ITERATIVE MULTISCALE FINITE VOLUME METHOD FOR MULTIPHASE FLOW IN POROUS MEDIA WITH COMPLEX PHYSICS

A dissertation submitted to

ETH ZURICH

for the degree of

Doctor of Sciences

presented by

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2011
Abstract

In this thesis, the multiscale finite-volume (MSFV) method for the solution of elliptic problems is extended to an efficient iterative algorithm that converges to the fine-scale numerical solution. The localization errors in the MSFV method are systematically reduced by updating the local boundary conditions with global information. This iterative multiscale finite-volume (i-MSFV) method allows the conservative reconstruction of the velocity field after any iteration, and the MSFV method is recovered if the velocity field is reconstructed after the first iteration. Both the i-MSFV and the MSFV methods lead to substantial computational savings, where an approximate but locally conservative solution of an elliptic problem is required. In contrast to the MSFV method, the i-MSFV method allows a systematic reduction of the errors in the multiscale approximation. Numerical convergence of the method is verified for various test cases. To demonstrate the efficiency of the method for multiphase transport in porous media, it is shown that it is sufficient to apply the iterative smoothing procedure only infrequently, i.e. not every time step. This result shows that the overall efficiency of the i-MSFV algorithm is comparable with that of the original MSFV method. At the same time, the solutions are significantly improved. Next, the MSFV method which was originally developed for the solution of heterogeneous elliptic problems is extended for parabolic problems arising from compressible flows. The proposed general MSFV method, opposed to previous methods, is constructed based on the basis and correction functions which are solutions of full governing equations in localized domains. At the same time, to enhance the computational efficiency of the scheme, the basis functions are kept pressure independent in a consistent way. This general approach requires no additional assumptions and its high efficiency and accuracy is demonstrated for various challenging test cases. In addition, to improve the quality of the results and also to extend the scheme for highly anisotropic heterogeneous problems, it is combined with the iterative MSFV (i-MSFV) method for parabolic problems. As one iterates, the i-MSFV solutions of compressible multiphase problems (parabolic problems) converge to the corresponding fine-scale reference solutions in the same way as demonstrated previously for incompressible cases (elliptic problems). Studies of its efficiency are also presented for many test cases. Furthermore, an i-MSFV method is devised for the simulation of multiphase flow in fractured porous media in the context of a hierarchical fracture modeling framework. Motivated by the small pressure change inside highly conductive fractures, the fully coupled system is split into smaller systems, which are then sequentially solved. This splitting technique results in only one additional degree of freedom (DOF) for each
connected fracture network appearing in the matrix system. For the solution of the resulting algebraic system, an i-MSFV method is introduced. In addition to the previously developed local basis and correction functions, local fracture functions are introduced to accurately capture the fractures at the coarse scale. In this multiscale approach there exists one fracture function per network and local domain, and in the coarse scale problem there appears only one additional DOF per connected fracture network. Numerical results are presented for validation and verification of the method and to investigate its computational efficiency. Finally, it is demonstrated that the new method is an effective, efficient multiscale approach to simulate realistic multiphase flows in fractured heterogeneous porous media. Also, the MSFV and i-MSFV methods are extended to account for non-matching coarse grids arising from faulted porous media. To obtain accurate quantities at the coarse scale, basis and correction functions are computed in extended local domains near the fault regions. With the new framework it is shown that the MSFV results are very close to the fine-scale reference solutions. Convergence histories of the new i-MSFV method are also shown for different test cases. In addition to the above mentioned developments, a space-time adaptive i-MSFV (ai-MSFV) method is introduced. It is shown how to improve the MSFV results adaptively in space and simulation time. The fine-scale smoother in the i-MSFV framework is also applied locally. Also, for multiphase flow problems, two criteria are investigated for adaptively updating the MSFV interpolation functions: (1) a criterion based on the total mobility change for the transient coefficients and (2) a criterion based on the pressure equation residual for the accuracy of the results. For various challenging test cases it is demonstrated that iterations are required only in small sub-domains and not everywhere. Finally, the i-MSFV method is extended to sequential implicit simulations of time dependent problems. To control the error of the coupled saturation-pressure system, the transport error due to an approximate velocity field is analyzed. Based on the analysis, an error control strategy is proposed which is based on the residual of the pressure equation. At the beginning of simulation the i-MSFV iterations are employed until a specified accuracy is achieved. To minimize the number of iterations in a multiphase flow problem, the solution at the previous time step is utilized to improve the localization assumption at the current time step. Additional iterations are employed only when the residual becomes larger than a specified threshold value. Numerical results show that only few iterations in average are necessary to improve the MSFV results significantly even for very challenging problems.
Zusammenfassung

Acknowledgements

I would like to express my especial thanks to Prof. Dr. Patrick Jenny for being an exceptional supervisor for me during this PhD project at the Institute of Fluid Dynamics (IFD). During the past four years, he has been proposing talent ideas when I was facing to problems for which I could not find a solution in a reasonable time. Four years ago, he interviewed me over phone and gave me a chance to join his team and prove myself. I enjoyed every single day working under his supervision at the IFD. I also thank him for giving me freedom in my research and letting me express my points of views and advising me in a very friendly way once a decision had to be taken.

I thank Chevron/Schlumberger Intersect Alliance for the financial support of my PhD project. In particular, I wish to thank Dr. Seong Hee Lee for being my co-advisor and together with Dr. Wen Chen giving me the opportunity to work for Chevron ETC. Working at Chevron for about nine months was a big chance for me, especially because I could work in a team of hardworking talented people in a very friendly and diverse atmosphere. It was also an opportunity for me to learn more about real world problems in the field of multiphase flow in porous media. I would like to also thank Prof. Dr. Hamdi Tchelepi of Stanford University and Dr. Bradley Mallison of Chevron ETC for the fruitful discussions we had during the time I was at Chevron. I am also honored that Prof. Dr. Rainer Helmig accepted to be my co-advisor and examiner.

I wish to thank Ms Bianca Maspero and Mr. Hans Peter Carpez for all their efforts regarding administrative and IT supports. I spent a memorable time with my colleagues at the IFD. I hope our paths cross in future again.

Last but not least, I would like to thank my family. I truly owe my parents for all they did for me to be whom I am today. My sister and my brother deserve my warm appreciations. My words cannot express my thanks to Hanieh, my lovely wife, who let me stay late at nights in my office and devoted a lot for letting me work hard to succeed in my professional career. I hope I return all their supports by heading back home in near future and do my best to better life of the people in my beloved country, Iran.

Zürich, 26 July 2011

Hadi Hajibeygi
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Fine-scale reference results at 0.2 PVI (gas injection) obtained on a grid with $220 \times 60$ cells: pressure (top-left) and saturation (top-right) maps. Also shown are the original non-iterative MSFV results at 0.2 PVI using a $20 \times 12$ coarse grid: pressure (bottom-left) and saturation (bottom-right) maps. (Case 2)

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Part I

Introduction

1 Multiphase Flow in Porous Media

In many practically important processes flow and transport through connected pores inside a medium play an important role. Such fluid mechanics phenomena, i.e. flow in porous media, can be observed in, e.g. underground water flow, food drying processes, and petroleum reservoirs.

A single-phase flow problem describes the flow of a fluid (component) at a given thermodynamic condition (phase) [11]. In this case, the only resistance for the flow is due to the interaction (momentum exchange) of the fluid with the medium (rock) at interfaces. For oil reservoirs, which are the main focus of the present thesis study, the case is typically more complex. In this case, the fluid mainly consists of three fluid components of water, oil, and gas in three different phases of aqua, liquid, and gaseous, respectively [6].

The phases are usually compressible and also there exist mass exchange between them due to dissolution and flow dynamics such as pressure drop. The term ”phase” generally denotes a fraction of a fluid filling the void volume of the porous media at a specific thermodynamic state which describes its unique physical parameters such as density, viscosity, compressibility, etc. In a multiphase flow system, flow resistance is larger than that of the single phase flow because a phase interacts not only with the medium but also with other existing phases.

Based on the mentioned statements, oil reservoir simulations generally involve multiphase flow problems with different types of physical assumptions based on the nature of the reservoir and its phases. For a water flooding problem, e.g., which consists of water and oil phases, the compressibility of phases (which are in aqua and liquid state) does not play a major role and therefore can be neglected. As a result, many works in the literature do not involve compressibility effects in their water flooding studies [6]. For depletion of a reservoir which contains gas (highly compressible), however, the compressibility plays a major role. Moreover, to describe the enhanced recovery techniques for a reservoir which consists of water, oil, and gas with mass exchange, a black-oil model has been proposed [6]. In this model, compressibility of the phases are accounted in a weak form, i.e. a linear relation is used to describe compressibility effects due to the fact that the phases are slightly compressible [6].

Hence, to understand and manage the performance of a reservoir it is an
important factor to use proper constitutive relations to describe the physics and properties of the phases in the formation.

Description of the flow and transport in a porous medium entails proper mathematical formulations. These mathematical formulations are governed based on the mass, momentum, and energy conservation equations [6]. The result of this approach is a set of partial differential equations (PDEs) describing the flow and transport with physical properties appearing as coefficients. Unfortunately, the extremely complicated geometry of the pores and large length scales of the problem make the solution beyond the scope of any analytical and numerical approach. Therefore, a simplified equation is commonly employed as a replacement to the momentum conservation equation, i.e., the Darcy’s law [11, 6].

The Darcy’s law introduces a linear relationship between the velocity and the pressure gradient with experimentally obtained modeling parameters. The law makes the governing equations much simpler, but at the same time, it adds parameter complexities into the framework. It is very important to note that the Darcy’s law is a valid equation for some certain range of scales (field scales) and flow dynamics. Other extensions to this law have been proposed which are classified as the non-Darcy’s laws [6]. In this work the length scales and the dynamics of the flow allow the Darcy’s law be employed. The final governing PDEs (using the Darcy’s law) do not have general analytical solutions, therefore, numerical solutions are required to solve them in order to describe and predict a reservoir dynamics.

The numerical complexities associated with the governing PDEs are due to the complexity of the coefficients introduced in the equations along with the large length scales and complex geometries of the domain. More precisely, the permeability factor which is a measure of the hydraulic conductivity of a reservoir rock is highly heterogeneous, anisotropic tensor. Note that honoring the integrated field data provided by geoscientists is important to obtain accurate simulations such that the performance of the reservoir is correctly understood and predicted. Therefore, many numerical approaches have been devised during the past decades aiming to reduce computational complexity of the problem. In this PhD thesis, one of these appealing approaches is considered and extended to be applicable for reservoir simulations with realistic physics and geometry.

Next the mathematical formulations describing the flow and transport in porous media is described. Then, the numerical simulation strategies are discussed. An overview of the present thesis is also provided at the end of this section.
1.1 Mathematical Formulations

In this subsection, first the governing equations for a compressible multiphase flow problem are reviewed. Then, as a simplified case, the incompressible multiphase flow equations are discussed.

1.1.1 Compressible Flow

For simplicity, no dissolution and no capillary effects are considered here. In this case, Darcy’s formulation of multiphase flow in porous media leads to the mass conservation law

\[
\frac{\partial}{\partial t} (\phi \rho_a S_a) - \nabla \cdot (\rho_a \lambda_a \cdot (\nabla p - \rho_a g \nabla z)) = \rho_a q_a \quad \forall \alpha \in \{1, \ldots, \gamma\} \tag{1}
\]

of phase \(\alpha\), where \(\phi\) denotes the porosity, \(\rho_a\) the density, \(S_a\) the saturation, \(g\) the gravitational acceleration, and \(q_a\) source terms. Moreover, \(\lambda_a = K k_{ra}/\mu_a\) is the phase mobility with the positive definite permeability tensor \(K\), the relative permeability \(k_{ra}\), and the phase viscosity \(\mu_a\). Porosity, density and viscosity are given as algebraic functions of the pressure \(p\) and \(k_{ra}\) is a function of the saturations. Phase saturations are subject to the constraint

\[
\sum_{\alpha=1}^{\gamma} S_a = 1. \tag{2}
\]

Note that Eqs. (1) and (2) form a system for \(\gamma + 1\) unknowns \((S_1, \ldots, S_\gamma, p)\) [6].

Implicit Euler time integration leads to

\[
\frac{\phi^{n+1} \rho_a^{n+1} S_a^{n+1} - \phi^n \rho_a^n S_a^n}{\Delta t} - \nabla \cdot (\rho_a^{n+1} \lambda_a \cdot (\nabla p^{n+1} - \rho_a^{n+1} g \nabla z)) = \rho_a^{n+1} q_a, \tag{3}
\]

which is now divided by the density \(\rho_a^{n+1}\) in order to obtain a decoupled pressure equation. This is achieved by then taking the sum over all phases \(\alpha\) resulting in

\[
\frac{\phi^{n+1}}{\Delta t} - \frac{\phi^n}{\Delta t} \sum_{\alpha=1}^{\gamma} B_a^{n+1} \rho_a^n S_a^n - \sum_{\alpha=1}^{\gamma} B_a^{n+1} \nabla \cdot (\rho_a^{n+1} \lambda_a \cdot (\nabla p^{n+1} - \rho_a^{n+1} g \nabla z)) = q, \tag{4}
\]

where \(q = \sum_{\alpha=1}^{\gamma} q_a\) denotes the total volumetric source term and \(B_a = 1/\rho_a\) is the volume formation factor. Linearization of this equation results in the iterative linear pressure equation

\[
\frac{C}{\Delta t} (p^{v+1} - p^v) - \sum_{\alpha=1}^{\gamma} B_a^v \nabla \cdot (\rho_a^v \lambda_a \cdot \nabla p^{v+1}) = R^v, \tag{5}
\]
where

\[ R^\nu = -\frac{\phi^\nu}{\Delta t} + \frac{\phi^n}{\Delta t} \sum_{\alpha=1}^\gamma B^\nu_\alpha \rho_\alpha^n S^n_\alpha + q - \sum_{\alpha=1}^\gamma B^\nu_\alpha \nabla \cdot (\rho_\alpha^2 \lambda_\alpha g \cdot \nabla z) \] (6)

which converges to Eq. (4) as \( \nu \to \infty \). The superscripts \( \nu \) and \( \nu + 1 \) denote quantities at the old and new iteration levels, respectively, and

\[ C = \frac{\partial \phi}{\partial p} \big| ^\nu - \phi^n \sum_{\alpha=1}^\gamma \frac{\partial B_\alpha}{\partial p} \big| ^\nu \rho_\alpha^n S^n_\alpha \] (7)

is the compressibility coefficient. Note that the coefficients \( B_\alpha \) and \( \rho_\alpha \) in the convective and gravitational terms lag one iteration behind. This discretized formulation was already used in previous studies to solve the fine-scale problem (see e.g. [89, 60]).

In this PhD research work, it will be explained later how a new consistent formulation for the discrete form of the compressible flow is devised which is suited for the multiscale simulations.

### 1.1.2 Incompressible Flow

Governing equations for incompressible flow can be simply derived by considering phase densities as constant pressure independent values. The mass conservation equation for each phase \( \alpha \) can be then normalized by its associate density \( \rho_\alpha \). Therefore a volume balance equation is considered for the phases which reads

\[ \frac{\partial}{\partial t} (\phi S_\alpha) - \nabla \cdot (\lambda_\alpha \cdot (\nabla p - \rho_\alpha g \nabla z)) = q_\alpha \quad \forall \alpha \in \{1, \ldots, \gamma\} \] (8)

To obtain the pressure equation, the same procedure as the compressible problem is taken. Here, however, the conservative equations are already normalized by the phase densities. This results in the pressure equation

\[ -\nabla \cdot (\lambda_\alpha \cdot \nabla p) = q, \] (9)

where no gravity effects are considered. This elliptic pressure equation is sequentially coupled with the \((\gamma - 1)\) transport equations for phase saturations. The saturation of the \( \gamma \)-th phase is explicitly obtained by the constraint (2).

Note that the transport equations can be solved implicitly or explicitly. In the case of implicit integration, the pressure equation can be solved with the new converged saturation values and the procedure can be repeated until the full convergence is achieved.
For most part of this PhD work, a higher order TVD explicit time integration scheme [62] is used for the saturation transport equation. Therefore an implicit pressure explicit saturation (IMPES) strategy is followed and explained in the next subsection for the fine-scale reference approach. Also, an overview and literature survey for the upscaling and multiscale approaches for pressure equations are presented in separate subsections. Note that a sequentially fully implicit strategy is also used in a section where error estimate and control is presented. The discretization of this approach is therefore explained in that section where this strategy is used.

1.2 Numerical Simulation Strategies

The governing equations described in the previous section do not have a general analytical solutions. Therefore, a numerical method is required to solve them in order to predict and manage the performance of a reservoir [6]. Here, the discretization and IMPES simulation strategy is briefly reviewed. Details of numerical discretization techniques suitable for these type of problems can be found in many books, e.g. [6, 18].

1.2.1 Reference Fine-Scale Approach

In an IMPES flow-transport coupling strategy, first the pressure equation is solved with all saturation dependent coefficients explicitly calculated based the old time step values. As explained before, depending on the physical assumptions the pressure equation is either elliptic, i.e. Eq. (9), or parabolic, i.e. Eq. (4). A discretization technique which provides mass conservative velocity fields is required to be used for the solution of the pressure equation. This is an important constraint because transport errors highly depend on the mass conservation errors of the velocity field. Detailed discussions of this important dependency is presented in this thesis later in section 10. Once the pressure solution is obtained, phase velocities \( \mathbf{u}_\alpha \), i.e.

\[
\mathbf{u}_\alpha = -\lambda_\alpha \cdot (\nabla p - \rho_\alpha g \nabla z).
\]  

(10)

are required to solve transport equations. This is achieved by first computing the total velocity \( \mathbf{u}_t \) from the pressure equation, i.e.

\[
\mathbf{u}_t = -\sum_{\alpha=1}^{\gamma} \lambda_\alpha \cdot (\nabla p - \rho_\alpha g \nabla z).
\]  

(11)

Then by using fractional flow functions

\[
f_\alpha = \frac{\lambda_\alpha}{\lambda_t}
\]  

(12)
the phase velocity \( u_\alpha \) for a two-phase system \( \{\alpha, \beta\} \) reads
\[
u_\alpha = f_\alpha (u_t + \lambda_\beta g \frac{(\rho_\alpha - \rho_\beta) \nabla z}{\Delta \rho}).
\] (13)

Once the phase velocities are obtained, the transport equations can be solved using a higher order TVD or first-order upwind scheme [62].

1.2.2 Why Multiscale Methods?

As explained before, mathematical formulations of multiphase flow in porous media lead to partial differential equations of different types with highly heterogeneous anisotropic tensorial factors \( \lambda \) representing the physical properties of the domain. The highly detailed information provided by geoscientists for these factors should be fully taken into account for accurate simulations and investigations of flow and transport. Unfortunately, with respect to the computational cost, it is not possible to use classical simulation techniques for this type of problems.

To reduce the computational complexity, various multiscale methods have been developed during the past decade [44, 46, 45, 24, 3, 4, 17, 47, 26, 1, 32, 2, 53]. The aim of these methods is to reduce the computational complexity by incorporating the fine-scale variation of the coefficients into the coarse-scale operator; similar to upscaling methods [87, 80, 28, 15, 14, 21]. Upscaling methods aim at coarse-scale descriptions based on effective, tensorial coefficients [20]. In addition, multiscale methods allow to reconstruct a fine-scale velocity field from a coarse-scale pressure solution [30].

In the multiscale methods, the reduction in the computational cost is achieved by introducing sets of basis functions, which are numerically computed on local small domains, and by mapping the fine-scale coefficients into the coarse scale operator (restriction step). The approximate fine-scale solution is then constructed from the coarse scale solution using the basis functions for interpolation (prolongation step). It has been demonstrated that for a wide range of test cases the solution obtained by multiscale methods is in good agreement with the results obtained by solving the global fine-scale system.

While the multiscale solutions differ from the reference solutions computed with the same standard numerical scheme on the fine grid, convergence with respect to coarse-grid refinement was proved for permeability fields characterized by two separable scales [17, 32, 45]. On the other hand, it was shown that MSFE methods do not converge for problems without scale separation [32, 45, 73, 24].

Note that since the basis functions are locally computed, they are locally updated during a multiphase flow simulations. In another words, the basis
functions are computed at the beginning of the simulation and are updated only in local domains with large total mobility changes \cite{47, 48, 83, 50}. This is the main reason why multiscale simulations are more efficient than fine-scale simulations.

Among the proposed multiscale methods the Multiscale Finite Volume (MSFV) \cite{47, 48, 83, 50} and the Mixed Multiscale Finite Element (MMSFE) \cite{17, 1, 2, 4} methods provide locally conservative velocity fields. As mentioned before, conservative velocity fields are crucial for accurately solving the saturation transport equations. Note that the MSFV method constructs smaller coarse systems than the MMSFE method.

1.3 Motivation and outline of this Thesis

Several studies have extended the MSFV method to involve more physics, i.e. capillary, gravity, compressibility, etc. \cite{68, 88, 51, 66, 89, 60, 36, 37}. The extensions were shown effective for many applications. They, however, entail limitations in terms of accuracy and applicability. For example, it is known that the MSFV method fails to provide good results in presence of highly anisotropic heterogeneous permeability field (see e.g. \cite{57}). Also, all of the previously introduced extensions of the MSFV method for compressible problems cannot correctly capture the transient state described by the nonlinear pressure equation (see e.g. \cite{89}). Moreover, it is known that about \%60 of the oil reservoirs are fractured \cite{33}. At the beginning of this project, there existed no MSFV method capable of solving fractured porous media \cite{51}. Above all, the MSFV method has always been implemented on Cartesian rectangular coarse grids. No strategy for non-matching coarse grids which can be of interest due to faulted reservoirs was proposed previously. All these limitations motivated the present PhD project. Detailed studies of these previously introduced MSFV methods will be discussed later during this thesis. Therefore, for the sake of short manuscript and avoiding redundancy, their advantages and limitations are not discussed more here.

In this thesis, a brief discretion of the MSFV method is presented first. Then some limitations are discussed. These limitations motivate the introduction of the iterative MSFV (i-MSFV) method which is explained and discussed in detail in a following section. Then the MSFV and i-MSFV methods are extended for the solution of compressible multiphase flow problems. In that chapter a discussion about failure of other methods are also presented. Then the MSFV and i-MSFV methods are extended for the solution of fractured and faulted porous media, followed by a section where the applicability of the i-MSFV method is addressed. In the applicability discussion an adaptive iterative MSFV (ai-MSFV) method is introduced which
lead to local improvement of the i-MSFV results. Also in that chapter, the necessity (frequency and number) of the i-MSFV iterations for obtaining a good solution in a sequentially fully implicit framework is discussed. Finally, the thesis is concluded with reviewing the main achievements and proposals for future works.
Part II
MSFV Method

2 MSFV Method for Elliptic Problems

In this section, the MSFV method for elliptic problems is presented. It is explained how the fine-scale variations of the pressure equation coefficients can be mapped onto a coarser level in the MSFV conservative framework. The good quality of the MSFV result for a heterogeneous test case is also illustrated. Finally, the limitations of the method are discussed which then motivates the next section where the i-MSFV method is devised.

2.1 Algorithm

To explain the MSFV method, we consider the elliptic problem

$$-\nabla \cdot (\lambda \cdot \nabla p) = q \tag{14}$$

on the domain $\Omega$ with the boundary conditions $\nabla p \cdot n = \zeta_1$ and $p(x) = \zeta_2$ at $\partial \Omega_1$ and $\partial \Omega_2$, respectively. Note that $\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2$ is the whole boundary of the domain $\Omega$ and $n$ is the outward unit normal vector. The mobility tensor $\lambda$ is positive definite and the right-hand sides (RHS) $q, \zeta_1$, and $\zeta_2$ are specified fields.

The MSFV method is designed to efficiently compute approximate solutions of problem (14) for highly heterogeneous coefficients $\lambda$ and RHS $q$, e.g. for mobility fields, which depict a high variance, complex correlation structures and are governed by a large range of length scales.

The MSFV method relies on an imposed coarse grid (solid lines in Fig. 1) and on a dual coarse grid (dashed lines in Fig. 1). The former is composed of $N_c$ control volumes $\Omega_k$ ($k \in [1, N_c]$) and the latter of $N_d$ cells $\hat{\Omega}_h$ ($h \in [1, N_d]$). As illustrated in Fig. 1, each control volume $\Omega_k$ contains exactly one node $x_k$ of the dual coarse grid in its interior. Note that these two grids can be much coarser than the underlying fine grid on which the mobility field is represented. It is also emphasized that the concept is not limited to the simple grids shown in Fig. 1. In principle, very irregular grids or decompositions can be employed. The reduction of degrees of freedom (restriction step) describing the fine pressure $p_f$ (pressure field on the fine grid) is achieved through
Figure 1: A computational 2D domain $\Omega$ with a coarse grid (solid lines) and a dual coarse grid (dashed lines); bold solid and bold dashed lines indicate a selected coarse cell $\tilde{\Omega}_k$ and a selected dual coarse cell $\tilde{\Omega}_h^2$, respectively. An enlarged coarse cell, which contains $11 \times 11$ fine cells, is shown on the right side. Also shown on the left is an enlarged dual coarse cell. In the enlarged figures, the fine-scale dual boundary cells are highlighted in grey.

the approximation

$$p_f(x) \approx p'(x) = \sum_{h=1}^{N_d} \left[ \sum_{k=1}^{N_c} \Phi_h^k(x) \tilde{p}_k + \Phi^h(x) \right], \quad (15)$$

where $\tilde{p}_k$ are the pressure values at the nodes $x_k$. We refer to $\Phi_h^k$ as the basis functions and to $\Phi^h$ as the correction function. Opposed to classical finite-element methods, basis functions and correction functions are not analytical functions, but local numerical solutions of problem (14) on $\tilde{\Omega}_h^2$ without and with RHS, respectively. Localization can be achieved by employing reduced problem boundary conditions at $\partial \tilde{\Omega}_h^2$, which is equivalent to

$$(\mathbf{n}^h \cdot \nabla)(\lambda \cdot \nabla \Phi_h^k) \cdot \mathbf{n}^h = 0 \quad (16)$$

and

$$(\mathbf{n}^h \cdot \nabla)(\lambda \cdot \nabla \Phi^h) \cdot \mathbf{n}^h = r^h \quad (17)$$

at $\partial \tilde{\Omega}_h^2$ with $\mathbf{n}^h$ being the unit normal vector pointing out of $\tilde{\Omega}_h^2$ and $r^h = 0$. At the dual-grid nodes $x_l$ which belongs to $\tilde{\Omega}_h^2$, $\Phi_h^k(x_l) = \delta_{kl}$ and $\Phi^h(x_l) = 0$.

By construction, outside $\tilde{\Omega}_h^2$ the $\Phi_h^k$ and $\Phi^h$ values are set to zero. An illustration of 2D basis and correction functions is shown in Fig. 2. In addition, the stencil corresponding to the reduced problem boundary condition is illustrated in Fig. 3. The solutions of the reduced problems at the boundaries are used as Dirichlet condition for the internal cells of local problems.
2.1 Algorithm

Figure 2: Illustration of the basis function $\Phi_h^1$ (left) and the correction function $\Phi^h$ (right) for elliptic problems with a single fine-scale source term. The permeability map for the corresponding localized domain $\tilde{\Omega}_h$ is partly shown on the base surfaces.

Figure 3: The 2D original stencil (left) for a dual coarse cell boundary cells based on 2-point-flux-approximation scheme. Arrows (interface fluxes) depict interaction between neighboring cells. Also shown is the reduced problem stencil (right) used for computation of basis and correction functions where transversal fluxes are neglected and Dirichlet values are set at the corners.
To derive a linear system for the coarse pressure values \( \hat{p}_k \), one needs to introduce the flux contributions between the neighboring coarse cells. Therefore, we substitute expression (15) for \( p' \) into Eq. (14) and integrate over \( \hat{\Omega}_l \) which leads to

\[
- \int_{\hat{\Omega}_l} \nabla \cdot (\lambda \cdot \nabla p') d\Omega = - \int_{\hat{\Omega}_l} \nabla \cdot \left( \lambda \cdot \nabla \left( \sum_{h=1}^{N_d} \sum_{k=1}^{N_c} \Phi_{hk} \hat{p}_k + \Phi^h \right) \right) d\Omega \\
= \int_{\hat{\Omega}_l} q d\Omega. \tag{18}
\]

for all \( l \in [1, N_c] \). With the Gauss theorem one obtains

\[
- \int_{\partial \hat{\Omega}_l} (\lambda \cdot \sum_{h=1}^{N_d} \sum_{k=1}^{N_c} \hat{p}_k \nabla \Phi_{hk} + \nabla \Phi^h) \cdot \bar{n}_l d\Gamma = \sum_{h=1}^{N_d} \sum_{k=1}^{N_c} \int_{\partial \hat{\Omega}_l} (-\lambda \cdot \nabla \Phi_{hk} \cdot \bar{n}_l d\Gamma + \sum_{h=1}^{N_d} \int_{\partial \hat{\Omega}_l} (-\lambda \cdot \nabla \Phi^h) \cdot \bar{n}_l d\Gamma = \int_{\hat{\Omega}_l} q d\Omega, \tag{19}
\]

which results in the linear system

\[ A_{lk} \hat{p}_k = b_l \tag{20} \]

for \( \hat{p}_k \) with

\[ A_{lk} = \sum_{h=1}^{N_d} \int_{\partial \hat{\Omega}_l} (-\lambda \cdot \nabla \Phi_{hk} \cdot \bar{n}_l d\Gamma \tag{21} \]

and

\[ b_l = \int_{\hat{\Omega}_l} q d\Omega - \sum_{h=1}^{N_d} \int_{\partial \hat{\Omega}_l} (-\lambda \cdot \nabla \Phi^h) \cdot \bar{n}_l d\Gamma. \tag{22} \]

The unit normal vector \( \bar{n}_l \) points out of \( \hat{\Omega}_l \). Note that the RHS \( b_l \) also contains the effects of the fine-scale fluxes across \( \partial \hat{\Omega}_l \) induced by the correction functions \( \Phi^h \).

With \( \hat{p}_k \) and the superposition (15) one obtains the fine-scale pressure \( p' \) (prolongation step), which is an approximation of the fine-scale reference solution \( p_f \).

An interesting property of this MSFV method is that the difference between \( p' \) and \( p_f \) is solely due to the localization assumption (17) (see Fig. 3), i.e. with

\[ r^h = (\bar{n}^h \cdot \nabla) \left( (\lambda \cdot \nabla p_f) \cdot \bar{n}^h \right) \text{ at } \partial \hat{\Omega}^h \forall h \in [1, N_d] \tag{23} \]
the two fine-scale pressure fields $p_f$ and $p'$ become identical.

It has been shown for a wide range of test cases that the MSFV method with $r^h = 0$ leads to very accurate results. In other words: in general, the reduced problem boundary conditions provide a good localization assumption.

An example for a heterogeneous problem is provided in Fig. 4, where no-flow condition is applied on the whole boundary of the domain. Non-dimensional pressures are set to 1 and 0 for the (1,1) and (44,44) cells, respectively. Fine-scale reference solution and the MSFV results are obtained by employing $44 \times 44$ fine and $4 \times 4$ coarse grid cells, respectively.

![Figure 4: Fine-scale reference solution (left) of a heterogeneous problem with $44 \times 44$ grid cells. The MSFV approximate superposed solution (right) is obtained by employing $4 \times 4$ coarse grid cells. The permeability map is partly shown on the base surfaces. Also shown (right) is the coarse (solid black) and dual coarse (solid white) grids.](image)

For multiphase problems, a conservative fine-scale velocity field is required to honor mass balance of the transported phase saturations. While the velocity

$$ u' = -\lambda \cdot \nabla p' $$

fulfills this requirement for each coarse volume $\hat{\Omega}_k$, it is non-conservative at the fine-scale. Therefore, if one is interested in solving saturation transport on the fine grid, a further step is required.

To construct a conservative fine-scale velocity field $u''$, which is consistent with $u'$, the additional local problems

$$ -\nabla \cdot (\lambda \cdot \nabla p_{k}'') = q \quad \text{on} \quad \hat{\Omega}_k $$

(25)
with
\[(\lambda \cdot \nabla p''_k) \cdot \bar{n}_k = (\lambda \cdot \nabla p') \cdot \bar{n}_k \quad \text{at} \quad \partial \tilde{\Omega}_k \quad (26)\]
are solved. Note that the velocity field
\[u'' = \begin{cases} -\lambda \cdot \nabla p''_k & \text{on} \quad \tilde{\Omega}_k \\ -\lambda \cdot \nabla p' & \text{at} \quad \partial \tilde{\Omega}_k \end{cases} \quad (27)\]
for all \(k \in [1, N_c]\) is conservative (provided \(p''\) is obtained with a conservative scheme) and can be employed to solve transport equations on the fine grid [47].

For multiphase subsurface flow problems, for example, saturation transport may be calculated explicitly [49] or implicitly [50]. Since the mobility \(\lambda\) generally depends on the saturations, in the implicit version one has to iterate between the pressure equation (14), which is solved with the MSFV method, and the transport equations. Good efficiency is achieved, if the latter one is solved implicitly on the individual domains \(\tilde{\Omega}_k\). The local solutions are then coupled by a simple Schwarz overlap scheme [50, 79]. With this technique, which is very efficient for hyperbolic problems, the low computational complexity of the overall MSFV algorithm can be maintained for multiphase flow.

An example of the implicit-pressure-explicit-saturation (IMPES) simulation of a multiphase flow problem where injected phase is 10 times less viscose than the residing phase is considered. No-flow condition is applied on the whole boundary. The permeability field of the SPE 10 [19] top layer for a domain of \(220 \times 55\) with the natural logarithm mean and variance of 2.35 and 5.66, respectively, is used (Fig. 5). A 10 times less viscose phase is injected with the non-dimensional total rate of 10 at cell (1,1) while production occurs at constant pressure of 0 at cell (220,55). For the pressure equation, the fine-scale and MSFV simulators employ \(220 \times 55\) and \(20 \times 5\) grid cells, respectively. The transport equation is solved on fine scale for both simulations. Results (pressure and saturation) after 0.132 Pore-Volume-Injection (PVI) are depicted in Figs. 6 and 7.

An important property of the MSFV method is its adaptivity. For example, the conservative velocity reconstruction described above is only required in those coarse cells \(\tilde{\Omega}_k\) where fine-scale transport is of interest. Moreover, very importantly, the basis and correction functions can be stored and reused for subsequent time steps. They have to be recomputed only in those dual cells \(\tilde{\Omega}^h\) where changes of the coefficient \(\lambda\) or (for the correction functions) the RHS \(q\) exceed a specified limit [49, 50, 83].
2.1 Algorithm

Figure 5: Natural logarithm of SPE 10 [19] top layer permeability field for a domain of $220 \times 55$ cells.

Figure 6: Fine-scale (top) and MSFV (bottom) pressure fields for the SPE 10 top layer test case after 0.132 PVI. The fine-scale and MSFV solutions are obtained by $220 \times 55$ and $20 \times 5$ grid cells, respectively.

Figure 7: Fine-scale (top) and MSFV (bottom) saturation maps for the SPE 10 top layer test case after 0.132 PVI. The fine-scale and MSFV solutions are obtained by $220 \times 55$ and $20 \times 5$ grid cells, respectively.
2.2 Limitations

As mentioned before, in order to make the MSFV method applicable for realistic problems, it was extended to include compressibility [66, 60, 89], gravity [60, 68], and complex wells [88, 51]. All these extended versions of the MSFV method proved to be effective for a wide range of cases for which the multiscale and fine-scale solutions are in good agreement. There exist scenarios, however, which demonstrate some limitations of the MSFV method.

One involves extended structures with sharp mobility contrasts. Examples are extended and almost impermeable shale layers [65], and meanders as they exist in the bottom layers of the SPE comparative test case 10 [19]. Figure 8 depicts solutions (pressure and saturation) of the fine-scale and MSFV simulations after 0.15 PVI for a 2D homogeneous test case, where three shale layers exist. The permeability of the shale layers are $10^8$ times less than that of the domain. No-flow condition is applied on the whole boundary of the domain. Gas which is 10 times less viscous than the residing oil is injected at constant non-dimensional rate of 10 at cell (7,31) while the production occurs at constant pressure of 0 at cell (55,55). The fine-scale solutions are obtained using $55 \times 55$ cells while the MSFV results are obtained using $5 \times 5$ coarse cells. It is clear that the MSFV results are not acceptable for this challenging problem. In fact, there exists no general localization condition for problems with long coherent structures.

Another class of problems for which the MSFV method fails to give accurate solutions are cases with large cell aspect ratios or highly anisotropic mobility factors. As an example of these scenarios, a heterogeneous anisotropic test case is considered. The permeability field of SPE 10 top layer as presented in Fig. 5 is used, and no-flow condition is applied on the whole boundary of the domain. Fine-scale simulator employs a $220 \times 55$ grid cells while the MSFV simulator uses $20 \times 5$ coarse control volumes for the pressure equation. The grid aspect ratio and viscosity ratio are both 10.

Figures 9 and 10 present pressure and saturation maps, respectively, after 0.165 PVI. It is clear that the MSFV results are not matching the reference solutions. Although various ways to overcome this problem have been devised so far [67, 43], there still exists potential for a systematic and general improvement.

As the final test case, the permeability field of the SPE 10 [19] bottom layer is considered and shown in Fig. 11. The dimensionless pressure is set to 1 and 0 at corner cells (1,60) and (220,1), respectively. No-flow condition is employed at the whole boundary of the domain. The domain consists of $220 \times 60$ fine and $44 \times 12$ coarse control volumes with grid aspect ratio of 2. This is indeed
2.2 Limitations

Figure 8: Fine-scale (top) and MSFV (bottom) pressure (left) and saturation (right) maps for a homogeneous test case with shale layers after 0.15 PVI. The fine-scale and MSFV solutions are obtained by $55 \times 55$ and $5 \times 5$ fine and coarse grid cells, respectively. Shale layers are $10^8$ times less permeable than the medium.
2 MSFV METHOD FOR ELLIPTIC PROBLEMS

Figure 9: Fine-scale (top) and MSFV (bottom) pressure fields for the anisotropic SPE 10 top layer test case after 0.165 PVI. The fine-scale and MSFV solutions are obtained using $220 \times 55$ and $20 \times 5$ grid cells, respectively, with the grid aspect ratio of 10.

Figure 10: Fine-scale (top) and MSFV (bottom) saturation maps for the anisotropic SPE 10 top layer test case after 0.132 PVI. The fine-scale and MSFV solutions are obtained using $220 \times 55$ and $20 \times 5$ grid cells, respectively, with the grid aspect ratio of 10.
a challenging problem for the MSFV method, since the permeability field is anisotropic and entails nonlocal structures with high contrasts. Figure 12 shows the fine-scale and MSFV pressure fields. It is clear than nonphysical peaks exist in the MSFV solution, which then result in inaccurate transport problem.

![Figure 11](image1.png)

Figure 11: Natural logarithm of $k$ in the bottom layer of the three-dimensional SPE10 test case [19].

![Figure 12](image2.png)

Figure 12: Fine-scale (left) and MSFV (right) pressure maps for heterogeneous anisotropic problem. The permeability of the SPE 10 [19] bottom layer with grid aspect ration of 2 is used. There exist $220 \times 60$ fine and $44 \times 12$ coarse control volumes.

Furthermore, for problems where the MSFV results are in good agreement with the reference solutions, no control on the level of error introduced in this multiscale framework is possible. Finally, it is of great interest to have a method, which allows to improve the solution iteratively to any desired level. Next, we present an extension, which allows to iteratively improve MSFV solutions and control the error introduced in this framework. It will be shown
that the fine-scale reference solutions can be recovered even for those men-
tioned challenging cases, i.e. where the standard MSFV method experiences
difficulties.
Part III
Iterative MSFV Method

3 I-MSFV Method for Elliptic Problems

As already pointed out, the difference between the MSFV solution $p'$ and the fine scale reference pressure $p_f$ is solely due to the localization assumptions. In another words, $p'$ and $p_f$ become identical if the boundary conditions (17) are employed in fulfillment of requirement (23) for $r^h$. Unfortunately, Eq. (23) requires a priori knowledge of $p_f$. Existing methods of this type rely on an initial global fine-scale solution [2, 1, 57, 25]. However, for problems with high phase viscosity ratios, frequently changing boundary conditions or varying well rates, the value of such an approach is questionable.

Another possibility is to improve the coarse operator iteratively. The adaptive local-global (ALG) upscaling approach is based on global iterations to obtain a self-consistent coarse-grid description [15, 14]. It was shown that ALG leads to more accurate solutions than local upscaling methods. Recently, ALG was employed to improve the local boundary conditions in the multiscale finite volume element method (ALG-MSFVE) [21]. It was also shown that the ALG leads to asymptotic solutions for large numbers of iterations, but typically these solutions are different from standard fine-scale solutions and the error due to ALG is problem dependent [15].

In this section we device an iterative procedure for the $r^h$ which converges to the exact expression (23).

In the next subsection, we present the i-MSFV algorithm where we explain how an iterative procedure can be devised in the MSFV framework. The i-MSFV method can be explained as a multigrid (MG) method [85] in the sense that the restriction and prolongation steps are done using MSFV basis and correction functions. The fine-scale smoother which is employed generally a few times in each iteration step is used to resolve high frequency errors, similar as in MG methods. The low frequency errors are resolved by solving the multiscale coarse scale system. An interpretation of the i-MSFV method as a multigrid method is also presented in this section. Finally, extensive numerical studies are performed to test the efficiency of the presented i-MSFV method.
3.1 Algorithm

Here, we explain a convergent iterative procedure to improve the localization boundary conditions, which does not depend on $p_f$.

Instead of requirement (23), we consider the iterative improvement

$$r^h = (\tilde{n} \cdot \nabla)((\lambda \cdot \nabla p^\nu_s) \cdot \tilde{n}) \quad \text{at} \quad \partial \Omega^h \quad \forall h \in [1, N_d]$$

of $r^h$. The superscript $\nu$ denotes the iteration level and

$$p^\nu_s = S^n_s \cdot p^\nu + T = S^n_s \cdot \sum_{h=1}^{N_d} \sum_{k=1}^{N_r} \Phi^h \bar{p}^\nu_{k} + [\Phi^h]_{\nu-1} + T$$

is the smoothed MSFV fine-scale pressure approximation, where here $S$ is a linear smoothing operator, $T$ is the non-homogeneous part of the iterative smoother, and $n_s$ the number of smoothing steps. Note that the correction functions $[\Phi^h]_{\nu-1}$ are based on the local boundary conditions (17) with $r^h = r^{h,\nu-1}$.

Figure 13b illustrates an initial correction function $[\Phi^h]_{\nu=1}$ (with no iteration), which is obtained by setting $p^\nu_s = 0$. Moreover, Fig. 2c depicts an improved correction function for the same local domain obtained after some iterations, i.e. $\nu > 1$. Note that the basis functions (e.g. Fig. 2a) are computed once and do not contribute in the iterative procedure. Therefore, the i-MSFV method is computationally more efficient than those methods which correct the system matrix rather than the RHS vector (e.g. see [15, 14, 21]). On top of that, in contrast to the ALG-MSFVE [21] the i-MSFV method converges to the fine-scale reference solution.

For a more compact presentation of the iterative MSFV (i-MSFV) method, we order the fine-grid values of $p^\nu_s$, $p^\nu$, $\Phi^h$, and $\Phi^h$ in vectors $p^\nu_s$, $p^\nu$, $\Phi^h$, and $\Phi^h$ with entries $[p^\nu_s]_i$, $[p^\nu]_i$, $[\Phi^h]_i$, and $[\Phi^h]_i$, respectively. We then express in matrix form all linear equations involved in the iterative procedure described above and write

$$[\Phi^h]_{\nu-1} = C^h [p^\nu_s]_j + E^h_i,$$

$$A_{ik} = \frac{1}{Q_k} \left( \int_{\Omega_k} \sum_{n=1}^{N_d} \int_{\Omega_n} (-\lambda \cdot \nabla [\Phi^h]_j \cdot \tilde{n}) d\Gamma \right),$$

$$[p^\nu]_i = [S^n_s]_{ij} \sum_{h=1}^{N_d} ([\Phi^h]_j \bar{p}^\nu_{k} + [\Phi^h]_{\nu-1}]) + T_i.$$
Figure 13: Illustration of the basis function $\Phi^h_1$ (a), the initial (MSFV) correction function $\Phi^{h,\nu=1}$ (b), and the correction function after some iterations, i.e. $\Phi^{h,\nu>1}$, for the same localized domain $\tilde{\Omega}^h$ (c). The permeability map is partly shown on the base surfaces and there exists only a single source term inside the $\tilde{\Omega}^h$.

boundary condition (17) defined according to (28). The terms on the right-hand side express the linear dependence of $\Phi^{h,\nu-1}$ on the smoothed pressure field $p^{\nu-1}_s$ at the previous iteration step (due to the iterative boundary condition (28)) and on the source term $q$ of the elliptic problem, respectively. Equation (31) corresponds to the coarse-scale problem (18), and is equivalent to Eqs. (20)-(22). Finally, Eq. (32) expresses the iterative reconstruction formula (29). Combining Eqs. (30), (31), and (32) and indicating with $I$ the identity matrix, we obtain the linear relation

$$[p^{\nu}_s]_r = [S^{nu}]_{ij} \sum_{h=1}^{Nd} \left[ (\Phi^h_{kl}) A^{-1}_{kl} (Q_l + D_{lq} E^{h}_q) + E^{h}_j \right] + T_i \begin{array}{c} \vdots \end{array}$$

$$+ [S^{nu}]_{ij} \sum_{h=1}^{Nd} \left[ (\Phi^h_{kl}) A^{-1}_{kl} D_{lq} + I_{jq} C^h_{qr} \right] [p^{\nu-1}_s]_r$$

between the smoothed fine-scale pressure fields $p^{\nu-1}_s$ and $p^{\nu}_s$, at two consecutive iteration steps.

An algorithmic outline of the i-MSFV method is shown in Table 1. First, the fine-scale pressure is initialized, e.g. it is set to zero. Then, all basis functions are computed and the right-hand side of the elliptic pressure equation is integrated over each coarse volume. These steps have to be performed only once and are followed by the main iteration loop. At the beginning of each iteration, $n_s$ smoothing steps are applied and the smoothed fine-scale
pressure is employed to compute the correction functions. These are required to obtain the right-hand side of the linear system for the coarse pressure. At the end of each iteration, the coarse system is solved and the new fine-scale pressure approximation is reconstructed. Note that the components of the vector \( \bar{p} \) are the actual pressure values at the dual coarse-grid nodes.

\[
\text{initialize } p^{\nu=0} \\
\forall h : \forall k : \text{compute basis functions } \Phi_k^h \\
calculate \hat{Q}; \text{ Eq. (31)} \\
\text{for } \nu = 1 \text{ to number of i-MSFV iterations } \{ \\
p_{s,\nu-1} = p_{\nu-1} \\
\text{for } i = 1 \text{ to } n_s \{ \\
p_{s,\nu-1} = S \cdot p_{s,\nu-1} + T; \text{ smoothing step} \\
\} \\
\forall h : \text{compute correction function } \Phi_{h,\nu-1}; \text{ based on } p_{s,\nu-1} \\
calculate \hat{b}^{\nu-1} = Q + DC \cdot p_{s,\nu-1} + DE; \text{ Eq. (31)} \\
solve coarse system } A \cdot \bar{p}^{\nu} = \hat{b}^{\nu-1}; \text{ Eq. (31)} \\
\text{reconstruct } p^{\nu}; \text{ Eq. (15)} \\
\} \\
\]

Table 1: Algorithmic outline of the i-MSFV method.

3.2 Interpretation as a Multigrid Method

The operations depicted in table 2 illustrate how the i-MSFV algorithm can be interpreted as a multigrid method. First, \( n_s \) smoothing steps (iterative linear solver) are applied to improve the approximate fine-grid solution \( p_{\nu-1}^{\nu} \) by resolving high-frequency errors. A subsequent restriction step leads to the right-hand side \( \hat{b}^{\nu-1} \) of the coarse-grid system for the pressure values \( \bar{p}^{\nu} \) at the dual coarse-grid nodes. Note that this involves updating the correction functions \( \Phi_{h,\nu-1} \); the coarse-grid operator \( A \) on the other hand, which is based on the basis functions \( \Phi_k^h \), has to be constructed only once at the beginning. The coarse system can be solved with any suitable solver, but due to the typically extreme coarsening factors, the coarse problem may be small enough to be solved directly. The updated fine-grid solution \( p^{\nu} \) is obtained by prolongation, which is simply achieved by superimposing the correction functions plus the basis functions weighted with the new coarse pressure values. Note that this interpretation of the i-MSFV method as a multigrid method is different than what is presented in [89], where neither correction functions nor fine-scale smoother are considered there.
Although only shown here for two grid levels, the i-MSFV method can be extended for more complex cycles. Moreover, it can be seen from the abstract operations in tables 1 and 2 that no assumptions regarding the topologies of the fine- and coarse-scale grids are required. For example, the same methodology can be applied for unstructured fine grids, and instead of coarse grids one can employ appropriate domain decompositions. In any case, however, the smoothing scheme is critical for robustness and good convergence. For all the cases we considered it was found that consistent line-relaxation works very well (note that a different smoother would be required for unstructured fine grids). This might be mainly due to the effectiveness of line-relaxation to distribute the residual from the coarse dual-cell boundaries across the domain. Moreover, line-relaxation only depends weakly on the grid aspect ratio and on the level of anisotropy [85]. Next, the line relaxation scheme used for the studies presented in this paper is described.

### Table 2: The i-MSFV algorithm interpreted as a multigrid method.

3.3 Fine-Scale Smoothing

As already mentioned, line-relaxation (LR) is only one possibility to smooth the approximate fine-grid solution \( \mathbf{p}^{\nu-1} \). Here, we describe how LR is employed in our current i-MSFV implementation. Therefore, we consider the fine-scale system

\[
\mathbf{M} \cdot \mathbf{p}_f = \mathbf{R}_i, \tag{34}
\]

which results from a conservative finite-volume discretization of Eq. (14) on the fine grid. For simplicity we assume that the grid lines are parallel to
the $x$-, $y$- and $z$-directions of a Cartesian coordinate system. The extension of this algorithm to unstructured grids is a topic of future research and not within the scope of this paper. Then we split the linear operator as $M = M_x + M_y + M_z$, where $M_{x,y,z}$ represent the discretizations of the elliptic operator in the corresponding coordinate directions. If only one operator plus the diagonal components of the other ones are treated implicitly, one obtains the iterative scheme

$$ (M_x + di(M_y + M_z)) \cdot p^{v+1/3} = R - (M_y + M_z - di(M_y + M_z)) \cdot p^v, $$

(35)

$$ (M_y + di(M_x + M_z)) \cdot p^{v+2/3} = R - (M_x + M_z - di(M_x + M_z)) \cdot p^{v+1/3}, $$

(36)

$$ (M_z + di(M_x + M_y)) \cdot p^{v+1} = R - (M_x + M_y - di(M_x + M_y)) \cdot p^{v+2/3}, $$

(37)

where $p^v$ is the approximate solution after the $v$-th LR-step and $di(M_L)$ represents the matrix with the diagonal of $M_x$. This scheme, in which the three linear systems (35)-(37) are solved sequentially each iteration, is a slightly modified version of the alternating directions implicit (ADI) method by Peaceman and Rachford [77]. For a two-point flux approximation, the linear operators $M_{x,y,z}$ have a tri-diagonal structure and the systems (35)-(37) can be solved with the Thomas algorithm [6], which has a linear complexity. Moreover, these three operators can further be split into independent linear systems for each grid line, which is an important property for massive parallel computing. Note that this iterative LR solver is convergent, but for big problems the rate is extremely slow. In our framework, however, only very few LR-steps are required to smooth $p^{v-1}$ sufficiently for an effective improvement of the local boundary conditions. As demonstrated in the next subsection, the optimum number of smoothing steps per i-MSFV iteration is case dependent.

4 Numerical Results

4.1 Numerical Convergence Studies

Here, the convergence rate of the i-MSFV method is assessed. The first set of studies is based on a test case consisting of a rectangular 2D domain with constant pressure and no-flow conditions at the vertical and horizontal boundaries, respectively. For the discretization, an equidistant Cartesian fine grid with $44 \times 44$ cells was used and in addition, for the i-MSFV method, a
4.1 Numerical Convergence Studies

A 4×4 coarse grid was employed (Fig. 14). Since each coarse cell is composed of 11×11 fine cells, the upscaling factor is 11 in each coordinate direction. For the following studies, homogeneous and heterogeneous mobility fields and domains with different grid aspect ratios α (horizontal to vertical dimension) are considered. The size of each fine cell is ∆x × ∆y with ∆x = α∆y = 1. Note that a case with isotropic mobility and ∆x = α∆y is numerically identical to a case with ∆x = ∆y and a mobility which is larger by a factor of α² in the y-direction.

The homogeneous test cases with \( \lambda_{ij} = \delta_{ij} \) also include a source with \( q = 1/(\Delta x \Delta y) \) and a sink with \( q = -1/(\Delta x \Delta y) \) distributed over the fine cells (13,13) and (32,32), respectively. For the heterogeneous cases, the mobility field depicted in Fig. 14 with natural logarithm (ln) variance and mean of 6.66 and −0.29, respectively, which is a part of the top layer of the three-dimensional SPE10 test case [19], was used.

![Figure 14: Natural logarithm of the heterogeneous mobility field with the computational domain which consists of 44 × 44 fine and 4 × 4 (shown) coarse cells. Also shown are the employed boundary conditions.](image)

Figure 14 shows the base-10 logarithm (log) of the maximum error in the domain, i.e. \( \log(\epsilon) \) with \( \epsilon = \| p' - p_f \|_\infty \), as a function of i-MSFV iterations and smoothing steps (per iteration), \( n_s \), for the homogeneous (Figs. 15a and 15c) and heterogeneous (Figs. 15b and 15d) cases with \( \alpha = 1 \) (Figs. 15a and 15b) and \( \alpha = 10 \) (Figs. 15c and 15d). For all cases there exists a minimum \( n_s \), for which the i-MSFV method converges. The best convergence
can be observed for the homogeneous isotropic ($\alpha = 1$) case and the worst convergence for the heterogeneous case with $\alpha = 10$.

Figure 15: Numerical convergence study with the first set of test cases; (a): homogeneous - isotropic, (b): heterogeneous - isotropic, (c): homogeneous - anisotropic ($\alpha = 10$), (d): heterogeneous - anisotropic ($\alpha = 10$).

Figure 16a shows the convergence histories for the heterogeneous test case as a function of $\alpha$ with $n_s = 10$. The slope decreases as $\alpha$ increases, but eventually it approaches an asymptotic value. This observation is confirmed by the plot in Fig. 16b, which shows the convergence rate (average slope between $\log(\epsilon) = -2$ and $\log(\epsilon) = -6$) as a function of $\alpha$ and $n_s$ for the heterogeneous-anisotropic case. For $\alpha \geq 20$, the convergence dependence on the aspect ratio becomes negligible. This result is encouraging, since it demonstrates that the i-MSFV method can be applied for cases with very
large aspect ratios and/or extreme anisotropies. One reason for this is the following property of the LR solver, which acts as a smoother in the i-MSFV method. For comparison, Fig. 16b also shows the convergence rates (multiplied with 100) of the LR solver for cases with different aspect ratios. Note that virtually no sensitivity on $\alpha$ can be detected.

Figure 16: (a): convergence history of the i-MSFV method for the heterogeneous domain with $n_s = 10$ and different aspect ratios; (b): convergence rates of the pure LR solver (multiplied with 100) and the i-MSFV method for the heterogeneous; anisotropic domain with different $n_s$ as a function of the aspect ratio.

Figure 17a illustrates how the convergence rate increases with $n_s$. To estimate the optimal number of smoothing steps per iteration, we assume that the amount of computational work to calculate the correction functions, to solve the coarse problem, and to reconstruct $p'$ corresponds to $\beta$ times the computational work required for one smoothing step. This leads to the

$$\text{Effective Convergence Rate} = \frac{\text{Convergence Rate}}{1 + n_s/\beta}. \quad (38)$$

It is a measure for the error reduction, if the computational work equivalent to one MSFV iteration (without smoothing nor reconstruction of a conservative velocity field) is invested. Fig. 17b shows the effective convergence rates for various aspect ratios $\alpha$ as functions of $n_s$, where $\beta$ is assumed to be one.

To analyze the computational cost associated with the i-MSFV method as a function of the problem size, the number of fine cells in the homogeneous isotropic test case was increased successively by adding coarse-grid cells with
Figure 17: (a): effect of $n_s$ on the convergence rate of the i-MSFV method for the heterogeneous domain with different aspect ratios; (b): effective convergence rate for heterogeneous-anisotropic test cases.

11 × 11 fine cells each. Fig. 18 depicts the convergence rates for 2 × 2, 3 × 3, 4 × 4, 5 × 5, 6 × 6, 7 × 7, 8 × 8, 9 × 9, and 10 × 10 coarse grids, and the log − log plot clearly shows (dashed line) that the convergence rate (for constant $n_s = 10$) is insensitive to the fine-grid size; opposed to the convergence rate of the LR-solver (solid line). Moreover, all calculation steps in the i-MSFV algorithm, except solving the global coarse-scale problem, can be performed locally and independently. Therefore, since up to very large cases the cost for solving the coarse system is virtually negligible, the i-MSFV algorithm is a very efficient linear solver for large, stiff problems, and it is ideally suited for massive parallel computing.

Another interesting parameter is the upscaling factor. Fig. 19 shows the convergence rate for the heterogeneous case with upscaling factors $\Gamma$ of 11 × 11, 7 × 7, and 5 × 5. In Fig. 19a, the convergence rates are shown as functions of $\alpha$ with constant $n_s = 10$ and in Fig. 19b, they are depicted as functions of $n_s$ with constant $\alpha = 5$. Obviously, the optimal choice of $\Gamma$ depends on the size of the fine grid and on the computational cost of the individual algorithmic components; in particular of the coarse-scale solver.

To complete our numerical investigations of the i-MSFV convergence behavior, four sets of 20 realisations of log-normally distributed mobility fields with spherical variogram and dimensionless correlation lengths $\psi_1 = 0.5$ and $\psi_2 = 0.02$ are generated using sequential Gaussian simulations [16]. For each set, variance and mean of $ln(\lambda)$ are 2.0 and 3.0, respectively. As depicted in
4.1 Numerical Convergence Studies

Figure 18: Effect of domain size on the convergence rate of the i-MSFV method with $n_s = 10$ and on the convergence rate of LR (multiplied by 100) for the homogeneous - isotropic domain.

Figure 19: Convergence rate of the i-MSFV method for a heterogeneous domain versus aspect ratio (a) and number of smoothing steps (b) for different upscaling factors.
Fig. 20, the angles $\theta$ between the long correlation length and vertical domain boundaries (or vertical grid lines orientation) are $0^\circ$, $15^\circ$, $30^\circ$, and $45^\circ$. For each case, a $100 \times 100$ fine and a $20 \times 20$ coarse grid were employed. At the boundaries of the quadratic domain, no-flow conditions were applied and at the lower left and upper right corners (cells (3,3) and (97,97)), a source and a sink of equal strength ($q = \pm 1/(\Delta x \Delta y)$) were imposed (Fig. 21). Figs. 22a and 22b show the mean convergence rates as functions of $\theta$ for different $n_s$, $\alpha$, and $\Gamma$. As one can see, there is a significant difference in the convergence rates. However, in general the convergence rate decreases with increasing layering orientation angle $\theta$.

![Figure 20: Natural logarithm of one of the 20 different mobility field realizations used in this study (for different angles $\theta = 0^\circ$, $\theta = 15^\circ$, $\theta = 30^\circ$, and $\theta = 45^\circ$ from left to right).](image)

### 4.2 Spectral Analysis of the i-MSFV Method

We conclude the convergence assessment of the i-MSFV method by analysing the spectrum of the associated iteration matrix, i.e., according to Eq. (33), of $A^{(n_s)}$ in

$$p^\nu_s = A^{(n_s)} \cdot p^{\nu-1}_s + b^\nu.$$

Clearly, the iteration procedure converges, if and only if all eigenvalues of $A^{(n_s)}$ lay within the unit-disc of the complex plane.

Various spectra of $A^{(n_s)}$ for the homogeneous anisotropic test case with no-flow conditions at all boundaries are depicted in Fig. 23. Fine and coarse grids consist of $44 \times 44$ and $4 \times 4$ cells, respectively and the different figures refer to iteration matrices based on different $n_s$. These results confirm those presented in Fig. 15-a, according to which at least two smoothing steps are required for the homogeneous isotropic case. Notice that unlike the matrix
4.2 Spectral Analysis of the i-MSFV Method

Figure 21: Computational domain, which consists of $20 \times 20$ coarse cells (shown); each coarse cell contains $5 \times 5$ fine cells (not shown). The black squares mark the two wells with $q = \pm 1/(\Delta x \Delta y)$ distributed over the fine cells (3,3) and (97,97).

Figure 22: (a): mean convergence rate for different angles $\theta$, different numbers of smoothing steps, and $\Gamma = 5 \times 5$. Also shown are the error bars for the two test cases. (b): convergence rate for two different upscaling factors of $5 \times 5$ and $7 \times 7$. The convergence rate for $\Gamma = 7 \times 7$ is presented for different smoothing steps.
$M$ of the fine-scale problem (79), $A^{(n_s)}$ is not symmetric and possesses non-real eigenvalues. All eigenvalues of $A^{(n_s)}$ are clustered around the negative real axis, which implies that the approximate solution at successive iteration steps oscillates around the exact one.

The eigenfunctions $\tilde{p}$ associated with the largest eigenvalues are plotted in Fig. 24 together with the corresponding residual $\rho = \nabla \cdot \lambda \cdot \nabla \tilde{p}$ in the discrete fulfillment of Eq. (14) without right-hand side. Only the results for $n_s = 0$ (unstable) and $n_s = 2$ (stable) are shown. In both cases, the residuum is largest at the dual-cell boundaries and without smoothing it is zero everywhere else. This is in agreement with the fact that any non-smoothed solution $p'$ fulfills Eq. (14) exactly inside the coarse dual cells. The smoothing steps efficiently redistribute the residuum and reduce its maximum amplitude. Consequently, the eigenvectors of $A^{(n_s)}$ get amplified for $n_s = 0$ and damped for $n_s = 2$.

In Fig. 25, similar spectra of the iteration matrix $A^{(n_s)}$ can be observed for cases with heterogeneous mobility fields. In Figs. 26a and 26b, the largest values of the residua associated with the ten least stable eigenvectors for $n_s = 0$ and $n_s = 5$, respectively, are presented. Notice the discontinuous distribution in Fig. 26a for the case with $n_s = 0$. Clearly, the residuum gets
4.2 Spectral Analysis of the i-MSFV Method

Figure 24: Eigenvectors associated to the largest eigenvalues of the spectra from Fig. 23 (left) and corresponding residuum in the fulfillment of Eq. (14) (right) without right-hand side; (a): $n_s = 0$, (b): $n_s = 2$. Eigenvectors and residual are independently rescaled.
distributed by the $n_s = 5$ smoothing steps (Fig. 26b). Finally, Figs. 27a and 27b depict the least stable eigenvector for $n_s = 5$ and its residuum.

![Graphs showing eigenvalues](image)

Figure 25: Spectra of the iteration matrix $A^{s(n_s)}$ for the heterogeneous isotropic case with $44 \times 44$ fine cells, $4 \times 4$ coarse cells, no-flow boundary conditions on $\partial \Omega$ and different values for $n_s$; (a): $n_s = 0$, (b): $n_s = 5$.

4.3 Application to Subsurface Flow

In typical incompressible subsurface flow simulations, the pressure in the porous media is governed by Eq. (14). As in the examples of section 4, the mobility $\lambda$ typically has a complex distribution with high variance and sharp contrasts. It is a function of the rock permeability $k$, the fluid phase saturations and the fluid viscosities. For single-phase flow of a fluid with viscosity $\mu$ one can write $\lambda = k/\mu$. The expression for multiphase flow is based on the relative permeability concept and reads $\lambda = k \sum_{j=1}^{n_p} k_{r_j}/\mu_j$ ($n_p$ is the number of fluid phases). The relative permeabilities $k_{r_j}$ [6] have to be specified for each fluid phase $j$ as functions of the saturations. While $\lambda$ does not change with time in single-phase flow simulations, it evolves if multiple fluid phases are transported through the reservoir. For the following studies, the right-hand side of Eq. (14) is non-zero only at the well, i.e. no capillary pressure difference between the fluid phases and no gravity are considered.
4.3 Application to Subsurface Flow

Figure 26: Residua in the fulfillment of Eq. (14) without right-hand side for the eigenvectors associated to the 10 largest eigenvalues of the spectra from Fig. 25; (a): $n_s = 0$, (b): $n_s = 5$.

Figure 27: Eigenvector (left) and corresponding residuum distribution (right) for the largest eigenvalue of the spectrum from Fig. 25a for the procedure with $n_s = 5$. Eigenvector and residuum are independently rescaled.
4.3.1 Single-Phase Flow

Here, in addition to the examples discussed in section 4, the convergence behavior of the i-MSFV method for single-phase flow in particularly challenging reservoirs is investigated. The rectangular 2D domain is discretized by a Cartesian, equidistant $220 \times 55$ fine grid. No-flow conditions are applied at the bottom and top walls; at the left and right boundaries, constant dimensionless pressure values of 1 and 0 are applied, respectively. The convergence histories for the permeability fields from the top and bottom layers (Figs. 5 and 11) of the 3D SPE 10 test case [19] are shown in Figs. 28a and 28b, respectively, where a $20 \times 5$ coarse grid was employed. As previously, the error is defined as the logarithm of the maximum absolute difference between the approximate i-MSFV and the reference fine-scale pressure values. While for the top layer a good convergence rate is achieved with $n_s = 10$ (Fig. 28a), approximately 250 smoothing steps are required for optimal convergence with the bottom layer permeability field (Fig. 28b). However, also in this case many more smoothing steps are necessary, if LR is employed as an iterative linear solver ($\sim 10^5$ iterations are necessary in order to reduce the error by 5 orders of magnitude). Moreover, Fig. 29 illustrates that the number of smoothing steps can be reduced dramatically, if a coarsening factor of $\Gamma = 5 \times 5$ (and fine grid of $220 \times 60$) instead of $\Gamma = 11 \times 11$ (and fine grid of $220 \times 55$) is employed.

![Figure 28](image-url)

Figure 28: Numerical convergence histories with the permeability field of Fig. 11a (a) and Fig. 11b (b) for $\alpha = 1$, $\Gamma = 11 \times 11$, and a fine grid of $220 \times 55$. 
4.3 Application to Subsurface Flow

As a further test case, a rectangular domain with two almost impermeable shale layers is considered (Fig. 30a); the mobility in the shale layers is $10^{10}$ times smaller than in the rest of the domain. The equidistant Cartesian fine grid consists of $55 \times 55$ cells and the coarse grid for the i-MSFV method contains $5 \times 5$ volumes. Again, no-flow conditions are applied at the bottom and top boundaries and at the left and right sides the dimensionless pressure values are set equal to 1 and 0, respectively. Fig. 30b shows the convergence histories with $n_s = 10$ for different aspect ratios.

### 4.3.2 Multi-Phase Flow

As already pointed out, in multiphase flow simulations the mobility $\lambda$ and therefore the pressure field evolve with time as the phase saturations are transported through the domain. Obviously, this also affects the localization boundary conditions, which continuously experience changes in the whole domain, even where the mobility remains constant. Consequently, in a straightforward application of the i-MSFV method for multiphase flow, all correction functions have to be re-computed multiple times every time-step. Although the number of i-MSFV iterations is reduced by the good initial condition obtained from the previous time step, such an approach is significantly more expensive than the original MSFV method. Here it is shown that only infrequently updating the localization boundary conditions for the re-computation of the correction functions is sufficient to obtain highly accurate solutions. While at the beginning of a simulation a converged solution is computed,
NUMERICAL RESULTS

Figure 30: (a): shale layer test case together with 5 × 5 coarse grid. Each coarse cell contains 11 × 11 fine cells (not shown). The mobility in the domain is $10^{10}$ times higher than in the shale layers. (b): numerical convergence history of the test case for $n_s = 10$ and different $\alpha$.

the same localization boundary conditions are used for a number of subsequent time steps and are only updated infrequently, e.g. each 10th time step by applying one iteration. Therefore, for the major part of the simulation the original MSFV method with slightly modified correction functions is employed and both basis and correction functions need to be updated in regions only where the total mobility changes are significant [49, 50]. The computational cost of this algorithm is comparable with the one of the original MSFV method. But typically, as shown next, the accuracy of the solutions is dramatically improved.

The i-MSFV method with infrequently updating the localization conditions is tested for two two-phase flow scenarios with a viscosity ratio $\mu_2/\mu_1$ of 10 and the relative permeabilities $k_{r_{1,2}} = S_{r_{1,2}}^2$ ($S_{r_{1,2}} \in [0, 1]$ are the phase saturations) for the first test case and $k_{r_{1,2}} = S_{r_{1,2}}$ for the second one. The permeability fields of Figs. 11a and 30a are employed and the rectangular domains are discretized by 220 × 55 and 55 × 55 fine grids, respectively. In both cases, coarse grids consisting of volumes containing 11 × 11 fine cells with an aspect ratio of 10 are used and no-flow conditions are applied at the whole domain boundary. Initially, the domains are occupied with phase two and the less viscous phase is injected into the fine cell (1,1). In the first scenario, production occurs from cell (220,55) and in the second scenario from cell (55,55). For the numerical solution of the phase transport equation, an
explicit scheme was employed. Figs. 31 and 32 show the saturation maps for the two test cases after 0.165 pore volume injected (PVI). One can observe that the i-MSFV method with updating the correction function boundary conditions every 10th time step leads to results, which are virtually identical with the fine-scale reference solutions. On the other hand, the MSFV solutions of these challenging test cases show significant deviations from the reference.

Note that the required frequency and minimum number of employed iterations, which are required to obtain good solutions, are problem dependent. Moreover, as mentioned before, here, an explicit integration scheme is used for the transport problem. Particularly, for implicit simulations the numbers are expected to be different. Here, it is emphasized only on the fact that the old solution can be used to enhance localization condition for the next time steps and iterations can be employed once needed. Also, it is very important to illustrate that there is no need to converge in order to obtain conservative solutions. As mentioned before, the i-MSFV solutions are conservative after any iteration level. In section 9 an extensive study on the average number of additional iterations required to obtain good solutions in the sequentially fully implicit framework is presented. Also, of high applicability importance of the method, an adaptive strategy (ai-MSFV) and error control study in the i-MSFV framework will be presented in section 10.
Figure 31: Two-phase flow saturation maps for the SPE10 top layer test case with $\mu_2/\mu_1 = 10$, $k_{r_{1,2}} = S_{1,2}^2$, and $\alpha = 10$ after 0.165 PVI; (a): fine scale, (b): i-MSFV result with local boundary conditions updated every 10th time step by applying 1 iteration, (c): standard MSFV result.

Figure 32: Two-phase flow saturation maps for the shale layer test case with $\mu_2/\mu_1 = 10$, $k_{r_{1,2}} = S_{1,2}$, and $\alpha = 10$ after 0.165 PVI; (a): fine scale result, (b): i-MSFV result with local boundary conditions updated every 10th time step by applying 1 iteration, (c): standard MSFV result.
Part IV
Compressible Multiphase Flows

5 A General MSFV Method for Compressible Flows

In this section, first a general MSFV framework for parabolic problems (compressible flows) is presented. This MSFV framework is based on a new formulation of the flow problem which leads to a symmetric system matrix. As a result, in this method the basis functions are kept independent of the pressure field and are not required to be updated due to pressure change in the domain. The devised approach involves minimal assumptions and the full governing equations are solved in local domains with assumed localization conditions. Such an approach is not only more general and accurate than previous methods; at the same time it is also computationally more efficient. Moreover, to construct the coarse system, exactly the same procedure applies as for the incompressible (elliptic) MSFV method.

In the next section, the i-MSFV method for compressible multiphase flow in porous media (parabolic problems) is presented. Similar to the i-MSFV method for elliptic problems [34], the solution field is smoothed in each iteration step and used to improve the localization condition for the next iteration step. So far, only a limited set of smoothers has been explored. For highly anisotropic heterogeneous problems (or for simulations with a stretched grid), however, line-relaxation proved to be very effective [34] and hence was used as a smoother for this work.

This section is organized as follows. First the devised MSFV and i-MSFV methods are motivated by a brief discussion about limitations of the existing studies. Then, the new governing equations for compressible flows are presented. Based on that, the new MSFV method for parabolic problems is introduced. Numerical test cases are then presented for a wide range of test cases, followed by a section where the i-MSFV method for compressible multiphase flow is introduced. Finally, the numerical convergence of the i-MSFV method and its efficiency for multiphase flow simulations are studied for different test cases.

5.1 Limitations of the Existing Methods

The MSFV method has been extended for the solution of parabolic problems arising from compressible multiphase flow in porous media through three
major works [66, 89, 60]. The proposed approaches proved to be effective for many cases, in other ones they fail to give good results, however.

In the first approach [66] compressibility effects inside the localized domains are neglected in order to avoid frequent updates of the basis functions. In this approach, only the coarse-scale system accounts for the compressibility effects. This scheme is computationally efficient, but requires additional assumptions.

Later, an improved approach where compressibility effects were considered only in the elliptic part of the localized problems was introduced [89]. The approach was introduced in an algebraic framework called the Operator Based Multiscale Method (OBMM). The accumulation terms only appeared in the coarse-scale system. In some of the difficult cases, the results of this approach were in better agreement with the corresponding fine-scale solutions than those presented in [66]. However, neither the OBMM can properly capture the transient fine-scale behavior.

To illustrate the limitation of the previously devised MSFV methods a single phase ideal gas injection problem in a 1D domain is considered. Note that for 1D test cases there exists no localization assumption for the MSFV method and, therefore, the fine-scale and MSFV results should be identical. The domain contains 100 fine and 5 coarse cells. The pressure initially is constant and at non-dimensional time $\tau^* = 0$ the pressure at cell (1) is raised to 10 times higher value. At cell (100) the pressure is constant with the value equal to the initial pressure. The pressure field at different non-dimensional times is plotted in Fig. 33; of the fine-scale reference solution and of the previously devised MSFV approaches. It is clear from this figure that all previous approaches fail to correctly represent the transient behavior of the parabolic pressure equation. The OBMM [89], however, can capture correctly the steady state solution while the FSA [66] also entails errors in the steady state condition.

None of the above mentioned approaches involves correction functions. Recently, correction functions were employed to account for the accumulation terms in the localized domains [60]. Similar to the previous approach [89], however, basis functions were still numerical solutions of the elliptic part of the full parabolic problem and had to be recomputed in each iteration step. Although updating the basis functions adaptively helps to improve the computational efficiency of the scheme, it is favorable to keep the basis functions pressure level independent. Next it is shown how this is achieved in the new MSFV framework.
5.1 Limitations of the Existing Methods

Figure 33: 1D single phase ideal gas injection test case: pressure at three different non-dimensional times. Shown are the fine scale reference and previously devised MSFV solutions [89] using 100 fine and 5 coarse cells using 100 fine and 5 coarse cells. FSA based multiscale method was developed in [66] and OBMM method was developed in [89].
5.2 A Consistent Reformulation of the Pressure Equation Suited for the MSFV Framework

As explained in the subsection 1.1.1 the mass conservation law for phase $\alpha$, i.e. Eq. (1), together with the $\sum_{\alpha=1}^{\gamma} S_{\alpha} = 1$ constraint constructs the system for $\gamma + 1$ unknowns $(S_1, ..., S_\gamma, p)$ [6]. It was also explained in the same subsection that the implicit Euler time integration of the conservation equations divided by the density $\rho_n^{\alpha+1}$ after summation over all phases $\alpha$ results in a decoupled pressure equation reading

$$\frac{\phi^{n+1}}{\Delta t} - \frac{\phi^n}{\Delta t} \sum_{\alpha=1}^{\gamma} B_{\alpha}^{n+1} \rho_n^{\alpha} S_{\alpha}^{n+1} - \sum_{\alpha=1}^{\gamma} B_{\alpha}^{n+1} \nabla \cdot (\rho_n^{\alpha+1} \lambda_{\alpha} \cdot \nabla p_n^{\alpha+1} - \rho_n^{\alpha+1} g \nabla z)) = q.$$  \hspace{1cm} (40)

Linearization of this equation results in the convergent iterative linear pressure equation which in brief reads

$$\frac{C}{\Delta t} (p^{\nu+1} - p^{\nu}) - \sum_{\alpha=1}^{\gamma} B_{\alpha}^{\nu} \nabla \cdot (\rho_{\alpha}^{\nu} \lambda_{\alpha} \cdot \nabla p^{\nu+1}) = RHS_{\nu}. \hspace{1cm} (41)$$

For a detailed explanation of the terms appearing in Eqs. (40) and (41) readers are referred to the governing equation subsection of the introduction part of this thesis.

As mentioned before, this discretized formulation was already used to construct the MSFV framework in previous studies [89, 60]. Here, however, for the sake of a more efficient MSFV formulation, which will be explained later, the convective term is consistently modified as

$$\sum_{\alpha=1}^{\gamma} B_{\alpha}^{\nu} \nabla \cdot (\rho_{\alpha}^{\nu} \lambda_{\alpha} \cdot \nabla p^{\nu+1}) \approx \nabla \cdot (\lambda_t \cdot \nabla p^{\nu+1}) - \nabla \cdot (\lambda_t \cdot \nabla p^{\nu}) + \sum_{\alpha=1}^{\gamma} B_{\alpha}^{\nu} \nabla \cdot (\rho_{\alpha}^{\nu} \lambda_{\alpha} \cdot \nabla p^{\nu}),$$  \hspace{1cm} (42)

where $\lambda_t = \sum_{\alpha=1}^{\gamma} \lambda_{\alpha}$ is the total mobility. Note that $\nu = \nu + 1 = n + 1$, if converged. It is worth mentioning that for black-oil models, which are more realistic to describe multiphase flow in oil reservoirs [6], one can still use this expression. Additional effects are e.g. capillary pressure differences and mass exchange between phases, which are treated on the right hand side, i.e. the implicit convection operator has the same structure as in Eq. (42) [60].

Finally, the newly devised convergent iterative procedure for the fine-scale pressure equation reads

$$\frac{C}{\Delta t} (p^{\nu+1} - p^{\nu}) - \nabla \cdot (\lambda_t \cdot \nabla p^{\nu+1}) = RHS^{\nu},$$  \hspace{1cm} (43)
5.3 Algorithm

where

\[ RHS^\nu = \mathcal{R}^\nu - \nabla \cdot (\lambda_t \cdot \nabla p^\nu) + \sum_{\alpha=1}^{\gamma} B^\nu_\alpha \nabla \cdot (\rho^\nu_\alpha \lambda_\alpha \cdot \nabla p^\nu). \]  

(44)

Here, \( \lambda_\alpha \) is calculated based on \( S^u_\alpha \) leading to an IMPES (Implicit in Pressure, Explicit in Saturation) scheme. In other words, Eq. (43) is solved until convergence occurs, then the new pressure field is used to solve the saturation equation (3) explicitly using a second order upwind scheme. Note that alternatively, Eq. (3) can be solved implicitly, thus leading to a sequentially implicit scheme [66, 89, 60]. It is an important property of relation (43) that except for the accumulation term its structure is the same as for elliptic problems. Next, it is explained how one can solve this linearized pressure equation for compressible flows accurately using the MSFV method.

5.3 Algorithm

For incompressible flow, as explained in the previous section, basis and correction functions are local numerical solutions of an elliptic problem with and without right hand side, respectively. Here, this idea is consistently generalized for compressible multiphase flow. Therefore, the basis and correction functions are computed by solving the localized problems

\[ \frac{C}{\Delta t} \Phi^b_k - \nabla \cdot (\lambda_t \cdot \nabla \Phi^b_k) = 0 \]  

and

\[ \frac{C}{\Delta t} \Phi^c^\nu_k - \nabla \cdot (\lambda_t \cdot \nabla \Phi^c^\nu_k) = RHS^\nu + \frac{C}{\Delta t} p^\nu \]  

(46)

on the dual coarse cells \( \hat{\Omega}^b \); again subject to the reduced problem boundary condition as in the previously explained elliptic case (Eqs. (16) and (17)). In contrast to the previous works [89, 60], there is no iteration superscript for the basis function calculations, i.e. they have to be computed only once while the correction functions are computed iteratively and take care of all pressure dependencies. Note that in 3D there exist 8 times more basis than correction functions. When convergence occurs, the first term on the left hand side of Eq. (43) cancels out. Hence, the specific value of \( C \) does not affect the numerical value of the converged solution, but affects the convergence rate. As \( C \) is a monotone function of the pressure, it can be computed based on either \( p^\nu \) or \( p^n \) (or some reference pressure). In this work, to keep the basis functions totally out of the iteration loop, \( C \) is computed based on \( p^n \).
An illustration of the basis and correction functions for parabolic problems (Eq. (45) and (46)) is given in Fig. 34. Moreover, Fig. 34 (left) emphasizes the difference between a basis function, which is calculated based on the full parabolic equation, i.e. Eq. (43), and the elliptic part of it, i.e. without the \( \frac{C}{\Delta t} \Phi_k \) term (wireframe). Finally, the time size for calculation of basis and correction functions are the same as that of the global parabolic problem.

Figure 34: Illustration of the basis function \( \Phi_1^h \) (left) and correction function \( \Phi^h \) (right) for parabolic problems. The permeability map for the corresponding localized domain \( \tilde{\Omega}^h \) is partly shown on the base surfaces. Also shown (left) is the wire-frame of the basis function for the corresponding elliptic problem, i.e. without first term on the left-hand-side of Eq. (45).

To derive a linear system for the coarse pressure values \( \bar{p}_k \), we follow exactly the described procedure for incompressible flow. Since the basis and correction functions are solutions of the full parabolic problem in the localized domains, no further consideration in the coarse scale system, such as averaging the accumulation term, is necessary (opposed to [60]). Substituting expression (15) into Eq. (43) and integrating over \( \tilde{\Omega}_l \) leads to

\[
\int_{\tilde{\Omega}_l} \frac{C}{\Delta t} \left( \sum_{h=1}^N \left( \sum_{k=1}^M \bar{p}_{k+1}^\nu \Phi_k^h + \Phi^{h\nu} \right) - p^{\nu'} \right) d\Omega - \\
\int_{\partial \tilde{\Omega}_l} \left( \lambda \sum_{h=1}^N \left( \sum_{k=1}^M \bar{p}_{k+1}^\nu \nabla \Phi^h + \nabla \Phi^{h\nu} \right) \cdot \bar{n}_l \right) d\Gamma = \int_{\tilde{\Omega}_l} RHS^\nu d\Omega + \int_{\tilde{\Omega}_l} q d\Omega,
\]

(47)
which results in an iterative linear system

$$A_{lk} \bar{p}_k^{n+1} = b_l^n$$

(48)

for $\bar{p}_k^{n+1}$ with

$$A_{lk} = \sum_{h=1}^{N} \left( \int_{\Omega_h} \frac{C}{\Delta t} \Phi_h \, d\Omega - \int_{\partial \Omega_h} (\lambda_t \cdot \nabla \Phi_h) \cdot \vec{n} \, d\Gamma \right)$$

(49)

and

$$b_l^n = \int_{\Omega_l} \left( RHS^n + \frac{C}{\Delta t} p^n \right) d\Omega - \sum_{h=1}^{N} \left( \frac{C}{\Delta t} \Phi_h \, d\Omega - \int_{\partial \Omega_h} (\lambda_t \cdot \nabla \Phi_h) \cdot \vec{n} \, d\Gamma \right).$$

(50)

Using $\bar{p}_k^{n+1}$ for $\bar{p}$ in expression (15) gives the new fine-scale pressure field $p_{n+1}^{n+1}$. The iterative procedure is repeated until $p'$ converges, i.e. until $\|p^{n+1} - p^n\|_\infty < \epsilon$, where $\epsilon$ is a specified convergence parameter. As already mentioned, a conservative velocity field must be reconstructed in order to solve the saturation transport equations on the fine grid. We reconstruct it by solving Eq. (43) locally subject to the Neumann boundary conditions based on $p_{n+1}^{n+1}$; similar as in the incompressible case. The resulting pressure field $p_{n+1}^{n+1}$ is considered as the final solution at the time step $n+1$. Finally, this conservative total velocity field is employed to explicitly solve the saturation equation (3) using a second order upwind scheme.

From the parabolic basis function calculation (Eq. (45)) and the coarse-scale matrix $A$ (Eq. (49)) it is clear that the accumulation term (compressibility coefficient) appears as a stencil in the coarse-scale system, and not only as a diagonal contribution. This is an important factor for the high accuracy of the method, which will be illustrated for several test cases in the next section. An algorithmic overview of the proposed MSFV method for compressible multiphase flow problems is presented in Table 3.

5.4 Numerical Results

5.4.1 Single-phase Flow

To analyze the performance of the new scheme compared to the previous studies, first a one dimensional (1D) homogeneous single phase flow problem is studied. No gravity effects are considered and the domain is initially filled with ideal gas at atmospheric pressure (14.7 psi). The right boundary is kept at a constant pressure of 14.7 psi and at time $t = 0$, the pressure at the left
initialize $p^{(t=n)}$
\[ \forall h : \forall k : \text{compute basis functions } \Phi_h^k; \text{ Eq. (45)} \]
from $\nu = 1$ to convergence \{ 
\[ \forall h : \text{compute correction functions } \Phi_h^{\nu}; \text{ Eq. (46)} \]
\[ \text{solve coarse system } A \cdot \bar{p}^{\nu+1} = b^{\nu}; \text{ Eq. (48)} \]
\[ \text{reconstruct } p^{\nu+1}; \text{ Eq. (15)} \]
\[ \text{update properties } \nu \leftarrow \nu + 1 \]
\}
\[ \text{reconstruct } p^{m+1} \text{ and calculate conservative velocity field; Eq. (43)} \]
\[ \text{solve saturation transport equations; Eq. (3)} \]

Table 3: Algorithmic outline of one time step with the MSFV method for compressible multiphase flow.

boundary is increased to 10atm (147.0psi). We recall the state equation of ideal gas under isothermal conditions, i.e.
\[ \frac{\rho_g}{\rho_{g0}} = \frac{p}{p_0}, \]  
(51)

where $\rho_{g0}$ is the density at atmospheric pressure $p_0$. The computational domain contains 105 fine cells and 5 coarse cells. Figure 35 shows results at different non-dimensional times $t^* = t/\tau$, where
\[ \tau = \frac{\mu \phi L^2}{k(p_l - p_r)}. \]  
(52)

In Eq. (52), $k$ is the average permeability in the domain, $L$ is the length of the domain, and $(p_l - p_r)$ is the pressure difference between the left and right boundaries. It can be observed that the new MSFV method (Fig. 35 (left)) leads to results, which are virtually identical to the fine-scale reference solutions. It is remarkable that unlike in previous works [66, 89] (Fig. 35 (right)) no peculiarities in the results were observed for the early transient phase.

Since no localization assumption is required for 1D problems, the MSFV and fine-scale solutions are identical up to machine accuracy. Note that in 2D or 3D, convergence of the MSFV solution can be achieved (even for highly anisotropic heterogeneous problems) by iteratively improving the localization assumption, i.e. by improving the boundary condition of the local problems [34].
5.4 Numerical Results

Figure 35: 1D single phase gas injection test case: pressure at three different times. Shown are the new MSFV and fine scale reference solutions (left) using 105 fine and 5 coarse cells together with previous multiscale solutions presented in [89] (right) using 100 fine and 5 coarse cells. FSA based and OBMM multiscale method was developed in [66] and [89], respectively.

5.4.2 Depletion in 1D and 2D

As a second test case, depletion of a liquid-gas reservoir is studied. No gravity effects are considered and the reservoir pressure is initially at 10 atmosphere (147.0 psi). At \( t = 0 \) the pressure at the right boundary is decreased to one atmosphere (14.7 psi), while it is kept constant at the left boundary. At top and bottom, no-flow boundary conditions are applied. The domain is initially filled with 50% ideal gas and 50% water. Porosity of the medium and viscosity ratio \( \mu_g/\mu_w \) are set to 0.1 and \( 1.8 \times 10^{-2} \) everywhere, and the water compressibility is neglected. Water saturation \( S_w \) at the left boundary is kept constant at 0.5. For all test cases presented in this paper, relative permeability is modeled as a quadratic function of the corresponding phase saturation.

First, a homogeneous 1D domain containing 105 fine and 5 coarse cells is considered. Fig.36 shows the MSFV results (pressure and saturation) at different times and it can be observed that they are in perfect agreement with the fine-scale reference solutions.

Another study was considered with a 2D permeability field which was extracted from the top layer of the SPE comparative test case 10 [19] (Fig. 5). The grid consists of \( 220 \times 55 \) fine and \( 20 \times 5 \) coarse cells. MSFV pressure and saturation maps at \( t^* = 10^{-4}, t^* = 10^{-3}, \) and \( t^* = 10^{-2} \) are depicted on
the right of Figs. 37 and 38, respectively, and the corresponding fine-scale reference solutions are shown on the left. For better comparison, the saturation maps are only shown for $x \geq 160$. In addition, Figs. 39 and 40 show results at $t^* = 10^{-1}$ and $t^* = 1$, respectively. Again, the MSFV and fine-scale solutions are in excellent agreement.

![Figure 36: Depletion of a homogeneous 1D domain: fine-scale solutions (using 105 cells) compared with the MSFV results (using 5 coarse cells); pressure (left) and water saturation (right) at different non-dimensional times.](image)

### 5.4.3 Five-spot problem: ideal gas injection

As another test case, the injection of an ideal gas into a 2D heterogeneous domain with the same permeability as shown in Fig. 5 is studied. No-flow boundary conditions are applied everywhere. The domain is initially filled with oil at atmospheric pressure (14.7 psi) and no gravity effects are taken into account. At $t \geq 0$, an ideal gas is injected at a constant volumetric rate from the left-bottom corner (fine cell (1,1)) and production occurs at the upper-right corner (fine cell (220,55)) at a constant pressure of 14.7 psi. The viscosity ratio is 10 and it is assumed that the oil density obeys the isothermal linear equation of state, i.e. that

$$\frac{\rho_o}{\rho_{o0}} = 1 + 10^{-3}(p - p_0), \quad (53)$$

where $\rho_{o0}$ is the density at the reference pressure $p_0 = 14.7$ psi. Results are shown in Fig. 41 after 0.132 PVI (Pore Volume Injected). For better comparison, Fig. 42 shows the absolute pressure and saturation differences.
Figure 37: Heterogeneous 2D depletion test case: pressure filed obtained from the fine-scale simulation using $220 \times 55$ fine cells (left) and from the MSFV simulation using $20 \times 5$ coarse cells (right) at $t^* = 10^{-4}$ (top), $t^* = 10^{-3}$ (middle), and $t^* = 10^{-2}$ (bottom).

Figure 38: Heterogeneous 2D depletion test case: water saturation maps obtained from the fine-scale simulation using $220 \times 55$ fine cells (left) and for the MSFV simulation using $20 \times 5$ coarse cells (right) at $t^* = 10^{-4}$ (top), $t^* = 10^{-3}$ (middle), and $t^* = 10^{-2}$ (bottom). The results are only depicted for $x \geq 160$. 
Figure 39: Heterogeneous 2D depletion test case: water saturation maps obtained from the fine-scale simulation using $220 \times 55$ fine cells (left) and from the MSFV simulation using $20 \times 5$ coarse cells (right) at $t^* = 10^{-1}$.

Figure 40: Heterogeneous 2D depletion test case: water saturation maps obtained from the fine-scale simulation using $220 \times 55$ fine cells (left) and from the MSFV simulation using $20 \times 5$ coarse cells (right) at $t^* = 1$. 
between the fine-scale and MSFV solutions. Again, a very good agreement can be observed.

Figure 41: Heterogeneous 2D gas injection test case: pressure field (left) and gas saturation maps (right) obtained for the fine scale simulation using $220 \times 55$ fine cells (top) and from the MSFV simulation using $20 \times 5$ coarse cells (bottom) at 0.132 PVI. The permeability field of Fig. 5 was employed.

Figure 42: Heterogeneous 2D gas injection test case: absolute difference between the pressure (left) and saturation (right) fields obtained from the fine-scale and MSFV simulations; corresponding to the results shown in Fig. 41.

As another test case, the SPE10 bottom layer [19] (Fig. 11) with the same grid, phase properties and initial conditions as for the top layer test case is considered. Again, no-flow boundary conditions are applied everywhere. Gas is injected from the upper left corner (fine cell $(1,55)$) at a constant pressure of 147.0psi and oil is produced at the bottom-right corner (fine cell $(220,1)$) at a constant pressure of 14.7psi. Fine and MSFV results are presented in Fig. 43 after $t = 0.189\tau$ (based on oil viscosity $\mu_o = 1$, $L_x = 220$, $k = 580.438$, and $\Delta p = (147.0psi − 14.7psi)$). As for the previous test cases, a very good agreement can be observed. For better comparison, absolute differences between the fine-scale and MSFV pressure and saturation solutions are shown in Fig. 44.
Figure 43: Heterogeneous 2D gas injection test case: pressure field (left) and gas saturation maps (right) obtained from the fine scale simulation using $220 \times 55$ fine cells (top) and from the MSFV simulation using $20 \times 5$ coarse cells (bottom) at $t^* = 0.189$; the permeability field of Fig. 11 was employed.

Figure 44: Heterogeneous 2D gas injection test case: absolute difference between the pressure (left) and saturation (right) fields obtained from the fine-scale and MSFV simulations; corresponding to the results shown in Fig. 43.
5.4.4 Gas injection with gravity effects

As a final test case, the injection of an ideal gas into a vertical homogeneous 2D domain with strong gravity effects is considered. Initially, the domain is filled with oil (Eq. (53)) at a pressure of 147.0 psi. The problem is initialized by setting at the upper boundary a constant pressure of 147.0 psi (Fig. 45). Then, an ideal gas (Eq. (51)) is injected with constant rate from the bottom left corner (fine cell (1,1)), while production occurs from the top right of the domain (fine cell (44,44)) at a constant pressure of 147.0 psi. The density ratio at STC (Stock Tank Condition) $\rho_o/\rho_g$ is 100, the viscosity ratio $\mu_o/\mu_g$ is 5, and the porosity is 0.2. The grid contains $44 \times 44$ fine cells and $4 \times 4$ coarse cells with no-flow boundary condition everywhere (Fig. 45). Fine-scale and MSFV pressure and saturation solutions are presented in Fig. 46 after 0.08 PVI.

![Figure 45: Homogeneous 2D gas injection with gravity effects: initialized problem by setting at the upper boundary a constant pressure of 147.0 psi. Also shown are $4 \times 4$ coarse grid used in MSFV simulation.](image)

For better comparison, Fig. 47 depicts the absolute differences between the fine-scale and MSFV results. It can be observed that there is very good agreement.

6 I-MSFV Method for Compressible Flows

6.1 Limitations of the MSFV method

As shown in the previous sub-sections, the MSFV method provides highly accurate results for many challenging test cases. This is not always the case,
Figure 46: Homogeneous 2D gas injection with gravity effects: pressure (left) and gas saturation maps (right) obtained from the fine-scale simulation using $44 \times 44$ fine cells (top) and the MSFV simulation using $4 \times 4$ coarse cells (bottom) at 0.08 PVI.
6.1 Limitations of the MSFV method

however, which is demonstrated for the following problematic test cases. The first of these test cases employs the permeability distribution shown in Fig. 11 (from SPE10 bottom layer). The grids, initial, and boundary conditions correspond to the previously shown quarter-five-spot test case, but here only single phase gas flow is considered. Gas is injected at the upper-left corner at a constant rate and production occurs at the bottom-right corner. A time step size of $\Delta t = 2.657 \times 10^{-3}$ PVI is chosen and Fig. 48 shows fine-scale and MSFV pressure solutions after one time step. One can observe local pressure peaks in both the non-conservative and conservative MSFV pressure fields ($p'$ and $p''$, respectively), which are unphysical. Moreover, the size of these peaks grows with time.

The next test case is almost a copy of the previous one, except that the permeability distribution shown in Fig. 5 is employed (from SPE10 top layer) and that the grid aspect ratio $\alpha = \Delta x/\Delta y = 5$. Note that this results in an effective increase and decrease of the transmissibilities by the factor $\alpha$ in y- and x-directions, respectively. Fine-scale reference solutions along with the results obtained with the new MSFV method are presented in Fig. 49 after one time step $\Delta t = 0.011$ PVI. Again, one can observe local pressure peaks in the non-conservative and conservative MSFV pressure fields ($p'$ and $p''$, respectively), which are unphysical and the size of which grows with time.

The problem of the MSFV method with anisotropic permeability fields and large grid aspect ratios has been studied previously for incompressible (elliptic) cases [57, 67, 43, 34]. It was shown that this issue refers to an unstable
Figure 48: Heterogeneous 2D single phase gas injection test case: pressure fields obtained by (a): fine-scale reference approach $p_f$; (b): MSFV method $p'$ (non-conservative); (c): MSFV method $p''$ (conservative) after one time step $\Delta t = 2.657 \times 10^{-3}$ PVI. Fine-scale solution is obtained using $220 \times 55$ fine cells while the MSFV results are obtained using $20 \times 5$ coarse cells; permeability of Fig. 11 is employed.
6.1 Limitations of the MSFV method

Figure 49: Heterogeneous 2D single phase gas injection test case: pressure fields obtained by (a): fine-scale reference approach $p_f$; (b): MSFV method $p'$ (non-conservative); (c): MSFV method $p''$ (conservative) after one time step $\Delta t = 0.011$ PVI. Fine-scale solution is obtained using $220 \times 55$ fine cells while the MSFV results are obtained using $20 \times 5$ coarse cells. Permeability of Fig. 5 and aspect ratio $\alpha = 5$ are employed. For better comparison, results in x-direction are shown for the first six coarse cells.
localization assumption provided by the reduced problem boundary condition [67]. From another point of view, for anisotropic problems the monotonicity region of the MSFV coarse operator, which is based on a 9-point stencil, is very limited [43]. Hence, a compact MSFV method was proposed, which leads to better results [43]. Finally, an iterative MSFV method was devised [34]. The i-MSFV method converges to the fine-scale reference and improves the results of anisotropic cases dramatically after already a few iterations. Next, it is explained how the new MSFV method for parabolic problems can be extended to become a convergent iterative MSFV method.

6.2 Iterative MSFV method for parabolic problems

As mentioned before, while the MSFV method with reduced problem boundary conditions proved to be very accurate for many challenging test cases, it fails to give good results for problems with stretched grids, highly anisotropic permeability fields or permeability fields with sharp contrasts. As shown previously in this thesis, such solutions can dramatically be improved by a few iterations, in which the boundary conditions of the local problems leading to the correction functions are updated based on the most actual fine-scale pressure reconstruction [34].

The i-MSFV method [34] improves the boundary conditions of the localized problems iteratively and the solutions converge to the fine-scale reference. It can either be used as a multiscale method (no iteration) or as an efficient iterative linear solver (with enough iterations). In each iteration, the boundary conditions of the localized problems leading to the correction functions are updated based on a pressure field, which results from smoothing $p^\nu_s$, i.e. the reconstruction after the previous iteration. In other words, the localization condition (23) is replaced by the iterative expression

$$r^{h,\nu} = (\mathbf{n}^h \cdot \nabla) \left( (\lambda \cdot \nabla p^\nu_s) \cdot \mathbf{n}^h \right) \text{ at } \partial \Omega^h \ \forall h \in [1,N_d]. \quad (54)$$

Since in the new MSFV method for parabolic problems the correction functions are recomputed anyhow after each iteration step due to the pressure dependent parameters, little additional cost is involved to iteratively update the localization boundary conditions at the same time as described above. Note that no further modifications are necessary to apply the i-MSFV method for compressible problems.

The smoother is essential for convergence and one can think of many variants. Here, line-relaxation (LR) is used, since it proved to be very effective for cases with highly anisotropic permeability fields or stretched grids [34]. LR is applied $n_s$ times (inner loop) in all spacial directions. A pseudo code
6.3 Numerical Results

6.3.1 Numerical Convergence study

The performance of the i-MSFV method for compressible multiphase flow is studied for homogeneous-isotropic, heterogeneous-isotropic, homogeneous-anisotropic, and heterogeneous-anisotropic problems. All domains are initially filled with an ideal gas at atmospheric pressure (14.7 psi) and no gravity effects are considered. At $t = 0$, the pressure at the left boundary is elevated to 147.0 psi, while it is kept constant at the right boundary. No-flow boundary conditions are applied at the top and bottom walls. For the homogeneous test cases, the permeability is equal to 1 and the heterogeneous permeability fields are subdomains of the SPE10 [19] top layer ($ln(k)$ mean and variance are -0.29 and 6.66, respectively). The porosity $\phi$ is 0.3 for all test cases. The fine and coarse grids contain $44 \times 44$ and $4 \times 4$ coarse cells and to study the effect of anisotropy, a fine cell aspect ratio $\alpha$ (width over...
height) of 10 is employed. Figure 50 depicts the computational domain as well as the natural logarithm (ln) of the heterogeneous permeability field.

Figure 50: Test case for convergence study of the i-MSFV method: computational domain, which consists of 44 × 44 fine cells and 4 × 4 coarse cells (left), and natural logarithm of the heterogeneous permeability field (right).

Figure 51 shows the pressure maps after one non-dimensional time step $\Delta t^* = \Delta t/\tau$ ($\Delta t^* = 0.03$ for the homogeneous and $\Delta t^* = 0.01$ for the heterogeneous test cases). The time scale $\tau$ is computed based on $L_{\text{aniso.}} = \alpha L_{\text{iso.}}$, $k_{\text{hom.}} = 1$, and $k_{\text{het.}} = 11.2421$.

Figure 52 shows the base-10 logarithm ($\log$) of the maximum pressure error in the domain normalized with the pressure difference between the left and right boundaries, i.e. $\log(\epsilon) = \| p^{n+1}_i - p^{n+1}_f \|_{\infty}/(p_l - p_r)$, as a function of i-MSFV iterations and smoothing steps (per iteration) $n_s$ for the homogeneous (Figs. 52a and 52c) and heterogeneous (Figs. 52b and 52d) cases with $\alpha = 1$ (Figs. 52a and 52b) and $\alpha = 10$ (Figs. 52c and 52d).

The convergence rate of the i-MSFV method (average slope between $\log(\epsilon) = -3$ and $\log(\epsilon) = -9$) as a function of $\Delta t^*$ is shown in Fig. 53 for different $n_s$.

Note that the i-MSFV algorithm (Table 4) and the MSFV algorithm (Table 3) for compressible multiphase flow are iterative schemes converging to the original non-linear pressure equation (Eq. (40)) exactly and approximately, respectively. It is clear from Fig. 54 that the i-MSFV method converges to the fine-scale reference solutions faster than the MSFV method to approximate solutions. In other words, improvement of the localization condition
6.3 Numerical Results

Figure 51: Convergence study of the i-MSFV method: pressure maps after one non-dimensional time step $\Delta t^* = 0.03$ and $\Delta t^* = 0.01$ for the homogeneous and heterogeneous cases, respectively. From left to right: homogeneous-isotropic, heterogeneous-isotropic, homogeneous-anisotropic ($\alpha = 10$), heterogeneous-anisotropic ($\alpha = 10$).
Figure 52: Convergence study of the i-MSFV method: numerical convergence history for the (a) homogeneous - isotropic ($\Delta t^* = 0.03$), (b) heterogeneous - isotropic ($\Delta t^* = 0.01$), (c) homogeneous - anisotropic ($\Delta t^* = 0.03, \alpha = 10$), and the (d) heterogeneous - anisotropic ($\Delta t^* = 0.01, \alpha = 10$) cases.
6.3 Numerical Results

Figure 53: Convergence study of the i-MSFV method: convergence rates as a function of the non-dimensional time step size $\Delta t^*$ for the (a) homogeneous - isotropic, (b) heterogeneous - isotropic, (c) homogeneous - anisotropic ($\alpha = 10$), and the (d) heterogeneous - anisotropic ($\alpha = 10$) cases.
not only improves the quality of the results, but also enhances the convergence rate. The only additional cost for the i-MSFV method compared with the MSFV method is due to the smoothing step.

![Convergence study of the i-MSFV and the MSFV methods: convergence rates as a function of the non-dimensional time step size \( \Delta t^* \) for the i-MSFV homogeneous-isotropic (left) and heterogeneous-isotropic (right) test cases. Note that the MSFV does not converge to the fine-scale reference solution.](image)

**6.3.2 Multiphase flow simulations**

As described in the previous subsection, one may not be interested to use the i-MSFV method as a linear solver, but as an algorithm to improve the quality of the MSFV results. Therefore, in this subsection we consider the previously described quarter-five-spot test case with the permeability of SPE10 top layer (Fig. 5) and \( \alpha = 10 \). Remember that for this test case the MSFV method leads to unphysical solutions. Phase properties, initial and boundary conditions are the same as for the previously described test case. Gas is injected with a constant rate at the upper-right corner (fine cell (220,55)) and oil is produced from the bottom-left corner (fine cell (1,1)), where the pressure is kept constant at 14.7 psi (initial pressure). Here, only two i-MSFV iterations \( (n_\nu = 2) \) are performed in each time step with \( n_s = 10 \). The conservative pressure field \( p'' \) is computed once at the end of each time step. Fig. 55 shows the results after 0.2116 PVI. It is clear that results are virtually identical. One could also employ the idea of infrequently updating
the boundary conditions [34] to improve the computational efficiency, but this is beyond the scope of this paper.

Figure 55: i-MSFV method for multiphase flow with $\alpha = 10$ and the permeability field of Fig. 5: pressure field (left) and gas saturation maps (right) obtained for the fine scale simulation using $220 \times 55$ fine cells (top) and from the i-MSFV simulation using $20 \times 5$ coarse cells (bottom) at $t = 0.2116$ PVI. For the i-MSFV simulation, two iterations with $n_s = 10$ per time was employed.
Part V
Fractured and Faulted Porous Media

7 I-MSFV Method for Fractured Porous Media

7.1 Introduction

Mathematical formulations describing flow in natural porous media are typically governed by highly heterogeneous anisotropic tensorial coefficients (hydraulic conductivity) at different scales. Moreover, a considerable percentage of natural formations, e.g. carbonate reservoirs, are fractured in the sense that there exist highly conductive channels (with small apertures) at various length scales acting as phase transport highways. In addition to the complex geometries, the high contrast in the physical properties and length scales (compared to those of the matrix) results in very expensive fine scale simulations. Therefore, there have been extensive studies during the past five decades to reduce the problem complexity and as a result different modeling approaches and numerical strategies suitable for different types of fractures have been proposed [9, 86, 55, 56, 75, 8, 84, 7, 10, 59, 58, 64, 54, 72, 74, 33].

The dual porosity model approach and its divisions [9, 86, 55, 56] were proposed for naturally fractured porous media with many small fractures. More precisely, this method introduces effective coefficients for \((n-1)\) dimensional (D) fractures by mapping (upscaling) them into a continuum \(nD\) domain. This upscaling based strategy results in reasonably efficient simulations with the cost of additional assumptions. More specifically, this method is appropriate for problems with only small scale fractures. For problems with long scale fractures, however, this approach fails to provide good solutions. This is due to the fact that in this approach no general upscaling strategy is possible. To obtain more accurate simulations, the discrete fracture modeling approach was devised; see e.g. [75, 7, 54]. In that approach, geometry and locations of fractures are honored accurately by using complex unstructured gridding techniques [72]. The grid is generated with the constraints that the fracture elements are located at the matrix cell interfaces and that the matrix cells around fractures are small enough to capture the correct fracture geometries. The latter constraint often results in very small cells, especially near intersections. Besides the fact that small cells lead to
big linear systems, they also impose time step restrictions for multiphase transport simulations.

It is very important to keep in mind that this approach has limited applicability for realistic scenarios due to the complex conforming grids. Moreover, this approach is not suited for dynamic fracture network problems, as e.g. in simulations of enhanced geothermal systems, where the grid has to be updated frequently due to generations of new fractures. In such applications, it is preferable to work with independent discretizations for the discrete large fractures and the damaged matrix.

Motivated by the previously discussed issues, a hierarchical fracture modeling approach was introduced [59, 58, 64]. In this approach, small scale fractures are homogenized and treated as a continuous damaged matrix with effective coefficients. Large-scale fractures, on the other hand, are explicitly represented by a coupled discrete fracture model. More precisely, simple structured \( n \times D \) and \((n-1) \times D \) grids are independently generated for matrix and fractures, respectively. Note that neither grid alignment nor any other constraints apply.

In this work, a hierarchical fracture modeling approach which is suited for multiscale methods [44, 46, 45, 24, 3, 4, 17, 47, 48, 26, 1, 32, 2, 50, 53] is introduced. Moreover, as will be demonstrated later in this paper, for highly conductive fractures the proposed strategy leads to enhanced convergence rates independent of the MSFV method. One additional constraint per fracture network is added to the matrix equations, which is crucial to ensure enough coupling to achieve good convergence rates. Combining a hierarchical fracture model to a multiscale method for reservoir simulation is of interest, since multiscale methods are capable of honoring fine-scale transmissibility variations with much fewer degrees of freedom (DOF) than classical simulators. Such fine-scale variations become an even bigger issue in highly fractured reservoirs. In order to properly deal with transport it is highly desirable to work with a multiscale method, which delivers conservative fine-scale velocity fields; see e.g. [41, 42]. Therefore, here the multiscale finite-volume (MSFV) method is favored, which requires fewer DOFs than mixed multiscale methods and still is conservative opposed to e.g. the multiscale finite-element methods. Moreover, as shown recently, in this framework the error can systematically be reduced [34, 37, 12, 70, 13, 90]; adaptive in space and time [38]. Opposed to classical iterative solvers like algebraic multigrid (AMG) [85], conservative solutions can be constructed after any iteration, i.e. it is not required to fully converge.

In this work, the i-MSFV method is extended for the solution of the hierarchical fracture problem using the proposed sequential coupling strategy. To capture fractures accurately at the coarse scales, local fracture functions
Hierarchical Fracture Modeling Approach

7.2 Hierarchical Fracture Modeling Approach

In this section the hierarchical fracture modeling approach is explained. First, the governing equations together with the model parameters, and then a previously presented fully coupled numerical simulation strategy \[59, 58\] are explained.

7.2.1 Governing equations and modeling parameters

Here, Darcy’s law for incompressible multiphase flow without capillary nor gravity effects is assumed. In that case total volume balance reads

\[-\nabla \cdot (\lambda \cdot \nabla p) = q_t \quad \text{on} \quad \Omega \subset \mathbb{R}^n \tag{55}\]

with \(\nabla p \cdot n = \zeta_1\) and \(p = \zeta_2\) at the boundaries \(\partial \Omega_1\) and \(\partial \Omega_2\), respectively. Note that \(\partial \Omega = \partial \Omega_1 \cup \partial \Omega_2\) is the whole boundary of the domain \(\Omega\) and
\[ \partial \Omega_1 \cap \partial \Omega_2 = \emptyset. \] Moreover, \( \mathbf{n} \) is the unit normal vector at \( \partial \Omega \) pointing outwards, \( p \) is the pressure, \( \lambda_t \) the total mobility, \( q \) the total volumetric source term, and \( \zeta_1 \) and \( \zeta_2 \) are specified quantities at the boundary. In a system with \( N_{ph} \) phases note that \( \lambda_t = \sum_{\alpha=1}^{N_{ph}} \lambda_\alpha \), where \( \lambda_\alpha = K k_{r\alpha}/\mu_\alpha \) is the phase mobility with the positive definite permeability tensor \( K \), the saturation dependent relative permeability \( k_{r\alpha} \), and the phase viscosity \( \mu_\alpha \).

In fractured porous media the length scales and permeability fields entail high contrasts. Therefore, in the hierarchical fracture modeling approach small scale fractures are homogenized and represented as part of the continuum described by the effective matrix coefficients \( K^m \in \mathbb{R}^n \). For large fractures a discrete lower dimensional representation with effective conductivities \( K^f \in \mathbb{R}^{n-1} \) is employed. Note that the effective conductivity is related to the fracture aperture \( a \) as \( K^f = a^2/12 \mathbf{I} \), with \( \mathbf{I} \) being the identity matrix. Equation (55) is then separated for the matrix and long fractures, i.e.

\[ -\nabla \cdot (\lambda_t \cdot \nabla p)^m + \Psi^m = q^m \text{ on } \Omega^m \subset \mathbb{R}^n \] (56)

and

\[ -\nabla \cdot (a\lambda_t \cdot \nabla p)^f + a\Psi^f = aq^f \text{ on } \Omega^f \subset \mathbb{R}^{n-1}, \] (57)

where \( \Psi^m \) and \( \Psi^f \) are the flux interactions between large scale fractures and matrix, and superscripts \( m \) and \( f \) denote matrix and fracture quantities, respectively. Note that

\[ \int_{\Omega} \Psi^m \delta(d)dx = \int_{\Omega} a\Psi^f \delta(d)dx \] (58)

holds, where \( \delta(d) \) is the Dirac delta.

It is worth re-emphasizing that the fracture equation is solved on the lower dimensional fracture manifolds, while the homogenized small scale fractures are treated by the matrix equation [64].

The flux interaction is modeled similar as in classical well models [78], e.g. in a volume \( V \) intersecting with a fracture interface \( A \) one obtains the volume averaged source

\[ \Psi_V^m = CI \lambda_t^m (p^m - p^f) / V = \eta^m (p^m - p^f) \] (59)

in the discretized matrix pressure equation and the areal averaged source

\[ a\Psi_A^m = CI \lambda_t^m (p^f - p^m) / A = \eta^f (p^f - p^m) \] (60)

in the discretized fracture pressure equation. Analogous to well productivity indices [78], the connectivity index \( CI \) is a measure of the total flux between
7.2 Hierarchical Fracture Modeling Approach

a fracture element of area $A$ and a matrix element of volume $V$; normalized by the pressure difference and the matrix parameter $\lambda^m$. By definition total flux balance between fractures and matrix is guarantied, i.e.

$$\int_V \Psi^m_V dV = - \int_A a \Psi^m_A dA. \quad (61)$$

Note that the connectivity indices $CI$ ($[CI] = L^1$) are grid dependent quantities. To illustrate how they can be computed, a 2D problem with one fracture network consisting of three connected fracture lines is considered (see Fig. 56). The matrix is discretized by a Cartesian grid and the fractures are independently divided into connected fracture segments. Obviously, $\Psi^m_V$ is non-zero only in those matrix cells, which overlap with at least one fracture segment, and the value of $CI_{i-j}$ can be calculated based on the area fraction $A_{i-j}$ of the fracture segment $i$ inside the matrix cell $j$, i.e.

$$CI_{i-j} = \frac{A_{i-j}}{\langle d \rangle}. \quad (62)$$

The average distance between matrix cell $j$ and fracture element $i$ is denoted by $\langle d \rangle$, which can be calculated as

$$\langle d \rangle = \frac{\int_V x_n(x')d\mathbf{x}'}{V}, \quad (63)$$

where $x_n$ is the distance from the fracture [64]. For highly connected fractures and matrix cells, the effect of $\langle d \rangle$ is small, as long as it is larger than a minimal value. In general, $\langle d \rangle$ has to be computed numerically, in many cases, however, analytical expressions exist; see some examples in figure 57.

Once the pressure equations are solved, the total velocity $\mathbf{u}_t = -\lambda_t \cdot \nabla p$ and the phase velocities $\mathbf{u}_\alpha = f_\alpha \mathbf{u}_t$ are computed, where $f_\alpha = \lambda_\alpha / \lambda_t$ is a fractional flow function.

The transport equations are solved either explicitly or implicitly. In this work, an explicit scheme is employed leading to a sequential implicit-pressure-explicit-saturation (IMPES) overall solution strategy, whereas the fractional flow functions are interpolated to the cell interfaces using a second-order TVD scheme [62].

7.2.2 Discretized linear system

A problem with $N_{fn}$ disconnected fracture networks is considered. Each fracture network may be governed by one or several connected fracture lines (planes in 3D). Employing a computational grid, the matrix and the $i$-th
Figure 56: 2D matrix with a 1D fracture network consisting of 3 connected fracture lines. The matrix control volumes overlapping with fractures are highlighted in grey.

\[ \langle d \rangle = dx/4 \]

\[ \langle d \rangle = (dx_2^2 + dx_1^2)/(2dx) \]

\[ \langle d \rangle = dx/3\sqrt{2} \]

\[ \langle d \rangle = dx dy/(3\sqrt{dx^2 + dy^2}) \]

Figure 57: Analytical expressions of \( \langle d \rangle \) for some selected 2D scenarios.
7.3 Consistent matrix-fracture coupling based on scale separation

Fracture network consists of \( N_m \) and \( N_{f,i} \) grid cells, respectively. Note that the computational grids for the matrix and each fracture network are all totally independent. If a classical finite-volume scheme is used to solve the coupled Eqs. (56) and (57) discretized on the computational grid, a system of equations for fracture and matrix pressure vector \( P \), i.e.

\[
BP = Q, \tag{64}
\]

is obtained. Here, \( B \) and \( Q \) are the system matrix and the right-hand-side (RHS) vector, respectively. Moreover, system (64) is ordered in a way that first the matrix equations and then fracture network equations appear. More precisely, for \( N_{f_n} \) fracture networks, system (64) reads

\[
\begin{pmatrix}
B_{MM} & B_{MF}^1 & \cdots & B_{MF}^{N_{f_n}} \\
B_{FM}^1 & B_{FF}^1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
B_{FM}^{N_{f_n}} & 0 & \cdots & B_{FF}^{N_{f_n}}
\end{pmatrix}
\begin{pmatrix}
p_m^n \\
p_{f,1}^1 \\
\vdots \\
p_{f,N_{f_n}}^{N_{f_n}}
\end{pmatrix}
= \begin{pmatrix}
Q_m^n \\
Q_{f,1}^1 \\
\vdots \\
Q_{f,N_{f_n}}^{N_{f_n}}
\end{pmatrix}, \quad (65)
\]

where \( B_{MM} = A_{MM} + D_{MF} \) and \( B_{FF}^i = A_{FF}^i + D_{FM}^i \). The matrices \( A_{MM} \) and \( A_{FF}^i \) account for matrix-matrix and fracture-fracture transmissibilities, respectively, while \( D_{MF}, D_{FM}^i \) and \( B_{MF}^i = B_{FM}^i \) account for matrix-fracture connections. The diagonal matrices \( D_{MF} \) and \( D_{FM}^i \) reflect the matrix-fracture connections in the diagonal of the system.

The size of the problem is

\[
N_{t,tight} = N_m + \sum_{i=1}^{N_{f_n}} N_{f,i}, \tag{66}
\]

which can become huge and thus lead to expensive simulations. Moreover, as explained before, the convergence rate of iterative linear solvers may be negatively affected by the high contrast between matrix-matrix and fracture-fracture connections. Therefore, it is a desirable goal to develop multi-scale methods which allow to approach the global problem with much less degrees of freedom. The next section is devoted to this goal, i.e. a consistent, but alternative coupling strategy suited for the MSFV method is devised.

7.3 Consistent matrix-fracture coupling based on scale separation

To make the hierarchical approach suitable for the MSFV method, a consistent matrix-fracture coupling based on scale separation of the fracture
pressure is devised here. For each fracture network, the pressure distribution must ensure mass balance. In other words, for incompressible systems without external source terms it has to be such that total volumetric exchange rate with the matrix is zero. This can be achieved by adjusting the reference pressure levels in each fracture network accordingly. Therefore, the pressure distribution \( p_{f,i} \) inside a network \( i \) is split into an average value \( \bar{p}_{f,i} \) plus a remainder \( \hat{p}_{f,i} \), i.e.

\[
p_{f,i} = (\bar{p}_{f,i} + \hat{p}_{1,i}, \hat{p}_{2,i}, ..., \hat{p}_{N_{fn},i})^T.
\]

Equation (65) can then be reformulated as

\[
\begin{pmatrix}
B_{MM} & c \Sigma B_{MF}^1 & \cdots & c \Sigma B_{MF}^{N_{fn}} \\
\bar{r} \Sigma B_{FM}^1 & \bar{r} \Sigma B_{FF}^1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\bar{r} \Sigma B_{FM}^{N_{fn}} & 0 & \cdots & \bar{r} \Sigma B_{FF}^{N_{fn}}
\end{pmatrix}
\begin{pmatrix}
p_m \\
\bar{p}_{f,1} \\
\vdots \\
\bar{p}_{f,N_{fn}}
\end{pmatrix}
= Q,
\]

where

\[
Q = \begin{pmatrix}
Q_m - \frac{1}{r} \sum_{i=1}^{N_{fn}} (B_{MF}^i \hat{p}_{f,i}) \\
\bar{r} \Sigma Q_{f,1} \\
\vdots \\
\bar{r} \Sigma Q_{f,N_{fn}}
\end{pmatrix}
\]

and the operators \( ^r \Sigma \), \( ^r \Sigma \) and \( ^r \Sigma \) indicate summation over columns, rows and both, respectively. Note that this is a system for matrix and mean fracture network pressures only. To obtain the fracture pressure remainders, one also has to solve the systems

\[
B_{FF}^i \hat{p}_{f,i} = Q_{f,i} - \left( B_{FM}^i c \Sigma B_{FF}^i \right) \begin{pmatrix}
p_m \\
\bar{p}_{f,1} \\
\vdots \\
\bar{p}_{f,N_{fn}}
\end{pmatrix} \forall i \in \{1, \cdots, N_{fn}\}
\]

for each fracture network independently. The systems (68) and (70) are then solved sequentially until convergence. Note that the size of system (68) is

\[
N_{m,loose} = N_m + N_{fn}
\]

which generally is much smaller than that of system (65), since it only contains one degree of freedom (DOF) per fracture network. This additional DOF is essential, since it strictly ensures total mass balance for each fracture network at any iteration level, which is an important property to achieve
good convergence rates of the iterative solver. Note that this splitting approach can be refined by allowing for spatial variations of \( \bar{p}^f_{i,i} \) within network \( i \), which automatically results in additional DOFs; this, however, is subject of future research. In general it can be stated that the smaller the \( \bar{p}^f_{i,i} \) contributions the better the expected convergence rate of the sequential procedure. As a direct consequence, as shown later in this paper, the approximation by the mean fracture network pressure is better in cases with high fracture connectivities (small pressure gradients). But we emphasize that this property is not our main motivation for this approach; as already mentioned it was our goal to extend our MSFV framework for hierarchical fracture modeling.

In the next section we devise an iterative MSFV (i-MSFV) method for Eq. (68), which generally is the most expensive part of the proposed sequential solution strategy.

7.4 I-MSFV method for hierarchical fracture modeling

As explained before, numerical simulation of flow and transport in natural porous media is a challenging problem due to the heterogeneous and anisotropic properties of the formations at different scales. Fractures add even more physical and computational complexities to the problem and therefore we devise an i-MSFV method, which can account for such scenarios. As shown earlier for non-fractured problems, the i-MSFV method is conservative at any iteration level [34], allows to reduce errors and thus can be used for efficient simulations of multiphase flow with rigorous local control of the error [38, 42].

In this section, we first explain the new i-MSFV framework for fractured reservoirs and then talk more specifically about the use of an iterative linear solver to reduce high frequency errors and thus achieve convergence.

7.4.1 I-MSFV framework for fractured reservoirs

The i-MSFV method is based on the approximation

\[
p_{m,\nu+1} \approx \Phi_{m,\nu+1} = \sum_{h=1}^{N_d} \left( \sum_{k=1}^{N_c} \Phi_{k,\nu+1}^{h,k} + \sum_{\beta=1}^{N_f} \bar{p}^{f,\beta,\nu+1} \Phi_{f,\beta}^{h} + \Phi_{h,\nu}^{h,\nu} \right) \tag{72}
\]
as a solution of system (68), which can be written as
\[
-\nabla \cdot (K \lambda_t \cdot \nabla p)^{m,\nu+1} + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta}(p^m - \bar{p}^{f,\beta})^{\nu+1} = q^m_t + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta} \hat{p}^{f,\beta,\nu} \tag{73}
\]
and
\[
\int_{A_{\beta}} \eta^{f,\beta}(p^m,\nu+1 - \bar{p}^{f,\beta,\nu+1} - \hat{p}^{f,\beta,\nu}) \, dx = \int_{A_{\beta}} q^{f,\beta}_t \, dx. \tag{74}
\]
Here basis functions $\Phi^h_k$, fracture functions $\Phi^h_{f,\beta}$, and correction functions $\Phi^h_{\nu}$, which are computed locally on dual coarse grid cells $\tilde{\Omega}^h$, are introduced. Moreover, $\bar{p}^{\nu+1}_k$ is the pressure at the new iteration level $\nu + 1$ in the center of the $k$-th coarse grid cell $\tilde{\Omega}^k$.

Mathematically, in this framework the effect of fracture networks on the matrix flow equations is very similar to that of rate constraint wells. Similar to the recently developed MSFV method for complex wells [51], here we integrate fractures into the MSFV framework by introducing local fracture functions. Therefore, the basis, fracture and correction functions belonging to $\tilde{\Omega}^h$ are numerically computed by solving the local problems
\[
-\nabla \cdot (K \lambda_t \cdot \nabla \Phi^h_k)^m + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta} \Phi^h_k = 0, \tag{75}
\]
\[
-\nabla \cdot (K \lambda_t \cdot \nabla \Phi^h_{f,\gamma})^m + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta}(\Phi^h_{f,\gamma} - \delta_{\beta\gamma}) = 0, \tag{76}
\]
and
\[
-\nabla \cdot (K \lambda_t \cdot \nabla \Phi^{h,\nu})^m + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta} \Phi^{h,\nu} = q^m_t + \sum_{\beta=1}^{N_{fn}} \eta^{m,\beta} \hat{p}^{f,\beta,\nu}, \tag{77}
\]
where $\delta_{\beta\gamma}$ is the Kronecker delta. Equation (75) is solved for all overlapping coarse cells, i.e. $\forall k \in \{1, \ldots, N_c\}|\tilde{\Omega}_k \cap \tilde{\Omega}^h \neq \emptyset$, and similarly Eq. (76) for all overlapping fracture networks, i.e. $\forall \gamma \in \{1, \ldots, N_{fn}\}|\Omega^f_\gamma \cap \tilde{\Omega}^h \neq \emptyset$. Note that if the dual coarse cell $\tilde{\Omega}^h$ does not overlap with the primary coarse cell $\tilde{\Omega}_k$, i.e. if $\tilde{\Omega}_k \cap \tilde{\Omega}^h = \emptyset$, then $\Phi^h_k = 0$; and also if it does not overlap with the $\gamma$-th fracture network, i.e. if $\Omega^f_{\gamma} \cap \tilde{\Omega}^h = \emptyset$, then $\Phi^h_{f,\gamma} = 0$. Equations (75) and (76) are solved subject to reduced problem boundary conditions, while Eq. (77) is solved using an iteratively improved boundary condition, i.e.
\[
- (\nabla \cdot n^h)((K \lambda_t \cdot \nabla \Phi^{h,\nu})^m \cdot n^h) = - (\nabla \cdot n^h)((K \lambda_t \cdot \nabla p^{\nu}_t)^m \cdot n^h), \tag{78}
\]
where $p^{m \nu}_{s \nu}$ is the matrix pressure field at the previous iteration level $\nu$, which is smoothed (improved) using a classical global or local iterative scheme [34, 37, 38].

An illustration of four basis functions associated to a homogeneous dual cell partially perforated by a fracture line is shown in Fig. 58. Furthermore, Fig. 59 illustrates a fracture and a correction function. Also shown is the summation of all basis and fracture functions inside the dual cell, which equals one. Note that the number of fracture functions associated with a dual cell is equal to the number of disconnected fracture networks overlapping with that cell.

7.4.2 Use of an iterative linear solver to reduce high frequency errors

As can be seen from Eq. (78), the localization boundary conditions for the correction functions in the i-MSFV method can iteratively be improved using the fine-scale solution $p^{m \nu}_{s \nu}$ from the previous iteration $\nu$, which is obtained by applying $n_s$ iterations of a consistent linear solver starting with $p^{m \nu}$. More-
Figure 59: Illustration of a fracture (a) and a correction (b) function in 2D belonging to a homogeneous dual coarse cell with a partially overlapping fracture line. Also shown is the summation of all 4 basis functions (c) and of all basis and fracture functions (d). Note that the correction function is plotted for $\nu > 0$, where already an improved solution is used to enhance the localization condition.
over, the overall convergence rate of the sequential solution strategy can be improved by also updating the fine-scale fracture pressure values by the iterative solver (smoother). Note that generally the smoothed (not converged) solution obtained by such an iterative fine-scale solver is not conservative, which would have a negative impact on the convergence rate. Therefore, after each iteration an additional step is required to ensure correct total flux balance for each fracture network.

Here, we chose a line-relaxation (LR) method to smooth the matrix pressure field; similar as presented in [34]. The smoothing procedure can be summarized as follows:

1. One LR iteration is applied on the matrix pressure $p^m$, whereas $\bar{p}^f$ is treated explicitly on the RHS.

2. The reference pressures $\bar{p}^f$ in each fracture network is adjusted such that total mass balance is honored.

3. The fluctuating pressure values $\hat{p}^f$ are improved either with a direct solver (for small fracture network systems) or iteratively, e.g. by a few LR iterations.

4. Steps 1-3 are repeated $n_s$ times.

Note that LR is only one possible option to reduce high frequency errors in the MSFV solution and indeed other methods can be applied to achieve convergence [70, 90], which is beyond the scope of this paper.

7.5 Numerical results

In this section, first the hierarchical fracture modeling approach is validated, i.e. comparison is made with a simulation in which the fractures are resolved by a very fine grid. Second, consistency between the new i-MSFV method and a fine-scale hierarchical fractured reservoir simulator is demonstrated. Third, convergence rates of the new i-MSFV method and a fully coupled fine-scale solver using an algebraic multi-grid (AMG) method are compared. Finally, the performance of the i-MSFV method is investigated for a wide range of heterogeneous fractured problems. For all test cases isotropic permeabilities, i.e. $K^m = k^m I$ and $K^f = k^f I$, are used.

7.5.1 Validation of the hierarchical approach

Although it is not the main scope of this paper, in this subsection the quality of the hierarchical fracture modeling approach is demonstrated for
a single phase flow test case. For this reason, a homogeneous 2D problem is considered (see Fig. 60), where a "+\\"-shaped fracture network with $k^* = k_f/k_m = 10^5$ located at the center of the quadratic $L \times L$ domain. No-flow boundary conditions are applied at the top and bottom, while the pressure values are set to $p = 1$ and $p = 0$ at the left and right boundaries. According to the length scales specified in Fig. 60 (i.e. with aperture $a = L/225$), a fully resolved fracture simulation can be achieved by employing $225 \times 225$ grid cells. For the simulation with the hierarchical approach $27 \times 27$ matrix grid and 32 fracture segments are used. Note that the effective conductivity ratio $(k_f \times a)/(k_m \times L)$ is $10^5/225$.

Figure 61 illustrates the solutions from the fully resolved simulation and the hierarchical approach. Note that the role of the vertical fracture is less important than that of the horizontal line one. Compared to the fully resolved fine-scale solution, the total flow rate in the hierarchical fracture modeling framework is under-predicted by only 1.6%.

7.5.2 Verification of the i-MSFV method for fractured reservoirs

The second test case is to verify the consistency of our proposed i-MSFV framework for fractured reservoirs with a fully coupled hierarchical fine scale model. This time, an "X"-shaped fracture network at the center of the same 2D domain as in the first test case with the same boundary conditions and $k^* = 10^3$ is considered. For the matrix, a $99 \times 99$ equidistant Cartesian grid is employed at the fine-scale (misaligned with the fractures). For the i-MSFV method also a $9 \times 9$ equidistant coarse grid is used and for both simulations the fractures consist of 141 segments. Note that the i-MSFV
Figure 61: Pressure maps corresponding to the problem set up of Fig. 60 obtained by a fully resolved simulation with a $225 \times 225$ grid (a) and the hierarchical fracture modeling approach using a $27 \times 27$ matrix grid and 32 fracture segments (b), respectively. Note that results are matching well and the total flow rate discrepancy is only 1.6%.

7.5.3 Convergence studies

The reference results shown in Fig. 62a are obtained by applying an AMG solver of Sandia National Lab. [31]. The convergence histories of the i-MSFV method and the fine-scale simulation with AMG corresponding to the results of Figs. 62a and 62b are presented in Fig. 63a. Note that the sequential strategy used by the i-MSFV method results in much smaller global systems. Convergence histories for $k^* = 10$ are shown in Fig. 63b, which clearly demonstrates that the convergence rate of the sequential strategy decreases as the conductivity of the fractures is reduced, which is confirmed by the convergence histories shown in Fig. 64. In fact, the lower the fracture conductivity, the higher the pressure gradient. Hence, the remainder $\hat{p}_f$ of the fracture pressure, which is updated sequentially, becomes more important and more outer iterations are required to achieve convergence.

Note that the devised i-MSFV method can be interpreted as a domain decomposition method [82, 76] with the important property that conservative velocity fields can be obtained after any iteration. Latter can be of great importance for practical applications, where one may be only interested in
improved MSFV results rather than in fully converged solutions. The reason why this i-MSFV method is so efficient for highly conductive fractures is also due to the fact that the representation of individual fracture networks by single average (coarse-scale) pressure values corresponds to extreme coarsening, or in AMG terminology to agglomeration of highly connected DOFs [85]. However, it is not our main intention here to compare convergence rates of AMG, i-MSFV and other domain decomposition methods.

Finally, a test case with the same setup as for the previous one, but a heterogeneous permeability field (see Fig. 65a) is considered. The solution for $k^* = k^f / \bar{k}^m = 10^3$, where $\bar{k}^m$ is the average matrix permeability, is shown in Fig. 65b, and convergence histories (for i-MSFV simulations) with different $k^*$ are depicted in Fig. 66.

### 7.5.4 Multiphase flow problems

Here results of two phase flow in homogeneous and heterogeneous porous media obtained with the devised i-MSFV method are presented. First, a homogeneous domain with one fracture line is considered. A fracture element is $3 \times 10^3$ more conductive than the porous media per unit area. The fracture contains 57 segments and matrix grid $99 \times 99$ fine cells; for the i-MSFV method a $9 \times 9$ coarse matrix grid is employed. No-flow conditions are applied at all boundaries, an injector well with a constant rate of 1 is located in fine
7.5 Numerical results

Figure 63: Convergence histories of the fine-scale simulation using AMG and i-MSFV for a homogeneous test case with $k^* = 10^3$ (a) and $k^* = 10$ (b). GS denotes Gauss-Seidel iterations employed by AMG as a smoother.

Figure 64: Convergence histories of the i-MSFV method for a homogeneous test case with different $k^*$. 
Figure 65: Heterogeneous test case: natural-logarithm of permeability with an "X"-shaped fracture at the center (a) and the pressure solution for $k^* = 10^3$ (b).

Figure 66: Convergence histories of the i-MSFV method for a heterogeneous test case with different $k^*$. 
Figure 67: Simulations of two phase flow in a homogeneous matrix with one fracture: pressure (a) and saturation (b) maps after 0.04 PVI obtained with the i-MSFV method using a $9 \times 9$ coarse matrix grid.

cell (1, 1) and production occurs out of fine cell (99, 99), where the pressure value is kept constant at 0. The injected phase is 10 times less viscous than the displaced one. Quadratic relative permeabilities, i.e. $k_r = S^2/\mu$, are used for flow in the matrix; in the fracture on the other hand, $k_r = S/\mu$ is employed. Figure 67 shows pressure and saturation maps obtained with the i-MSFV method after 0.04 PVI (pore volume injection).

Finally, results of a similar test case with a heterogeneous matrix (see Fig. 65a) are shown in Fig. 68.

Note that for both of the previous test cases fine-scale and multiscale results are virtually identical, since here i-MSFV iterations were applied until the maximum fracture pressure change becomes extremely small ($10^{-4}$ in the homogeneous and $10^{-7}$ in the heterogeneous case).

8 I-MSFV Method for Faulted Porous Media

8.1 Introduction

So far, the MSFV and i-MSFV methods have been implemented for problems with regular geometries and conforming coarse grids. In practical applications, however, reservoirs entail complex geometries. Of particular interest is the extension of the method for faulted reservoirs with non-conforming coarse grids. In this work, the MSFV and i-MSFV methods are extended to account for faulted reservoirs [35]. The concept of locally computed basis
and correction functions is generalized and adjusted for the fault region by introducing extended local domains, which overlap with all coarse control volumes adjacent to the fault region. Numerical results ranging from homogeneous isotropic to heterogeneous anisotropic problems are presented and it is shown that the new i-MSFV method is very flexible and can deal with faults efficiently.

The description of the new MSFV method for faulted reservoirs are presented in the next subsection. Then numerical results for some selective test cases are presented.

8.2 The i-MSFV Method for Non-Conforming Coarse Grids

As mentioned before, so far the MSFV method has been used for elliptic and parabolic problems on regular conforming coarse grids. The method, however, is not suited for faulted reservoirs and therefore it was the motivation for the present work to generalize the concept accordingly and also allow for non-matching coarse grids. In the following we avoid any discussion about fine-scale discretization across the fault and rather discuss how it is possible to account for non-conforming coarse grids in the i-MSFV method. Figure 69 depicts an example of the kind of faulted reservoirs which are considered in this study. Obviously, in order to apply the MSFV method, it is required to allow for irregular coarse cells near the fault region. Here, we devise a MSFV
Figure 69: Illustration of a faulted reservoir with 2 faults, where extended dual cells are highlighted in red. The coarse cells with red nodes are adjacent to the faults, while the green nodes belong to the other coarse cells. Note that there exist eight and seven basis functions associated to the left and right extended dual coarse cells, respectively, and only one correction function for each dual coarse cell including the extended ones.

domains are obtained in the same way as in regular dual coarse cells, but there may exist more than four (eight in 3D) nodes per dual coarse cell and thus accordingly many basis functions. To compute the basis function $\Phi^k_h$, for example, the value at node $k$ is set to one and at all other nodes to zero; at the boundary between nodes reduced problem boundary condition is applied. For the correction function $\Phi^h$, at all nodes the values are set to zero. Figure 70 shows basis and correction functions associated with a dual coarse cell covering a fault region; shown are a basis function for a homogeneous, one for a heterogeneous, and a correction function for a homogeneous permeability distribution. As clearly shown in Fig. 70, the correction functions take care of the source term and improved localization condition. Once basis and correction functions are computed, the rest of the i-MSFV procedure is identical as explained above.

8.3 Numerical Results

As a first test case, a homogeneous problem is considered. The domain contains $220 \times 55$ fine cells and is divided into $20 \times 5$ primary coarse cells for the i-MSFV. Two source terms cover the fine cells $(10, 10)$ and $(210, 45)$, no-flow conditions are applied at the top and bottom boundaries, and the
Figure 70: Illustration of basis functions belonging to an extended homogeneous (top) and heterogeneous (middle) dual cell overlapping with a fault region. Also shown is a correction function after some iterations $\nu > 0$ for a homogeneous extended dual cell with two source terms (bottom). Coarse nodes are also shown (filled dark-red circles).
8.3 Numerical Results

dimensionless pressure is set to one and zero at the left and right boundaries, respectively. A fault, which is shifted by 15 fine cells, is located at \( x = 110 \).

Figure 71 depicts the problem geometry along with the fine-scale and MSFV (without iterating) solutions. Convergence histories of the i-MSFV method

Figure 71: Fine-scale reference (left) and original non-iterative MSFV (right) pressure fields. The fine-scale solution is obtained using \( 220 \times 55 \) grid cells, while the MSFV result is obtained using \( 20 \times 5 \) coarse cells. Also shown on the base plane is the homogeneous permeability field and the geometry of the problem.

for different line-relaxation smoothing steps \( n_s \) (per multiscale iteration) are shown in Fig. 72 for an isotropic and an anisotropic permeability. Note that the anisotropic problem is equivalent to the homogeneous one with a fine-cell aspect ratio \( \alpha = \Delta x / \Delta y = 10 \) opposed to \( \alpha = 1 \).

The second test case is a heterogeneous problem with same geometry, same fine grid, same coarse grid and same boundary conditions as in the first test case, but without sources. The permeability field is taken from the top layer of the SPE 10 comparative test case presented by [19]. Fine-scale reference and MSFV (without iterations) solutions are shown in Fig. 73. Also shown on the base surfaces is the permeability field. Moreover, Fig. 74 shows the convergence history of the i-MSFV method for different line-relaxation smoothing steps \( n_s \) (per multiscale iteration).
Figure 72: Convergence histories of the i-MSFV method for the homogeneous isotropic (left) and anisotropic (right) test case corresponding to Fig. 71. Grid aspect ratio, i.e. \( \alpha = \Delta x / \Delta y \), for the isotropic and anisotropic case is 1 and 10, respectively. Results are shown for different line-relaxation smoothing steps \( n_s \) (each iteration).

Figure 73: Fine-scale reference (left) and original non-iterative MSFV (right) pressure fields for the heterogeneous test case. The fine-scale solution is obtained using \( 220 \times 55 \) grid cells, while the MSFV result is obtained using \( 20 \times 5 \) coarse cells. Also shown on the base plane is the SPE 10 top layer permeability field ([19]) used for this test case and the geometry of the problem.
8.3 Numerical Results

Figure 74: Convergence histories of the i-MSFV method for the heterogeneous isotropic (left) and anisotropic (right) test cases corresponding to Fig. 73. For the isotropic heterogeneous case ($\alpha = 1$) results are shown for different line-relaxation smoothing steps $n_s$ (each iteration). For anisotropic heterogeneous problems, results are presented for different grid aspect ratios $\alpha = \Delta x / \Delta y$ and $n_s = 10$. 
Part VI
Applicability of the i-MSFV Method

9 Adaptive i-MSFV Method

9.1 Introduction

In this section, we introduce an adaptive iterative MSFV method (ai-MSFV) as an efficient way to improve MSFV results by applying i-MSFV iterations adaptively in space and time. This is motivated by the fact that MSFV results are usually acceptable except in some small local sub-regions, where the reduced problem boundary condition does not adequately honor the fine-scale pressure variations. Such areas are typically found near wells, long coherent structures with high permeability contrast and nearly impermeable shale layers.

With the ai-MSFV method we show that the error introduced in the MSFV method can be reduced by applying i-MSFV iterations only in small sub-regions, where the residual corresponding to the global fine-scale system is too high. For multiphase flow problems, the interpolation functions are locally (and infrequently) updated during the simulation based on two criteria: one criterion is based on the total mobility change due to changing coefficients [48, 83, 50] and the second criterion is based on the residual in fulfillment of the fine-scale system to improve the localization boundary conditions for the correction function calculations. Moreover, due to local improvement of the results, the ai-MSFV method is as good as the i-MSFV method for parallel processing, expect that load balancing becomes more challenging. Latter is due to the fact that the sub-domain, where the correction functions have to be updated by the ai-MSFV method changes dynamically. Regarding the residual as an error indicator turns out to be reasonable and efficient. Reasonable, since the residual is a direct measure of the local divergence error and efficient since the fine-scale transmissibilities are available and the residual is obtained by local matrix-vector multiplication.

The chapter is organized as follows. In section 9.2, the ai-MSFV method is introduced and strategies to efficiently improve the results are explained. Numerical studies are presented in section 9.3, which is divided into two subsections. In the first subsection performance tests for single-phase flow scenarios are presented and in the second subsection the approach is illus-
trated for multiphase flow problems.

9.2 The ai-MSFV algorithm

Initially, the fine-scale solution $p'$ is estimated by applying the MSFV method without iterating. Then, based on any desired error criterion, which will be discussed in the next subsection, critical fine cells are detected and all dual volumes overlapping with coarse volumes containing at least one critical fine cell are declared as critical. Finally, $n_s$ smoothing steps are applied to $p'$ within the small critical sub-domain $S$ consisting after critical dual volumes and this smoothed field is then used to improve the correction functions. An illustration of the strategy is shown in Fig. 75.

![Diagram showing critical fine cells, boundary cells, critical coarse cells, and critical sub-domain S](image)

Figure 75: Illustration of the strategy for finding critical sub-domain $S$. Two critical fine-cells are found based on the residual criterion (grey filled cells). Then, the critical coarse cell (solid bold line) is found, and finally its adjacent dual cells construct the critical sub-domain $S$ (dashed bold line), where the smoother is applied, and the boundary conditions are improved (hashed cells).

The improved correction functions result in a change of the coarse system (right-hand side), which is solved again leading to an improved MSFV solution. If the residual now is small enough, a conservative fine-scale solution is constructed and used for saturation transport; else further improvement steps have to be performed. A flowchart of the described ai-MSFV method is
shown in Fig. 76. Moreover, in the chart it is shown that after recalculating the critical correction functions a new sub-domain $S$ is determined and again the smoother is applied locally to improved the boundary conditions for the correction function calculations in the next time step.

It is clear that total additional cost of the ai-MSFV method with respect to the MSFV non-iterative method is mainly due to $\sum_{i=1}^{n_i-1} f_i \cdot M$ recalculation of correction functions, $n_i - 1$ coarse system recalculations, and $\sum_{i=1}^{n_i} f_i \cdot n_s$ times smoothing the pressure field. Here, $M$ is the number of dual coarse cells, $f_i$ is the area fraction of the critical sub-domain and $n_i$ is the number of iterations. For the studies presented in this chapter, mostly at $n_i = 3$ the $f_3$ becomes zero. Therefore, here $n_i$ is set to 2, i.e. the correction functions in the critical sub-domain are recalculated only once. It is worth mentioning that in general, there is no need for setting a $n_i$ value; the iterations terminate once the residual is bellow the specific threshold value and $f_i = 0$.

Figure 76: Flowchart overview of the ai-MSFV method for single-phase flow problems.

In the following subsection it is explained how to identify critical fine cells.
as well as strategies for multiphase flow simulations.

9.2.1 Identification of critical fine cells

The fine-scale system

\[ M \cdot p_f = F \]  \hspace{1cm} (79)

is considered, which originates from a conservative finite-volume discretization of Eq. (55). By substituting the MSFV solution \( p' \) for \( p_f \) in Eq. (79), one obtains the residual

\[ R = F - M \cdot p' \]  \hspace{1cm} (80)

and can identify all critical fine cells \( i \) as

\[ \frac{|r_i|}{||R||} > \epsilon_a \] \hspace{1cm} (81)

where \( r_i \) is the corresponding component of the vector \( R \), \( \epsilon_a \) is a critical value and \( ||R|| \) is some norm of the residual vector (in this paper, the infinity norm is considered). Eq. (81) is checked during the simulation and the localization condition is improved inside the adaptive critical sub-domain \( S \) only. It is also shown in the next section that with this criterion the MSFV method leads to very accurate results.

Besides adaptively improving the localization conditions, basis and correction functions also have to be updated (locally and periodically) due to changing coefficients \( \lambda_t \) and right-hand-sides \( q \) [48]. Therefore, during simulations two criteria are decisive for adaptively updating the interpolation functions at the end of a time step:

- All basis and correction functions in dual volumes containing at least one fine cell violating the criteria

\[ \frac{1}{1 + \epsilon_b} \frac{\lambda_t^n}{\lambda_t^*} < 1 + \epsilon_b \] \hspace{1cm} (82)

or

\[ \frac{1}{1 + \epsilon_b} \frac{q^n}{q^*} < 1 + \epsilon_b, \] \hspace{1cm} (83)

where \( \epsilon_b \) is a threshold value, are updated.

- All correction functions in the critical sub-domain \( S \) are updated with improved boundary conditions.
The superscripts $n$ and $\ast$ denote the current state and the values employed for the last basis and correction function update. Note that only the correction functions have to be recomputed with the improved localization condition, which is illustrated in Fig. 76. It is worth mentioning that if the mobility tensor $\lambda_i$ inside a dual coarse cell is a constant which changes (for the whole cells) after a time step, there is no need to update the basis functions. However, correction functions for those dual cells with non-zero RHS or non-zero improved boundary conditions should still be recalculated.

9.3 Numerical Results

To study the performance of the ai-MSFV method, first incompressible single phase flow and then also multiphase flow are considered. It is shown that with the ai-MSFV method one can locally reduce the error introduced by the MSFV localization assumption.

9.3.1 The ai-MSFV method for single-phase flow

For the first test case, a homogeneous isotropic domain with some nearly impermeable shale layers is considered. The grid contains $220 \times 55$ fine cells and $20 \times 5$ coarse cells. No-flow conditions are employed at the whole domain boundary. Figure 77 shows the domain with the coarse grid and the shale layers. The permeability in the domain is $10^8$ times larger than that of the shale layers. An injector with constant injection rate of $1/(\Delta x \Delta y)$ is located at the fine cell (1,1) and a producer with constant pressure $p = 0$ is located at the fine cell (220,55).

Figure 77: Homogeneous isotropic problem with shale layers (black rectangles) and the $20 \times 5$ grid. The fine grid contains $220 \times 55$ cells and the permeability of the shale layers is $10^8$ times smaller than that of the remaining domain. Also shown on the right are the indexes of the fine cells located at the corners of the shale layers.

The fine-scale reference pressure (normalized by the injector pressure) is shown in Fig. 78a and the residual $R$ of the MSFV solution is plotted in Fig.
Since the superposition of basis and correction functions is a numerical solution of the governing equation inside the dual coarse cells, residual are only non-zero at the dual cell interfaces $N$, where the localization assumption is applied. The infinity norm and the root-mean-square (RMS) average, i.e.

$$||R||_{\text{rms}} = \sqrt{\frac{\sum (r_i - \bar{r})^2}{N}}, \quad (84)$$

of the MSFV residual vector are 0.1053 and 0.0126, respectively. The average residual $\bar{r}$ in Eq. (84) is always zero in the MSFV framework. Critical sub-domain $S$, which is shown in Fig. 78c, is obtained based on the threshold parameter $\epsilon_a = 0.5$ and it is 11.2% of the whole domain. One ai-MSFV iteration ($n_i = 2$) is now applied on $S$ only (i.e. one re-calculation of the correction functions), which involves $n_s = 10$ smoothing steps. The residual map of the ai-MSFV result is plotted in Fig. 78d and one can observe significant local improvements compared to Fig. 78b. The infinity norm and RMS of the ai-MSFV residual vector are 0.0299 and 0.0074, respectively.

![Figure 78](image.png)

**Figure 78:** Homogeneous isotropic problem with shale layers: (a) normalized fine-scale reference pressure field, (b): residual map of the MSFV result, (c): the critical sub-domain $S$ for $\epsilon_a = 0.5$, (d): residual map of the ai-MSFV method with $n_i = 2$ and $n_s = 10$.

Figure 79a shows the RMS norm of the ai-MSFV residual vector versus the area fraction of the critical sub-domain $S$, i.e. $f_1$, for $n_i = 2$ and different number of smoothing steps $n_s$. Different areas of $S$ are obtained by employing different values for $\epsilon_a$. The MSFV result corresponds to the area fraction equal to 0 and the i-MSFV for two iterations, i.e. $n_i = 2$, corresponds to that of 1. Moreover, Fig. 79b depicts the RMS of the residual vector versus the total area fraction of the critical sub-domain, i.e. $\sum f_i$. The iterations are employed until no critical sub-domain is detected.
9.3 Numerical Results

Figure 79: Homogeneous isotropic problem with shale layers: (a) RMS of residual vector versus area fraction of the critical sub-domain $S$, i.e. $f_1$ for $n_i = 2$ and different $n_s$. Values of the corresponding threshold parameter $\epsilon_a$ are also shown next to the symbols. (b) RMS of residual vector versus total area fraction of the critical sub-domain, i.e. $\sum f_i$ for $n_s = 10$. The ai-MSFV iterations are employed till no critical cells exist in the domain. Also shown in colorful bars are the critical area fractions at each iteration level $f_i$. The iteration level at which no critical cell is detected with the corresponding $\epsilon_a$ are also shown in brackets $[n_i, \epsilon_a]$. 
As another test case, a 2D heterogeneous isotropic domain is considered. The permeability field is extracted from the top layer of the SPE comparative test case 10 \cite{19} (Fig. 5). The fine and coarse grids and the boundary conditions are the same as for the previously studied test case.

The fine scale reference solution normalized by the pressure at the injection cell (1,1) is shown in Fig. 80a. The infinity and RMS norm of the MSFV residual (Fig. 80b) are 0.1762 and 0.0217, respectively. Based on $\epsilon_a = 0.5$, 29\% of the domain is selected as the critical sub-domain $S$, which is shown in Fig. 80c. Applying the ai-MSFV method with $n_i = 2$ and $n_s = 10$ results in a residual with infinity and RMS norms of 0.0853 and 0.0132, respectively (Fig. 80d). Similar to the previous test case, results are improved where the ai-MSFV iterations are employed.

Figure 80: Heterogeneous isotropic SPE10 top layer problem: (a) normalized fine-scale reference pressure field, (b): residual map of the MSFV result, (c): the critical sub-domain $S$ for $\epsilon_a = 0.5$, (d): residual map of the ai-MSFV method with $n_i = 2$ and $n_s = 10$.

Figure 81a shows the RMS norm of the residual versus the area fraction of the critical sub-domain $S$ for different number of smoothing steps $n_s$. The different area fractions are obtained by setting different values for the threshold parameter $\epsilon_a$. Again, the original MSFV (no improvement) and the i-MSFV (improvement for all dual cells once) results correspond to the area fraction of 0 and 1, respectively. Moreover, Fig. 81b depicts the RMS of the residual vector versus the total critical area fraction, i.e. $\sum f_i$. The iterations are employed until no critical sub-domain is detected.

### 9.3.2 The ai-MSFV method for multiphase flow

As illustrated in the previous sub-section, it is possible to improve MSFV results locally. In this section, the ai-MSFV method for multiphase flow
### 9.3 Numerical Results

#### Area fraction of critical sub-domain: $f_1$ and $||R||_{\text{rms}}$

| $n_s$ | $||R||_{\text{rms}}$ |
|-------|---------------------|
| 5     | 0.002               |
| 10    | 0.004               |
| 15    | 0.006               |

Values of the corresponding threshold parameter $\epsilon_a$ are also shown next to the symbols.

#### Total area fraction of critical sub-domain: $\Sigma f_i$

<table>
<thead>
<tr>
<th>$n_s$</th>
<th>$\Sigma f_i$</th>
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</thead>
<tbody>
<tr>
<td>10</td>
<td>0.002</td>
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The ai-MSFV iterations are employed till no critical cell is detected with the corresponding $\epsilon_a$. Also shown in colorful bars are the critical area fractions at each iteration level $f_i$. The iteration level at which no critical cell is detected with the corresponding $\epsilon_a$ are also shown in brackets $[n_i, \epsilon_a]$.

**Figure 81**: Heterogeneous isotropic SPE10 top layer problem: (a) RMS of residual vector versus area fraction of the critical sub-domain $S$ for different $n_s$. Values of the corresponding threshold parameter $\epsilon_a$ are also shown next to the symbols. (b): RMS of residual vector versus total area fraction of the critical sub-domain, i.e. $\sum_{i=1}^{n_s} f_i$, for $n_s = 10$. The ai-MSFV iterations are employed till no critical cells exist in the domain. Also shown in colorful bars are the critical area fractions at each iteration level $f_i$. The iteration level at which no critical cell is detected with the corresponding $\epsilon_a$ are also shown in brackets $[n_i, \epsilon_a]$. 
problems is studied. In i-MSFV method with infrequently updated boundary conditions [34], the i-MSFV iterations are applied at the beginning of the simulation to provide the exact localization condition, and then during the multiphase simulation, the localization conditions are infrequently updated for the whole domain. Here, a value for threshold parameters $\epsilon_a$ and $\epsilon_b$ are set and the basis and correction functions are updated in those local domains where the criteria (81) and (82) are not satisfied. In addition, only the correction functions inside the sub-domain $S$ are recomputed periodically. For the following test cases $n_i$ and $n_s$ are set to 2 and 10, respectively, and residual vectors are computed based on the adaptively updated transmissibility field which is used to calculate basis and correction functions. Moreover, the RMS norm of the initial MSFV residual vector is used to normalize the residual vectors during the simulation. The next test case is the same as the previously studied homogeneous shale layer problem, but this time gas is injected to displace oil. The viscosity ratio ($\mu_o/\mu_g$) is 10 and the reservoir is initially filled with oil. Figures 82a and b show the gas saturation maps after 0.25 PVI of the fine-scale reference and the original MSFV simulations ($\epsilon_b = 0$), respectively.

![Figure 82: Homogeneous isotropic problem with shale layers: fine-scale (a) and MSFV (b) saturation maps after 0.25 PVI.](image)

For better comparison, Fig. 83a shows the absolute saturation error of the MSFV result obtained by setting $\epsilon_b = 0.02$ which results in 37.5% update of basis and correction functions due to transient mobility coefficients. Note that the value of $\epsilon_b$ could be increased; e.g. for practical applications choose it between 0.1 and 0.2, which leads to virtually the same results as those without adaptivity, but dramatically reduces the updating frequency of basis and correction functions. With the same $\epsilon_b = 0.02$, Figs. 83b, c, and d depict the absolute saturation error of the ai-MSFV results obtained with $\epsilon_a = 0.08$ (area fraction is 0.28%), 0.05 (area fraction is 0.84%), and 0.02 (area fraction is 3.3%), respectively. By comparing Figs. 83a and d, one finds that little additional cost for adaptive improvement leads to much better results.

As another test case, the permeability field of the SPE10 top layer of Fig. 5 is used. The injection and production conditions are the same as in
Figure 83: Homogeneous isotropic problem with shale layers: absolute saturation (left column) and pressure (right column) error maps after 0.25 PVI: (a) MSFV result $\epsilon_b = 0.02$, (b) ai-MSFV result $\epsilon_b = 0.02$, $\epsilon_a = 0.08$, (c) ai-MSFV result $\epsilon_b = 0.02$, $\epsilon_a = 0.05$, (d) ai-MSFV result $\epsilon_b = 0.02$, $\epsilon_a = 0.02$. 
previously studied test case. The fine-scale reference solution together with the original MSFV saturation map ($\epsilon_b = 0$) after 0.18 PVI are presented in Figs. 84a and b, respectively.

![Figure 84: Heterogeneous isotropic problem of SPE10 top layer: fine-scale (left) and MSFV saturation maps after 0.18 PVI.](image)

Even though the MSFV solution seems to be in a good agreement with that obtained by fine-scale simulation, Fig. 85a shows that there exist big errors, especially near the saturation front. Figures 85b and c depict absolute errors of the ai-MSFV simulations with $\epsilon_a = \epsilon_b = 0.05$ and $\epsilon_a = \epsilon_b = 0.02$, respectively. The result of Figs. 85b and c are obtained by applying smoothing-iterations for 5.2% and 6.97% of the domain with recalculation of 3.5% and 4.97% of the correction functions, respectively. From these figures one finds that applying space-time ai-MSFV iterations leads to considerable improvements of the results with very little additional computational cost.

![Figure 85: Heterogeneous isotropic problem of SPE10 top layer: absolute saturation (left column) and pressure (right column) error maps after 0.18 PVI: (a) MSFV result $\epsilon_b = 0$, (b) ai-MSFV result $\epsilon_b = \epsilon_a = 0.05$, (c) ai-MSFV result $\epsilon_b = \epsilon_a = 0.02$.](image)
To emphasize again that the presented improvements are obtained by reducing the residual, Fig. 86 depicts the residual maps of the MSFV ($\epsilon_b = 0.02$) and the ai-MSFV ($\epsilon_a = \epsilon_b = 0.02$, corresponding to Fig. 85c) at 0.18 PVI after injection.

Figure 86: Heterogeneous isotropic problem of SPE10 top layer: residual of the results of the MSFV with $\epsilon_b = 0.02$ (a) and the ai-MSFV (b) with $\epsilon_a = \epsilon_b = 0.02$, corresponding to Fig. 85c, at 0.18 PVI after injection. Residual maps are calculated using the fine-scale system derived based on adaptively updated saturation field which is used for calculation of the interpolation functions.

Figures 87a and 87b depict the second norms of the saturation errors for different $\epsilon_a$ values (area fraction of the critical sub-domain) corresponding to the previous 2 test cases.

So far, the performance of the ai-MSFV method has been investigated for heterogeneous test cases, where the original MSFV method results are acceptable. It was shown how to reduce the error locally by improving the localization condition only where the residual is high. For anisotropic heterogeneous problems, however, the original MSFV method fails to give acceptable results [57, 67, 43, 34, 37]. As an example, the SPE10 top layer test case of Fig. 5 with the same boundary conditions, but with a grid aspect ratio $\alpha = \Delta x/\Delta y$ of 10 is considered. Figures 88a and 88b show the streamlines obtained by fine-scale and MSFV simulations for single phase flow and huge artifacts in the MSFV results can be observed almost everywhere in the domain.

For these test cases, the ai-MSFV method starts with the exact solution by applying enough i-MSFV iterations. Then, during the simulation, the localization conditions will be enhanced only for those dual cells, where the residual is not acceptable.

As a further test case, the same SPE10 top layer with a grid aspect ratio of 10 is considered. Gas with 10 times less viscosity than oil at a constant rate of $1/(\Delta x \Delta y)$ is injected from cell (1,1), while oil is produced from cell (220,55) at a constant pressure of 0. Figure 89 shows the fine-scale and MSFV results (pressure and saturation maps) after 0.18 PVI. It is clear from
Figure 87: Second norm of the saturation error for the MSFV and ai-MSFV methods corresponding to the previously studied test cases: shale layer after 0.25 PVI (a) and SPE10 top layer after 0.18 PVI (b). Results are presented for $\epsilon_b = 0.02$ and different $\epsilon_a$ which results in different area fraction of critical sub-domain.

Figure 88: Heterogeneous anisotropic problem of SPE10 top layer with a grid aspect ratio of 10: fine-scale (a) and MSFV (b) streamlines for a single phase oil flow problem.
this figure that the MSFV solutions are not acceptable due to inappropriate localization provided by the reduced problem boundary conditions.

Figure 89: Heterogeneous anisotropic problem of SPE10 top layer with grid aspect ratio of 10 after 0.18 PVI: (a) fine-scale reference saturation map, (b): fine-scale reference pressure field, (c): MSFV saturation map, (d): MSFV pressure field.

For better comparison, Fig. 90a shows the error, i.e. the difference between saturation and pressure of the MSFV and fine-scale reference solutions. In addition, results of the ai-MSFV are shown in Figs. 90b - d for \( \epsilon_a = \epsilon_b = 0.02 \) (critical area fraction is 0.36%), \( \epsilon_a = \epsilon_b = 0.01 \) (critical area fraction is 0.8%), and \( \epsilon_a = \epsilon_b = 0.005 \) (critical area fraction is 1.52%). It is clear from Fig. 90d that applying smoothing-iterations in only 1.52% of the domain and recalculating only 1.21% of the correction functions significantly improves the quality of the MSFV results. Figure 91 depicts the fine-scale and ai-MSFV saturation and pressure maps for \( \epsilon_a = \epsilon_b = 0.005 \) corresponding to the result of Fig. 90d. The second norm of the saturation errors versus percentage of area fraction of critical sub-domain are shown in Fig. 92.

As the final test case, the permeability field of the SPE10 bottom layer is considered. The first grid contains \( 220 \times 55 \) fine and \( 20 \times 5 \) coarse cells (Fig. 11) and the second one consists of \( 220 \times 60 \) fine and \( 44 \times 12 \) coarse cells. The grid aspect ratios of these test cases are 1 and 2 respectively. The viscosity ratio and the boundary conditions are the same as in the previously studied test cases. Gas is injected at the upper left fine cell at a constant pressure of 1, while oil is produced from the lower right cell at a constant pressure of 0.

The dimensionless time \( t^* = t/\tau \), where

\[
\tau = \frac{\mu \phi L^2}{k(p_i - p_f)},
\]

(85)
Figure 90: Heterogeneous anisotropic problem of SPE10 top layer: absolute saturation (left column) and pressure (right column) error maps after 0.18 PVI: (a) MSFV with $\epsilon_b = 0$, (b) ai-MSFV with $\epsilon_b = \epsilon_a = 0.02$, (c) ai-MSFV with $\epsilon_b = \epsilon_a = 0.01$, (d) ai-MSFV with $\epsilon_b = \epsilon_a = 0.005$.

Figure 91: Heterogeneous anisotropic problem of SPE10 top layer with grid aspect ratio of 10 after 0.18 PVI: (a) fine-scale reference saturation map, (b): fine-scale reference pressure field, (c): ai-MSFV saturation map, (d): ai-MSFV pressure field. The ai-MSFV results are obtained by applying smoothing-iterations for $\%1.52$ of the domain and recalculation of $\%1.21$ of the correction functions.
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Figure 92: Second norm of the saturation errors for the MSFV and ai-MSFV methods. The first 4 symbols correspond to results of Figs. 90a - d, respectively, and the last one is obtained by setting \( \epsilon_a = \epsilon_b = 0.002 \).

is introduced, where \( \bar{k} \) is the average permeability in the domain, \( \phi \) is the porosity, \( L \) a reference length, and \( (p_i - p_p) \) is the pressure difference between injection and production points.

Figure 93 shows solutions (pressure and saturation) obtained from the fine-scale and MSFV (\( \epsilon_b = 0.05 \)) simulations for the first problem after \( t^* = 1.0 \) with \( \tau \) computed based on \( L_x = 220 \), \( \mu_o = 1 \), and \( (p_i - p_p) = 1 \).

Figures 94a and 94b shows the absolute difference between the MSFV and fine-scale results corresponding to the results of Figs. 93c and 93d. In addition, Figs. 94c and 94d depict the error of the ai-MSFV results obtained with \( \epsilon_a = \epsilon_b = 0.025 \), which leads to 2.8% average area fraction of the critical sub-domain and recalculation of 1.24% of the correction functions. The number of smoothing steps used for this test case is \( n_s = 250 \), which is relatively large. This is required due to high permeability contrasts at the dual cell boundaries [34]. For these type of problems, one can employ a modified version of to the i-MSFV method [12], which converges with only a few smoothing steps. However, applying \( n_s = 250 \) line relaxation steps in only 2.8% of the domain is not expensive.

Figure 95 shows the residual of the MSFV (\( \epsilon_b = 0.05 \)) and the ai-MSFV (\( \epsilon_a = \epsilon_b = 0.025 \)) corresponding to the results of Fig. 94. The fine-scale sys-
Figure 93: Heterogeneous SPE10 bottom layer with grid aspect ratio of 1 after $t^* = 1.0$: (a) fine-scale reference saturation map, (b): fine-scale reference pressure field, (c): MSFV saturation map, (d): MSFV pressure. The MSFV results are obtained based on $\epsilon_b = 0.05$.

Figure 94: Heterogeneous SPE10 bottom layer with grid aspect ratio of 1 after $t^* = 1.0$: error map of the (a) MSFV saturation, (b): MSFV pressure, (c): ai-MSFV saturation, (d): ai-MSFV pressure. The MSFV results are obtained based on $\epsilon_b = 0.05$ and the ai-MSFV results with $\epsilon_a = \epsilon_b = 0.025$. 
9.3 Numerical Results

tem used to calculate the residual maps are based on the adaptively updated mobility field, which is used to calculate the basis and correction functions. This leads to a non-zero residual only at the boundary of the dual coarse cells.

![Figure 95: Heterogeneous isotropic problem of SPE10 top layer: residual of the results of the MSFV with $\epsilon_b = 0.05$ (a) and the ai-MSFV (b) with $\epsilon_a = \epsilon_b = 0.025$ corresponding to the results of Fig. 94.](image)

Figure 96 shows solutions (pressure and saturation) obtained from the fine-scale and the MSFV ($\epsilon_b = 0.02$) simulations for the second problem after $t^* = 0.5$ with $\tau$ computed based on $L_x = 440$, $\mu_o = 1$, and $(p_i - p_p) = 1$.

![Figure 96: Heterogeneous SPE10 bottom layer with grid aspect ratio of 2 after $t^* = 0.5$: (a) fine-scale reference saturation map, (b): fine-scale reference pressure field, (c): MSFV saturation map, (d): MSFV pressure. The MSFV results are obtained based on $\epsilon_b = 0.02$.](image)

Figures 97a and 97b show the absolute difference of the MSFV and the fine-scale results corresponding to the results of Figs. 96c and 96d. In addition, Figs. 97c and 97d depict the error of the ai-MSFV results obtained with $\epsilon_a = \epsilon_b = 0.02$, which leads to 4.6% average area fraction of the critical sub-domain and recalculation of 1.9% of the correction functions. The number of smoothing steps used for this test case is $n_s = 30$. 

Figure 97: Heterogeneous SPE10 bottom layer with grid aspect ratio of 2 after $t^* = 0.5$: error map of the (a) MSFV saturation, (b): MSFV pressure, (c): ai-MSFV saturation, (d): ai-MSFV pressure. The MSFV results are obtained based on $\epsilon_b = 0.02$ and the ai-MSFV results with $\epsilon_a = \epsilon_b = 0.02$.

Figure 98 shows the residuals for the MSFV ($\epsilon_b = 0.02$) and the ai-MSFV ($\epsilon_a = \epsilon_b = 0.02$) methods, corresponding to the results of Fig. 97.

Figure 98: Heterogeneous SPE10 bottom layer with grid aspect ratio of 2 after $t^* = 0.5$: residual of the results of the MSFV with $\epsilon_b = 0.02$ (a) and the ai-MSFV (b) with $\epsilon_a = \epsilon_b = 0.02$, corresponding to results of Fig. 97.

10 Error Control in the i-MSFV Framework

10.1 Introduction

In this chapter, the i-MSFV method is adaptively employed to solve the flow (pressure) equation in the sequentially fully implicitly coupled flow-transport systems [50]. For simplicity we study incompressible multiphase flow in porous media that is similar to the case studied in [41]. First we analyze the saturation error introduced by the approximate but locally conservative MSFV (and adaptive i-MSFV) velocity field; then we illustrate the
importance of a conservative velocity field to obtain an accurate solution of the transport problem. Numerical test cases are also presented to illustrate the conclusions which are made in this regards.

In investigating the accuracy-cost trade-off in numerical simulations, the transport error term is correlated with an a priori measurable residual in the original pressure equation. With several numerical test cases it is demonstrated that the MSFV pressure residual is a reliable measure to control saturation errors and that this error control strategy is very efficient in solving flow-transport equations. In the conservative MSFV framework, only a few iterations are applied to keep the residual norm below a threshold value, which results in very efficient simulations of multiphase flow in heterogeneous reservoirs. It is also studied how the saturation error depends on the residual threshold value in the pressure equation and on the number of additional iterations employed during the simulations.

The chapter is constructed as follows. In section 10.2 the discretization and the simulation strategy is explained. In section 10.3 the i-MSFV method in domain decomposition (DD) notations is presented, followed by section 10.4 where the i-MSFV method for sequentially fully implicit simulations of multiphase problems is introduced. Error analysis is presented in section 10.5, where also the importance of conservative velocity field is also discussed. Finally, numerical results are provided in section 10.6.

10.2 Discretization and Sequential Implicit Coupling

For simplicity, a two-phase system is considered such that only one transport equation needs to be solved and the mobility is a function of the saturation of one phase only, i.e. of $S_\alpha$. The discrete form of Eq. (9) can be written as

$$A p = r,$$  \hspace{1cm} (86)

where $p$ and $r$ are the pressure and right-hand-side (RHS) vectors, respectively. On a rectangular two-dimensional (2D) Cartesian grid the coefficient matrix $A$ is a penta-diagonal matrix (seven diagonal for 3D problems) if a two-point flux approximation is used and the equations are appropriately ordered. Note that the coefficient matrix depends on saturations via phase mobilities.

For the time-dependent saturation equation an implicit scheme is employed:

$$\frac{\phi}{\Delta t} (S^{n+1}_\alpha - S^n_\alpha) + \nabla \cdot (f^{n+1}_\alpha u) - q^{n+1}_\alpha = 0,$$  \hspace{1cm} (87)
where \( n \) and \( n + 1 \) denote the properties at the old and new times steps, respectively, and \( \Delta t \) is the time increment. Since the fractional flow function is a nonlinear function of the saturation, \( f_{\alpha}^{n+1} \) is linearized as

\[
f_{\alpha}^{n+1} \approx f_{\alpha} + \frac{df_{\alpha}}{dS_{\alpha}} (S_{\alpha}^{n+1} - S_{\alpha}^n),
\]

which leads to the Newton scheme:

\[
\frac{\phi}{\Delta t} (S_{\alpha}^{n+1} - S_{\alpha}^n) + \nabla \cdot \left( [f_{\alpha} + \frac{df_{\alpha}}{dS_{\alpha}} (S_{\alpha}^{n+1} - S_{\alpha}^n)] u_t \right) = q_{\sigma}.
\]

Here, \( \sigma \) denotes the iteration level and \( \frac{df_{\alpha}}{dS_{\alpha}} \) is evaluated at \( S_{\alpha}^\sigma \).

Equations (86) and (89) are coupled by a sequential fully implicit scheme [6, 50]. In this coupled scheme, first the pressure equation is solved. From the pressure solution the total velocity and upwind direction are calculated, and then the saturation equation is solved with a Newton scheme (inner-loop). The new saturation is then used to update the matrix \( A \) and the RHS \( r \) and the pressure equation is solved again. This procedure is repeated (outer-loop) until convergence is achieved.

### 10.3 I-MSFV Method in Domain Decomposition Formulation

As explained in the previous section, Eq. (9) has to be solved frequently in multiphase flow simulation as saturation changes modify the total mobility and the source term. In this situation it is crucial to use an efficient pressure solver because a large elliptic problem (or a parabolic problem in case of compressible flow) must be solved several times each time step in a sequential implicit scheme. As mentioned before, to reduce this computational cost, the MSFV method provides an approximate solution of Eq. (9) rather than a fully converged one.

The MSFV method can be described as a domain decomposition (DD) method [82] as explained in [76], where no correction functions are considered, however. Here, the formulation presented in [69] with the correction functions is reviewed.

First, Eq. (86) is re-ordered according to the dual coarse grid such that \( N_i \) internal-cell \( (i) \) equations appear first, then the \( N_e \) equations for edge cells \( (e) \), and finally the \( N_n \) equations for node cells \( (n) \) (see Fig. 99). With this ordering, Eq. (86) can be written as

\[
\begin{bmatrix}
A_{ii} & A_{ie} & 0 \\
A_{ei} & A_{ee} & A_{en} \\
0 & A_{ne} & A_{nn}
\end{bmatrix}
\begin{bmatrix}
p_i \\
p_e \\
p_n
\end{bmatrix}
=
\begin{bmatrix}
r_i \\
r_e \\
r_n
\end{bmatrix}.
\]

(90)
10.3 I-MSFV Method in Domain Decomposition Formulation

Figure 99: Illustration of a 2D domain with $10 \times 10$ fine (light lines) and $2 \times 2$ coarse grids (bold lines). In the partition introduced by the dual grid, fine cells are labeled as internal (white), edge (light-blue), and node (dark-blue) cells. Arrows represent the stencils used in the fine-scale (left) and MSFV systems (right): an arrow indicates that the pressure value of the starting point affects the mass balance of the end point.

Note that the dimension of the whole system is $N$ which is equal to $N_i + N_e + N_n$.

The MSFV pressure solution, i.e. Eq. (15), can be written in the DD form,

$$\tilde{p} = B\tilde{p}_n + C r,$$

(91)

where $B$ is the basis-function operator (an $N \times N_n$ matrix), which interpolates the node pressure (or coarse pressure) $p_n$; and $C$ the correction-function operator (an $N \times N$ matrix), which describes the fine-scale effects of the source term $r$. The edge pressure $p_e$ is computed by solving reduced (one-dimensional) problems along the edges with the node pressure as boundary condition and appropriate source terms, i.e. Eqs. (16) and (17).

Once the edge pressure is obtained, the pressure in the internal cells, $p_i$, can be calculated by using $p_e$ as boundary condition. This procedure yields the following operators:

$$B = \begin{bmatrix} A_{ii}^{-1} A_{ie} M_{ee}^{-1} A_{en} \\ -M_{ee}^{-1} A_{en} \\ I_{nn} \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} A_{ii}^{-1} & -A_{ii}^{-1} A_{ie} M_{ee}^{-1} & 0 \\ 0 & M_{ee}^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

(92)

where $M_{ee} = A_{ee} + \text{diag}(\sum_i A_{ei})$ is the reduced-problem operator and $I_{nn}$ is the $N_n \times N_n$ identity matrix.
The coarse-pressure problem is constructed using the primal coarse cells as control volume. If $\chi$ is the integral operator that sums up all the values pertaining to fine-scale cells that belong to the same primal coarse cell (an $N_n \times N_n$ matrix), the coarse scale problem (Eq. (20)) can be written in DD notation as

$$M_{nn} \tilde{p}_n = (\chi AB) \tilde{p}_n = (\chi - \chi AC)r$$

(93)

where $M_{nn} = \chi AB$ is the coarse scale operator. Consequently, the approximate pressure solution can be represented as the solution of the problem

$$M \tilde{p} = Qr,$$

(94)

where $M$ is the multiscale operator and $Q$ is the operator that constructs the MSFV right hand side. The inverse of $M$ and the operator $Q$ are given by

$$M^{-1} = BM_{nn}^{-1}R + C,$$

(95)

$$Q = I - R^T R + R^T (\chi - \chi AC).$$

(96)

Here, $R$ is a restriction operator defined by $R = [0 \ 0 \ I_{nn}]$ to yield $p_n = Rp$.

As described earlier, the error introduced in the approximate Eq. (94) is only due to the localization assumption introduced to solve the reduced problems along the edges, i.e. Eq. (17). The reduced-problem operator, $M_{ee}$, neglects the fluxes transversal to the dual boundaries. To reduce the localization error and improve the quality of MSFV results, several iterative MSFV (i-MSFV) schemes have been proposed. The first i-MSFV method devised for incompressible and compressible problems (see sections 3 and 6) used multiscale operator in combination with a fine-scale iterative solver (smoother) [34, 37]. Later, another class of i-MSFV method was introduced [70], where the MSFV operator was used as a preconditioner for a Krylov subspace method [81]. Recently, the MSFE operator has also been considered in combination with the MSFV operator for more robust iterations in the presence of a challenging permeability fields [13, 90]. Here, the i-MSFV procedure based on Krylov iterations is reviewed which employs the Generalized Minimal Residual (GMRES) method [81], i.e.

$$\tilde{p}_\nu^{\nu+1} = \tilde{p}_\nu^\nu + w^\nu M^{-1} Q(r - A \tilde{p}_\nu^\nu),$$

(97)

where the vector of the relaxation parameters $w^\nu$ is computed by the GMRES method [70].

The iterative scheme above converges to the fine-scale reference solution, if enough iterations are employed. However, with respect to other iterative
methods, the i-MSFV method has a particular property that a conservative velocity field can be constructed after any iteration step. This allows the iterative scheme to be interrupted before convergence and still obtain a velocity field that can be safely used in a transport problem.

To obtain an approximate conservative velocity field, the approximate pressure from Eq. (97), i.e. \( \tilde{p} \), is used to compute the Neumann boundary conditions that are used in solving local problems on primal-coarse cells (see Eq. (25)). The \( N_f \) fine-scale faces at which one has to compute the velocity can be divided into \( N_b \) faces belonging to the boundary of the primal coarse cells and \( N_h \) faces that do not belong to the coarse boundaries. The conservative velocity at the boundaries of the primal coarse cells, \( \tilde{u}_b \), is then obtained from the gradient of the approximate pressure, \( \tilde{p} \); whereas elsewhere the conservative velocity, \( \tilde{u}_h \), is obtained from the gradient of the newly computed pressure solution.

Therefore, if \( H \) is the operator that solves the localized problems in the primal-coarse cells the conservative velocity (Eq. (27)) in the DD notation is given by

\[
\tilde{u} = \begin{bmatrix} \tilde{u}_b \\ \tilde{u}_h \end{bmatrix} = \begin{bmatrix} D_b & 0 \\ -D_b H^{-1}(A - H) & D_h H^{-1} \end{bmatrix} \tilde{p} + \begin{bmatrix} 0 \\ D_h H^{-1} \end{bmatrix} r
\]

(98)

where \( D_b \) is the Darcy operator that computes the velocity at the boundary of the primal coarse grids and \( D_h \) the Darcy operator that computes the velocity at the other faces of the fine-scale cells; the pressure is obtained from Eq. (97). Defining the \( N_f \times N_h \) restriction operator \( R_b \) such that \( \tilde{u}_b = R_b \tilde{u} \), one can write Eq. (98) in the form

\[
\tilde{u} = R_b^T R_b D \tilde{p} + (I_{ff} - R_b^T R_b) D H^{-1} [r - (A - H) \tilde{p}] \]

(99)

where \( D \) is the exact Darcy operator (an \( N \times N_f \) matrix), and \( I_{ff} \) is \( N_f \times N_f \) the identity operator for the fine-scale faces.

Here, the MSFV and i-MSFV methods in DD notation were briefly reviewed because in the next subsection an adaptive strategy for employing the i-MSFV iterations is devised for the sequential fully implicit coupling approach. This adaptive strategy involves improving the MSFV system based on the old solution which is also presented in the DD notation. More information and discussion about the DD notation of the MSFV method falls beyond the scope of this thesis and can be found in [69].

10.4 I-MSFV Method for Time Dependent Problems

In case of time dependent problems, the pressure equation has to be solved several times because of the coupling between pressure and saturation equa-
10 ERROR CONTROL IN THE I-MSFV FRAMEWORK

It is important that the iterative scheme, proposed to improve the localization condition in the MSFV method, does not deteriorate the efficiency of the MSFV method by requiring a large number of iterations during the simulation.

This can be achieved by using the solution at the previous time step, i.e. $p^n$, to provide the first estimate of the fluxes transversal to the dual-cell boundaries as explained in [34]. Hence, the first MSFV solution at the new time step, i.e. $p^{n+1,0}$, is obtained by solving

$$M^{n+1} \tilde{p}^{n+1,0} = Q^{n+1} r^{n+1} + Q^{n+1} (M^n \tilde{p}^n - Q^n r^n), \quad (100)$$

where the second term on the RHS computes the estimate of transversal fluxes from the old pressure solution, and the superscripts $n$ and $n+1$ denote the operators at old and new time steps, respectively. The $\tilde{p}^{n+1,0}$ will be used as an initial pressure estimate for the time step $n+1$ in the iterative procedure of Eq. (97): i.e.,

$$\tilde{p}^{n+1,\nu+1} = \tilde{p}^{n+1,\nu} + w^{\nu} (M^{n+1})^{-1} Q^{n+1} (r^{n+1} - A^{n+1} \tilde{p}^{n+1,\nu}). \quad (101)$$

10.5 Solution Error due to Approximate Velocity Field

For efficient simulations of multiphase flow, a minimum number of i-MSFV iterations should be applied to generate an approximate but locally conservative velocity field that is within the desired error tolerance. In the sequential fully implicit framework, the transport equations at the fine-scale are solved accurately by a Newton scheme. The only source for the saturation error is, therefore, the approximate velocity field that is computed from the incompletely converged i-MSFV solution.

To analyze the effect of inaccurate velocity fields on the saturation solution, the approximate velocity is written as

$$u_t = u^*_t + \delta u_t, \quad (102)$$

where the superscript $*$ denotes the exact variable and the symbol $\delta$ denotes the error component. Eq. (8) is then rewritten as

$$\phi \frac{\partial S_\alpha}{\partial t} + u^*_t \cdot \nabla f_\alpha - g_\alpha = -f_\alpha \nabla \cdot \delta u_t - \delta u_t \cdot \nabla f_\alpha, \quad (103)$$

where the conservation property of the exact velocity, i.e. $\nabla \cdot u^*_t = 0$, is employed. In Eq. (103) one can identify two terms that contribute to the saturation error, i.e.,

$$\zeta = f_\alpha \nabla \cdot \delta u_t \quad (104)$$
and

\[ \eta = \delta u_t \cdot \nabla f_\alpha = \delta u_t \cdot \sum_{i=1}^{n} \frac{df_\alpha}{dS_i} \nabla S_i. \]  

(105)

The velocity error incurs two different types of errors in the transport equation: (1) the \( \zeta \) in Eq. (104) is due to the total mass conservation error and (2) the \( \eta \) in Eq. (105) is proportional to the total velocity error, \( \delta u_t \), and represents an advection of the phase with a wrong velocity.

Equations (104) and (105) clearly indicate the dependence of saturation errors on \( f_\alpha \). The term \( \zeta \) occurs in a relatively large region in which the fractional flow function \( f_\alpha \) is significant; this region grows with the injection time. The term \( \eta \), however, is localized only in regions where large gradients of the fractional flow function exist.

Both the original MSFV and the adaptive i-MSFV methods deliver a conservative velocity field such that \( \nabla \cdot \delta u_t = 0 \), and the term \( \zeta \) always vanishes. However, in non-conservative methods, such as the Multiscale Finite Element method (MSFE) (see e.g. [26]) or Algebraic Multigrid method (AMG) ([85]) with incomplete convergence, saturation errors can be generated in regions in which \( \zeta \neq 0 \). A non-physical source term, as a result, will appear in the saturation transport equations which may lead to a non-physical saturation solution with numerical values less than 0 or larger than 1. Note that, if a general linear solver such as AMG is used, only the fully converged solution is mass conservative, and the intermediate solutions do not honor the local mass conservation exactly. In fact, the level of total mass conservation discrepancy in the solution of these solvers after any iteration level can be computed by the residual, i.e. \( \mathbf{r} - \mathbf{A} \tilde{\mathbf{p}} \). As explained before, to construct the conservative approximate velocity field in the MSFV method, an additional step is taken by solving local problems on coarse grid cells subject to the Neumann boundary conditions based on the primal pressure field, i.e. Eq. (99).

In the MSFV framework one needs to estimate the value of \( \delta u_t \cdot \nabla f_\alpha \) in order to control the saturation error during a simulation. Unfortunately, direct computation of \( \delta u_t \) is not possible, since it entails a priori knowledge of the fine-scale reference solution. On the other hand, the quality of the conservative velocity \( u_t \) depends only on the quality of the local Neumann boundary conditions provided by the non-conservative i-MSFV velocity field, \( -\mathbf{\lambda}_t \cdot \nabla \tilde{p} \). As mentioned before, the error in \( \tilde{p} \) is only due to the neglected transversal fluxes, which are equivalent to unphysical source terms at the boundaries of dual coarse cells. Hence, the evolution of saturation errors in
the MSFV conservative framework becomes
\[ \eta = \delta u_t \cdot \nabla f_\alpha \approx \{-\lambda_t \cdot \nabla \delta p^\nu\} \cdot \nabla f_\alpha. \]  
(106)

This expression results in
\[ \eta \approx \{-\lambda_t \cdot \nabla \left[ ( - \nabla \cdot (\lambda_t \cdot \nabla) \right]^{-1} (q_t + \nabla \cdot (\lambda_t \cdot \nabla p^\nu)) \} \cdot \nabla f_\alpha. \]  
(107)

which can concisely be written in the DD form as:
\[ \eta \approx [DM^{-1}Q(r - A\tilde{p}^\nu)]^T F, \]  
(108)

where \( D \) is the Darcy operator and \( F \) is the vector containing the discrete gradient of the fractional flow function, \( \nabla f_\alpha \). In addition, the \( A^{-1} \) is replaced by the MSFV operator \( M^{-1}Q \) for the sake of computational efficiency. Since the pressure residual is the only pressure dependent term in Eqs. (107) and (108), in this work, the saturation error is controlled by keeping the residual \( \ell \)-th norm (2nd in this work) below a specified threshold value, i.e.
\[ \frac{\| r - A\tilde{p}^\nu \|_\ell}{\| r \|_\ell} < \epsilon. \]  
(109)

Note that the residual is a measure of total mass balance for the MSFV dual pressure field, i.e. \( r - A\tilde{p}^\nu \), which affects the saturation solution. Therefore, keeping the residual norm below a specified threshold value does not necessarily result in small pressure errors \( \delta \tilde{p}^\nu \). In fact, in very low permeability (transmissibility) regions \( \delta \tilde{p}^\nu \) can be large, but with a small effect on the residual \( r - A\tilde{p}^\nu \) = \( A\delta \tilde{p}^\nu \). On the other hand, there will be low flow rates in those regions with minimal influence on the quality of the transport solution. This justifies the residual based error control strategy.

### 10.6 Numerical Results

To test the performance of the adaptive i-MSFV method for time-dependent problems with residual-based error control, three challenging problems were constructed from the SPE 10th Comparative Problem by [19]: (1) the SPE 10 top layer with \( k_y = k_x \), (2) the SPE 10 top layer with \( k_y = 10 k_x \), and (3) the SPE bottom layer which includes a channelized permeability field with high contrast. No-flow boundary condition is applied at the boundaries of all the test cases. Gas is injected from cell \((1, 60)\) at the constant, non-dimensional rate of 10, while oil is produced from cell \((220, 1)\) at the fixed dimensionless pressure 0. The viscosity ratio \( \mu_o/\mu_g \) is 10. A grid with \( 220 \times 60 \) cells was employed for the fine-scale simulations and \( 20 \times 12 \) coarse control volumes for the MSFV method.
Figure 100: SPE 10 [19] top layer permeability distribution ($ln(k_x)$). The fine-scale simulator uses a $220 \times 60$ grid while the MSFV method employs a $20 \times 12$ coarse grid.

### 10.6.1 Case 1

For a wide range of heterogeneous test cases the original MSFV results (with no iterations) are in good agreement with the fine-scale reference solutions. As discussed earlier in this thesis, one needs to control the level of accuracy of the MSFV results by the proposed adaptive error control strategy. In this first test case the SPE 10 top layer with isotropic permeability field, i.e. $k_x = k_y$, is considered. The permeability distribution is depicted in Fig. 100. Fine-scale reference solutions (pressure and saturation) are presented at 0.2 PVI in Fig. 101 together with the original non-iterative MSFV results and absolute error maps.

The MSFV error maps are shown again in Fig. 102 with a smaller interval for the contour color map. The proposed error control strategy in the i-MSFV framework is used to adaptively employ iterations. The quality of the adaptive i-MSFV results clearly depends on the threshold values used to control the pressure residual. For $\epsilon = 10^{-2}$ and $5 \times 10^{-3}$ the error maps of i-MSFV solutions (pressure and saturation) are also depicted in Fig. 102 at the same scale of the MSFV error plots. The i-MSFV results for $\epsilon = 10^{-3}$ are obtained by only 0.56 and 0.97 additional iterations in average per pressure solver call, respectively.

Figure 103 shows the history of the second norm of the saturation error during the simulation for the MSFV and i-MSFV results obtained with different values of $\epsilon$. Moreover, to illustrate the importance of the reconstruction step to obtain conservative fine-scale velocity fields, the error histories for $\epsilon = 10^{-3}$ without and with reconstruction step are depicted in Fig. 104. Note that if the reconstruction step is skipped, the i-MSFV solution is conservative only at the coarse scale. The nonconservative i-MSFV result for $\epsilon = 10^{-3}$ is obtained by employing 3.46 iterations in average in the simulation, while the conservative i-MSFV solution for the same value of $\epsilon$ is obtained by 2.2 ad-
Figure 101: Fine-scale reference results at 0.2 PVI (gas injection) obtained on a grid with 220 × 60 cells: pressure (top-left) and saturation (top-right) maps. Also shown are the original non-iterative MSFV results at 0.2 PVI using a 20 × 12 coarse grid: pressure (middle-left) and saturation (middle-right) maps and the absolute differences: pressure error (bottom-left) and saturation error (bottom-right) maps. (Case 1)

Figure 102: Pressure (left) and saturation (right) error plots for the MSFV (top) and i-MSFV with $\epsilon = 10^{-2}$ (middle) and $\epsilon = 5 \times 10^{-4}$ (bottom) at 0.2 PVI (gas injection). Note that the i-MSFV results for $\epsilon = 10^{-2}$ and $\epsilon = 5 \times 10^{-3}$ are obtained with only 0.56 and 0.97 additional iterations in average, respectively. (Case 1)
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Figure 103: Saturation error histories for the i-MSFV simulations with $\epsilon = 1 \times 10^{-2}$, $\epsilon = 5 \times 10^{-3}$, and $\epsilon = 1 \times 10^{-3}$ which result in 0.56, 0.97, and 2.2 additional iterations per pressure solver call, respectively. (Case 1)

Additional iterations in average. From the saturation error plots it is also clear that the reconstruction step improves the quality of the results significantly. In other words, even with a similar pressure residual norm, the reconstruction step in the MSFV framework improves the saturation solutions significantly. Also, the error of conservative i-MSFV solutions (with reconstruction step) for $\epsilon = 10^{-2}$, which is obtained by only 0.56 additional iterations, is shown again in this figure. Note that the quality of the conservative i-MSFV solutions with $\epsilon = 10^{-2}$ is comparable with that of the nonconservative one with $\epsilon = 10^{-3}$. It is also worth to be mentioned that the nonconservative simulation experiences a stability limitation in the flow-transport coupling for the cases of $\epsilon = 5^{-3}$ and $10^{-2}$.

The iteration history of the i-MSFV results for different values of $\epsilon$ is presented in Fig. 105. Also, the iteration history of the nonconservative simulation for $\epsilon = 10^{-3}$ is shown in this figure. Note that more outer iterations are employed by the nonconservative simulation due to the stability limitations mentioned above, which exist even for a low pressure residual norm.

The aim of this chapter is to optimize the cost-accuracy trade-off in multiphase flow simulations. The smaller the threshold value $\epsilon$, the higher the accuracy (lower error) and bigger the computational cost. Figure 106 illustrates the dependency of the average saturation error during the simulation on the numerical values of $\epsilon$ (residual norm) and on the average number of employed additional iterations. In this figure, a wider range of $\epsilon$ values are used.
Figure 104: Saturation error histories for the i-MSFV simulations with and without including the reconstruction step. The result for $\epsilon = 1 \times 10^{-3}$ without including the reconstruction step (nonconservative at fine-scale) is obtained by employing 3.46 additional iterations while that obtained with reconstruction step (conservative at fine-scale) is obtained by 2.2 additional iterations. Note that non-conservative simulations did not converge for $\epsilon = 5 \times 10^{-3}$ and $10^{-2}$ due to stability limitations. Also, the quality of the i-MSFV result (conservative) for $\epsilon = 10^{-2}$, which is obtained by only 0.56 additional iterations in average, is comparable to the quality of nonconservative simulations with a 10 times lower threshold value, i.e. $\epsilon = 10^{-3}$ . (Case 1)
Figure 105: I-MSFV iteration histories (number of iterations employed in each pressure solver call, i.e. outer loop) for different values of $\epsilon$. The results corresponding to the nonconservative simulation, i.e. without reconstruction step, with $\epsilon = 10^{-3}$ is also shown (bottom-right). This figure clearly shows that more outer iterations are employed by the nonconservative simulator due to stability limitations even for such a low pressure residual norms. (Case 1)
10.6.2 Case 2

In the second test case, the \( k_x \) permeability distribution of the SPE 10 top layer is used (see Fig. 100); like in the previous case. However, in contrast to the previous case, an anisotropic permeability field with \( k_y = 10 \ k_x \) is considered. This anisotropic heterogeneous field makes the problem challenging for the original non-iterative MSFV method. In general, the MSFV method exhibits large inaccuracies in the pressure solution in the presence of highly anisotropic heterogeneous permeability fields. This fact was observed and studied in several previous works, e.g. see [57], [67], and [43]. To resolve this limitation of the original MSFV, the i-MSFV method was devised to improve the localization assumptions systematically in a convergent iterative procedure; e.g. see [34] and [70].

Reference solutions (pressure and saturation) from fine-scale simulation are presented in Fig. 107 together with the original, non-iterative MSFV results. As mentioned before, for this highly anisotropic heterogeneous problem, the original MSFV methods unfortunately fails to provide accurate results.

The i-MSFV method is used to improve the original MSFV results at the beginning of the simulation until the residual is below \( 10^{-2} \), i.e. \( \epsilon = 10^{-3} \). Then, for the subsequent time steps, the error control approach is used to adaptively apply iterations once the normalized residual becomes larger than a specified value, i.e. \( \epsilon \). Note that number of the additional MSFV iterations can be reduced by using the solution from the previous time step as an initial
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Figure 107: Fine-scale reference results at 0.2 PVI (gas injection) obtained on a grid with $220 \times 60$ cells: pressure (top-left) and saturation (top-right) maps. Also shown are the original non-iterative MSFV results at 0.2 PVI using a $20 \times 12$ coarse grid: pressure (bottom-left) and saturation (bottom-right) maps. (Case 2)

Figure 108 shows saturation and pressure error maps of the i-MSFV results obtained with $\epsilon = 4 \times 10^{-2}$, (i.e. the 2-norm norm of the residual is below 0.4), and $2 \times 10^{-2}$, (i.e. the 2-norm of the residual is below 0.2). The error is defined as the absolute differences between the MSFV (or i-MSFV) and fine-scale reference solutions. For the two threshold values, the i-MSFV results are obtained by applying only 0.7 and 1.4 additional iterations in average per pressure solver call, respectively.

Figure 109 shows the 2-norm of the saturation error for MSFV and i-MSFV simulations with different $\epsilon$ values. It was noted that the big saturation errors in the original MSFV results are due to the non-physical circulations in the velocity field resulting from inaccurate localization conditions in the presence of highly anisotropic heterogeneous permeability fields.

A nonconservative simulation with controlled pressure residual norm is also considered for this test case; like in the previous test case. The nonconservative solutions are obtained by skipping (excluding) the reconstruction of a conservative velocity field in the MSFV method. Therefore, the solutions satisfy the local mass conservation only at the coarse scale, but not at the fine-scale. The nonconservative solutions with controlled residual norm suffer from instability of the coupled flow-transport system. Therefore, the nonconservative simulation with small $\epsilon = 10^{-3}$ required an additional outer loop convergence criterion; the outer loop stops once the number of iterations exceeds 5. The saturation error histories of the nonconservative and conservative i-MSFV simulations with the same $\epsilon$ are compared in Fig. 110. The error histories for conservative simulations with different $\epsilon$ are also included.
Figure 108: Error maps of the MSFV results corresponding to the results of Fig. 107: pressure error (top-left) and saturation error (top-right) maps. Also shown in the middle and bottom rows are the error maps of the i-MSFV results with $\epsilon = 4 \times 10^{-2}$ and $2 \times 10^{-2}$, respectively: pressure errors (left) and saturation errors (bright) maps. Note that the error is defined as absolute difference with respect to the fine-scale reference solution. (Case 2)

Figure 109: 2-norm of saturation error for the MSFV and i-MSFV simulations with $\epsilon = 4 \times 10^{-2}$, $\epsilon = 2 \times 10^{-2}$, and $\epsilon = 1 \times 10^{-2}$ which result in 0.7, 1.4, and 2.3 additional iterations per pressure solver call, respectively. (Case 2)
Figure 110: Saturation error history for the i-MSFV simulations including (conservative) and excluding (nonconservative) the reconstruction step. The nonconservative solutions are obtained by employing 10.2 additional iterations while the i-MSFV results are obtained by applying 8.14, 3.6, and 2.3 additional iterations for $\epsilon = 10^{-3}$, $5 \times 10^{-3}$, and $10^{-2}$, respectively. (Case 2)

in the figure. The nonconservative simulator employed 10.2 additional iterations while the conservative i-MSFV simulator 8.14 additional iterations. Clearly the quality of the results is significantly improved once a conservative velocity field is reconstructed to solve transport equations. Figure 111 depicts the history of iterations employed in the conservative and nonconservative simulations.

Finally, the average saturation error as a function of $\epsilon$ and additional number of iterations are depicted in Fig. 112. Note that for practical applications the i-MSFV with less than 1 additional iteration improves the solution significantly.

10.6.3 Case 3

For the last test case, the permeability field of SPE 10 bottom layer presented by [19] is used (Fig. 11). This permeability field includes long coherent structures with high permeability contrasts. Therefore, it is a challenging test case for MSFV simulation. Fig. 113 depicts the fine-scale and MSFV solution maps (pressure and saturation). The figure clearly indicates the non-physical pressure errors in the low permeability areas. In addition, Fig. 114 shows the MSFV and i-MSFV solution error maps. The i-MSFV results are obtained by setting $\epsilon = 5 \times 10^{-2}$ which leads to 2.1 additional iterations per pressure call during the simulation. It is clear that these 2.1 (in average) additional iterations yield a significantly improved solution. The i-MSFV
Figure 111: I-MSFV iteration histories for different values of $\epsilon$. Note that with the outer loop convergence criterion $||S^{n+1} - S^n||_{\infty} < 10^{-3}$ loosening the $\epsilon$ value results in more outer iterations. This effect becomes minimal if slightly looser convergence criteria are used. Also shown (bottom-right) is the iteration history for a nonconservative (excluding the reconstruction step) i-MSFV simulation showing that much more outer iterations are employed due to lack of stability. (Case 2)

Figure 112: Average saturation error as a function of $\epsilon$ and additional iterations. (Case 2)
10.6 Numerical Results

Figure 113: Fine-scale (top) and MSFV (bottom) pressure (left) and gas saturation (right) maps. The non-physical peaks in the MSFV pressure solution mainly occur in low permeability regions. (Case 3)

Figure 114: Pressure (left) and gas saturation (right) error maps for the MSFV and i-MSFV simulations. The i-MSFV results are obtained with $\epsilon = 5 \times 10^{-2}$ leading to 2.1 additional iterations per pressure solver in average. (Case 3)

solution, however, still contains small pressure peaks in low permeability regions. Nevertheless, the adaptive i-MSFV successfully removes most pressure and saturation errors. Obviously one can further improve the quality of solutions by tightening the residual tolerance criterion. It was found that $\epsilon = 5 \times 10^{-2}$ tends to produce solutions that are accurate enough for practical applications.
Part VII

Conclusions

11 Conclusions

In this thesis, an iterative multiscale finite-volume (i-MSFV) method was devised. In each iteration, the MSFV pressure solution gets improved by a number of smoothing steps. The new approximation is then used to obtain better local boundary conditions for the new correction functions, which are required for the RHS of the coarse system. The smoother removes high-frequency errors which in combination with the MSFV operator (resolves low-frequency errors) leads to an efficient scalable iterative multiscale solver. It is demonstrated for a wide range of difficult test cases that the i-MSFV method combined with a simple line-relaxation (LR) smoother works well, especially for cases with high grid aspect ratios. In each LR step, an independent tri-diagonal system has to be solved for each grid line, e.g. with the Thomas algorithm which scales linearly with the problem size. Moreover, as shown earlier, LR is insensitive to the grid aspect ratio and the level of anisotropy, which is the reason that the i-MSFV method is very efficient even for highly anisotropic problems.

The number of iterations required by the overall solution algorithm is problem size independent, i.e. the i-MSFV method is a scalable iterative linear solver similar to multigrid methods [85]. On the other hand, more i-MSFV iterations are required as the coarse-grid-cell aspect ratio increases. The number of iterations, however, reaches an asymptotic value beyond an aspect ratio of approximately 20.

A further favorable property of the i-MSFV algorithm is that it can be operated anywhere between a multiscale method (without iterating it becomes identical to the MSFV method) and an efficient linear solver. For example, if the iteration procedure is terminated before full convergence is achieved, the method still delivers a conservative fine-scale velocity field. This is of relevance, e.g. if transport is solved on the fine grid.

It was also shown that the i-MSFV method can be interpreted as a multigrid method which allows for extreme coarsening factors. An important property here is that no explicit upscaling is required. In other words, MSFV basis and correction functions are used for the restriction and prolongation steps.

Convergence studies with test cases involving shale layers, high coarse-cell aspect ratios, and layered permeability fields with high contrasts demonstrate the robustness and efficiency of the new method.
Another important result is related to multiphase flow, where the mobility experiences changes due to evolving phase saturations. As a preliminary study, for the examples considered in section 4.3.2 the i-MSFV method with one iteration every 10th time step leads to accurate solutions that are in very good agreement with the corresponding fine-scale solutions. This indicates that the cost of the i-MSFV method is similar to that of the MSFV method for complex two-phase flow problems. However, the i-MSFV solutions are more accurate and problems with high coarse-cell aspect ratios, strong anisotropy, and shale layers are avoided. More investigations for a wider range of multiphase flow scenarios with implicit time integration schemes are also considered in section 10.

For compressible problems, a new iterative formulation for the linearized pressure equation is proposed. The system matrix is symmetric and can very easily be implemented. In addition, based on this new formulation a general MSFV method for compressible problems was devised. In this scheme, the basis and correction functions are calculated based on the full governing equation, and not just based on its elliptic part.

At the same time, the coefficients appearing in the basis function equations are pressure independent (due to the new formulation). This is crucial for high computational efficiency, since it allows to keep the basis functions for multiple iterations and time steps.

During the iterative solution of the linearized pressure equation, only the correction functions have to be updated (in 3D, there exist 8 times more basis functions than correction functions).

It is shown for highly compressible scenarios that the new scheme leads to results which are in very good agreement with those obtained with a classical fine-scale solver (reference solutions).

In particular, during transient phases the new method outperforms existing multiscale methods. The reason for the high accuracy of the proposed MSFV method is due to the minimal assumptions that are made, i.e. the full governing equation is solved at the fine-scale to obtain the correction functions. The only assumption applied in this framework is the localization condition, which is trivial and inevitable for multiscale methods. Therefore, for 1D problems, where no localization assumptions are required, results of the proposed MSFV method and the reference fine-scale solutions are identical at any time, i.e. during transient and steady states.

Further, an i-MSFV method for compressible problems is described. The i-MSFV method ensures that the multiscale solution converges to the fine-scale reference. Numerical convergence of the method is demonstrated for many test cases for different numbers of smoothing steps $n_s$ and different
non-dimensional time step sizes $\Delta t^\ast$. It is shown that the i-MSFV method (converging to fine-scale reference solutions) is more efficient than the MSFV method (converging to approximate solutions). The only additional cost is due to the smoothing step, which is minimal and can be reduced by updating the boundary conditions only infrequently, i.e. not every time steps. Furthermore, an i-MSFV method for the solution of fractured porous media using a hierarchical fracture modeling approach is presented. The fracture modeling approach was extended to become suitable for the MSFV framework. This extension involves splitting of the fracture pressure into an average value and a deviation. This splitting technique results in only one additional DOF for each connected fracture network being implicitly coupled with the matrix system. This results in much smaller coupled systems compared to those resulting from the previously proposed hierarchical approach. For the fracture deviation pressure a classical solver is employed, since the corresponding systems tend to be much smaller than the globally coupled one.

To reduce the computational cost required for the solution of the global system with its additional DOFs, an i-MSFV is devised. Local fracture functions are introduced to accurately capture fractures at the coarse scale; similar as well functions in [51]. In each coarse dual cell there exists only one local fracture function per overlapping network and one DOF per fracture network is added to the coarse scale system.

Efficiency and accuracy of the proposed i-MSFV approach for fractured porous media was assessed for a wide range of test cases and corresponding results are documented in this thesis. These demonstrate that the combination of the new i-MSFV method with sequential coupling of the global coarse system with the small systems for the fracture deviation pressure is very effective; especially for highly conductive fractures. Improving the overall convergence rate for problems with less conductive fractures will be subject of future research.

The MSFV method is also extended to account for non-matching coarse grids arising from faulted reservoirs. The devised extension is based on more general local domains to accommodate the fault regions. These new domains are obtained by agglomerating neighboring regular dual control volumes overlapping with a fault, which leads to an extended dual coarse cell with typically more than four (eight in 3D) associated nodes and equally many associated basis functions. The rest of the steps are the same as in the MSFV method for conforming grids. Numerical results show that this extended MSFV method is suitable for practical problems involving complex geometries, e.g. faulted reservoirs.

Moreover, to improve the quality of the results and to extend the applicabil-
ity of the method for highly anisotropic heterogeneous problems, the i-MSFV method as in [34] is extended for non-matching coarse grids. Numerical results are presented for different test cases and it is demonstrated that this i-MSFV method is a convergent iterative solver with the advantageous property that a conservative velocity field can be recovered after any multiscale iteration. It is also worth mentioning that the new MSFV method can deal with non-matching fine grids, provided the fine-scale solver employed to compute basis and correction functions has this capability.

In this thesis, also a space-time adaptive iterative multiscale finite-volume (ai-MSFV) method is devised. It is shown that the error introduced by the MSFV framework can locally be reduced by applying i-MSFV iterations in small critical sub-domains. The critical sub-domains are found based on residual maps, which is a computationally cheap task and results in an efficient improvement of the results. Numerical studies with a wide range of test cases demonstrate the efficiency of the ai-MSFV method for single phase flow, where an elliptic problem is solved once, and for multiphase flow scenarios, where elliptic problems are solved several times with different coefficients. Furthermore, for anisotropic heterogeneous problems, where the standard MSFV method leads to unphysical results, the ai-MSFV method proved to be very efficient.

Moreover, the ai-MSFV method is employed to optimize the accuracy-cost trade off in the i-MSFV framework. Clearly, the optimum value of the parameter $\epsilon_a$ is application dependent. Based on the presented test cases, it was found that generally the choice of $\epsilon_a \approx 0.01$ leads to good results and efficient simulations; i.e. localization conditions are improved only infrequently and adaptively.

Further error criteria based on different parameters such as permeability, flow field, vorticity, etc., which are common in the upscaling community (see e.g. [29, 63, 23, 22, 71, 5]) are also applicable in the ai-MSFV context. A very important advantage of the ai-MSFV framework with respect to upscaling techniques is that here the results are improved via the boundary conditions for the correction function calculations. Neither local grid refinement, nor coarse grid adjustment, nor any other grid-based treatments, which are common in the upscaling literature, are applied here. Hence, this adaptive strategy has no effect on the time-step size, which is a drawback of classical local grid refinements, when IMPES schemes are employed. The strategy of the space-time ai-MSFV method is the same for compressible multiphase flow scenarios. However, further studies for the optimum error criterion in these cases are necessary, which is a topic of future research.

An efficient adaptive strategy in the i-MSFV framework is devised for
sequential implicit simulation of multiphase flow. The saturation errors due to an inaccurate velocity field are analyzed. We also showed the importance of the conservative velocity field obtained from the MSFV method; in comparison with nonconservative velocity fields. Based on the analysis of the saturation error estimate, a residual-based strategy for adaptive iterations is introduced. Its performance was validated for a numerically difficult problem (heterogeneous anisotropic multiphase flow with high viscosity ratio). Further it is shown how the previous solution can be used to improve the initial estimate of the i-MSFV result at the new time. This strategy minimizes the number of i-MSFV iterations required to obtain good solutions. Numerical results showed that the residual-based error control strategy in the i-MSFV framework leads to an adaptive solver that greatly improves the accuracy with a small increase of the computational cost.

Note that this part of the present work was aimed to optimize the accuracy-cost trade off in multiphase flow simulations. The overall efficiency of the strategy could further be improved by also solving the transport equations with a MSFV method [61], and by using the adaptive MSFV residual improvement [38]. In the latter case, based on the analysis presented in this work, one should define more strict threshold values near front regions and moderate ones for the rest of the domain.

Finally, the reader should note that the proposed MSFV methods are applicable not only for porous media problems, but in all other fields where linear and non-linear elliptic problems are to be solved.

12 Closing Remarks and Future Works

When this PhD project started, i.e. in June 2007, the MSFV method could not be used for realistic problems. This is due to the fact that no error control and reduction was possible at that time. In the first step of this PhD project, the i-MSFV method was devised. The i-MSFV method opened a new chapter in the applicability of the MSFV method. With the i-MSFV method, a general systematic reduction of MSFV errors became possible. Dealing with more realistic phase and reservoir behavior, motivated the next steps of this work, where the i-MSFV method was extended for the solution of compressible problems with fractures and faults. To close the loop, an extensive study for adaptive error reduction and error analysis was done. Finally, to the author’s understanding, the whole package is ready to be implemented in a commercial reservoir simulator. This step is in progress by the main financial supporter of this PhD work, i.e. Chevron/Schlumberger Intersect Alliance Technology.
There exist many possibilities to extend the current work, which are mainly concerned with the efficiency of the proposed method. Several i-MSFV methods have been proposed after the publication of the one presented in this thesis; most of them employ GMRES [81] to enhance the convergence rate. So far, no method has been devised, which exploits permeability data for error reduction and quality optimization. In other words, the coarse grid and dual coarse grid cells may be utilized in a way that the reduced problem boundary conditions are very accurate. This step is very important and subject of the future research.

So far, the coarse scale MSFV problem has been solved using a direct (or iterative) numerical solver (to machine accuracy). For very large problems, however, the coarse problem may be huge itself. This is a good motivation to develop a multilevel MSFV method similar to multigrid methods [85] which allows to solve larger problems.

There is potential to improve the efficiency of the i-MSFV method for fractured porous media with low conductive fractures [40, 39]. As mentioned earlier, an improvement may be achieved by introducing more than one DOF per fracture network to capture the gradient of the pressure. The fracture basis functions are multiplied with additional DOFs at the coarse scale which leads to enhanced coarse scale systems. A similar approach could be applied to enhance the coarse scale condition number in a consistent way. Opposed to the previously developed strategies [27], sum of basis and additional local functions (which have to be scaled with a DOF) is one.

As a general closing remark, one should aim for a multiscale framework which can honor different physics at different scales. There exist scenarios, in which the physical assumptions and mathematical formulations are different depending on the length scales of observation. For example, one cannot employ Darcy’s law at the pore scale. Coupling pore-scale and the Darcy scale simulations in a multi-scale multi-physics framework could be of great importance for the community.
REFERENCES

References


REFERENCES


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REFERENCES


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References

Patents


Journal Publications


Conference Proceedings/Abstracts

REFERENCES


REFERENCES

on Challenges of Porous Media, Kaiserslautern, Germany, March 11-14, 2009.


• H. Hajibeygi, G. Bonfigli, M.A. Hesse, and P. Jenny: "Iterative multiscale finite-volume method", The XVII International Conference on Computational Methods in Water Resources (CMWR 2008), San Francisco, CA, USA.


