularization of the codimension-two area functional. Indeed, we stress that the Ginzburg–Landau energies should be understood first and foremost as a relaxation of the Dirichlet energy for singular maps to S^1, and while the limiting singularities of critical points may coincide with minimal varieties, the associated variational problems exhibit substantial qualitative differences at both large and small scales.

In the present paper, we consider instead the self-dual Yang–Mills–Higgs energy

$$E(u, \nabla) := \int_M \Big(|\nabla u|^2 + |F_\nabla|^2 + W(u)\Big) \tag{1.1}$$

and its rescalings (for $\epsilon \in (0, 1]$)

$$E_\epsilon(u, \nabla) := \int_M \Big(|\nabla u|^2 + \epsilon^2 |F_\nabla|^2 + \epsilon^{-2} W(u)\Big), \tag{1.2}$$

for couples (u, ∇) consisting of a section u of a given Hermitian line bundle $L \to M$, and a metric connection ∇ on L. Here, the nonlinear potential $W : L \to \mathbb{R}$ is given by

$$W(u) := \frac{1}{4}(1 - |u|^2)^2, \tag{1.3}$$

while $F_\nabla \in \Omega^2(\mathrm{End}(L))$ denotes the curvature of ∇.

For the trivial bundle $L = \mathbb{C} \times \mathbb{R}^2$ on the plane $M = \mathbb{R}^2$, a detailed study of the functional (1.1) and its critical points can be found in the doctoral work of Taubes [35,36]. In [36], all finite-energy critical points (u, ∇) of (1.1) in the plane are shown to solve the first order system[1]

$$\nabla_{\partial_1} u \pm i \nabla_{\partial_2} u = 0; \quad *F_\nabla = \pm \frac{1}{2}(1 - |u|^2) \tag{1.4}$$

known as the *vortex equations*—a two-dimensional counterpart of the instanton equations in four-dimensional Yang–Mills theory. In particular, all such

[1] Here and elsewhere, we implicitly identify F_∇ with the two-form ω given by $F_\nabla(X, Y) = -i\omega(X, Y)$.

solutions (u, ∇) minimize energy among pairs (u, ∇) with fixed vortex number

$$N := \frac{1}{2\pi}\int_{\mathbb{R}^2} *F_\nabla \in \mathbb{Z},$$

and carry energy exactly $E(u, \nabla) = 2\pi|N|$. In [35], Taubes shows moreover that there exist solutions of (1.4) with any prescribed zero set

$$u^{-1}(0) = \{z_1, \dots, z_N\} \subset \mathbb{R}^2,$$

which are unique up to gauge equivalence, so that [35,36] together give a complete classification of finite-energy critical points of (1.1) in the plane.

In [18], Hong, Jost, and Struwe initiate the study of the rescaled functionals (1.2) in the limit $\epsilon \to 0$ for line bundles $L \to \Sigma$ over a closed Riemann surface Σ. The main result of [18] shows that, for solutions $(u_\epsilon, \nabla_\epsilon)$ of the rescaled vortex equations (given by replacing $\frac{1}{2}(1-|u|^2)$ with $\frac{1}{2\epsilon^2}(1-|u_\epsilon|^2)$ in (1.4)), the curvature $*\frac{1}{2\pi}F_{\nabla_\epsilon}$ converges as $\epsilon \to 0$ to a finite sum of Dirac masses of total mass $|\deg(L)|$, away from which ∇_ϵ converges to a flat connection ∇_0, and u_ϵ to a unit section u_0 with $\nabla_0 u_0 = 0$, up to change of gauge. While the authors of [18] focus on the vortex equations over Riemann surfaces, they suggest that the asymptotic analysis of the rescaled functionals E_ϵ may also yield interesting results in higher dimension, pointing to similarities with the Allen–Cahn functionals for scalar-valued functions.

In the present paper, we develop the asymptotic analysis as $\epsilon \to 0$ for critical points of E_ϵ associated to line bundles $L \to M$ over Riemannian manifolds M^n of arbitrary dimension $n \geq 2$. The bulk of the paper is devoted to the proof of the following theorem, which describes the limiting behavior as $\epsilon \to 0$ of the energy measures

$$\mu_\epsilon := \frac{1}{2\pi} e_\epsilon(u_\epsilon, \nabla_\epsilon)\ \mathrm{vol}_g$$

and curvatures F_{∇_ϵ} for critical points $(u_\epsilon, \nabla_\epsilon)$ satisfying a uniform energy bound.

Theorem 1.1 *Let $L \to M$ be a Hermitian line bundle over a closed, oriented Riemannian manifold M^n of dimension $n \geq 2$, and let $(u_\epsilon, \nabla_\epsilon)$ be a family of critical points for E_ϵ satisfying a uniform energy bound*

$$E_\epsilon(u_\epsilon, \nabla_\epsilon) \leq \Lambda < \infty.$$

Then, as $\epsilon \to 0$, the energy measures

$$\mu_\epsilon := \frac{1}{2\pi} e_\epsilon(u_\epsilon, \nabla_\epsilon) \text{ vol}_g$$

converge subsequentially, in duality with $C^0(M)$, to the weight measure μ of a stationary, integral $(n-2)$-varifold V. Also, for all $0 \leq \delta < 1$,

$$\text{spt}(V) = \lim_{\epsilon \to 0} \{|u_\epsilon| \leq \delta\}$$

in the Hausdorff topology. The $(n-2)$-currents dual to the curvature forms $\frac{1}{2\pi} F_{\nabla_\epsilon}$ converge subsequentially to an integral $(n-2)$-cycle Γ, with $|\Gamma| \leq \mu$.

As will be clear from the proofs, orientability will be assumed only to show the statement concerning the current Γ.

Roughly speaking, Theorem 1.1 says that the energy of the critical points concentrates near the zero sets $u_\epsilon^{-1}(0)$ of u_ϵ as $\epsilon \to 0$, which converge to a (possibly rather singular) minimal submanifold of codimension two. In the case $\dim(M) = 3$, for instance, it follows from the results above and work of Allard and Almgren [3] that energy concentrates along a stationary geodesic network with integer multiplicities. The convergence of the curvature, moreover, to an integral cycle Poincaré dual to $c_1(L)$, with mass bounded above by $\lim_{\epsilon \to 0} E_\epsilon(u_\epsilon, \nabla_\epsilon)$, provides a higher dimensional analog to the limiting behavior described in two dimensions by Hong–Jost–Struwe [18].

At first glance, the obvious advantages of Theorem 1.1 over analogous results for the complex Ginzburg–Landau equations (cf., e.g., [8,28,33]) are the *integrality* of the limit varifold V, and the concentration of the *full energy measure* to V, independent of the topology of M. Indeed, Theorem 1.1 and the analysis leading to its proof align much more closely with the work of Hutchinson and Tonegawa [19] on the Allen–Cahn equations than they do with related results (e.g. [8,28]) for the complex Ginzburg–Landau equations. The parallels between the analysis presented here and that of the Allen–Cahn equations in [19] are in fact quite striking in places—a point to which we will draw the reader's attention throughout the paper.

Remark 1.2 We warn the reader, however, that while the qualitative analysis of the Allen–Cahn functionals does not depend on the precise choice of the double-well potential W, the analysis of the abelian Yang–Mills–Higgs functionals (1.1)–(1.2) seems to depend *quite strongly* on the choice $W(u) = \frac{1}{4}(1 - |u|^2)^2$. Indeed, already in two dimensions, replacing W with a potential $W_\lambda(u) := \frac{\lambda}{4}(1 - |u|^2)^2$ for some $\lambda \neq 1$ yields a dramatically different qualitative behavior, breaking the symmetry which leads to the first-order equations (1.4), and introducing interactions between disjoint components of the zero set (see, e.g., [21, Chapters I–III]). This should serve as one indication that the analysis of the abelian Higgs model is somewhat more delicate than that of related semilinear scalar equations, in spite of the strong parallels.

To get some idea of the role played by gauge invariance, note that unit sections of a Hermitian line bundle are indistinguishable up to change of gauge (when no preferred connection has been selected) and, for a given unit section u of L, one can always choose locally a connection with respect to which u appears constant. Thus, while most of the energy of solutions v_ϵ to the complex Ginzburg–Landau equations falls on annular regions—relatively far from the zero set—where v_ϵ resembles a harmonic S^1-valued map, the energy $e_\epsilon(u_\epsilon, \nabla_\epsilon)$ of a critical pair $(u_\epsilon, \nabla_\epsilon)$ for the abelian Yang–Mills–Higgs energy instead concentrates near the zero set $u_\epsilon^{-1}(0)$, with $|\nabla_\epsilon u_\epsilon|$ vanishing rapidly outside this region.

Of course, the results of Theorem 1.1 would be of limited interest if nontrivial critical points $(u_\epsilon, \nabla_\epsilon)$ could be found only in a few special settings. After completing the proof of Theorem 1.1, we therefore establish the following general existence result, showing that nontrivial families satisfying the hypotheses of Theorem 1.1 arise naturally on any line bundle (including, importantly, the trivial bundle) over any Riemannian manifold M^n, from variational constructions.

Theorem 1.3 *For any Hermitian line bundle $L \to M$ over an arbitrary closed base manifold M^n, there exists a family $(u_\epsilon, \nabla_\epsilon)$ satisfying the hypotheses of Theorem 1.1, with nonempty zero sets $u_\epsilon^{-1}(0) \neq \varnothing$. In particular, the energy μ_ϵ of these families concentrates (subsequentially) on a nontrivial stationary integral $(n-2)$-varifold V as $\epsilon \to 0$.*

For nontrivial bundles $L \to M$, this follows from a fairly simple argument, showing that the minimizers $(u_\epsilon, \nabla_\epsilon)$ of E_ϵ satisfy uniform energy bounds as $\epsilon \to 0$. For these energy-minimizing solutions, we expect moreover that the limiting minimal variety $\mu = \theta \mathcal{H}^{n-2} \llcorner \Sigma$, i.e. the weight measure $|V|$ of V, coincides with the weight measure $|\Gamma|$ of the limiting $(n-2)$-cycle $\Gamma = \lim_{\epsilon \to 0} * \frac{1}{2\pi} F_{\nabla_\epsilon}$, and that Γ minimizes $(n-2)$-area in its homology class. While we do not take up this question here, we believe that it would be interesting to study the convergence of the functionals (1.2) to the $(n-2)$-area functional in a Γ-convergence framework. Let us mention that an asymptotic study for *minimizers* of the Ginzburg–Landau functional, on a domain with boundary, was successfully carried out by Lin and Rivière [27], who were able to identify the concentration measure with the weight of an integral current. (See also [1,22] for related Γ-convergence results in that setting.)

Remark 1.4 We remark that a very special class of minimizers for E_ϵ are given by solutions $(u_\epsilon, \nabla_\epsilon)$ of the first-order vortex equations in Kähler manifolds (M^{2n}, ω_K) of higher dimension; these generalize the system (1.4) from the two-dimensional setting by replacing $*F_\nabla$ in (1.4) by the inner product $\langle F_\nabla, \omega_K\rangle$ with the Kähler form ω_K, and requiring additionally that $F_\nabla^{0,2} = 0$. As in the two-dimensional setting, solutions of this first-order system minimize the energy E_ϵ in appropriate line bundles on Kähler manifolds, and it was shown by Bradlow[2] [9] that the moduli space of solutions corresponds to the space of complex subvarieties in M (of complex codimension one) via the zero locus $(u_\epsilon, \nabla_\epsilon) \mapsto u_\epsilon^{-1}(0)$.

In particular, the zero loci $u_\epsilon^{-1}(0)$ in this case are already area-minimizing subvarieties, before passing to the limit $\epsilon \to 0$. Note that the analysis of the vortex equations plays a key role in the study of Seiberg–Witten invariants of Kähler surfaces [39], and a similar analysis figures crucially into Taubes's work relating the Seiberg–Witten and Gromov–Witten invariants of symplectic four-manifolds [37]. For a concise introduction to the higher-dimensional vortex equations and connections to Seiberg–Witten theory, we refer the interested reader to the survey [13] by García–Prada.

For the trivial bundle $L \cong \mathbb{C} \times M$, we prove Theorem 1.3 by applying min–max methods to the functionals (1.2), to produce nontrivial families $(u_\epsilon, \nabla_\epsilon)$ satisfying a uniform energy bound as $\epsilon \to 0$. While we consider only one min–max construction in the present paper, we remark that many more may be carried out in principle, due to the rich topology of the space

$$\mathcal{M} := \{(u, \nabla) : 0 \not\equiv u \in \Gamma(\mathbb{C} \times M),\ \nabla \text{ a Hermitian connection}\}/\mathcal{G},$$

where $\mathcal{G} := \mathrm{Maps}(M, S^1)$ is the gauge group. Indeed, on a closed oriented manifold M, one can show that the homotopy groups $\pi_i(\mathcal{M})$ are given by

$$\pi_1(\mathcal{M}) \cong H^1(M; \mathbb{Z}),\ \pi_2(\mathcal{M}) \cong \mathbb{Z},\ \text{and } \pi_i(\mathcal{M}) = 0 \text{ for } i \geq 3;$$

it may be of interest to note that these are isomorphic to the homotopy groups of the space $\mathcal{Z}_{n-2}(M; \mathbb{Z})$ of integral $(n-2)$-cycles in M, as computed by Almgren [4].

As an application of Theorem 1.3, we obtain a new proof of the existence of stationary integral $(n-2)$-varifolds in an arbitrary Riemannian manifold—a result first proved by Almgren in 1965 [5] using a powerful, but rather involved geometric measure theory framework. As already mentioned, similar constructions for the Allen–Cahn equations have been carried out successfully by Guaraco [16] and Gaspar–Guaraco [14], yielding new proofs of the existence of minimal hypersurfaces of optimal regularity, and leading to other recent breakthroughs in the min–max theory of minimal hypersurfaces (e.g., [11]).

[2] The precise form of the energies considered by Bradlow in [9] differs slightly from the functionals E_ϵ considered here, but the analysis is essentially the same.

In [11,16] (building on results of [38]), the stability properties of the min–max critical points for the Allen–Cahn functionals play a central role in controlling the regularity and multiplicity of the limit hypersurface. To obtain an improved understanding of min–max families $(u_\epsilon, \nabla_\epsilon)$ and the associated minimal varieties in the abelian Higgs setting, it would likewise be very interesting to refine the conclusions of Theorem 1.1 under the assumption that the families $(u_\epsilon, \nabla_\epsilon)$ satisfy a uniform Morse index bound as $\epsilon \to 0$. We hope to take up this line of investigation in future work.

1.1 Organization of the paper

In Sect. 2 we fix notation and record some basic properties satisfied by critical pairs (u, ∇) for the energies E_ϵ.

In Sect. 3, we record some useful Bochner identities for the gauge-invariant quantities $|u|^2$, $|F_\nabla|^2$, and $|\nabla u|^2$, and use them to establish an initial rough estimate on $\xi_\epsilon := \epsilon|F_\nabla| - \frac{(1-|u|^2)}{2\epsilon}$, whose role should be compared to that of the *discrepancy function* in the Allen–Cahn setting. Under suitable assumptions on the curvature of M, the fact that $\xi_\epsilon \leq 0$ follows quickly from the aforementioned Bochner identities and the maximum principle. Without the curvature assumptions, some nontrivial additional work is required to obtain the pointwise upper bound $\xi_\epsilon \leq C(M, E_\epsilon(u, \nabla))$. This estimate is the key ingredient to obtain the sharp $(n-2)$-monotonicity of the energy, and relies on the specific choice of coupling constants appearing in the *self-dual* Yang–Mills–Higgs functionals.

In Sect. 4 we derive the stationarity equation for inner variations, from which an obvious $(n-4)$-monotonicity property of the energy follows rather immediately. Using our rough initial bounds on ξ_ϵ from Sect. 3, we deduce an intermediate $(n-3)$-monotonicity; we use this to reach the pointwise bound $\xi_\epsilon \leq C(M, E_\epsilon(u, \nabla))$, from which we finally infer the sharp $(n-2)$-monotonicity.

In Sect. 5 we show that, similar to the Allen–Cahn setting, the energy density $e_\epsilon(u, \nabla)$ decays exponentially away from the set $u^{-1}(0)$—more precisely, away from $\{|u|^2 \geq 1 - \beta_d\}$ for some β_d independent of ϵ.

Section 6, which constitutes the main part of the paper, contains an initial description of the limiting varifold, showing that it is stationary, $(n-2)$-rectifiable, and has a lower density bound on the support. Then we establish its integrality with a blow-up analysis, employing the estimates from the preceding sections to reduce the problem to a statement for entire planar solutions, already contained in the work of Jaffe and Taubes [21]. We then use this analysis to show that the level sets $u_\epsilon^{-1}(0)$ converge to the support of V in the Hausdorff topology, and conclude the section with a discussion of the asymptotics for the curvature forms $\frac{1}{2\pi} F_{\nabla_\epsilon}$.

In Sect. 7, we show that E_ϵ satisfies a variant of the Palais–Smale property on suitable function spaces, allowing us to produce critical points via classical min–max methods. We provide a variational construction to get nontrivial critical points satisfying the assumptions of our main theorem, with energy bounded from above and below, both for nontrivial and trivial line bundles.

Finally, the "Appendix" addresses the issue of showing regularity of critical points, as obtained from Sect. 7, when they are read in a local or global Coulomb gauge.

2 The Yang–Mills–Higgs equations on $U(1)$ bundles

Let M be a closed, oriented Riemannian manifold, and let $L \to M^n$ be a complex line bundle over M, endowed with a Hermitian structure $\langle\cdot,\cdot\rangle$. Denote by $W : L \to \mathbb{R}$ the nonlinear potential

$$W(u) := \frac{1}{4}(1-|u|^2)^2.$$

For a Hermitian connection ∇ on L, a section $u \in \Gamma(L)$ and a parameter $\epsilon > 0$, denote by $E_\epsilon(u,\nabla)$ the scaled Yang–Mills–Higgs energy

$$E_\epsilon(u,\nabla) := \int_M \Big(|\nabla u|^2 + \epsilon^2|F_\nabla|^2 + \epsilon^{-2}W(u)\Big), \tag{2.1}$$

where F_∇ is the curvature of ∇. Throughout, we will identify the curvature F_∇ with a closed real two-form ω via

$$F_\nabla(X,Y)u = [\nabla_X,\nabla_Y]u - \nabla_{[X,Y]}u = -i\omega(X,Y)u. \tag{2.2}$$

In computing inner products for two-forms, we follow the convention

$$|\omega|^2 = \sum_{1\le j<k\le n} \omega(e_j,e_k)^2 = \frac{1}{2}\sum_{j,k=1}^n \omega(e_j,e_k)^2 \tag{2.3}$$

with respect to a local orthonormal basis $\{e_j\}_{j=1}^n$ for TM.

Note that E_ϵ enjoys the $U(1)$ gauge invariance

$$E_\epsilon(u,\nabla) = E_\epsilon(e^{i\theta}u, \nabla - i d\theta),$$

for any (smooth) $\theta : M \to \mathbb{R}$. More generally, we have

$$E_\epsilon(u,\nabla) = E_\epsilon(\varphi u, \nabla - i\varphi^*(d\theta)),$$

for any $\varphi : M \to S^1$, identifying S^1 with the unit circle in $\mathbb{C}$.

It is easy to check that the smooth pair (u, ∇) gives a critical point for the energy E_ϵ, with respect to smooth variations, if and only if it satisfies the system

$$\nabla^*\nabla u = \frac{1}{2\epsilon^2}(1 - |u|^2)u, \tag{2.4}$$

$$\epsilon^2 d^*\omega = \langle \nabla u, iu \rangle. \tag{2.5}$$

We denote $\Delta_H = dd^* + d^*d$ the usual positive definite Hodge Laplacian on differential forms and note that, in our convention, the adjoint to $d : \Omega^1(M) \to \Omega^2(M)$ is

$$(d^*\omega)(e_k) = -\sum_{j=1}^{n}(D_{e_j}\omega)(e_j, e_k).$$

Since the curvature form ω is closed, taking the exterior derivative of (2.5) gives

$$\begin{aligned}\epsilon^2(\Delta_H\omega)(e_j, e_k) &= (d\langle \nabla u, iu\rangle)(e_j, e_k)\\ &= \langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle - \langle i\nabla_{e_k}u, \nabla_{e_j}u\rangle\\ &\quad + \langle iu, F_\nabla(e_j, e_k)u\rangle\\ &= \psi(u)(e_j, e_k) - |u|^2\omega(e_j, e_k);\end{aligned}$$

i.e.,

$$\epsilon^2\Delta_H\omega = -|u|^2\omega + \psi(u), \tag{2.6}$$

where

$$\psi(u)(e_j, e_k) := 2\langle i\nabla_{e_j}u, \nabla_{e_k}u\rangle.$$

For future reference, we record the simple bound

$$|\psi(u)| \le |\nabla u|^2. \tag{2.7}$$

To confirm (2.7), fix $x \in M$ and note that the linear map $\nabla u(x) : T_xM \to L_x$ has a kernel of dimension at least $n - 2$. Take an orthonormal basis $\{e_j\}$ of T_xM such that $e_j \in \ker \nabla u(x)$ for $j > 2$. We compute at x that

$$|\psi(u)| = 2|\langle i\nabla_{e_1}u, \nabla_{e_2}u\rangle| \le 2|\nabla_{e_1}u||\nabla_{e_2}u| \le |\nabla_{e_1}u|^2 + |\nabla_{e_2}u|^2,$$

which gives (2.7).

3 Bochner identities and preliminary estimates

From the Eqs. (2.6) and (2.4), we apply the standard Bochner–Weitzenböck formulas to obtain some identities which will play a central role in our analysis. For the curvature two-form ω, it will be useful to record the Bochner identity

$$\Delta\frac{1}{2}|\omega|^2 = |D\omega|^2 + \epsilon^{-2}(|u|^2|\omega|^2 - \langle\psi(u), \omega\rangle) + \mathcal{R}_2(\omega, \omega), \tag{3.1}$$

where D is the Levi–Civita connection and $\mathcal{R}_2$ denotes the Weitzenböck curvature operator for two-forms on the base Riemannian manifold M. For any $\delta > 0$ we have

$$(|\omega|^2 + \delta^2)^{1/2}\Delta(|\omega|^2 + \delta^2)^{1/2} + |D|\omega||^2 \geq \Delta\frac{1}{2}(|\omega|^2 + \delta^2) = \Delta\frac{1}{2}|\omega|^2.$$

Since $|D|\omega||^2 \leq |D\omega|^2$, (3.1) implies

$$(|\omega|^2 + \delta^2)^{1/2}\Delta(|\omega|^2 + \delta^2)^{1/2} \geq \epsilon^{-2}(|u|^2|\omega|^2 - \langle\psi(u), \omega\rangle) + \mathcal{R}_2(\omega, \omega).$$

Dividing by $(|\omega|^2 + \delta^2)^{1/2}$ and letting $\delta \to 0$, we obtain

$$\Delta|\omega| \geq \epsilon^{-2}(|u|^2|\omega| - |\psi(u)|) - |\mathcal{R}_2^-||\omega|, \tag{3.2}$$

in the distributional sense (and classically on $\{|\omega| > 0\}$). Note that, by (2.7), the relation (3.2) also gives us the cruder subequation

$$\Delta|\omega| \geq \epsilon^{-2}|u|^2|\omega| - \epsilon^{-2}|\nabla u|^2 - |\mathcal{R}_2^-||\omega|. \tag{3.3}$$

For the modulus $|u|^2$ of the Higgs field u, we record

$$\Delta\frac{1}{2}|u|^2 = |\nabla u|^2 - \frac{1}{2\epsilon^2}(1 - |u|^2)|u|^2, \tag{3.4}$$

and observe that a simple application of the maximum principle yields the pointwise bound

$$|u|^2 \leq 1 \quad \text{on } M.$$

For the energy density $|\nabla u|^2$ of the Higgs field u, we see that

$$\begin{aligned}\Delta\frac{1}{2}|\nabla u|^2 &= |\nabla^2 u|^2 - \langle\nabla(\nabla^*\nabla u), \nabla u\rangle + \langle d^*\omega, \langle iu, \nabla u\rangle\rangle\\ &\quad - 2\langle\omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u)\end{aligned}$$

$$= |\nabla^2 u|^2 - 2\langle \omega, \psi(u)\rangle + \frac{1}{\epsilon^2}|\langle iu, \nabla u\rangle|^2$$
$$- \frac{1}{2\epsilon^2}(1-|u|^2)|\nabla u|^2 + \frac{1}{\epsilon^2}|\langle u, \nabla u\rangle|^2 + \mathcal{R}_1(\nabla u, \nabla u)$$
$$= |\nabla^2 u|^2 + \frac{1}{2\epsilon^2}(3|u|^2 - 1)|\nabla u|^2 - 2\langle \omega, \psi(u)\rangle + \mathcal{R}_1(\nabla u, \nabla u),$$

where at $p \in M$ we let $\mathcal{R}_1(\nabla u, \nabla u) = \mathrm{Ric}(e_i, e_j)\langle \nabla_{e_i} u, \nabla_{e_j} u\rangle$ and $\nabla^2_{e_i,e_j} u = \nabla_{e_i}(\nabla_{e_j} u)$, for any local orthonormal frame $\{e_i\}_{i=1}^n$ with $De_i(p) = 0$.

Next, we introduce the function

$$\xi_\epsilon := \epsilon|F_\nabla| - \frac{1}{2\epsilon}(1-|u|^2), \tag{3.5}$$

and combine (3.3) with (3.4) to see that

$$\Delta\xi_\epsilon \ge \epsilon^{-1}|u|^2|\omega| - \epsilon^{-1}|\nabla u|^2 - \epsilon|\mathcal{R}_2^-||\omega| + \epsilon^{-1}|\nabla u|^2 - \frac{1}{2\epsilon^3}(1-|u|^2)|u|^2$$
$$\ge \epsilon^{-2}|u|^2\xi_\epsilon - \epsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|.$$

If $\mathcal{R}_2 > 0$, we can actually replace the term $-\epsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|$ with $c\epsilon|\omega|$, for some positive constant $c = c(M)$; from a simple application of the maximum principle, in this case we get $\xi_\epsilon \le 0$ everywhere on M, and consequently (cf. [21, Theorem III.8.1])

$$\epsilon^2|F_\nabla|^2 \le \frac{W(u)}{\epsilon^2} \text{ pointwise, provided } \mathcal{R}_2 > 0 \text{ on } M. \tag{3.6}$$

This balancing of the Yang–Mills and potential terms, which should be compared with Modica's gradient estimate in the asymptotic analysis of the Allen–Cahn equations (cf. [19, Proposition 3.3]), will play a key role in our analysis, allowing us to upgrade the obvious $(n-4)$-monotonicity typical of Yang–Mills–Higgs problems to the much stronger $(n-2)$-monotonicity $\frac{d}{dr}(r^{2-n}\int_{B_r} e_\epsilon(u, \nabla)) \ge 0$.

Remark 3.1 We remark that the analog of the identity $\Delta\xi_\epsilon \ge \epsilon^{-2}|u|^2\xi_\epsilon - \epsilon\|\mathcal{R}_2^-\|_{L^\infty}|\omega|$—and, consequently, the sharp $(n-2)$-monotonicity result—fails for choices of coupling constants other than those corresponding to the *self-dual* Yang–Mills–Higgs functionals considered here.

Without the positive curvature assumption, we may still employ the subequation

$$\Delta\xi_\epsilon \ge \frac{|u|^2}{\epsilon^2}\xi_\epsilon - C(M)\epsilon|F_\nabla|, \tag{3.7}$$

to obtain strong estimates for the positive part ξ_ϵ^+ of ξ_ϵ. To begin, denote by $G(x, y)$ the nonnegative Green's function for the Laplacian on M, unique up to additive constant, so that $\Delta_x G(x, y) = \frac{1}{\mathrm{vol}(M)} - \delta_y$, and set

$$h_\epsilon(x) := \int_M G(x, y)\epsilon|F_\nabla|(y)\,dy \geq 0, \tag{3.8}$$

so that

$$\Delta h_\epsilon(x) = \frac{1}{\mathrm{vol}(M)}\|\epsilon F_\nabla\|_{L^1} - \epsilon|F_\nabla|(x). \tag{3.9}$$

Taking C' to be the constant appearing in (3.7), for the difference $\xi_\epsilon - C'h_\epsilon$ we then have

$$\begin{aligned}\Delta(\xi_\epsilon - C'h_\epsilon) &\geq \frac{|u|^2}{\epsilon^2}(\xi_\epsilon - C'h_\epsilon) + C'\frac{|u|^2}{\epsilon^2}h_\epsilon - C'\frac{\|\epsilon F_\nabla\|_{L^1}}{\mathrm{vol(M)}}\\ &\geq \frac{|u|^2}{\epsilon^2}(\xi_\epsilon - C'h_\epsilon) - C'\frac{\|\epsilon F_\nabla\|_{L^1}}{\mathrm{vol}(M)}.\end{aligned} \tag{3.10}$$

Observe that the L^1 norm of $\xi_\epsilon - C'h_\epsilon$ is bounded by the energy:

$$\begin{aligned}\|\xi_\epsilon - C'h_\epsilon\|_{L^1} &\leq \|\xi_\epsilon\|_{L^1} + C(M)\|h_\epsilon\|_{L^1}\\ &\leq \|\xi_\epsilon\|_{L^1} + C(M)\|\epsilon F_\nabla\|_{L^1}\\ &\leq C(M)E_\epsilon(u, \nabla)^{1/2}.\end{aligned} \tag{3.11}$$

(Where the constant $C(M)$ may of course change from line to line.)

Integrating (3.10) against the positive part $\zeta := (\xi_\epsilon - C'h_\epsilon)^+$ and bounding $\|\epsilon F_\nabla\|_{L^1} \leq C(M)E_\epsilon(u, \nabla)^{1/2}$, we get

$$\begin{aligned}\int_M |d\zeta|^2 &\leq -\int_M \frac{|u|^2}{\epsilon^2}\zeta^2 - C(M)E_\epsilon(u, \nabla)^{1/2}\int_M \zeta\\ &\leq -C(M)E_\epsilon(u, \nabla)^{1/2}\int_M \zeta.\end{aligned}$$

Applying (3.11), this gives $\|d\zeta\|_{L^2} \leq C(M)E_\epsilon(u, \nabla)$.

Thus, applying Moser iteration, namely integrating (3.10) against powers ζ^γ with increasing exponents $\gamma > 1$, we deduce that

$$\xi_\epsilon - C'h_\epsilon \leq \zeta \leq C(M)E_\epsilon(u, \nabla)^{1/2}. \tag{3.12}$$

As a simple application of (3.12), we note that by definition (3.8) of h_ϵ and the standard estimate (see, e.g., [7, Section 4.2])

$$G(x, y) \le C(M) d(x, y)^{2-n}$$

if $n \ge 3$ (or $G(x, y) \le -C(M)\log(d(x, y)) + C(M)$ if $n = 2$), we have the L^∞ estimate

$$\|h_\epsilon\|_{L^\infty} \le C(M)\|\epsilon F_\nabla\|_{L^{n-1}}$$

(with 2 replacing $n-1$ when $n = 2$). If $n = 2$, this inequality and (3.12) give a pointwise bound

$$\|\xi_\epsilon^+\|_{L^\infty} \le C(M)\|\epsilon F_\nabla\|_{L^2} + C(M)E_\epsilon(u, \nabla)^{1/2} \le C(M)E_\epsilon(u, \nabla)^{1/2}.$$

In the sequel, we assume $n \ge 3$ and aim for a similar pointwise bound. We have

$$\|h_\epsilon\|_{L^\infty} \le C(M)\|\epsilon F_\nabla\|_{L^{n-1}} \le C\epsilon \|F_\nabla\|_{L^\infty}^{\frac{n-3}{n-1}} \|F_\nabla\|_{L^2}^{\frac{2}{n-1}}.$$

Using this in (3.12), we compute at a maximum point for $|F_\nabla|$ to see that

$$\|\epsilon F_\nabla\|_{L^\infty} - \frac{1}{2\epsilon}(1-|u|^2) = \xi_\epsilon \le C\|\epsilon F_\nabla\|_{L^\infty}^{\frac{n-3}{n-1}} E_\epsilon(u, \nabla)^{\frac{1}{n-1}} + CE_\epsilon(u, \nabla)^{1/2},$$

and, by an application of Young's inequality, it follows that

$$(1 - C\delta)\|\epsilon F_\nabla\|_{L^\infty} \le \frac{1}{2\epsilon} + C\delta^{\frac{3-n}{2}} E_\epsilon(u, \nabla)^{1/2}$$

for any $\delta \in (0, 1)$. Taking $\delta = \epsilon^{2/n}$, we arrive at the crude preliminary estimate

$$\begin{aligned}\|\epsilon F_\nabla\|_{L^\infty} &\le \frac{1}{1 - C\epsilon^{2/n}}\left(\frac{1}{2\epsilon} + C\epsilon^{3/n}\epsilon^{-1}E_\epsilon(u, \nabla)^{1/2}\right)\\ &\le \frac{1}{2\epsilon} + \frac{\alpha(\epsilon)}{2\epsilon}(1 + E_\epsilon(u, \nabla)^{1/2}),\end{aligned}$$

where $\alpha(\epsilon) \to 0$ as $\epsilon \to 0$.

Now, consider the function

$$f := \epsilon|\omega| - \frac{1 + \alpha(\epsilon)(1 + E_\epsilon(u, \nabla)^{1/2})}{2\epsilon}(1 - |u|^2).$$

By virtue of the preceding estimate for $\|F_\nabla\|_{L^\infty}$, we then see that

$$f \leq \frac{1+\alpha(\epsilon)(1+E_\epsilon(u,\nabla)^{1/2})}{2\epsilon}|u|^2$$

pointwise. Appealing once again to (3.4) and (3.3), we see that

$$\Delta f \geq \frac{|u|^2}{\epsilon^2} f - C\epsilon|F_\nabla|,$$

so at a point where f achieves its maximum we have

$$\frac{|u|^2}{\epsilon^2} f \leq C\epsilon|F_\nabla| \leq \frac{C(1+E_\epsilon(u,\nabla)^{1/2})}{\epsilon}.$$

On the other hand, we know that $|u|^2 \geq \frac{\epsilon}{C(1+E_\epsilon(u,\nabla)^{1/2})} f$ everywhere, so the preceding computations yield an estimate of the form

$$\frac{(\max f)^2}{\epsilon} \leq \frac{C(M, E_\epsilon(u,\nabla))}{\epsilon},$$

provided $\max f \geq 0$, and we deduce that $f \leq C(M, E_\epsilon(u,\nabla))$ everywhere. Putting all this together, we arrive at the following lemma.

Lemma 3.2 *Let* (u,∇) *solve* (2.4) *and* (2.5) *on a line bundle* $L \to M$*, and suppose* $E_\epsilon(u,\nabla) \leq \Lambda$*. Then there exist a constant* $C(M,\Lambda)$ *and a function* $\alpha(M,\Lambda,\epsilon)$*, with* $\alpha(\epsilon) \to 0$ *as* $\epsilon \to 0$*, such that*

$$\xi_\epsilon \leq \alpha(\epsilon)\frac{(1-|u|^2)}{\epsilon} + C. \tag{3.13}$$

In the next section, we will improve the rough preliminary estimate of Lemma 3.2 to a uniform pointwise bound of the form $\xi_\epsilon \leq C(M,\Lambda)$, but this will require some additional effort.

4 Inner variations and improved monotonicity

In this section, we derive the inner variation equation for solutions of (2.4)–(2.5), and explore the scaling properties of the energy $E_\epsilon(u_\epsilon,\nabla_\epsilon)$ over balls of small radius. Under the assumption that the curvature operator $\mathcal{R}_2$ appearing in (3.3) is positive-definite (so that (3.6) holds), the analysis simplifies considerably, leading with little effort to the desired monotonicity of the $(n-2)$-energy density. Without this curvature assumption, more work is required, first building on the cruder estimates of the preceding section to obtain a uniform pointwise bound for ξ_ϵ.

Fixing notation, with respect to a local orthonormal basis $\{e_i\}$ for TM, define the (0, 2)-tensors $\nabla u^*\nabla u$ and $\omega^*\omega$ by

$$(\nabla u^*\nabla u)(e_i, e_j) := \langle \nabla_{e_i} u, \nabla_{e_j} u\rangle, \tag{4.1}$$

$$\omega^*\omega(e_i, e_j) := \sum_{k=1}^{n} \omega(e_i, e_k)\omega(e_j, e_k). \tag{4.2}$$

Note that $\mathrm{tr}(\nabla u^*\nabla u) = |\nabla u|^2$ and $\mathrm{tr}(\omega^*\omega) = 2|\omega|^2$. Denote by $e_\epsilon(u, \nabla)$ the energy integrand

$$e_\epsilon(u, \nabla) := |\nabla u|^2 + \epsilon^2 |F_\nabla|^2 + \frac{W(u)}{\epsilon^2}.$$

The fact that $d\omega = 0$ reads

$$D\omega(e_i, e_j) = D_{e_i}\omega(\cdot, e_j) + D_{e_j}\omega(e_i, \cdot),$$

where D is the Levi–Civita connection of M. Using this identity, it is straightforward to check that

$$\begin{aligned} de_\epsilon(u, \nabla) &= 2\,\mathrm{div}(\nabla u^*\nabla u) + 2\langle \nabla u, \nabla^*\nabla u\rangle + d\frac{W(u)}{\epsilon^2} \\ &\quad + 2\omega(\langle iu, \nabla u\rangle^\#, \cdot) + 2\epsilon^2\,\mathrm{div}(\omega^*\omega) - 2\epsilon^2\omega((d^*\omega)^\#, \cdot). \end{aligned}$$

In particular, defining the stress-energy tensor $T_\epsilon(u, \nabla)$ by

$$T_\epsilon(u, \nabla) := e_\epsilon(u, \nabla)g - 2\nabla u^*\nabla u - 2\epsilon^2\omega^*\omega, \tag{4.3}$$

for (u, ∇) solving (2.4) and (2.5) it follows that

$$\mathrm{div}(T_\epsilon(u, \nabla)) = 0, \tag{4.4}$$

meaning that $\sum_i (D_{e_i} T_\epsilon)(e_i, \cdot) = 0$. Integrating (4.4) against a vector field X on some domain $\Omega \subseteq M$, we arrive at the usual inner-variation equation

$$\int_\Omega \langle T_\epsilon(u, \nabla), DX\rangle = \int_{\partial\Omega} T_\epsilon(u, \nabla)(X, \nu), \tag{4.5}$$

where we identify $T_\epsilon(u, \nabla)$ with a (1, 1)-tensor and denote by ν the outer unit normal to Ω. Taking $\Omega = B_r(p)$ to be a small geodesic ball of radius r about a point $p \in M$, and taking $X = \text{grad}(\frac{1}{2}d_p^2)$, where d_p is the distance function to p, (4.5) gives

$$
\begin{aligned}
r\int_{\partial B_r(p)} & (e_\epsilon(u,\nabla) - 2|\nabla_\nu u|^2 - 2\epsilon^2|\iota_\nu\omega|^2) \\
&= \int_{B_r(p)} \langle T_\epsilon(u,\nabla), DX\rangle \\
&= \int_{B_r(p)} \langle T_\epsilon(u), g\rangle + \int_{B_r(p)} \langle T_\epsilon(u), DX - g\rangle \\
&= \int_{B_r(p)} (ne_\epsilon(u,\nabla) - 2|\nabla u|^2 - 4\epsilon^2|F_\nabla|^2) \\
&\quad + \int_{B_r(p)} \langle T_\epsilon(u), DX - g\rangle.
\end{aligned}
$$

Now, by the Hessian comparison theorem, we know that

$$|DX - g| \le C(M)d_p^2;$$

applying this in the relations above, we see that

$$
\begin{aligned}
r\int_{\partial B_r(p)} e_\epsilon(u,\nabla) \ge 2r & \int_{\partial B_r(p)} (|\nabla_\nu u|^2 + \epsilon^2|\iota_\nu\omega|^2) \\
& + \int_{B_r(p)} \Big((n-2)|\nabla u|^2 + (n-4)\epsilon^2|F_\nabla|^2 + n\frac{W(u)}{\epsilon^2}\Big) \\
& - C'(M)r^2\int_{B_r(p)} e_\epsilon(u,\nabla).
\end{aligned}
$$

Setting

$$f(p,r) := e^{C'r^2}\int_{B_r(p)} e_\epsilon(u,\nabla), \tag{4.6}$$

it follows from the computations above (temporarily throwing out the additional nonnegative boundary terms) that

$$\frac{\partial f}{\partial r} \ge \frac{e^{C'r^2}}{r}\int_{B_r(p)} \Big((n-2)|\nabla u|^2 + (n-4)\epsilon^2|F_\nabla|^2 + n\frac{W(u)}{\epsilon^2}\Big). \tag{4.7}$$

At this point, one easily observes that the right-hand side of (4.7) is bounded below by $\frac{n-4}{r}f(p,r)$, to obtain the monotonicity of the $(n-4)$-energy density

$$\frac{\partial}{\partial r}(r^{4-n}f(p,r)) \ge 0.$$

For general Yang–Mills and Yang–Mills–Higgs problems, this codimension-four energy growth is well known to be sharp (cf., e.g., [32,40]). For solutions of (2.4) and (2.5) on Hermitian line bundles, however, we show now that this can be improved to (near-) monotonicity of the $(n-2)$-density $r^{2-n}f(p,r)$ on small balls, which constitutes a key technical ingredient in the proof of Theorem 1.1.

To begin, we rearrange (4.7), to see that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq \frac{n-2}{r}f(r)+\frac{2e^{C'r^2}}{r}\int_{B_r(p)}\Big(\frac{W(u)}{\epsilon^2}-\epsilon^2|F_\nabla|^2\Big)\\ &=\frac{n-2}{r}f(r)-\frac{2e^{C'r^2}}{r}\int_{B_r(p)}\xi_\epsilon\Big(\epsilon|F_\nabla|+\frac{1}{2\epsilon}(1-|u|^2)\Big),\end{aligned}$$

recalling the notation $\xi_\epsilon := \epsilon|F_\nabla|-\frac{1}{2\epsilon}(1-|u|^2)$. Now, by Lemma 3.2, assuming $E_\epsilon(u,\nabla)\leq\Lambda$, we have the pointwise bound

$$\begin{aligned}\xi_\epsilon\Big(\epsilon|F_\nabla|+\frac{1}{2\epsilon}(1-|u|^2)\Big) &\leq 2\Big(C+\alpha(\epsilon)\frac{1-|u|^2}{\epsilon}\Big)e_\epsilon(u,\nabla)^{1/2}\\ &\leq Ce_\epsilon(u,\nabla)^{1/2}+C\alpha(\epsilon)e_\epsilon(u,\nabla).\end{aligned}$$

Applying this in our preceding computation for $\frac{\partial f}{\partial r}$, we deduce that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq \frac{n-2}{r}f(r)-\frac{e^{C'r^2}}{r}\int_{B_r(p)}Ce_\epsilon(u,\nabla)^{1/2}-\alpha(\epsilon)\frac{e^{C'r^2}}{r}\int_{B_r(p)}Ce_\epsilon(u,\nabla)\\ &\geq \frac{n-2-C\alpha(\epsilon)}{r}f(r)-\frac{e^{C'r^2}}{r}Cr^{n/2}\Big(\int_{B_r(p)}e_\epsilon(u,\nabla)\Big)^{1/2}\\ &\geq \frac{n-2-C''\alpha(\epsilon)}{r}f(r)-C''r^{n/2-1}f(r)^{1/2}\end{aligned}$$

for some constant $C''(M,\Lambda)$ and $0<r<c(M)$. Taking ϵ sufficiently small, we arrive next at the following coarse estimate for the $(n-3)$-energy density, which we will then use to establish an improved bound for ξ_ϵ.

Lemma 4.1 *For $\epsilon\leq\epsilon_m(M,\Lambda)$ sufficiently small, we have a uniform bound*

$$\sup_{0<r<\mathrm{inj}(M)} r^{3-n}\int_{B_r(p)}e_\epsilon(u,\nabla)\leq C(M,\Lambda). \tag{4.8}$$

Proof The statement is trivial if $n = 2, 3$, so assume $n \geq 4$. In the preceding computation, take $\epsilon \leq \epsilon_m(M, \Lambda)$ sufficiently small that $C''\alpha(\epsilon) < \frac{1}{2}$. Then the estimate gives

$$f'(r) \geq \frac{n-2-1/2}{r} f(r) - C'' r^{n/2-1} f(r)^{1/2},$$

from which it follows that, for $0 < r < c(M)$,

$$\begin{aligned} \frac{d}{dr}(r^{3-n} f(r)) &\geq r^{3-n} f'(r) + (3-n) r^{2-n} f(r) \\ &\geq r^{2-n} \Big(\Big(n - \frac{5}{2} \Big) f(r) - C r^{n/2} f(r)^{1/2} + (3-n) f(r) \Big) \\ &\geq r^{2-n} \Big(\frac{1}{2} f(r) - C r^{n/2} f(r)^{1/2} \Big). \end{aligned}$$

If $r^{3-n} f(r)$ has a maximum in $(0, c(M))$, it follows that $f(r) \leq C r^{n/2} f(r)^{1/2}$ there, and therefore $r^{3-n} f(r) \leq C r^3 \leq C$. Obviously the desired estimate holds at $r = 0$ and $r = c(M)$, so (4.8) follows. □

With Lemma 4.1 in hand, we can now improve the bounds of Lemma 3.2 to a uniform pointwise estimate, as follows.

Proposition 4.2 *Let (u, ∇) solve (2.4)–(2.5) on a line bundle $L \to M$, with the energy bound $E_\epsilon(u, \nabla) \leq \Lambda$ and $\epsilon \leq \epsilon_m$. Then there is a constant $C(M, \Lambda)$ such that*

$$\xi_\epsilon := \epsilon |F_\nabla| - \frac{1}{2\epsilon}(1 - |u|^2) \leq C(M, \Lambda). \tag{4.9}$$

Proof We can assume $n \geq 3$, as we already obtained the claim for $n = 2$ in Sect. 3. Recall from that section the function

$$h_\epsilon(x) := \int_M G(x, y) \epsilon |F_\nabla|(y)\, dy,$$

where G is the nonnegative Green's function on M. As discussed in Sect. 3, we can deduce from (3.7) a pointwise estimate of the form

$$\xi_\epsilon \leq C(M) h_\epsilon + C(M) E_\epsilon(u, \nabla)^{1/2}. \tag{4.10}$$

Thus, to arrive at the desired bound (4.9), it will suffice to establish a pointwise bound of the same form for h_ϵ.

To this end, recall again that $G(x, y) \leq C(M) d(x, y)^{2-n}$, so that by definition we have

$$\begin{aligned}h_\epsilon(x) &\le C\int_M d(x,y)^{2-n}\epsilon|F_\nabla|(y)\,dy\\&\le C\int_M d(x,y)^{2-n}e_\epsilon(u,\nabla)^{1/2}(y)\,dy\\&\le C\int_M (d(x,y)^{-n+1/2}+d(x,y)^{3-n+1/2}e_\epsilon(u,\nabla))\,dy,\end{aligned}$$

where the last line is a simple application of Young's inequality. Since the integral $\int_M d(x,y)^{-n+1/2}\,dy$ is finite, it follows that

$$\begin{aligned}h_\epsilon(x) \le C(M)+C(M)\Lambda+C(M)&\int_0^{\mathrm{inj}(M)} r^{3-n+1/2}\Big(\int_{\partial B_r(x)} e_\epsilon(u,\nabla)\Big)\,dr\\=C(M,\Lambda)+C(M)&\int_0^{\mathrm{inj}(M)}\frac{d}{dr}\Big(r^{-n+7/2}\int_{B_r(x)} e_\epsilon(u,\nabla)\Big)\,dr\\+(n-7/2)C(M)&\int_0^{\mathrm{inj}(M)} r^{3-n-1/2}\Big(\int_{B_r(x)} e_\epsilon(u,\nabla)\Big)\,dr\\\le C(M,\Lambda)+C(M)&\int_0^{\mathrm{inj}(M)} r^{3-n-1/2}\Big(\int_{B_r(x)} e_\epsilon(u,\nabla)\Big)\,dr.\end{aligned}$$

On the other hand, by Lemma 4.1, we know that $r^{3-n}\int_{B_r(x)} e_\epsilon(u,\nabla)\le C(M,\Lambda)$ for every r, so we see finally that

$$h_\epsilon(x)\le C(M,\Lambda)+C(M,\Lambda)\int_0^{\mathrm{inj}(M)} r^{-1/2}\,dr\le C(M,\Lambda),$$

as desired. □

Applying (4.9) in our original computation for $f'(r)$, we see now that

$$\begin{aligned}\frac{\partial f}{\partial r}&\ge\frac{n-2}{r}f(r)-\frac{2e^{C'r^2}}{r}\int_{B_r(p)}\xi_\epsilon\Big(\epsilon|F_\nabla|+\frac{1}{2\epsilon}(1-|u|^2)\Big)\\&\ge\frac{n-2}{r}f(r)-\frac{2e^{C'r^2}}{r}\int_{B_r(p)}C(M,\Lambda)e_\epsilon(u,\nabla)^{1/2}\\&\ge\frac{n-2}{r}f(r)-C(M,\Lambda)r^{\frac{n-2}{2}}f(r)^{1/2}.\end{aligned}$$

In fact, bringing in the extra boundary terms that we have been neglecting, and applying Young's inequality to the term $r^{\frac{n-2}{2}}f(r)^{1/2}$, we see that

$$\begin{aligned}\frac{\partial f}{\partial r} &\geq 2e^{C'r^2}\int_{\partial B_r(p)}(|\nabla_\nu u|^2+\epsilon^2|\iota_\nu F_\nabla|^2)\\ &\quad+\frac{n-2}{r}f(r)-Cr^{\frac{n-2}{2}}f(r)^{1/2}\\ &\geq 2e^{C'r^2}\int_{\partial B_r(p)}(|\nabla_\nu u|^2+\epsilon^2|\iota_\nu F_\nabla|^2)\\ &\quad+\frac{n-2}{r}f(r)-Cf(r)-Cr^{n-2}.\end{aligned}$$

With this differential inequality in place, a straightforward computation leads us finally to one of our key technical theorems, the monotonicity formula for the $(n-2)$-density.

Theorem 4.3 *Let (u,∇) solve (2.4)–(2.5) on a Hermitian line bundle $L\to M$, with an energy bound $E_\epsilon(u,\nabla)\leq\Lambda$. Then there exist positive constants $\epsilon_m(M,\Lambda)$ and $C_m(M,\Lambda)$ such that the normalized energy density*

$$\widetilde{E}_\epsilon(x,r):=e^{C_m r}r^{2-n}\int_{B_r(x)}e_\epsilon(u,\nabla)\tag{4.11}$$

satisfies

$$\widetilde{E}_\epsilon'(r)\geq 2r^{2-n}\int_{\partial B_r(x)}(|\nabla_\nu u|^2+\epsilon^2|\iota_\nu F_\nabla|^2)-C_m,\tag{4.12}$$

for $0<r<\mathrm{inj}(M)$ and $\epsilon\leq\epsilon_m$.

As a simple corollary of the monotonicity result (together with a pointwise bound for $|\nabla u|$ derived in the following section), we deduce that (u,∇) must have positive $(n-2)$-energy density wherever $|u|$ is bounded away from 1.

Corollary 4.4 (clearing-out) *Let (u,∇) solve (2.4)–(2.5) on a line bundle $L\to M$, with $E_\epsilon(u,\nabla)\leq\Lambda$ and $\epsilon\leq\epsilon_m$. Given $0<\delta<1$, if*

$$r^{2-n}\int_{B_r(x)}e_\epsilon(u,\nabla)\leq\eta(M,\Lambda,\delta)$$

with $x\in M$ and $\epsilon<r<\mathrm{inj}(M)$, then we must have $|u(x)|>1-\delta$.

Proof For $\epsilon\leq\epsilon_m$, Theorem 4.3 gives

$$\epsilon^{2-n}\int_{B_\epsilon(x)}e_\epsilon(u,\nabla)\leq C(M,\Lambda)\eta+C(M,\Lambda)r.$$

The gradient bound (5.3) in Proposition 5.1 of the following section gives $|d|u||\leq C\epsilon^{-1}$. Hence, if $|u(x)|\leq 1-\delta$ then $|u(y)|<1-\frac{\delta}{2}$ on $B_{\epsilon\delta/(2C)}(x)$, so that $1-|u(y)|^2 \geq 1-|u(y)| > \frac{\delta}{2}$. We deduce that

$$\frac{\delta^2}{16}\operatorname{vol}(B_{\epsilon\delta/(2C)}(x)) \leq \int_{B_\epsilon(x)} W(u) \leq \epsilon^2 \int_{B_\epsilon(x)} e_\epsilon(u,\nabla) \leq C\epsilon^n(\eta+r).$$

Since $\operatorname{vol}(B_{\epsilon\delta/(2C)}(x))$ is bounded below by $c(M,\Lambda,\delta)\epsilon^n$, we can choose $\widetilde{\eta}(M,\Lambda,\delta) \leq \operatorname{inj}(M)$ so small that we get a contradiction if $r,\eta \leq \widetilde{\eta}$. On the other hand, if $r > \widetilde{\eta}$ then

$$\widetilde{\eta}^{2-n}\int_{B_{\widetilde{\eta}}(x)} e_\epsilon(u,\nabla) \leq \left(\frac{\operatorname{inj}(M)}{\widetilde{\eta}}\right)^{n-2}\eta.$$

Hence, setting $\eta := \left(\frac{\widetilde{\eta}}{\operatorname{inj}(M)}\right)^{n-2}\widetilde{\eta} \leq \widetilde{\eta}$, we can reduce to the previous case (replacing r with $\widetilde{\eta}$), reaching again a contradiction. □

5 Decay away from the zero set

Again, let (u,∇) solve (2.4)–(2.5) on a line bundle $L \to M$, with the energy bound $E_\epsilon(u,\nabla) \leq \Lambda$. In the preceding section, we obtained the pointwise estimate

$$|F_\nabla| \leq \frac{1}{2\epsilon^2}(1-|u|^2) + \frac{1}{\epsilon}C(M,\Lambda) \tag{5.1}$$

when $\epsilon \leq \epsilon_m$. As a first step toward establishing strong decay of the energy away from the zero set of u, we show in the following proposition that the full energy density $e_\epsilon(u,\nabla)$ is controlled by the potential $\frac{W(u)}{\epsilon^2}$.

Proposition 5.1 *For (u,∇) as above, we have the pointwise estimates*

$$\epsilon^2|F_\nabla|^2 \leq C(M,\Lambda)\frac{W(u)}{\epsilon^2} + C(M,\Lambda)\epsilon \tag{5.2}$$

and

$$|\nabla u|^2 \leq C(M,\Lambda)\frac{W(u)}{\epsilon^2} + C(M,\Lambda)\epsilon^2, \tag{5.3}$$

provided $\epsilon \leq \epsilon_d$, for some $\epsilon_d = \epsilon_d(M,\Lambda)$.

Proof To begin, let $C_1 = C_1(M,\Lambda)$ be the constant from (5.1), and consider the function

$$f := \epsilon|F_\nabla| - \frac{1+2C_1\epsilon}{2\epsilon}(1-|u|^2) = \xi_\epsilon - C_1 + C_1|u|^2.$$

Similar to the proof of Lemma 3.2, observe that $C_1|u|^2 \geq f$ pointwise, by (5.1), while the computations from Sect. 3 give

$$\Delta f \geq \frac{|u|^2}{\epsilon^2} f - C'(M)\epsilon|F_\nabla|.$$

By (5.1) we have $|F_\nabla| \leq \frac{1}{2\epsilon^2} + \frac{C_1}{\epsilon}$, so at a positive maximum for f it follows that

$$0 \geq \frac{|u|^2}{\epsilon^2} f - C'\epsilon|F_\nabla| \geq \frac{f^2}{C_1\epsilon^2} - \frac{C(M,\Lambda)}{\epsilon},$$

so that

$$(\max f)^2 \leq C\epsilon$$

(provided $\max f \geq 0$), and consequently $f \leq C\epsilon^{1/2}$ everywhere. As a consequence, at any point, we have either $f < 0$, in which case

$$\epsilon^2|F_\nabla|^2 \leq (1+2C_1\epsilon)^2 \frac{W(u)}{\epsilon^2},$$

or $f \geq 0$, in which case

$$\begin{aligned}\epsilon^2|F_\nabla|^2 &\leq 2f^2 + 2(1+2C_1\epsilon)^2 \frac{W(u)}{\epsilon^2} \\ &\leq C\epsilon + 2(1+2C_1\epsilon)^2 \frac{W(u)}{\epsilon^2}.\end{aligned}$$

In either scenario, we obtain a bound of the desired form (5.2).

To bound $|\nabla u|^2$, recall from Sect. 3 the identity

$$\Delta\frac{1}{2}|\nabla u|^2 = |\nabla^2 u|^2 + \frac{1}{2\epsilon^2}(3|u|^2-1)|\nabla u|^2 - 2\langle\omega,\psi(u)\rangle + \mathcal{R}_1(\nabla u,\nabla u). \tag{5.4}$$

In view of the estimate (5.1) for $|F_\nabla| = |\omega|$ and (2.7), we can estimate the term $2\langle\omega,\psi(u)\rangle$ from above by

$$2|F_\nabla||\nabla u|^2 \leq \frac{1}{\epsilon^2}(1-|u|^2)|\nabla u|^2 + \frac{C}{\epsilon}|\nabla u|^2,$$

to obtain the existence of $C_2(M,\Lambda)$ such that

$$\Delta\frac{1}{2}|\nabla u|^2 \geq |\nabla^2 u|^2 + \frac{1}{2\epsilon^2}(5|u|^2-3)|\nabla u|^2 - \frac{C_2}{\epsilon}|\nabla u|^2.$$

For $\Delta|\nabla u|$, this then gives

$$\Delta|\nabla u| \geq \frac{1}{2\epsilon^2}(5|u|^2 - 3)|\nabla u| - \frac{C_2}{\epsilon}|\nabla u|. \tag{5.5}$$

Recalling once again the Eq. (3.4) for $\Delta\frac{1}{2}|u|^2$, we define

$$w := |\nabla u| - \frac{1}{\epsilon}(1 - |u|^2),$$

and observe that

$$\begin{aligned}\Delta w &\geq \frac{1}{2\epsilon^2}(5|u|^2 - 3)|\nabla u| - \frac{C_2}{\epsilon}|\nabla u| \\ &\quad + \frac{2}{\epsilon}|\nabla u|^2 - \frac{1}{\epsilon^3}|u|^2(1 - |u|^2) \\ &= \frac{|u|^2}{\epsilon^2}w + |\nabla u|\Big(\frac{2}{\epsilon}|\nabla u| - \frac{3}{2}\frac{(1 - |u|^2)}{\epsilon^2} - \frac{C_2}{\epsilon}\Big) \\ &= \frac{|u|^2}{\epsilon^2}w + \frac{|\nabla u|}{\epsilon}\Big(2w + \frac{1}{2\epsilon}(1 - |u|^2) - C_2\Big).\end{aligned}$$

We then have

$$\Delta w \geq \frac{|u|^2}{\epsilon^2}w + \frac{1}{\epsilon}\Big(w + \frac{1}{\epsilon}(1 - |u|^2)\Big)\Big(2w + \frac{1}{2\epsilon}(1 - |u|^2) - C_2\Big). \tag{5.6}$$

If w has a positive maximum, it follows that

$$2w + \frac{1}{2\epsilon}(1 - |u|^2) \leq C_2$$

at this maximum point; in particular, we deduce then that

$$|u|^2 \geq 1 - 2C_2\epsilon$$

at this point, and see from (5.6) that here

$$0 \geq \frac{1 - 2C_2\epsilon}{\epsilon^2}w - \frac{1}{\epsilon}\Big(w + \frac{1}{\epsilon}(1 - |u|^2)\Big)C_2 \geq \frac{1 - 3C_2\epsilon}{\epsilon^2}w - 2\frac{C_2^2}{\epsilon}.$$

If $\epsilon \leq \epsilon_d(M, \Lambda)$ is small enough, it follows that $\max w \leq C\epsilon$; as a consequence, we check that

$$|\nabla u|^2 \leq C\frac{W(u)}{\epsilon^2} + C\epsilon^2,$$

completing the proof of (5.3). □

As a simple consequence of the estimates in Proposition 5.1, we obtain the following corollary.

Corollary 5.2 *There exist constants* $0 < \beta_d(M, \Lambda) < 1$ *and* $C(M, \Lambda)$ *such that, for* (u, ∇) *as above, we have*

$$\Delta \frac{1}{2}(1 - |u|^2) \geq \frac{1}{4\epsilon^2}(1 - |u|^2) - C\epsilon^2 \tag{5.7}$$

on the set $Z_{\beta_d}(u) := \{|u|^2 \geq 1 - \beta_d\}$.

Proof By the formula (3.4) for $\Delta \frac{1}{2}|u|^2$, we know that

$$\Delta \frac{1}{2}(1 - |u|^2) = \frac{1}{2\epsilon^2}|u|^2(1 - |u|^2) - |\nabla u|^2.$$

Combining this with the estimate (5.3) for $|\nabla u|^2$, we then deduce the existence of a constant $\widehat{C} = \widehat{C}(M, \Lambda)$ such that

$$\Delta \frac{1}{2}(1 - |u|^2) \geq |u|^2 \frac{1}{2\epsilon^2}(1 - |u|^2) - \widehat{C}\frac{(1 - |u|^2)^2}{2\epsilon^2} - C\epsilon^2.$$

By taking $\beta_d = \beta_d(M, \Lambda) > 0$ sufficiently small, we can arrange that

$$|u|^2 - \widehat{C}(1 - |u|^2) \geq 1 - \beta_d - \widehat{C}\beta_d \geq \frac{1}{2}$$

on $\{|u|^2 \geq 1 - \beta_d\}$, from which the claimed estimate follows. □

Next, we employ the result of Corollary 5.2 to show that the quantity $(1 - |u|^2)$ vanishes rapidly away from $Z_{\beta_d}(u)$ (compare [21, Sections III.7–III.8]).

Proposition 5.3 *Let* (u, ∇) *be as before, with* $\epsilon \leq \epsilon_d$*, and define the set*

$$Z_{\beta_d} := \{x \in M : |u(x)|^2 \leq 1 - \beta_d\},$$

where $\beta_d(M, \Lambda)$ *is the constant provided by Corollary 5.2. Defining* $r : M \to [0, \infty)$ *by*

$$r(p) := \text{dist}(p, Z_\beta),$$

we have an estimate of the form

$$(1 - |u|^2)(p) \leq Ce^{-a_d r(p)/\epsilon} + C\epsilon^4 \tag{5.8}$$

for some $C = C(M, \Lambda)$ *and* $a_d = a_d(M) > 0$.

Proof Fix a point $p \in M$, and let $r = r(p) = \mathrm{dist}(p, Z_\beta)$ as above. We can clearly assume $r(p) < \frac{1}{2}\mathrm{inj}(M)$. On the ball $B_r(p)$, for some constant $a = a_d > 0$ to be chosen later, consider the function

$$\varphi(x) := e^{(a/\epsilon)(d_p(x)^2+\epsilon^2)^{1/2}},$$

where $d_p(x) := \mathrm{dist}(p, x)$. A straightforward computation then gives

$$\begin{aligned}\Delta\varphi &= \frac{a}{\epsilon}\varphi\left(\frac{(a/\epsilon)d_p^2}{d_p^2+\epsilon^2} - \frac{d_p^2}{(d_p^2+\epsilon^2)^{3/2}}\right)\\ &\quad + \frac{a}{2\epsilon}\varphi\frac{\Delta d_p^2}{(d_p^2+\epsilon^2)^{1/2}}\\ &\leq \frac{a^2}{\epsilon^2}\varphi + \frac{a}{2\epsilon}\varphi\frac{\Delta d_p^2}{(d_p^2+\epsilon^2)^{1/2}}\\ &\leq \frac{a^2+C_1 a}{\epsilon^2}\varphi\end{aligned}$$

for some $C_1 = C_1(M)$. Now, fix some constant $c_2 > 0$ to be chosen later, and let

$$f := \frac{1}{2}(1-|u|^2) - c_2\varphi.$$

Combining the preceding computation with (5.7), we see that, on $B_r(p)$,

$$\begin{aligned}\Delta f &\geq \frac{1}{4\epsilon^2}(1-|u|^2) - C(M,\Lambda)\epsilon^2 - \frac{a^2+C_1 a}{\epsilon^2}c_2\varphi\\ &= \frac{1}{2\epsilon^2}f + \frac{1-2a^2-2C_1 a}{2\epsilon^2}c_2\varphi - C(M,\Lambda)\epsilon^2.\end{aligned}$$

Choosing $a = a_d(M) > 0$ sufficiently small, we can arrange that $2a^2 + 2C_1 a \leq 1$, so that the above computation gives

$$\Delta f \geq \frac{f}{2\epsilon^2} - C\epsilon^2. \tag{5.9}$$

On the boundary of the ball $\partial B_r(p)$, it follows from definition of $r = r(p)$ that $|u|^2 \geq 1 - \beta_d$, and therefore

$$f(x) \leq \frac{\beta_d}{2} - c_2\varphi \leq \frac{\beta_d}{2} - c_2 e^{ar/\epsilon} \quad \text{on } \partial B_r(p).$$

Taking $c_2 := \beta_d e^{-ar/\epsilon}$, it then follows that $f < 0$ on $\partial B_r(p)$, so we can apply the maximum principle with (5.9) to deduce that

$$f \leq C\epsilon^4 \quad \text{in } B_r(p).$$

Evaluating at p, this gives

$$C\epsilon^4 \geq f(p) = \frac{1}{2}(1-|u|^2)(p) - \beta_d e^{-ar(p)/\epsilon} e^a,$$

so that

$$(1-|u|^2)(p) \leq C(M,\Lambda)e^{-ar(p)/\epsilon} + C(M,\Lambda)\epsilon^4,$$

as desired. □

Combining these estimates with those of Proposition 5.1, we arrive immediately at the following decay estimate for the energy integrand $e_\epsilon(u,\nabla)$.

Corollary 5.4 *Defining Z_{β_d} and $r(p) = \operatorname{dist}(p, Z_{\beta_d})$ as in Proposition 5.3, there exist $a_d(M) > 0$ and $C_d(M,\Lambda)$ such that*

$$e_\epsilon(u,\nabla)(p) \leq C_d \frac{e^{-a_d r(p)/\epsilon}}{\epsilon^2} + C_d\epsilon. \tag{5.10}$$

6 The energy-concentration varifold

This section is devoted to the proof of the main result of the paper, which we recall now.

Theorem 6.1 *Let $(u_\epsilon, \nabla_\epsilon)$ be a family of solutions to (2.4)–(2.5) satisfying a uniform energy bound $E_\epsilon(u_\epsilon,\nabla_\epsilon) \leq \Lambda$ as $\epsilon \to 0$. Then, as $\epsilon \to 0$, the energy measures*

$$\mu_\epsilon := \frac{1}{2\pi} e_\epsilon(u_\epsilon,\nabla_\epsilon)\ \mathrm{vol}_g$$

converge subsequentially, in duality with $C^0(M)$, to the weight measure of a stationary, integral $(n-2)$-varifold V. Also, for all $0 \leq \delta < 1$,

$$\operatorname{spt}(V) = \lim_{\epsilon\to 0}\{|u_\epsilon| \leq \delta\}$$

in the Hausdorff topology. The $(n-2)$-currents dual to the curvature forms $\frac{1}{2\pi}\omega_\epsilon$ converge subsequentially to an integral $(n-2)$-cycle Γ, with $|\Gamma| \leq \mu$.

6.1 Convergence to a stationary rectifiable varifold

Let $(u_\epsilon, \nabla_\epsilon)$ be as in Theorem 6.1, and pass to a subsequence $\epsilon_j \to 0$ such that the energy measures μ_{ϵ_j} converge weakly-* to a limiting measure μ, in duality with $C^0(M)$.

Note that, for $0 < r < R < \operatorname{inj}(M)$, Theorem 4.3 yields

$$\begin{aligned} e^{CR}R^{2-n}\mu(\overline{B}_R(x)) + CR &\geq \limsup_{\epsilon\to 0} e^{CR}R^{2-n}\mu_\epsilon(\overline{B}_R(x)) + CR \\ &\geq \liminf_{\epsilon\to 0} e^{Cr}r^{2-n}\mu_\epsilon(B_r(x)) + Cr \\ &\geq e^{Cr}r^{2-n}\mu(B_r(x)) + Cr \end{aligned}$$

with $C = C_m$, so approximating R with smaller radii we deduce

$$e^{CR}R^{2-n}\mu(B_R(x)) + CR \geq e^{Cr}r^{2-n}\mu(B_r(x)) + Cr, \tag{6.1}$$

and in particular the $(n-2)$-density

$$\Theta_{n-2}(\mu, x) := \lim_{r\to 0}(\omega_{n-2}r^{n-2})^{-1}\mu(B_r(x))$$

is defined. As a first step toward the proof of Theorem 6.1, we show that this density is bounded from above and below on the support $\operatorname{spt}(\mu)$.

Proposition 6.2 *There exists a constant* $0 < C = C(M, \Lambda) < \infty$ *such that*

$$C^{-1} \leq r^{2-n}\mu(B_r(x)) \leq C \quad \text{for } x \in \operatorname{spt}(\mu),\ 0 < r < \operatorname{inj}(M), \tag{6.2}$$

and thus $C^{-1} \leq \Theta_{n-2}(\mu, x) \leq C$ *for all* $x \in \operatorname{spt}(\mu)$.

Proof The upper bound follows from (6.1), which gives (when $R = \operatorname{inj}(M)$)

$$\begin{aligned} r^{2-n}\mu(B_r(x)) &\leq e^{C_m r}r^{2-n}\mu(B_r(x)) + C_m r \\ &\leq C(M, \Lambda)\mu(M) + C(M, \Lambda)\operatorname{inj}(M) \\ &\leq C(M, \Lambda). \end{aligned}$$

To see the lower bound, let $\beta_d = \beta_d(M, \Lambda) \in (0, 1)$ be the constant given by Corollary 5.4, and again set

$$Z_\beta(u_\epsilon) := \{x \in M : |u_\epsilon(x)|^2 \leq 1 - \beta\}.$$

Let Σ be the set of all limits $x = \lim_\epsilon x_\epsilon$, with $x_\epsilon \in Z_{\beta_d}(u_\epsilon)$; that is, take

$$\Sigma := \bigcap_{\eta>0} \overline{\bigcup_{0<\epsilon<\eta} Z_{\beta_d}(u_\epsilon)}.$$

We then claim that

$$\operatorname{spt}(\mu) \subseteq \Sigma \tag{6.3}$$

and

$$\mu(B_r(x)) \geq c(M, \Lambda) r^{n-2} \quad \text{for } x \in \Sigma,\ 0 < r < \operatorname{inj}(M). \tag{6.4}$$

Once both (6.3) and (6.4) are established, the lower bound in (6.2) follows immediately.

To establish (6.3), fix some $p \in M \setminus \Sigma$; by definition of Σ, there must exist $\delta = \delta(p) > 0$ such that

$$\operatorname{dist}(p, Z_{\beta_d}(u_\epsilon)) \geq 2\delta$$

for all ϵ sufficiently small. Applying Corollary 5.4 for all $x \in B_\delta(p)$, we deduce that

$$\begin{aligned}\mu(B_\delta(p)) &\leq \liminf_{\epsilon\to 0} \frac{1}{2\pi} \int_{B_\delta(p)} e_\epsilon(u_\epsilon, \nabla_\epsilon)\\ &\leq \lim_{\epsilon\to 0} \int_{B_\delta(p)} (C\epsilon^{-2} e^{-a\delta/\epsilon} + C\epsilon)\\ &= 0.\end{aligned}$$

In particular, $p \in M \setminus \operatorname{spt}(\mu)$, confirming (6.3).

To see (6.4), let $x \in \Sigma$. Note that, by definition of Σ, there exist points $x_\epsilon \in Z_{\beta_d}(u_\epsilon)$ with $x_\epsilon \to x$ as $\epsilon \to 0$ (along a subsequence). We then see that

$$|u_\epsilon(x_\epsilon)|^2 \leq 1 - \beta_d$$

and Corollary 4.4 gives $c(M, \Lambda)$ such that

$$\mu_\epsilon(B_r(x_\epsilon)) \geq c(M, \Lambda) r^{n-2}$$

for $\epsilon < r < \operatorname{inj}(M)$. Since for any $\delta > 0$ we have $B_r(x_\epsilon) \subseteq \overline{B}_{r+\delta}(x)$ eventually, it follows that $\mu(\overline{B}_{r+\delta}(x)) \geq cr^{n-2}$, hence

$$\mu(B_r(x)) \geq cr^{n-2}$$

for $0 < r < \operatorname{inj}(M)$, which is (6.4). □

With Proposition 6.2 in place, we will invoke a result by Ambrosio and Soner [6] to conclude that the limiting measure $\mu = \lim_{\epsilon\to 0}\mu_\epsilon$ coincides with the weight measure of a stationary, rectifiable $(n-2)$-varifold. Recall from Sect. 4 the stress-energy tensors

$$T_\epsilon = e_\epsilon(u_\epsilon, \nabla_\epsilon)g - 2\nabla_\epsilon u_\epsilon^* \nabla_\epsilon u_\epsilon - 2\epsilon^2 F_{\nabla_\epsilon}^* F_{\nabla_\epsilon}.$$

We record first the following lemma; in its statement, we canonically identify (and pair with each other) tensors of rank (2, 0), (1, 1), and (0, 2), using the underlying metric g.

Lemma 6.3 *As $\epsilon \to 0$, the tensors T_ϵ converge (subsequentially) as* $\mathrm{Sym}(TM)$*-valued measures, in duality with $C^0(M, \mathrm{Sym}(TM))$, to a limit T satisfying*

$$\langle T, DX\rangle = 0 \quad \textit{for all vector fields } X \in C^1(M, TM), \tag{6.5}$$

$$\frac{1}{2\pi}\langle T, \varphi g\rangle \geq (n-2)\langle \mu, \varphi\rangle \quad \textit{for every } 0 \leq \varphi \in C^0(M), \tag{6.6}$$

and

$$-\int_M |X|^2\, d\mu \leq \frac{1}{2\pi}\langle T, X\otimes X\rangle \leq \int_M |X|^2\, d\mu \quad \textit{for all } X \in C^0(M, TM). \tag{6.7}$$

Proof For each $\epsilon > 0$, note that, by definition of T_ϵ, for every continuous vector field $X \in C^0(M, TM)$ we have

$$\int_M \langle T_\epsilon, X\otimes X\rangle = \int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)|X|^2 - \int_M 2|(\nabla_\epsilon)_X u_\epsilon|^2 - \int_M 2\epsilon^2 |\iota_X F_{\nabla_\epsilon}|^2.$$

Evaluating (2.3) in an orthonormal basis such that X is a multiple of e_1, we see that $|\iota_X F_{\nabla_\epsilon}|^2 \leq |F_{\nabla_\epsilon}|^2|X|^2$, while $|(\nabla_\epsilon)_X u_\epsilon|^2 \leq |\nabla_\epsilon u_\epsilon|^2|X|^2$. We deduce that

$$-\int_M |X|^2 e_\epsilon(u_\epsilon, \nabla_\epsilon) \leq \int_M \langle T_\epsilon, X\otimes X\rangle \leq \int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)|X|^2. \tag{6.8}$$

As an immediate consequence, we see that the uniform energy bound $E_\epsilon(u_\epsilon, \nabla_\epsilon) \leq \Lambda$ gives a uniform bound on $\|T_\epsilon\|_{(C^0)^*}$ as $\epsilon \to 0$, so we can indeed extract a weak-* subsequential limit $T \in C^0(M, \mathrm{Sym}(TM))^*$, for which (6.7) follows from (6.8).

The stationarity condition (6.5) for the limit T follows from (4.5). It remains to establish the trace inequality (6.6). For this, we simply compute, for non-negative $\varphi \in C^0(M)$,

$$\begin{aligned}\int_M \langle T_\epsilon, \varphi g\rangle &= \int_M \varphi(ne_\epsilon(u_\epsilon, \nabla_\epsilon) - 2|\nabla_\epsilon u_\epsilon|^2 - 4\epsilon^2|F_{\nabla_\epsilon}|^2)\\ &= \int_M (n-2)\varphi e_\epsilon(u_\epsilon, \nabla_\epsilon) + 2\int_M \varphi\Big(\frac{W(u_\epsilon)}{\epsilon^2} - \epsilon^2|F_{\nabla_\epsilon}|^2\Big)\\ &\geq 2\pi(n-2)\langle \mu_\epsilon, \varphi\rangle - 4\pi\int_M \varphi e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2}\Big(\epsilon|F_{\nabla_\epsilon}| - \frac{(1-|u_\epsilon|^2)}{2\epsilon}\Big)^+.\end{aligned}$$

Recalling from Proposition 4.2 that

$$\epsilon|F_{\nabla_\epsilon}| - \frac{(1-|u_\epsilon|^2)}{2\epsilon} \leq C(M, \Lambda),$$

we then see that

$$\langle T, \varphi g\rangle = \lim_{\epsilon\to 0}\int_M \langle T_\epsilon, \varphi g\rangle \geq 2\pi(n-2)\langle \mu, \varphi\rangle - C\lim_{\epsilon\to 0}\int_M \varphi e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2}.$$

In particular, (6.6) will follow once we show that $\lim_{\epsilon\to 0}\int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} = 0$.

But this is straightforward: from Proposition 6.2 we know that for $0 < \delta < \mathrm{inj}(M)$ we have

$$\mu(B_\delta(x)) \geq c(M, \Lambda)\delta^{n-2} \quad \text{for } x \in \Sigma = \mathrm{spt}(\mu).$$

Since $\mathrm{vol}(B_{5\delta}(x)) \leq C(M)\delta^n$, a simple Vitali covering argument then implies that the δ-neighborhood $B_\delta(\Sigma)$ of Σ satisfies a volume bound

$$\mathrm{vol}(B_\delta(\Sigma)) \leq C(M, \Lambda)\delta^2.$$

With this estimate in hand, we then see that

$$\begin{aligned}\int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} &= \int_{B_\delta(\Sigma)} e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} + \int_{M\setminus B_\delta(\Sigma)} e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2}\\ &\leq \mathrm{vol}(B_\delta(\Sigma))^{1/2}\Lambda^{1/2} + C(M)\mu_\epsilon(M\setminus B_\delta(\Sigma))^{1/2}.\end{aligned}$$

Fixing δ and taking the limit as $\epsilon \to 0$, we have $\mu_\epsilon(M\setminus B_\delta(\Sigma)) \to 0$. Since $\mathrm{vol}(B_\delta(\Sigma)) \leq C\delta^2$, we find that

$$\limsup_{\epsilon\to 0}\int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} \leq C\delta\Lambda^{1/2}.$$

Finally, taking $\delta \to 0$, we conclude that $\int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} \to 0$ as $\epsilon \to 0$, completing the proof. □

Estimate (6.7) says that $|T|$ is absolutely continuous with respect to μ, so by the Radon–Nikodym theorem we can write the limiting $\mathrm{Sym}(TM)$-valued measure T from Lemma 6.3 as

$$\frac{1}{2\pi}\langle T, S\rangle = \int_M \langle P(x), S(x)\rangle \, d\mu(x) \tag{6.9}$$

for some L^∞ (with respect to μ) section $P : M \to \mathrm{Sym}(TM)$. Moreover, it follows from (6.6) and (6.7) that $-g \le P(x) \le g$ and $\mathrm{tr}(P(x)) \ge n-2$ at μ-a.e. $x \in M$, so that $\frac{1}{2\pi}T$ defines in a natural way a generalized $(n-2)$-varifold in the sense of Ambrosio and Soner, namely a Radon measure on the bundle

$$A_{n,n-2}(M) := \{S \in \mathrm{Sym}(TM) : -ng \le S \le g,\ \mathrm{tr}(S) \ge n-2\}. \tag{6.10}$$

We refer the reader to [6, Section 3]. Note that in [6] the authors work in the Euclidean space and require the trace to be equal to $n-2$ in (6.10); however, the main result on generalized varifolds, namely [6, Theorem 3.8], still holds in our setting. Indeed, in the proof of part (a) of that theorem, the condition $\sum_{i=1}^{m+1} \lambda_i = m$ that the authors obtain becomes $\sum_{i=1}^{m+1} \lambda_i \ge m$ in our setting (with $m = n-2$), and the constraint $\lambda_i \le 1$ still ensures the conclusion $\lambda_i \ge 0$ for all i. Similarly, for part (b), the condition $\sum_{i=1}^{m} \lambda_i = m$ has to be replaced by $\sum_{i=1}^{m} \lambda_i = m$, and this still implies $\lambda_i = 1$ for all $i = 1, \dots, m$.

Hence, in view of the stationarity condition (6.5) and the density bounds of Proposition 6.2, we can apply [6, Theorem 3.8(c)] to conclude that $\frac{1}{2\pi}T$ can be identified with a stationary, rectifiable $(n-2)$-varifold with weight measure μ (so, in particular, $\mathrm{spt}(\mu)$ is $(n-2)$-rectifiable), and that $P(x)$ is given μ-a.e. by the orthogonal projection onto the $(n-2)$-subspace $T_x\,\mathrm{spt}(\mu) \subset T_xM$. We collect this information in the following statement.

Proposition 6.4 *For a family $(u_\epsilon, \nabla_\epsilon)$ satisfying the hypotheses of Theorem 6.1, after passing to a subsequence, there exists a stationary, rectifiable $(n-2)$-varifold $V = v(\Sigma^{n-2}, \theta)$ such that*

$$\lim_{\epsilon \to 0} \frac{1}{2\pi} \int_M \langle T_\epsilon(u_\epsilon, \nabla_\epsilon), S\rangle = \int_\Sigma \theta(x)\langle T_x\Sigma, S(x)\rangle \, d\mathcal{H}^{n-2} \tag{6.11}$$

for every continuous section $S \in C^0(M, \mathrm{Sym}(TM))$. The energy measure μ is given by $\mu = \theta\mathcal{H}^{n-2} \llcorner \Sigma$. Also, we can choose $\Sigma := \mathrm{spt}(\mu)$ and $\theta(x) := \Theta_{n-2}(\mu, x)$.

6.2 Integrality of the limit varifold and convergence of level sets

We now show that the varifold V is integer rectifiable. Given $x \in \mathrm{spt}(\mu)$ and $s > 0$, we define $M_{x,s}$ to be the ball of radius $s^{-1}\,\mathrm{inj}(M)$ in the Euclidean space $(T_x M, g_x)$ and define $\iota_{x,s} : M_{x,s} \to M$ by $\iota_{x,s}(y) := \exp_x(sy)$. We endow $M_{x,s}$ with the smooth metric $g_{x,s} := s^{-2}\iota_{x,s}^* g$, which converges locally smoothly to the Euclidean metric g_x as $s \to 0$.

By rectifiability, for μ-a.e. x the dilated varifolds $V_{x,s} := (\iota_{x,s}^{-1})_*(V \llcorner B_{\mathrm{inj}(M)}(x))$ in $M_{x,s}$ satisfy

$$V_{x,s} \rightharpoonup v(T_x\Sigma, \Theta_{n-2}(x)) \tag{6.12}$$

as $s \to 0$, in duality with $C_c(T_x M)$. Fix $x \in \mathrm{spt}(\mu)$ such that (6.12) holds. The integrality of V will follow once we prove that $\Theta = \Theta_{n-2}(\mu, x)$ is an integer.

We can identify $(T_x M, g_x)$ with $\mathbb{R}^n$ by a linear isometry such that $T_x\Sigma = \{0\} \times \mathbb{R}^{n-2}$. We also call $\mu_{x,s}$ the mass measure of $V_{x,s}$; equivalently,

$$\mu_{x,s} := s^{2-n}(\iota_{x,s}^{-1})_*(\mu \llcorner B_{\mathrm{inj}(M)}(x)).$$

With a diagonal selection, changing our sequence $\epsilon \to 0$ accordingly, we can find scales $s_\epsilon \to 0$ such that we have the convergence of Radon measures

$$\lim_{\epsilon\to 0} \widehat{\mu}_\epsilon = \lim_{s\to 0} \mu_{x,s} = \Theta\mathcal{H}^{n-2} \llcorner T_x\Sigma,$$

where $(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)$ is the pullback of $(u_{s_\epsilon\epsilon}, \nabla_{s_\epsilon\epsilon})$ by means of ι_{x,s_ϵ}, and $\widehat{\mu}_\epsilon$ is the associated energy measure. Note that $(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)$ is stationary for E_ϵ in the line bundle $\iota_{x,s_\epsilon}^* L$, with respect to the base metric g_{x,s_ϵ}. We introduce the notation

$$e_\epsilon^T(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) := \sum_{i=3}^{n} (|(\nabla_\epsilon)_{\partial_i}\widehat{u}_\epsilon|^2 + \epsilon^2|\iota_{\partial_i} F_{\widehat{\nabla}_\epsilon}|^2).$$

Balls will be denoted by $\mathcal{B}_r(y)$ or $B_r^n(y)$, depending on whether they are with respect to g_{x,s_ϵ} or $g_{\mathbb{R}^n}$, respectively. The volume $|E|$ of a set E will be always understood with respect to the Euclidean metric.

The next proposition, which exploits quantitatively the monotonicity formula, is similar to an estimate in the proof of [26, Lemma 2.1].

Proposition 6.5 *As* $\epsilon \to 0$ *we have*

$$\lim_{\epsilon\to 0} \int_{B_2^2\times B_2^{n-2}} e_\epsilon^T(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) = 0.$$

Proof Let C_m be the constant in Theorem 4.3. We first note that, given $y \in \{0\} \times \mathbb{R}^{n-2}$,

$$\lim_{\epsilon \to 0} \widehat{\mu}_\epsilon(\mathcal{B}_r(y)) = \Theta\omega_{n-2}r^{n-2};$$

indeed, for any $\eta > 0$, $B^n_{r-\eta}(y) \subseteq \mathcal{B}_r(y) \subseteq B^n_{r+\eta}(y)$ eventually. Setting $y_\epsilon := \iota_{x,s_\epsilon}(y) \in M$, we deduce that

$$\begin{aligned}
&\lim_{\epsilon\to 0}(e^{C_m s_\epsilon r}(s_\epsilon r)^{2-n}\mu_{s_\epsilon \epsilon}(B_{s_\epsilon r}(y_\epsilon)) + C_m s_\epsilon r) \\
&\quad = \lim_{\epsilon\to 0}(e^{C_m s_\epsilon r} r^{2-n}\widehat{\mu}_\epsilon(\mathcal{B}_r(y)) + C_m s_\epsilon r) \\
&\quad = \Theta\omega_{n-2}.
\end{aligned} \tag{6.13}$$

Pick $3 \le i \le n$ and fix $R > 0$. Choosing $y := -2Re_i$, we can apply (4.12) between the radii $s_\epsilon R$ and $3s_\epsilon R$ to obtain that

$$\begin{aligned}
&\int_{B_{3s_\epsilon R}(p_i)\setminus B_{s_\epsilon R}(p_i)} d_{p_i}^{2-n}(|\nabla_{\nu_{R,i}}u_{s_\epsilon\epsilon}|^2 + s_\epsilon^2\epsilon^2|\iota_{\nu_{R,i}}F_{\nabla_{s_\epsilon\epsilon}}|^2) \\
&\quad \le (e^{C_m(3s_\epsilon R)}(3s_\epsilon R)^{2-n}\mu_{s_\epsilon\epsilon}(B_{3s_\epsilon R}(p_i)) + C_m(3s_\epsilon R)) \\
&\qquad - (e^{C_m(s_\epsilon R)}(s_\epsilon R)^{2-n}\mu_{s_\epsilon\epsilon}(B_{s_\epsilon R}(p_i)) + C_m(s_\epsilon R)),
\end{aligned}$$

where $p_i := \iota_{x,s_\epsilon}(-2Re_i)$ and $\nu_{R,i} := \operatorname{grad} d_{p_i}$. Now (6.13) and the comparability of g_{x,s_ϵ} with $g_{\mathbb{R}^n}$ give

$$\lim_{\epsilon\to 0}\int_{\mathcal{B}_{3R}(-2Re_i)\setminus\mathcal{B}_R(-2Re_i)}(|\nabla_{\widetilde{\nu}_{R,i}}\widehat{u}_\epsilon|^2 + \epsilon^2|\iota_{\widetilde{\nu}_{R,i}}F_{\widehat{\nabla}_\epsilon}|^2) = 0,$$

where $\widetilde{\nu}_{R,i}$ is the gradient of the distance function d_{-2Re_i}, both with respect to the metric g_{x,s_ϵ}. Since eventually $\mathcal{B}_{3R}(-2Re_i)\setminus\mathcal{B}_R(-2Re_i)$ includes $B^2_2 \times B^{n-2}_2$ for R big enough, we get

$$\lim_{\epsilon\to 0}\int_{B_2^2\times B_2^{n-2}}(|\nabla_{\widetilde{\nu}_{R,i}}\widehat{u}_\epsilon|^2 + \epsilon^2|\iota_{\widetilde{\nu}_{R,i}}F_{\widehat{\nabla}_\epsilon}|^2) = 0. \tag{6.14}$$

By monotonicity, as $\epsilon \to 0$ we have

$$\begin{aligned}
\limsup_{\epsilon\to 0}\int_{B_2^2\times B_2^{n-2}} e_\epsilon(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) &\le \limsup_{\epsilon\to 0} s_\epsilon^{2-n}\int_{B_{5s_\epsilon}(x)} e_{s_\epsilon\epsilon}(u_{s_\epsilon\epsilon}, \nabla_{s_\epsilon\epsilon}) \\
&\le C(M, \Lambda).
\end{aligned} \tag{6.15}$$

The smooth convergence $g_{x,s_\epsilon} \to g_{\mathbb{R}^n}$ gives $\widetilde{\nu}_{R,i}(y) \to Y_{R,i}(y) := \frac{y+2Re_i}{|y+2Re_i|}$ uniformly on $B_2^2 \times B_2^{n-2}$. Hence, the bound (6.15) and (6.14) give

$$\lim_{\epsilon\to 0} \int_{B_2^2 \times B_2^{n-2}} (|\nabla_{Y_{R,i}} \widehat{u}_\epsilon|^2 + \epsilon^2 |\iota_{Y_{R,i}} F_{\widehat{\nabla}_\epsilon}|^2) = 0. \tag{6.16}$$

Now $Y_{R,i} \to e_i = \partial_i$ as $R \to \infty$, and the statement follows from (6.16) and the uniform bound (6.15). □

We now state the main technical result of the section, which will be shown later. Fix a cut-off function $\chi \in C_c^\infty(B_2^2)$ with $\chi(z) = 1$ for $|z| \le \frac{3}{2}$ and $0 \le \chi \le 1$, and let $\widehat{\chi}(z,t) := \chi(z)$.

Proposition 6.6 *There exists $F_\epsilon \subseteq B_1^{n-2}$ with $|F_\epsilon| \ge \frac{1}{4}|B_1^{n-2}|$ such that*

$$\sup_{t\in F_\epsilon} \operatorname{dist}\left(\int_{\mathbb{R}^2\times\{t\}} \chi(z) e_\epsilon(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)(z,t), 2\pi\mathbb{N} \right) \to 0 \quad \text{as } \epsilon \to 0. \tag{6.17}$$

Before giving the proof, let us see how this implies the integrality of V.

Proof of Theorem 6.1 As $\epsilon \to 0$, we have both (6.17) and

$$\lim_{\epsilon\to 0} \frac{1}{2\pi} \int_{\mathbb{R}^2\times B_1^{n-2}} \widehat{\chi} e_\epsilon(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^2\times B_1^{n-2}} \widehat{\chi}\, d\widehat{\mu}_\epsilon = \omega_{n-2}\Theta, \tag{6.18}$$

$$\int_{\mathbb{R}^2\times B_2^{n-2}} |d\widehat{\chi}|\, d\widehat{\mu}_\epsilon \le C\widehat{\mu}_\epsilon((B_2^2\setminus B_1^2) \times B_1^{n-2}) \to 0, \tag{6.19}$$

as $\widehat{\mu}_\epsilon \rightharpoonup \Theta\mathcal{H}^{n-2} \llcorner \{0\} \times \mathbb{R}^{n-2}$.

In view of (6.15) and (6.19), for any vector field $(Y^3, \ldots, Y^n) \in C_c^\infty(B_2^{n-2}, \mathbb{R}^{n-2})$ we can integrate (4.4) against $\widehat{\chi}(\sum_{i=3}^n Y^i \partial_i)$ and obtain, in the Euclidean metric,

$$\left| \int_{\mathbb{R}^2\times B_2^{n-2}} \widehat{\chi} \langle T_\epsilon(u_\epsilon, \nabla_\epsilon), dY^i \otimes \partial_i \rangle \right| \le \lambda_\epsilon(\|Y\|_{L^\infty} + \|DY\|_{L^\infty})$$

for some sequence $\lambda_\epsilon \to 0$, thanks to the smooth convergence $g_{x,s_\epsilon} \to g_{\mathbb{R}^n}$.

Invoking Proposition 6.5 and noting that $\|Y\|_{L^\infty} \le 2\|DY\|_{L^\infty}$, we can conclude that the nonnegative function $f_\epsilon(t) := \frac{1}{2\pi} \int_{\mathbb{R}^2\times\{t\}} \widehat{\chi} e_\epsilon(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)$ satisfies

$$\left| \int_{B_2^{n-2}} f_\epsilon \operatorname{div}(Y) \right| \le \lambda_\epsilon \|DY\|_{L^\infty}$$

for a possibly different sequence $\lambda_\epsilon \to 0$. Applying the Hahn–Banach theorem to the subspace $\{DY \mid Y \in C_c^\infty(B_2^{n-2}, \mathbb{R}^{n-2})\} \subseteq C_0(B_2^{n-2}, \mathbb{R}^{n-2} \otimes \mathbb{R}^{n-2})$ (C_0 denoting the closure of C_c), we can find real measures $(\nu_\epsilon)^i_j$ such that

$$\partial_j f_\epsilon = \sum_{i=3}^{n} \partial_i (\nu_\epsilon)^i_j \quad \text{for all } j = 3, \dots, n$$

as distributions and $|(\nu_\epsilon)^i_j|(B_2^{n-2}) \to 0$. Allard's strong constancy lemma [2, Theorem 1.(4)] gives then

$$\left\| f_\epsilon - \frac{1}{\omega_{n-2}} \int_{B_1^{n-2}} f_\epsilon \right\|_{L^1(B_1^{n-2})} \to 0.$$

Since the sets F_ϵ of Proposition 6.6 have positive measure, there clearly exists $t_\epsilon \in F_\epsilon$ such that

$$\left| f_\epsilon(t_\epsilon) - \frac{1}{\omega_{n-2}} \int_{B_1^{n-2}} f_\epsilon \right| \le \frac{1}{|F_\epsilon|} \left\| f_\epsilon - \frac{1}{\omega_{n-2}} \int_{B_1^{n-2}} f_\epsilon \right\|_{L^1(B_1^{n-2})} \to 0.$$

Recalling (6.17), we deduce that

$$\operatorname{dist}\left(\frac{1}{\omega_{n-2}} \int_{B_1^{n-2}} f_\epsilon, 2\pi \mathbb{N} \right) \to 0.$$

Hence, by (6.18), we get $\operatorname{dist}(\Theta, \mathbb{N}) = 0$, which concludes the proof that V is integral. □

Proof of Proposition 6.6 Taking into account Proposition 6.5, the classical Hardy–Littlewood weak-(1,1) maximal estimate (applied to the function $t \mapsto \int_{B_2^2 \times \{t\}} e_\epsilon^T(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)$) gives

$$\frac{1}{r^{n-2}} \int_{B_2^2 \times B_r^{n-2}(t)} e_\epsilon^T(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) \le C(n) \int_{B_2^2 \times B_2^{n-2}} e_\epsilon^T(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon) \to 0 \tag{6.20}$$

for all $t \in B_1^{n-2} \setminus E_1^\epsilon$ and $0 < r < 1$, where E_1^ϵ is a Borel set with $|E_1^\epsilon| \le \frac{1}{4}|B_1^{n-2}|$. Similarly, (6.15) and (6.19) give

$$\frac{1}{r^{n-2}} \widehat{\mu}_\epsilon(B_2^2 \times B_r^{n-2}(t)) \le C(M, \Lambda), \tag{6.21}$$

$$\frac{1}{r^{n-2}} \widehat{\mu}_\epsilon((B_2^2 \setminus B_1^2) \times B_r^{n-2}(t)) \le C(n) \widehat{\mu}_\epsilon((B_2^2 \setminus B_1^2) \times B_2^{n-2}) \to 0 \tag{6.22}$$

for $t \in B_1^{n-2} \setminus (E_2^\epsilon \cup E_3^\epsilon)$ and $0 < r < 1$, with $|E_2^\epsilon|, |E_3^\epsilon| \le \frac{1}{4}|B_1^{n-2}|$.

Pick any $t^\epsilon \in B_1^{n-2} \setminus (E_1^\epsilon \cup E_2^\epsilon \cup E_3^\epsilon)$ and, for $0 < r < 1$, define

$$\mathcal{V}^\epsilon(r) := \{z \in B_1^2 : \operatorname{dist}((z, t^\epsilon),\ Z_{\beta_d/2}(\widehat{u}_\epsilon)) < r\}$$

(with the Euclidean distance), where $Z_{\beta_d/2}(\widehat{u}_\epsilon) = \{|\widehat{u}_\epsilon|^2 \le 1 - \beta_d/2\}$. In other words, $\mathcal{V}^\epsilon$ is the t^ϵ-slice of the neighborhood $B_r^n(Z_{\beta_d/2}(\widehat{u}_\epsilon))$.

We claim that, for $0 < r < \frac{1}{2}$, $\mathcal{V}^\epsilon(r)$ satisfies a uniform area bound

$$|\mathcal{V}^\epsilon(r)| \le C(M, \Lambda) r^2, \tag{6.23}$$

provided $\epsilon < r$ and ϵ is small enough. Indeed, $\mathcal{V}^\epsilon(r) \times \{t^\epsilon\}$ is covered by the balls $B_r^n(y)$ with $y \in (B_{3/2}^2 \times B_r^{n-2}(t^\epsilon)) \cap Z_{\beta_d/2}(\widehat{u}_\epsilon)$. Vitali's covering lemma gives a disjoint collection $\{B_r^n(y_j) \mid j \in J\}$ such that $\mathcal{V}^\epsilon(r) \times \{t^\epsilon\} \subseteq \bigcup_j B_{5r}^n(y_j)$. By Corollary 4.4, we have a bound on the cardinality $|J|$:

$$\widehat{\mu}_\epsilon(B_2^2 \times B_{2r}^{n-2}(t^\epsilon)) \ge \sum_{j \in J} \widehat{\mu}_\epsilon(B_r^n(y_j)) \ge \sum_{j \in J} \widehat{\mu}_\epsilon(\mathcal{B}_{r/2}(y_j)) \ge c(M, \Lambda) r^{n-2} |J|$$

(since $\frac{1}{4} g_{\mathbb{R}^n} \le g_{x,s_\epsilon} \le 4 g_{\mathbb{R}^n}$ for ϵ sufficiently small). Using also (6.21), we get $|J| \le C(M, \Lambda)$. Hence, writing $y_j = (z_j, t_j)$, we obtain

$$|\mathcal{V}^\epsilon(r)| \le \sum_{j \in J} |B_{5r}^2(z_j)| \le 25\pi |J| r^2 \le C(M, \Lambda) r^2,$$

confirming (6.23).

Given $R > 0$, let $\{z_1^\epsilon, \dots, z_{N(R,\epsilon)}^\epsilon\}$ be a maximal subset of $\mathcal{V}^\epsilon(R\epsilon)$ with $|z_k^\epsilon - z_\ell^\epsilon| \ge 2\epsilon$. Since $\bigcup_k (B_1^2 \cap B_\epsilon^2(z_k)) \subseteq \mathcal{V}^\epsilon((R+1)\epsilon)$ and the balls $B_\epsilon^2(z_k)$ are disjoint, (6.23) gives a uniform bound on $N(R, \epsilon)$ independent of ϵ (eventually), so up to subsequences we can assume that $N(R) = N(R, \epsilon)$ is constant and that $\epsilon^{-1}|z_k^\epsilon - z_\ell^\epsilon|$ has a limit $r_{k\ell}$ as $\epsilon \to 0$, for each k, l.

We say that $k \sim \ell$ if $r_{k\ell} < \infty$; this is evidently an equivalence relation (as $r_{km} \le r_{k\ell} + r_{\ell m}$), so we can pick a set of representatives $\{k_1, \dots, k_P\}$ of the distinct equivalence classes $[k_1], \dots, [k_P]$ and conclude that

$$\mathcal{V}^\epsilon(R\epsilon) \subseteq \bigcup_{j=1}^P B_{S\epsilon}^2(z_{k_j}^\epsilon)$$

eventually, for any fixed $S \ge S_0(R) := \max\{\sum_{\ell \in [k_j]} r_{k_j \ell} + 2 \mid j = 1, \dots, P\}$.

Fix such an S which is also bigger than the constants C in (6.21) and a_d^{-1}, C_d in Corollary 5.4. For any fixed $\delta > 0$, (6.20) and (6.21) show that, for ϵ sufficiently small, Proposition 6.7 below applies to $\widehat{u}_\epsilon(z^\epsilon_{k_j}+\epsilon\cdot, t^\epsilon+\epsilon\cdot)$ (with $\beta:=\beta_d$). Writing $K=K(\beta_d,\delta,S)>S$, note that the balls $B^2_{K\epsilon}(z_{k_j})$ are eventually disjoint and included in $\{\chi=1\}$. Hence, Proposition 6.7 and (6.22) give

$$\begin{aligned}\operatorname{dist}\Big(\int_{\mathbb{R}^2\times\{t^\epsilon\}}\widehat{\chi}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon),2\pi\mathbb{N}\Big)&\le P\delta+\int_{B^2_2\setminus\bigcup_{j=1}^P B^2_{K\epsilon}(z^\epsilon_{k_j})}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)(\cdot,t^\epsilon)\\&\le P\delta+\int_{B^2_2\setminus\mathcal{V}^\epsilon(R\epsilon)}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)(\cdot,t^\epsilon)\\&\le(P+1)\delta+\int_{B^2_1\setminus\mathcal{V}^\epsilon(R\epsilon)}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)(\cdot,t^\epsilon)\end{aligned}$$

(for ϵ sufficiently small). Choosing $\delta=\delta(R)\le\frac{1}{(P+1)R}$, we arrive at the estimate

$$\operatorname{dist}\Big(\int_{\mathbb{R}^2\times\{t^\epsilon\}}\widehat{\chi}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon),2\pi\mathbb{N}\Big)\le\frac{1}{R}+\int_{B^2_1\setminus\mathcal{V}^\epsilon(R\epsilon)}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)(\cdot,t^\epsilon).$$

To conclude the proof, it suffices to show that

$$\lim_{R\to0}\limsup_{\epsilon\to0}\int_{B^2_1\setminus\mathcal{V}^\epsilon(R\epsilon)}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)(\cdot,t^\epsilon)\to0. \tag{6.24}$$

Once we have this, we infer that

$$\liminf_{\epsilon\to0}\operatorname{dist}\Big(\int_{\mathbb{R}^2\times\{t^\epsilon\}}\widehat{\chi}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon),2\pi\mathbb{N}\Big)=0$$

for the original sequence (t^ϵ). Noting that the choice of t^ϵ in $F_\epsilon:=B^{n-2}_1\setminus E^\epsilon_1\cup E^\epsilon_2\cup E^\epsilon_3$ was arbitrary, we get

$$\liminf_{\epsilon\to0}\sup_{t\in F_\epsilon}\operatorname{dist}\Big(\int_{\mathbb{R}^2\times\{t\}}\widehat{\chi}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon),2\pi\mathbb{N}\Big)=0.$$

Since the argument applies to an arbitrary subsequence $\epsilon_j\to0$, the proposition then follows.

To show (6.24), note that for $z\in B^2_1$ the distance of $\iota_{x,s_\epsilon}((z,t^\epsilon))$ to the set $Z_{\beta_d/2}(u_{s_\epsilon\epsilon})$ is (eventually) bounded below by $\frac{s_\epsilon}{2}\min\{1,r_\epsilon(z)\}$, where $r_\epsilon(z)$ is the (Euclidean) distance of (z,t^ϵ) to $Z_{\beta_d/2}(\widehat{u}_\epsilon))$. Since $Z_{\beta_d/2}(u_{s_\epsilon\epsilon})\supseteq Z_{\beta_d}(u_{s_\epsilon\epsilon})$, for any $R>1$ Corollary 5.4 gives

$$\int_{B^2_1\setminus\mathcal{V}^\epsilon(R\epsilon)}e_\epsilon(\widehat{u}_\epsilon,\widehat{\nabla}_\epsilon)\le C\epsilon^{-2}\int_{B^2_1\setminus\mathcal{V}^\epsilon(R\epsilon)}e^{-a_dr_\epsilon(z)/(2\epsilon)}+C\epsilon^{-2}e^{-a_d/(2\epsilon)}+Cs_\epsilon\epsilon$$

$$= C\epsilon^{-3}\int_{B_1^2\setminus\mathcal{V}^\epsilon(R\epsilon)}\int_{r_\epsilon(z)}^{\infty}\frac{a_d}{2}e^{-a_d r/(2\epsilon)}\,dr\,dz + C\epsilon^{-2}e^{-a_d/(2\epsilon)} + Cs_\epsilon\epsilon$$
$$= C\epsilon^{-3}\int_{R\epsilon}^{\infty}\frac{a_d}{2}e^{-a_d r/(2\epsilon)}|\mathcal{V}^\epsilon(r)|\,dr + C\epsilon^{-2}e^{-a_d/(2\epsilon)} + Cs_\epsilon\epsilon$$
$$\leq C\epsilon^{-3}\int_{R\epsilon}^{\infty}e^{-a_d r/(2\epsilon)}r^2\,dr + C\epsilon$$
$$= C\int_{R}^{\infty}e^{-a_d t/2}t^2\,dt + C\epsilon,$$

where we used Fubini's theorem in the second equality. The statement follows. □

The following key technical proposition, used in the proof of Proposition 6.6, relies ultimately on the quantization phenomenon for the energy of entire solutions in the plane, presented in [21, Chapter III]. For the reader's convenience, we give a self-contained proof, including the relevant arguments from [21].

Proposition 6.7 *Given $0 < \beta, \delta < \frac{1}{2}$ and $S > 1$, there exist $K(\beta, \delta, S) > S$ and $0 < \kappa(\beta, \delta, S, n) < K^{-1}$ such that the following is true. Assume (u, ∇) is smooth and solves (2.4) and (2.5), with $|u| \leq 1$ and $\epsilon = 1$, on a line bundle L over a cylinder (Q, g), with $Q = B^2_{\kappa^{-1}} \times B^{n-2}_{\kappa^{-1}}$. If we have*

$$Z_{\beta/2}(u) \cap (B^2_{\kappa^{-1}} \times \{0\}) \subseteq \overline{B}^2_S \times \{0\}, \tag{6.25}$$

the energy bounds

$$e_1(u, \nabla) \leq S, \tag{6.26}$$

$$\sum_{i=3}^{n}\int_{B^2_{\kappa^{-1}}\times B^{n-2}_r}(|\nabla_{\partial_i}u|^2 + |\iota_{\partial_i}F_\nabla|^2) \leq \kappa r^{n-2} \quad \text{for all } 0 < r < \kappa^{-1}, \tag{6.27}$$

as well as the decay

$$e_1(u, \nabla)(p) \leq Se^{-S^{-1}r} + \kappa \quad \text{whenever } \mathcal{B}_r(p) \subset\subset Q\setminus Z_\beta, \tag{6.28}$$

and $\|g - g_{\mathbb{R}^n}\|_{C^2} \leq \kappa$, then

$$\left|\int_{B^2_K\times\{0\}} e_1(u, \nabla) - 2\pi|p|\right| < \delta,$$

where p is the degree of $\frac{u}{|u|}(S\cdot, 0)$, as a map from the circle to itself.

Proof To begin with, fix a real number $K(\beta, \delta, S) > S$ so big that

$$\int_K^\infty (2\pi r) S e^{-S^{-1}(r-S)} < \delta. \tag{6.29}$$

Arguing by contradiction, assume there exists a sequence $\kappa_j \to 0$ such that the statement admits a counterexample (u_j, ∇_j) (for $\kappa = \kappa_j$) for a (necessarily trivial) line bundle L_j over $Q_j = B^2_{\kappa_j^{-1}} \times B^{n-2}_{\kappa_j^{-1}}$, with respect to a metric $g = g_j$ satisfying $\|g - g_{\mathbb{R}^n}\|_{C^2} \le \kappa_j$. Fixing a trivialization of L_j over Q_j, we can write $\nabla_j = d - iA_j$ for some real one-form A_j.

By virtue of the uniform pointwise estimate (6.28) for $e_1(u_j, \nabla_j) \ge |d|u_j||^2$, we see that the functions $|u_j|$ are locally equi-Lipschitz. In particular, we can apply the Arzelà–Ascoli theorem to extract a subsequence $|u_j|$ converging in C^0_{loc} to a continuous function $\rho_\infty : \mathbb{R}^n \to \mathbb{R}$.

Since $|\partial_k |u_j|| \le |(\nabla_j)_{\partial_k} u_j|$ for all k, (6.27) implies that ρ_∞ depends only on the first two variables. Moreover, (6.25) gives $\rho_\infty^2 \ge 1 - \frac{\beta}{2} > 1 - \beta$ outside $B^2_S \times \mathbb{R}^{n-2}$. In particular, setting

$$R_j := \max\left\{ r \le \kappa_j^{-1} : (B^2_r \setminus B^2_S) \times B^{n-2}_1 \subseteq \left\{ |u_j| > \frac{1}{2} \right\} \right\},$$

we have $R_j \to \infty$. Let $w_j := \frac{u_j}{|u_j|}$ on $\{|u_j| > \frac{1}{2}\}$.

The degree p_j is uniformly bounded as, for $r \ge S$ and $t \in \mathbb{R}^{n-2}$,

$$2\pi p_j = \int_{\partial B^2_r \times \{t\}} w_j^*(d\theta) = \int_{B^2_r \times \{t\}} dA_j + \int_{\partial B^2_r \times \{t\}} (w_j^*(d\theta) - A_j)$$

for j sufficiently large, so averaging over $S < r < 2S$ and $t \in B^{n-2}_1$ we get

$$\begin{aligned} 2\pi |p_j| &\le C(S) \int_{B^2_{2S} \times B^{n-2}_1} |dA_j| + C(S) \int_{(B^2_{2S} \setminus B^2_S) \times B^{n-2}_1} |w_j^*(d\theta) - A_j| \\ &\le C(\beta, S) \Big(\int_{B^2_{2S} \times B^{n-2}_1} e_1(u_j, A_j) \Big)^{1/2}, \end{aligned}$$

as $|u_j||w_j^*(d\theta) - A_j| \le |\nabla_j u_j|$. Thus, up to subsequences we can assume $p_j = p$ is constant.

We now claim that, up to change of gauge, $(u_j, A_j) \to (u_\infty, A_\infty)$ subsequentially in $C^1_{loc}(\mathbb{R}^2 \times B^{n-2}_1)$. Let $\widetilde{u}_j = e^{i\theta_j} u_j$ and $\widetilde{A}_j = A_j + d\theta_j$ be the section and the connection in the Coulomb gauge on the domain $(\overline{B}^n_{5S}, g_j)$, with $\widetilde{A}_j(\nu) = 0$ on the boundary (as described in the "Appendix"). Note that B^n_{5S} includes the cylinder $Q' := B^2_{4S} \times B^{n-2}_1$, and observe that, on $Q'' := (B^2_{4S} \backslash B^2_S) \times B^{n-2}_1$, $\widetilde{u}_j$ has the form

$$\widetilde{u}_j(re^{i\theta}, t) = |u_j| e^{ip\theta + i\psi_j}$$

for a unique real function ψ_j with $0 \le \psi_j(2S, 0) < 2\pi$.

Hence, $u_j = |u_j| e^{i(p\theta + \psi_j - \theta_j)}$ on Q'' and we can extend $\psi_j - \theta_j$ uniquely to a function $\sigma_j : (B^2_{R_j} \backslash B^2_S) \times B^{n-2}_1 \to \mathbb{R}$ so that $u_j = |u_j| e^{ip\theta + i\sigma_j}$ holds true on all the domain of σ_j. Finally, we replace (u_j, A_j) with $(e^{i\tau_j} u_j, A_j + d\tau_j)$, where

$$\tau_j(z, t) := \begin{cases} \theta_j - \chi(|z|)\psi_j & |z| < 4S \\ -\sigma_j & |z| > 3S \end{cases}$$

for a fixed smooth function $\chi : [0, \infty) \to [0, 1]$ such that $\chi = 0$ on $[0, 2S]$ and $\chi = 1$ on $[3S, \infty)$. Observe that, in the cylinder $Q' = B^2_{4S} \times B^{n-2}_1$, the new couple equals

$$(\widetilde{u}_j e^{-\chi(|z|)\psi_j}, \widetilde{A}_j - d(\chi(|z|)\psi_j)).$$

The function ψ_j obeys uniform local $W^{2,q}$ bounds, on (the interior of) Q'', for all $1 \le q < \infty$, thanks to the Coulomb gauge specification (per Proposition A.1 in the "Appendix"). Hence, the new couple (u_j, A_j) has uniform local $W^{2,q}$ bounds on Q'.

Moreover, in the exterior annular region $\mathcal{A}_j := (B^2_{R_j} \backslash \overline{B}^2_{3S}) \times B^{n-2}_1$, we have that $u_j(re^{i\theta}, t) = |u_j| e^{pi\theta}$ and we can obtain local $W^{2,q}$ bounds noting that

$$pd\theta - A_j = |u_j|^{-2} \langle \nabla_j u_j, iu_j \rangle.$$

Indeed, since the right-hand side is bounded by $2e_1(u_j, \nabla_j)^{1/2} \le 2S^{1/2}$ and $pd\theta$ is a fixed smooth one-form, we immediately obtain uniform L^∞ bounds for A_j locally in $\mathcal{A}_j$. Next, note that the identity (3.4) applies to give us an estimate

$$|\Delta |u_j|^2| \le Ce_1(u_j, \nabla_j) + C \le CS$$

in $\mathcal{A}_j$, from which it follows that the modulus $|u_j|$ satisfies uniform $W^{2,q}$ bounds for every $q \in (1, \infty)$ locally in $\mathcal{A}_j$. Multiplying (2.4) by $e^{-pi\theta}$ and taking the imaginary part gives

$$|u_j| d^*(pd\theta - A_j) = 2\langle d|u_j|, pd\theta - A_j \rangle,$$

from which it follows that d^*A_j satisfies uniform L^∞ bounds locally in $\mathcal{A}_j$ as well; together with the obvious pointwise bound $|dA_j| \le e_1(u_j, \nabla_j)^{1/2} \le S^{1/2}$, this in particular yields uniform bounds on the full derivative $\|DA_j\|_{L^q}$ for every $q \in (1, \infty)$ on fixed compact subsets of $\mathcal{A}_j$ (this follows, e.g., from [20, Lemma 4.7] and a cut-off argument).

Finally, writing (2.5) as

$$\Delta_H A_j = dd^*A_j + |u_j|^2(pd\theta - A_j),$$

the preceding chain of identities and estimates give a uniform L^q bound on the right-hand side over any fixed compact subset of $\mathcal{A}_j$, for any $q \in (1, \infty)$; in particular, this gives us the desired uniform local $W^{2,q}$ bounds for A_j (while we already have the desired $W^{2,q}$ bounds for $u_j = |u_j|e^{pi\theta}$).

Thanks to the compact embedding $W^{2,q} \hookrightarrow C^1$ on bounded regular domains (for $q > n$), we obtain a limit couple (u_∞, A_∞) on $\mathbb{R}^2 \times B_1^{n-2}$, as claimed, which solves (2.4) and (2.5) with respect to the flat metric. Also, $|u_\infty| = \rho_\infty$ and

$$(\nabla_\infty)_{\partial_k} u_\infty = 0, \quad \iota_{\partial_k} dA_\infty = 0 \quad \text{for } k = 3, \dots, n. \tag{6.30}$$

The second part of (6.30) implies that we can find a function $\alpha \in C^1(\mathbb{R}^2 \times B_1^{n-2})$ with $\alpha(z, 0) = 0$ and $\partial_k \alpha = (A_\infty)_k$, for all $z \in \mathbb{R}^2$ and all $k \ge 3$. Set $\widetilde{u}_\infty := e^{-i\alpha} u_\infty$ and $\widetilde{A}_\infty := A_\infty - d\alpha$, so that

$$(\widetilde{A}_\infty)_k = 0, \quad \partial_k(\widetilde{A}_\infty)_\ell = \partial_k(A_\infty)_\ell - \partial^2_{k\ell}\alpha = \partial_\ell(A_\infty - d\alpha)_k = 0$$

for all $k = 3, \dots, n$ and $\ell = 1, \dots, n$ [using again the second part of (6.30)]. The first part gives instead $\partial_k \widetilde{u}_\infty = 0$ for $k = 3, \dots, n$. Hence, $(\widetilde{u}_\infty, \widetilde{A}_\infty)$ depends only on the first two variables and therefore corresponds to a planar solution of (2.4) and (2.5).

Also, from (6.28) we deduce that

$$e_1(\widetilde{u}_\infty, \widetilde{A}_\infty)(z, t) = e_1(u_\infty, A_\infty)(z, t) = \lim_{j \to \infty} e_1(u_j, A_j)(z, t) \le S e^{-S^{-1}(|z| - S)} \tag{6.31}$$

for $|z| > S$, as eventually $\overline{B}^n_{|z|-S}(z, t) \cap Z_\beta(u_j) = \emptyset$.

Integrating (4.4) on $\mathbb{R}^2 = \mathbb{R}^2 \times \{0\}$ against the position vector field we get

$$\int_{\mathbb{R}^2} |d\widetilde{A}_\infty|^2 = \int_{\mathbb{R}^2} W(\widetilde{u}_\infty).$$

Thanks to the decay of $e_1(\widetilde{u}_\infty, \widetilde{A}_\infty)$, we can repeat the proof of (3.6): starting from

$$\Delta\widetilde{\xi}_\infty \geq |\widetilde{u}_\infty|^2\widetilde{\xi}_\infty, \text{ with } \widetilde{\xi}_\infty := |d\widetilde{A}_\infty| - \frac{1-|\widetilde{u}_\infty|^2}{2},$$

and applying the maximum principle, we deduce that the decaying function $\widetilde{\xi}_\infty$ is nonpositive. We then obtain $|d\widetilde{A}_\infty| \leq \sqrt{W(\widetilde{u}_\infty)}$, so we must have $|d\widetilde{A}_\infty| = \sqrt{W(\widetilde{u}_\infty)}$ everywhere (cf. [21, Section III.10]).

Observe that, by (3.4) and the strong maximum principle, $|\widetilde{u}_\infty| < 1$ (unless $|\widetilde{u}_\infty| = 1$ everywhere, in which case $|d\widetilde{A}_\infty| = \sqrt{W(\widetilde{u}_\infty)} = 0$ and $|\widetilde{\nabla}_\infty\widetilde{u}_\infty| = 0$ by (3.4), thus $e_1(\widetilde{u}_\infty, \widetilde{A}_\infty) = 0$ and $p = 0$; so the statement of the proposition holds eventually, contradiction). As a consequence, $| * d\widetilde{A}_\infty| = W(\widetilde{u}_\infty) > 0$ and we get either $\frac{1-|\widetilde{u}_\infty|^2}{2} = *d\widetilde{A}_\infty$ everywhere or $\frac{1-|\widetilde{u}_\infty|^2}{2} = - * d\widetilde{A}_\infty$ everywhere. Thus, integrating by parts and using (2.4), as well as the decay of $|pd\theta - \widetilde{A}_\infty|$,

$$\begin{aligned}
\int_{\mathbb{R}^2} e_1(\widetilde{u}_\infty, \widetilde{A}_\infty) &= \int_{\mathbb{R}^2} (|\widetilde{\nabla}_\infty\widetilde{u}_\infty|^2 + 2W(\widetilde{u}_\infty)) \\
&= \int_{\mathbb{R}^2} (\langle \widetilde{\nabla}_\infty^*\widetilde{\nabla}_\infty\widetilde{u}_\infty, \widetilde{u}_\infty\rangle + 2W(\widetilde{u}_\infty)) \\
&= \int_{\mathbb{R}^2} \frac{1-|\widetilde{u}_\infty|^2}{2} = \pm\int_{\mathbb{R}^2} d\widetilde{A}_\infty = \pm \lim_{r\to\infty}\int_{\partial B_r^2} \widetilde{A}_\infty \\
&= \pm \lim_{r\to\infty}\int_{\partial B_r^2} pd\theta = \pm 2\pi p.
\end{aligned}$$

Hence, the energy of the two-dimensional solution $(\widetilde{u}_\infty, \widetilde{A}_\infty)$ is $2\pi|p|$. Our choice of K, namely (6.29), together with (6.31), then ensures that

$$\operatorname{dist}\left(\int_{B_K^2\times\{0\}} e_1(u_\infty, A_\infty), 2\pi\mathbb{N}\right) < \delta.$$

As a consequence, this must hold eventually also for (u_j, A_j), giving the desired contradiction. □

Remark 6.8 As a consequence, one also finds that

$$\int_{B_K^2\times\{0\}} e_1(u, \nabla) < \delta$$

if $|u| > 0$ everywhere on the cylinder Q. Indeed, if $|u| > 0$ everywhere, then the degree p in the statement of Proposition 6.7 clearly must vanish.

We are now able to address the statement on the convergence of level sets.

Proposition 6.9 *For any* $0 \le \delta < 1$ *we have* $\mathrm{spt}(\mu) = \lim_{\epsilon\to 0}\{|u_\epsilon| \le \delta\}$*, in the Hausdorff topology.*

Proof If $x = \lim_{\epsilon\to 0} x_\epsilon$, for points $x_\epsilon \in \{|u_\epsilon| \le \delta\}$ defined along a subsequence, then the same argument used in the proof of Proposition 6.2 shows that $x \in \mathrm{spt}(\mu)$. Hence, for all $\eta > 0$, eventually $\{|u_\epsilon| \le \delta\}$ is included in the η-neighborhood of $\mathrm{spt}(\mu)$.

To conclude the proof, it suffices to show that the converse inclusion $\mathrm{spt}(\mu) \subseteq B_\eta(\{u_\epsilon = 0\})$ holds eventually. Arguing by contradiction, assume that there are points $p_\epsilon \in \mathrm{spt}(\mu)$ whose distance from $\{u_\epsilon = 0\}$ is at least η, along some subsequence (not relabeled). Up to further subsequences, let $p_\epsilon \to p_0 \in \mathrm{spt}(\mu)$.

Since μ is $(n-2)$-rectifiable, there exists a point $q \in \mathrm{spt}(\mu)$ with $\mathrm{dist}(p_0, q) < \frac{\eta}{2}$, and such that μ blows up to $\Theta_{n-2}(\mu, q)\mathcal{H}^{n-2} \llcorner T_q\Sigma$ at q. Observe that eventually we have

$$\mathrm{dist}(q, \{u_\epsilon = 0\}) \ge \frac{\eta}{2}. \tag{6.32}$$

Now, repeating all the preceding blow-up analysis at q, in view of Remark 6.8 we can improve (6.17) to the uniform convergence

$$\int_{\mathbb{R}^2\times\{t\}} \chi(z) e_\epsilon(\widehat{u}_\epsilon, \widehat{\nabla}_\epsilon)(z,t) \to 0$$

for $t \in F_\epsilon$, which implies that $\Theta_{n-2}(\mu, q) = 0$. However, since $q \in \mathrm{spt}(\mu)$, this is impossible, by Proposition 6.2. □

6.3 Limiting behavior of the curvature

As before, we identify the curvature F_{∇_ϵ} with a closed two-form ω_ϵ by $F_{\nabla_\epsilon}(X,Y) = -i\omega_\epsilon(X,Y)$. Recall that the cohomology class $[\frac{1}{2\pi}\omega_\epsilon]$ represents the (rational) first Chern class $c_1(L) \in H^2(M;\mathbb{R})$ of the complex line bundle $L \to M$.

Theorem 6.10 *Let* $(u_\epsilon, \nabla_\epsilon)$ *be a family as in Theorem 6.1. The curvature forms* $\frac{1}{2\pi}\omega_\epsilon$ *can be identified with* $(n-2)$*-currents that converge (weakly), as* $\epsilon \to 0$*, to an integer rectifiable cycle* Γ *which is Poincaré dual to* $c_1(L)$*, and whose mass measure* $|\Gamma|$ *satisfies* $|\Gamma| \le \mu$*.*

Proof Recall from Sect. 2 that

$$d\langle\nabla_\epsilon u_\epsilon, iu_\epsilon\rangle = \psi(u_\epsilon) - |u_\epsilon|^2\omega_\epsilon,$$

where $\psi(u_\epsilon) = \langle 2i\nabla u_\epsilon, \nabla_\epsilon u_\epsilon\rangle$ is a two-form satisfying $|\psi(u_\epsilon)| \le |\nabla_\epsilon u_\epsilon|^2$ pointwise. In particular, denoting by $J(u_\epsilon, \nabla_\epsilon)$ the two-form

$$J(u_\epsilon, \nabla_\epsilon) := \psi(u_\epsilon) + (1 - |u_\epsilon|^2)\omega_\epsilon,$$

we can rewrite this identity as

$$J(u_\epsilon, \nabla_\epsilon) - \omega_\epsilon = d\langle \nabla_\epsilon u_\epsilon, iu_\epsilon\rangle, \tag{6.33}$$

and observe that

$$|J(u_\epsilon, \nabla_\epsilon)| \le |\nabla_\epsilon u_\epsilon|^2 + \epsilon^2|\omega_\epsilon|^2 + \frac{1}{4\epsilon^2}(1 - |u_\epsilon|^2)^2 = e_\epsilon(u_\epsilon, \nabla_\epsilon). \tag{6.34}$$

The dual $(n-2)$-currents given by

$$\langle \Gamma_\epsilon, \zeta\rangle := \frac{1}{2\pi}\int_M J(u_\epsilon, \nabla_\epsilon) \wedge \zeta,$$

for any $(n-2)$-form $\zeta \in \Omega^{n-2}(M)$, are thus bounded in mass by $\frac{1}{2\pi}\Lambda$. (Here we compute the mass with the ℓ^2 norm on exterior algebras; for the limit current, by rectifiability this will coincide with the usual mass, dual to the comass.) Up to subsequences, we can take a weak limit Γ. The bound $|\Gamma_\epsilon| \le \mu_\epsilon$ implies that also $|\Gamma| \le \mu$.

From (6.33) and integration by parts we get

$$\int_M \omega_\epsilon \wedge \zeta = \int_M J(u_\epsilon, \nabla_\epsilon) \wedge \zeta - \int_M \langle \nabla_\epsilon u_\epsilon, iu_\epsilon\rangle \wedge d\zeta.$$

Since (as discussed in the proof of Proposition 6.2)

$$\int_M |\langle \nabla_\epsilon u_\epsilon, iu_\epsilon\rangle| \le \int_M e_\epsilon(u_\epsilon, \nabla_\epsilon)^{1/2} \to 0$$

as $\epsilon \to 0$, it follows that

$$\langle \Gamma, \zeta\rangle = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_M J(u_\epsilon, \nabla_\epsilon) \wedge \zeta = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_M \omega_\epsilon \wedge \zeta \tag{6.35}$$

for every smooth $(n-2)$-form $\zeta \in \Omega^{n-2}(M)$.

Since the two-forms ω_ϵ are closed, for any $\xi \in \Omega^{n-3}(M)$ we have

$$\langle \partial\Gamma, \xi\rangle = \langle \Gamma, d\xi\rangle = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_M \omega_\epsilon \wedge d\xi = \frac{1}{2\pi}\lim_{\epsilon\to 0}\int_M d(\omega_\epsilon \wedge \xi) = 0,$$

so Γ is a cycle. Since μ is $(n-2)$-rectifiable, Γ must be a rectifiable $(n-2)$-current: this can be seen by blow-up, applying [25, Proposition 7.3.5]. By (6.35), Γ is Poincaré dual to $c_1(L)$.

To complete the proof, it remains to show that Γ has integer multiplicity. By means of a diagonal selection of a subsequence, as in the previous subsection, we can deduce integrality at those points $p \in \mathrm{spt}(\mu)$ where μ and Γ blow up respectively to $\Theta_{n-2}(\mu, p)\mathcal{H}^{n-2} \llcorner T_p\Sigma$ and a multiple of $[T_p\Sigma]$, using the following lemma. Note that its hypotheses are verified thanks to Corollary 5.4 and the fact that $Z_{\beta_d}(u_\epsilon)$ necessarily converges to a subset of $T_p\Sigma$ in the local Hausdorff topology, after rescaling (see the proof of Proposition 6.2).

Since μ is $(n-2)$-rectifiable, we deduce that the limiting current Γ has integer multiplicity $\mathcal{H}^{n-2}$-a.e. on its support, as claimed. $\square$

Lemma 6.11 *On the Euclidean ball B_4^n, let $(u_\epsilon, \nabla_\epsilon)$ be a sequence of sections and connections in a trivial line bundle $L \to B_4^n$ (not necessarily satisfying any equation) for which $E_\epsilon(u_\epsilon, \nabla_\epsilon) \le \Lambda$, $e_\epsilon(u_\epsilon, \nabla_\epsilon) \to 0$ in $C^0_{loc}(B_4^n \setminus P)$ and $*\omega_\epsilon \to \theta_1[P]$ in $\mathcal{D}_{n-2}(B_4^n)$, where $P = \{0\} \times \mathbb{R}^{n-2}$. Then $\theta_1 \in 2\pi\mathbb{Z}$.*

Proof To begin, fix a test function $\varphi \in C^1_c(B_1^2 \times B_1^{n-2})$ of the form $\varphi(x^1, \dots, x^n) = \psi(x^1, x^2)\eta(x^3, \dots, x^n)$, with $\psi(x^1, x^2) = 1$ for $|(x^1, x^2)| \le \frac{1}{2}$. In the sequel, we shall omit the domain of integration when it equals $\mathbb{R}^n$. By assumption, we then have

$$\theta_1 \int_P \eta dx^3 \wedge \cdots \wedge dx^n = \lim_{\epsilon \to 0} \int \varphi \omega_\epsilon \wedge dx^3 \wedge \cdots \wedge dx^n.$$

Fixing trivializations of L over B_2^n, we write $\nabla_\epsilon = d - iA_\epsilon$ for some one-forms A_ϵ, so that $\omega_\epsilon = dA_\epsilon$, and the right-hand term in the preceding limit becomes

$$\begin{aligned}\int \omega_\epsilon \wedge (\varphi dx^3 \wedge \cdots \wedge dx^n) &= \int d(\varphi A_\epsilon \wedge dx^3 \wedge \cdots \wedge dx^n) \\ &\quad + \int A_\epsilon \wedge d\varphi \wedge dx^3 \wedge \cdots \wedge dx^n \\ &= \int \eta |u_\epsilon|^2 A_\epsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^n \\ &\quad + \int \eta(1 - |u_\epsilon|^2) A_\epsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^n.\end{aligned}$$

On B_2^n we can choose our trivializations so that $d^*A_\epsilon = 0$, and $A_\epsilon(\nu) = 0$ on ∂B_2^n (see the "Appendix"). We then have the L^2 control

$$\int_{B_2^n} |A_\epsilon|^2 \le C \int_{B_2^n} |dA_\epsilon|^2 \le C\epsilon^{-2}\Lambda \tag{6.36}$$

(see, e.g., [20, Theorem 4.8]), and consequently

$$\begin{aligned}\left|\int \eta(1-|u_\epsilon|^2)A_\epsilon \wedge d\psi \wedge dx^3 \wedge \cdots \wedge dx^n\right| &\leq C\|1-|u_\epsilon|^2\|_{C^0(\mathrm{spt}(\eta d\psi))}\|A_\epsilon\|_{L^1(B_2^n)}\\ &\leq C\Lambda^{1/2}\|\epsilon^{-1}(1-|u_\epsilon|^2)\|_{C^0(\mathrm{spt}(\eta d\psi))}\\ &\leq C\Lambda^{1/2}\|e_\epsilon(u_\epsilon,\nabla_\epsilon)\|^{1/2}_{C^0(\mathrm{spt}(\eta d\psi))}\\ &\to 0\end{aligned}$$

as $\epsilon \to 0$, where we have used the fact that $d\psi(x^1,x^2)=0$ for $|(x^1,x^2)|\leq \frac{1}{2}$, and the assumption that $e_\epsilon(u_\epsilon,\nabla_\epsilon)\to 0$ in $C^0_{loc}(B_2^n\setminus P)$.

Combining our computations thus far, we have arrived at the identity

$$\theta_1\int_P \eta dx^3\wedge\cdots\wedge dx^n = \lim_{\epsilon\to 0}\int \eta|u_\epsilon|^2 A_\epsilon\wedge d\psi\wedge dx^3\wedge\cdots\wedge dx^n.$$

Noting next that

$$||u_\epsilon|^2A_\epsilon - \langle du_\epsilon, iu_\epsilon\rangle| = |\langle\nabla_\epsilon u_\epsilon, iu_\epsilon\rangle| \leq e_\epsilon(u_\epsilon,\nabla_\epsilon)^{1/2},$$

and using again the hypothesis that $e_\epsilon(u_\epsilon,\nabla_\epsilon)\to 0$ uniformly on $\mathrm{spt}(\eta d\psi)$, the preceding identity yields

$$\begin{aligned}\theta_1\int_P \eta dx^3\wedge\cdots\wedge dx^n &= \lim_{\epsilon\to 0}\int \eta\langle du_\epsilon, iu_\epsilon\rangle\wedge d\psi\wedge dx^3\wedge\cdots\wedge dx^n\\ &= \lim_{\epsilon\to 0}\int \eta|u_\epsilon|^2(u_\epsilon/|u_\epsilon|)^*(d\theta)\wedge d\psi\wedge dx^3\wedge\cdots\wedge dx^n\\ &= \lim_{\epsilon\to 0}\int \eta(u_\epsilon/|u_\epsilon|)^*(d\theta)\wedge d\psi\wedge dx^3\wedge\cdots\wedge dx^n.\end{aligned}$$

Finally, since the one-form $(u_\epsilon/|u_\epsilon|)^*(d\theta)$ is closed on $\{u_\epsilon\neq 0\}$ and $d\eta\wedge dx^3\wedge\cdots\wedge dx^n=0$, integrating by parts on $(\mathbb{R}^2\setminus B^2_{1/2})\times\mathbb{R}^{n-2}$ we see that

$$\begin{aligned}&\int \eta(u_\epsilon/|u_\epsilon|)^*(d\theta)\wedge d\psi\wedge dx^3\wedge\cdots\wedge dx^n\\ &= \int_{\mathbb{R}^{n-2}}\eta(t)\int_{\partial B^2_{1/2}\times\{t\}}(u_\epsilon/|u_\epsilon|)^*(d\theta)\,dt\\ &= 2\pi\deg(u_\epsilon,P)\int_P\eta,\end{aligned}$$

where $\deg(u_\epsilon,P)$ stands for the degree of $(u_\epsilon/|u_\epsilon|)(\frac{1}{2}e^{i\theta},0)$. The statement follows. □

7 Examples from variational constructions

The goal of this section is to show that, for every closed manifold M and every line bundle $L \to M$ endowed with a Hermitian metric, there exist critical couples $(u_\epsilon, \nabla_\epsilon)$ for the Yang–Mills–Higgs functional E_ϵ, for ϵ small enough, in such a way that

$$0 < \liminf_{\epsilon \to 0} E_\epsilon(u_\epsilon, \nabla_\epsilon) \le \limsup_{\epsilon \to 0} E_\epsilon(u_\epsilon, \nabla_\epsilon) < \infty. \tag{7.1}$$

This will be easier when the line bundle is nontrivial, as in this case we can just take $(u_\epsilon, \nabla_\epsilon)$ to be a global minimizer for E_ϵ. The upper and lower bounds in (7.1) have the following immediate consequence—proved previously by Almgren [5] using GMT methods.

Corollary 7.1 *Every closed Riemannian manifold* (M^n, g) *supports a nontrivial stationary, integral* $(n-2)$*-varifold.*

Proof We can always equip M with the trivial line bundle $L := \mathbb{C} \times M$. As shown in the next subsection, there exists a sequence of critical couples $(u_\epsilon, \nabla_\epsilon)$ satisfying (7.1). The statement now follows from Theorem 6.1. □

7.1 Min–max families for the trivial line bundle

In this section we will show how min–max methods may be applied to the functionals E_ϵ to produce nontrivial critical points in the trivial bundle $L = \mathbb{C} \times M$ on an arbitrary closed manifold M of dimension $n \ge 2$. The min–max construction that we consider here is based on two-parameter families parametrized by the unit disk, similar to the constructions employed in [10,33] for the Ginzburg–Landau functionals—with several technical adjustments to account for the gauge-invariance and other features particular to the Yang–Mills–Higgs energies.

One can show that the families we consider induce a nontrivial class in $\pi_2(\mathcal{M})$ for the quotient

$$\mathcal{M} := \{(u, \nabla) \mid 0 \not\equiv u \in \Gamma(L),\ \nabla \text{ a Hermitian connection}\}/\{\text{gauge transformations}\},$$

and the analysis that follows can be reformulated in terms of min–max methods applied directly to $\mathcal{M}$, which can be given the structure of a Banach manifold.

Without loss of generality, we assume henceforth that M is connected. In some proofs we will also implicitly assume that $n = \dim(M) \ge 3$, leaving the obvious changes for $n = 2$ to the reader.

Definition 7.2 Fix $n = \dim(M) < p < \infty$. In what follows, $\widehat{X}$ will denote the Banach space of couples (u, A), where $u \in L^p(M, \mathbb{C})$ and $A \in \Omega^1(M, \mathbb{R})$, both of class $W^{1,2}$, with the norm

$$\|(u, A)\| := \|u\|_{L^p} + \|du\|_{L^2} + \|A\|_{L^2} + \|DA\|_{L^2}.$$

Denote by $X := \{(u, A) \in \widehat{X} : d^*A = 0\}$ the subspace consisting of those couples for which the connection form A is co-closed.

Note that, for $(u, A) \in X$, the full covariant derivative $\int_M |DA|^2$ is bounded by $C(M) \int_M (|A|^2 + |dA|^2)$: see, e.g., [20, Theorem 4.8] for a proof.

Definition 7.3 Given a form $A \in \Omega^1(M, \mathbb{R})$ in L^2, we denote by $h(A)$ the harmonic part of its Hodge decomposition, or equivalently the orthogonal projection of A onto the (finite-dimensional) space $\mathcal{H}^1(M)$ of harmonic one-forms.

Remark 7.4 Selection of a Coulomb gauge gives a continuous retraction $\mathcal{R} : \widehat{X} \to X$: namely, given a couple $(u, A) \in \widehat{X}$, consider the unique solution $\theta \in W^{2,2}(M, \mathbb{R})$ to the equation

$$\Delta\theta = d^*A,$$

with $\int_M \theta = 0$, and set

$$\mathcal{R}((u, A)) := (e^{i\theta}u, A + d\theta).$$

Note that the continuity of $(u, A) \mapsto d(e^{i\theta}u) = e^{i\theta}(du + iud\theta)$, from $\widehat{X}$ to L^2, follows from the fact that $L^p \cdot L^{2^*} \subseteq L^2$, where $2^* = \frac{2n}{n-2}$.

Throughout this section, $W(u) = f(|u|)$ will be a smooth radial function given by $W(u) = \frac{(1-|u|^2)^2}{4}$ for $|u| \le 3/2$, and satisfying $W(u), W'(u)[u] > 0$ for all $|u| > 1$. For technical reasons, we also find it convenient to require that

$$W(u) = |u|^p \quad \text{for } |u| \ge 2, \tag{G}$$

which evidently gives the additional estimates $|u|f'(|u|) + |u|^2 f''(|u|) \le C|u|^p$ for $|u| \ge 2$, for some constant C. For future use, observe also that the potential $W(u)$ then satisfies a simple bound of the form

$$(1 - |u|)^2 \le CW(u). \tag{7.2}$$

Proposition 7.5 *The functional E_ϵ is of class C^1 on $\widehat{X}$. Moreover, a couple (u, A) is critical in $\widehat{X}$ for E_ϵ if and only if $\mathcal{R}((u, A))$ is critical in X. Critical points are smooth up to change of gauge.*

Proof Given a point $(u, A) \in \widehat{X}$ and a pair $(v, B) \in \widehat{X}$ with $\|(v, B)\|_{\widehat{X}} \leq 1$, direct computation gives

$$E_\epsilon(u+v, A+B) = E_\epsilon(u, A) + 2\int_M \langle du - iuA, dv - ivA - iuB\rangle + 2\epsilon^2 \int_M \langle dA, dB\rangle + \epsilon^{-2}\int_M W'(u)[v] + O(\|(v, B)\|_{\widehat{X}}^2),$$

where we are using the fact that $\widehat{X} \cdot \widehat{X} \subseteq L^n \cdot L^{2^*} \subseteq L^2$ to see that

$$\|vA\|_{L^2}^2 + \|uB\|_{L^2}^2 + \|vB\|_{L^2}^2 + E_\epsilon(u, A)^{1/2}\|vB\|_{L^2} = O(\|(v, B)\|_{\widehat{X}}^2),$$

and we invoke our assumptions on the structure of W to see that

$$\int_M (W(u+v) - W(u)) = \int_M W'(u)[v] + O(\|(v, B)\|_{\widehat{X}}^2)$$

for fixed $(u, A) \in \widehat{X}$. It follows immediately that E_ϵ is C^1 on $\widehat{X}$, with differential

$$dE_\epsilon(u, A)[v, B] = \int_M (2\langle du - iuA, dv - ivA - iuB\rangle + 2\epsilon^2\langle dA, dB\rangle + \epsilon^{-2} W'(u)[v]).$$

To confirm the second statement, assume without loss of generality that v and B are smooth, and observe that

$$\mathcal{R}((u+tv, A+tB)) = (e^{ti\psi}\widetilde{u} + te^{i\theta+ti\psi}v, \widetilde{A} + tB + td\psi),$$

where $(\widetilde{u}, \widetilde{A}) := \mathcal{R}((u, A)) = (e^{i\theta}u, A + d\theta)$ and ψ solves $\Delta\psi = d^*B$. This easily gives

$$\mathcal{R}((u+tv, A+tB)) = \mathcal{R}((u, A)) + t(e^{i\theta}v + i\psi\widetilde{u}, B + d\psi) + o(t) \quad \text{in } X$$

and, using the gauge invariance $E_\epsilon = E_\epsilon \circ \mathcal{R}$, we deduce that

$$dE_\epsilon(u, A)[v, B] = dE_\epsilon(\widetilde{u}, \widetilde{A})[e^{i\theta}v + i\psi\widetilde{u}, B + d\psi]. \tag{7.3}$$

It follows that if $(\widetilde{u}, \widetilde{A})$ is critical for E_ϵ in X then (u, A) is critical for E_ϵ in $\widehat{X}$, as claimed. The converse is similar.

Finally, if (u, A) is critical for E_ϵ (in either $\widehat{X}$ or X), then applying the above formula for the differential with $v = (|u|-1)^+u/|u| \in W^{1,2}$ and $B = 0$ we get

$$0 = \int_M 2\langle (d - iA)u, (d - iA)v\rangle + \epsilon^{-2}\int_M W'(u)[v]$$
$$\geq \epsilon^{-2}\int_M |u|^{-1}(|u|-1)^+ W'(u)[u],$$

where we used the fact that $\langle u \otimes d((|u|-1)^+/|u|), \nabla u\rangle$ equals $|u|^{-1}|d|u||^2 \geq 0$ a.e. on $\{|u| > 1\}$ and vanishes elsewhere. Since $W'(u)[u] > 0$ on $\{|u| > 1\}$ by our assumption on W, we deduce that $|u| \leq 1$. Together with Proposition A.1 and Remark A.3 in the "Appendix", this implies that (u, A) is smooth in an appropriate (Coulomb) gauge. □

We next show that the functionals E_ϵ satisfy a suitable variant of the Palais–Smale condition on X, giving compactness of critical sequences for E_ϵ after an appropriate change of gauge. (Cf. [23] for similar results in the Seiberg–Witten setting.)

Proposition 7.6 *The functional E_ϵ satisfies the following form of the Palais–Smale condition: every sequence (u_j, A_j) in X with bounded energy and $dE_\epsilon(u_j, A_j) \to 0$ in X^* admits a subsequence converging strongly in X to a critical couple (u_∞, A_∞), up to possibly replacing (u_j, A_j) with*

$$v_j \cdot (u_j, A_j) := (v_j u_j, A_j + v_j^*(d\theta))$$

for suitable smooth harmonic functions $v_j : M \to S^1$.

Proof First, we show that the boundedness of $E_\epsilon(u_j, A_j)$ implies the boundedness of the sequence in X, up to a change of gauge as in the statement. The assumption (G) on the potential W gives

$$\int_M |u_j|^p \leq C + \int_M W(u_j) \leq C + E_\epsilon(u_j, A_j) \leq C, \tag{7.4}$$

that is, u_j is uniformly bounded in L^p.

Denote by $\Lambda \subseteq \mathcal{H}^1(M)$ the lattice in the space of harmonic one-forms given by

$$\Lambda := \{-v_j^*(d\theta) \mid v_j : M \to S^1 \text{ harmonic}\}$$
$$= \left\{h \in \mathcal{H}^1(M) : \int_\gamma h \in 2\pi\mathbb{Z} \text{ for every } \gamma \in C^1(S^1, M)\right\},$$

and let $\lambda_j \in \Lambda$ be a closest integral harmonic one-form to $h(A_j)$ (with respect to the L^2 norm, say, on $\mathcal{H}^1(M)$). Then $\lambda_j = -v_j^*(d\theta)$ for a suitable harmonic map $v_j : M \to S^1$, and

$$\|\lambda_j - h(A_j)\|_{L^2} \leq C(M).$$

Replacing (u_j, A_j) with the change of gauge $(v_j u_j, A_j - \lambda_j) \in X$, we can then assume that $h(A_j)$ is bounded.

By standard Hodge theory we can write

$$A_j = h(A_j) + d^*\xi_j$$

for some closed $\xi_j \in W^{2,2}$ satisfying $\Delta_H \xi_j = dA_j$ and $\|d^*\xi_j\|_{W^{1,2}} \leq C(M)\|dA_j\|_{L^2}$. Thus, given the energy bound $E_\epsilon(u_j, A_j) \leq C$, we see that

$$\|A_j\|_{W^{1,2}}^2 \leq C + 2\|d^*\xi_j\|_{W^{1,2}}^2 \leq C + C\|dA_j\|_{L^2}^2 \leq C,$$

whereby A_j is bounded in $W^{1,2}$ and, consequently, in L^{2^*}. As a consequence, we see next that

$$\begin{aligned}\|du_j\|_{L^2}^2 &\leq 2\int_M |du_j - iu_jA_j|^2 + 2\int_M |u_jA_j|^2\\ &\leq C + C\|u_j\|_{L^p}^2\|A_j\|_{L^{2*}}^2\\ &\leq C + C\|u_j\|_{L^p}^p;\end{aligned}$$

taking into account (7.4), we infer then that $\|du_j\|_{L^2}$ is also bounded as $j \to \infty$.

We have therefore shown that (u_j, A_j) is uniformly bounded in X as $j \to \infty$, so passing to subsequences we can assume that (u_j, A_j) converges pointwise a.e. and weakly (in X) to a limiting couple (u_∞, A_∞).

In particular, defining r by

$$\frac{1}{r} := \frac{1}{2} - \frac{1}{q} > \frac{1}{2} - \frac{1}{n} = \frac{1}{2^*},$$

where $n < q < p$ is an arbitrary fixed exponent, it follows from the compactness of the embedding $W^{1,2} \hookrightarrow L^r$ that

$$A_j \to A_\infty \text{ strongly in } L^r.$$

Moreover, the boundedness of u_j in L^p and the pointwise convergence to u_∞ give

$$u_j \to u_\infty \text{ strongly in } L^q. \tag{7.5}$$

By definition of r, this implies in particular that

$$\lim_{j,k\to\infty} u_j A_k = u_\infty A_\infty \text{ strongly in } L^2.$$

Next, compute

$$\begin{aligned} &dE_\epsilon(u_j, A_j)[u_j - u_k, A_j - A_k] \\ &\quad = \int_M 2\langle (d - iA_j)u_j, (d - iA_j)(u_j - u_k) - iu_j(A_j - A_k)\rangle \\ &\qquad + \int_M (2\epsilon^2\langle dA_j, d(A_j - A_k)\rangle + \epsilon^{-2}W'(u_j)[u_j - u_k]), \end{aligned}$$

and observe that, due to the L^2 convergence $u_j A_k \to u_\infty A_\infty$, the right-hand side equals

$$\int_M (2\langle (d - iA_j)u_j, d(u_j - u_k)\rangle + 2\epsilon^2\langle dA_j, d(A_j - A_k)\rangle + \epsilon^{-2}W'(u_j)[u_j - u_k]) + o(1)$$

as $j, k \to \infty$. For the difference

$$D_{j,k} := dE_\epsilon(u_j, A_j)[u_j - u_k, A_j - A_k] - dE_\epsilon(u_k, A_k)[u_j - u_k, A_j - A_k],$$

we then see that

$$\begin{aligned} D_{j,k} = \int_M &(2|d(u_j - u_k)|^2 + 2\epsilon^2|d(A_j - A_k)|^2 \\ &+ \epsilon^{-2}(W'(u_j) - W'(u_k))[u_j - u_k]) + o(1) \end{aligned}$$

as $j, k \to \infty$.

Now, by our assumption (G) on the structure of $W(u)$, it is not difficult to check (see, e.g., [17, Corollary 1]) that the zeroth order term in our computation for $D_{j,k}$ satisfies a lower bound

$$(W'(u_j) - W'(u_k))[u_j - u_k] \geq C^{-1}|u_j - u_k|^p - C|u_j - u_k|$$

for some constant $C > 0$. In particular, it follows now from the preceding computations and the L^1 convergence $u_j \to u_\infty$ that

$$D_{j,k} \geq \int_M (2|d(u_j - u_k)|^2 + 2\epsilon^2|d(A_j - A_k)|^2 + C^{-1}\epsilon^{-2}|u_j - u_k|^p) + o(1)$$

as $j, k \to \infty$. On the other hand, since $dE_\epsilon(u_j, A_j) \to 0$ and $(u_j - u_k, A_j - A_k)$ is bounded in X, we know also that

$$D_{j,k} \to 0 \quad \text{as } j, k \to \infty,$$

and it then follows that (u_j, A_j) is Cauchy in X. In particular, (u_j, A_j) converges strongly to (u_∞, A_∞), which necessarily satisfies

$$dE_\epsilon(u_\infty, A_\infty) = \lim_{j\to\infty} dE_\epsilon(u_j, A_j) = 0.$$

□

Having confirmed that the energies E_ϵ satisfy a Palais–Smale condition, we now argue in roughly the same spirit as [10,33] to produce nontrivial critical points via min–max methods. To begin, note that the space X splits as $\mathbb{C} \oplus Y$, where $\mathbb{C}$ is identified with the set of constant couples $(\alpha, 0)$ and

$$Y := \left\{(u, A) \in X : \int_M u = 0\right\}.$$

Definition 7.7 Let Γ denote the set of continuous families of couples $F : \overline{D} \to X$ parametrized by the closed unit disk $\overline{D}$, with

$$F(e^{i\theta}) = (e^{i\theta}, 0)$$

for all $\theta \in \mathbb{R}$. Equivalently, under the above identification $\mathbb{C} \subset X$, we require $F|_{\partial D} = \mathrm{id}$. We denote by $\omega_\epsilon(M)$ the "width" of Γ with respect to the energy E_ϵ, namely

$$\omega_\epsilon(M) := \inf_{F\in\Gamma} \max_{y\in\overline{D}} E_\epsilon(F(y)).$$

Thanks to Proposition 7.6, we can apply classical min–max theory for C^1 functionals on Banach spaces (see e.g. [15, Theorem 3.2]) to conclude that ω_ϵ is achieved as the energy of a smooth critical couple (u_ϵ, A_ϵ). In the following proposition, we show that $\omega_\epsilon(M)$ is positive, so that the corresponding critical couples (u_ϵ, A_ϵ) are nontrivial.

Proposition 7.8 *We have* $\omega_\epsilon(M) > 0$.

Proof We argue by contradiction, though the proof could be made quantitative. Since we are proving only the positivity $\omega_\epsilon(M) > 0$ at this stage—making no reference to the dependence on ϵ—in what follows we take $\epsilon = 1$ for convenience. Assume that we have a family $F \in \Gamma$ with $\max_{y \in \overline{D}} E(F(y)) < \delta$, with δ very small. Writing $F(y) = (u, A)$, this implies that

$$\|A - h(A)\|_{W^{1,2}} \leq C\|dA\|_{L^2} < C\delta^{1/2}, \quad \|DA\|_{L^2} \leq C(\delta^{1/2} + \|h(A)\|). \tag{7.6}$$

When $b_1(M) \neq 0$, some additional work is required to deduce that the harmonic part $h(A)$ of A must also be small for all couples $(u, A) = F(y)$ in the family. In particular, we will need to employ the following lemma, showing that $h(A)$ lies close to the integral lattice $\Lambda \subset \mathcal{H}^1(M)$ when $E(u, A) < \delta$.

Lemma 7.9 *There exists $C(M) < \infty$ such that if $(u, A) \in X$ satisfies $E(u, A) < \delta$, with δ small enough, then*

$$\mathrm{dist}(h(A), \Lambda) \leq C\delta^{1/2}.$$

Proof As in [33], it is convenient to define a box-type norm $|\cdot|_b$ on the space $\mathcal{H}^1(M)$ of harmonic one-forms as follows. Fix a collection $\gamma_1, \ldots, \gamma_{b_1(M)} \in C^\infty(S^1, M)$ of embedded loops generating $H_1(M; \mathbb{Q})$ and, for $h \in \mathcal{H}^1(M)$, set

$$|h|_b := \max_{1 \leq i \leq b_1(M)} \left| \int_{\gamma_i} h \right|. \tag{7.7}$$

Since $\mathcal{H}^1(M)$ is finite-dimensional, this is of course equivalent to any other norm on $\mathcal{H}^1(M)$. Assuming for simplicity that M is orientable, we may fix a collection of diffeomorphisms $\Phi_i : B_1^{n-1}(0) \times S^1 \to T(\gamma_i)$ onto tubular neighborhoods $T(\gamma_i)$ of γ_i, such that $\Phi_i(0, \theta) = \gamma_i(\theta)$. For every $t \in B_1^{n-1}$, set $\gamma_i^t(\theta) := \Phi_i(t, \theta)$.

Suppose now that $(u, A) \in X$ satisfies the energy bound

$$E(u, A) = \int_M (|du - iuA|^2 + |dA|^2 + W(u)) < \delta. \tag{7.8}$$

As a consequence of the curvature bound $\|dA\|_{L^2} \leq \delta^{1/2}$ and the definition of X, it follows that

$$\|A - h(A)\|_{L^2}^2 \leq C\delta$$

as well. As in the proof of Proposition 7.6, applying a gauge transformation $\phi \cdot (u, A)$ by an appropriate choice of harmonic map $\phi : M \to S^1$, we may assume moreover that

$$|h(A)|_b = \mathrm{dist}_b(h(A), \Lambda) \leq \pi,$$

which together with the energy bound (7.8) and the definition of X leads us to the estimate

$$\int_M |A|^2 \leq C(M). \tag{7.9}$$

(Note that making a harmonic change of gauge preserves not only the energy $E(u, A)$, but also the distance $\mathrm{dist}_b(h(A), \Lambda)$, so it indeed suffices to establish the desired estimate in this gauge.)

Combining these estimates with a simple Fubini argument, we see that there exists a nonempty set S of $t \in B_1^{n-1}$ for which

$$\int_{\gamma_i^t} (|du - iuA|^2 + |dA|^2 + W(u)) < C\delta, \tag{7.10}$$

$$\int_{\gamma_i^t} |A - h(A)|^2 < C\delta, \tag{7.11}$$

and

$$\int_{\gamma_i^t} |A|^2 \leq C. \tag{7.12}$$

Recalling the pointwise bound (7.2) for $W(u)$, observe next that

$$|d(1 - |u|)^2| = 2(1 - |u|)|d|u|| \leq CW(u) + |du - iuA|^2,$$

so that, along a curve γ_i^t satisfying (7.10), it follows that

$$\|(1 - |u|)^2\|_{C^0} \leq C\|(1 - |u|)^2\|_{W^{1,1}} \leq C\delta. \tag{7.13}$$

Now, choose $\delta < \delta_1(M)$ sufficiently small that (7.13) gives

$$\|1 - |u|\|_{C^0} \leq \eta < \frac{1}{2}$$

on γ_i^t, so that $\phi := u/|u|$ defines there an S^1-valued map $\phi : \gamma_i^t \to S^1$, whose degree is given by

$$2\pi \deg(\phi) = \int_{\gamma_i^t} |u|^{-2}\langle du, iu\rangle.$$

When (7.10)–(7.12) hold, we observe next that

$$\int_{\gamma_i^t} |u|^2|A - |u|^{-2}\langle iu, du\rangle| = \int_{\gamma_i^t} |\langle iu, iuA - du\rangle| \leq C\delta^{1/2}.$$

Since $|u| \geq \frac{1}{2}$ on γ_i^t, it follows that

$$\left|2\pi \deg(\phi) - \int_{\gamma_i^t} A\right| \leq \int_{\gamma_i^t} |A - |u|^{-2}\langle iu, du\rangle| \leq C\delta^{1/2} \tag{7.14}$$

as well. Combining this with (7.11), we then deduce that

$$\left|2\pi \deg(\phi) - \int_{\gamma_i^t} h(A)\right| \leq C\delta^{1/2}. \tag{7.15}$$

On the other hand, we already made a gauge transformation so that

$$\left|\int_{\gamma_i} h(A)\right| = \left|\int_{\gamma_i^t} h(A)\right| \leq \pi.$$

So, for δ chosen sufficiently small that $C\delta^{1/2} < \pi$, it follows that the degree $\deg(\phi) = 0$. In particular, we can now conclude that

$$|h(A)|_b = \max_i \left|\int_{\gamma_i} h(A)\right| \leq C\delta^{1/2},$$

giving the desired estimate. □

Remark 7.10 If M is not orientable, we have the weaker conclusion $\operatorname{dist}(h(A), \frac{1}{2}\Lambda) \leq C\delta^{1/2}$ (still sufficient for the sequel): indeed, whenever γ_i reverses the orientation, we can still parametrize a double cover of $T(\gamma_i)$ in the same way, with γ_i^t homotopic to γ_i traveled twice; in this case, the bound (7.15) implies that $2\int_{\gamma_i} h(A) = \int_{\gamma_i^t} h(A)$ has distance to $2\pi\mathbb{Z}$ bounded by $C\delta^{1/2}$, from which the claim follows.

Returning to the proof of Proposition 7.8, suppose again that we have a family $\overline{D} \ni y \mapsto F(y) \in X$ in Γ with

$$\max_{y \in \overline{D}} E(F(y)) < \delta.$$

For $\delta < \delta_1(M)$ sufficiently small, it follows from the lemma that $\operatorname{dist}_b(h(A), \Lambda) < \pi$ for every couple $(u, A) = F(y)$ in the family. In particular, since the assignment $(u, A) \mapsto h(A)$ gives a continuous map $X \to \mathcal{H}^1(M)$, and since $h(A) = A = 0$ for $y \in \partial\overline{D}$, it follows that 0 is the nearest point in the lattice Λ to $h(A)$ for every $y \in \overline{D}$, and the estimate therefore becomes

$$\|h(A)\| \leq C\delta^{1/2}.$$

In particular, combining this with (7.6), we see now that

$$\|A\|_{W^{1,2}} \le C\delta^{1/2} \tag{7.16}$$

for every couple $(u, A) = F(y)$ in the family.

Now, for $(u, A) = F(y)$, our structural assumption (G) on $W(u)$ gives

$$\|u\|_{L^p}^p \le C + E(u, A) \le C + \delta,$$

which together with the smallness

$$\|A\|_{L^{2^*}} \le C\|A\|_{W^{1,2}} \le C\delta^{1/2}$$

of A in L^{2^*} (recalling that $p > n$) gives

$$\int_M |uA|^2 \le C\delta.$$

Combining this with the fact that $\int_M |du - iuA|^2 \le E(u, A) < \delta$ by assumption, we then deduce that

$$\int_M |du|^2 \le C\delta$$

as well.

Finally, by (7.2) and the Poincaré inequality, we have

$$\begin{aligned} 1 - \left|\frac{1}{\mathrm{vol}(M)}\int_M u\right| &\le C\int_M |1 - |u|| + C\int_M \left|u - \frac{1}{\mathrm{vol}(M)}\int_M u\right| \\ &\le C\Big(\int_M W(u)\Big)^{1/2} + C\Big(\int_M |du|^2\Big)^{1/2} \\ &\le C\delta^{1/2}. \end{aligned}$$

As a consequence, we find that $\int_M u_y$ is nonzero for all $(u_y, A_y) = F(y)$ in the family. But then the averaging map

$$\overline{D} \to \mathbb{C}, \quad y \mapsto \frac{\int_M u_y}{|\int_M u_y|} \tag{7.17}$$

gives a retraction $\overline{D} \to \partial\overline{D}$, whose nonexistence is well known. This gives the desired contradiction. □

Having shown positivity $\omega_\epsilon(M) > 0$ of the min–max energies, we can now deduce the lower bound in (7.1) from the following simple fact.

Proposition 7.11 *There exist $c(M) > 0$ and $\epsilon_0(M) > 0$ such that the following holds, for $\epsilon \leq \epsilon_0$. If (u, ∇) is critical for the functional E_ϵ, then either $E_\epsilon(u, \nabla) \geq c$ or $E_\epsilon(u, \nabla) = 0$.*

Remark 7.12 For future reference, we make the obvious observation that the trivial case $E_\epsilon(u, \nabla) = 0$ can only occur when the bundle L is trivial.

Proof By Proposition 7.5, critical points are smooth up to change of gauge. We claim that, whenever $E_\epsilon(u, \nabla) > 0$, u has to vanish at some point $x_0 \in M$. Once we have this, assume e.g. $E_\epsilon(u, \nabla) \leq 1$; Corollary 4.4 (with $\Lambda = 1$) gives a constant $\epsilon_0 > 0$ such that $r^{2-n} E_\epsilon(u, \nabla, B_r(x_0))$ has a lower bound independent of ϵ and r, for any radius $\epsilon < r < \operatorname{inj}(M)$, provided that $\epsilon \leq \epsilon_0$.

We show the contrapositive, namely we assume that u is nowhere vanishing and show that the energy is zero. Note that L must be trivial and we can use the section $\frac{u}{|u|}$ to identify L isometrically with the trivial line bundle $\mathbb{C} \times M$, equipped with the canonical Hermitian metric. Under this identification, $u : M \to \mathbb{C}$ takes values into positive real numbers. Writing $\nabla = d - iA$ and observing that $\langle \nabla u, iu \rangle = -|u|^2 A$, (2.5) becomes

$$\epsilon^2 d^* dA + |u|^2 A = 0.$$

Integrating against A we get $\int_M (\epsilon^2 |dA|^2 + u^2 |A|^2) = 0$, so $A = 0$ and ∇ is the trivial connection. At a minimum point y_0 for u, (3.4) gives

$$0 \leq \frac{1}{2} \Delta |u|^2 = |du|^2 - \frac{1}{2\epsilon^2}(1 - |u|^2)|u|^2 = -\frac{1}{2\epsilon^2}(1 - u^2)u^2,$$

which forces $u(y_0) \geq 1$ and thus $u = 1$ everywhere, giving $E_\epsilon(u, \nabla) = 0$. □

Finally, we turn to the uniform upper bound. In the next statement, $L \to M$ is a Hermitian line bundle with a fixed Hermitian reference connection ∇_0. We identify any other Hermitian connection ∇ with the real one-form A such that $\nabla s = \nabla_0 s - is \otimes A$ for all sections s.

Proposition 7.13 *Given a smooth section $u : M \to L$, we can find a smooth couple (u', A') such that*

$$E_\epsilon(u', A') \leq C\epsilon^{-2} \operatorname{vol}\left(\left\{|u| \leq \frac{1}{2}\right\}\right) + C(1 + \epsilon^2 \|\nabla_0 u\|_{L^\infty}^2) \int_{\{|u| \leq \frac{1}{2}\}} |\nabla_0 u|^2 + C\epsilon^2 \int_M |\omega_0|^2 \tag{7.18}$$

for a universal constant C.

Proof On $\{u \neq 0\}$ we let

$$w := \frac{u}{|u|}, \quad iw \otimes A := \nabla_0 w.$$

Note that the compatibility of ∇_0 with the Hermitian metric on L forces $\langle \nabla_0 w, w \rangle = 0$, so that A is a real one-form.

We fix a smooth function $\rho : [0, \infty] \to [0, 1]$ with

$$\rho(t) = 0 \text{ for } t \leq \frac{1}{4}, \quad \rho(t) = 1 \text{ for } t \geq \frac{1}{2}$$

and we set

$$(u', A') := \rho(|u|)(w, A),$$

where the right-hand side is meant to be zero on $\{u = 0\}$.

Writing $F_{\nabla_0} = -i\omega_0$, observe that $(\nabla_0 - iA)w = 0$, hence

$$|dA + \omega_0| = |F_A| = 0 \quad \text{on } \{u \neq 0\}.$$

In particular, $e_\epsilon(u', A') = 0$ on $\{|u| > \frac{1}{2}\}$.

From the estimates $|d|u|| \leq |\nabla_0 u|$ and $|A| = |\nabla_0 w| \leq 2|u|^{-1}|\nabla_0 u|$, it follows that also

$$\begin{aligned} |\nabla_0 u'| &\leq C|\nabla_0 u|, \\ |A'| &\leq C|\nabla_0 u|, \\ |dA'| &\leq |\rho'(|u|)d|u| \wedge A| + |\omega_0| \leq C|\nabla_0 u||d|u|| + |\omega_0|, \end{aligned}$$

and the statement follows immediately. □

Proof of (7.1) The method used in [33, Section 3] gives a continuous map $H : \overline{D} \to W^{1,2} \cap C^0(M, \mathbb{C})$ such that $H(y) \equiv y$ for $y \in \partial D$ and

$$\begin{aligned} \|dH(y)\|_{L^\infty} &\leq C\epsilon^{-1}, \\ \int_{\{|H(y)| \leq \frac{3}{4}\}} |dH(y)|^2 &\leq C, \\ \operatorname{vol}\left(\left\{|H(y)| \leq \frac{3}{4}\right\}\right) &\leq C\epsilon^2 \end{aligned} \tag{7.19}$$

for all $y \in \overline{D}$—the full Dirichlet energy having a worse bound $\int_M |dH(y)|^2 \leq C \log \epsilon^{-1}$, which is the natural one in the setting of Ginzburg–Landau. By approximation, we can assume that H takes values in $C^\infty(M, \mathbb{C})$, continuously in y, and still satisfies the same uniform bounds (7.19) (possibly increasing C and replacing $\frac{3}{4}$ with $\frac{1}{2}$).

To each section $H(y)$ of the trivial line bundle, Proposition 7.13 assigns in a continuous way an element $F(y) \in X$. From the way $F(y)$ is constructed, it is clear that $F \in \Gamma$. Finally, combining (7.18) with (7.19) gives

$$\omega_\epsilon(M) \leq \max_{y \in \overline{D}} E_\epsilon(F(y)) \leq C.$$

□

7.2 Minimizers for nontrivial line bundles

Suppose now that L is a nontrivial line bundle, equipped with a Hermitian metric. Fix a smooth Hermitian connection ∇_0 and identify any other Hermitian connection ∇ with the real one-form A such that

$$\nabla = \nabla_0 - iA.$$

We can define $\widehat{X}$ and X as in the previous subsection. With this notation, observe that the curvature of ∇ is given by

$$F_\nabla = F_{\nabla_0} - i dA.$$

Hence, writing $F_{\nabla_0} = -i\omega_0$, we have

$$E_\epsilon(u, \nabla) = \int_M |\nabla_0 u - iu \otimes A|^2 + \epsilon^{-2} \int_M W(u) + \epsilon^2 \int_M |\omega_0 + dA|^2.$$

Definition 7.14 For a fixed $n < p < \infty$, we define $\widehat{X}$ to be the Banach space of couples (u, A), where $u : M \to L$ is an L^p section and $A \in \Omega^1(M, \mathbb{R})$, both of class $W^{1,2}$, with the norm

$$\|(u, A)\| := \|u\|_{L^p} + \|\nabla_0 u\|_{L^2} + \|A\|_{L^2} + \|DA\|_{L^2}.$$

We let $X := \{(u, A) \in \widehat{X} : d^* A = 0\}$.

The analogous statements to Remark 7.4 and Propositions 7.5 and 7.6 hold, with identical proofs (replacing du and uA with $\nabla_0 u$ and $u \otimes A$, respectively).

Arguing as in the proof of Proposition 7.6, it is easy to see that a minimizing sequence for E_ϵ in X converges weakly—up to change of gauge—to a global minimizer (u_ϵ, A_ϵ). We now show that the energy of these minimizers enjoys uniform upper and lower bounds as $\epsilon \to 0$.

Proof of (7.1) The lower bound in (7.1) follows directly from Proposition 7.11 and Remark 7.12. In order to obtain the upper bound, pick a smooth section $s : M \to L$ transverse to the zero section (see, e.g., [24, Theorem IV.2.1]) and let $N := \{s = 0\}$, which is a smooth embedded $(n-2)$-submanifold of M. Proposition 7.13 applied to $\epsilon^{-1}s$ gives a couple $(u'_\epsilon, A'_\epsilon)$ with

$$E_\epsilon(u'_\epsilon, A'_\epsilon) \le C\epsilon^{-2}\,\mathrm{vol}\left(\left\{|\epsilon^{-1}s| \le \frac{1}{2}\right\}\right) + C\epsilon^2 \int_M |\omega_0|^2.$$

By transversality of s, the set $\{|s| \le \frac{\epsilon}{2}\}$ is contained in a $C(s)\epsilon$-neighborhood of N, whose volume is bounded by $C(s)\epsilon^2$. We infer that

$$E_\epsilon(u_\epsilon, A_\epsilon) \le E_\epsilon(u'_\epsilon, A'_\epsilon) \le C\epsilon^{-2}\,\mathrm{vol}\left(\left\{|s| \le \frac{\epsilon}{2}\right\}\right) + C \le C.$$

□

Remark 7.15 When M is oriented, N can be oriented in such a way that $[N] \in H_{n-2}(M, \mathbb{R})$ is Poincaré dual to the Euler class $e(L) \in H^2(M, \mathbb{R})$ of the line bundle, which equals the first Chern class $c_1(L)$. The fact that the energy of our competitors concentrates along N suggests that, given a sequence of global minimizers (u_ϵ, A_ϵ), up to subsequences the corresponding energy concentration varifold is induced by an integral mass-minimizing current whose homology class is Poincaré dual to $c_1(L)$. Theorem 6.10 provides the natural candidate Γ, which also satisfies $|\Gamma| \le \mu$.

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Appendix: Interior regularity in the Coulomb gauge

In this short appendix, we describe the essential ingredients needed to establish local regularity in the Coulomb gauge for finite-energy critical points (u, A) of the $(\epsilon = 1)$ abelian Higgs energy $E(u, A)$, collecting some estimates which will be of use elsewhere in the paper.

Consider the manifold with boundary $(\overline{\Omega}^n, g)$ given by a smooth, contractible domain $\Omega^n \subset\subset \mathbb{R}^n$ equipped with a C^2 metric g, and let $L \cong \mathbb{C} \times \Omega$ be the trivial line bundle over Ω, with the standard Hermitian structure. With respect to the metric g, we then define the Yang–Mills–Higgs energy

$$E(u, A) := \int_\Omega e(u, A) = \int_\Omega |du - iu \otimes A|^2 + |dA|^2 + W(u)$$

as in the preceding section. By (the first part of) Proposition 7.5, it is easy to see that a pair (u, A) in $W^{1,2}$ with

$$|u| \leq 1 \tag{A.1}$$

is a critical point for E (with respect to smooth perturbations supported in Ω) if and only if the equations

$$d^* dA = \langle du - iu \otimes A, iu \rangle, \tag{A.2}$$

$$\Delta u = 2\langle i du, A \rangle + |A|^2 u - \frac{1}{2}(1 - |u|^2)u - i(d^* A)u \tag{A.3}$$

are satisfied distributionally in Ω, where all geometric quantities and operators are defined with respect to the metric g.

Now, given a pair (u, A) in $W^{1,2}$ satisfying (A.2)–(A.3) and

$$E(u, A) \leq \Lambda < \infty, \tag{A.4}$$

we can select a *local Coulomb gauge* adapted to Ω as follows. Denote by $\theta \in W^{2,2}(\Omega, \mathbb{R})$ the unique solution of the Neumann problem

$$\Delta \theta = d^* A \text{ in } \Omega; \quad \frac{\partial \theta}{\partial \nu} = -A(\nu) \text{ on } \partial\Omega \tag{A.5}$$

with zero mean $\int_\Omega \theta = 0$. Then the gauge-transformed pair

$$(\widetilde{u}, \widetilde{A}) := (e^{i\theta} u, A + d\theta)$$

lies in $W^{1,2}$ and continues to satisfy (A.2)–(A.3), with

$$E(\widetilde{u},\widetilde{A})=E(u,A)\le\Lambda,$$

but now with the additional constraints

$$d^*\widetilde{A}=0 \text{ on } \Omega;\quad \widetilde{A}(\nu)=0 \text{ on } \partial\Omega. \tag{A.6}$$

For the remainder of the section, we will assume that the pair (u, A) is already in the Coulomb gauge on Ω, so that A satisfies (A.6). Note that (A.2)–(A.3) then become

$$\Delta u=2\langle idu,A\rangle+|A|^2u-\frac{1}{2}(1-|u|^2)u, \tag{A.7}$$

$$\Delta_H A=\langle du-iu\otimes A,iu\rangle. \tag{A.8}$$

We now establish the local regularity for critical points (u, A) in the Coulomb gauge, giving in particular local estimates for (u, A) in $W^{2,q}$ norms.

Proposition A.1 *Let* (u, A) *solve* (A.2)–(A.3) *in the Coulomb gauge* (A.6) *on* (Ω, g), *with* $|u|\le 1$. *If*

$$E(u,A;\Omega)\le\Lambda \tag{A.9}$$

and

$$\|g\|_{C^2}+\|g^{-1}\|_{C^2}\le\Lambda, \tag{A.10}$$

then for every compactly supported subdomain $\Omega'\subset\subset\Omega$ *and* $q\in(1,\infty)$ *there exists* $C_q(\Lambda,\Omega,\Omega')<\infty$ *such that*

$$\|u\|_{W^{2,q}(\Omega')}+\|A\|_{W^{2,q}(\Omega')}\le C_q. \tag{A.11}$$

Proof To begin, note that (A.8) and standard Bochner–Weitzenböck identities give the (weak) subequation

$$\begin{aligned}\Delta\frac{1}{2}|A|^2&=-\langle\Delta_H A,A\rangle+|DA|^2+\mathrm{Ric}(A,A)\\&\ge-|du-iu\otimes A||A|+|DA|^2-C(\Lambda)|A|^2\end{aligned} \tag{A.12}$$

for $|A|^2$. On the other hand, as in Sect. 3, we also obtain from (A.3) the relation

$$\Delta\frac{1}{2}|u|^2=|du-iu\otimes A|^2-\frac{1}{2}(1-|u|^2)|u|^2. \tag{A.13}$$

Recalling that $|u| \leq 1$ and using Young's inequality, we can combine (A.12)–(A.13) to find an estimate of the form

$$\frac{1}{2}\Delta(|A|^2 + |u|^2) \geq \alpha(|DA|^2 + |du|^2) - C(\alpha, \Lambda)|A|^2 - C(\Lambda), \quad \text{(A.14)}$$

for any $0 < \alpha < 1$.

By standard estimates for one-forms A satisfying (A.6) (see, e.g., [20, Theorem 4.8]), we have the global L^2 bound

$$\|A\|_{W^{1,2}(\Omega)} \leq C(\Lambda, \Omega)\|dA\|_{L^2(\Omega)} \leq C(\Lambda, \Omega),$$

hence $|u|$, $|A|$ are both bounded in $W^{1,2}$ in terms of Λ (and Ω).

Note that (A.8) gives a local $W^{2,2}$ bound on A, by standard elliptic regularity. This, together with Sobolev embedding and (A.7), gives

$$\|u\|_{W^{2,p}(\Omega_0)} + \|A\|_{W^{2,2}(\Omega_0)} + \||A|^p\|_{W^{1,2}(\Omega_0)} \leq C(\Lambda, \Omega, \Omega_0) \quad \text{(A.15)}$$

for all $\Omega_0 \subset\subset \Omega$ and some $1 < p < 2$, depending only on n. We need the following observation, stated and proved separately for the sake of clarity.

Lemma A.2 *Defining* $f \in W^{1,2}(\Omega)$ *by*

$$f := (1 + |A|^2 + |u|^2)^{1/2},$$

we have the subequation

$$\Delta f^p \geq -C(p, \Lambda) f^p \quad \text{(A.16)}$$

and, for all $\Omega_0 \subset\subset \Omega$,

$$\|f^p\|_{W^{1,2}(\Omega_0)} \leq C(\Lambda, \Omega, \Omega_0).$$

Proof Since $u \in L^\infty \cap W^{1,2} \cap W^{2,p}_{loc}$ and $A \in W^{2,2}_{loc}$, a standard approximation argument shows that $|u|^2, |A|^2 \in W^{2,1}_{loc}$, so that (A.14) holds pointwise a.e.

Likewise, we have $f \in W^{2,1}_{loc}$ and the chain rule applies, giving

$$\Delta f = f^{-1}(|DA|^2 + |du|^2 - \langle A, D^*DA\rangle + \langle u, \Delta u\rangle) - f^{-1}|df|^2$$

pointwise. The first term equals $f^{-1}\Delta\frac{1}{2}f^2$, so recalling (A.14) we obtain

$$\Delta f \geq \alpha f^{-1}(|DA|^2 + |du|^2) - C(\alpha, \Lambda) f - f^{-1}|df|^2.$$

Also, since $f \in W^{1,2} \cap W^{2,p}_{loc}$, we have the pointwise inequalities

$$\begin{aligned}\Delta f^p &= p(p-1) f^{p-2}|df|^2 + pf^{p-1}\Delta f\\ &\geq p\alpha f^{p-2}(|DA|^2 + |du|^2) - C(\alpha, \Lambda) f^p + p(p-2) f^{p-2}|df|^2\\ &\geq p(\alpha + p - 2) f^{p-2}|df|^2 - C(\alpha, \Lambda) f^p.\end{aligned}$$

Choosing $\alpha := 2 - p$, inequality (A.16) follows. The second claim is an easy consequence of (A.15) and the fact that $|u| \leq 1$. □

Returning to the proof of Proposition A.1, we can now apply Moser iteration to (A.16), obtaining in particular that

$$\|A\|_{L^\infty(\Omega_1)} \leq C(\Lambda, \Omega, \Omega_1) \tag{A.17}$$

for any $\Omega_1 \subset\subset \Omega$.

Now, fixing some intermediate domain $\Omega' \subset\subset \Omega_1 \subset\subset \Omega$ between Ω' and Ω, (A.7) together with the $L^\infty(\Omega_1)$ estimate for A give pointwise bounds of the form

$$|\Delta u| \leq C(\Lambda, \Omega, \Omega_1)(|du| + 1) \quad \text{in } \Omega_1. \tag{A.18}$$

And since

$$|du| \leq |du - iu \otimes A| + |A| \leq e(u, A) + C$$

in Ω_1, we obtain from the energy bound $E(u, A) \leq \Lambda$ and (A.18) the simple estimate

$$\|\Delta u\|_{L^2(\Omega_1)} \leq C(\Lambda, \Omega, \Omega_1),$$

and consequently

$$\|u\|_{W^{2,2}(\Omega_2)} \leq C$$

for any $\Omega' \subset\subset \Omega_2 \subset\subset \Omega_1$. Returning to the pointwise bound (A.18), we can now employ a simple iteration argument—combining L^q regularity theory with the Sobolev embedding $W^{2,r} \hookrightarrow W^{1,\frac{rn}{n-r}}$—over successive domains between Ω' and Ω, to arrive at the desired $W^{2,q}$ estimates for u.

Returning finally to (A.8), it therefore follows from the preceding estimates that

$$\|A\|_{L^\infty(\Omega'')} + \|\Delta_H A\|_{L^\infty(\Omega'')} \leq C(\Lambda, \Omega, \Omega'')$$

for some intermediate domain $\Omega' \subset\subset \Omega'' \subset\subset \Omega$. In particular, this gives us upper bounds for $\|\Delta A\|_{L^q(\Omega'')}$ for every $q \in (1, \infty)$, and L^q regularity theory therefore gives us the desired estimates for A in $W^{2,q}(\Omega')$. □

Finally, we remark that higher regularity of u and A in the Coulomb gauge follows in a standard way—e.g., via Schauder theory—from the $W^{2,q}$ estimates obtained in the preceding proposition.

Remark A.3 With local regularity established, note that it is easy to find a globally smooth couple $(\widetilde{u}, \widetilde{\nabla})$ gauge equivalent to any critical pair (u, ∇) for E_ϵ on $L \to M$. Indeed, for any critical pair (u, ∇) with $u \in W^{1,2} \cap L^\infty$ and $\nabla = \nabla_0 - iA$ (where ∇_0 is a smooth reference connection and $A \in W^{1,2}$), it follows from the local regularity results above that the gauge-invariant objects $|u|^2$ and $dA = F_\nabla - F_{\nabla_0}$ are smooth globally. Making a change of gauge $(u, \nabla) \to (\widetilde{u}, \widetilde{\nabla} = \nabla_0 - i\widetilde{A})$ such that

$$d\widetilde{A} = dA \quad \text{and} \quad d^*\widetilde{A} = 0,$$

it follows from the smoothness of dA that the new connection $\widetilde{\nabla} = \nabla_0 - i\widetilde{A}$ is smooth. And since $\widetilde{u}$ satisfies

$$\widetilde{\nabla}^*\widetilde{\nabla}\widetilde{u} = \frac{1}{2\epsilon^2}(1 - |u|^2)\widetilde{u}$$

where both $\widetilde{\nabla}$ and $|u|^2$ are smooth, standard results for linear elliptic equations imply that $\widetilde{u} \in \Gamma(L)$ is a smooth section as well.

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