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Variations on the Center Transversal Theorem

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Variations on the Center Transversal Theorem

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Abstract

We consider several variants of the Center Transversal Theorem and its boundary cases, the Centerpoint Theorem and the Ham-Sandwich Theorem. The Center Transversal Theorem states that given any $k$ mass distributions in $\mathbb{R}^d$, there is a $(k-1)$-flat $g$ with the property that every half-space which contains $g$ contains at least a $\frac{1}{d-k+2}$-fraction of each mass distribution. The statements for $k = 1$ and $k = d$ are the above mentioned Centerpoint Theorem and Ham-Sandwich Theorem, respectively.

The variants of the Center Transversal Theorem studied in this thesis can be grouped into two variations. For the first variation, instead of considering mass distributions in $\mathbb{R}^d$, we consider continuous assignments of mass distributions to all $n$-dimensional linear subspaces of $\mathbb{R}^d$. This could for example be the intersections of sufficiently smooth mass distributions with the considered subspaces. The main result of this first variation is that there always exists a subspace in which we can find a center transversal for the same number of masses as we could for mass distributions in the total space.

As a special case of this result, we also show that for any 3-colored line arrangement in $\mathbb{R}^3$, there exists an additional line $\ell$ such that exactly half of the lines of each color pass above $\ell$, answering a question by Luis Barba. For this result we give two proofs. The first proof uses
topological methods and generalizes to higher dimensions. The second proof uses only elementary tools from discrete geometry and allows for an efficient algorithm.

The second variation considers more complicated transversals consisting of several flats or half-flats. Here we study three different settings, one for the case of Centerpoints and two for the case of Ham-Sandwich cuts. For the first setting, we investigate the following generalization of the Centerpoint Theorem: given some mass distribution in $\mathbb{R}^d$ and values $\alpha_1, \ldots, \alpha_k$, we want to find $k$ points with the property that every half-space which contains $j$ of them contains an $\alpha_j$-fraction of the mass. We give bounds on the values of $\alpha_1, \ldots, \alpha_k$ that are possible in this setting and an algorithm computing such points for $d = k = 2$ when the mass distribution is a finite point set.

As for Ham-Sandwich cuts, the first extension that we study considers bisections with several hyperplanes. Here an arrangement of hyperplanes defines a partition of $\mathbb{R}^d$ into two parts according to the natural 2-coloring of the faces of the arrangement. A mass distribution is then bisected by the arrangement if exactly half of the mass is in either part of this partition. It was conjectured by Stefan Langerman that any $n d$ mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by an arrangement of $n$ hyperplanes. We answer this conjecture in the positive for the case $n = d = 2$ and give an algorithm to find two bisecting lines when the masses are finite point sets. We further give a positive answer to this conjecture for any $n$ and $d$ in a relaxed setting and a related result about Ham-Sandwich cuts after projective transformations.

The second extension of Ham-Sandwich cuts considers partitions with $k$-fans, that is, $k$ half-hyperplanes emanating from a common $(d - 2)$-dimensional apex. In this setting we extend a planar result by Báránny and Matoušek for 2-fans to general dimensions. We also reproduce a result by Makeev and extend it to partitions where different parts of the fans contain different fractions of the masses.
Zusammenfassung

Wir betrachten mehrere Varianten des Zentrums-Transversalen-Satzes und seiner Randfälle, des Zentrumspunkt-Satzes und des Schinken-Sandwich-Satzes. Der Zentrums-Transversalen-Satz besagt, dass es für beliebige $k$ Masseverteilungen in $\mathbb{R}^d$ eine $(k-1)$-Fläche $g$ gibt mit der Eigenschaft, dass jeder Halbraum, welcher $g$ enthält, auch mindestens einen Bruchteil von $\frac{1}{d-k+2}$ jeder Masseverteilung enthält. Die Aussagen für $k = 1$ respektive $k = d$ sind der oben erwähnte Zentrumspunkt-Satz, respektive der Schinken-Sandwich-Satz.

Die Erweiterungen des Zentrums-Transversalen-Satzes, welche in dieser Arbeit untersucht werden, können in zwei Variationen gruppiert werden. Für die erste Variation betrachten wir anstelle von Masseverteilungen in $\mathbb{R}^d$ stetige Zuweisungen von Masseverteilungen auf alle $n$-dimensionalen linearen Teilräume von $\mathbb{R}^d$. Dies können beispielsweise die Schnitte hinreichend glatter Masseverteilungen mit den betrachteten Teilräumen sein. Das Hauptresultat dieser ersten Variation ist, dass es immer einen Teilraum gibt, in welchem wir eine Zentrums-Transversale für die selbe Anzahl Masseverteilungen wie im gesamten Raum finden können.

Als Spezialfall dieses Resultates beantworten wir eine Frage von Luis Barba und zeigen, dass es für jedes 3-gefärbcte Geradenarrangement in $\mathbb{R}^3$ eine zusätzliche Gerade $\ell$ gibt, so dass genau die Häfle der Geraden
jeder Farbe über \( \ell \) liegen. Für dieses Resultat präsentieren wir zwei Beweise. Der erste Beweis verwendet topologische Methoden und lässt sich für höhere Dimensionen verallgemeinern. Der zweite Beweis benutzt lediglich grundlegende Hilfsmittel der diskreten Geometrie und ermöglicht einen effizienten Algorithmus.

Die zweite Variation betrachtet kompliziertere Transversalen bestehend aus mehreren Flächen oder Halbflächen. Hier untersuchen wir zwei verschiedene Varianten, eine für Zentrumspunkte und zwei für Schinken-Sandwich-Schnitte. In der ersten Variante studieren wir die folgende Verallgemeinerung des Zentrumspunkt-Satzes: gegeben eine Masseverteilung in \( \mathbb{R}^d \) und Werte \( \alpha_1, \ldots, \alpha_k \) ist es unser Ziel, \( k \) Punkte zu finden mit der Eigenschaft, dass jeder Halbraum, welcher dieser Punkte enthält, auch einen Bruchteil von \( \alpha_j \) der Masse enthält. Wir präsentieren Schranken für die möglichen Werte von \( \alpha_1, \ldots, \alpha_k \) und einen Algorithmus, welcher solche Punkte für \( d = k = 2 \) berechnet, wenn die Masseverteilung eine endliche Punktemenge ist.

Für Schinken-Sandwich-Schnitte handelt die erste Erweiterung, welche wir betrachten, von Halbierungen mit mehreren Hyperebenen. Hierbei definiert ein Hyperebenenarrangement eine Partition von \( \mathbb{R}^d \) in zwei Teile gemäß der natürlichen 2-Färbung der Gebiete des Arrangements. Eine Masseverteilung wird dann von einem Arrangement halbiert wenn sich in beiden Teilen dieser Partition genau die Hälfte der Masse befindet. Stefan Langerman hat die Vermutung aufgestellt, dass jegliche \( nd \) Masseverteilungen in \( \mathbb{R}^d \) gleichzeitig von einem Arrangement von \( n \) Hyperebenen halbiert werden können. Wir beweisen diese Vermutung im Fall \( n = d = 2 \) und entwickeln einen Algorithmus, welcher zwei halbierende Geraden berechnet, wenn die Masseverteilungen endliche Punktemengen sind. Weiter beweisen wir eine schwächere Variante der Vermutung für beliebige \( n \) und \( d \) und präsentieren ein verwandtes Resultat über Schinken-Sandwich-Schnitte nach projektiven Transformationen.

Die zweite Erweiterung von Schinken-Sandwich-Schnitten behandelt Aufteilungen mit \( k \)-Fächern, also \( k \) Halbhyperebenen, welche von einem
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The *Center Transversal Theorem* is a celebrated result from the 1990’s, discovered independently by Dol’nikov [37], as well as Zivaljević and Vrećica [95]. It interpolates between two of the most famous results in discrete geometry: the *Centerpoint Theorem* and the *Ham-Sandwich Theorem*. Let us first discuss these two fundamental results.

The Centerpoint Theorem, established by Rado in 1947 [80], can be regarded as a generalization of a *median* to higher dimensions. Recall that a median of a mass distribution $\mu$ in $\mathbb{R}$ is a point $q$ which has the following property: every half-line which contains $q$ contains at least half of the mass $\mu$. The Centerpoint Theorem now certifies that this idea can be carried to higher dimensions, replacing half-lines with half-spaces, and still guaranteeing that they all contain a large fraction of the mass.

**Theorem 1.1** (Centerpoint Theorem). Let $\mu$ be a mass distribution
in \( \mathbb{R}^d \). Then there exists a point \( q \in \mathbb{R}^d \) such that for every closed half-space \( H \) which contains \( q \) we have

\[
\mu(H) \geq \frac{\mu(\mathbb{R}^d)}{d + 1}.
\]

The Ham-Sandwich Theorem gets its name from the following illustration: imagine that you have a 3-dimensional sandwich consisting of bread, ham and cheese, which you want to share with your friend. The Ham-Sandwich Theorem, proven in its general version by Stone and Tukey in 1942 [89], tells you, that you can slice the sandwich with a single straight cut, such that afterwards both parts contain exactly half of each ingredient. This can even be done if the cheese is still in the fridge.¹ More generally, you can always fairly divide \( d \) ingredients in \( d \) dimensions:

**Theorem 1.2 (Ham-Sandwich Theorem).** Let \( \mu_1, \ldots, \mu_d \) be \( d \) mass distributions in \( \mathbb{R}^d \). Then there exists a hyperplane \( h \) such that

\[
\mu_i(h^+) \geq \frac{\mu(\mathbb{R}^d)}{2} \quad \text{and} \quad \mu_i(h^-) \geq \frac{\mu(\mathbb{R}^d)}{2} \quad \text{for} \quad i = 1, 2, \ldots, d,
\]

where \( h^+ \) and \( h^- \) are the two closed half-spaces defined by \( h \).

Note that we could phrase this differently: for every closed half-space \( H \) which contains \( h \) we have \( \mu_i(H) \geq \frac{\mu_i(\mathbb{R}^d)}{2} \) for \( i = 1, 2, \ldots, d \). In particular, for \( d = 1 \), the Centerpoint Theorem and the Ham-Sandwich Theorem are exactly the same statement, namely the existence of a median. In higher dimensions, the two theorems differ in three points: the number of masses, the dimension of the affine subspace which is contained in all relevant half-spaces, and the fraction of each mass in the relevant subspaces. For the Centerpoint Theorem the number of masses and the dimension of the subspace is constant, but the fraction depends on the dimension \( d \). For the Ham-Sandwich Theorem the fraction is constant, but the number of masses and the dimension of

¹This illustration is inspired by Herbert Edelsbrunner [40].
1.1. Definitions

the subspace depend on \(d\). The Center Transversal Theorem formalizes how these three points depend on each other.

**Theorem 1.3 (Center Transversal Theorem).** Let \(\mu_1, \ldots, \mu_k\) be \(k\) mass distributions in \(\mathbb{R}^d\), where \(k \leq d\). Then there is a \((k - 1)\)-dimensional affine subspace \(g\) such that for every closed half-space \(H\) which contains \(g\) we have

\[
\mu_i(H) \geq \frac{\mu_i(\mathbb{R}^d)}{d - k + 2} \quad \text{for } i = 1, 2, \ldots, k.
\]

We call such an affine subspace a \((k - 1, d)\)-center transversal. Setting \(k = 1\) and \(k = d\), we retrieve the Centerpoint Theorem and the Ham-Sandwich Theorem, respectively, as boundary cases. In this thesis, the three results above will appear in various guises in every chapter.

### 1.1 Definitions

Before delving into details, let us discuss the most important definitions.

A *hyperplane* in \(\mathbb{R}^d\) is a \((d - 1)\)-dimensional affine subspace of \(\mathbb{R}^d\), that is, it is a set \(\{x \in \mathbb{R}^d \mid ax = b\}\), for some (nonzero) vector \(a \in \mathbb{R}^d\) and some number \(b \in \mathbb{R}\). Any such \(a\) and \(b\) also define a *closed half-space* as the set \(\{x \in \mathbb{R}^d \mid ax \leq b\}\), and an *open half-space* as the set \(\{x \in \mathbb{R}^d \mid ax < b\}\). Note that a hyperplane also bounds the closed half-space \(\{x \in \mathbb{R}^d \mid ax \geq b\}\), which can also be written in the above form as \(\{x \in \mathbb{R}^d \mid -ax \leq -b\}\).

Generalizing the notion of a hyperplane, a *\(k\)-flat* is a \(k\)-dimensional affine subspace of \(\mathbb{R}^d\). In particular, a 0-flat is a point and a \(k\)-dimensional linear subspace is a \(k\)-flat which contains the origin. We say that a \(k\)-dimensional linear subspace \(h\) of \(\mathbb{R}^d\) is \(m\)-horizontal (for \(m \leq d\)), if it contains \(e_1, \ldots, e_m\), where \(e_i\) denotes the \(i\)'th unit vector of \(\mathbb{R}^d\).
The Stiefel manifold $V_k(\mathbb{R}^d)$ is the space of ordered orthonormal $k$-tuples of vectors in $\mathbb{R}^d$. It is a manifold of dimension $dk - \frac{1}{2}k(k + 1)$ and inherits a subspace topology from $\mathbb{R}^{d \times k}$.

Similarly, the Grassmann manifold $G_k(\mathbb{R}^d)$ is the space consisting of all $k$-dimensional linear subspaces of $\mathbb{R}^d$. The Grassmann manifold is a quotient space of the Stiefel manifold and inherits a topology from it. Note that the manifold $G_1(\mathbb{R}^d)$ is also called projective space.

We further denote the space of all $m$-horizontal, $k$-dimensional subspaces of $\mathbb{R}^d$ by $Hor^m_k(\mathbb{R}^d)$. The space $Hor^m_k(\mathbb{R}^d)$ also inherits a topology from the Stiefel manifold.

A flag $F$ in $\mathbb{R}^d$ is an increasing sequence of linear subspaces of the form $F : \{0\} = V_0 \subset V_1 \subset \cdots \subset V_k = \mathbb{R}^d$. To each flag we can assign a signature vector of dimensions of the subspaces. A flag with $\text{dim}V_i = i$ for all $i < k$ is called a $(k-1)$-flag. A flag is a complete flag if $\text{dim}V_i = i$ for all $i$ (and thus $k = d$). A flag manifold $\mathcal{F}$ is the set of all flags with the same signature vector. It is a quotient space of the Stiefel manifold and inherits a topology from it. We denote the complete flag manifold, that is, the manifold of complete flags, by $\tilde{V}_{d,d}$. More generally, we denote the manifold of $(k-1)$-flags by $\tilde{V}_{d,k-1}$.

A (d-dimensional) mass distribution $\mu$ on $\mathbb{R}^d$ is a measure on $\mathbb{R}^d$ such that all open subsets of $\mathbb{R}^d$ are measurable, $0 < \mu(\mathbb{R}^d) < \infty$ and $\mu(S) = 0$ for every lower-dimensional subset $S$ of $\mathbb{R}^d$. An intuitive example of a mass distribution is, for example, the volume of some full-dimensional geometric object of finite volume in $\mathbb{R}^d$.

For some topological space $X$ consisting of $k$-dimensional subspaces of $\mathbb{R}^d$ we define a mass assignment on $X$ as a continuous assignment $\mu : X \to M_k$, where $M_k$ denotes the space of all $k$-dimensional mass distributions, taken with the usual weak topology. We will only consider mass assignments on $G_k(\mathbb{R}^d)$ and $Hor^m_k(\mathbb{R}^d)$. Examples of such mass assignments include projections of higher dimensional mass distributions to linear subspaces $h$ or the volume of intersections of $h$ with (sufficiently smooth) higher dimensional geometric objects. Also, mass
distributions in $\mathbb{R}^d$ can be viewed as mass assignments on $G_d(\mathbb{R}^d)$. In fact, in this thesis we will use the letter $\mu$ both for mass distributions as well as for mass assignments.

For $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_1 + \ldots + \alpha_m = 1$ a partition of $\mathbb{R}^d$ into $m$ parts $A_1, \ldots, A_m$ is said to $\alpha$-equipartition a mass distribution $\mu$ if $\mu(A_i) = \alpha_i \cdot \mu(\mathbb{R}^d)$ for all $i \in \{1, \ldots, m\}$. For $m = 2$ and $\alpha_1 = \alpha_2 = \frac{1}{2}$ we also use the term bisection. If the regions $A_1 \ldots, A_m$ are defined by some geometric object $O$, we also say that $O \alpha$-equipartitions $\mu$. For example, if $h^+$ and $h^-$ are the two half-spaces defined by a hyperplane $h$ and we have $\mu(h^+) = \mu(h^-)$ the we say that $h$ bisects $\mu$. Finally, a partition $A_1, \ldots, A_m$ simultaneously $\alpha$-equipartitions a family $\mu_1, \ldots, \mu_n$ of mass distributions if it $\alpha$-equipartitions each $\mu_i$.

Replacing mass distributions with finite point sets, we say that a partition of $\mathbb{R}^d$ into $m$ parts $A_1, \ldots, A_m$ $\alpha$-equipartitions a finite point set $P$ of size $n$ if for each $i \in \{1, \ldots, m\}$ the interior of $A_i$ contains at most $\alpha_i \cdot n$ points of $P$. With this definition, all results in this thesis for $\alpha$-equipartitions of mass distributions imply the analogous results for $\alpha$-equipartitions of point sets by replacing each point with an arbitrarily small disk and then taking the area of the disks as the mass distribution, see [70] for more details.

One family of geometric objects whose induced partition of space we will study are arrangements of hyperplanes. For them, a partition of space into two parts is given by the natural 2-coloring of the cells of the arrangement. More precisely, let $\mathcal{L}$ be a set of oriented hyperplanes. For each $\ell \in \mathcal{L}$, let $\ell^+$ and $\ell^-$ denote the positive and negative side of $\ell$, respectively. For every point $p \in \mathbb{R}^d$, define $\lambda(p) := |\{\ell \in \mathcal{L} \mid p \in \ell^+\}|$ as the number of hyperplanes that have $p$ in their positive side. Let $R^+ := \{p \in \mathbb{R}^d \mid \lambda(p) \text{ is even}\}$ and $R^- := \{p \in \mathbb{R}^d \mid \lambda(p) \text{ is odd}\}$. More intuitively, this definition can also be understood the following way: if $C$ is a cell in the hyperplane arrangement induced by $\mathcal{L}$, and $C'$ is another cell sharing a facet with $C$, then $C$ is a part of $R^+$ if and only if $C'$ is a part of $R^-$. 
In accordance to previous terminology we now say that \( \mathcal{L} \) bisects a mass distribution \( \mu \) if \( \mu(R^+) = \mu(R^-) \). Note that reorienting one hyperplane just maps \( R^+ \) to \( R^- \) and vice versa. In particular, if a set \( \mathcal{L} \) of oriented hyperplanes simultaneously bisects a family of mass distributions \( \mu_1, \ldots, \mu_n \), then so does any set \( \mathcal{L}' \) of the same hyperplanes with possibly different orientations. Thus we can ignore the orientations and say that a set \( \mathcal{L} \) of (unoriented) hyperplanes simultaneously bisects a family of mass distributions if some orientation of the hyperplanes does.

Other geometric objects whose partition of space we will consider are wedges and cones: a (spherical) cone in \( \mathbb{R}^d \) is defined by an apex \( a \in \mathbb{R}^d \), a central axis \( \ell \), which is a one-dimensional ray emanating from the apex \( a \), and an angle \( \alpha \). The cone is now the set of all points that lie on a ray \( \vec{r} \) emanating from \( a \) such that the angle between \( \vec{r} \) and \( \ell \) is at most \( \alpha \). Note that for \( \alpha = 90^\circ \), the cone is a half-space. Also, for \( \alpha > 90^\circ \), the cone is not convex, which we explicitly allow. Finally note that the closure of the complement of a cone is also a cone and that a cone or its complement are convex.

Let now \( H_k \) be some \( k \)-dimensional linear subspace of \( \mathbb{R}^d \) and let \( \pi : \mathbb{R}^d \to H_k \) be the natural (orthogonal) projection. A \( k \)-cone \( C \) is now a set \( \pi^{-1}(C_k) \), where \( C_k \) is a cone in \( H_k \). The apex \( a \) of \( C \) is the set \( \pi^{-1}(a_k) \), where \( a_k \) is the apex of \( C_k \). It has dimension \( d - k \). Again, note that the complement of a \( k \)-cone is again a \( k \)-cone and that one of the two is convex. Also, a 2-cone is either the intersection or the union of two half-spaces, that is, a so-called wedge. Further, a \( d \)-cone is just a spherical cone. Alternatively, we could also define a \( k \)-cone by a \( (d - k) \)-dimensional apex and a \( (d - k + 1) \)-dimensional half-flat \( h \) emanating from it. The \( k \)-cone would then be the union of points on all \( (d - k + 1) \)-dimensional half-flats emanating from the apex such that their angle with \( h \) is at most \( \alpha \).

Finally, we can get a partition of space into more than two parts using several wedges with a common apex: More precisely, we define a \( k \)-fan in \( \mathbb{R}^d \) as a \( (d - 2) \)-flat \( a \), which we call apex, and \( k \) semi-hyperplanes
emanating from it. Each $k$-fan partitions $\mathbb{R}^d$ into $k$ wedges, which can be given a cyclic order $W_1, \ldots, W_k$.

1.2 Problems and Contributions

In this section, we will give a brief overview of the problems that are considered in this thesis. All of the results presented are based on the following papers, which appeared or are submitted to international conferences and journals and are also available on arXiv:

1. **Bisecting three classes of lines**, by Alexander Pilz and Patrick Schnider [79];

2. **Ham-Sandwich Cuts and Center Transversals in Subspaces**, by Patrick Schnider [85] [86];

3. **Extending the Centerpoint Theorem to Multiple Points**, by Alexander Pilz and Patrick Schnider [77] [78];

4. **Sharing a pizza: bisecting masses with two cuts**, by Luis Barba, Alexander Pilz and Patrick Schnider [13] [14];

5. **Equipartitions with Wedges and Cones**, by Patrick Schnider [84].

The chapters in this thesis correspond roughly to the above papers, but some of the results have been moved to make the chapters more coherent. The problems investigated and contributions made in this thesis can be divided into two variations of the Center Transversal Theorem.

1.2.1 Variation 1: Mass assignments in subspaces

Consider the following question, posed by Luis Barba [12]:

**Question 1.4.** *Given three sets $R, B$ and $G$ of lines in $\mathbb{R}^3$ in general position, each with an even number of lines, is there a line $\ell$ in $\mathbb{R}^3$ such*
that \( H \) lies below exactly \( |R|/2 \) lines of \( R \), \( |B|/2 \) lines of \( B \) and \( |G|/2 \) lines of \( G \)? That is, is there some Ham-Sandwich line that simultaneously bisects (with respect to above-below relation) the lines of \( R, B \) and \( G \)?

The question can also be phrased in a slightly different terminology: Given three sets \( R, B \) and \( G \) of lines in \( \mathbb{R}^3 \) in general position, each with an even number of lines, is there a vertical plane \( h \) not parallel to any line such that \( R \cap h, B \cap h \) and \( G \cap h \) can be simultaneously bisected by a line in \( h \)?

As we will see in Chapter 3, this question can be answered in the positive. In fact, we will even show that the vertical plane \( h \) can be chosen to contain the origin. The main idea of the proof is as follows: the intersection of \( h \) with the lines defines a 3-colored point set in \( h \), which moves continuously as \( h \) rotates (with the exception of the degenerate cases when \( h \) is parallel to one of the lines). At any point during the rotation, any two of the colors can be bisected using the Ham-Sandwich Theorem. The additional degree of freedom which we get from the rotation of \( h \) can now be used to show that somewhere during the rotation the point set in \( h \) is sufficiently degenerate so that all three colors can be bisected with a single line.

There is no reason to use this argument only for the intersections of lines with planes. We will consider a more general setting where we continuously assign mass distributions to each vertical plane. This can further be extended to higher dimensions, in Chapter 3 we prove the following:

**Theorem 1.5.** Let \( \mu_1, \ldots, \mu_{d-k+2} \) be mass assignments on \( \text{Hor}^{k-1}_k(\mathbb{R}^d) \), where \( 2 \leq k \leq d \). Then there exists a \( k \)-dimensional \((k-1)\)-horizontal linear subspace \( h \) for which \( \mu_h^1, \ldots, \mu_{d-k+2}^h \) have a common Ham-Sandwich cut.

Here \( \mu_i^h \) is a shorthand for \( \mu_i(h) \). The answer to Barba’s question then follows (after some work to get rid of the degeneracies) from \( d = 3, k = 2 \). This will be done in Chapter 3. There we will also give
an alternative proof which does not use topological methods and an
efficient algorithm to compute a bisecting line.

Note that the higher $k$ is chosen, the weaker the above result. In fact,
for $k > \frac{d}{2} + 1$, our result is weaker than what we would get from the
Ham-Sandwich Theorem.

On the other hand, restricting ourselves to $(k-1)$-horizontal subspaces,
we have given up some degrees of freedom which could potentially
be used to bisect more masses. Studying mass assignments on the
Grassmann manifold $G_k(\mathbb{R}^d)$, we see in Chapter 4 that this is indeed
the case, even for Center Transversals:

**Theorem 1.6.** Let $\mu_1, \ldots, \mu_{n+d-k}$ be mass assignments on $G_k(\mathbb{R}^d)$,
where $n \leq k \leq d$. Then there exists a $k$-dimensional linear subspace $h$
such that $\mu_h^1, \ldots, \mu_h^{n+d-k}$ have a common $(n-1, k)$-center transversal.

### 1.2.2 Variation 2: Transversals with multiple flats

The second variation considers natural extensions of the two boundary
cases of the Center Transversal Theorem to several points and hyper-
planes, respectively.

Recall that a Centerpoint can be viewed as a generalization of a me-
dian. In other words, it can be interpreted as a good representative for
the underlying mass distribution. But what if we allow more than one
representative? For example in one-dimensional data sets, often cer-
tain quantiles are chosen as representatives instead of the median. In
Chapter 5 we present a possible extension of the concept of quantiles to
higher dimensions. The idea is to find a set $Q$ of (few) points such that
every half-space that contains one point of $Q$ contains a large fraction
of the underlying mass distribution $\mu$ and every half-space that con-
tains more of $Q$ contains an even larger fraction of $\mu$. This concept is
based on the following interpretation of quantiles: the $\frac{1}{3}$-quantile and
the $\frac{2}{3}$-quantile form a set of two points in $\mathbb{R}$ such that every half-line
that contains one of them contains at least $\frac{1}{3}$ of the underlying mass
distribution $\mu$. Furthermore, a half-line containing both of the points contains at least $\frac{2}{3}$ of $\mu$.

More precisely, we introduce the following generalization of Tukey depth for a set $Q$ of multiple points with respect to some mass distribution $\mu$:

$$
gtd_{\mu}(Q) := \min_{h \in H: Q \cap h \neq \emptyset} \left\{ \frac{\mu(h)}{|h \cap Q|} \right\},$$

where $H$ denotes the set of closed half-spaces. We prove that there is always a set $Q$ of $k$ points that has generalized Tukey depth $\frac{1}{kd+1}$. In fact, we prove the following, more general statement:

**Theorem 1.7.** Let $\mu$ be a mass distribution in $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) = 1$. Let $\alpha_1, \ldots, \alpha_k$ be non-negative real numbers such that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$ and for every $i, j$ with $i + j \leq k + 1$ we have that $(d - 1)\alpha_k + \alpha_i + \alpha_j \leq 1$. Then there are $k$ points $p_1, \ldots, p_k$ in $\mathbb{R}^d$ such that for each closed half-space $h$ containing $j$ of the points $p_1, \ldots, p_k$ we have $\mu(h) \geq \alpha_j$.

Note that, for $d = 1$ and $k = 2$, the points $p_1$ and $p_2$ correspond to the $\alpha_1$-quantile and the $(1 - \alpha_1)$-quantile; for $\alpha_j = \frac{j}{kd+1}$ we get our bound on the generalized Tukey depth, and for $\alpha_1 = \ldots = \alpha_k$, the result implies the Centerpoint Theorem.

The second result in Chapter 5 is motivated by interpreting the $\frac{1}{3}$-quantile and the $\frac{2}{3}$-quantile not as two points, but as a one-dimensional simplex. We then have that every halfline that contains a part of the simplex contains at least $\frac{1}{3}$ of $\mu$ and every halfline that contains the whole simplex contains at least $\frac{2}{3}$ of $\mu$. For this interpretation we give a generalization to two dimensions:

**Theorem 1.8.** Let $\mu$ be a mass distribution in $\mathbb{R}^2$ with $\mu(\mathbb{R}^2) = 1$. Let $\alpha$ and $\beta$ be real numbers such that $0 < \alpha \leq \beta$ and $\alpha + \beta = \frac{2}{3}$. Then there is a triangle $\Delta$ in $\mathbb{R}^2$ such that

1. for each closed half-plane $h$ containing one of the vertices of $\Delta$ we have $\mu(h) \geq \alpha$ and

2. for each closed half-plane $h$ fully containing $\Delta$ we have $\mu(h) \geq \beta$. 
Note that this again generalizes Centerpoints in the plane for $\alpha = \beta$. However, this result does not give bounds on the generalized Tukey depth of these sets, as, e.g., a half-space containing two points may still only contain an $\alpha$-fraction of the mass.

Finally, we give algorithms to compute two points satisfying the two-dimensional case of Theorem 1.7 and three points satisfying Theorem 1.8.

As for the Ham-Sandwich Theorem, recall the illustration of simultaneously bisecting the ingredients of a Sandwich with a single cut. However, if two people want to share a pizza, this result will not help them too much, as pizzas generally consist of more ingredients than sandwiches. One option to overcome this issue is to cut the pizza more than once. The cuts then define an arrangement of hyperplanes. Langerman [62] (see also [13]) conjectured the following:

**Conjecture 1.9.** Any $\alpha n$ mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by an arrangement of $n$ hyperplanes.

For $n = 1$, this is again the Ham-Sandwich theorem. For $d = 1$, this conjecture is also true, this result is known as the Necklace splitting theorem [5, 52]. In the plane, Bereg et al. [16] have previously shown that 3 point sets can be simultaneously bisected by 2 lines. In Chapter 6 Section 6.1 we first prove the above conjecture for $d = n = 2$ and give an algorithm to find two bisecting lines. By now, the conjecture is settled for all dimensions $d$, where $d$ is a power of 2, and in many other dimensions, slightly weaker bounds are known [20, 53]. Further, for $n = 2$, it is known that $2d - O(\log(d))$ masses can be simultaneously bisected by 2 hyperplanes [21].

We then consider the following relaxed version: We say that a hyperplane arrangement $\mathcal{L}$ almost bisects $\mu$ if there is an $\ell \in \mathcal{L}$ such that either $\mathcal{L}$ or $\mathcal{L} \setminus \{\ell\}$ bisects $\mu$. For a family of mass distributions $\mu_1, \ldots, \mu_k$ we say that $\mathcal{L}$ almost simultaneously bisects $\mu_1, \ldots, \mu_k$ if for every $i \in \{1, \ldots, k\}$ $\mathcal{L}$ almost bisects $\mu_i$. See Figure 1.1 for an illustration. In this relaxed setting, we are able to prove the above conjecture:
Theorem 1.10. Let $\mu_1, \ldots, \mu_{dn}$ be $dn$ mass distributions in $\mathbb{R}^d$. Then there are $n$ hyperplanes that almost simultaneously bisect $\mu_1, \ldots, \mu_{dn}$.

To finish Chapter 6 we return to the case $n = 2$. Note that alternatively, bisections with two hyperplanes can also be viewed as Ham-Sandwich cuts after a projective transformation: if $h_1$ and $h_2$ simultaneously bisect some masses we can find a projective transformation $\varphi$ which sends $h_1$ to the hyperplane at infinity. After this transformation, $h_2$ is a Ham-Sandwich cut of the transformed masses. In general, only $d$ masses can be bisected with a single hyperplane, thus the transformed masses are in a sense very degenerate. Instead of making one family of many masses very degenerate, we could also try to make several families of fewer masses “slightly degenerate”. This is made precise in Section 6.3, where we will prove the following:

Theorem 1.11. Let $P_1^1, \ldots, P_{d+1}^1, P_1^2, \ldots, P_{d+1}^2, \ldots, P_d^{d+1}$ be $d$ families each containing $d + 1$ point sets in $\mathbb{R}^d$ such that their union is in general position. Then there exists a projective transformation $\varphi$
such that $\varphi(P^i_1), \ldots, \varphi(P^i_{d+1})$ can be simultaneously bisected by a single hyperplane for every $i \in \{1, \ldots, d\}$.

We will actually prove a very similar statement for mass distributions, but in odd dimensions we need some technical restrictions.

Finally, we end with Chapter 7 where we consider mass partitions with multiple semi-hyperplanes. In Section 7.1 we prove the following:

**Theorem 1.12.** Let $p$ be an odd prime.

1. Any $\left\lceil \frac{2d-1}{p-1} \right\rceil + 1$ mass distributions in $\mathbb{R}^d$, where $d$ is odd, can be simultaneously $(\frac{1}{p} \ldots, \frac{1}{p})$-equipartitioned by a $p$-fan;

2. Any $\left\lceil \frac{2d+1}{p-1} \right\rceil$ mass distributions in $\mathbb{R}^d$, where $d$ is even, can be simultaneously $(\frac{1}{p} \ldots, \frac{1}{p})$-equipartitioned by a $p$-fan;

3. Let $(a_1, \ldots, a_q) \in \mathbb{N}^q$ with $q < n$ and $a_1 + \ldots + a_q = p$. Then any $\left\lceil \frac{2d}{p-1} \right\rceil + 1$ mass distributions in $\mathbb{R}^d$, where $d$ is odd, can be simultaneously $(\frac{a_1}{p} \ldots, \frac{a_q}{p})$-equipartitioned by a $q$-fan;

4. Let $(a_1, \ldots, a_q) \in \mathbb{N}^q$ with $q < n$ and $a_1 + \ldots + a_q = p$. Then any $\left\lceil \frac{2d+1}{p-1} \right\rceil$ mass distributions in $\mathbb{R}^d$, where $d$ is even, can be simultaneously $(\frac{a_1}{p} \ldots, \frac{a_q}{p})$-equipartitioned by a $q$-fan.

Items 1 and 2 of the above result were already proved by Makeev (see [60], Thm. 57). In fact, in even dimensions, Makeev’s result is slightly stronger than the above. However, there does not seem to be a publicly available proof of this result, so for the sake of completeness we still include our proof. On the other hand, not much is known about partitions with $k$-fans in higher dimensions when $k$ is not an odd prime. As for 2-fans, note that a hyperplane is a 2-fan, so any $d$ mass distributions can be simultaneously bisected by a 2-fan. We improve this by giving a result that extends to $k$-cones:

**Theorem 1.13.** Let $\mu_1, \ldots, \mu_{d+1}$ be $d+1$ mass distributions in $\mathbb{R}^d$ and let $1 \leq k \leq d$. Then there exists a $k$-cone $C$ that simultaneously bisects $\mu_1, \ldots, \mu_{d+1}$.
CHAPTER 2

Background

To put our results into a wider context, we give a brief overview over the literature that is related to the topics discussed in this thesis. After this, we introduce the most important techniques that will be used in the following chapters. Along the way, we will prove the Center Transversal theorem, as well as give separate proofs of the Ham-Sandwich and Centerpoint Theorems. The main ideas of these proofs are at the core of many of the results in subsequent chapters.

For a deeper treatment, we refer to the following books: *Lectures in discrete geometry* by Jiří Matoušek [72] for Section 2.2, *A Course in Topological Combinatorics* by Mark de Longueville [35] as well as *Using the Borsuk-Ulam Theorem* by Jiří Matoušek [70] for Section 2.3, and finally *Characteristic Classes* by John Milnor and James Stasheff [74] for Section 2.4. We further assume that the reader is familiar with the most important notions of algebraic topology such as degrees of
maps, homology and cohomology. There are many excellent books about algebraic topology. Among those, we mention the works by Glen Bredon [25] and Allen Hatcher [48].

2.1 Related work

Despite the fact that the Center Transversal Theorem is a relatively new result, there are a few direct extensions and variants of it. One of the questions that has been studied is, whether a larger fraction of each mass can always be guaranteed, see [22, 28, 29, 67]. The Center Transversal Theorem has further been extended to a projective setting [59] and to hyperplane arrangements [38]. From an algorithmic point of view, a Center Transversal line for two point sets of total size $n$ in $\mathbb{R}^3$ can be computed in time $O(n^{6+\varepsilon})$ [1].

On the other hand, both the Ham Sandwich Theorem and the Centerpoint Theorem are at the basis of a vast body of research on mass partitions and representatives for mass distributions, respectively. In the following two sections, we will give a very brief overview of some of the most important results.

2.1.1 Mass partitions

There is a rich history of the study of partitions of mass distributions with several objects, starting with the Ham-Sandwich Theorem, which Stone and Tukey [89] already phrased in a more general version: any $\binom{n+d}{d} - 1$ mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by an algebraic surface of degree $n$. This statement is known as the polynomial Ham-Sandwich Theorem, and it has many applications, see e.g. [47] for some examples. The standard Ham-Sandwich Theorem can be retrieved by setting $n = 1$. In fact, the polynomial Ham-Sandwich Theorem can be proved using the standard Ham-Sandwich Theorem by lifting the masses to some Veronese space of the correct dimension.
2.1. Related work

A straightforward application of the 2-dimensional Ham-sandwich Theorem is that any mass distribution in the plane can be partitioned into four equal parts with 2 lines. It is also possible to partition a mass distribution in \( \mathbb{R}^3 \) into 8 equal parts with three planes, but for \( d \geq 5 \), it is not always possible to partition a mass distribution into \( 2^d \) equal parts using \( d \) hyperplanes [40]. The case \( d = 4 \) is still open. Several different types of partitions with arrangements of hyperplanes have since been proposed, see e.g. [19, 61, 81].

For \( \mathbb{R}^1 \), partitions with hyperplanes (or in this case points) are known as the necklace splitting problem. Imagine some thieves that have stolen an (open) necklace with different type of valuable beads. They want to distribute the stolen goods fairly, but they also want to minimize the number times they need to cut the necklace. Hobby and Rice [52] have shown, that two thieves can always fairly distribute \( k \) types of beads using at most \( k \) cuts. A different proof was later found by Alon and West [5], and Alon [4] further generalized the result to fit larger groups of thieves, by showing that dividing \( k \) types of beads among \( q \) thieves can be done using \( k(q - 1) \) cuts.

A result by Buck and Buck [27] states that a mass distribution in the plane can be partitioned into 6 equal parts by 3 lines passing through a common point. This has sparked significant interest in partitions of mass distributions in the plane with \( k \)-fans. Note that 3 lines going through a common point can be viewed as a 6-fan, thus the previously mentioned result shows that any mass partition in the plane can be equipartitioned by a 6-fan. Motivated by a question posed by Kaneko and Kano [57], several authors have shown independently that 2 mass distributions in the plane can be simultaneously partitioned into 3 equal parts by a 3-fan [18, 54, 83]. The analogous result for 4-fans holds as well [11]. Partitions into non-equal parts have also been studied [10, 94]. In higher dimensions, the problem of partitions by \( k \)-fans was studied by Makeev (see [60], Thm. 57).

All these results give a very clear description of the sets used for the partitions. If we allow for more freedom, much more is possible. In
particular, Soberón [88] as well as Karasev, Hubard and Aronov [58] have shown independently that any $d$ mass distributions in $\mathbb{R}^d$ can be simultaneously equipartitioned into $k$ equal parts by $k$ convex sets.

From an algorithmic point of view, for fixed $d$, a Ham-Sandwich cut of $d > 1$ point sets in $\mathbb{R}^d$ can be computed in time $O(n^{d-1})$ [65]. Some other types of partitions have also been studied from the algorithmic point of view, mainly in the plane, such as partitions with 2-fans [15], convex sets [2] and partitions of one mass into for and six parts with 2 and 3 concurrent lines, respectively [82].

2.1.2 Generalizations of medians

The idea of small point sets representing a larger point set has been studied in many different settings. One of the most famous of those is the concept of $\varepsilon$-nets, introduced by Haussler and Welzl [50]. For a range space $(X, R)$, consisting of a set $X$ and a set $R$ of subsets of $X$, an $\varepsilon$-net on $P \subset X$ is a subset $N$ of $P$ with the property that every $r \in R$ with $|r \cap P| \geq \varepsilon |P|$ intersects $N$. It is known that for range spaces with bounded VC-dimension $d$, for any point set $P$ there exists an $\varepsilon$-net of size $O\left(\frac{d}{\varepsilon} \log \frac{d}{\varepsilon}\right)$ [50]. In particular, this bound does not depend on the size of $P$. Further, an $\varepsilon$-net of this size can be computed in time $O(d^{3d}) \cdot \left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)^d \cdot |P|$ [26]. Note that an $\varepsilon$-net is required to be a subset of $P$. If this condition is dropped, we arrive at the concept of weak $\varepsilon$-nets. The fact that the points for the weak $\varepsilon$-net can be chosen anywhere in $\mathbb{R}^d$ allows for very small weak $\varepsilon$-nets for many range spaces. There has been some work on weak $\varepsilon$-nets of fixed small size. For half-planes in $\mathbb{R}^2$ for example, Aronov et al. [8] have shown that there is always a weak $\frac{1}{2}$-net of two points. They also consider many other range spaces, such as convex sets, disks and rectangles. Similarly, Babazadeh and Zarrabi-Zadeh [9] construct weak $\frac{1}{2}$-nets of size 3 for half-spaces in $\mathbb{R}^3$. For two-dimensional convex sets, Mustafa and Ray [76] have shown that there is always a weak $\frac{4}{7}$-net of two points; Shabbir [87] shows how to find two such points in $O(n \log^4 n)$.
2.1. Related work

time.

Another related concept is the concept of $\varepsilon$-approxi-
mations: For a range space $(X, R)$ an $\varepsilon$-approximation on $P \subset X$ is a subset $N$ of $P$ with the property that for every $r \in R$ we have $|r \cap P| |P| - |r \cap N| |N| \leq \varepsilon$. Again, the restriction that $N$ has to be a subset of $P$ can be dropped to get the notion of weak $\varepsilon$-approximations. Just as for $\varepsilon$-nets, there has been a lot of work on $\varepsilon$-approxi-
mations and weak $\varepsilon$-approximations, see [75] for a recent survey. In particular it was shown that for halfspaces in $\mathbb{R}^d$, there always is an $\varepsilon$-approximation of size $O(\frac{1}{\varepsilon^{2-2/(d+1)}})$ [69, 71].

Another related setting is the concept of one-sided $\varepsilon$-approxi-
mants, recently introduced by Bukh and Nivasch [30]: For a range space $(X, R)$, a one-sided $\varepsilon$-approximant on $P \subset X$ is a subset $N$ of $P$ with the property that for every $r \in R$ we have $\frac{|r \cap P|}{|P|} - \frac{|r \cap N|}{|N|} \leq \varepsilon$. Once again, the restriction that $N$ has to be a subset of $P$ can be dropped to get the notion of weak one-sided $\varepsilon$-approximations. Note that the only difference to the definition of $\varepsilon$-approximations is that one does not take the absolute value of the difference. In particular, if $\frac{|r \cap N|}{|N|} > \frac{|r \cap P|}{|P|}$, i.e., if $r$ contains many points of $N$ despite containing only few points of $P$, the difference is negative, and thus smaller than $\varepsilon$. In their paper, Bukh and Nivasch [30] consider the range space of convex sets. They show that any point set in $\mathbb{R}^d$ allows a one-sided $\varepsilon$-approximant for convex ranges of size $g(\varepsilon, d)$, where $g(\varepsilon, d)$ only depends on $\varepsilon$ and $d$, but not on the size of $P$.

Another possibility of generalizing the idea of a median is by so-called depth measures. Depth measures are a tool to capture how deep a point lies in a given point set in $\mathbb{R}^d$. There are many well-studied depth measures such as Tukey depth [92] or simplicial depth [64], see [6] for an overview. Many of these depths have also been studied from an algorithmic point of view. Let us just mention one result here: a centerpoint in the plane can be computed in linear time [55]. This is particularly notable as checking whether a point in the plane is a centerpoint requires $\Theta(n \log n)$ time [7].
There have also been approaches in generalizing quantiles to higher dimensions, e.g. \cite{32}.

### 2.2 Helly’s Theorem

The Centerpoint Theorem is a corollary of Helly’s theorem \cite{51}:

**Theorem 2.1** (Helly’s theorem). Let $X = \{X_1, \ldots, X_n\}$ be a family of convex subsets of $\mathbb{R}^d$ with the property that any $d + 1$ of them have a common intersection. Then the whole family has a common intersection.

The statement also holds for infinite families, assuming compactness of the sets, see e.g. \cite{72}.

Let us now prove the Centerpoint Theorem using Helly’s theorem.

**Proof of the Centerpoint Theorem.** Let $\mu$ be a mass distribution and assume without loss of generality that $\mu(\mathbb{R}^d) = 1$. Recall that we want to show that there is a point $q \in \mathbb{R}^d$ such that for every closed half-space $H$ which contains $q$ we have $\mu(H) \geq \frac{1}{d+1}$. This is equivalent to the statement that every open half-space $\bar{H}$ with $\mu(\bar{H}) > \frac{d}{d+1}$ contains $q$: assume there is such a half-space $\bar{H}$ which does not contain $q$. Then its complement is a closed halfspace $H$ which contains $q$ but for which $\mu(H) < \frac{1}{d+1}$, which is a contradiction. The same argument holds in the other direction, showing the equivalence of the two statements.

Let now $\mathcal{H}$ be the set of all open half-spaces which contain more than $\frac{d}{d+1}$ of $\mu$. Consider any $d + 1$ elements $H_1, \ldots, H_{d+1}$ of $\mathcal{H}$. As $\mu(H_i) > \frac{d}{d+1}$ for $1 \leq i \leq d+1$ we deduce that $\mu(\bigcap_{i=1}^{d+1} H_i) > 0$ and in particular $\bigcap_{i=1}^{d+1} H_i \neq \emptyset$.

We would now like to apply Helly’s theorem to deduce that there is indeed a point in the intersection of all elements of $\mathcal{H}$. However, $\mathcal{H}$ is an infinite family of open half-spaces, and these are not compact. This
2.3. The CS/TM scheme

can be fixed by the following observation: for every open half-space $H$ with $\mu(H) > c$ there exists a compact set $C \subseteq H$ with $\mu(C) > c$. All the above arguments still hold when replacing each open half-space $H$ with a corresponding compact set $C$, and Helly’s theorem can now be applied.

In the remainder of this thesis we will sometimes leave away the last argument and apply Helly’s theorem directly to infinite families of open half-spaces.

### 2.3 The CS/TM scheme

The configuration space/test map scheme, or CS/TM scheme for short, is a powerful tool to prove geometric and combinatorial results. Its main idea is to consider a configuration space $X$ which encodes all possible configurations in the considered problem (e.g. the space of all half-spaces or the space of all $k$-fans) and a test map $t$ which distinguishes desired configurations (e.g. half-spaces containing exactly half of each mass) from the other ones by mapping them to a subspace $Z \subset V$ of the test space $V$. If we can now find a group $G$ acting on $X$ and on $V$ (keeping $Z$ invariant), and if $t$ is $G$-equivariant (i.e. $t(g \cdot x) = g \cdot t(x)$ for all $g \in G$ and $x \in X$), then the existence of a configuration with the desired property follows from the non-existence of a $G$-equivariant map $X \to V \setminus Z$. This non-existence of a $G$-equivariant map is then normally shown by topological methods.

In many cases, there are several possible configuration spaces that might be considered, and sometimes, choosing the right one can make proofs significantly simpler. In the following, we will apply the CS/TM scheme to give two proofs of the Ham-Sandwich Theorem, using different configuration spaces. Variations of both of these configuration spaces will reappear later on in other proofs.
First proof of the Ham-Sandwich Theorem. Let $\mu_1, \ldots, \mu_d$ be $d$ mass distributions in $\mathbb{R}^d$. We look at the space of all half-spaces in $\mathbb{R}^d$ and claim that it can be parameterized using the sphere $S^d \subset \mathbb{R}^{d+1}$. Indeed, every point $p = (p_0, \ldots, p_d) \in S^d$ defines a hyperplane $h(p) : p_1 x_1 + \ldots + p_d x_d + p_0 = 0$ with corresponding half-spaces $h^+(p)$ and $h^-(p)$, defined by the inequalities $p_1 x_1 + \ldots + p_d x_d + p_0 \geq 0$ and $p_1 x_1 + \ldots + p_d x_d + p_0 \leq 0$, respectively. Note that for $p = (1, 0, \ldots, 0)$ we have $h^+(p) = \mathbb{R}^d$ and $h^-(p) = \emptyset$, and vice-versa for $p = (-1, 0, \ldots, 0)$. On the other hand, for every half-space we can normalize the coefficients of the defining hyperplane to get a point on the sphere $S^d$. As all these maps are continuous, the space of all half-spaces in $\mathbb{R}^d$ is thus homeomorphic to $S^d$, and we choose $S^d$ as our configuration space.

For each mass distribution $\mu_i$ define the function $f_i : S^d \to \mathbb{R}$ as $f_i(p) = \mu_i(h^+(p)) - \mu_i(h^-(p))$. The assumptions on $\mu_i$ guarantee that $f_i$ is continuous. Also, the hyperplane defined by $p$ bisects $\mu_i$ if and only if $f_i(p) = 0$. Further, these functions define a map $f = (f_1, \ldots, f_d) : S^d \to \mathbb{R}^d$ and the hyperplane defined by $p$ is a Ham-Sandwich cut if and only if $f(p) = 0$. We take $f$ as our test map, with the test space $\mathbb{R}^d$, where the solution subspace $Z$ is the origin.

Finally, there is a natural $\mathbb{Z}_2$-action on $S^d$ and on $\mathbb{R}^d$ given by $p \mapsto -p$ and $x \mapsto -x$, respectively. We note that $f$ is $\mathbb{Z}_2$-equivariant with respect to these actions: it follows from the above definitions that $h^+(-p) = h^-(p)$ and thus $f(-p) = -f(p)$. Further, $-0 = 0$, that is, the action on $\mathbb{R}^d$ keeps $Z$ invariant. In order to show the existence of a Ham-Sandwich cut it is thus sufficient to prove that there is no $\mathbb{Z}_2$-equivariant map $S^d \to \mathbb{R}^d \setminus \{0\}$.

To show this, we note that by normalizing, such a map would induce a $\mathbb{Z}_2$-equivariant map $S^d \to S^{d-1}$. The non-existence of such a map is given by the famous Borsuk-Ulam theorem [24] (see also [70]).

In the second proof, we will not consider all hyperplanes, but only the ones that already bisect a mass.
Second proof of the Ham-Sandwich Theorem. Let \( \mu_1, \ldots, \mu_d \) be \( d \) mass distributions in \( \mathbb{R}^d \). Consider the space of all oriented lines through the origin and note that this space is homeomorphic to \( S^{d-1} \subset \mathbb{R}^d \). For each such oriented line \( \ell \) move an orthogonal (oriented) hyperplane \( h \) to the midpoint of the unique interval on \( \ell \) where it bisects the mass distribution \( \mu_d \) and let \( h^+(\ell) \) and \( h^- (\ell) \) be the two half-spaces defined by \( h \). We now have that the space of all (oriented) hyperplanes that bisect \( \mu_d \) is homeomorphic to \( S^{d-1} \). For each \( \ell \), define \( f_i : S^{d-1} \to \mathbb{R} \) as \( f_i(\ell) = \mu_i(h^+(\ell)) - \mu_i(h^-(\ell)) \), \( 1 \leq i \leq d-1 \). As before, the assumptions on \( \mu_i \) guarantee that \( f_i \) is continuous. Like in the previous proof, these functions define a map \( f = (f_1, \ldots, f_{d-1}) : S^{d-1} \to \mathbb{R}^{d-1} \) and the hyperplane defined by \( \ell \) is a Ham-Sandwich cut if and only if \( f(\ell) = 0 \). We again have the same natural \( \mathbb{Z}_2 \)-actions, and the result again follows from the Borsuk-Ulam theorem, but in one dimension lower.

While we ended up using the Borsuk-Ulam theorem in both of the above proofs, there are also examples where different configurations spaces can lead to very different equivariant maps, such as for bisections with line arrangements in Chapter 6.

Note that in \( \mathbb{R}^2 \) the second proof is nicer in the following sense: the non-existence of a \( \mathbb{Z}_2 \)-equivariant map \( S^1 \to S^0 \) follows from much more elementary methods than the Borsuk-Ulam theorem, such as the mean value theorem, so the proof idea can easily be explained to people that are not familiar with any algebraic topology. On the other hand, it is crucial that in each direction we could choose in a continuous fashion a unique hyperplane bisecting \( \mu_d \). This is not possible anymore if we consider charges (mass distributions, which may be locally negative) instead of mass distributions. The first proof, however, also works for charges.
2.4 Vector Bundles

Let us again take a look at the second proof of the Ham-Sandwich Theorem. For each oriented line through the origin, we chose a unique hyperplane orthogonal to it, and this hyperplane was the same for both orientations of the line. In particular, we chose a unique hyperplane for each unoriented line through the origin. Taking the intersection of the hyperplane and the line, this is the same as choosing a unique point on each line. We did this using the mass \( \mu_d \), but we could do it for any other mass as well, defining \( d \) points on each line. We could now also prove the Ham-Sandwich Theorem by showing that there is some line where all these points coincide.

This idea can be made more formal using vector bundles. A vector bundle consists of a base space \( B \), a total space \( E \), and a continuous projection map \( \pi : E \to B \). Furthermore, for each \( b \in B \), the fiber \( \pi^{-1}(b) \) over \( b \) has the structure of a vector space over the real numbers. Finally, a vector bundle satisfies the local triviality condition, meaning that for each \( b \in B \) there is a neighborhood \( U \subset B \) containing \( p \) such that \( \pi^{-1}(U) \) is homeomorphic to \( U \times \mathbb{R}^d \).

For example, both an infinite cylinder and an infinite Möbius strip are vector bundles whose base space is a circle, but the Möbius strip has an additional ”twist”. Also in the case of lines through the origin we get a natural vector bundle. Even more generally, recall that we denote by \( G_m(\mathbb{R}^d) \) the Grassmann manifold consisting of all \( m \)-dimensional subspaces of \( \mathbb{R}^d \). We let \( \gamma_m^d \) be the canonical bundle over \( G_m(\mathbb{R}^d) \). The bundle \( \gamma_m^d \) has a total space \( E \) consisting of all pairs \( (L,v) \), where \( L \) is an \( m \)-dimensional subspace of \( \mathbb{R}^d \) and \( v \) is a vector in \( L \), and a projection \( \pi : E \to G_m(\mathbb{R}^d) \) given by \( \pi((L,v)) = L \).

A section of a vector bundle is a continuous mapping \( s : B \to E \) such that \( \pi s \) equals the identity map, i.e., \( s \) maps each point of \( B \) to its fiber. In the above example, we have constructed \( d \) sections \( s_1, \ldots, s_d \) of \( \gamma_1^d \), and we would like to show that for some line, they all coincide.
Note that as each fiber is a vector space, the difference of two sections is well-defined and again a section. We can thus equivalently show that the $d - 1$ sections $s_1 - s_d, \ldots, s_{d-1} - s_d$ are all zero in some line. For this, we will need two ingredients: first, we want to interpret the $d - 1$ sections as a single section in some bundle which has the same base space, but where each fiber is the product of the original fibers. This can be achieved by the construction called Whitney sum. Secondly, we will need some tool to show that for a considered vector bundle every section has a zero. For this, we use Stiefel-Whitney classes.

### 2.4.1 Whitney sums

Let $\xi = (E, B, \pi)$ be a vector bundle and let $B'$ be a topological space. Consider a map $f : B' \to B$. Then we can define an induced bundle $f^*\xi$ over $B'$ as follows: The total space $E'$ of $f^*\xi$ is a subset $E' \subseteq B' \times E$ which consists of all pairs $(b, e)$ with $f(b) = \pi(e)$, and the projection map is given by $\pi'(b, e) = b$. In other words, to each $b \in B'$, we attach the fiber over $f(b)$. It is not hard to show that $f^*\xi$ is indeed a vector bundle, we refer to [74] for more details.

Given two vector bundles $\xi_1 = (E_1, B_1, \pi_1)$ and $\xi_2 = (E_2, B_2, \pi_2)$, we can define their Cartesian product $\xi_1 \times \xi_2$ as the vector bundle with total space $E_1 \times E_2$, base space $B_1 \times B_2$ and projection map $\pi_1 \times \pi_2$.

Consider now two vector bundles $\xi_1 = (E_1, B, \pi_1)$ and $\xi_2 = (E_2, B, \pi_2)$ over the same base space $B$. Let $d : B \to B \times B$ be the diagonal embedding. The Whitney sum of $\xi_1$ and $\xi_2$, denoted by $\xi_1 \oplus \xi_2$ is now defined as the induced bundle $d^*(\xi_1 \times \xi_2)$. With this definition, each fiber is isomorphic to the direct sum of the original fibers and, in particular, as a section in $\xi_1$ together with a section in $\xi_2$ defines a unique section in $\xi_1 \oplus \xi_2$. 
2.4.2 Stiefel-Whitney classes

Stiefel-Whitney classes are topological invariants of a vector bundle which describe the obstructions to constructing everywhere independent sets of sections. In particular, they can be applied to decide orientability of a vector bundle (e.g. a tangent bundle). They are named after the Swiss mathematician Eduard Stiefel, who worked at ETH Zürich\footnote{Diving a bit into the history of ETH, one finds out that Eduard Stiefel was the head of the newly formed Institute for Applied Mathematics at ETH in 1950. Under his tenure the institute acquired a Z4 computer, making ETH the first university on the European mainland to own a programmable computer. In 1981, a part of the institute split of to form the Department of Computer Science. Among the first professors in this Department was Peter Läuchli, one of Eduard Stiefel’s students. When Peter Läuchli retired, his position was filled by Emo Welzl, who is the advisor of the author of this thesis.}, and his American colleague Hassler Whitney.

The Stiefel-Whitney classes of a vector bundle $\xi = (E, B, \pi)$ are a sequence of cohomology classes

$$w_i(\xi) \in H^i(B; \mathbb{Z}_2),$$

that satisfy the following axioms:

1. (Rank) We have $w_0 = 1 \in H^0(B; \mathbb{Z}_2)$ and $w_i = 0$ for $i > d$, where $d$ is the fiber dimension;

2. (Naturality) If $B'$ is a topological space and $f : B' \to B$, then

$$w_i(f^*\xi) = f^*w_i(\xi);$$

3. (Whitney product formula) If $\xi_1$ and $\xi_2$ are bundles over the same base space, then

$$w_k(\xi_1 \oplus \xi_2) = \sum_{i=0}^{k} w_i(\xi_1) \cup w_{k-i}(\xi_2),$$

where $\cup$ denotes the cup product;
2.5. Proof of the Center Transversal Theorem

4. (Normalization) The class $w_1(\gamma_1^1)$ is non-trivial.

In the following, we will omit the cup product symbol and just write $w_i(\xi_1)w_j(\xi_2)$ for the cup product instead.

The existence and uniqueness of Stiefel-Whitney classes is shown for example in [74]. They are useful for our purpose because of the following result (§4, Proposition 4 in [74]):

**Proposition 2.2.** If $\xi$ is a vector bundle with fiber dimension $d$ which possesses a nowhere zero section, then $w_d(\xi) = 0$.

This means that in order to show that a section must have a zero, it suffices to show $w_d(\xi) \neq 0$.

2.5 Proof of the Center Transversal Theorem

Using the above tools, we can now finish our third proof on the Ham-Sandwich by showing that $w_{d-1}(\bigoplus_{i=1}^{d-1} \gamma_1^d) \neq 0$. This is indeed true, and the following more general statement, proven independently by Dol’nikov [37], as well as Zivaljević and Vrećica [95], is the topological basis for the Center Transversal Theorem:

**Lemma 2.3.** Let $\xi = \bigoplus_{i=1}^{k} \gamma_{n+k}^i$, i.e. $\xi$ is a vector bundle with fiber dimension $kn$. Then $w_{kn}(\xi) \neq 0$.

**Proof.** By the Whitney product formula we have $w_{kn}(\xi) = w_n(\gamma_n^{n+k})^k$. Recall that the Flag manifold $\hat{V}_{n+k,n}$ is the space of all $n$-flags in $\mathbb{R}^{n+k}$. We have the projection $f : \hat{V}_{n+k,n} \to G_n(\mathbb{R}^{n+k})$ which maps $(V_0, \ldots, V_n, \mathbb{R}^{n+k})$ to $V_n$. We further have the inclusions $i : \hat{V}_{n+k,n} \to \hat{V}_{\infty,n}$ and $j : G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^{\infty})$. This gives the following commu-
tative diagram

\[
\begin{array}{ccc}
\tilde{V}_{n+k,n} & \xrightarrow{i} & \tilde{V}_{\infty,n} \\
\downarrow f & & \downarrow g \\
G_n(\mathbb{R}^{n+k}) & \xrightarrow{j} & G_n(\mathbb{R}^\infty)
\end{array}
\]

which on the \(\mathbb{Z}_2\)-cohomology level induces the commutative diagram

\[
\begin{array}{ccc}
H^*(\tilde{V}_{n+k,n}) & \xleftarrow{i^*} & H^*(\tilde{V}_{\infty,n}) \\
\uparrow f^* & & \uparrow g^* \\
H^*(G_n(\mathbb{R}^{n+k})) & \xleftarrow{j^*} & H^*(G_n(\mathbb{R}^\infty))
\end{array}
\]

It is known that \(H^*(\tilde{V}_{\infty,n})\) is a polynomial algebra \(\mathbb{Z}_2[t_1, \ldots, t_n]\) and that \(g^*\) maps \(H^*(G_n(\mathbb{R}^\infty))\) injectively onto the algebra \(\mathbb{Z}_2[\sigma_1, \ldots, \sigma_n] \subset \mathbb{Z}_2[t_1, \ldots, t_n]\), where \(\sigma_i\) denotes the \(i\)'th symmetric polynomial in the variables \(t_1, \ldots, t_n\) \([74, 49]\). Since \(w_n(\gamma_n^{n+k}) = j^*(\sigma_n)\) (see \([74]\)), if \(w_n(\gamma_n^{n+k})^k = 0\), this would imply that \((\sigma_n)^k \in \ker i^*\). But this is a contradiction to Proposition 3.16 in \([44]\).

Together with the above discussion and Proposition 2.2 this gives the following:

**Lemma 2.4.** Let \(s_1, \ldots, s_{k+1}\) be \(k+1\) sections of the canonical bundle \(\gamma_n^{n+k}\). Then there is an \(n\)-dimensional linear subspace \(h \subset \mathbb{R}^{n+k}\) such that \(s_1(h) = \ldots = s_{k+1}(h)\).

With this tool at hand, we can now finally give a proof of the Center Transversal Theorem.

**Proof of the Center Transversal Theorem.** Let \(\mu_1 \ldots, \mu_k\) be \(k \leq d\) mass distributions in \(\mathbb{R}^d\). Consider the Grassmann manifold \(G_{d-k+1}(\mathbb{R}^d)\) of all \((d - k + 1)\)-dimensional linear subspaces. For each subspace \(h \in G_{d-k+1}(\mathbb{R}^d)\), consider the projection \(\pi_h : \mathbb{R}^d \to h\) and denote the projection of \(\mu_i\) to \(h\) by \(\mu_i^h\). Note that \(\mu_i^h\) is a \((d - k + 1)\)-dimensional mass distribution. In particular, by the Centerpoint Theorem, there
is a compact convex centerpoint region. Let $c^h_i$ denote the (unique) barycenter of this region. Note that $c^h_i$ varies continuously with $h$ and that $\pi_h^{-1}(c^h_i)$ is a $(k - 1)$-flat with the property that for every half-space $H$ containing it we have $\mu_i(H) \geq \frac{\mu_i(\mathbb{R}^d)}{d-k+2}$. This means that if we find a subspace $h \in G_{d-k+1}(\mathbb{R}^d)$ for which $c^h_1 = \ldots = c^h_k$, the result follows.

To this end, note that for each $i$ the points $c^h_i$ define a section in the canonical bundle $\gamma^d_{d-k+1}$. We thus have $k$ sections in $\gamma^d_{d-k+1}$, and it follows from Lemma 2.4 (replacing $k$ with $k - 1$ and $n$ with $d - k + 1$) that there is indeed a subspace $h$ on which they all coincide. □
Recall the question by Barba whether for any three sets of lines in $\mathbb{R}^3$ there is a line which simultaneously bisects the classes of lines with respect to the above-below relation. In this chapter, we will answer this question in the positive and give an algorithm to compute such a Ham-Sandwich line:

**Theorem 3.1.** Let $\mathcal{L} = (R \cup B \cup G)$ be an arrangement of $n$ lines in $\mathbb{R}^3$ in general position colored red, blue and green, none of which intersect the $z$-axis. Then, there exists a plane $h$ containing the $z$-axis that defines a cross-section in which there exists a line $\ell$ that simultaneously bisects the three point sets $R \cap h$, $B \cap h$ and $G \cap h$. Further, $h$ and $\ell$ can be computed in time $O(n^2 \log^2(n))$.

We say that the arrangement $\mathcal{L}$ is in general position if no two lines are parallel or intersect and no line is vertical (parallel to the $z$-axis).
Note that this does not correspond exactly to the setting of Barbas question: the number of lines in each color class are not required to be even, and we allow a more general notion of bisections. In particular, \( \ell \) might intersect some of the lines of \( L \) or be parallel to one of them. However, a solution in the above setting implies a solution in the setting of Barbas question:

**Corollary 3.2.** Given three sets \( R, B \) and \( G \) of lines in \( \mathbb{R}^3 \) in general position, each with an even number of lines, there is a line \( \ell \) in \( \mathbb{R}^3 \) such that \( \ell \) lies below exactly \( |R|/2 \) lines of \( R \), \( |B|/2 \) lines of \( B \) and \( |G|/2 \) lines of \( G \).

**Proof.** We can assume without loss of generality that for any four lines in \( L := R \cup B \cup G \), if there is a line intersecting all of them, the (unique) vertical plane containing this common intersecting line does not go through the origin. In particular, in any cross-section defined by a vertical plane at most three points are collinear. Let now \( h \) be a vertical plane and \( \ell \) be a line which bisects all three classes of points in the cross-section defined by \( h \), as given by Theorem 3.1. We will distinguish the cases whether there is a line in \( L \) which is parallel to \( h \).

If no line is parallel to \( h \), then all three point sets in the cross-section have even size. Thus, as \( \ell \) bisects every class simultaneously and no 4 points are collinear, \( \ell \) either passes through two points or through no points. If \( \ell \) passes through no points, then it is a desired solution. If it passes through two points, then the two points must be of the same color. In this case, \( \ell \) can be rotated slightly so that it still bisects all the point sets but doesn’t pass through any points.

So, assume now that there is a line parallel to \( h \) and assume without loss of generality that it is in \( R \), that is, colored red. Then, the number of red points in the cross-section is odd and thus \( \ell \) must pass through one of them. Further, as at most three points are collinear, \( \ell \) passes through either one or three points. If \( \ell \) only passes through a single (red) point, we can slightly rotate \( h \) to get an additional red point in the cross section, without loss of generality above \( \ell \). Slightly moving \( \ell \)
upwards then gives a desired solution.

If \( \ell \) passes through three points, then either they are all red or one of them is red and the other two are of the same color, without loss of generality blue. If the middle point of the three points on \( \ell \) is red, we can thus slightly rotate \( \ell \) to pass only through the middle point and we are again in the above case. Thus, the only remaining case is if the middle point is blue. In this case, again rotate \( h \) slightly to get an additional red point which is again without loss of generality above \( \ell \). We can now rotate \( \ell \) around the midpoint of the two blue points to get a desired solution.

It should be mentioned that Barba has shown that the analogous statement for four sets of lines is false \cite{12}. As this result is not planned to be published, let us quickly explain the main idea here: consider three skew lines, each colored with a different color. The set of lines which meet all three colored lines is a regulus \( A \). Now, replace each colored line by a bundle of lines of the same color to get three sets \( R, B \) and \( G \) of lines. Any Ham-Sandwich line must now essentially be a line on \( A \). By placing a fourth bundle of lines sufficiently far away from \( A \), it follows that not all four sets can be simultaneously bisected.

We will prove the existence of a Ham-Sandwich line \( \ell \) in Theorem \ref{thm:existence} in two different ways. In Section \ref{sec:topological} we use topological methods to prove a more general statement on bisections of mass assignments to horizontal subspaces (Theorem \ref{thm:topological}) and then show that the existence of \( \ell \) follows. However, this proof is purely existential and does not provide any algorithm. We resolve this issue in Section \ref{sec:algorithmic} by giving a proof which uses only standard tools from discrete geometry. The structure of this combinatorial proof allows us to devise an efficient algorithm for the problem. We do this in Section \ref{sec:framework} by providing a general framework, from which we expect that it can be applied to solve similar problems on cross-sections and kinetic points.
3.1 Ham-Sandwich cuts in horizontal subspaces

Recall the theorem that we want to prove:

**Theorem 1.5.** Let $\mu_1, \ldots, \mu_{d-k+2}$ be mass assignments on $\text{Hor}_k^{k-1}(\mathbb{R}^d)$, where $2 \leq k \leq d$. Then there exists a $k$-dimensional $(k-1)$-horizontal linear subspace $h$ for which $\mu_h^1, \ldots, \mu_h^{d-k+2}$ have a common Ham-Sandwich cut.

In order to prove this, we establish a few preliminary lemmas. Consider the following space, which we denote by $F_{\text{hor}}$: the elements of $F_{\text{hor}}$ are pairs $(h, \overrightarrow{\ell})$, where $h$ is an (unoriented) $k$-dimensional $(k-1)$-horizontal linear subspace of $\mathbb{R}^d$ and $\overrightarrow{\ell}$ is an oriented 1-dimensional linear subspace of $h$, that is, an oriented line in $h$ through the origin. The space $F_{\text{hor}}$ is a quotient space of a subspace of the Stiefel manifold and thus inherits a topology from it. Furthermore, inverting the orientation of $\overrightarrow{\ell}$ is a free $\mathbb{Z}_2$-action, giving $F_{\text{hor}}$ the structure of a free $\mathbb{Z}_2$-space.

We will first give a different description of the space $F_{\text{hor}}$. Define

$$F' := S^{d-k} \times S^{k-2} \times [0,1]/(\approx_0, \approx_1),$$

where $(x, y, 0) \approx_0 (x, y', 0)$ for all $y, y' \in S^{k-2}$ and $(x, y, 1) \approx_1 (-x, y, 1)$ for all $x \in S^{d-k}$. Further, define a free $\mathbb{Z}_2$-action on $F'$ by $-(x, y, t) := (-x, -y, t)$. We claim that the $\mathbb{Z}_2$-space $F'$ is “the same” as $F_{\text{hor}}$.

**Lemma 3.3.** There is a $\mathbb{Z}_2$-equivariant homeomorphism between $F'$ and $F_{\text{hor}}$.

**Proof.** Consider the subspace $Y \subset \mathbb{R}^d$ spanned by $e_1, \ldots, e_{k-1}$. The space of unit vectors in $Y$ is homeomorphic to $S^{k-2}$. Similarly let $X \subset \mathbb{R}^d$ be spanned by $e_k, \ldots, e_d$. Again, the space of unit vectors in $X$ is homeomorphic to $S^{d-k}$. In a slight abuse of notation, we will write $y$ and $x$ both for a unit vector in $Y$ and $X$ as well as for the corresponding points in $S^{k-2}$ and $S^{d-k}$, respectively.
We first construct a map \( \varphi \) from \( S^{d-k} \times S^{k-2} \times [0, 1] \) to \( F_{\text{hor}} \) as follows: for every \( x \in S^{d-k} \) let \( h(x) \) be the unique \((k-1)\)-horizontal subspace spanned by \( x, e_1, \ldots, e_{k-1} \). See Figure 3.1 for an illustration. Note that \( h(-x) = h(x) \). Further, define \( v(x, y, t) := (1-t)x + ty \) and let \( \overrightarrow{\ell}(x, y, t) \) be the directed line defined by the vector \( v(x, y, t) \). Note that \( \overrightarrow{\ell}(x, y, t) \) lies in the plane spanned by \( x \) and \( y \) and thus also in \( h(x) \). Finally, set \( \varphi(x, y, t) := (h(x), \overrightarrow{\ell}(x, y, t)) \). Both \( h \) and \( v \) are both open and closed continuous maps, and thus so is \( \varphi \). Also, we have that \( v(-x, -y, t) = -(1-t)x - ty = -v(x, y, t) \), so \( \varphi \) is \( \mathbb{Z}_2 \)-equivariant.

Note that for \( t = 0 \) we have \( v(x, y, 0) = x \), so \( \varphi(x, y, 0) \) does not depend on \( y \), and in particular \( \varphi(x, y, 0) = \varphi(x, y', 0) \) or all \( y, y' \in S^{k-2} \). Similarly, for \( t = 1 \) we have \( v(x, y, 1) = y \) and \( h(-x) = h(x) \), and thus \( \varphi(x, y, 1) = \varphi(-x, y, 1) \) for all \( x \in S^{d-k} \). Hence, \( \varphi \) induces a map \( \varphi' \) from \( F' \) to \( F_{\text{hor}} \) which is still open, closed, continuous and \( \mathbb{Z}_2 \)-equivariant. Finally, it is easy to see that \( \varphi' \) is bijective. Thus, \( \varphi' \) is a \( \mathbb{Z}_2 \)-equivariant homeomorphism between \( F' \) and \( F_{\text{hor}} \), as required. 

We now prove a Borsuk-Ulam-type statement for \( F_{\text{hor}} \).
Lemma 3.4. There is no $\mathbb{Z}_2$-map $f : F_{\text{hor}} \to S^{d-k}$.

Proof. Assume for the sake of contradiction that $f$ exists. Then, by Lemma 3.3, $f$ induces a map $F : S^{d-k} \times S^{k-2} \times [0, 1] \to S^{d-k}$ with the following properties:

1. $F(-x, -y, t) = -F(x, y, t)$ for all $t \in (0, 1)$;
2. $F(x, y, 0) = F(x, y', 0)$ for all $y, y' \in S^{k-2}$ and $F(-x, y, 0) = -F(x, y, 0)$ for all $x \in S^{d-k}$;
3. $F(x, -y, 1) = -F(x, y, 1)$ for all $y \in S^{k-2}$ and $F(-x, y, 1) = F(x, y, 1)$ for all $x \in S^{d-k}$.

In particular, $F$ is a homotopy between $f_0(x, y) := F(x, y, 0)$ and $f_1(x, y) := F(x, y, 1)$. Fix some $y_0 \in S^{k-2}$. Then $F$ induces a homotopy between $g_0(x) := f_0(x, y_0)$ and $g_1(x) := f_1(x, y_0)$. Note that $g_0 : S^{d-k} \to S^{d-k}$ has odd degree by property (2). On the other hand, $g_1 : S^{d-k} \to S^{d-k}$ has even degree by property (3). Thus, $F$ induces a homotopy between a map of odd degree and a map of even degree, which is a contradiction.

We now have all tools that are necessary to prove Theorem 1.5.

Proof of Theorem 1.5. For each $\mu_i$ and $(h, \ell)$, consider the point $v_i$ on $\ell$ for which the orthogonal hyperplane bisects $\mu_i^h$. (If $v_i$ is not unique, the set of all possible such points is an interval, in which case we choose $v_i$ as the midpoint of this interval.) This induces a continuous $\mathbb{Z}_2$-map $g : F_{\text{hor}} \to \mathbb{R}^{d-k+2}$. For $i \in \{1 \ldots, d-k+1\}$, set $w_i := v_i - v_{d-k+2}$. The $w_i$'s then induce a continuous $\mathbb{Z}_2$-map $f : F_{\text{hor}} \to \mathbb{R}^{d-k+1}$. We want to show that there exists $(h, \ell)$ where $v_1 = v_2 = \ldots = v_{d-k+2}$, or equivalently, $w_1 = \ldots = w_{d-k+1} = 0$, i.e., $f$ has a zero. Assume that this is not the case. Then normalizing $f$ induces a $\mathbb{Z}_2$-map $f' : F_{\text{hor}} \to S^{d-k}$, which is a contradiction to Lemma 3.4.

Note that the higher $k$ is chosen, the smaller the number of bisected
masses compared to $d$. In fact, for $k > \frac{d}{2} + 1$, our result is weaker than what we would get from the Ham-Sandwich theorem. We conjecture that this trade-off is not necessary:

**Conjecture 3.5.** Let $\mu_1, \ldots, \mu_d$ be mass assignments on $\text{Hor}_k^{k-1}(\mathbb{R}^d)$ and $k \geq 2$. Then there exists a $k$-dimensional $(k-1)$-horizontal linear subspace $h$ such that $\mu^h_1, \ldots, \mu^h_d$ have a common Ham-Sandwich cut.

Let us now apply Theorem 1.5 to show the existence of a Ham-Sandwich line $\ell$ as in Theorem 3.1. For this, we need to define a mass assignment. To this end, we replace every line $r$ in $R$ by a very thin infinite cylinder of radius $\varepsilon$, centered at $r$. Denote the collection of cylinders obtained this way by $R^\ast$. Define $B^\ast$ and $G^\ast$ analogously. For each vertical plane $h$ through the origin, let $D^h_K$ be a disk in $h$ centered at the origin, with some (very large) radius $K$. Define $\mu^h_R$ as $(R^\ast \cap h) \cap D^h_K$. It is straightforward to show that $\mu^h_R$ is a mass assignment. Analogously we can define mass assignments $\mu^h_B$ and $\mu^h_G$. From Theorem 1.5, where we set $e_1$ to be the unit vector on the $z$-axis, we deduce that there is a vertical plane $h_0$ and a line $\ell \in h_0$ such that $\ell$ simultaneously bisects $\mu^h_R$, $\mu^h_B$ and $\mu^h_G$. We claim that this $\ell$ is a Ham-Sandwich line.

To show this, we distinguish two cases: The first case is that all the cylinders in $R^\ast \cup B^\ast \cup G^\ast$ intersect $D^h_{K_0}$. In this case, it is a standard argument to show that $\ell$ is a Ham-Sandwich cut of the point set $(R \cup B \cup G) \cap h_0$. The second case is that some cylinders in $R^\ast \cup B^\ast \cup G^\ast$ do not intersect $D^h_{K_0}$. By the general position assumption that no two lines are parallel, choosing $K$ sufficiently large, we can assume that exactly one cylinder $c^\ast$ does not intersect $D^h_{K_0}$. Without loss of generality, let $c^\ast \in R^\ast$, defined by some line $c \in R$. If $K$ is chosen sufficiently large, as no line intersects the $z$-axis we can further assume that $c$ is parallel to $h_0$. Thus, similar to above, $\ell$ is a Ham-Sandwich cut of the point set $((R \setminus \{c\}) \cup B \cup G) \cap h_0$. 

3.2 A Combinatorial proof

We now give an alternative proof of Theorem 3.1. The proof relies on an analysis of the changes in the combinatorial structure of the cross-sections when rotating a plane containing the z-axis. In particular, we will show that during a rotation of the plane by $180^\circ$, one of the cross-sections must be such that there exists a line that simultaneously bisects all three classes of points. Further, the method used to prove this allows us to algorithmically decide, for some moment in the rotation, whether a solution appears before or after this moment. This sidedness oracle is a crucial element for the algorithm described in Section 3.3.

3.2.1 Dual interpretation in cross-sections

For every directed plane $h$ containing the $z$-axis, denote by $S(h)$ the induced cross-section. Consider the dual line arrangement $A(h)$, under the standard point-line-duality. (For example, a point $(x_p, y_p)$ is mapped to the line with equation $y = x_p x + y_p$.) For an arrangement of $m$ lines, the $k$-level is the set of points that has at most $k$ lines below it and at most $m - k - 1$ lines above it. If there is no vertical line, the $k$-level is a piece-wise linear $x$-monotone curve. The $\left\lfloor \frac{m}{2} \right\rfloor$-level is called the median level. For simplicity, we will in general assume that the number of lines in each color class is odd. (This assumption may be dropped at the cost of a more involved exposition.) Denote by $r(h)$, $b(h)$ and $g(h)$ the median levels of the red, green and blue lines in $A(h)$, respectively. A line that bisects all three point sets in a cross-section $S(h)$ corresponds to an intersection point of the three median levels $r(h)$, $b(h)$ and $g(h)$. Thus, we want to show that there exists some plane $h$ such that in the induced line arrangement $A(h)$ the three median levels $r(h)$, $b(h)$ and $g(h)$ have a common intersection.

To this end, we start with some plane $h_0$ through the $z$-axis and rotate it around that axis, keeping track of the changes in the induced
arrangement. After a rotation of angle $\pi$, we will arrive again at $h_0$, but oriented the other way; we will call this oriented plane $h_1$. Note that $S(h_1)$ is $S(h_0)$ mirrored at the $z$-axis. Similarly, $A(h_1)$ is $A(h_0)$ mirrored at the $z$-axis. (The first coordinate of a point corresponds to the slope of the dual line; hence, mirroring along the $z$-axis, which is the $y$-axis of the plane, corresponds to inverting all slopes, which in turn is equivalent to mirroring along the $z$-axis.) We will now study what type of changes can occur in the intersection pattern of the middle levels during this rotation.

As we assume that each color class of $L$ contains an odd number of lines, all three median levels are $x$-monotone curves along lines of the arrangement, except in the degenerate cases where the plane is parallel to one of the lines. We will look at these degenerate cases later. For now, suppose no line is parallel to $h$. We walk along the green median level $g(h)$ (which is now just a curve) and look at the order of intersections with $r(h)$ and $b(h)$ from left to right. We say that a level $m$ intersects $g(h)$ from below if $m$ is below $g(h)$ before the intersection and above afterwards. Similarly, $m$ can intersect $g(h)$ from above. Note that if an intersection of $m$ and $g(h)$ is from below, then the next intersection has to be from above. We will call intersections of $r(h)$ with $g(h)$ red intersections and intersections of $b(h)$ with $g(h)$ blue intersections. Further note that when reorienting $h$, the order of intersections reverses, and all intersections from above become intersections from below and vice versa.

When rotating the plane $h$, the dual line arrangement and thus the median levels $r(h)$, $b(h)$, and $g(h)$ change continuously as long as no line becomes parallel to $h$. Note that the order of intersection along $g(h)$ can only change when the order type of $S(h)$ (and thus also $A(h)$) changes. There are three ways in which the order type of $A(h)$ can change. The first is when a line moves over the intersection of two other lines. In this case, the order of intersection along $g(h)$ can be affected in two ways, described in items (1) and (2). The second change in the order type of $A(h)$ occurs when two lines become parallel. The
effects of this are described in item (3) below. Finally, a last type of change occurs in the degenerate case when the plane becomes parallel to one of the lines. This situation is described in item (4) below. In any of these changes, it is of course possible that the order of intersection along \( g(h) \) is not affected. Thus, there are the four ways in which the order of intersection along \( g(h) \) can change during a rotation of \( h \).

(1) The first is that a red intersection and a blue intersection switch places (i.e., the relative order in which they appear along \( g(h) \)). In this case, the two intersections coincide at some point in time, and the point at which they coincide is an intersection that we are looking for; in particular, this means that three lines of \( A(h) \) meet in a single point. See Figure 3.2 for an illustration.

(2) The second case happens when (without loss of generality) two red intersections, the first from below, the second from above, coincide. Then, rotating further, the two intersections vanish. This type of change does not indicate a solution. See Figure 3.3 for an illustration.

(3) The third case happens when the last segments of \( g(h) \) and (without loss of generality) \( r(h) \) become parallel. Then the last red intersection vanishes, but it reappears as the first red intersection as the rotation of \( h \) continues. Furthermore, if this intersection was from above, it is now from below, and vice versa. Also this type of change will not give us a solution. See Figure 3.4 for an illustration. (This case does not give a change when considering \( h \) as a projective plane; however, as we prefer to have an order on the intersections along \( g(h) \), we consider \( h \) to be Euclidean.)

(4) Finally, consider the case when the plane becomes parallel to one of the lines; call it \( \ell \) and assume without loss of generality that it is red. See Figure 3.5 for an illustration. Let \( m \) be the number of red lines. In this case, the point of intersection of \( \ell \) and \( h \) moves towards infinity, vanishes when \( h \) is parallel to \( \ell \), and then enters from the other side. In the dual, this corresponds to a
3.2. A Combinatorial proof

Figure 3.2: A red intersection and a blue intersection switching places.

line \(a\) getting steeper and steeper, until it vanishes when \(h\) is parallel to \(\ell\), and then rotating further, changing the sign of its slope. (It is useful to consider \(a\) to become vertical at this limit instead of vanishing.) In this situation, the median level \(r(h)\) changes quite drastically: assume without loss of generality that \(a\) has negative slope before it becomes orthogonal and positive slope afterwards. Then, shortly before \(a\) vanishes, it has the smallest slope of all red lines. Afterwards, it has the largest slope of all red lines. Furthermore, the new median level (i.e., \(\lfloor \frac{m}{2} \rfloor\)-level) of the red arrangement is the former \((\lfloor \frac{m}{2} \rfloor - 1)\)-level to the left of the intersection of \(a\) and \(r(h)\), and it is the former \((\lfloor \frac{m}{2} \rfloor + 1)\)-level to the right of this intersection. In particular, several pairs of red intersections might have vanished and several red and blue intersections might have switched places. However, at the moment when \(a\) is vertical (or actually vanishes), it plays no role for the median level. Thus, the red median level is actually the whole area between the former \((\lfloor \frac{m}{2} \rfloor - 1)\)-level and the former \(\lfloor \frac{m}{2} \rfloor\)-level to the left of \(a\), and the area between the former \(\lfloor \frac{m}{2} \rfloor\)-level and the former \((\lfloor \frac{m}{2} \rfloor + 1)\)-level to the right of \(a\). Thus, if a red and a blue intersection switch places, the blue intersection lies inside the red median level at the moment when \(a\) is vertical, and we again have a solution. (In \(S(h)\), this intersection corresponds to a line through a green and a blue point, bisecting an even number of red points.) The same argument holds if \(\ell\) is blue and a similar argument applies if \(\ell\) is green.

In conclusion, we can track the intersections of \(r(h)\) and \(b(h)\) along \(g(h)\) and we find a solution whenever a red intersection and a blue
intersection switch places. Furthermore, we know that after rotating all the way from $h_0$ to $h_1$ the order of intersections has reversed and all intersections that were from above are now from below and vice versa. Thus, what we want to show is that at some point during the rotation, a red and a blue intersection need to switch places. We will do this in a purely combinatorial setting in the next section.

### 3.2.2 Bi-chromatic Sign Sequences

We define a *bi-chromatic sign sequence* $S$ as a finite sequence of signs $s \in \{+,-\}$ that satisfies the following conditions:

1. each symbol $s$ is either red or blue;
2. the subsequence of all symbols of one color has odd length and alternates between $+$ and $-$. 

![Figure 3.5: A red line becoming vertical.](image)
3.2. A Combinatorial proof

Figure 3.6: Three median levels and their corresponding bi-chromatic sign sequence.

Note that the red and blue intersections along \( g(h) \) define a bi-chromatic sign sequence when writing intersections from above as + and intersections from below as −. See Figure 3.6 for an illustration.

We define the inverse of a sign \( s \), denoted by \( \bar{s} \) as the other sign, but in the same color. We further define the reverse of \( S = s_1, \ldots, s_k \), denoted by \( \overbar{S} \), as \( \bar{s}_k, \ldots, \bar{s}_1 \). Note that if \( S \) is the sequence of red and blue intersections for some \( h \), then reorienting \( h \) gives rise to \( \overbar{S} \).

We consider the following operations on bi-chromatic sign sequences:

1. **Switch:** Let \( s_i \) and \( s_{i+1} \) be two consecutive elements of \( s_1, \ldots, s_k \) of different color. Then we can switch \( s_i \) and \( s_{i+1} \) to get a new sequence \( s_1, \ldots, s_{i-1}, s_{i+1}, s_i, \ldots, s_k \).

2. **Insertion/Deletion:** Let \( s_i \) and \( s_{i+1} \) be two consecutive elements of \( s_1, \ldots, s_k \) of the same color. (Hence, their signs are different.) Then we can delete \( s_i \) and \( s_{i+1} \) to get a new sequence \( s_1, \ldots, s_{i-1}, s_{i+2}, \ldots, s_k \). Similarly, let either (i) \( s_i \) and \( s_j \) be two
elements of the same color that are consecutive in the subsequence of their color, or (ii) $s_j$ be the first element of its color in $S$ and $s_j = \overline{s_i}$ or (iii) $s_i$ be the last element of its color in $S$ and $s_i = \overline{s_j}$. Then we can insert $\overline{s_i}$ and $\overline{s_j}$ consecutively (w.r.t. the entire sequence) somewhere after $s_i$ before $s_j$ to get a new sequence $s_1, \ldots, s_i, \ldots, \overline{s_i}, \overline{s_j}, \ldots, s_j, \ldots, s_k$. (Case (ii) and (iii) correspond to an insertion at the beginning or the end of $S$, respectively.)

3. **Transfer:** Let $s_k$ be the last element of the sequence. Then we can transfer $s_k$ to the beginning of the sequence to get a new sequence $\overline{s_k}, s_1, \ldots, s_{k-1}$. Similarly, we can transfer the first element $s_1$ to the end of the sequence to get a new sequence $s_2, \ldots, s_k, \overline{s_1}$.

These operations describe exactly the changes that can happen for a sequence of red and blue intersections: Case 1 translates to a switch, Case 2 gives an insertion or a deletion, Case 3 corresponds to a transfer. For Case 4, we need the following lemma:

**Lemma 3.6.** Consider the degenerate case when the plane becomes parallel to one of the lines. Assume that there is no solution at this point of rotation. Then the change in the bi-chromatic sign sequence defined by the red and blue intersections along $g(h)$ can be described as a sequence of insertions, deletions and transfers.

**Proof.** Assume without loss of generality that the line $\ell$ that the plane becomes parallel to is red. In the dual of the cross-section, we will, as above, consider the dual line $a$ to become vertical. Assume again without loss of generality that $a$ has negative slope before it becomes orthogonal and positive slope afterwards. Let $r_0(h)$ and $r_1(h)$ be the red median levels right before and just after $a$ is vertical. From the assumption that there is no solution at this point of rotation, we conclude that all intersections of $b(h)$ with $g(h)$ must be either below or above both $r_0(h)$ and $r_1(h)$. In particular, the only changes that can happen between two consecutive intersections of $b(h)$ with $g(h)$ are insertions and deletions. It remains to analyze the changes before the first and
after the last intersection of $b(h)$ with $g(h)$. For this, we distinguish two cases. In the first case, both $r_0(h)$ and $r_1(h)$ start on the same side of $g(h)$, without loss of generality above. Note that then both $r_0(h)$ and $r_1(h)$ need to end below $g(h)$. In particular, the same argument as above also holds before the first and after the last intersection of $b(h)$ with $g(h)$. In the second case, $r_0(h)$ starts above $g(h)$, whereas $r_1(h)$ starts below $g(h)$. Then, $r_0(h)$ ends below $g(h)$ and $r_1(h)$ ends above $g(h)$. In particular, without loss of generality, the bi-chromatic sign sequence for $r_0(h)$ starts with a red $+$, while the bi-chromatic sign sequence for $r_1(h)$ ends with a red $-$. This change can be described by a transfer. Again, by the above arguments, all other changes correspond to insertions and deletions. The analogous proof holds when the line $\ell$ that the plane becomes parallel to is blue and similar arguments apply when $\ell$ is green.  

In fact, it can be shown that Case 4 can always be described as a sequence of insertions, deletions, transfers and switches, but the above lemma is enough for our purposes. Let us summarize our findings so far:

**Summary 1.** Let $\mathcal{L}$ be an arrangement of $n$ lines in $\mathbb{R}^3$ colored red, green, and blue, none of which intersect the $z$-axis. For every oriented plane $h$ containing the $z$-axis, the induced cross-section defines a bi-chromatic sign sequence. Further, if $h_0$ and $h_1$ are two oriented planes and $S_0$ and $S_1$ are their induced bi-chromatic sign sequences, rotating $h_0$ until it coincides with $h_1$ gives rise to a sequence of bi-chromatic sign sequences from $S_0$ to $S_1$, where two consecutive sign sequences differ by a switch, insertion, deletion or transfer. Finally, every switch used in this transformation from $S_0$ to $S_1$ corresponds to a cross-section for which there exists a line that simultaneously bisects all three classes of points.

We will show that it is not possible to transform a sequence $S$ to $\overline{S}$ using only insertions, deletions and transfers. To this end, we first define the *reduced sign sequence* of a bi-chromatic sign sequence: look at a
consecutive subsequence of elements of the same color (i.e., elements of one color such that no element of the other color is between them). If this subsequence has even length, remove it. Iteratively removing such consecutive mono-chromatic subsequences of even length leaves us with a sequence that alternates between the colors. Further, it is easy to check that this sequence is still a bi-chromatic sign sequence. Finally, as both the subsequence of red elements as well as the subsequence of blue elements have odd length, the new sequence is not empty and it starts with a different color than it ends with (a fact that will be crucial in proving the following lemmas). We call the sequence obtained this way the reduced sign sequence of a bi-chromatic sign sequence $S$, and denote it by $\rho(S)$.

At last, we arrive at our final definition. For a reduced sign sequence $\rho(S)$, we consider two functions. The first, denoted by $\Sigma(\rho(S))$, is the sum of “+”-symbols minus the number of “−”-symbols in $\rho(S)$. Note that the value of this function is always in \{-2, 0, 2\}. The second function, denoted by $\lambda(\rho(S))$, is 2 whenever the first symbol of $\rho(S)$ is red and 0 otherwise. We now define the trace $\tau$ of a bi-chromatic sign sequence as follows:

$$\tau(S) := \frac{\Sigma(\rho(S)) + \lambda(\rho(S))}{2} \mod 2.$$

We continue by proving a few crucial lemmas about traces.

**Lemma 3.7.** The trace of a bi-chromatic sign sequence is invariant under insertions, deletions and transfers.

**Proof.** It follows immediately from the definitions that the reduced sign sequences, and thus the trace, is invariant under insertions and deletions. As for transfers, consider the two sequences $S = s_1, \ldots, s_k$ and $S' = s_2, \ldots, s_k, \bar{s}_1$. If their reduced sign sequences are equal, so are their traces, so assume that their reduced sign sequences are different. In this case we claim that we either have (a) $\rho(S) = r_1, r_2, \ldots, r_m$ and $\rho(S') = r_2, \ldots, r_m, \bar{r}_1$ or (b) $\rho(S) = r_1, r_2, \ldots, r_m$ and $\rho(S') =$
\(r_m, r_1, \ldots, r_{m-1}\). Assume without loss of generality that \(r_1\) is red, and thus \(r_m\) is blue. We will distinguish the cases where \(s_1\) is red or blue. If \(s_1\) is red we can assume without loss of generality that \(s_1 = r_1\). In particular, \(s_1\) will be the last element of \(\rho(S')\). Further, as all blocks of the same color in \(S\) except the first and last one do not change their length with a transfer, all other elements in \(\rho(S)\) stay the same and we are in case (a). If \(s_1\) is blue, the transfer will either create or destroy an odd-length blue block in the beginning of \(S\). In particular, \(\rho(S')\) will begin with a blue element \(r_0\). On the other hand, the transfer will either create or destroy an even-length blue block at the end of \(S\) and thus \(\rho(S')\) will end with the red element \(r_{m-1}\). Since \(\rho(S')\) is still a bi-chromatic sign sequence, we conclude that \(r_0 = r_m\) and we are in case (b). In both cases we deduce that \(\Sigma(\rho(S')) = \Sigma(\rho(S)) \pm 2\) and \(\lambda(\rho(S')) = \lambda(\rho(S)) \pm 2\). Thus \(\tau(S') := \frac{1}{2}(\Sigma(\rho(S)) + \lambda(\rho(S)) \pm 4) \mod 2 = \tau(S) \pm 2 \mod 2 = \tau(S)\).

Note that the above in general does not hold for switch operations. While there is no change in the sum of the signs in the sign sequence by a switch operation, this does not hold for the reduced sign sequence, as the parities of adjacent substrings of the same color changes.

**Lemma 3.8.** The traces of a bi-chromatic sign sequence \(S\) and its reverse \(\overline{S}\) are different.

*Proof.* As in the reverse of a sequence all signs change, we have \(\Sigma(\rho(\overline{S})) = -\Sigma(\rho(S))\). Further, the first color in \(\rho(\overline{S})\) is the last color in \(\rho(S)\) and thus different from the first color of \(\rho(S)\). Hence, we have \(\lambda(\rho(\overline{S})) = \lambda(\rho(S)) \pm 2\). Plugging this into the definition of \(\tau\), we conclude that \(\tau(\overline{S}) \neq \tau(S)\). \(\square\)

Combining Lemma 3.7 and Lemma 3.8, we conclude the following:

**Corollary 3.9.** Any sequence of operations transforming a bi-chromatic sign sequence \(S\) into its reverse \(\overline{S}\) contains at least one switch.

Combining this with Summary 1, we obtain a proof of the existential
part of Theorem 3.1. In particular, knowing the traces of $S$ and $\overline{S}$, computing the trace for a cross-section between $S$ and $\overline{S}$ also tells us in which direction a switch has to occur. In order to have a sidedness oracle for a plane in our original problem, we thus need to efficiently compute the trace of the reduced sign sequence of the cross-section determined by this plane.

**Lemma 3.10.** The reduced sign sequence of a cross-section of $n$ lines can be obtained in $O(n^{4/3} \log^{1+\epsilon}(n))$ time.

*Proof.* We first compute each color class of the bi-chromatic sign sequence separately. To this end, we walk the median level of the line arrangement of the two involved color classes, keeping track of the levels of them (note that every intersection between the two median levels lies in the median level of their total arrangement). That is, we walk the median level of the red and the green lines, and then the median level of the blue and the green lines. This can be done in $O(m \log^{1+\epsilon}(n))$ time \[31, 42\], where $m$ is the complexity of the median level. The current best bound for the complexity of a median level of an arrangement of $n$ lines is $O(n^{4/3})$ \[36\]. Further, the above argument shows that the number of elements of each color class in the bi-chromatic sign sequence is bounded by the complexity of the corresponding median level of the arrangement of two colors. Given the two subsequences, we can easily merge them to the order in which they appear from left to right into the bi-chromatic sign sequence. The time this requires is linear in the length of that sequence, which we have just observed to be in $O(n^{4/3})$. Finally, given a bi-chromatic sign sequence, its reduced sign sequence can easily be computed in time linear in the length of the bi-chromatic sign sequence.

All that remains now for the sidedness oracle is to compute the trace of the sign sequence, which can again be easily done in time linear in the length of the reduced sign sequence.

**Corollary 3.11.** Given a plane through the $z$-axis, we can determine in which direction of rotation there exists a bisecting cross-section in
3.3 The Algorithm

We may now use the above insight to devise an algorithm to find the defining plane of a cross-section with a simultaneously bisecting line. Whether there is a bisecting line for a cross-section only depends on its order type (rather than on the actual coordinates of the individual points). We first observe that the points of a cross-section belonging to three lines become collinear only a constant number of times during the rotation of $h$: W.l.o.g., let $\vec{n} = (t, 1, 0)^T$ be the normal vector of $h$ at time $t$. Then the coordinates of the intersection point of a line and $h$ is a function in $t$. Three points $(p, q, r)$ become collinear if the determinant

\[
\begin{vmatrix}
  x_p & x_q & x_r \\
  y_p & y_q & y_r \\
  1 & 1 & 1
\end{vmatrix} = 0
\]

is zero. Since we have a rational function of constant degree, the number of times at which three points of a cross-section become collinear is constant. Further, there is a change in the order type if one of the lines is parallel to $h$, which happens only once. In total, such a cross-section of $n$ lines thus determines $O(n^3)$ different order types. (See [3, 66] for similar observations regarding translated planes.) Checking all of them would already give a polynomial time algorithm, but we want to do something faster.

3.3.1 A general framework

We will now develop an algorithmic framework that allows us to to find the defining plane of a cross-section with a simultaneously bisecting line efficiently. For our framework we will assume the following input:
1. a continuously changing family of point sets $P_t, t \in [0, 1]$ of size $n$ with the property that each triple of points changes their orientation only a constant number of times and for each triple these events can be found in constant time. This family contains some solution point sets $P_x$.

2. a sidedness oracle $S$ which for every $t \in [0, 1]$ computes in time $T(n)$ a value $S(t) \in \{-1, 0, 1\}$ with the following properties:
   
   (a) $S(0) = -1$;
   (b) $S(1) = 1$;
   (c) if $S(t) = 0$ then $P_t$ is a solution point set;
   (d) for every $a < b$ such that $S(a) = -1$ and $S(b) = 1$ or vice versa, the interval $[a, b]$ contains some $t$ with $S(t) = 0$.

For example, the family of point sets could be defined by points in two colors, red and blue, moving at constant speed along trajectories that are described by polynomials of constant degree, the blue points starting left and ending right of the $y$-axis, the red points starting right and ending left of the $y$-axis. The solution point sets could then be all point sets where both the blue and the red Tukey median regions can be intersected by a single vertical line. A sidedness oracle could then be constructed by computing both Tukey median regions and checking if they are separated by a vertical line, returning 0 if not, or $-1$ or 1, depending on which color is to the left of the vertical separating line. The runtime of the sidedness oracle in this example would be $O(n \log^3(n))$, see [63].

We will use Megiddo’s [73] parametric search technique in the following way. Let $P_a$ and $P_b$ be two point sets with $S(a) = -1$ and $S(b) = 1$ or vice versa (we may start with $P_a = P_0$ and $P_b = P_1$). We therefore know that there is at least one solution point set $P_x$ between them. The high-level idea is to construct a representation of the order type of the (unknown) point set $P_x$ by simulating a parallel sorting algorithm for each point of $P_x$. In each iteration, the sorting algorithm produces a
3.3. The Algorithm

batch of $O(n)$ comparisons that need to be answered. Each comparison is related to a constant number of events. There is a total order in which these events occur when going from $P_a$ to $P_b$. By selecting the median of these events in time $O(n)$, we can answer approximately half of the events in each batch by applying the sidedness oracle once. We repeat this $O(\log(n))$ times to answer all the events of the batch. A batch therefore can be processed in $O((n + T(n)) \log(n))$ time. In addition, we update $P_a$ and $P_b$ at each application of the sidedness oracle to narrow down the interval containing our solution $P_x$. For one run of the sorting algorithm (requiring $O(\log(n))$ batches), this amounts to $O((n + T(n)) \log^2(n))$ time. In total, we require $n$ runs of the sorting algorithm, giving the total running time of $O((n + T(n))n \log^2(n))$.

After this high-level exposition, we now turn to the detailed description of our algorithm, where we will also describe how to further speed it up.

**Order types and local sequences.** The combinatorial representation of the order type we will use is the set of local sequences of unordered switches (for short, *local sequences*), devised by Goodman and Pollack [46]. (See also the work by Streinu [90] and by Felsner and Weil [45] on that topic.) For any point $p$ of a point set $S$, consider the rotating line with pivot $p$; the order in which this line passes over the other points of $S$ is the local sequence of $p$ in $S$. (While the local sequence is cyclic, it can be linearized by letting it start at the element of $S \setminus \{p\}$ with, say, the smallest label.) The local sequences of all the points of $S$ determine the order type of $S$ [46]. We construct these sequences by considering $n$ different (but not independent) sorting problems: for each point $p \in S$, sort the points of $S \setminus \{p\}$ around $p$ as they appear along the local sequence.

**Determining the order type on $P_x$.** Let $P = P_x$ be the unknown solution point set. For each point $p \in P$, we run a parallel sorting algorithm that sorts the points of $P \setminus \{p\}$ according to the unsigned
local sequence of $p$. Pick an arbitrary point $q \in P \setminus \{p\}$ with which the sequence should start (recall that it is actually circular). This sorting algorithm has $O(\log(n))$ iterations. In each iteration, we get a batch $B$ of $(n - 1)/2$ point pairs of $S \setminus \{p\}$; such a pair $(a, b)$ corresponds to a comparison of $a$ and $b$ in the order around $p$, that can be answered by knowing the order type of the four points $p, q, a, b$. Between $P_a$ and $P_b$, there are at most a constant number of events at which this order type changes (all of which can be obtained in constant time). Thus, for the batch $B$ of comparisons, we get a list $E$ of such events, whose size is in $O(|B|) = O(n)$. Each event is associated with a value $t$ in the interval $[a, b]$. As soon as we know for each event the side on which the solution $P$ lies, we can answer all the comparisons of the batch $B$. We pick the median of these events in $O(n)$ time, which gives us a point set $P'$ at which the event happens. We apply the sidedness oracle to determine the side of $P'$ containing $P$ (or find a solution, if the oracle returns 0). Also, we update $P_a$ or $P_b$ by setting one to $P'$, maintaining the invariant that $P$ is between $P_a$ and $P_b$. Hence, in $T(n) + O(n)$ time, we can process approximately half of the events; after $O(\log(n))$ iterations of picking the median of the remaining events and narrowing the interval between $P_a$ and $P_b$, all events $E$ of the batch $B$ have been answered. That is, we are left with an interval between $P_a$ and $P_b$ that contains the solution $P$ and within which the orientation of the point triples involved in the comparisons does not change. In total, the parallel sorting algorithm requires us to answer $O(\log(n))$ batches to sort the points around $p$. Hence, after $O((n + T(n)) \log^2(n))$ time, the local sequence of a point $p$ in the point set $P$ is determined. We do this for all $n$ points of $P$, further narrowing down the interval between $P_a$ and $P_b$. Thus after a total of $O((n + T(n))n \log^2(n)) = O(n^2 \log^2(n) + T(n)n \log^2(n))$ steps, the local sequence of all the points and thus the order type of $P$ is determined. Further, in the final interval $[a, b]$, this order type does not change, meaning that a solution is found.

**Speeding up.** As an anonymous reviewer of the paper [79] noted, the above algorithm can be sped up by using a slight variation of the
3.3. The Algorithm

The parallel sorting algorithm: by bundling together $O(n)$ batches of all $n$ runs of the sorting algorithm, we get $O(\log(n))$ batches, each of size $O(n^2)$. For each such batch, we pick the median of the events in $O(n^2)$ time and apply the sidedness oracle to determine the side of $P'$ containing $P$ (or find a solution, if the oracle returns 0). Hence, in $T(n) + O(n^2)$ time, we can process approximately half of the events, leaving us again with $O(\log(n))$ iterations to answer all events of the batch. As we again have $O(\log(n))$ batches, the local sequence of all the points and thus the order type of $P$ can thus be determined in time $O((n^2 + T(n)) \log^2(n)) = O(n^2 \log^2(n) + T(n) \log^2(n))$.

This algorithm proves the following:

**Theorem 3.12.** Given an input to the framework described above, a solution point set $P_x$ can be found in time $O(n^2 \log^2(n) + T(n) \log^2(n))$.

3.3.2 Applying the framework

We may now use the above framework to devise an algorithm to find the defining plane of a cross-section with a simultaneously bisecting line. We first recall that the existence of a bisecting line depends only on the order type of the cross-section. In this terminology, we observe that there exists a bisection only if three points of the cross-section become collinear (i.e., when the median levels of the three classes intersect in a single point), or when a line of $\mathcal{L}$ becomes parallel to the defining plane $h$ (and thus the intersection of two median levels is inside the two-dimensional region defined by the third one). In particular, the latter events constitute a change in the order type that is not captured by our framework. When rotating the plane $h_0$ by an angle of $\pi$, there are $n$ events in which a line becomes collinear to the plane. We will now describe how we handle these events.

We start by arbitrarily choosing $h_0$ (and thus $h_1$). For each line in $\mathcal{L}$, we can determine in constant time the plane between $h_0$ and $h_1$ that is parallel to it. For each of these $n$ planes, we would like to compute the
reduced sign sequence. If the solution is at one of these \( n \) events, we are done. Otherwise, the sign sequences give us two planes between which there is a solution, and between which there is no event in which a line becomes parallel to the rotating plane. This interval (or a solution) can be found using binary search in \( O(n \log(n) + T(n) \log(n)) \) time. We let \( h_a \) and \( h_b \) be these two planes.

From the observations above, it follows that the number of the other events is in \( O(n^3) \); in particular, recall that the number of times a point triple in the cross-section changes its order type is constant. Further, Corollary 3.11 gives us a sidedness oracle for these events. We can thus apply the above framework where \( P_0 \) and \( P_1 \) are the cross-sections defined by \( h_a \) and \( h_b \), respectively. The runtime of the framework dominates the above computations, thus the total runtime of our algorithm is \( O(n^2 \log^2(n) + T(n) \log^2(n)) \). Recall that from Corollary 3.11 we have \( T(n) \in O(n^{4/3} \log^{1+\epsilon}(n)) \), hence the algorithmic statement of Theorem 3.1 follows.
In this chapter we consider the more general case of assignments of mass distributions to all linear subspaces and we generalize the Center Transversal Theorem to this setting. Recall the main idea of the proof of the Center Transversal Theorem: for a mass and for each subspace, consider the centerpoint of the projection of the mass to this subspace. This defines a section in the canonical bundle over the Grassmann manifold. In total, we thus get a section for each mass, and for the correct numbers of masses, it can be shown, that on some subspace all the sections coincide in a single point. Taking the orthogonal complement of the subspace through that point is then a Center transversal.

This idea generalizes to mass assignments, the only difference being that we have to consider flags instead of subspaces: we first look at all subspaces of dimension $k$ on which the mass assignments are defined, and in each of them, we project the mass to every $n$-dimensional sub-
space, where \( n \leq k \). Considering the centerpoints of these projections, we get sections in the canonical \( n \)-dimensional bundle over some flag manifold. We show that still, many of these sections coincide on some flag, even for the more general case of complete flags.

Recall that the complete flag manifold \( \tilde{V}_{n,n} \) is the manifold of all complete flags of \( \mathbb{R}^n \). In Section 1.1, we have defined a canonical bundle for each \( V_i \), which we denoted by \( \vartheta_i^n \).

**Lemma 4.1.** Let \( s_1, \ldots, s_{m+1} \) be \( m+1 \) sections of the canonical bundle \( \vartheta_{m+l}^l \). Then there is a flag \( F \in \tilde{V}_{m+l,m+l} \) such that \( s_1(F) = \ldots = s_{m+1}(F) \).

This Lemma is a generalization of Proposition 2 in [95] and Lemma 1 in [37]. Our proof follows the proof in [95].

**Proof.** Consider the sections \( q_i := s_{m+1} - s_i \). We want to show that there exists a flag \( F \) for which \( q_1(F) = \ldots = q_m(F) = 0 \). The sections \( q_1, \ldots, q_m \) determine a unique section in the \( m \)-fold Whitney sum of \( \vartheta_{m+l}^l \), which we denote by \( W \). Note that \( W \) has base \( \tilde{V}_{m+l,m+l} \) and fiber dimension \( ml \). We will show that \( W \) does not admit a nowhere zero section. For this, it suffices to show that the highest Stiefel-Whitney class \( w_{ml}(W) \) is nonzero (see [74], §4, Proposition 4).

By the Whitney product formula we have \( w_{ml}(W) = w_l(\vartheta_{m+l}^l)^m \). Note that the projection \( f : \tilde{V}_{m+l,m+l} \to G_l(\mathbb{R}^{m+l}) \) defined by \( (V_0, \ldots, V_l, \ldots, V_{m+l}) \mapsto V_l \) induces a bundle map from \( \vartheta_{m+l}^l \) to \( \gamma_{m+l}^l \). Thus by the naturality of Stiefel-Whitney classes we have \( w_l(\vartheta_{m+l}^l)^m = f^*(w_l(\gamma_{m+l}^l)^m) = f^*(w_l(\gamma_{m+l}^l))^m \). Further, we have the following commutative diagram

\[
\begin{array}{ccc}
\tilde{V}_{m+l,m+l} & \xrightarrow{i} & \tilde{V}_{\infty,m+l} \\
\downarrow f & & \downarrow g \\
G_l(\mathbb{R}^{m+l}) & \xrightarrow{j} & G_l(\mathbb{R}^{\infty})
\end{array}
\]
where $i$ and $j$ are inclusions and $g$ is the canonical map from $\tilde{V}_{\infty,m+l}$ to $G_l(\mathbb{R}^{\infty})$ (see e.g. [49, 95]). In $\mathbb{Z}_2$-cohomology, we get the following diagram:

$$
\begin{array}{ccc}
H^*(\tilde{V}_{m+l,m+l}) & \xleftarrow{i^*} & H^*(\tilde{V}_{\infty,m+l}) \\
\uparrow{f^*} & & \uparrow{g^*} \\
H^*(G_l(\mathbb{R}^{m+l})) & \xleftarrow{j^*} & H^*(G_l(\mathbb{R}^{\infty}))
\end{array}
$$

It is known that $H^*(\tilde{V}_{\infty,m+l})$ is a polynomial algebra $\mathbb{Z}_2[t_1, \ldots, t_{m+l}]$ and that $g^*$ maps $H^*(G_l(\mathbb{R}^{\infty}))$ injectively onto the algebra

$$
\mathbb{Z}_2[\sigma_1, \ldots, \sigma_l] \subset \mathbb{Z}_2[t_1, \ldots, t_l] \subset \mathbb{Z}_2[t_1, \ldots, t_{m+l}],
$$

where $\sigma_i$ denotes the $i$'th symmetric polynomial in the variables $t_1, \ldots, t_l$ [49, 74]. Further, $H^*(\tilde{V}_{m+l,m+l})$ is a polynomial algebra

$$
H^*(\tilde{V}_{m+l,m+l}) \simeq \mathbb{Z}_2[t_1, \ldots, t_{m+l}]/(\sigma_1, \ldots, \sigma_{m+l})
$$

and $i^*$ is the corresponding quotient map. Since $w_l(\gamma_l^{m+l}) = j^*(\sigma_l)$, we have $w_l(\phi_l^{m+l})^m = f^*(j^*(\sigma_l))^m$ and in particular $w_l(\phi_l^{m+l})^m = 0$ would imply that $(\sigma_l)^m \in \ker i^*$, i.e. $(t_1 \cdots t_l)^m$ is in the ideal $(\sigma_1, \ldots, \sigma_{m+l})$. But this is a contradiction to Proposition 2.21 in [93].

Consider now a continuous map $\mu : \tilde{V}_{m+l,m+l} \to M_l$, which assigns an $l$-dimensional mass distribution to $V_l$ for every flag. We call such a map an $l$-dimensional mass assignment on $\tilde{V}_{m+l,m+l}$.

**Corollary 4.2.** Let $\mu_1, \ldots, \mu_{m+1}$ be $l$-dimensional mass assignments on $\tilde{V}_{m+l,m+l}$. Then there exists a flag $\mathcal{F} \ni V_l$ such that some point $p \in V_l$ is a centerpoint for all $\mu_1^F, \ldots, \mu_{m+1}^F$.

**Proof.** For every $\mu_i$ and every flag $\mathcal{F}$, the centerpoint region of $\mu_i^F$ is a convex compact region in the respective $V_l$. In particular, for each $\mu_i$ we get a multivalued, convex, compact section $s_i$ in $\phi_l^{m+l}$. Using Proposition 1 from [95], which states that under the condition that there is no nowhere zero section there is also no nowhere zero
multivalued, convex, compact section, Lemma 4.1 implies that there is a Flag in which all $s_i$’s have a common point $p$. □

Alternatively, we could avoid multivalued sections by considering the barycenter of the centerpoint regions.

We can now deduce Theorem 1.6.

**Theorem 1.6.** Let $\mu_1, \ldots, \mu_{n+d-k}$ be mass assignments on $G_k(\mathbb{R}^d)$, where $n \leq k \leq d$. Then there exists a $k$-dimensional linear subspace $h$ such that $\mu^h_1, \ldots, \mu^h_{n+d-k}$ have a common $(n-1,k)$-center transversal.

**Proof.** Note that a $(n-1,k)$-center transversal in a $k$-dimensional space is a common centerpoint of the projection of the masses to a $k-(n-1)$-dimensional subspace. Consider a flag $\mathcal{F} = (V_0, \ldots, V_d)$. For each mass assignment $\mu_i$ define $\mu_i'(\mathcal{F}) := \pi_{k-(n-1)}(\mu_i^V_k)$, where $\pi_{k-(n-1)}$ denotes the projection from $V_k$ to $V_{k-(n-1)}$. Every $\mu_i'$ is an $(k-(n-1))$-dimensional mass assignment on $\tilde{V}_{d,d}$.

The result now follows from Corollary 4.2 by setting $l = k - (n-1)$ and $m = d - k + n - 1$. □
CHAPTER 5

A Generalization of the Centerpoint Theorem

The main goal of this chapter is to prove Theorem 1.7:

**Theorem 1.7.** Let $\mu$ be a mass distribution in $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) = 1$. Let $\alpha_1, \ldots, \alpha_k$ be non-negative real numbers such that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_k$ and for every $i, j$ with $i + j \leq k + 1$ we have that $(d-1)\alpha_k + \alpha_i + \alpha_j \leq 1$. Then there are $k$ points $p_1, \ldots, p_k$ in $\mathbb{R}^d$ such that for each closed half-space $h$ containing $j$ of the points $p_1, \ldots, p_k$ we have $\mu(h) \geq \alpha_j$.

To that end, we will start with a simpler problem in Section 5.1, where the underlying mass distribution is a point set in the plane, and we want to find just two representatives $p_1$ and $p_2$ for fixed $\alpha_1 = \frac{1}{5}$ and $\alpha_2 = \frac{2}{3}$. Even though this result is a special case of Theorem 1.7, we still show its proof for two reasons: first, the algorithm presented in Section 5.4 relies heavily on the presented proof and, secondly, the proof already illustrates the main ideas for the proof of Theorem 1.7. We then generalize the ideas in Section 5.2 and prove Theorem 1.7.
In Section 5.3 we prove Theorem 1.8. We finish the chapter by giving algorithms in the plane in both considered settings.

5.1 Two points

We prove the following result:

**Theorem 5.1.** Let \( P \) be a set of \( n \) points in general position in the plane. Then there are two points \( p_1 \) and \( p_2 \) in \( \mathbb{R}^2 \) such that

1. each closed half-plane containing one of the points \( p_1 \) and \( p_2 \) contains at least \( \frac{n}{5} \) of the points of \( P \) and

2. each closed half-plane containing both \( p_1 \) and \( p_2 \) contains at least \( \frac{2n}{5} \) of the points of \( P \).

*Proof.* Note that condition (1) is equivalent to the condition that every open half-plane containing more than \( \frac{4n}{5} \) of the points of \( P \) must contain both \( p_1 \) and \( p_2 \). Similarly, condition (2) is equivalent to the condition that every open half-plane containing more than \( \frac{3n}{5} \) of the points of \( P \) must contain one of \( p_1 \) and \( p_2 \). We will now construct two points \( p_1 \) and \( p_2 \) satisfying both these conditions.

Let \( C \) be the intersection of all open half-planes containing more than \( \frac{4n}{5} \) of the points of \( P \). Clearly \( C \) is convex. Also, note that \( C \) is closed. The centerpoint region is a strict subset of \( C \) and thus \( C \) has a non-empty interior. In order to satisfy condition (1), both \( p_1 \) and \( p_2 \) have to be placed in \( C \).

Let now \( H \) be the set of all open half-planes containing more than \( \frac{3n}{5} \) of the points of \( P \). For any \( h_i \) in \( H \) let \( c_i \) be the intersection of \( h_i \) and \( C \). In order to also satisfy condition (2), we need to find two points \( p_1 \) and \( p_2 \) such that every \( c_i \) contains at least one of them. To this end, we partition \( H \) into two subsets \( A \) and \( B \). The set \( A \) contains all half-planes that lie above their respective boundary lines. Analogously, \( B \) contains all half-planes that lie below their respective boundary lines.
For a half-plane $h_i$ that has a vertical boundary line, we put $h_i$ in $A$ if and only if it lies to the left of its boundary line.

Note that any three half-planes in $A$ have a non-empty intersection: Consider the intersection of all half-planes $h \in A$ with vertical boundary line. This is a closed half-plane $h'$ with vertical boundary line. Let $r$ be the upper intersection of the boundary line with the boundary of the convex hull of $P$. We claim that $r$ is in any half-plane of $A$. Indeed, if there was a half-plane in $A$ not containing $r$, it would contain a strict subset of the intersection of the convex hull of $P$ with $h'$; however, then we could also find a half-plane with vertical boundary line which does not contain $r$ and this would contradict the construction of $h'$. The analogous holds for $B$.

We will now show that for any two half-planes $h_1$ and $h_2$ in $A$, their corresponding regions $c_1$ and $c_2$ have a non-empty intersection. The same arguments hold for any two half-planes in $B$. Assume for the sake of contradiction that $c_1$ and $c_2$ do not intersect. As $C$ and $h_1 \cap h_2$ are convex, this means that there is an open half-plane $g$ containing more than $\frac{4n}{5}$ of the points of $P$ such that the intersection of the boundary lines of $h_1$ and $h_2$ lies in $\overline{g}$, the complement of $g$ (see Figure 5.1). In particular, $g \cap h_1$ is a strict subset of $\overline{h_2}$. As $\overline{g}$ contains strictly fewer than $\frac{n}{5}$ of the points of $P$ and $\overline{h_1}$ contains strictly fewer than $\frac{2n}{5}$ of the points of $P$, $g \cap h_1$ must contain strictly more than $\frac{2n}{5}$ of the points of $P$. However, being a subset of $\overline{h_2}$, which also contains strictly fewer than $\frac{2n}{5}$ of the points of $P$, this is a contradiction. Thus, by contradiction, $c_1$ and $c_2$ intersect.

As neither three half-planes in $A$ nor two half-planes in $A$ and $C$ have an empty intersection, Helly’s Theorem entails that there exists a point in both $C$ and all half-planes in $A$, i.e., all $c_i$s associated to $A$ have a non-empty intersection $D_A$. Again, the same holds for $B$, with a non-empty intersection $D_B$. Placing $p_1$ in $D_A$ and $p_2$ in $D_B$, we have thus constructed two points such that the conditions (1) and (2) hold.

This result is tight in the following sense: There is a point set for
A Generalization of the Centerpoint Theorem

which it is not possible to improve both conditions at the same time, that is, it is not possible to find two points such that any half-plane containing one of them contains strictly more than \( \frac{n}{5} \) of the points and any half-plane containing both of them contains strictly more than \( \frac{2n}{5} \) of the points. For this consider a set of \( n = 5k \) point arranged in the following way. Partition the points into 5 sets \( P_1, \ldots, P_5 \) of \( k \) points each. Place \( P_1, \ldots, P_5 \) in such a way that the convex hull of each \( P_i \) is disjoint from the convex hull of the union of the other four sets (see Figure 5.2).

Figure 5.1: Two \( c_i \)'s associated to \( A \) must intersect.

Denote by \( S_{i,j} \) the convex hull \( \text{CH}(P_i \cup P_j) \) of \( P_i \cup P_j \). Let \( \ell \) be a line through \( \text{CH}(P_i) \) and \( \text{CH}(P_j) \). Note that any other set \( P_m \) is not separated by \( \ell \) (i.e., lies entirely on one side). Let \( A_{i,j} \) be the side of \( \ell \)

Figure 5.2: A construction for which the bounds of Theorem 5.1 cannot be improved.
containing fewer of the other sets and let $B_{i,j}$ be the other side. For any point $q$ in $\text{CH}(P_1 \cup \ldots \cup P_5)$ we say that $q$ is above $S_{i,j}$ if it is not in $S_{i,j}$ but it is in $A_{i,j}$. Similarly, for any point $q$ in $\text{CH}(P_1 \cup \ldots \cup P_5)$ we say that $q$ is below $S_{i,j}$ if it is not in $S_{i,j}$ but it is in $B_{i,j}$. Suppose, for the sake of contradiction, that there exist two points $p_1$ and $p_2$ such that any half-plane containing one of them contains strictly more than $k$ of the points of $P_1 \cup \ldots \cup P_5$ and any half-plane containing both of them contains strictly more than $2k$ of the points of $P_1 \cup \ldots \cup P_5$. Consider two sets $P_i$ and $P_j$ such that $A_{i,j}$ contains exactly one other set. First we note that neither $p_1$ nor $p_2$ can lie above $S_{i,j}$ as otherwise we can find a half-plane containing that point and only one of the sets, i.e., only $k$ points. Similarly, we cannot place both $p_1$ and $p_2$ below $S_{i,j}$, as otherwise we can find a half-plane containing both points and only two of the sets, i.e., only $2k$ points. Also, we must clearly place both $p_1$ and $p_2$ in $\text{CH}(P_1 \cup \ldots \cup P_5)$. Thus, for any two sets $P_i$ and $P_j$ such that $A_{i,j}$ contains exactly one other set, $S_{i,j}$ must contain at least one of $p_1$ and $p_2$. However, there are five such $S_{i,j}$ and $P_1, \ldots, P_5$ can be placed in such a way that no three of them have a common intersection. So no matter how we place $p_1$ and $p_2$, one of the $S_{i,j}$ will be empty.

5.2 An arbitrary number of points

We now strengthen Theorem 5.1 in four ways: we allow for arbitrarily many query points, we extend it to higher dimensions, we consider mass distributions instead of point sets, and we give a range of possible bounds. We use the following observation, which follows from the fact that for an empty intersection of $d + 1$ half-spaces, any neighborhood with non-zero mass is completely contained in at most $d$ such half-spaces.

Observation 5.2. Let $\mu$ be a mass distribution in $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) = 1$. Let $h_1, \ldots, h_{d+1}$ be $d+1$ open half-spaces with $h_1 \cap \ldots \cap h_{d+1} = \emptyset$. Then $\mu(h_1) + \ldots + \mu(h_{d+1}) \leq d$. 
Proof of Theorem 1.7. The result is straightforward for $d = 1$, so assume $d \geq 2$. Again the condition that for each closed half-space $h'$ containing $j$ of the points $p_1, \ldots, p_k$ we have $\mu(h') \geq \alpha_j$ is equivalent to the condition that every open half-space $h$ with $\mu(h) > 1 - \alpha_j$ must contain at least $k - j$ of the points $p_1, \ldots, p_k$. Let $\alpha_0 = 0$. For $1 \leq j \leq k$, we call an open half-space $h$ a $j$-half-space if $1 - \alpha_{k-j+1} < \mu(h) \leq 1 - \alpha_{k-j}$. We thus want to find $k$ points such that each $j$-half-space contains at least $j$ of them.

Consider the $x_1$-$x_2$-plane, denoted by $X$, and for each vector $v = (v_1, v_2, \ldots, v_d)$ in $\mathbb{R}^d$ let $\pi(v) = (v_1, v_2, 0, \ldots, 0)$ be the projection of $v$ to $X$. Let $v_1, \ldots, v_k$ be $k$ unit vectors in $X$ with the property that the angle between any $v_i$ and $v_{i+1}$ is $\frac{2\pi}{k}$. Note that this implies that also the angle between $v_k$ and $v_1$ is $\frac{2\pi}{k}$. For each $v_i$ we construct a principal set $V_i$ of half-spaces as follows: For each $j$, consider all $j$-half-spaces. For any such half-space $h$, let $n(h)$ be the normal vector to its bounding hyperplane that points into $h$. Let $h$ be in $V_i$ if the angle between $\pi(n(h))$ and $v_i$ is at most $\frac{j\pi}{k}$. If $\pi(n(h)) = 0$, place $h$ arbitrarily in $j$ of the $V_i$’s. Note that with this construction each $j$-half-space is contained in exactly $j$ principal sets. Thus, if for each principal set we can pick a point in all its half-spaces, then each $j$-half-space contains $j$ points.

It remains to show that the half-spaces in each principal set have a common intersection. Let $h_1, \ldots, h_{d+1}$ be $d + 1$ half-spaces in $V_i$ and assume for the sake of contradiction that they have no common intersection. Then the positive hull (conical hull) of their projected normal vectors must be $X$, and in particular there are three of them, w.l.o.g. $h_1$, $h_2$ and $h_3$, whose projected normal vectors already have $X$ as their positive hull. Further, among those three half-spaces, there are two of them, w.l.o.g. $h_1$ and $h_2$, such that the angles between their projected normal vectors and $v_i$ sum up to more than $\pi$. If $h_1$ is a $j_1$-half-space, then by construction of $V_i$ we have that the angle between $\pi(n(h_1))$ and $v_i$ is at most $\frac{j_1\pi}{k}$. Analogously, if $h_2$ is a $j_2$-half-space, the angle between $\pi(n(h_2))$ and $v_i$ is at most $\frac{j_2\pi}{k}$. By the choice of $h_1$ and $h_2$
we thus have $\frac{(j_1+j_2)\pi}{k} > \pi$, which is equivalent to $j_1 + j_2 > k$, and to $j_1 + j_2 \geq k+1$, as $j_1$ and $j_2$ are integers. By definition of a $j$-half-space we have

$$\mu(h_1) + \mu(h_2) > 1 - \alpha_{k+1-j_1} + 1 - \alpha_{k+1-j_2}.$$  

Furthermore we have $\mu(h_i) > 1 - \alpha_k$ for every $i \in \{1, \ldots, d+1\}$, and thus

$$\mu(h_1) + \ldots + \mu(h_{d+1}) > 1 - \alpha_{k+1-j_1} + 1 - \alpha_{k+1-j_2} + (d-1)(1-\alpha_k),$$

which is equivalent to

$$(d-1)\alpha_k + \alpha_{k+1-j_1} + \alpha_{k+1-j_2} > d + 1 - (\mu(h_1) + \ldots + \mu(h_{d+1})).$$

As $k + 1 - j_1 + k + 1 - j_2 = 2k + 2 - (j_1 + j_2) \leq k + 1$, we have that $(d-1)\alpha_k + \alpha_{k+1-j_1} + \alpha_{k+1-j_2} \leq 1$ and thus $\mu(h_1) + \ldots + \mu(h_{d+1}) > d$, which is a contradiction to Observation \ref{obs:5.2}. $\Box$

Setting $\alpha_j = \frac{j}{kd+1}$, we get a bound for the generalized Tukey depth:

**Corollary 5.3.** Let $\mu$ be a mass distribution in $\mathbb{R}^d$ with $\mu(\mathbb{R}^d) = 1$. Then there exist $k$ points $p_1, \ldots, p_k$ in $\mathbb{R}^d$ with generalized Tukey depth $gtd_\mu(\{p_1, \ldots, p_k\}) = \frac{1}{kd+1}$.

## 5.3 Triangles

Plugging $d = 1$, $k = 2$, $\alpha_1 = \frac{1}{3}$ and $\alpha_2 = \frac{2}{3}$ into Theorem \ref{thm:1.7}, the two solution points are the $\frac{1}{3}$-quantile and the $\frac{2}{3}$-quantile. As mentioned in Chapter \ref{chap:1}, the $\frac{1}{3}$-quantile and the $\frac{2}{3}$-quantile can also be interpreted as a one-dimensional simplex with the property that every halfline that contains a part of the simplex contains at least $\frac{1}{3}$ of the underlying data set and every halfline that contains the whole simplex contains at least $\frac{2}{3}$ of the underlying data set. For this interpretation, we give a generalization to two dimensions. As above, we first give a proof for point sets instead of mass distributions and for fixed values of $\alpha$ and $\beta$, which will be the setting of the algorithm in Section \ref{sec:5.4}.
Theorem 5.4. Let $P$ be a set of $n$ points in general position in the plane. Then there are three points $p_1, p_2$ and $p_3$ in $\mathbb{R}^2$ such that

$(1)$ each closed half-plane containing one of the points $p_1, p_2$ and $p_3$ contains at least $\frac{n}{6}$ of the points of $P$ and

$(2)$ each closed half-plane containing all of $p_1, p_2$ and $p_3$ contains at least $\frac{n}{2}$ points of $P$.

Note that this can also be interpreted as an instance of Theorem 1.7 with $\alpha_1 = \alpha_2 = \frac{1}{6}$ and $\alpha_3 = \frac{1}{2}$. However, as $\alpha_3 + \alpha_3 + \alpha_1 > 1$, the precondition of Theorem 1.7 does not apply. The proof of this result uses similar ideas as the above proofs.

Proof. Let $C$ be the intersection of all open half-planes containing more than $\frac{5n}{6}$ of the points of $P$. Just as in the proof of Theorem 5.1, condition (1) is equivalent to $p_1, p_2$ and $p_3$ lying in $C$. Similarly, condition (2) is equivalent to the following statement: for every open half-plane $h$ containing more than $\frac{n}{2}$ of the points of $P$, $h$ contains at least one of $p_1, p_2$ and $p_3$. For the latter, let $n_i$ be a vector and let $N_i$ be the set of all of these half-planes that have $n_i$ as their normal vector. The intersection of all elements of $N_i$ is a closed half-space $h_i$. Let $K$ be the set of all of these $h_i$, i.e., for every possible direction of $n_i$. Then, condition (2) is equivalent to the following statement: for every half-plane $h$ in $K$, $h$ contains at least one of $p_1, p_2$ and $p_3$. For each such half-plane $h_i$, let $c_i$ be the intersection of $h_i$ and $C$. Note that $c_i$ is compact. We thus want to show the claim that we can find three points $p_1, p_2$ and $p_3$ such that each $c_i$ contains at least one of them. Let $H$ be the set of $c_i$s that are minimal with respect to set inclusion. Clearly, it is enough to show the claim just for the elements of $H$.

First we show that among any three elements of $H$, two of them intersect. Let $c_1, c_2$ and $c_3$ be elements of $H$, and let $h_1, h_2$ and $h_3$ be their associated half-planes. Assume for the sake of contradiction that $c_1, c_2$ and $c_3$ are pairwise disjoint. Let $n_i$ be the normal vector of $h_i$. Let $A$ be the positive hull of $n_1, n_2$ and $n_3$. Then $A$ is either a cone or the whole plane. See Figure 5.3. If $A$ is the whole plane, then $h_1, h_2$ and
Figure 5.3: Disjoint $c_1$, $c_2$, $c_3$, where the positive hull of the normal vectors spans the whole plane (left) or not (right).

$h_3$ have no common intersection. Otherwise, if $A$ is a cone, then one of $n_1$, $n_2$ and $n_3$ can be described as a positive linear combination of the other two. In particular, $h_1$, $h_2$ and $h_3$ have a common intersection and one of them is redundant in the description of $h_1 \cap h_2 \cap h_3$. We thus consider two cases, namely whether $h_1$, $h_2$ and $h_3$ have a common intersection or not.

First, assume that $h_1$, $h_2$ and $h_3$ have no common intersection. Then $h_1$, $h_2$ and $h_3$ partition the plane into seven regions (see Figure 5.4): $h_i \cap h_j$ for $i \neq j$, $h_i \setminus (h_j \cup h_k)$, for $i, j, k$ all different and $h_1^c \cap h_2^c \cap h_3^c$. Note that each $h_i \cap h_j$ contains strictly fewer than $\frac{n}{6}$ of the points of $P$, as otherwise the corresponding $c_i$ and $c_j$ intersect. In particular, $h_2 \setminus h_1$ contains more than $\frac{n}{2} - \frac{n}{6} = \frac{n}{3}$ points of $P$. It follows that $(h_1 \cup h_2)^c$ and thus also $h_3 \setminus (h_1 \cup h_2)$ contains strictly fewer than $\frac{n}{6}$ of the points of $P$. The number of points in $h_3$ is the sum of the number of points in $h_3 \cap h_1$, $h_3 \cap h_2$ and $h_3 \setminus (h_1 \cup h_2)$. All of these sets contain strictly fewer than $\frac{n}{6}$ of the points of $P$, implying that $h_3$ contains fewer than $\frac{n}{2}$ of the points of $P$, which is a contradiction. This concludes the first case.

For the second case assume that $h_1$, $h_2$ and $h_3$ have a common intersection and one of the half-planes is redundant in the description of $h_1 \cap h_2 \cap h_3$; assume without loss of generality that it is $h_3$. Just as in
the first case, each $h_i \cap h_j$ contains strictly fewer than $\frac{n}{6}$ of the points of $P$. Again it follows that $h_3 \setminus (h_1 \cup h_2)$ contains strictly fewer than $\frac{n}{6}$ of the points of $P$. The sets $h_3 \cap h_1$, $h_3 \cap h_2$ and $h_3 \setminus (h_1 \cup h_2)$ cover $h_3$, implying that $h_3$ contains fewer than $\frac{n}{2}$ of the points of $P$, which is again a contradiction. This concludes the proof that among any three elements of $H$, two intersect.

It remains to show that we can find three points $p_1$, $p_2$ and $p_3$ such that each element of $H$ contains at least one of them. This can be achieved by picking one element of $H$ and placing two points $p_1$ and $p_2$ at the extreme intersection points with the boundary of $C$; since any three elements of $H$ intersect, any two elements not containing $p_1$ and $p_2$ must intersect and we may apply Helly’s theorem in dimension one. However, we actually have more flexibility in choosing $p_1$. Note that the normal vectors pointing into the half-planes defining the elements of $H$ define a circular order on $H$. Place $p_1$ at a topmost point of the boundary of $C$. Let $h_1$ be the first element of $H$ in counterclockwise direction whose defining half-plane does not contain $p_1$ in its interior. Place $p_2$ at the intersection of the defining line of $h_1$ with the boundary of $C$ that is furthest in counterclockwise direction from $p_1$. Since $h_1$ is minimal, any element of $H$ intersecting $h_1$ contains either $p_1$ or $p_2$. Therefore, all elements of $H$ that do not intersect $h_1$ have a common intersection, in which we place $p_3$. \hfill \Box

The general statement can be proved analogously:
Theorem 1.8. Let \( \mu \) be a mass distribution in \( \mathbb{R}^2 \) with \( \mu(\mathbb{R}^2) = 1 \). Let \( \alpha \) and \( \beta \) be real numbers such that \( 0 < \alpha \leq \beta \) and \( \alpha + \beta = \frac{2}{3} \). Then there is a triangle \( \Delta \) in \( \mathbb{R}^2 \) such that

1. for each closed half-plane \( h \) containing one of the vertices of \( \Delta \) we have \( \mu(h) \geq \alpha \) and
2. for each closed half-plane \( h \) fully containing \( \Delta \) we have \( \mu(h) \geq \beta \).

5.4 Algorithms

In this section, we describe algorithms for constructing the points described in Theorem 5.1 and Theorem 5.4. For this, we will work in the dual setting under the point-line duality. Let us first discuss the relevant dual objects.

Recall the definition of \( C \) in the proof of Theorem 5.1: it is a closed convex region defined as the intersection of all open half-planes containing more than \( \frac{4n}{5} \) of the points of \( P \). More generally, we define the \( \alpha n \)-hull of a point set \( P \) as the closed convex region defined as the intersection of all open half-planes containing more than \( (1 - \alpha)n \) of the points of \( P \). Equivalently, the \( \alpha n \)-hull is the set of points \( q \in \mathbb{R}^2 \) with the property that every closed half-plane that contains \( q \) contains at least \( \alpha n \) points of \( P \). In this language, the Centerpoint Theorem says that the \( \frac{n}{3} \)-hull is not empty.

Using the second definition of an \( \alpha n \)-hull, we can now describe its dual: a dual line \( q^* \) is in the dual \( \alpha n \)-hull if for every point \( x \) on the line \( q^* \) we have that at least \( \alpha n \) lines of the dual arrangement \( P^* \) pass above and below \( x \). In other words, \( q^* \) separates the \( [\alpha n] \)-level and the \( [(1 - \alpha)n] \)-level of the arrangement. More precisely, for a \( k \)-level in a line arrangement, define its lower hull \( H_L(k) \) as the convex hull of all points that lie on or below the \( k \)-level. Analogously, the upper hull \( H_U(k) \) is the convex hull of all points that lie on or above the \( k \)-level. If \( k \) is not an integer, we define \( H_L(k) \) as \( H_L(\lceil k \rceil) \) and \( H_U(k) \)
as \( H_U([k]) \). Note that \( H_L(k-1) \subset H_L(k) \) and \( H_U(k+1) \subset H_U(k) \). A dual line \( q^* \) is now in the dual \( \alpha n \)-hull if it lies completely in the closure of \( \mathbb{R}^2 \setminus (H_L(\alpha n) \cup H_U((1-\alpha)n)) \).

Going back to the proof of Theorem 5.1, we analogously see that the set \( D_A \) (the intersection of all half-spaces with more that \( \frac{3n}{5} \) above them corresponds to the closure of \( \mathbb{R}^2 \setminus H_L(\frac{2n}{5}) \), and \( D_B \) corresponds to the closure of \( \mathbb{R}^2 \setminus H_U(\frac{3n}{5}) \).

To find the solution point in \( D_A \), we thus want to find a line which does not intersect \( H_L(\frac{2n}{5}) \) (the dual point is in \( D_A \)) and also does not intersect \( H_L(\frac{n}{5}) \) and \( H_U(\frac{4n}{5}) \) (the dual point is in the \( \frac{n}{5} \)-hull). As \( H_L(\frac{n}{5}) \subset H_L(\frac{2n}{5}) \), it is thus enough to find a line which separates \( H_L(\frac{2n}{5}) \) and \( H_U(\frac{4n}{5}) \), for example a bitagent, that is, a line which is tangent to both \( H_L(\frac{2n}{5}) \) and \( H_U(\frac{4n}{5}) \). In order to find such a line, we will use a variation of the following lemma by Matoušek [68]:

**Lemma 5.5** (Matoušek [68, Lemma 3.2]). In an arrangement of \( n \) lines, let \( \gamma \) be the boundary of \( H_L(k) \). Given the arrangement, \( k \), and a point \( p \), one can find the tangent to \( \gamma \) passing through \( p \) and touching \( \gamma \) to the right of \( p \) (if it exists) in time \( O(n \log^2 n) \).

Matoušek uses this result as a subroutine to compute the \( k \)-hull of a point set in time \( O(n \log^4 n) \). In our case, the tangent does not have to pass through a given point, but it should separate two levels or, even more generally, it should separate a level from a part of an other level:

**Lemma 5.6.** Given an arrangement of \( n \) lines and two numbers \( k < l \leq n \), as well as a half-plane \( h \), a line separating the \( k \)-level from the intersection of \( h \) with the \( l \)-level can be found in \( O(n \log^3 n) \) time, if it exists. The separating line is tangent to both level parts and, from left to right, first intersects the \( k \)-level and then the relevant part of the \( l \)-level.

**Proof.** Let \( \gamma \) be the boundary of \( H_L(k) \), and let \( \nu \) be the intersection of \( h \) with the \( l \)-level. Note that \( \nu \) might not be connected. Suppose we want our line to be the counterclockwise bitangent of \( \gamma \) and \( \nu \) (i.e., from
left to right, it first intersects \( \gamma \), which has no point above it, and then \( \nu \). Our algorithm works by obtaining tangents to \( \nu \) through points on \( \gamma \). Matoušek’s \( O(n \log^2 n) \) algorithm for determining the tangent to a level through a given point that is to the right of that point \cite{68, Lemma 3.2} (our Lemma 5.5) also directly works for parts of a level such as \( \nu \): It requires a sub-algorithm that decides in \( O(n \log n) \) time whether a given line \( \ell \) intersects the level (or, in our case, the partial level \( \nu \)). This can be done by sorting the intersection of the lines of the arrangements along \( \ell \) (see also \cite{68, Lemma 3.1}) as well as along the line bounding \( h \); \( \ell \) either intersects the relevant part of \( \nu \), or we can compare the intersection of \( h \) with \( \ell \) to the intersections of \( h \) with \( \nu \) to determine whether there is a point of \( \nu \) below \( \ell \).

Suppose first we are given \( \gamma \). (It requires \( O(n \log^4 n) \) time though to obtain it, so we eventually get rid of this assumption.) The convex hull of a level is known to have at most \( n \) vertices \cite{68, Lemma 2.1}. For a point \( p \) on \( \gamma \), we can find in \( O(n \log^2 n) \) time the point \( q \) on \( \nu \) such that the line \( pq \) has no point on \( \nu \) below it. We can thus find, by binary search on the \( O(n) \) vertices of \( \gamma \), a vertex \( p \) with \( q \) on \( \nu \) such that \( pq \) separates \( \gamma \) and \( \nu \). This gives an \( O(n \log^4 n) \) time algorithm for obtaining the bitangent. To improve on that bound, we need to get rid of the explicit construction of \( \gamma \) to find the tangents to \( \nu \).

To this end, we consider Matoušek’s algorithm for constructing the convex hull boundary \( \gamma \) of a level and compute only the relevant part (see \cite{68, Section 4}). In particular, the algorithm works by finding, for a constant \( c \) and two vertical lines, \((c - 1)\) further vertical lines between the given ones such that there are at most \( n^2/c \) crossings of the arrangement between two of these verticals. This can be done in \( O(n) \) time (as described in \cite{69}). The tangents on \( \gamma \) at the intersection points with the vertical lines can be computed in \( O(n \log^3 n) \) time \cite{68, Lemma 3.3}. It is shown in \cite{68} that, when choosing \( c = 64 \), there are at most \( n/2 \) lines of the arrangement relevant for the construction of \( \gamma \) between two such vertical lines, and these lines can be found in \( O(n) \) time. The original algorithm proceeds recursively within each interval defined by
two neighboring vertical lines after removing the non-relevant lines. In our adaption, however, we find the interval that contains the point \( p \) on \( \gamma \) such that a tangent to \( \gamma \) through the vertex \( p \) with \( q \) on \( \nu \) such that \( pq \) separates \( \gamma \) and \( \nu \). (We do this by considering the tangent to \( \gamma \) at each of the constant number of intersection of a vertical line with \( \gamma \).) When we have found this interval, we can prune \( n/2 \) of the lines and recurse inside this interval. Note, however, that we cannot prune the set of lines when looking for a tangent to \( \nu \). Thus, in each recursive call, we need \( O(n \log^2 n) \) time for computing the tangent. As the recursion depth is \( O(\log n) \), this amounts to \( O(n \log^3 n) \) in total. Also, for \( n_i \) lines during the \( i \)th recursion, we need \( O(n_i \log^3 n_i) \subseteq O(n_i \log^3 n) \) time for determining the intervals. As \( n_i \) decreases geometrically, this also amounts to \( O(n \log^3 n) \). This is the total running time for finding the bitangent, as claimed.

We call such a line the **counterclockwise bitangent** of the two subsets of the plane (i.e., the intersection with the region not above it has smaller \( x \)-coordinate than the intersection with the region not below it). Note that by mirroring the plane horizontally or vertically, the lemma also provides other types of bitangents. Lemma 5.6 can now be used to obtain the following result.

**Theorem 5.7.** Given a set \( P \) of \( n \) points in the plane, two points satisfying the conditions of Theorem 5.1 can be constructed in time \( O(n \log^3 n) \).

**Proof.** As argued before, to find a point \( p_1 \) in the intersection of \( C \) and \( D_A \), it suffices to compute a bitangent to \( H_L(\frac{2n}{5}) \) and \( H_U(\frac{4n}{5}) \). This can be done in \( O(n \log^3 n) \) time using Lemma 5.6 (with \( h = \mathbb{R}^2 \)). We obtain \( p_2 \) analogously as a bitangent to \( H_L(\frac{n}{5}) \) and \( H_U(\frac{3n}{5}) \).

**Theorem 5.8.** Three points as described in Theorem 5.4 can be computed in time \( O(n \log^3 n) \).

**Proof.** Consider the dual line arrangement of the point set. The points \( p_1, p_2, p_3 \) dualize to three lines \( p_1^*, p_2^*, p_3^* \) that separate \( H_L(\lceil \frac{n}{6} \rceil) \) and
5.4. Algorithms

$H_U\left(\left\lfloor \frac{5n}{6} \right\rfloor \right)$ s.t. every point on the middle level has at least one of these lines above it and one of these lines below it. (We assume for simplicity that $n$ is odd and the middle level is the $\left\lfloor \frac{n}{2} \right\rfloor$-level of the arrangement; if $n$ is even, one has to consider the points between the $\frac{n}{2}$-level and the $(\frac{n}{2} + 1)$-level.) Theorem 5.4 asserts that such lines exist, and its proof tells us that we can choose one of these lines to be an arbitrary tangent of one of the levels not intersecting the interior of the other one. We denote by $\gamma_l$ and $\gamma_u$ the boundaries of $H_L\left(\left\lceil \frac{n}{6} \right\rceil \right)$ and $H_U\left(\left\lfloor \frac{5n}{6} \right\rfloor \right)$, respectively.

We let $p_1^*$ be the clockwise bitangent of $\gamma_l$ and $\gamma_u$, which we can obtain in $O(n \log^3 n)$ time using Lemma 5.6. For simplicity of explanation, we also compute the counterclockwise bitangent $\ell$. (This step may be omitted in an actual implementation, but assuming it to be given facilitates the explanation and does not change the asymptotic running time.)

The line $p_1^*$ intersects the middle level of the arrangement. Let $\mu_1$ be the parts of the middle level below $p_1^*$, and $\mu_2$ be the part above it. Note that each of these parts may be disconnected. Using Lemma 5.6, we search for the counterclockwise bitangent between $\gamma_l$ (or, equivalently, the $\left\lceil \frac{n}{6} \right\rceil$-level) and $\mu_1$ (which is the intersection of the middle level with a half-space defined by $p_1^*$) in $O(n \log^3 n)$ time. If it exists, and its intersection point with $\gamma_l$ is between the intersections of $\gamma_l$ with $p_1^*$ and $\ell$, we choose this line to be $p_2^*$. Otherwise, we continue our search from the other side in the same way (i.e., we look for the counterclockwise bitangent between $\gamma_l$ and $\mu_1$). The line $p_3^*$, which separates $\gamma_u$ and $\mu_2$, can be found in an analogous manner. $\square$
A Generalization of the Centerpoint Theorem
Recall the conjecture by Stefan Langerman \[62\] that any \(dn\) mass distributions in \(\mathbb{R}^d\) can be simultaneously bisected by an arrangement of \(n\) hyperplanes. In Section \[6.1\] we prove the conjecture to be true for \(d = n = 2\) and we give an algorithm to find such a partition for 4 point sets. The range of values, for which the conjecture is true has since been extended to any \(n\) when \(d\) is a power of 2 \[20, 53\].

In Section \[6.2\] we consider the relaxed setting of \textit{almost bisections}. Recall that we say that an arrangement of hyperplanes almost simultaneously bisects a family of mass distributions if for each mass in the family either the whole arrangement bisects the mass or there is a hyperplane whose removal gives an arrangement which does. In this setting, the conjecture turns out to be true:

**Theorem 1.10.** Let \(\mu_1, \ldots, \mu_{dn}\) be \(dn\) mass distributions in \(\mathbb{R}^d\). Then there are \(n\) hyperplanes that almost simultaneously bisect \(\mu_1, \ldots, \mu_{dn}\).
Note that we might remove different hyperplanes for different masses. However, the proof of the above will give us some control on which hyperplanes are removed: we will divide the masses into $n$ families of $d$ masses each, each family $F_i$ corresponding to a hyperplane $h_i$. For each mass in a family $F_i$, we will see that it is either bisected by the whole arrangement or after removing $h_i$.

For the last result of this chapter, we will expand on the idea of separately bisecting several families of masses. We conjecture the following:

**Conjecture 6.1.** Let $\mu_1^1, \ldots, \mu_{d+k}^1, \mu_1^2, \ldots, \mu_{d+k}^2, \ldots, \mu_{d+k}^m$ be $m = \left\lfloor \frac{d}{k} \right\rfloor$ families each containing $d + k$ mass distributions in $\mathbb{R}^d$. Then there exists a hyperplane $g$ and hyperplanes $h_1, \ldots, h_m$ such that for each $i \in \{1, \ldots, m\}$ the mass distributions $\mu_i^1, \ldots, \mu_i^{d+k}$ are simultaneously bisected by $\{g, h_i\}$.

Note that for $k = d$, this is equivalent to Langerman’s conjecture for 2 hyperplanes (i.e., $n = 2$). In Section 6.3 we study the other extreme case $k = 1$ and we almost settle the conjecture in this case: if the dimension is odd, we are only able to show that the difference of the masses in the two regions defined by $g$ and $h_i$ is at most $\varepsilon$, for any arbitrarily small $\varepsilon$.

Let us mention that Conjecture 6.1 would be tight with respect to the number of families that are bisected: Consider $m + 1$ families each containing $d + k$ mass distributions in $\mathbb{R}^d$ where each mass is densely centered around some point. Place these points in such a way that no hyperplane passes through $d + 1$ of them. Look at one family of $d + k$ masses. If $(h_1, h_2)$ are to simultaneously bisect the masses in this family, each mass must be transversed by either $h_1$ or $h_2$, or both. In particular, as $h_2$ can pass through at most $d$ masses, $h_1$ must pass through at least $k$ masses. This is true for all $m$ families, which means that $h_1$ must pass through at least $(m + 1)k > d$ masses in total, which cannot happen by our construction of the masses.

By considering $g$ as the hyperplane that gets sent to infinity by a projective transformation, the above conjecture can also be stated as follows:
Conjecture 6.2. Let \(\mu_1^1, \ldots, \mu_{d+k}^1, \mu_1^2, \ldots, \mu_{d+k}^2, \ldots, \mu_{d+k}^m\) be \(m = \lfloor \frac{d}{k} \rfloor\) families each containing \(d + k\) mass distributions in \(\mathbb{R}^d\). Then there exists a projective transformation \(\varphi\) such that \(\varphi(\mu_1^i), \ldots, \varphi(\mu_{d+k}^i)\) can be simultaneously bisected by a single hyperplane for every \(i \in \{1, \ldots, m\}\).

6.1 Four masses in the plane

The main result of this section is the following:

Theorem 6.3. Let \(\mu_1, \mu_2, \mu_3, \mu_4\) be four mass distributions in \(\mathbb{R}^2\). Then there exist two lines \(\ell_1, \ell_2\) such that \(\{\ell_1, \ell_2\}\) simultaneously bisects \(\mu_1, \mu_2, \mu_3, \mu_4\).

The proof is very similar to the first proof of the Ham-Sandwich Theorem, as discussed in Section 2.3. The main idea is to allow for partitions with quadratic curves, in which case we could bisect 5 masses, and then sacrifice one of the masses to enforce degeneracy of the quadratic curve.

Proof. For each \(p = (a, b, c, d, e, g) \in S^5\) consider the bivariate polynomial \(c_p(x, y) = ax^2 + by^2 + cxy + dx + ey + g\). Note that \(c_p(x, y) = 0\) defines a conic section in the plane. Let \(R^+(p) := \{(x, y) \in \mathbb{R}^2 \mid c_p(x, y) \geq 0\}\) be the set of points that lie on the positive side of the conic section and let \(R^-(p) := \{(x, y) \in \mathbb{R}^2 \mid c_p(x, y) \leq 0\}\) be the set of points that lie on its negative side. Note that for \(p = (0, 0, 0, 0, 0, 1)\) we have \(R^+(p) = \mathbb{R}^2\) and \(R^-(p) = \emptyset\), and vice versa for \(p = (0, 0, 0, 0, 0, -1)\). Also note that \(R^+(p) = R^-(p)\). We now define four functions \(f_i : S^5 \to \mathbb{R}\) as follows: for each \(i \in \{1, \ldots, 4\}\) define \(f_i := \mu_i(R^+(p)) - \mu_i(R^-(p))\). From the previous observation it follows immediately that \(f_i(-p) = -f_i(p)\) for all \(i \in \{1, \ldots, 4\}\) and \(p \in S^5\). Also, the functions are continuous. Further let

\[A(p) := \det \begin{pmatrix} a & c/2 & d/2 \\ c/2 & b & e/2 \\ d/2 & e/2 & g \end{pmatrix}.\]

It is well-known that the conic section \(c_p(x, y) = 0\) is degenerate if and
only if \( A(p) = 0 \). Furthermore, being a determinant of a \( 3 \times 3 \)-matrix, \( A \) is continuous and \( A(-p) = -A(p) \). Hence, setting \( f_5(p) := A(p) \)
\( f := (f_1, \ldots, f_5) \) is an antipodal mapping from \( S^5 \) to \( \mathbb{R}^5 \), and thus by the Borsuk-Ulam theorem, there exists \( p^* \) such that \( f(p^*) = 0 \).
For each \( i \in \{1, \ldots, 4\} \) the condition \( f_i(p^*) = 0 \) implies by definition that \( \mu_i(R^+(p^*)) = \mu_i(R^-(p^*)) \). The condition \( f_5(p^*) = 0 \) implies that \( c_p(x,y) = 0 \) describes a degenerate conic section, i.e., two lines (possibly one of them at infinity), a single line of multiplicity 2, a single point or the empty set. For the latter three cases, we would have \( R^+(p^*) = \mathbb{R}^2 \) and \( R^-(p^*) = \emptyset \) or vice versa, which would contradict \( \mu_i(R^+(p^*)) = \mu_i(R^-(p^*)) \). Thus \( f(p^*) = 0 \) implies that \( c_p(x,y) = 0 \) indeed describes two lines that simultaneously bisect \( \mu_1, \mu_2, \mu_3, \mu_4 \).

Using similar ideas, we can put more restrictions on the cut, at the expense of bisecting fewer masses:

**Theorem 6.4.** Let \( \mu_1, \mu_2, \mu_3 \) be three mass distributions in \( \mathbb{R}^2 \). Given any line \( \ell \) in the plane, there exist two lines \( \ell_1, \ell_2 \) such that \( \{\ell_1, \ell_2\} \) simultaneously bisects \( \mu_1, \mu_2, \mu_3 \) and \( \ell_1 \) is parallel to \( \ell \).

We note here that a line at infinity is parallel to any other line.

**Proof.** Assume without loss of generality that \( \ell \) is parallel to the \( x \)-axis; otherwise rotate \( \mu_1, \mu_2, \mu_3 \) and \( \ell \) to achieve this property. Consider the conic section defined by the polynomial \( ax^2 + by^2 + cxy + dx + ey + g \).
If \( a = 0 \) and the polynomial decomposes into linear factors, then one of the factors must be of the form \( \beta y + \gamma \). If \( \beta = 0 \), then this factor corresponds to a line at infinity, which is by definition parallel to any line, thus also the \( x \)-axis. Otherwise, the line defined by this factor is parallel to the \( x \)-axis. Thus, we can modify the proof of Theorem 6.3 in the following way: we define \( f_1, f_2, f_3 \) and \( f_5 \) as before, but set \( f_4 := a \). It is clear that \( f \) still is an antipodal mapping and thus has a zero. As \( f_5 = 0 \), we know, like above, that the polynomial defining the conic section decomposes into linear factors, i.e., two lines. Since \( f_4 = a = 0 \), we thus also know that one of the two corresponding lines is parallel to the \( x \)-axis. Finally, \( f_1 = f_2 = f_3 = 0 \) implies that the two
lines simultaneously bisect the three mass distributions, which proves the result.  

Another natural condition on a line is that it has to pass through a given point.

**Theorem 6.5.** Let \( \mu_1, \mu_2, \mu_3 \) be three mass distributions in \( \mathbb{R}^2 \) and let \( q \) be a point. Then there exist two lines \( \ell_1, \ell_2 \) such that \( \{\ell_1, \ell_2\} \) simultaneously bisects \( \mu_1, \mu_2, \mu_3 \) and \( \ell_1 \) goes through \( q \).

**Proof.** Assume without loss of generality that \( q \) coincides with the origin; otherwise translate \( \mu_1, \mu_2, \mu_3 \) and \( q \) to achieve this. Consider again the conic section defined by the polynomial \( ax^2 + by^2 + cxy + dx + ey + g \). If \( g = 0 \) and the polynomial decomposes into linear factors, then one of the factors must be of the form \( \alpha x + \beta y \). In particular, the line defined by this factor goes through the origin. We can again modify the proof of Theorem 6.3 in the following way: we define \( f_1, f_2, f_3 \) and \( f_5 \) as before, but set \( f_4 := g \). It is clear that \( f \) still is an antipodal mapping. The zero of this mapping now implies the existence of two lines simultaneously bisecting three mass distributions, one of them going through the origin, which proves the result.  

Note that the above results complement each other when considering the projective plane: Theorem 6.5 makes \( \ell_1 \) pass through a point \( q \) in the affine plane, and Theorem 6.4 covers the case in which \( q \) is on the line at infinity.

At the cost of another mass distribution, we can also enforce the intersection of the two lines to be at a given point.

**Theorem 6.6.** Let \( \mu_1, \mu_2 \) be two mass distributions in \( \mathbb{R}^2 \) and let \( q \) be a point. Then there exist two lines \( \ell_1, \ell_2 \) such that \( \{\ell_1, \ell_2\} \) simultaneously bisects \( \mu_1, \mu_2 \), and both \( \ell_1 \) and \( \ell_2 \) go through \( q \).

**Proof.** Assume without loss of generality that \( q \) coincides with the origin; otherwise translate \( \mu_1, \mu_2 \) and \( q \) to achieve this. Consider the conic
section defined by the polynomial $ax^2 + by^2 + cxy$, i.e., the conic section where $d = e = g = 0$. If this conic section decomposes into linear factors, both of them must be of the form $\alpha x + \beta y = 0$. In particular, both of them pass through the origin. Furthermore, as $d = e = g = 0$, the determinant $A(p)$ vanishes, which implies that the conic section is degenerate. Thus, we can modify the proof of Theorem 6.3 in the following way: we define $f_1, f_2$ as before, but set $f_3 := d$, $f_4 := e$ and $f_5 := g$. It is clear that $f$ still is an antipodal mapping. The zero of this mapping now implies the existence of two lines simultaneously bisecting two mass distributions, both of them going through the origin, which proves the result.

Going back to the general case, instead of considering four mass distributions $\mu_1, \ldots, \mu_4$, one can think of having four finite sets of points $P_1, \ldots, P_4 \subset \mathbb{R}^2$. The question is now to find an efficient algorithm to compute two bisecting lines given any four sets $P_1, \ldots, P_4$ with a total of $n$ points. A trivial $O(n^5)$ time algorithm can be applied by looking at all pairs of combinatorially different lines. While this running time can be reduced using known data structures, it still goes through $\Theta(n^4)$ different pairs of lines. An algorithm that does not consider all combinatorially different pairs of lines is described in the proof of the following theorem.

**Theorem 6.7.** Given any four planar point sets $P_1, \ldots, P_4$ with a total of $n$ points, one can find two lines $\ell_1, \ell_2$ such that $\{\ell_1, \ell_2\}$ simultaneously bisects $P_1, \ldots, P_4$ in $O(n^{10/3} \log^{1+\varepsilon}(n))$ time.

**Proof.** We know from Theorem 6.3 that a solution exists. Given a solution, we can move one of the lines to infinity using a projective transformation. After this transformation, the remaining line simultaneously bisects the four transformed point sets. In other words, given any four planar point sets $P_1, \ldots, P_4$, we can always find a projective transformation $\varphi$ such that $\varphi(P_1), \ldots, \varphi(P_4)$ can be simultaneously bisected by a single line. Checking whether four point sets can be simultaneously bisected by a line can be done by first building the dual
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line arrangement of the union of the four sets in $O(n^2)$ time \[33, 41\]. We can then walk along the middle level of the arrangement, keeping track of how many of the dual lines of each point set are above and below the middle level, which tells us whether somewhere along the middle level exactly half of the dual lines of every point set are above. For the starting point of our walk, we count the number of dual lines above and below the middle level in linear time and every update only needs constant time. Thus, the time needed after building the arrangement is, as in Chapter 3, bounded by $O(m \log^{1+\varepsilon}(n))$, where $m$ is the complexity of the middle level, which is at most $O(n^{\frac{4}{3}})$, as shown by Dey \[36\]. The choice of the line at infinity for a projective transformation of a point set corresponds to choosing the north pole (i.e., the point at vertical infinity, which is dual to the line at infinity) in the dual. The north pole is contained in one of the $O(n^2)$ cells of the dual arrangement. So in order to check for every possible projective transformation of a point set whether $\varphi(P_1), \ldots, \varphi(P_4)$ can be simultaneously bisected by a line, it suffices to build the dual arrangement once; after that, we can check whether $\varphi(P_1), \ldots, \varphi(P_4)$ can be simultaneously bisected by a line for every combinatorially different choice of the line at infinity in time $O(n^{\frac{4}{3}} \log^{1+\varepsilon}(n))$ per choice. As there are $O(n^2)$ combinatorially different choices for the line at infinity of a projective transformation (i.e., cells in the dual arrangement), the running time of $O(n^{2+\frac{4}{3}} \log^{1+\varepsilon}(n)) = O(n^{\frac{10}{3}} \log^{1+\varepsilon}(n))$ follows. \[36\]

The analysis of the above algorithm heavily depend on Dey’s result \[36\] on the middle level in arrangements. The current best lower bound on the complexity of the middle level is $\Omega(n \log n)$ \[43\]. Note that in the analysis of our algorithm we implicitly use an upper bound of $O(n^{\frac{10}{3}})$ for the complexity of all projectively different middle levels. More formally, let $c$ be a cell in the dual line arrangement $\mathcal{A}$ and let $m(c)$ be the complexity of the middle level when the north pole lies in $c$. Then $\sum_{c \in \mathcal{A}} m(c)$ is upper bounded by $O(n^{\frac{10}{3}})$. However, this bound does not take into account that many of the considered middle levels could be significantly smaller than $O(n^{\frac{4}{3}})$. This gives rise to the following
Question 6.8. What is the total complexity \( \sum_{c \in \mathcal{A}} m(c) \) of all projectively different middle levels?

Any improvement on the bound \( O(n^{10}) \) would immediately improve the bound of the running time of our algorithm. The above question has recently been studied for the more general \( k \)-levels [34]. It was shown that the total complexity of all different \( k \)-levels is in \( O(k^4) \). Unfortunately, for large \( k \), and in particular for middle levels, this bound is worse than our above bound. It would be interesting to improve the bounds in [34]. So far, it is not even known whether there are arrangements for which the total complexity of the middle levels is supercubic.

The idea used for the algorithm can also be used to get algorithms for Theorem 6.4 and Theorem 6.5.

**Theorem 6.9.** Given any three planar point sets \( P_1, P_2, P_3 \) with a total of \( n \) points and a line \( \ell \), one can find two lines \( \ell_1, \ell_2 \) such that \( \{\ell_1, \ell_2\} \) simultaneously bisects \( P_1, \ldots, P_3 \) and \( \ell_1 \) is parallel to \( \ell \) in time \( O(n^{\frac{2}{3}} \log^{1+\varepsilon}(n)) \).

**Proof.** We know from Theorem 6.4 that a solution exists, in which we again may move one line to infinity, namely \( \ell_1 \). The duals of the family of lines that are parallel to \( \ell \) defines a family of points that are exactly the points on a vertical line \( v \) in the dual, which passes through the dual point \( \ell^* \) of the line \( \ell \). This means that, by fixing \( \ell_1 \), we place the north pole in a cell intersected by the line \( v \). As in the previous proof, we consider combinatorially different placements of \( \ell_1 \) and walk through the respective middle level. However, the line \( v \) intersects the interior of only \( n \) cells, so we only have to walk along a linear number of middle levels in order to find a solution. (By the Zone theorem [17], the cells containing \( v \) can be traversed in total \( O(n) \) time.) This implies the runtime of \( O(n^{\frac{2}{3}} \log^{1+\varepsilon}(n)) \).

While for Theorem 6.4 the intercept is the only parameter for \( \ell_1 \) (while
the slope is fixed to be the one of \( \ell \), for Theorem 6.5 the only parameter for \( \ell_1 \) is its slope. The dual of the lines through the given point \( q \) are exactly the points on the dual line \( q^* \) of \( q \). If instead of placing the north pole only in cells intersected by the line \( v \), we place it only in cells intersected by the line \( q^* \), an algorithm for Theorem 6.5 follows.

**Theorem 6.10.** Given any three planar point sets \( P_1, P_2, P_3 \) with a total of \( n \) points and a point \( q \), one can find two lines \( \ell_1, \ell_2 \) such that \( \{ \ell_1, \ell_2 \} \) simultaneously bisects \( P_1, \ldots, P_3 \) and \( \ell_1 \) goes through \( q \) in time \( O(n^{\frac{2}{3}} \log^{1+\varepsilon}(n)) \).

We conclude this section by giving an algorithm for our last result in two dimensions, Theorem 6.6.

**Theorem 6.11.** Given any two planar point sets \( P_1 \) and \( P_2 \) with a total of \( n \) points and a point \( q \), one can find two lines \( \ell_1, \ell_2 \) such that \( \{ \ell_1, \ell_2 \} \) simultaneously bisects \( P_1 \) and \( P_2 \) and both \( \ell_1 \) and \( \ell_2 \) go through \( q \) in time \( O(n \log n) \).

**Proof.** We know from Theorem 6.6 that a solution exists. Let \( \ell \) be any (non-vertical) line through \( q \), not passing through any point in \( P = P_1 \cup P_2 \). For any point \( p \in P \) that lies below \( \ell \), reflect \( p \) at \( q \). Clearly, this can be done in constant time for each point, so the overall runtime for this step is \( O(n) \). Let \( P' \) be the point set obtained this way. The crucial observation is that any solution for \( P' \) is also a solution for \( P \). Order the points in \( P' \) along the radial order around \( q \) in \( O(n \log n) \) time. It now remains to find an interval \( I \) in this sequence of points such that \( I \) contains exactly half of the points of each point set. As the size of this interval has to be \( |P|/2 \), there are only linearly many possible intervals, so it is an easy task to find \( I \) in linear time. The runtime of the algorithm is therefore dominated by the sorting step. \( \Box \)
6.2 Almost bisections

We now consider the relaxed setting of almost bisections. We will first prove a result where we enforce that all bisecting hyperplanes contain the origin. The general version then follows from lifting the problem one dimension higher. The proof is based on the following idea: for each mass, \( n - 1 \) of the hyperplanes define two regions, one we take with positive sign, the other with negative sign. This defines a so called charge (a mass distribution, which unfortunately may be locally negative, which is why we will need the relaxed setting). The \( n' \)th hyperplane should now bisect this new mass distribution. However, this \( n' \)th hyperplane now again changes the other mass distributions, so in the end we want to guarantee that there are \( n \) hyperplanes such that all of them correctly bisect the masses. More precisely, let \( G_{d-1}(\mathbb{R}^d)^n \) be the space of all sets of \( n \) hyperplanes containing the origin (i.e., linear subspaces) in \( \mathbb{R}^d \). Similar to before, we define a mass assignment \( \mu \) on \( G_{d-1}(\mathbb{R}^d)^n \) as a continuous assignment \( G_{d-1}(\mathbb{R}^d)^n \to M_d \), where \( M_d \) again denotes the space of all \( d \)-dimensional mass distributions. An example of such mass assignments could be the intersection of a fixed \( d \)-dimensional mass distribution with the Minkowski sum of the hyperplanes with a unit ball.

We show that given \( (d-1)n \) such mass assignments, there is a \( p \) such that each \( h_i \) bisects \( d - 1 \) of their images. The idea is the following: we assign \( d - 1 \) masses to each \( h_i \). For every \( p \), we now sweep a copy of \( h_i \) along a line \( \ell \) orthogonal to \( h_i \) and for every mass assigned to \( h_i \) we look at the point on \( \ell \) for which the swept copy through that point bisects the mass. We want to show that for some \( p \), all these points coincide with the origin.

**Lemma 6.12.** Consider the vector bundle \( \xi := (\gamma_m^d)^k \) (the \( k \)-fold Cartesian product of \( \gamma_m^d \)) over the space \( B := G_m(\mathbb{R}^d)^k \). Let \( q := d - m \). Then for any \( q \) sections \( s_1, \ldots, s_q \) of \( \xi \) there exists \( b \in B \) such that
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$s_1(b) = \ldots = s_q(b) = 0$.

This Lemma is another generalization of Proposition 2 in [95] and Lemma 1 in [37]. Our proof follows the proof in [37].

**Proof.** The sections $s_1, \ldots, s_q$ determine a unique section in the $q$-fold Whitney sum of $\xi$, which we denote by $\xi^q$. The bundle $\xi^q$ has base $B$ and fiber dimension $kqm$. We want to show that $\xi^q$ does not allow a nowhere zero section. For this, it is again enough to show that the highest Stiefel-Whitney class $w_{kqm}(\xi^q)$ does not vanish. Denote by $\Gamma_m^d$ the $q$-fold Whitney sum of $\gamma_m^d$ and consider the vector bundle $\zeta := (\Gamma_m^d)^k$. Note that $\zeta$ also has base $B$ and fiber dimension $kqm$. Furthermore, there is a natural bundle map from $\zeta$ to $\xi^q$, and as they have the same base space, $\zeta$ and $\xi^q$ are isomorphic (see [74], §3, Lemma 3.1). Thus, it is enough to show that the highest Stiefel-Whitney class $w_{kqm}(\zeta)$ does not vanish. The Stiefel-Whitney classes of a Cartesian product of vector bundles can be computed as the cross product of the Stiefel-Whitney classes of its components in the following way (see [74], §4, Problem 4-A):

$$w_j(\eta_1 \times \eta_2) = \sum_{i=0}^{j} w_i(\eta_1) \times w_{j-i}(\eta_2).$$

It was shown by Dol’nikov [37] that

$$w_{qm}(\Gamma_m^d) = 1 \in \mathbb{Z}_2 = H^{qm}(G_m(\mathbb{R}^d); \mathbb{Z}_2).$$

By the Künneth theorem and induction it follows that $w_{kqm}((\Gamma_m^d)^k) = 1 \in \mathbb{Z}_2 = H^{kqm}((G_m(\mathbb{R}^d))^k; \mathbb{Z}_2)$. \hfill $\square$

In the following, we will use Lemma 6.12 only for the case $m = 1$, i.e., for products of line bundles. This case could also be proved using a Borsuk-Ulam-type result on product of spheres (Theorem 4.1 in [39], for $n_1 = \ldots = n_r = d - 1$, see also [81]). Consider now $B := G_1(\mathbb{R}^d)^n$, i.e., all $n$-tuples of lines in $\mathbb{R}^d$ through the origin. Further, for every $i \in \{1, \ldots, n\}$ we define $\xi_i$ as the following vector bundle: the base
space is $B$, the total space $E_i$ is the set of all pairs $(b, v)$, where $b = (\ell_1(b), \ldots, \ell_n(b))$ is an element of $B$ and $v$ is a vector in $\ell_i(b)$, and the projection $\pi$ is given by $\pi((b, v)) = b$. It is straightforward to show that this is indeed a vector bundle. In other words, we consider one line to be marked and the fiber over an $n$-tuple of lines is the 1-dimensional vector space given by the marked line.

Using Lemma 6.12, we can now prove the following:

**Theorem 6.13.** Let $\mu_1, \ldots, \mu_{(d-1)n}$ be $(d-1)n$ mass assignments on $G_{d-1}(\mathbb{R}^d)^n$. Then there exists $p = (h_1, \ldots, h_n) \in G_{d-1}(\mathbb{R}^d)^n$ such that for every $i \in \{1, \ldots, n\}$, the hyperplane $h_i$ simultaneously bisects $\mu_{(d-1)(i-1)+1}, \ldots, \mu_{(d-1)i}$.

**Proof.** Consider $\xi = (\gamma_1^d)^n$. Recall that $B = G_1(\mathbb{R}^d)^n$. For an element $b = (\ell_1(b), \ldots, \ell_n(b))$ of $B$, consider for every $i \in \{1, \ldots, n\}$ the $(d-1)$-dimensional hyperplane through the origin that is orthogonal to $\ell_i(b)$ and denote it by $h_i(b)$. Similarly, for every $(b, v) \in E_i$, let $g_i(b, v)$ be the hyperplane through $v$ orthogonal to $\ell_i(b)$. Note that $g_i(b, 0) = h_i(b)$.

Consider now the mass $\mu_1$. For any fixed $b \in B$, the set of vectors $v \in \ell_i(b)$ for which $(g_1(b, v), h_2(b), \ldots, h_n(b))$ bisects $\mu_1$ is an interval. Thus, after taking midpoints, the set of all pairs $(b, v)$ such that $(g_1(b, v), h_2(b), \ldots, h_n(b))$ bisects $\mu_1$ defines a section $s_1$ in $\xi_1$. Analogously, use $\mu_{(d-1)(j-1)+1}$ to define $s_j$ for all $j \in \{2, \ldots, n\}$. Then $s^1 := (s_1^1, \ldots, s_n^1)$ is a section in $(\gamma_1^d)^n$. Similarly, for $i \in \{2, \ldots, d-1\}$, using the masses $\mu_{(d-1)(j-1)+i}$ for all $j \in \{2, \ldots, n\}$ define a section $s^i$ in $(\gamma_1^d)^n$.

We have thus defined $d-1$ sections in $(\gamma_1^d)^n$. Hence, by applying Lemma 6.12, we get that there is a point $b_0$ in $B$ such that $s_1^1(b_0), \ldots, s_{d-1}^1(b_0) = 0$. In particular, all orthogonal hyperplanes $g_i(b, v)$ contain the origin, so their collection is an element of $G_{d-1}(\mathbb{R}^d)^n$. Further, it follows from the definition of the sections $s^i$ that $h_i$ simultaneously bisects $\mu_{(d-1)(i-1)+1}, \ldots, \mu_{(d-1)i}$. $\square$
We are now ready to prove Theorem 1.10. Before we dive into the technicalities, let us briefly discuss the main ideas. We first show that any \((d - 1)n\) mass distributions in \(\mathbb{R}^d\) can be almost simultaneously bisected by \(n\) hyperplanes through the origin. The idea of this proof is very similar to the proof of Theorem 6.13: consider some mass \(\mu\) and assume that \(n - 1\) of the hyperplanes are fixed. Sweep the last hyperplane along a line through the origin and stop when the resulting arrangement of \(n\) hyperplanes almost bisects \(\mu\). We do the same for every mass, one hyperplane is swept, the others are considered to be fixed. Each hyperplane is swept for \((d - 1)\) masses. Using Lemma 6.12, we want to argue, that there is a solution, such that all the swept hyperplanes are stopped at the origin. The only problem with this approach is, that the points where we can stop the hyperplane are in general not unique. In fact, the region of possible solutions for one sweep can consist of several connected components, so in particular, it is not a section, and we cannot use Lemma 6.12 directly. We will therefore need another lemma, that says that we can find find a section in this space of possible solutions. This lemma is actually the only reason why our approach only works for the relaxed setting: we need to sometimes ignore certain hyperplanes to construct such a section. However, constructing a section that lies completely in the space of solutions is stronger than what we would need to use Lemma 6.12. It would be enough to argue, that assuming no almost simultaneous bisection exists, we could find a nowhere zero section contradicting Lemma 6.12. It is thus possible that our approach could be strengthened to prove Conjecture 1.9.

Let us now start by stating the aforementioned result for bisections with hyperplanes containing the origin:

**Theorem 6.14.** Let \(\mu_1, \ldots, \mu_{(d-1)n}\) be \((d - 1)n\) mass distributions in \(\mathbb{R}^d\). Then there are \(n\) hyperplanes, all containing the origin, that almost simultaneously bisect \(\mu_1, \ldots, \mu_{(d-1)n}\).

As mentioned, in order to prove this result, we need a few additional observations. In the following, by a limit antipodal function we mean a continuous function \(f : \mathbb{R} \mapsto \mathbb{R}\) with the following two properties:
1. \( \lim_{x \to \infty} f = - \lim_{x \to -\infty} f \),

2. the set of zeroes of \( f \) consists of finitely many connected components.

See Figure 6.1 for an illustration. Note that these two conditions imply that if \( \lim_{x \to \infty} f \neq 0 \) and if the graph of \( f \) is never tangent to the \( x \)-axis, the zero set consists of an odd number of components. For any subset \( A \) of a vector bundle \( \xi = (E, B, \pi) \), denote by \( Z(A) \) the set of base points on whose fiber \( A \) contains 0 or \( A \) is unbounded. In particular, for any section \( s \), \( Z(s) \) denotes the set of zeroes of the section (as a section is a single point on every fiber, and thus never unbounded).

Consider again \( B := G_1(\mathbb{R}^d)^n \), i.e., all \( n \)-tuples of lines in \( \mathbb{R}^d \) through the origin and the vector bundles \( \xi_i \). Note that \( \xi_i \) has a natural orientable cover \( \xi'_i = (E', B', \pi') \) where all the lines are oriented. Denote by \( p \) the covering map from \( \xi'_i \) to \( \xi_i \).

Assume now that we are given a continuous function \( f : E' \to \mathbb{R} \) with the following properties:

(a) for every point \( b' \in B \), the restriction of \( f \) to the fiber \( \pi^{-1}(b') \), denoted by \( f_{b'} \), is a limit antipodal function;

(b) for any point \( b \in B \) and any two lifts \( b'_1, b'_2 \in p^{-1}(b) \) we have either \( f_{b'_1}(x) = f_{b'_2}(x) \) or \( f_{b'_1}(x) = -f_{b'_2}(x) \) or \( f_{b'_1}(x) = f_{b'_2}(-x) \) or \( f_{b'_1}(x) = -f_{b'_2}(-x) \).

Let \( V'_f := \{ e \in E' | f(e) = 0 \} \) be the zero set of \( f \). Note that the second condition ensures that \( V'_f \) is the lift of a set \( V_f \subseteq E \). We call \( V_f \) a
quasi-section in $\xi_i$. Further note that $Z(V_f)$ consists of the base points where $f_b(0) = 0$ or $\lim_{x \to \infty} f_b = 0$.

**Lemma 6.15.** Let $V_f$ be a quasi-section in $\xi_i$. Then there is a section $s$ such that $Z(s) \subset Z(V_f)$. In particular, if $Z(V_f) = \emptyset$, then $\xi_i$ allows a nowhere zero section.

Before proving this lemma, we show how to apply it to prove Theorem 6.14.

**Proof of Theorem 6.14.** Define $h_i(b)$ and $g_i(b, v)$ as in the proof of Theorem 6.13. Consider now the mass $\mu_1$. For each $b \in B$, choose some orientations of $h_2(b), \ldots, h_n(b)$ and an orientation of $\ell_1(b)$ arbitrarily. Then for each $v \in \ell_1(b)$, we have well-defined regions $R^+(b, v)$ and $R^-(b, v)$. In particular, taking $\mu_1(R^+(b, v)) - \mu_1(R^-(b, v))$ for all orientations defines a function $f_1 : E' \to \mathbb{R}$ which satisfies condition (b) from above. We can also assume without loss of generality that it satisfies condition (a): we certainly have $\lim_{x \to \infty} f_b = -\lim_{x \to -\infty} f_b$, and if the set of zeroes consists of infinitely many connected components, then either $\lim_{x \to \infty} f_b = 0$ and thus $b \in Z(V_f)$, or there is a finite number of convergence points and we can replace the function in a sufficiently small neighborhood around each convergence point by a function that is zero only at this convergence point. Let $V_1$ be the set of all pairs $(b, v)$ such that $(g_1(b, v), h_2(b), h_3(b), \ldots, h_n(b))$ bisects $\mu_1$. As this is exactly the set of pairs $(b, v)$ for which $f_1(b, v) = 0$, it follows that $V_1$ is a quasi-section.

Let now $s^1_1$ be a section in $\xi_1$ with $Z(s_1) \subset Z(V_1)$, the existence of which we get from Lemma 6.15. Analogously, use $\mu_i$ to define $V_i$ and $s^1_i$ for all $i \in \{2, \ldots, n\}$. Then $s^1 := (s^1_1, \ldots, s^1_n)$ is a section in $(\gamma^d_1)^n$. Similarly, for $k \in \mathbb{N}$, using the masses $\mu_{(k-1)n+1}, \ldots, \mu_{kn}$ define a section $s^k$ in $(\gamma^d_1)^n$.

We have thus defined $d-1$ sections in $(\gamma^d_1)^n$. Hence, by applying Lemma 6.12, we get that there is a point $b_0$ in $B$ such that $s^1(b_0), \ldots, s^{d-1}(b_0) =$
We claim that \( H := (h_1(b_0), \ldots, h_n(b_0)) \) almost simultaneously bisects \( \mu_1, \ldots, \mu_{(d-1)n} \): without loss of generality, consider the mass \( \mu_1 \). As \( s_1^1(b_0) = 0 \), we know by the definition of \( s_1^1 \) that \( (b_0, 0) \) is in \( Z(V_1) \). By the definition of \( Z(V_1) \) this means that \( V_1 \cap \pi^{-1}(b_0) \) (1) contains \( (b_0, 0) \) or (2) is unbounded.

In case (1), we get that \( (g_1(b_0, 0), h_2(b_0), \ldots, h_n(b_0)) \) bisects \( \mu_1 \). But since \( g_i(b_0, 0) = h_i(b_0) \), this set is exactly \( H \). In case (2), we notice that \( V_1 \) is unbounded on \( \pi^{-1}(b_0) \) if and only if \( \lim_{x \to \infty} f_{1,b_0} = 0 \). But this means that \( (h_2(b_0), \ldots, h_n(b_0)) \) bisects \( \mu_1 \). Thus, \( H \) indeed almost bisects \( \mu_1 \).

From Theorem 6.14 we also deduce the main result of this section:

**Theorem 1.10.** Let \( \mu_1, \ldots, \mu_{dn} \) be \( dn \) mass distributions in \( \mathbb{R}^d \). Then there are \( n \) hyperplanes that almost simultaneously bisect \( \mu_1, \ldots, \mu_{dn} \).

The idea is to lift the problem to one dimension higher. We will use the same argument again in the next chapter without proof.

**Proof.** Map \( \mathbb{R}^d \) to the hyperplane \( p : x_{d+1} = 1 \) in \( \mathbb{R}^{d+1} \). This induces an embedding of the masses \( \mu_1, \ldots, \mu_{dn} \). By defining \( \mu_i'(S) = \mu(S \cap p) \) for every full-dimensional open subset of \( \mathbb{R}^{d+1} \), we get \( dn \) mass distributions \( \mu_1', \ldots, \mu_{dn}' \) in \( \mathbb{R}^{d+1} \). By Theorem 6.14, there are \( n \) hyperplanes \( \ell_1', \ldots, \ell_n' \) of dimension \( d \) through the origin that almost simultaneously bisect \( \mu_1', \ldots, \mu_{dn}' \). Define \( \ell_i := \ell_i' \cap p \). Note that each \( \ell_i \) is a hyperplane of dimension \( d - 1 \). By the definition of \( \mu_i' \), the hyperplanes \( \ell_1, \ldots, \ell_n \) then almost simultaneously bisect \( \mu_1, \ldots, \mu_{dn} \). \( \square \)

It remains to prove Lemma 6.15.

**Proof of Lemma 6.15.** Consider again the bundle \( \xi_i' = (E', B', \pi') \), which is a cover of \( \xi_i \). The set \( Z(V_f') \) partitions \( B' \setminus Z(V_f') \) into connected components. Consider two lifts \( b_1', b_2' \) of a point \( b \in B \) with the property that the marked line \( \ell_i \) is oriented differently in \( b_1' \) than in \( b_2' \). We will call a pair of such lifts **antipodal**. We claim that if \( b_1', b_2' \notin Z(V_f') \) then \( b_1' \) and \( b_2' \) are not in the same connected component. If this is true, then
we can assign 1 or \(-1\) to each connected component in such a way that for any antipodal pair \(b'_1, b'_2\), whenever we assign 1 to the connected component containing \(b'_1\) we assign \(-1\) to the connected component containing \(b'_2\). We define \(s'\) as follows: for every \(b'_i\), let \(d(b'_i)\) be the distance to the boundary of its connected component (note that there are several ways to define distance measures on \(B'\), any of them is sufficient for our purposes). Place a point at distance \(d(b'_i)\) from the origin on the positive side of \(\ell_i\) if the connected component containing \(b'_i\) was assigned a 1, and on the negative side otherwise. This gives a section on \(\xi'\). Further, for any two antipodal lifts \(b'_1, b'_2\), we have \(s(b'_1) = -s(b'_2)\). Also, for any two lifts \(b'_3, b'_4\), that are not antipodal, that is, \(\ell_i\) is oriented the same way for both of them, we have \(s(b'_3) = s(b'_4)\). Thus, \(s'\) projects to a section \(s\) in \(\xi\) with the property that \(s(b) = 0\) only if \(b \in Z(V_f)\), which is what we want to prove.

Hence, we only need to show that a pair \(b'_1, b'_2\) of antipodal lifts is not in the same connected component. To this end, we will show that every path in \(B'\) from \(b'_1\) to \(b'_2\) crosses \(Z(V_f)\). Let \(\gamma\) be such a path. Then \(\gamma\) induces a continuous family of limit antipodal functions \(f_t, t \in [0, 1]\), with \(f_0 = f_{b'_1}\) and \(f_1 = f_{b'_2}\). Further, as \(b'_1\) and \(b'_2\) are antipodal, we have \(f_0(x) = \pm f_1(-x)\). If for any \(t\) we have \(\lim_{x \to \infty} f_t = 0\) we are done, so assume otherwise. Then, it is not possible that \(f_0(x) = f_1(-x)\), as in this case \(\lim_{x \to \infty} f_0 = -\lim_{x \to \infty} f_1\), so by continuity, there must be a \(t\) with \(\lim_{x \to \infty} f_t = 0\). Thus, assume that we have \(f_0(x) = -f_1(x)\).

The set of zeroes of the \(f_t\) defines a subset of \(\mathbb{R} \times [0, 1]\), which we denote by \(W\). See Figure 6.2 for an illustration. In general \(W\) is not connected, but has finitely many connected components, as by the second condition for limit antipodality each \(f_t\) has finitely many connected components of zeroes. We say that a connected component \(W_i\) of \(W\) has full support if for every \(t \in [0, 1]\), \(f_t\) has a zero in \(W_i\). It can be deduced from the limit antipodality of the \(f_t\)'s that \(W\) has an odd number of connected components with full support, denoted by \(W_1, \ldots, W_{2k+1}\). Consider the median component \(W_{k+1}\). Without loss of generality, \(W_{k+1}\) is a path in \(\mathbb{R} \times [0, 1]\) from \((x, 0)\) to \((-x, 1)\). By
a simple continuity argument, we see that $W_{k+1}$ must cross the line $(0,t), t \in [0,1]$. At this crossing, we are at a base point $b' \in Z(V_f)$, which concludes the proof.

In order to prove Conjecture 1.9, we would like to choose $Z(V_f)$ as the set of base points where $f_b(0) = 0$. Let us briefly give an example where our arguments fail for this definition. Consider $\mu$ as the area of a unit disk in $\mathbb{R}^2$. If we want to simultaneously bisect $\mu$ with two lines $\ell_1, \ell_2$ through the origin, these lines need to be perpendicular. Further, any single line through the origin bisects $\mu$ into two equal parts. Imagine now the line $\ell_1$ to be fixed, and consider the limit antipodal function $f_b$ defined by sweeping $\ell_2$ along an oriented line perpendicular to $\ell_1$. Without loss of generality, this function can be written as

$$
f_b(x) = \begin{cases} 
0 & x \in (-\infty, -1] \\
1 + x & x \in [-1, 0] \\
1 - x & x \in [0, 1] \\
0 & x \in [1, \infty].
\end{cases}
$$
Note that this holds whenever \( \ell_1 \) and the sweep line for \( \ell_2 \) are perpendicular, so in particular, continuously rotating the arrangement by 180° induces a path between two antipodal lifts in the cover. Further, along this path we never had \( f_\beta(0) = 0 \), so the two antipodal lifts would be in the same connected component, which would break the proof of Lemma \[6.15\] under this definition of \( Z(V_f) \). Thus, Conjecture \[1.9\] remains open for now.

### 6.3 Ham-Sandwich cuts after projective transformations

Before proving Theorem \[1.11\], we will prove a more general statement about bisections of mass distributions with two hyperplanes. Let us first explain in more detail how bisections with two hyperplanes can be regarded as Ham-Sandwich cuts after a projective transformation: Let \( \mu \) be a mass distribution in \( \mathbb{R}^d \) and let \((h_1, h_2)\) be two hyperplanes that bisect \( \mu \). Use gnomonic projection to map \( \mu \) and \((h_1, h_2)\) to the upper hemisphere of \( S^d \subseteq \mathbb{R}^{d+1} \). \[1\] Now, antipodally copy \( \mu \) and \((h_1, h_2)\) to the lower hemisphere. Note that both \( h_1 \) and \( h_2 \) are oriented \((d - 1)\)-dimensional great circles on \( S^d \), so we can extend them to oriented hyperplanes through the origin in \( \mathbb{R}^{d+1} \), which we denote as \( H_1 \) and \( H_2 \), respectively. Also, we will denote the defined measure on \( S^d \) by \( \mu_S \). Note now that \( \mu_S(S^d) = 2\mu(\mathbb{R}^d) \) and that \((H_1, H_2)\) bisects \( \mu_S \). Further, the above is invariant under rotations of the sphere, thus we can rotate the sphere until \( H_1 \) is one of the two orientations of the hyperplane.

\[1\] Gnomonic projection is a projection \( \pi \) of the upper hemisphere \( S^+ \) of a (unit) sphere to its tangent space \( T \) at the north pole, and vice-versa. It works as follows: for some point \( p \) on \( S^+ \), let \( \ell(p) \) be the line through \( p \) and the origin. The projection \( \pi(p) \) of \( p \) is then defined as the intersection of \( \ell(p) \) and \( T \). Note that this is a bijection from the (open) upper hemisphere to the tangent space. Gnomonic projection maps great circles to lines. More generally, we say that a great \( k \)-circle on a sphere \( S^d \) is the intersection on \( S^d \) with a \( k + 1 \)-dimensional linear subspace (i.e., a \((k + 1)\)-flat containing the origin). In particular, gnomonic projection then maps great \( k \)-circles to \( k \)-flats.
Using gnomonic projection to map the upper hemisphere to \( \mathbb{R}^d \), we get a projective transformation \( \varphi \) of \( \mathbb{R}^d \) with the property that \( \varphi(h_1) \) is the sphere at infinity and that \( \varphi(h_2) \) bisects \( \varphi(\mu) \). Thus, we get the following:

**Lemma 6.16.** Let \( \mu_1, \ldots, \mu_k \) be mass distributions in \( \mathbb{R}^d \) and let \((h_1, h_2)\) be two hyperplanes which simultaneously bisect \( \mu_1, \ldots, \mu_k \). Then there is a projective transformation \( \varphi \) of \( \mathbb{R}^d \) with the property that \( \varphi(h_1) \) is the sphere at infinity and that \( \varphi(h_2) \) simultaneously bisects \( \varphi(\mu_1), \ldots, \varphi(\mu_k) \).

In the following we will now prove results about bisections of different families with different double wedges which still share one of the hyperplanes.

**Lemma 6.17.** Let \( \mu_1^1, \ldots, \mu_d^1, \mu_1^2, \ldots, \mu_d^2, \ldots, \mu_d^{d-1} \) be \( d-1 \) families each containing \( d \) mass distributions in \( \mathbb{R}^d \), where \( d \) is odd. Then there exists a hyperplane \( h_1 \) containing the origin and \( d-1 \) hyperplanes \( h_i^2 \), \( i \in \{1, \ldots, d-1\} \), all containing the origin, such that \((h_1, h_i^2)\) simultaneously bisects \( \mu_1^i, \ldots, \mu_d^i \).

**Proof.** The space of pairs \((h_1, h_2)\) of oriented hyperplanes in \( \mathbb{R}^d \) containing the origin is \( S^{d-1} \times S^{d-1} \). For some mass distribution \( \mu \), we can thus define a function \( f : S^{d-1} \times S^{d-1} \to \mathbb{R} \) by \( f(h_1, h_2) := \mu(R^+) - \mu(R^+) \), where \( R^+ \) is defined as in Section 6.1. Note that \( f(-h_1, h_2) = f(h_1, -h_2) = -f(h_1, h_2) \) and that \( f(h_1, h_2) = 0 \) if and only if \((h_1, h_2)\) bisects \( \mu \). Further, for each \( h_1 \in S^{d-1} \) we get a function \( f_{h_1} := f(h_1, \cdot) : S^{d-1} \to \mathbb{R} \).

Let us now fix some \( h_1 \) and consider the family \( \mu_1^1, \ldots, \mu_d^1 \). Assume that \( \mu_1^1, \ldots, \mu_d^1 \) cannot be simultaneously bisected by a double wedge defined by \( h_1 \) and some other hyperplane through the origin. In particular, defining a function as above for each mass yields a map \( g_{h_1} : S^{d-1} \to \mathbb{R}^d \) which has no zero. Thus, after normalizing, we get a map \( g_{h_1} : S^{d-1} \to S^{d-1} \). In particular, this map has a degree. As we have \( g_{h_1}(-h_2) = -g_{h_1}(h_2) \), this degree is odd and thus non-zero. Further, varying \( h_1 \) again, we note that \( g_{-h_1}(h_2) = -g_{h_1}(h_2) \), and thus, as \( d-1 \) is even, we have \( \deg(g_{-h_1}) = -\deg(g_{h_1}) \). In particular, any path from
$-h_1$ to $h_1$ defines a homotopy between two maps of different degree, which is a contradiction. Thus, along every path from $-h_1$ to $h_1$ we encounter a hyperplane $h_1^*$ such that $g_{h_1^*}$ has a zero. In particular, this partitions $S^{d-1}$ into regions where $g_{h_1}$ has a zero and where it does not. Let $Z \subseteq S^{d-1}$ be the region where $g_{h_1}$ has a zero. We note that all regions are antipodal (i.e., $g_{h_1}$ has a zero if and only if $g_{-h_1}$ does) and no connected component of $S^{d-1} \setminus Z$ contains two antipodal points. Hence, we can define a map $t_1 : S^{d-1} \to \mathbb{R}$ as follows: for each $h_1 \in Z$, set $t_1(h_1) = 0$. Further, for each $h_1 \in S^{d-1} \setminus Z$, set $t_1(h_1) = \deg(g_{h_1}) \cdot d(h_1, Z)$, where $d(h_1, Z)$ denotes the distance from $h_1$ to $Z$. Note that $t_1$ is continuous and $t_1(-h_1) = -t_1(h_1)$ and $t_1(h_1) = 0$ if and only if $g_{h_1}$ has a zero.

We can do this for all families to get an antipodal map $t := (t_1, \ldots, t_{d-1}) : S^{d-1} \to \mathbb{R}^{d-1}$. By the Borsuk-Ulam theorem, this map has a zero. This zero gives us a hyperplane $h_1$ through the origin which, by construction, has the property that for each family of masses $\mu^1_1, \ldots, \mu^1_d$ there exists another hyperplane $h_2$ through the origin such that $(h_1, h_2)$ simultaneously bisects $\mu^1_1, \ldots, \mu^1_d$.

Lifting to one dimension higher, we get the following:

**Corollary 6.18.** Let $\mu^1_1, \ldots, \mu^1_{d+1}, \mu^2_1, \ldots, \mu^2_{d+1}, \ldots, \mu^d_{d+1}$ be $d$ families each containing $d + 1$ mass distributions in $\mathbb{R}^d$, where $d$ is even. Then there exists a hyperplane $h_1$ and $d$ hyperplanes $h^i_2$, $i \in \{1, \ldots, d\}$, such that $(h_1, h^i_2)$ simultaneously bisects $\mu^i_1, \ldots, \mu^i_{d+1}$.

Note that for the argument $\deg(g_{-h_1}) = -\deg(g_{h_1})$ we require that the considered sphere has even dimension. This means, that if we want to prove Lemma 6.17 for even dimensions, we have to use different arguments. In the following we try to do this, but at the expense that we will only be able to 'almost' bisect the masses. More precisely, we say that two hyperplanes $(h_1, h_2) \varepsilon$-bisect a mass $\mu$ if $|\mu(R^+) - \mu(\overline{R}^+)| < \varepsilon$. Similarly we say that $(h_1, h_2)$ simultaneously $\varepsilon$-bisects $\mu_1, \ldots, \mu_k$ if it $\varepsilon$-bisects $\mu_i$ for every $i \in \{1, \ldots, k\}$. In the following we will show
Lemma 6.17 for even dimensions, with 'bisect' replaced by 'ε-bisect'. For this we first need an auxiliary lemma.

**Lemma 6.19.** Let $\delta > 0$ and let $\mu_1^1, \ldots, \mu_{d-1}^1, \mu_1^2, \ldots, \mu_{d-1}^2, \mu_{d-2}^1, \ldots, \mu_d^{d-2}$ be $d-2$ families each containing $d-1$ mass distributions in $\mathbb{R}^d$, where $d$ is odd. Then there exists a hyperplane $h_1$ containing the $x_d$-axis, a point $p$ at distance $\delta$ to the $x_d$-axis and $d-2$ hyperplanes $h_i^2$, $i \in \{1, \ldots, d-1\}$, all containing the origin, such that $(h_1, h_2^i)$ simultaneously bisects $\mu_1^i, \ldots, \mu_{d-1}^i$ and $h_2^i$ contains $p$.

The proof is very similar to the proof of Lemma 6.17.

**Proof.** The space of pairs $(h_1, h_2)$ of oriented hyperplanes in $\mathbb{R}^d$ with $h_1$ containing the $x_d$-axis and $h_2$ containing the origin is $S^{d-2} \times S^{d-1}$. Let us again fix some $h_1$ and consider the family $\mu_1^1, \ldots, \mu_{d-1}^1$. As above, we can define $d-1$ functions which give rise to a function $g_{h_1} : S^{d-1} \to \mathbb{R}^{d-1}$, which has a zero if and only if $\mu_1^1, \ldots, \mu_{d-1}^1$ can be simultaneously bisected by two hyperplanes, including $h_1$. Further, for each $h_1$ we can define in a continuous fashion a point $p$ which lies in the positive side of $h_1$ and on the upper hemisphere of $S^{d-1} \subseteq \mathbb{R}^d$, and which has distance $\delta$ to the $x_d$-axis. Define now $d_{h_1} := I_{p \in R^+} \cdot d(p, h_2)$, where $d(p, h_2)$ denotes the distance from $p$ to $h_2$ and $I_{p \in R^+} = 1$ if $p$ lies in $R^+$ and $I_{p \in R^+} = -1$ otherwise. Note that $d_{h_1}$ is continuous, $d_{h_1} = 0$ if and only if $p$ is on $h_2$, and $d_{h_1}(-h_2) = -d_{h_1}(h_2)$. Thus, together with the $d-1$ functions defined by the masses, we get a map $g_{h_1} : S^{d-1} \to \mathbb{R}^d$, which has a zero if and only if $\mu_1^1, \ldots, \mu_{d-1}^1$ can be simultaneously bisected by a double wedge using $h_1$ and an $h_2$ passing through $p$. Again, assuming this map has no zero, we get a map $g_{h_1} : S^{d-1} \to S^{d-1}$, which, because of the antipodality condition, has odd degree. Again, we have $g_{-h_1}(h_2) = -g_{h_1}(h_2)$, and thus along every path from $-h_1$ to $h_1$ we encounter a hyperplane $h_1^*$ such that $gh_1^*$ has a zero. As above, this partitions the sphere into antipodal regions, the only difference being that this time we only consider the sphere $S^{d-2}$. In particular, the Borsuk-Ulam theorem now gives us a hyperplane $h_1$ containing the $x_d$-axis which, by construction, has the property that for each family of masses $\mu_1^1, \ldots, \mu_{d-1}^1$ there exists another hyperplane $h_2^i$ through the
origin such that \((h_1, h_2^i)\) simultaneously bisects \(\mu_1^i, \ldots, \mu_{d-1}^i\). Further, the hyperplane \(h_1\) also defines a point \(p\) at distance \(\delta\) to the \(x_d\)-axis with the property that each \(h_2^i\) contains \(p\).

After projection, we thus get hyperplanes \((h_1, h_2^i)\) that simultaneously bisect the masses \(\mu_1^i, \ldots, \mu_{d-1}^i\) and such that \(h_1^i\) contains the origin and the distance from \(h_2^i\) is at most \(\delta\). If we now translate \(h_2^i\) to contain the origin, by continuity we get that there is some \(\varepsilon > 0\) such that \((h_1, h_2^i)\) simultaneously \(\varepsilon\)-bisects the masses \(\mu_1^i, \ldots, \mu_{d-1}^i\). In particular, for every \(\varepsilon > 0\) we can choose \(\delta > 0\) in the above lemma such that after lifting to one dimension higher we get the following:

**Corollary 6.20.** Let \(\varepsilon > 0\).

1. Let \(\mu_1^1, \ldots, \mu_1^{d-1}, \mu_2^1, \ldots, \mu_2^{d-1}, \ldots, \mu_d^{d-1}\) be \(d-1\) families each containing \(d\) mass distributions in \(\mathbb{R}^d\), where \(d\) is even. Then there exists a hyperplane \(h_1\) containing the origin and \(d-1\) hyperplanes \(h_2^i, i \in \{1, \ldots, d-1\}\), all containing the origin, such that \((h_1, h_2^i)\) simultaneously \(\varepsilon\)-bisects \(\mu_1^i, \ldots, \mu_{d-1}^i\).

2. Let \(\mu_1^1, \ldots, \mu_{d+1}^1, \mu_2^1, \ldots, \mu_{d+1}^2, \ldots, \mu_d^{d+1}\) be \(d\) families each containing \(d+1\) mass distributions in \(\mathbb{R}^d\), where \(d\) is odd. Then there exists a hyperplane \(h_1\) and \(d\) hyperplanes \(h_2^i, i \in \{1, \ldots, d\}\), such that \((h_1, h_2^i)\) simultaneously \(\varepsilon\)-bisects \(\mu_1^i, \ldots, \mu_{d+1}^i\).

By the remarks after the definition of bisections in Chapter 1 (see also [70]), a bisection partition result for mass distributions also implies the analogous result for point sets in general position. Further, for point sets in general position, we can choose \(\varepsilon\) small enough to get an actual bisection. Thus, Theorem 1.11 now follows from Lemma 6.16, Corollary 6.18 and the second part of Corollary 6.20.

**Theorem 1.11.** Let \(P_1^1, \ldots, P_{d+1}^1, P_1^2, \ldots, P_{d+1}^2, \ldots, P_d^{d+1}\) be \(d\) families each containing \(d+1\) point sets in \(\mathbb{R}^d\) such that their union is in general position. Then there exists a projective transformation \(\varphi\) such that \(\varphi(P_1^i), \ldots, \varphi(P_{d+1}^i)\) can be simultaneously bisected by a single hyperplane for every \(i \in \{1, \ldots, d\}\).
PARTITIONS WITH FANS AND CONES

For the final chapter, we turn our attention to fans and cones. We first generalize a result by Makeev about equipartitions with \( p \)-fans (see [60], Theorem 57) to partitions where not every wedge in the fan contains the same fraction. As there does not seem to be a publicly available proof of Makeev’s result, for the sake of completeness we also give a proof for equipartitions. In even dimensions our result is slightly weaker than the one by Makeev, it would be interesting to find out what techniques he used and to what extent they can be adapted to more general partitions.

Finally, for 2-fans, we give tight results about bisections of masses, which extend to \( k \)-cones. The result at the bottom of this is Lemma 4.1 which we will rephrase as a Borsuk-Ulam-type theorem for flag manifolds.
7.1 Partitions with fans

Similar to the previous chapters, we will again first prove all results with the apex containing the origin. We will call such a fan a fan through the origin. For technical reasons, we distinguish whether the dimension is even or odd. The general results will then again follow from lifting to one dimension higher. Our proofs are very similar to those in [10].

Lemma 7.1. Let $p$ be an odd prime and let $d$ be odd.

1. Any $\left\lceil \frac{2d-3}{p-1} \right\rceil + 1$ mass distributions in $\mathbb{R}^d$ can be simultaneously \((\frac{1}{p}, \ldots, \frac{1}{p})\)-equipartitioned by a $p$-fan through the origin;

2. Let \((a_1, \ldots, a_q) \in \mathbb{N}^q\) with $q < d$ and $a_1 + \ldots + a_q = p$. Then any $\left\lceil \frac{2d-2}{p-1} \right\rceil + 1$ mass distributions in $\mathbb{R}^d$ can be simultaneously \((\frac{a_1}{p}, \ldots, \frac{a_q}{p})\)-equipartitioned by a $q$-fan through the origin.

Proof. We start with equipartitions. Assume without loss of generality that for each mass distribution $\mu_i$ we have $\mu_i(\mathbb{R}) = 1$. Let $k := \left\lceil \frac{2d-3}{p-1} \right\rceil$. Consider the Stiefel manifold $V_2(\mathbb{R}^d)$ of all pairs $(x, y)$ of orthonormal vectors in $\mathbb{R}^d$. To each $(x, y) \in V_2(\mathbb{R}^d)$ we assign a $p$-fan $F(x, y)$ as follows: Let $h$ by the linear subspace spanned by $(x, y)$ and let $\pi : \mathbb{R}^d \rightarrow h$ be the canonical projection. The apex of the $p$-fan $F(x, y)$ is then $\pi^{-1}(0)$. Further, note that $(x, y)$ defines an orientation on $h$, so we can consider a ray on $h$ rotating in clockwise direction. Start this rotation at $x$, and let $r_1$ be the (unique) ray such that the area between $x$ and $r_1$ is the projection of a wedge $W_1$ which contains exactly a $\frac{1}{p}$-fraction of the total mass. Analogously, let $r_i$ be the (unique) ray such that the area between $r_{i-1}$ and $r_i$ is the projection of a wedge $W_i$ which contains exactly a $\frac{1}{p}$-fraction of the total mass. This construction thus continuously defines a $p$-fan $F(x, y)$ through the origin for each $(x, y) \in V_2(\mathbb{R}^d)$. Further note that there is a natural $\mathbb{Z}_p$ action on $V_2(\mathbb{R}^d)$, defined by $(W_1, W_2, \ldots, W_p) \mapsto (W_p, W_1, \ldots, W_{p-1})$, i.e., by turning by one sector.
For a mass distribution \( \mu_i, i \in \{1, \ldots, k\} \) we introduce a test map \( f_i(x, y) : V_2(\mathbb{R}^d) \to \mathbb{R}^p \) by
\[
f_i(x, y) := \left( \mu_i(W_1) - \frac{1}{p}, \mu_i(W_2) - \frac{1}{p}, \ldots, \mu_i(W_p) - \frac{1}{p} \right).
\]
Note that the image of \( f_i \) is contained in the hyperplane \( Z = \{ y \in \mathbb{R}^p : y_1 + y_2 + \ldots + y_p = 0 \} \) of dimension \( p-1 \) and that \( f_i(x, y) = 0 \) implies that \( f_i(x, y) = 0 \) for all \( i \in \{1, \ldots, k\} \), then \( F(x, y) \) simultaneously equipartitions \( \mu_1, \ldots, \mu_k \), and thus, as \( F(x, y) \) equipartitions the total mass by construction, it also equipartitions \( \mu_{k+1} \). We thus want to show that all test maps have a common zero. To this end, we note that the \( \mathbb{Z}_p \)-action on \( V_2(\mathbb{R}^d) \) induces a \( \mathbb{Z}_p \)-action on \( B \mathbb{Z}_p \) by \( \nu(y_1, y_2, \ldots, y_p) = (y_p, y_1, \ldots, y_{p-1}) \). Further, as \( p \) is prime, this action is free on \( \mathbb{Z}_p \{0\} \). Thus, if we assume that the test maps do not have a common zero, they induce a \( \mathbb{Z}_p \)-map \( f : V_2(\mathbb{R}^d) \to Z^k \setminus \{0\} \). We will now show that there is no such map.

To this end, we first note that the dimension of \( Z^k \) is \((p - 1)k\) and that, after normalizing, \( f \) induces a map \( f' : V_2(\mathbb{R}^d) \to S^{(p-1)k-1} \). We will use the following strengthening of Dold’s theorem due to Jelic [56]:

**Theorem 7.2** ([56], Thm. 2.1). Let \( G \) be a finite group acting freely on a cell \( G \)-complex \( Y \) of dimension at most \( n \), and let \( X \) be a \( G \)-space. Let \( R \) be a commutative ring with unit such that \( H^{n+1}(BG; R) \neq 0 \) and \( \tilde{H}^k(X; R) = 0 \) for \( 0 \leq k \leq n \). Then there is no \( G \)-equivariant map \( g : X \to Y \).

Setting \( G = R = \mathbb{Z}_p \), we get that \( H^k(B\mathbb{Z}_p; \mathbb{Z}_p) \neq 0 \) for all \( k \), see [91], Theorem III 2.5. Further, for \( d \) odd, we have \( H^i(V_2(\mathbb{R}^d); \mathbb{Z}) \cong \mathbb{Z} \) for \( i \in \{0, 2d-3\} \), \( H^{d-1}(V_2(\mathbb{R}^d); \mathbb{Z}) \cong \mathbb{Z}_2 \), and all other cohomology groups are trivial (see [23], Prop. 10.1). From the universal coefficients theorem we thus get \( H^i(V_2(\mathbb{R}^d); \mathbb{Z}_p) \cong \mathbb{Z}_p \) for \( i \in \{0, 2d-3\} \), and \( H^i(V_2(\mathbb{R}^d); \mathbb{Z}_p) = 0 \) otherwise. As
\[
(p - 1)k - 1 = (p - 1)\left( \frac{2d - 3}{p - 1} \right) - 1 \leq 2d - 3 - 1 = 2d - 4,
\]
we can thus apply Theorem 7.2 with \( n = 2d - 4 \), where \( Y = S^{(p-1)k-1} \), to show that \( f' \), and thus \( f \), cannot exist. This finishes the proof for the equipartitions.

As for the more general partitions, we let \( k := \lfloor \frac{2d - 2}{p-1} \rfloor \). We take the same configuration space and test maps, only that we now have more possible solutions to exclude from \( Z^k \). In particular, let \( L_i := \{ y \in \mathbb{R}^p : y_1 + \ldots + y_{a-1} = 0, y_{a+1} + \ldots + y_{a+a-1} = 0, \ldots, y_{a+\ldots+a-1+a-1} + \ldots + y_p = 0 \} \). Further let \( L_i := \{ L_i, \nu(L_i), \nu^2(L_i), \ldots, \nu^{p-1}(L_i) \} \). We now want to show that there is no \( Z_p \)-equivariant map \( f : V_2(\mathbb{R}^d) \to Z^k \setminus \bigcup_{1 \leq i \leq k} L_i \).

By Lemma 6.1 in [10], \( f \) would induce a map \( f' : V_2(\mathbb{R}^d) \to M \), where \( M \) is a \(((p-1)k-2)\)-dimensional manifold on which \( Z_p \) acts freely. As \((p-1)k-2 = (p-1)\left\lfloor \frac{2d - 2}{p-1} \right\rfloor - 2 \leq 2d - 2 - 2 = 2d - 4\),

the non-existence of \( f \) again follows from Theorem 7.2. \( \square \)

When \( d \) is even, we cannot use the above method, as any Cohomology of \( V_2(\mathbb{R}^d) \) has too many non-trivial groups. However, we can again use lifting to the upper hemisphere of \( S^d \subset \mathbb{R}^{d+1} \) and use some degrees of freedom to enforce that the apex of the equipartitioning fan contains the \( x_{d+1} \)-axis. Projecting back to \( \mathbb{R}^d \) then gives us an equipartitioning fan through the origin. For this we need the following lemma:

**Lemma 7.3.** Let \( F^n_p \) be the space of all \( p \)-fans through the origin in \( \mathbb{R}^n \), endowed with the natural \( \mathbb{Z}_p \)-action. Then there exists a \( \mathbb{Z}_p \)-equivariant map \( z : F^n_p \to \mathbb{R}^{p-1} \) such that \( z(F') = 0 \) if and only if the apex of \( F \) contains the \( x_n \)-axis.

**Proof.** Let \( P \) be the north pole of the sphere \( S^{n-1} \), that is, the point \((0, \ldots, 0, 1) \in \mathbb{R}^n \). Let \( F \) be a \( p \)-fan through the origin \( O \) whose wedges \( W_1, \ldots, W_p \) have angles \( \alpha_1, \ldots, \alpha_p \). Let \( a \) be the apex of \( F \) and let \( h \) be the orthogonal complement of \( a \). Project \( F \) and \( P \) to \( h \). This defines a two-dimensional \( p \)-fan, which we again denote by \( F \), with apex \( O \) and a point, which we again denote by \( P \). We will again denote the wedges of \( F \) by \( W_1, \ldots, W_p \), and they still have angles \( \alpha_1, \ldots, \alpha_p \). We now
construct a \((p - 1)\)-valued function \(z\) which is \(\mathbb{Z}_p\)-equivariant and for which \(z(F) = 0\) if and only if \(P = O\). As the projection is continuous, such a function implies the lemma.

Fix some small \(\varepsilon > 0\) and let \(D := \min(d(P, O), 1 - \varepsilon)\), where \(d(P, O)\) denotes the distance from \(O\) to \(P\). Assume that \(O \neq P\). Consider the circle \(C\) with radius \(d(P, O)\) and center \(O\). On this circle, draw circular segments of length \((1 - D)\pi\) to both sides of \(P\). (See Figure 7.1 for an illustration). This defines a circular segment \(S\). Note that as \(d(P, O)\) converges to 0, the circular segment \(S\) converges to a full circle. Let \(s_i\) denote the length of the circular segment \(S \cap W_i\). Define

\[
g(W_i) := \frac{s_i}{\alpha_i} \cdot \frac{1}{p} \cdot \left(\sum_{j=1}^{p} \frac{s_j}{\alpha_j} \cdot \frac{1}{p}\right)^{-1}.
\]

Note that \(\sum_{i=1}^{p} g(W_i) = 1\) and that, assuming \(O \neq P\), there is always a wedge \(W_i\) for which \(\frac{s_i}{\alpha_i} < 1\). Further note that \(g(W_i)\) converges to \(\frac{1}{p}\) when \(d(P, O)\) converges to 0. Thus, we can continuously extend \(g\) by setting \(g(W_i) := \frac{1}{p}\) for \(d(P, O) = 0\).

Let now \(z(F) := (\frac{1}{p} - g(W_1), \ldots, \frac{1}{p} - g(W_p))\). Note that \(z\) carries a natural \(\mathbb{Z}_p\)-action. Further note that the image of \(z\) is contained in the hyperplane \(\{y \in \mathbb{R}^d : y_1 + \ldots + y_p = 0\}\), i.e., we can view \(z\) as a map to \(\mathbb{R}^{p-1}\). Finally, it follows from the construction that \(z(F) = 0\) if and only if \(P = O\).

Using the above lemma, we can now prove the following:

**Lemma 7.4.** Let \(p\) be an odd prime and let \(d\) be even. Let \((a_1, \ldots, a_q) \in \mathbb{N}^q\) and \(a_1 + \ldots + a_q = p\). Then any \(\left\lfloor \frac{2d-1}{p-1} \right\rfloor\) mass distributions in \(\mathbb{R}^d\) can be simultaneously \((\frac{a_1}{p}, \ldots, \frac{a_q}{p})\)-equipartitioned by a \(q\)-fan through the origin.

Note that here we do not distinguish between equipartitions and general partitions. The reason for this is that in odd dimensions, we used
Lemma 6.1 in [10] to reduce the dimension of the target space. However, this lemma requires that the linear span of the excluded solution is the whole target space, which is now not the case anymore, as for the function $z$ we only exclude the origin of $\mathbb{R}^p$. Otherwise, the proof is essentially the same as above, so we will only sketch the main differences.

**Proof.** Lift the mass distributions to the upper hemisphere of $S^d \subseteq \mathbb{R}^{d+1}$. We will show that there is a $q$-fan whose apex contains the $x_{d+1}$-axis which simultaneously $(\frac{a_1}{p}, \ldots, \frac{a_q}{p})$-equipartitions all mass distributions. After projecting back, we then get the desired result.

Let $k := \lfloor \frac{2d-1}{p-1} \rfloor - 1$. Consider the same configuration space and test maps as in the proof of Lemma 7.1. Further, add the function $z$ from Lemma 7.3 as an additional test map. Assuming there is no required $q$-fan, we get a $\mathbb{Z}_p$-equivariant map $f' : V_2(\mathbb{R}^{d+1}) \to S^l$, where $l =$
(p - 1)k + (p - 1) - 1 = (p - 1)(k + 1) - 1. As

(p - 1)(k + 1) - 1 = (p - 1)[\frac{2d - 1}{p - 1}] - 1 \leq 2d - 1 - 1 = 2(d + 1) - 4,

such a map cannot exist by Jelic’s extension of Dold’s theorem.

Analogously to the previous sections, Theorem 1.12 now follows again from lifting to one dimension higher:

**Theorem 1.12.** Let $p$ be an odd prime.

1. Any $\lfloor \frac{2d-1}{p-1} \rfloor + 1$ mass distributions in $\mathbb{R}^d$, where $d$ is odd, can be simultaneously $\left(\frac{1}{p} \ldots, \frac{1}{p}\right)$-equipartitioned by a $p$-fan;

2. Any $\lfloor \frac{2d+1}{p-1} \rfloor$ mass distributions in $\mathbb{R}^d$, where $d$ is even, can be simultaneously $\left(\frac{1}{p} \ldots, \frac{1}{p}\right)$-equipartitioned by a $p$-fan;

3. Let $(a_1, \ldots, a_q) \in \mathbb{N}^q$ with $q < n$ and $a_1 + \ldots + a_q = p$. Then any $\lfloor \frac{2d}{p-1} \rfloor + 1$ mass distributions in $\mathbb{R}^d$, where $d$ is odd, can be simultaneously $\left(\frac{a_1}{p} \ldots, \frac{a_q}{p}\right)$-equipartitioned by a $q$-fan;

4. Let $(a_1, \ldots, a_q) \in \mathbb{N}^q$ with $q < n$ and $a_1 + \ldots + a_q = p$. Then any $\lfloor \frac{2d+1}{p-1} \rfloor$ mass distributions in $\mathbb{R}^d$, where $d$ is even, can be simultaneously $\left(\frac{a_1}{p} \ldots, \frac{a_q}{p}\right)$-equipartitioned by a $q$-fan.

**7.2 Bisections with $k$-cones**

We start by rephrasing a special case of Lemma 4.1 as a Borsuk-Ulam type result for flag manifolds. We will consider flag manifolds with oriented lines: a flag manifold with oriented lines $\mathcal{F}$ is a flag manifold where each flag contains a 1-dimensional subspace (that is, a line), and

\footnote{From a purely mathematical point of view this is of course not necessary. However, with this rephrasing, the proof about bisections with cones resembles the proof of the Ham-Sandwich Theorem even more closely and might thus be more accessible for readers not familiar with vector bundles.}
this line is oriented. In particular, each flag manifold with oriented lines $\mathcal{F}$ is a double cover of the underlying flag manifold $\mathcal{F}$. Further, reorienting the line defines a natural antipodal action. We denote the complete flag manifold with oriented lines by $\mathcal{V}_{n,n}$.

**Theorem 7.5** (Borsuk-Ulam for flag manifolds). Let $\mathcal{F}$ be a flag manifold with oriented lines in $\mathbb{R}^{d+1}$. Then every antipodal map $f : \mathcal{F} \to \mathbb{R}^d$ has a zero.

Let us briefly mention how this implies the Borsuk-Ulam theorem: the simplest flags containing a line are those of the form $F = \{0\} \subset \ell \subset \mathbb{R}^{d+1}$. The corresponding flag manifold is the manifold of all lines through the origin in $\mathbb{R}^{d+1}$, that is, the projective space. Orienting the lines, we retrieve the manifold of all oriented lines through the origin in $\mathbb{R}^{d+1}$, which is homeomorphic to the sphere $S^d$. The above theorem now says that every antipodal map from $S^d$ to $\mathbb{R}^d$ has a zero, which is one of the versions of the Borsuk-Ulam theorem.

Note that any antipodal map $f : \mathcal{F} \to \mathbb{R}^d$ can be extended to an antipodal map $g : \mathcal{V}_{d+1,d+1} \to \mathbb{R}^d$ by concatenating it with the natural projection from $\mathcal{V}_{d+1,d+1}$ to $\mathcal{F}$, thus it is enough to show the statement for $\mathcal{V}_{d+1,d+1}$.

Recall that from Lemma 4.1 we know that for any $m + 1$ sections of the canonical bundle $\vartheta^m_1$ there is a flag $F \in \tilde{V}_{m+1,m+1}$ in which they all coincide. We will now show how this implies Theorem 7.5.

**Proof of Theorem 7.5**. Let $f : \mathcal{V}_{d+1,d+1} \to \mathbb{R}^d$ be antipodal. Write $f$ as $(f_1, \ldots, f_d)$, where each $f_i : \mathcal{V}_{d+1,d+1} \to \mathbb{R}$ is an antipodal map. In particular, each $f_i$ defines a section $s_i$ in $\vartheta^d_1$. Let $s_0$ be the zero section in $\vartheta^d_1$. Thus, $s_0, s_1, \ldots, s_d$ are $d + 1$ sections in $\vartheta^d_1$ and thus by Lemma 4.1 there is a flag $F$ on which the coincide. As $s_0$ is the zero section, this means that $f_i(F) = 0$ for all $i \in \{1, \ldots, d\}$ and thus $f(F) = 0$, which is what we wanted to show.

We will now use the Borsuk-Ulam theorem for flag manifolds to show
the existence of bisections with $k$-cones.

**Theorem 7.6.** Let $\mu_1, \ldots, \mu_d$ be $d$ mass distributions in $\mathbb{R}^d$, let $p \in \mathbb{R}^d$ be a point and let $1 \leq k \leq d$. Then there exists a $k$-cone $C$ whose apex contains $p$ and that simultaneously bisects $\mu_1, \ldots, \mu_d$.

**Proof.** Without loss of generality, let $p$ be the origin. Let $\overrightarrow{\mathcal{F}}$ be the flag manifold with oriented lines defined by the flags $(0, \overrightarrow{\mathcal{L}}, V_k, \mathbb{R}^d)$, where $\overrightarrow{\mathcal{L}}$ has dimension 1 and $V_k$ has dimension $k$. Each $(0, \overrightarrow{\mathcal{L}}, V_k, \mathbb{R}^d)$ defines a unique $k$-cone that bisects the total mass $\mu_1 + \ldots + \mu_d$ and whose projection to $V_k$ is a cone $C$ with central axis $\overrightarrow{\mathcal{L}}$ and apex $p$. For $i \in \{1, \ldots, d-1\}$, define $f_i := \mu_i(C) - \mu_i(\overline{C})$, where $\overline{C}$ denotes the complement of $C$. Then $f := (f_1, \ldots, f_{d-1})$ is a map from $\overrightarrow{\mathcal{F}}$ to $\mathbb{R}^{d-1}$. Further, as every $C$ bisects the total mass, so does the complement $\overline{C}$. In particular, $\overline{C}$ is the unique $k$-cone that we get when switching the orientation of $\overrightarrow{\mathcal{L}}$. Thus, $f$ is antipodal, and by Theorem 7.5 it has a zero. Let $C_0$ be the $k$-cone defined by this zero. By the definition of $f$ we have that $C_0$ simultaneously bisects $\mu_1, \ldots, \mu_{d-1}$. By construction, $C_0$ also bisects the total mass, thus, it must also bisect $\mu_d$. $\square$

As for previous results, Theorem 1.13 now follows by lifting to one dimension higher.

**Theorem 1.13.** Let $\mu_1, \ldots, \mu_{d+1}$ be $d+1$ mass distributions in $\mathbb{R}^d$ and let $1 \leq k \leq d$. Then there exists a $k$-cone $C$ that simultaneously bisects $\mu_1, \ldots, \mu_{d+1}$.

This is tight in the sense that there is a family of $d+2$ mass distributions in $\mathbb{R}^d$ that cannot be simultaneously bisected: place $d+1$ point-like masses at the vertices of a $d$-dimensional simplex and a last point-like mass in the interior of the simplex. Assume that there is a $k$-cone $C$ which simultaneously bisects all the masses and assume without loss of generality that $C$ is convex (otherwise, just consider the complement of $C$). As it simultaneously bisects all masses, $C$ must now contain all vertices of the simplex, so by convexity $C$ also contains the interior of the simplex, and thus all of the last mass. Hence $C$ cannot simulta-
neously bisect all masses. Still, while we cannot hope to bisect more masses, in some cases we are able to enforce additional restrictions: for \( d \)-cones in odd dimensions, we can always enforce the apex to lie on a given line.

**Theorem 7.7.** Let \( \mu_1, \ldots, \mu_{d+1} \) be \( d+1 \) mass distributions in \( \mathbb{R}^d \), where \( d \) is odd and let \( g \) be a line. Then there exists a \( d \)-cone \( C \) whose apex \( a \) lies on \( g \) that simultaneously bisects \( \mu_1, \ldots, \mu_{d+1} \).

**Proof.** Without loss of generality, let \( g \) be the \( x_d \)-axis. Place \( a \) somewhere on \( g \). We will move \( a \) along \( g \) from \( -\infty \) to \( +\infty \). Consider all directed lines through \( a \). Each such line \( \overrightarrow{\ell} \) defines a unique \( d \)-cone \( C \) bisecting the total mass. For each \( i \in \{1, \ldots, d\} \), define \( f_i := \mu_i(C) - \mu_i(\overline{C}) \), where \( \overline{C} \) again denotes the complement of \( C \). Thus, for each choice of \( a \), we get a map \( f_a = (f_1, \ldots, f_d) : S^{d-1} \to \mathbb{R}^d \). We claim that for some \( a \) this map has a zero. Assume for the sake of contradiction that none of the maps have a zero. Then we can normalize them to get maps \( f_a : S^{d-1} \to S^{d-1} \). In particular, each such map has a degree. Further, all of the maps are antipodal, implying that their degree is odd, and thus non-zero. Finally, note that \( f_{-\infty} = -f_{+\infty} \), and thus \( \deg(f_{-\infty}) = -\deg(f_{+\infty}) \). (Here we require that \( d-1 \) is even.) In particular, as the degrees are non-zero, \( f_{-\infty} \) and \( f_{+\infty} \) have different degrees. But moving \( a \) along \( g \) from \( -\infty \) to \( +\infty \) defines a homotopy from \( f_{-\infty} \) to \( f_{+\infty} \), which is a contradiction. Thus, there exists some \( a \) such that \( f_a \) has a zero and analogous to above this zero defines a \( d \)-cone \( C \) that simultaneously bisects \( \mu_1, \ldots, \mu_{d+1} \). \( \Box \)
Bibliography


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