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Some effective estimates for André–Oort in $Y(1)^n$

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Abstract. Let $X \subset Y(1)^n$ be a subvariety defined over a number field \mathbb{F} and let $(P_1, \dots, P_n) \in X$ be a special point not contained in a positive-dimensional special subvariety of X . We show that if a coordinate P_i corresponds to an order not contained in a single exceptional Siegel–Tatuzawa imaginary quadratic field K_* , then the associated discriminant $|\Delta(P_i)|$ is bounded by an effective constant depending only on $\deg X$ and $[\mathbb{F} : \mathbb{Q}]$. We derive analogous effective results for the positive-dimensional maximal special subvarieties. From the main theorem we deduce various effective results of André–Oort type. In particular, we define a genericity condition on the leading homogeneous part of a polynomial, and give a fully effective André–Oort statement for hypersurfaces defined by polynomials satisfying this condition.

1. Introduction

1.1. Notations. We identify $Y(1) \simeq \mathbb{A}_{\mathbb{Q}}^1$ by means of the j -invariant. A point in $p \in Y(1)$ is said to be *special* if the corresponding elliptic curve E_p admits complex multiplication. In this case we denote by $K(p)$ the quadratic field generated by the periods E_p , and by $\Delta(p)$ the discriminant of the endomorphism ring of E_p in the ring of integers of K (see Section 2.1). A point $P = (P_1, \dots, P_n) \in Y(1)^n$ is called special if each coordinate P_i is special. We denote $D(P) := \max_i |\Delta(P_i)|$.

For every $N > 1$ there is a *modular polynomial* $\Phi_N(x, y) \in \mathbb{Z}[x, y]$ whose zero locus in $Y(1)^2$ is the set of pairs (x, y) corresponding to N -isogenous elliptic curves. Let $S_0 \cup \dots \cup S_w$ be a partition of $\{1, \dots, n\}$ with S_0 only permitted to be empty. Let $p_i \in Y(1)$ be special for $i \in S_0$. Let $s_i \in S_i$ be minimal for $i > 0$ and for each $s_i \neq j \in S_i$ choose a positive integer N_{ij} . Then a w -dimensional *special subvariety* V of $Y(1)^n$ is an irreducible component of

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the variety

$$(1.1) \{(y_1, \dots, y_n) \in X : y_i = p_i, i \in S_0, \Phi_{N_{ij}}(y_{s_i}, y_j) = 0, s_i \neq j \in S_i, i = 1, \dots, w\}.$$

Following [25], we will call a special subvariety V of $Y(1)^n$ *strongly special* if $S_0 = \emptyset$. We denote $P(V) := (p_i)_{i \in S_0}$ and set $D(V) = D(P(V))$. If V is defined as above but without the condition that p_i is special, we say that V is *weakly-special*.

1.2. Results for general varieties. To state our main result, we let K_* denote a universally fixed quadratic field and Δ_* the discriminant of its ring of integers. The field K_* arises from the theorem of Siegel–Tatuzawa and its definition is given in Section 2.2. We note that it is possible that such an exceptional field does not exist (for instance, this is the case if one assumes the generalized Riemann hypothesis for imaginary quadratic fields). In this case we may formally set $K_* = \emptyset$ below, and our results then become a fully effective solution of the André–Oort conjecture for $Y(1)^n$. We define $D_*(P)$ and by analogy with $D(P)$, but with the maximum taken only over those P_i where $K(P_i) \neq K_*$ (and similarly for $D_*(V)$).

For a variety $X \subset Y(1)^n$ we denote by \mathbb{F}_X its field of definition. We denote by X^{sp} the set of all maximal special subvarieties of X . We also denote by X^{spp} the set of all special points in X that are not contained in a positive-dimensional special subvariety of X . We denote by $\deg X$ the degree of X with respect to the projective embedding $Y(1)^n \simeq \mathbb{A}^n \subset \mathbb{P}^n$ (more precisely we take the sum of the degrees of all irreducible components). If X is a hypersurface given as the zero locus of a polynomial F over a number field, then we denote by $H(F) = H(X)$ the maximal Weil height of any of its coefficients.

As a general convention throughout the paper we denote by $c(\cdot)$ some effectively computable constant depending on a set of parameters. If the constant does not depend on any parameters, we use “const” to avoid confusion. Our main result is as follows.

Theorem 1. *Let $X \subset Y(1)^n$ be defined over a number field and let $V \in X^{\text{sp}}$. Then*

$$\deg V \leq c(\deg X, [\mathbb{F}_X : \mathbb{Q}]), \quad D_*(V) \leq c(\deg X, [\mathbb{F}_X : \mathbb{Q}]).$$

For $D(V)$ we have the weaker estimate

$$D(V) \leq c(\Delta_*, \deg X, [\mathbb{F}_X : \mathbb{Q}]).$$

To state some consequences of Theorem 1 we introduce the following terminology. A special variety $V \subset Y(1)^n$ is called a **-variety* if all associated quadratic fields of $P(V)$ equal K_* . A variety X' is called a *special section* of a variety X if

$$X' = \pi(X \cap \{x_i = P_i \text{ for all } i \in \Sigma\}), \quad \pi(x_1, \dots, x_n) = (x_i)_{i \notin \Sigma}$$

for some $\Sigma \subset \{1, \dots, n\}$ and some special points $P_i \in Y(1)$ for $i \in \Sigma$.

Theorem 1 reduces the problem of effectively computing X^{sp} to the problem of computing the *-varieties in X_α^{sp} , where X_α ranges over an effectively constructed set of special sections of X . Indeed, for each set $\Sigma \subset \{1, \dots, n\}$ and each choice of P_i with discriminant at most¹⁾ $c(\deg X, [\mathbb{F}_X : \mathbb{Q}])$ define X_α to be the corresponding special section. Then every $V \in X^{\text{sp}}$ corresponds to a *-variety in some X_α^{sp} .

¹⁾ We remark that there exist only a finite number of special points in $Y(1)$ with a given discriminant, and it is straightforward to effectively enumerate them.

The following corollary is a generalization of [17, Theorem 3, Corollary 2] from $n = 2$ to the general case. It provides a uniform bound on the discriminants of special points where at most one coordinate corresponds to K_* .

Corollary 1. *Let $X \subset Y(1)^n$ be defined over a number field and let $V \in X^{\text{sp}}$. Suppose that at most one of the associated quadratic fields of $P(V)$ equal K_* . Then*

$$D(V) \leq c(\deg X, [\mathbb{F}_X : \mathbb{Q}]).$$

1.3. Results for degree non-degenerate varieties. In some cases Theorem 1 can be combined with additional arguments to produce a fully effective André–Oort-type statement. To state such a result we introduce the following terminology.

Definition 2 (Degree non-degenerate polynomials). A polynomial $F \in \mathbb{C}[x_1, \dots, x_n]$ is called *degree non-degenerate* (or dnd) if for every $i = 1, \dots, n$ we have either

$$\deg_{x_i} F = \deg F$$

or

$$\deg_{x_i} F = 0.$$

The zero locus of such F is called a dnd hypersurface.

We say that F is *hereditarily dnd* (or hdnd) if it is dnd, and if for each $i \neq j$ the restriction $F|_{x_i=x_j}$, viewed as a polynomial in $n - 1$ variables, is hdnd. The zero locus of such F is called an hdnd hypersurface.

Our key technical result concerning dnd hypersurfaces is as follows.

Corollary 3. *Let $X \subset Y(1)^n$ be a dnd hypersurface defined over a number field and let $P \in X^{\text{sp}}$. Then either $P_i = P_j$ for some $i \neq j$, or*

$$D(P) \leq c(\deg X, H(X), [\mathbb{F}_X : \mathbb{Q}]).$$

In the case that X is hdnd we derive from Corollary 3 the following effective André–Oort statement. We call a special variety *linear* if the modular relations $\Phi_N(x_i, x_j)$ involved in its definition are all of the form $x_i \equiv x_j$.

Corollary 4. *Let $X \subset Y(1)^n$ be an hdnd hypersurface defined over a number field and let $V \in X^{\text{sp}}$. Then V is linear and*

$$D(V) \leq c(\deg X, H(X), [\mathbb{F}_X : \mathbb{Q}]).$$

Corollary 4 provides a wealth of examples in arbitrary dimension and degree, where X^{sp} can be effectively computed – indeed an open dense set of examples in every dimension and degree. In particular, Corollary 4 applies when X is a linear hypersurface. More generally Corollary 3 gives another proof of the key technical ingredient [1, Lemma 3] in the recent effective solution of André–Oort for arbitrary linear subvarieties due to Bilu–Kühne. Our proof of Corollary 3 is based on an idea inspired by [1] involving asymptotics around the cusp, but modulo Theorem 1 the argument becomes quite simple.

Remark 5. It is instructive to consider an example where the conclusion of Corollary 4 fails. Perhaps the simplest such example is the modular polynomial Φ_2 given by

$$\begin{aligned} \Phi_2(x, y) := & x^3 + y^3 - x^2y^2 + 1488xy(x + y) - 162 \cdot 10^3(x^2 + y^2) \\ & + 40773375xy + 8748 \cdot 10^6(x + y) - 157464 \cdot 10^9. \end{aligned}$$

As guaranteed by Corollary 4, the 2-modular curve is not hdnd since we have the inequality $\deg_x \Phi_2 = 3 < 4 = \deg \Phi$.

1.4. Overview of the proof. We start by reviewing the proof of Theorem 1. The proof follows the general approach of Pila [25]. We assume for the purposes of this overview that the reader is familiar with this approach, as there are already several good surveys available (e.g., [27]) in addition to the original paper. There are three main sources of ineffectivity in the proof of [25], as follows:

- (1) The Siegel bound that is used to produce lower bounds $h(d) \gg |d|^{\frac{1}{2}-\varepsilon}$ for the class number $h(d)$ is ineffective.
- (2) The Pila–Wilkie bound for sets definable in $\mathbb{R}_{\text{an,exp}}$, which is used to provide a competing upper bound for Galois orbits, is ineffective.
- (3) The process by which one reduces the problem of controlling the maximal special varieties X^{sp} to the problem of controlling the maximal special points X^{spp} employs \mathfrak{o} -minimal finiteness properties and is ineffective.

In Section 1.4.1 we review the effectivization of the upper bound for special points in X^{spp} , corresponding roughly to items (1)–(2) above. In Section 1.4.2 we review the main idea used to effectively reduce the computation of X^{sp} to that of X^{spp} . Finally, in Section 1.4.3 we review the proof of the fully effective result for hdnd hypersurfaces.

1.4.1. Effectivizing the upper bound for $P \in X^{\text{spp}}$. To deal with the ineffectivity of the lower bound, we appeal to a result of Tatzawa [28] stating that the constant in Siegel’s bound can be made effective for all discriminants except those corresponding to orders in a single imaginary quadratic field K_* . The possibility of using this result was already mentioned in [25, Section 13.3], where it was noted that if one could effectivize the requisite Pila–Wilkie statement this would lead to a bound for the number of points in X^{spp} whose coordinates all correspond to fields other than K_* . Note however that we produce bounds for the discriminants of non-exceptional coordinates even when some coordinates do correspond to K_* . This extra generality does not follow straightforwardly since the height of such points is determined by the coordinates of largest discriminant, which may be associated to K_* .

The main result of [4] can be used to effectivize the Pila–Wilkie bounds needed in Pila’s proof with a significant caveat: the results apply to any compact subdomain of the fundamental domain, but cannot be used to obtain effective bounds uniformly over the entire (non-compact) fundamental domain. To overcome this difficulty we appeal to Duke’s equidistribution theorem [9, Theorem 1.i]. This implies that the Galois orbit of each coordinate P_i is equidistributed in $Y(1)$, and in particular that a large portion of it is contained in a fixed compact subset. Duke’s result is itself ineffective because of the same use of Siegel’s ineffective bound. It can however be made effective using Tatzawa’s theorem, with the same exception for discriminants of K_* . A sketch of the proof of this variant of Duke’s theorem is provided for completeness in Appendix A by E. Kowalski.

Let K_*^{tf} denote the field generated by all ring class fields associated to K_* . Our idea for dealing with the case that some coordinates of P correspond to K_* is to repeat Pila’s argument *relativized* over K_*^{tf} . More specifically, we use a result of Cohn [8, Proposition 8.3.12] showing that K_*^{tf} is almost disjoint from the ring class fields of non- K_* discriminants. We show that this implies that the number of conjugates of P under $\text{Gal}(\bar{\mathbb{Q}}/K_*^{\text{tf}})$ remains large, and that a large part of this orbit remains in a compact subset as predicted by the equidistribution theorem. One can therefore *fix* the values of all K_* -coordinates and apply Pila’s strategy to the resulting section, obtaining an upper bound for all discriminants of non- K_* coordinates. A crucial point here is that the effective Pila–Wilkie theorem of [4] gives constants that are independent of the chosen section.

1.4.2. Effectivizing the upper bound for X^{SP} . As in Pila’s approach, our idea is to inductively reduce the study of X^{SP} to the study of X^{SPP} by analyzing the possible types of the positive-dimensional maximal special subvarieties of X . The new ingredient in our approach is a transformation of this question, via a differential algebraic construction, to a completely algebraic question. The reduction is based on the following observation: while the special subvarieties are defined by algebraic conditions $\Phi_N(x_i, x_j) = 0$ of (a-priori) unbounded degrees, their preimages under the universal covering map $\pi : \mathbb{H}^n \rightarrow Y(1)^n$ satisfy essentially linear relations $\tau_i = g \cdot \tau_j$, where $g \in \text{GL}_2^+(\mathbb{Q})$. After describing the graph of π using a rational vector field encoding the differential equation satisfied by the j -function, the problem reduces to describing the points where the trajectory of a vector field satisfies such a linear condition identically.

The problem above is almost amenable to methods of differential equations. In particular, it may be studied using *multiplicity estimates* for the maximal order of vanishing of a polynomial on the trajectory of a vector field. More precisely, the problem can be reduced to a purely algebraic problem if we relax the condition $g \in \text{GL}_2^+(\mathbb{Q})$ to $g \in \text{GL}_2(\mathbb{C})$. Luckily the functional transcendence results of [25] can be used to show that the existence of any such algebraic dependence actually implies the existence of a dependence with rational coefficients.

We remark that the methods applied here yield explicit and fairly sharp bounds, and can be used to treat similar problems in far greater generality, including similar questions in the context of abelian varieties (treated by other methods in [6, Theorem 1.a]) and Shimura varieties more general than $Y(1)$. In a joint work in progress with Christopher Daw we apply these ideas to the study of optimal varieties in the sense of the Zilber–Pink conjecture for these various contexts.

1.4.3. The effective result for hdnd hypersurfaces. In light of Theorem 1, one can essentially reduce to the case of finding an upper bound for the $*$ -points on an hdnd hypersurface. In other words we may assume that all coordinates are associated to the fundamental discriminant Δ_* , say $\Delta_i = f_i^2 \Delta_*$. Assume without loss of generality that $|\Delta_1|$ is maximal among these.

We use a simple idea borrowed from [1, 17]: conjugating P_1 to the point closest to the cusp within its orbit, we see from the asymptotic expansion of the j -function around the cusp that $P_1 \simeq e^{\pi f_1 \sqrt{|\Delta_*|}}$. Any point P_i with $f_i < f_1$ is similarly majorated by $e^{\pi f_i \sqrt{|\Delta_*|}}$, while any point $P_i \neq P_1$ with the same discriminant ($f_i = f_1$) turns out to be majorated by $e^{\frac{1}{2}\pi f_1 \sqrt{|\Delta_*|}}$. Thus, if no point P_i equals P_1 , we see that P_1^n becomes asymptotically dominant over all other terms in our hdnd equation assuming that Δ_* is sufficiently large.

If we assume on the contrary that Δ_* is bounded by some (effective) constant, then Theorem 1 already becomes fully effective. This finishes the proof of Corollary 3. Corollary 4 is then proved using an induction over the dimension by intersecting with all possible diagonals $x_i = x_j$.

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2. Auxiliary results

2.1. Special points on $Y(1)$ and their discriminants. Recall that we are identifying $Y(1) \simeq \mathbb{A}_{\mathbb{Q}}^1$ by means of the j -invariant. A point $p \in Y(1)$ is special if and only if $p = j(\tau)$ for some quadratic number $\tau \in \mathbb{H}$. In this case we denote by $K(p) := \mathbb{Q}(\tau)$ the quadratic field generated by τ , by $\mathcal{O}_{\mathbb{Q}(\tau)}$ the maximal order of $\mathbb{Q}(\tau)$, and by \mathcal{O}_{τ} the order of $\mathbb{Q}(\tau)$ given by the endomorphism ring of $\mathbb{Z}[\tau]$. The *discriminant* of τ is defined to be

$$\Delta(\tau) := \text{disc } \mathcal{O}_{\tau} = f_{\tau}^2 \text{disc } \mathcal{O}_{\mathbb{Q}(\tau)},$$

where f_{τ} is the conductor of \mathcal{O}_{τ} with respect to $\mathcal{O}_{\mathbb{Q}(\tau)}$. We have $\mathcal{O}_{\tau} = \mathbb{Z} + f\mathcal{O}_{\mathbb{Q}(\tau)}$.

Fix an imaginary quadratic field K . For each $f \in \mathbb{N}$ denote by $\mathcal{O}_{K,f} := \mathbb{Z} + f\mathcal{O}_K$ the order of conductor f and by $K[f]/K$ the *ring class field* associated to $\mathcal{O}_{K,f}$. Then $K[f]/\mathbb{Q}$ is a Galois extension, and $\text{Gal}(K[f]/K) \simeq \text{Pic}(\mathcal{O}_{K,f})$.

Denote $d := \text{disc } \mathcal{O}_{K,f} = f^2 \text{disc } \mathcal{O}_K$ and define the *class number* $h(d) := \#\text{Pic}(\mathcal{O}_{K,f})$. If $\tau \in \mathbb{H}$ with $\Delta(\tau) = d$, then $K[f] = K(j(\tau))$. In particular, this is independent of the choice of τ . The number of different τ satisfying $\Delta(\tau) = d$, up to $\text{SL}_2(\mathbb{Z})$ -equivalence, is $h(d)$. Moreover, $\{j(\tau) : \Delta(\tau) = d\}$ forms a complete set of Galois conjugates in $K[f]/K$.

2.2. The Siegel–Tatuzawa theorem. Let χ_K denote the Dirichlet character associated to an imaginary quadratic field K and $L(s, \chi_K)$ the associated Dirichlet L-function. The Siegel–Tatuzawa [28] theorem implies that for any $\varepsilon > 0$ we have

$$(2.1) \quad L(1, \chi_K) \geq c(\varepsilon) \text{disc}(\mathcal{O}_K)^{-\varepsilon}$$

with the possible exception of a single imaginary quadratic field $K_*(\varepsilon)$.

Note that, consistent with our general convention, the constant $c(\varepsilon)$ in (2.1) is effective. Without this extra stipulation of effectivity the same statement holds for every imaginary quadratic field K by a classical result of Siegel.

By using the Dirichlet’s class number formula, inequality (2.1) implies that for any $K \neq K_*(\varepsilon)$ we have

$$\#\text{Pic}(\mathcal{O}_K) \geq c(\varepsilon) \text{disc}(\mathcal{O}_K)^{\frac{1}{2}-\varepsilon} \quad \text{for any } \varepsilon > 0.$$

This can be extended to an arbitrary imaginary quadratic order \mathcal{O} not contained in $K_*(\varepsilon)$,

$$(2.2) \quad \#\text{Pic}(\mathcal{O}) \geq c(\varepsilon) \text{disc}(\mathcal{O})^{\frac{1}{2}-\varepsilon} \quad \text{for any } \varepsilon > 0.$$

For a proof see [1, equation (17)]. We note that in this reference the authors give an explicit constant with $\varepsilon = \frac{1}{12}$, but it is clear that the proof extends for any $\varepsilon > 0$.

Finally, for definiteness of our notation we set $\varepsilon_* = 0.01$ and $K_* = K(\varepsilon_*)$.

2.3. Galois group action for the intersection of two ring class fields. Following [18], for an imaginary quadratic field K we denote by $K^{\text{tf}} := \bigcup_{f \in \mathbb{N}} K[f]$ the union of the ring class fields associated to all orders in K (the notation signifies the notion of a *transfer field*). Write $L = K \cdot K_*^{\text{tf}}$. Then Cohn [8, Proposition 8.3.12] proves that the Galois group $\text{Gal}(K^{\text{tf}} \cap L/K)$ is annihilated by 2 (i.e. has exponent at most 2). The same result holds for any two quadratic fields, but we require it only with K_* . We remark that in [18] Kühne proves a similar result for the intersection of r different ring class fields, with the exponent 2 replaced by 2^{r+1} .

The following lemma will play a key role in our argument.

Lemma 6. *Let \mathcal{O} be an order of $K \neq K_*$ and set $L = K \cdot K_*^{\text{tf}}$. Then*

$$[K[\mathcal{O}] \cdot L : L] \geq c(\varepsilon_*) \text{disc}(\mathcal{O})^{\frac{1}{2}-\varepsilon_*}.$$

Proof. We follow some arguments of [18]. Consider the following diagram of abelian field extensions:

$$\begin{array}{ccccc} & & K[\mathcal{O}] \cdot L & & \\ & \swarrow & & \searrow & \\ K[\mathcal{O}] & & & & L \\ & \searrow & & \swarrow & \\ & & K[\mathcal{O}] \cap L & & \\ & & | & & \\ & & K & & \end{array}$$

By [19, Theorem VI.1.12] we have

$$[K[\mathcal{O}] \cdot L : L] = [K[\mathcal{O}] : K[\mathcal{O}] \cap L].$$

Since $\text{Gal}(K[\mathcal{O}]/K) \simeq \text{Pic}(\mathcal{O})$, we have by (2.2) the estimate

$$\#\text{Gal}(K[\mathcal{O}]/K) \geq c(\varepsilon_*) d^{\frac{1}{2}-\varepsilon_*}, \quad d := \text{disc}(\mathcal{O}).$$

By the result of Cohn mentioned above, $g \rightarrow 2g$ induces a group homomorphism

$$\text{Gal}(K[\mathcal{O}]/K) \rightarrow \text{Gal}(K[\mathcal{O}]/K[\mathcal{O}] \cap L).$$

Our claim will thus follow once we show that the kernel, i.e. the 2-torsion subgroup $\text{Pic}(\mathcal{O})[2]$ of $\text{Gal}(K[\mathcal{O}]/K) \simeq \text{Pic}(\mathcal{O})$, has size at most $c(\varepsilon_*) \text{disc}(\mathcal{O})^{\varepsilon_*}$.

In [29, Proposition 6.3] it is proved that

$$\dim_{\mathbb{F}_2} \text{Pic}(\mathcal{O})[2] \leq 1 + 2\omega(d),$$

where $\omega(d)$ denotes the number of prime divisors of d . Therefore we indeed have

$$\#\text{Pic}(\mathcal{O})[2] = 2^{\dim_{\mathbb{F}_2} \text{Pic}(\mathcal{O})[2]} \leq 2^{1+2\omega(d)} \leq c(\varepsilon_*)d^{\varepsilon_*},$$

where we used the elementary estimate $\omega(d) \leq c(\varepsilon_*) + \varepsilon_* \log_2 d$. \square

2.4. Equidistribution of CM-points. Let

$$\mathcal{F} := \left\{ \tau \in \mathbb{H} : -\frac{1}{2} \leq \text{Re } \tau < \frac{1}{2} \text{ and } |\tau| > 1 \right\} \cup \{ \text{Re } \tau \leq 0 \text{ and } |\tau| = 1 \}$$

denote the standard fundamental domain for the $\text{SL}_2(\mathbb{Z})$ -action on \mathbb{H} . We equip \mathbb{H} with the invariant measure $d\mu(x + iy) = \frac{3}{\pi} dx \frac{dy}{y^2}$ so $\mu(\mathcal{F}) = 1$. For each negative discriminant d denote by Λ_d the set of all $\tau \in \mathcal{F}$ with $\Delta(\tau) = d$, so that $\#\Lambda_d = h(d)$. In [9] Duke proves the following equidistribution theorem for d a fundamental discriminant.

Theorem ([9, Theorem 1.i]). *Suppose $\Omega \subset \mathcal{F}$ is convex with a piecewise smooth boundary. Then for some $\delta > 0$ depending only on Ω ,*

$$\frac{\#(\Lambda_d \cap \Omega)}{\#\Lambda_d} = \mu(\Omega) + O(|d|^{-\delta}),$$

where the asymptotic constant depends only on Ω , though ineffectively.

Duke’s result was extended in [7] to all discriminants. The principal source of ineffectivity is the use of Siegel’s ineffective estimate (see Section 2.2). However, if one restricts to discriminants associated to orders not contained in K_* , one can replace this by the effective result of Siegel–Tatuzawa, leading to an effective equidistribution result. Explicitly, we will use this result in the following form. Here and below we set $\Omega_R := \mathcal{F} \cap \{ \frac{1}{2}\sqrt{3} < \text{Im } \tau < R \}$ for any $1 < R < \infty$.

Theorem 2. *Let $1 < R < \infty$. There is an effective constant $c(R, \varepsilon_*)$ such that for any discriminant $d > c(R, \varepsilon_*)$ not associated to an order contained in K_* ,*

$$\frac{\#(\Lambda_d \cap \Omega_R)}{\#\Lambda_d} \geq 1 - 2A, \quad A := \mu(\mathcal{F} \setminus \Omega_R).$$

In particular, Theorem 2 implies that as $R \rightarrow \infty$, the proportion of points of Λ_d belonging to Ω_R tends to 1 (for discriminants d satisfying the hypotheses). A sketch of the proof of Theorem 2 is provided in Appendix A.

2.5. Effective Pila–Wilkie for $Y(1)^n$. Let $H(\alpha)$ denote the absolute multiplicative height of the algebraic number α . For $Z \subset \mathbb{H}^n$ we denote

$$Z(k, H) := \{ (\tau_1, \dots, \tau_n) \in \mathbb{H}^n : [\mathbb{Q}(\tau_i) : \mathbb{Q}] \leq k, H(\tau_i) \leq H \text{ for } i = 1, \dots, n \}.$$

We denote by Z^{alg} the union of all connected positive-dimensional semialgebraic sets contained in Z and set $Z^{\text{trans}} := Z \setminus Z^{\text{alg}}$. The Pila–Wilkie theorem [26], in the variant established

in [24], implies that for any set Z which is definable in an o-minimal structure and any $\varepsilon > 0$ the estimate $\#Z^{\text{trans}}(k, H) \leq C(\varepsilon, k, Z) \cdot H^\varepsilon$ holds. Note however that the constant $C(\varepsilon, k, Z)$ is ineffective, and in the vast generality of the Pila–Wilkie theorem it is not clear in what terms one could hope to effectively express this constant.

In [4] we establish an effective version of the Pila–Wilkie theorem for sets defined using *Noetherian functions* restricted to compact domains. We also show that $j : \mathbb{H} \rightarrow \mathbb{C}$ is Noetherian (with effective parameters) when restricted to any compact subset. Let $\pi : \mathcal{F}^n \rightarrow Y(1)^n$ be given coordinatewise by the j -function. As a consequence of [4] we have the following.

Theorem 3. *Let $X \subset Y(1)^n$ be an algebraic variety and $R < \infty$. Set*

$$Z := \Omega_R^n \cap \pi^{-1}(X).$$

Then

$$\#Z^{\text{trans}}(2, H) \leq c(R, \varepsilon, \deg X) \cdot H^\varepsilon$$

with an effective constant $c(R, \varepsilon, \deg X)$.

It will be important later that the constant above does not depend on \mathbb{F}_X , nor on the heights of the coefficients of the equations defining X .

3. Estimates for positive-dimensional special subvarieties

Let $X \subset Y(1)^n$ be an algebraic variety. Our goal in this section is to construct a collection of subvarieties $\{X_\alpha \subset X\}$ which control X^{sp} in the following sense:

- (i) Each subvariety X_α is given up to permutation of coordinates by the form $\tilde{X}_\alpha \times V_\alpha$, where $\tilde{X}_\alpha \subset Y(1)^{k_\alpha}$ is an algebraic subvariety and $V_\alpha \subset Y(1)^{n-k_\alpha}$ is a strongly special variety.
- (ii) Each maximal special subvariety $V \in X^{\text{sp}}$ is of the form $\{P\} \times V_\alpha$ with $P \in \tilde{X}_\alpha^{\text{spp}}$, for some α .

It is clear that constructing such a collection reduces the computation of X^{sp} to the computation of $\tilde{X}_\alpha^{\text{spp}}$. The existence of finite collections of this type follows easily from the results of [25]. Our goal, more explicitly, is to obtain such collections effectively. Specifically, we have the following result.

Theorem 4. *Let $X \subset Y(1)^n$ be an algebraic variety. There exists a collection $\{X_\alpha \subset X\}$ as above, with the number of subvarieties X_α and their degrees bounded by $\text{poly}_n(\deg X)$.*

We denote by X^{wsp} the union of all positive-dimensional weakly-special subvarieties of X . The results of [25] imply that X^{wsp} is Zariski closed, but do not yield an effective estimate on its degree. We will reduce the proof of Theorem 4 to the following lemma, whose proof occupies the remainder of this section.

Lemma 7. *Let $X \subset Y(1)^n$ be an algebraic variety. Then*

$$\deg X^{\text{wsp}} \leq c(n)(\deg X)^{16n^2}.$$

Our proof of Lemma 7 is based on methods of differential equations. We begin by demonstrating a general result providing effective bounds for the degree of the collection of trajectories of a rational vector field that belong to a given algebraic variety. We then describe the graph of $j(\tau)$ (and its first two derivatives) as trajectories of a rational vector field. Finally, we apply this result in combination with the functional independence results of Pila [25] to obtain an effective description of X^{wsp} .

We now show how Lemma 7 implies Theorem 4.

Proof of Theorem 4. We proceed by induction on $\dim X$. Let $V \in X^{\text{sp}}$. Then certainly $V \subset X^{\text{wsp}}$, and replacing X by the irreducible component of X^{wsp} that contains V , we may assume without loss of generality that X is irreducible and $X = X^{\text{wsp}}$ (the crucial bound on the degree in this reduction follows from Lemma 7).

Every point of X is contained in a weakly-special subvariety. Every weakly-special subvariety $W \subset X$ satisfies one of the following conditions:

- (1) Up to reordering the coordinates, W is of the form $W' \times Y(1)$.
- (2) The equation $\Phi_N(x_i, x_j)$ vanishes identically on W , for some $N \in \mathbb{N}$ and $i \neq j$.

If $W \subset X$ satisfies condition (1), then X contains the trajectory of $\frac{\partial}{\partial x_i}$ (for some i) through every point of W . The set of points where this happens is easily seen to be Zariski closed (see, e.g., Lemma 8, although this case is completely elementary). The zeros of $\Phi_N(x_i, x_j)$ are certainly also Zariski closed. We thus see that X is the union of a countable collection of Zariski closed subsets, and since X is irreducible we conclude that in fact one of conditions (1)–(2) holds identically with W replaced by X .

Suppose that the second condition is satisfied identically on X . Up to reordering the coordinates, we may suppose that $(i, j) = (n-1, n)$. Then the projection to first $n-1$ coordinates gives a finite map $\pi : X \rightarrow X'$, where $X' = \pi(X)$, and the degree of this map is bounded above by $\deg X$. We apply the inductive hypothesis to X' to obtain a collection $\{X'_\beta\}$. We define the collection $\{X_\alpha\}$ to be the collection of irreducible components of $\pi^{-1}(X'_\beta)$ for all β . Since $\Phi_N(x_n, x_{n-1})$ vanishes identically on X , it follows that indeed each such component is of the required special form (where x_n belongs to the same “block” as x_{n-1}). Since $\pi(V) \subset X'$ is special, it follows by induction that it belongs to some X'_β , and consequently V belongs to some X_α as required.

If X is of the form $X = X' \times Y(1)$ up to reordering the coordinates, then the claim follows by a similar (but simpler) induction over dimension for X' . \square

3.1. The collection of trajectories contained in an algebraic variety. Consider now $M = \mathbb{C}^m \setminus \Sigma$, where Σ is an algebraic hypersurface, and let ξ be a vector field whose coefficients are regular functions on M . Let W be a subvariety of M . Denote by $\mathcal{R}_\xi W$ the union of all trajectories of ξ that are contained in W . We have the following.

Lemma 8. *In the notation above, $\mathcal{R}_\xi W \subset M$ is an algebraic subvariety. Its Zariski closure in \mathbb{C}^m has degree at most $c(\xi) \deg(W)^{m^2}$.*

Proof. Replacing ξ by $f^N \xi$, where $\Sigma = (f)$, we may suppose that ξ is a polynomial vector field (note this scalar multiplication does not affect the trajectory structure of ξ , only the time parametrization). Let $\{F_\alpha\}$ be a collection of polynomial equations of degrees at

most $\deg W$ which define W set-theoretically. The condition $p \in \mathcal{R}_\xi W$ is equivalent to the condition the derivatives $\xi^k F_\alpha$ vanish for every $k \in \mathbb{N}$ and every α .

It is possible to give an effective upper bound for the maximal order of vanishing of a polynomial along the trajectory of a polynomial vector field (assuming that the polynomial does not vanish identically on the trajectory). Such results are known as *multiplicity estimates*, see, e.g., [3, 11, 23]. The sharpest such estimate presently known is as follows.

Theorem ([3, Corolary 1]). *Let p be a non-singular point of ξ and let F be a polynomial of degree d in m variables. Then the multiplicity of the zero of F when restricted to the trajectory of ξ through p , assuming it is finite, does not exceed*

$$\mu = 2^{m+1}(d + (m - 1)(\delta - 1))^m,$$

where δ is the degree of the vector field ξ .

Applying this to F_α , we see that the vanishing of $\xi^k F_\alpha$ for every k is equivalent to the vanishing of the $k = 1, \dots, \mu$ derivatives, where $\mu = c(\xi)d^m$ for $d = \deg W$. We thus have a system of polynomials of degrees at most $c(\xi)d^m$ which define $\mathcal{R}_\xi W$ set-theoretically. From this it is straightforward to deduce using the Bezout theorem that the degree of the Zariski closure of $\mathcal{R}_\xi W$ in \mathbb{C}^m is bounded as claimed. \square

3.2. The j -function as a trajectory of a rational vector field. We recall that the Schwartzian operator is defined by

$$S(f) = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

We introduce the differential operator

$$\chi(f) = S(f) + R(f)(f')^2, \quad R(f) = \frac{f^2 - 1968f + 2654208}{2f^2(f - 1728)^2},$$

which is a third-order algebraic differential operator vanishing on Klein's j -invariant j , see [20, p. 20]. As observed in [10] it is easy to check that the solutions of $\chi(f) = 0$ are exactly the functions of the form $j_g(\tau) := j(g \cdot \tau)$, where $g \in \mathrm{PGL}_2(\mathbb{C})$ acts on \mathbb{C} in the standard manner.

The differential equation above may be written in the form $f''' = A(f, f', f'')$, where A is a rational function. More explicitly, consider the ambient space $M := \mathbb{C} \times \mathbb{C}^3 \setminus \Sigma$ with coordinates $(\tau, y, \dot{y}, \ddot{y})$ where Σ consists of the zero loci of $y, y - 1728$ and \dot{y} . On this space the vector field

$$\xi := \frac{\partial}{\partial \tau} + \dot{y} \frac{\partial}{\partial y} + \ddot{y} \frac{\partial}{\partial \dot{y}} + A(y, \dot{y}, \ddot{y}) \frac{\partial}{\partial \ddot{y}}$$

encodes the differential equation above, in the sense that any trajectory is given by the graph of a function $j_g(\tau)$ and its first two derivatives. We remark that 0 and 1728 play a special role as the critical values of the j -function.

3.3. Description of X^{wsp} using vector fields. We will use the following characterization of the weakly-special varieties in terms of the j -function [25, Definition 6.7] (cf. (1.1)). Let $S_0 \cup \dots \cup S_w$ be a partition of $\{1, \dots, n\}$ with S_0 only permitted to be empty. Let $t_i \in \mathbb{H}$

for $i \in S_0$. Let $s_i \in S_i$ be minimal for $i > 0$ and for each $s_i \neq j \in S_i$ choose an element $g_{ij} \in \mathrm{GL}_2^+(\mathbb{Q})$. Then V is a w -dimensional weakly-special variety if and only if it is the image under j of the set

$$(3.1) \quad \{(\tau_1, \dots, \tau_n) \in \mathbb{H}^n : \tau_i = t_i, i \in S_0, y_j = g_{i,j} \cdot y_{s_i}, s_i \neq j \in S_i, i = 1, \dots, w\}.$$

A special variety is obtained exactly when each t_i for $i \in S_0$ is quadratic.

Let $S \subset \{1, \dots, n\}$ of size q . We consider the ambient space

$$\Omega = \mathbb{C}_\tau \times \Omega_1 \cdots \times \Omega_n,$$

where for $i \in S$ we set $\Omega_i = \mathbb{C}^3 \setminus \Sigma$ with Σ as in Section 3.2; and for $i \notin S$ we set $\Omega_i = \mathbb{C}$. We use the coordinates $\tau, (y_i)_{1 \leq i \leq n}$ and $(\dot{y}, \ddot{y})_{i \in S}$. On this space we consider the vector field ξ given on the Ω_i with $i \in S$ factors as in Section 3.2, and on remaining Ω_i factors as zero.

The trajectories of ξ are given by function of the following form. For each $i \in S$ fix some $g_i \in \mathrm{PGL}_2(\mathbb{C})$ and for each $i \notin S$ fix some $p_i \in \mathbb{C}$. Then the corresponding solution is the function $\tau \rightarrow (\tau, \gamma(\tau))$, where γ is given in the $i \in S$ coordinates by $j(g_i \cdot \tau)$ and in the $i \notin S$ coordinates by the constant p_i .

Let $X \subset Y(1)^n$ which we identify with \mathbb{C}^n with coordinates y_1, \dots, y_n using the j -function. Let $W = \pi^{-1}(X)$ where $\pi : \Omega \rightarrow \mathbb{C}^n$ is the projection map. We claim that the Zariski closure of $\pi(\mathcal{R}_\xi W)$ is a subset of X^{wsp} . Since X^{wsp} is Zariski closed it is enough to prove that $\pi(\mathcal{R}_\xi W) \subset X^{\mathrm{wsp}}$.

Let $P \in \pi(\mathcal{R}_\xi W)$ and let γ be the trajectory contained in W with $P \in \pi(\gamma)$. This means that V contains a curve given by $y_i = j(g_i \cdot \tau)$ in the $i \in S$ coordinate and by a constant in the remaining coordinates (for τ in some open neighborhood where the expressions above are all defined). Therefore the preimage of V in \mathbb{H}^n (by coordinate-wise application of j) contains an algebraic curve given by $\tau_i = g_i^{-1} g_j \tau_j$ for $i, j \in S$ and by a constant on the remaining coordinates. By a result of Pila [25, Theorem 6.8] any such algebraic curve belongs to the pre-image of a weakly-special variety contained in V . Thus $P \in X^{\mathrm{wsp}}$ as claimed.

We now claim that the union of the above constructed $\pi(\mathcal{R}_\xi W)$ over every choice of S contains X^{wsp} . Indeed, let $P \in X^{\mathrm{wsp}}$. Then X contains some positive-dimensional weakly-special variety V containing p . Let S be given by the block S_1 in the definition of V as in (3.1). Choose $g_{s_1} = \mathrm{id}$ and $g_j = g_{1,j}$ for $s_1 \neq j \in S_1$. For $i \notin S$ choose $p_i = P_i$. Then the trajectory γ corresponding to this data is contained in W by definition and its projection $\pi(\gamma)$ passes through P . Thus $P \in \pi(\mathcal{R}_\xi W)$. The proof is now concluded by application of Lemma 8 for each choice of S .

4. Proofs of the main results

In this section we give the proofs of Theorem 1 and consequently of Corollaries 1, 3 and 4.

4.1. Proof of Theorem 1. Using Theorem 4, we already have the required estimate for $\deg V$, and the general problem is reduced to producing the requisite upper bounds for $D_*(P)$ and $D(P)$, where $P \in X^{\mathrm{sp}}$. We fix such P and denote $K_i = K_i(P)$, $\Delta_i = \Delta_i(P)$ and $\mathcal{O}_i = \mathcal{O}_i(P_i)$.

We argue first assuming $\mathbb{F}_X = \mathbb{Q}$, indicating the very minor changes needed for the general case at the end. Denote by $O_P := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \cdot P$ the set of all Galois conjugates of P . By our assumption, $O_P \subset X$. To simplify our notations, we reorder the coordinates so that $K_1, \dots, K_m \neq K_*$ and $K_{m+1}, \dots, K_n = K_*$. Choose R so that the $\mu(\Omega_R) \geq 1 - \frac{1}{4m}$. We note that the Galois action on O_P factors as a transitive action on Λ_{Δ_i} when restricted to the i -th coordinate. Theorem 2 for each $i = 1, \dots, m$ thus implies that if $\Delta_i(P) > c(R, \varepsilon_*)$, then

$$(4.1) \quad \#(O_P \cap \{P_i \in j(\Omega_R)\}) \geq \left(1 - \frac{1}{2m}\right) \cdot \#O_P.$$

Increasing R if necessary so that Ω_R also contains every point $\tau \in \mathbb{H}$ of discriminant smaller than the (effective) constant $c(R, \varepsilon_*)$ above, we can in fact assume that (4.1) holds without the restriction on $\Delta_i(P)$. We set $\Omega = \Omega_R^m$ for this R and let $\pi : \mathcal{F}^m \rightarrow Y(1)^m$ be given coordinatewise by the j -function.

Taking intersection of (4.1) for $j = 1, \dots, m$, we see that

$$\#(O_P \cap [\pi(\Omega) \times Y(1)^{n-m}]) \geq \frac{1}{2} \cdot \#O_P.$$

By an averaging argument over the fibers of the projection to $Y(1)^{n-m}$, we see that there exists a point $Q = (Q_1, \dots, Q_{n-m}) \in K_*^{\text{tf}}$ such that, with $Y_Q := Y(1)^m \times \{Q\}$, we have

$$O_P \cap Y_Q \neq \emptyset$$

and

$$\#(O_P \cap [\pi(\Omega) \times \{Q\}]) \geq \frac{1}{2} \cdot \#(O_P \cap Y_Q).$$

For each $i = 1, \dots, m$ let $L_i := K_i \cdot K_*^{\text{tf}}$. The Galois group $\text{Gal}(K_i[\mathcal{O}_i] \cdot L_i/L_i)$ fixes Q and hence acts on the set $O_P \cap Y_Q$. Moreover, its action is faithful on the i -th coordinate since $K_i[\mathcal{O}_i]$ is generated over K_i by P_i . It follows that

$$(4.2) \quad \begin{aligned} \#(O_P \cap [\pi(\Omega) \times \{Q\}]) &\geq \frac{1}{2} \cdot \#(O_P \cap Y_Q) \\ &\geq \frac{1}{2} \cdot \#\text{Gal}(K_i[\mathcal{O}_i] \cdot L_i/L_i) \geq c(\varepsilon_*) |\Delta_i|^{\frac{1}{2} - \varepsilon_*}, \end{aligned}$$

where the final inequality follows from Lemma 6.

Let Δ denote the maximal discriminant among $\Delta_1, \dots, \Delta_m$. Set $X_Q := X \cap Y_Q$, which we view as a subvariety of $Y(1)^m$ in the obvious way, and let $Z := \pi^{-1}(X_Q)$. Each point of $O_P \cap Y_Q$ corresponds to a point of X_Q , and moreover since these points are Galois conjugate to P , none of them are contained in a special subvariety of positive dimension in X_Q (since this would also be a special subvariety of X). According to [25, Theorem 6.8], the π -preimage of each such point belongs to Z^{trans} . According to the proof of [25, Proposition 5.7] they each have height at most $\text{const} \cdot \Delta$. Combined with (4.2) we conclude that

$$\#(Z \cap \Omega)^{\text{trans}}(2, \text{const} \cdot |\Delta|) \geq c(\varepsilon_*) |\Delta|^{\frac{1}{2} - \varepsilon_*}.$$

On the other hand, Theorem 3 implies that

$$\#(Z \cap \Omega)^{\text{trans}}(2, \text{const} \cdot |\Delta|) \leq c(\varepsilon, \deg X) |\Delta|^\varepsilon.$$

For say $\varepsilon = \frac{1}{3}$ these two competing estimates imply that $|\Delta| < c(\deg X)$ as claimed.

If we allow our constants to depend on Δ_* , then both Lemma 6 and Theorem 2 become effective without any restriction on the discriminant, and the proof above yields an estimate on Δ_i with no restriction on K_i .

If \mathbb{F}_X is an arbitrary (say normal) number field, then we replace X by the union of its $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ -conjugates, which is a variety defined over \mathbb{Q} and of degree $[\mathbb{F}_X : \mathbb{Q}] \cdot \deg X$. The claim now follows from what was already proved (since if P is not contained in a special variety of positive dimension in X , then the same is true for each of its conjugates).

4.2. Proof of Corollary 1. The proof is similar to that of [17, Corollary 4]. We reduce to the case $\mathbb{F}_X = \mathbb{Q}$ as above. Further, using Theorem 4, we immediately reduce from the case of general V to the case $V = \{P\}$ for some $P \in X^{\text{spp}}$.

We first prove that the number of points P satisfying the conditions of the corollary is bounded by $c(\deg X, [\mathbb{F}_X : \mathbb{Q}])$. To see this, we reduce the computation of X^{spp} to the computation of $*$ -points in X_α^{spp} for the collection of special sections $\{X_\alpha\}$ described following the proof of Theorem 1. It is enough to show that the number of points satisfying the conditions of the corollary in each of these special sections is bounded by such a constant.

If the ambient dimension of X_α is greater than one, then none of the $*$ -points in X_α correspond to the points considered in the corollary. If the ambient dimension of X_α is zero, then it corresponds to a single point. Finally, suppose the ambient dimension is one. If X_α is zero-dimensional, then the total number of points in X_α is bounded by $\deg X$. If $X_\alpha = Y(1)$, then $X_\alpha^{\text{spp}} = \emptyset$.

Now let P be as in the conditions of the corollary. From the above we conclude that for $i = 1, \dots, n$, the number of Galois conjugates of P_i should be bounded by $c(\deg X, [\mathbb{F}_X : \mathbb{Q}])$. In other words, $h(\Delta_i) < c(\deg X, [\mathbb{F}_X : \mathbb{Q}])$. The effective bound on $|\Delta_i|$ now follows from the deep effective estimate

$$h(d) \geq c(\varepsilon)(\log |d|)^{1-\varepsilon} \quad \text{for all } \varepsilon > 0$$

due to Goldfeld–Gross–Zagier [12, 13].

4.3. Proof of Corollary 3. Note that we may assume that the polynomial defining X depends on all variables, since otherwise X^{spp} is empty. Suppose that we are in the case that the P_1, \dots, P_n are pairwise distinct. Suppose without loss of generality that Δ_1 is maximal among Δ_i . If $K_1 \neq K_*$, then Theorem 1 gives an upper bound for $|\Delta_1|$ and we are done. Therefore assume that $K_1 = K_*$ and write $\Delta_1 = f^2 \Delta_*$. Following [1], we note that for every discriminant Δ ,

$$\tau_\Delta := \frac{-b_\Delta + i\sqrt{|\Delta|}}{2}, \quad \{0, 1\} \ni b_\Delta \equiv \Delta \pmod{2},$$

is a CM-period of discriminant Δ , and moreover every other CM-period of discriminant Δ in \mathcal{F} has imaginary part at most $\frac{\text{Im } \tau_\Delta}{2}$. Recall the following estimate from [2, Lemma 1]:

$$||j(\tau)| - e^{2\pi \text{Im } \tau}| < 2079 \quad \text{for every } \tau \in \bar{\mathcal{F}}.$$

After applying a Galois conjugation, we may assume that $P_1 = j(\tau_{\Delta_1})$. From the above we deduce that

$$\begin{aligned} |P_1| &\geq \text{const} \cdot e^{\pi f |\Delta_*|^{\frac{1}{2}}}, \\ |P_i| &\leq c(\deg X, [\mathcal{F}_X : \mathbb{Q}]) \cdot e^{\pi(f-1)|\Delta_*|^{\frac{1}{2}}} \quad \text{for } i = 2, \dots, n. \end{aligned}$$

Indeed, for $K_i \neq K_*$ the estimate follows from Theorem 1. For $K_i = K_*$, if $\Delta_i = \Delta_1$, then since these points are distinct we have $\text{Im } \tau_i \leq \frac{1}{2} \text{Im } \tau_\Delta$; and otherwise $\Delta_i \leq (f-1)^2 \Delta_*$. In particular, we deduce that

$$\frac{|P_1|}{|P_i|} \geq c(\deg X, [\mathcal{F}_X : \mathbb{Q}]) \cdot e^{\pi|\Delta_*|^{\frac{1}{2}}} \quad \text{for } i = 2, \dots, n.$$

Let c_0 denote the (non-zero) absolute value of the coefficient of x_1^d in the polynomial F defining X , and let c_1 denote the maximum of the absolute values of all remaining coefficients. It is clear that for $|\Delta_*|$ larger than some effective function of c_0, c_1 and d , the term P_1^d becomes dominant and the equation $F(P) = 0$ cannot be satisfied. On the other hand if $\Delta_* < c(\deg X, H(X), [\mathbb{F}_X : \mathbb{Q}])$ for such an effective function, then Theorem 1 is already effective for K_* as well, thus concluding the proof.

4.4. Proof of Corollary 4. We first show that any special subvariety $V \subset X$ is linear. Suppose V is positive-dimensional. If V is not linear, then it contains a special curve which is not linear, so we may assume that V is a curve. If a coordinate function x_i is constant on V , then we may reduce to corresponding section of X (note that X remains hdnd upon taking such a section). Similarly, if $x_i - x_j \equiv 0$ on V , then we may pass to the diagonal $x_i = x_j$. After making these reductions, we may assume that each x_i is non-constant on V and each $x_i - x_j$ vanishes at a finite number of points on V .

Let K be a quadratic field. Since V is a special curve with no constant coordinates, it consists of a single block S_1 , and it follows that V contains a point P , where every coordinate x_j is a special point associated to K . Moreover, by the finiteness of the zeros of $x_i - x_j$ on V , we see that excluding a finite number of fields, the coordinates of P are distinct. The proof of Corollary 3 shows that a dnd hypersurface cannot contain points of this type for a sufficiently large fundamental discriminant, and we thus obtain a contradiction for an appropriate choice of K .

We now prove the estimate on $D(V)$. Since V is linear, it is given up to reordering the coordinates by $\{P\} \times V'$, where $P \in Y(1)^m$ and $V' \subset Y(1)^{n-m}$ is a strongly special linear variety. In particular, V' contains some universally fixed special point Q , say $Q := (j(i), \dots, j(i))$. Thus $P \in \tilde{X}^{\text{spp}}$ where \tilde{X} is the section obtained by setting $(x_{n-m+1}, \dots, x_n) = Q$. Furthermore, $\tilde{X} \subset Y(1)^m$ is again given by an hdnd polynomial.

The above reduces the problem of estimating $D(V)$ to the problem of estimating $D(P)$ for $P \in \tilde{X}^{\text{spp}}$. If the coordinates of P are pairwise distinct, this follows from Corollary 3. If $P_i = P_j$, then it follows by induction over dimension reducing to the diagonal $x_i = x_j$ (noting again that X remains the hdnd on the diagonal by definition).

A. Duke's theorem avoiding Tatzawa fields

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Duke's theorem [9, Theorem 1] for CM points states that as $d \rightarrow +\infty$, the CM points Λ_d of discriminant $-d$ become equidistributed in the modular curve $Y(1)$, in the quantitative form

$$\frac{|\Lambda_d \cap \Omega|}{|\Lambda_d|} = \mu(\Omega) + O(|d|^{-\delta})$$

for some $\delta > 0$ depending only on the domain $\Omega \subset Y(1)$, which is assumed to be convex with piecewise smooth boundary. Here μ is the hyperbolic area measure normalized so that $\mu(Y(1)) = 1$. The constant $\delta > 0$ and the implied constant depend only on Ω ; the former is effective (and explicit), but the latter is not, due to the use of Siegel’s lower bound for class numbers in the proof.

Tatuzawa [28, Theorem 1] has given a form of Siegel’s bound that, for a given $\varepsilon > 0$, gives an effective (explicit) constant $c(\varepsilon) > 0$ (in fact, proportional to ε) such that

$$|\Lambda_d| > c(\varepsilon)|d|^{\frac{1}{2}-\varepsilon}$$

for all fundamental discriminants d with at most one exception, which may depend on ε . We write d_ε^* for this exception.

Let F be the standard fundamental domain of $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$. We view Λ_d as a subset of F . We consider the regions

$$\Omega_R = \left\{ z \in F : \frac{\sqrt{3}}{2} < \mathrm{Im}(z) < R \right\}.$$

Note a minor issue even in a direct application of Duke’s theorem, without consideration of effectivity: this region is *not* convex in the hyperbolic sense.

The following is however a simple deduction from the principles of the proof, combined with Tatuzawa’s theorem.

Proposition 9. *Let $0 < \varepsilon < \frac{1}{16}$ be fixed. Let d_ε^* be the corresponding exceptional discriminant. Let $m \geq 1$ be an integer. There exists an effective constant $c(m, \varepsilon) > 0$ such that for $d > c(m, \varepsilon)$ and for $d \neq d_\varepsilon^*$, we have*

$$\frac{1}{|\Lambda_d|} |\{z \in \Lambda_d : z \in \Omega_{8m}\}| \geq 1 - \frac{1}{4m}.$$

Remark 10. (1) The choice $R = 8m$ is to have the hyperbolic area of Ω_R equal to $1 - \frac{1}{R}$.

(2) We sketch the proof with fundamental discriminants, but proceeding as in [7], it extends to all discriminants.

Sketch of proof. Recall that

$$|\Lambda_d| = \frac{1}{2\pi} w_d |d|^{\frac{1}{2}} L(1, \chi_d),$$

where w_d is the number of roots of unity in the quadratic field with discriminant d , by Dirichlet’s Class Number Formula (see, e.g., [16, (22.59)]).

Let $R = 8m$. Consider a smooth compactly supported function $\psi: Y(1) \rightarrow \mathbb{C}$ which satisfies $0 \leq \psi \leq 1$, is equal to 1 for $z \in F$ with imaginary part $\leq \frac{R}{2}$ (hence on $\Omega_{R/2}$) and vanishes for z with imaginary part $\geq R$, and such that the partial derivatives of ψ are bounded (by constants depending only on the order of the derivative).

Note that the choice of such a function depends on m . Observe that

$$\frac{1}{|\Lambda_d|} |\{z \in \Lambda_d : z \in \Omega_R\}| \geq \frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} \psi(z).$$

Now by the spectral decomposition in $L^2(Y(1))$ (see, e.g., [15, Theorem 4.7, Theorem 7.3]), we have

$$\psi(z) = \mu_\psi + \int_0^{+\infty} \langle \psi, E(\cdot, \frac{1}{2} + it) \rangle E(z, \frac{1}{2} + it) dt + \sum_j \langle \psi, u_j \rangle u_j(z),$$

where

$$\mu_\psi = \int_F \psi(z) d\mu(z),$$

the functions $E(z, s)$ are the Eisenstein series for $\mathrm{SL}_2(\mathbb{Z})$ and (u_j) runs over an orthonormal basis of the cuspidal subspace of $L^2(Y(1))$, which we may assume consists of Hecke eigenforms.

We have

$$\mu_\psi \geq \mu(\Omega_{R/2}) = 1 - \frac{2}{R},$$

hence

$$\frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} \psi(z) \geq 1 - \frac{1}{4m} + \mathcal{R},$$

where

$$\mathcal{R} = \int_0^{+\infty} \langle \psi, E(\cdot, \frac{1}{2} + it) \rangle \frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} E(z, \frac{1}{2} + it) dt + \sum_j \langle \psi, u_j \rangle \frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} u_j(z).$$

A classical formula (see references in [9, p. 88] or [16, (22.45)]) computes

$$\frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} E(z, \frac{1}{2} + it) = w_d \frac{\zeta(\frac{1}{2} + it) L(\chi_d, \frac{1}{2} + it)}{|d|^{\frac{1}{4} - \frac{it}{2}} L(1, \chi_d)},$$

where w_d is the number of roots of unity in the quadratic field. Combining an old result of Weyl for $\zeta(s)$ and a result of Heath-Brown [14], whose proof is effective, yields upper bounds

$$\zeta(\frac{1}{2} + it) \ll (1 + |t|)^{\frac{1}{6}}, \quad L(\chi_d, \frac{1}{2} + it) \ll |d|^{\frac{1}{6} + \eta} (1 + |t|)^{\frac{1}{6} + \eta}$$

for any $\eta > 0$, where the implied constant is effective and depends only on η .

On the other hand, using the Waldspurger formula (see the discussion of Michel and Venkatesh [21, (2.5)]), one finds a formula of the (similar) type

$$\left| \frac{1}{|\Lambda_d|} \sum_{z \in \Lambda_d} u_j(z) \right|^2 = \alpha \frac{L(u_j, \frac{1}{2}) L(u_j \times \chi_d, \frac{1}{2})}{|d|^{\frac{1}{2}} L(1, \chi_d)^2}$$

(where α is a constant) in terms of central values of twisted L -functions. We use the subconvexity estimate of Blomer and Harcos [5, Theorem 2] (although we could use also that of Michel and Venkatesh [22], or indeed any subconvex bound that has polynomial control in terms of the eigenvalue of u_j would suffice, and there are many more versions): we have

$$L(u_j \times \chi_d, \frac{1}{2}) \ll |d|^{\frac{3}{8}} (1 + |t_j|)^3, \quad L(u_j, \frac{1}{2}) \ll (1 + |t_j|)^3,$$

where the implied constants are effective and $\frac{1}{4} + t_j^2$ is the Laplace eigenvalue of the cusp form u_j (we have $t_j \in \mathbb{R}$ since it is known that there are no eigenvalues $< \frac{1}{4}$ for $Y(1)$).

Using “integration by parts”, namely writing

$$\langle \psi, u_j \rangle = \frac{1}{(\frac{1}{4} + t_j^2)^A} \langle \psi, \Delta^A u_j \rangle = \frac{1}{(\frac{1}{4} + t_j^2)^A} \langle \Delta^A \psi, u_j \rangle,$$

we obtain for any $A \geq 1$ the bound

$$|\langle \psi, u_j \rangle| \leq \frac{1}{(\frac{1}{4} + t_j^2)^A} \|\Delta^A \psi\| \ll_A \frac{R^{2A+1}}{(1 + |t_j|)^{2A}}$$

(since $\Delta^A \psi(z)$ vanishes unless $\frac{R}{2} \leq \text{Im}(z) \leq R$, and the derivatives are bounded). Similarly, one gets

$$\langle \psi, E(\cdot, \frac{1}{2} + it) \rangle \ll \frac{R^{2A+1}}{(1 + |t|)^{2A}}$$

for any $A > 0$, where the implied constant depends on A and is effective.

Taking A fixed and large enough to make the integral and series converge absolutely (e.g., $A = 3$), we derive the lower bound

$$\frac{1}{|\Lambda_d|} |\{z \in \Lambda_d : z \in \Omega_{8m}\}| \geq 1 - \frac{2}{R} + O(R^{2A+1} |d|^{\frac{1}{2} - \frac{1}{16}} |\Lambda_d|^{-1}),$$

where the implied constant is effective, and hence for $d \neq d_\varepsilon^*$, we obtain

$$\frac{1}{|\Lambda_d|} |\{z \in \Lambda_d : z \in \Omega_R\}| \geq 1 - \frac{2}{R} + O(R^{2A+1} |d|^{\varepsilon - \frac{1}{16}}),$$

where the implied constant is effective. The result now follows. \square

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