Kronecker limit formulas for parabolic, hyperbolic and elliptic Eisenstein series via Borcherds products

General Section

Kronecker limit formulas for parabolic, hyperbolic and elliptic Eisenstein series via Borcherds products

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ARTICLE INFO

ABSTRACT

The classical Kronecker limit formula describes the constant term in the Laurent expansion at the first order pole of the non-holomorphic Eisenstein series associated to the cusp at infinity of the modular group. Recently, the meromorphic continuation and Kronecker limit type formulas were investigated for non-holomorphic Eisenstein series associated to hyperbolic and elliptic elements of a Fuchsian group of the first kind by Jorgenson, Kramer and the first named author. In the present work, we realize averaged versions of all three types of Eisenstein series for $\Gamma_0(N)$ as regularized theta lifts of a single type of Poincaré series, due to Selberg. Using this realization and properties of the Poincaré series we derive the meromorphic continuation and Kronecker limit formulas for the above Eisenstein series. The corresponding Kronecker limit functions are then given by the logarithm of the absolute value of the Borcherds product associated to a special value of the underlying Poincaré series.

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1. Introduction

1.1. Non-holomorphic Eisenstein series

Let N be a positive integer. The classical non-holomorphic Eisenstein series associated to a cusp p of $\Gamma_0(N)$ is defined by

$$E_p^{\mathrm{par}}(z,s) = \sum_{M \in \Gamma_0(N)_p \backslash \Gamma_0(N)} \mathrm{Im}(\sigma_p^{-1} M z)^s$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix for the cusp p, that is $\sigma_p \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right) \sigma_p^{-1}$ generates the stabilizer $\Gamma_0(N)_p / \{\pm 1\}$ of the cusp p in $\Gamma_0(N) / \{\pm 1\}$. The Eisenstein series has a meromorphic continuation to the whole complex plane with a simple pole at $s = 1$, and the Kronecker limit formula describes the constant term in the Laurent expansion at this point. For example, for $N = 1$ it has the well-known form

$$E_\infty^{\mathrm{par}}(z,s) = \frac{3/\pi}{s-1} - \frac{1}{2\pi} \log \left(|\Delta(z)| \, \mathrm{Im}(z)^6 \right) + C + O(s-1) \qquad \text{as } s \to 1, \quad (1.1)$$

where $C = (6 - 72\zeta'(-1) - 6\log(4\pi))/\pi$. We call $E_p^{\mathrm{par}}(z,s)$ a parabolic Eisenstein series in order to distinguish it from the following two analogs:

Given a geodesic c in $\mathbb{H}$, i.e., a vertical line or a semi-circle centered at the real line, we define the hyperbolic Eisenstein series associated to c via

$$E_c^{\mathrm{hyp}}(z,s) = \sum_{M \in \Gamma_0(N)_c \backslash \Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(Mz, c))^{-s}$$

for $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here $\Gamma_0(N)_c$ denotes the stabilizer of the oriented geodesic c in $\Gamma_0(N)$, and $d_{\mathrm{hyp}}(z, c)$ denotes the hyperbolic distance from the point z to c. They are scalar valued analogs of the form valued hyperbolic Eisenstein series introduced by Kudla and Millson in their work [KM79]. The analytic continuation of $E_c^{\mathrm{hyp}}(z,s)$ for geodesics corresponding to hyperbolic elements of an arbitrary Fuchsian group of the first kind has been established by Jorgenson, Kramer and the first named author in [JKP10]. We note that we do not assume that the geodesic c comes from some hyperbolic element of the underlying group $\Gamma_0(N)$, that is, the stabilizer $\Gamma_0(N)_c$ is allowed to be trivial, which is equivalent to saying that the image of c in the modular curve $\Gamma_0(N) \backslash \mathbb{H}$ is an infinite geodesic.

For a point $w \in \mathbb{H}$, we define the elliptic Eisenstein series associated to w via

$$E_w^{\mathrm{ell}}(z,s) = \sum_{M \in \Gamma_0(N)_w \backslash \Gamma_0(N)} \sinh(d_{\mathrm{hyp}}(Mz, w))^{-s}$$

for $z \in \mathbb{H}$ not lying in the $\Gamma_0(N)$-orbit of w, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. These series were introduced by Jorgenson and Kramer in their unpublished work [JK04] (see also

[JK11]), and they have been investigated in detail for an arbitrary Fuchsian group of the first kind by the first named author in [Pip10] and [Pip16]. It was shown in [Pip16] that the elliptic Eisenstein series has a meromorphic continuation to $\mathbb{C}$ and admits a Kronecker limit formula at $s = 0$, which for $N = 1$ takes the beautiful form

$$E_w^{\text{ell}}(z, s) = -\log\left(|j(z) - j(w)|^{2/|\Gamma_0(N)_w|}\right) \cdot s + O(s^2). \tag{1.2}$$

Here $j(z)$ is the usual modular j-function. Further explicit examples are computed in [JPS17].

The main goal of the present work is to realize (averaged versions of) all three types of Eisenstein series for $\Gamma_0(N)$ as regularized theta lifts of certain non-holomorphic vector valued Poincaré series of weight $1/2$, and to use this representation to systematically derive explicit Kronecker limit type formulas for these averaged hyperbolic and elliptic Eisenstein series. For completeness, we also re-prove the classical Kronecker limit formula for the parabolic Eisenstein series using our theta lift approach.

Throughout this work, we assume that N is squarefree. Although the results of this work certainly hold for general positive integers N (with some minor modifications), this assumption greatly simplifies the exposition and allows us to make our results very explicit. Let us describe our results in some more detail.

1.2. Eisenstein series as theta lifts

For $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and a discriminant $D \in \mathbb{Z}$ with $D \equiv \beta^2 \bmod 4N$ we consider the non-holomorphic vector valued Poincaré series

$$P_{\beta,D}(\tau, s) = \frac{1}{2} \sum_{(M,\phi) \in \langle (T,1) \rangle \backslash \operatorname{Mp}_2(\mathbb{Z})} v^s e(D\tau/4N) \mathfrak{e}_\beta \Big|_{1/2,\rho} (M, \phi), \quad (\tau = u + iv \in \mathbb{H}),$$

where we refer to Section 2 for the notation. Note that this definition slightly differs from the one given in (3.1), namely by the relation $D = 4Nm$. The function $P_{\beta,D}(\tau, s)$ is a vector valued version of Selberg's Poincaré series introduced in [Sel65]. It transforms like a vector valued modular form of weight $1/2$ for the Weil representation ρ associated to a certain even lattice L of signature $(2, 1)$ and level $4N$, and it satisfies the differential equation

$$\Delta_{1/2} P_{\beta,D}(\tau, s) = s\left(\frac{1}{2} - s\right) P_{\beta,D}(\tau, s) + s\frac{\pi D}{N} P_{\beta,D}(\tau, s + 1),$$

where $\Delta_{1/2}$ denotes the weight $1/2$ hyperbolic Laplace operator, see (2.2). For $D = 0$, the Poincaré series is a non-holomorphic parabolic Eisenstein series of weight $1/2$, whose analytic properties are well understood by the fundamental work of Selberg and Roelcke (see [Sel56] and [Roe66,Roe67]). In particular, it has a meromorphic continuation in s to all of $\mathbb{C}$. For $D > 0$, using the spectral theory of automorphic forms, Selberg proved in

[Sel65] that $P_{\beta,D}(\tau, s)$ has a meromorphic continuation to $\mathbb{C}$, with poles corresponding to the eigenvalues of $\Delta_{1/2}$. By computing the Fourier expansion of $P_{\beta,D}(\tau, s)$ and employing the estimates of Goldfeld and Sarnak (see [GS83]) and Pribitkin (see [Pri00]) for the Kloosterman zeta functions appearing in this expansion, one obtains the meromorphic continuation of $P_{\beta,D}(\tau, s)$ also for $D < 0$. Carefully translating these classical results to our vector valued setting in Section 3, we can evaluate the Fourier expansion of the Poincaré series at $s = 0$, proving the following result (compare Theorem 3.7 and 3.10).

Theorem 1.1. *For each $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $D \in \mathbb{Z}$ with $D \equiv \beta^2 \bmod 4N$, the Poincaré series $P_{\beta,D}(\tau, s)$ has a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic at $s = 0$, yielding a harmonic Maass form $P_{\beta,D}(\tau, 0)$ of weight $1/2$.*

Next, we consider Borcherds' regularized theta lift (see [Bor98]) of $P_{\beta,D}(\tau, s)$, namely

$$\Phi(z, P_{\beta,D}(\,\cdot\,, s)) = \int\limits_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} \langle P_{\beta,D}(\tau, s), \Theta(\tau, z)\rangle v^{1/2}\frac{du\,dv}{v^2}, \quad (\tau = u + iv \in \mathbb{H}),$$

where $\Theta(\tau, z)$ is the Siegel theta function associated to the lattice L, which transforms like a vector valued modular form of weight $1/2$ for the Weil representation ρ in τ, and is $\Gamma_0(N)$-invariant in z. Note that the above integral has to be regularized as explained in Section 2.5.

For $\beta \in \mathbb{Z}/2N\mathbb{Z}$ and $D \in \mathbb{Z}$ with $D \equiv \beta^2 \bmod 4N$ we let $\mathcal{Q}_{\beta,D}$ be the set of integral binary quadratic forms $ax^2 + bxy + cy^2$ of discriminant $D = b^2 - 4ac$ with $N \mid a$ and $b \equiv \beta \bmod 2N$. The group $\Gamma_0(N)$ acts on $\mathcal{Q}_{\beta,D}$ from the right by $Q.M = M^t Q M$, with finitely many orbits if $D \neq 0$. Let $Q \in \mathcal{Q}_{\beta,D}$ with $Q(x, y) = ax^2 + bxy + cy^2$. If $D < 0$, then the order of the stabilizer of Q in $\Gamma_0(N)$ is finite, and there is an associated Heegner (or CM) point

$$z_Q = -\frac{b}{2a} + i\frac{\sqrt{|D|}}{2|a|},$$

which is characterized by $Q(z_Q, 1) = 0$. If $D > 0$, then the stabilizer of Q in $\Gamma_0(N)/\{\pm 1\}$ is trivial if D is a square, and infinite cyclic otherwise, and there is an associated geodesic in $\mathbb{H}$ given by

$$c_Q = \{z \in \mathbb{H} : a|z|^2 + bx + c = 0\}.$$

In both cases the actions of $\Gamma_0(N)$ on $\mathcal{Q}_{\beta,D}$ and $\mathbb{H}$ are compatible in the sense that $M.z_Q = z_{Q.M}$ and $M.c_Q = c_{Q.M}$ hold for all $M \in \Gamma_0(N)$. For $D < 0$ we let $H_{\beta,D}$ be the set of all Heegner points z_Q with $Q \in \mathcal{Q}_{\beta,D}$, and for $D \geq 0$ we let $H_{\beta,D} = \emptyset$. With the above notation we can now state one of our main results, which is proven in Section 4.

Theorem 1.2. *For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ the regularized theta lift $\Phi(z, P_{\beta,D}(\,\cdot\,, s))$ defines a real analytic function in $z \in \mathbb{H} \setminus H_{\beta,D}$ and a holomorphic function in s, which can be meromorphically continued to all $s \in \mathbb{C}$. It is holomorphic at $s = 0$ if $D \neq 0$, and has a simple pole at $s = 0$ if $D = 0$. Further, we have*

$$
\Phi(z, P_{\beta,D}(\,\cdot\,, s)) = \begin{cases} \dfrac{2\Gamma(s)}{(\pi D/N)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,D}/\Gamma_0(N)} E_{c_Q}^{\mathrm{hyp}}(z, 2s), & \text{if } D > 0, \\[2.5em] 4N^s \zeta^*(2s) \displaystyle\sum_{p \in C(\Gamma_0(N))} E_p^{\mathrm{par}}(z, 2s), & \text{if } D = 0, \\[2.5em] \dfrac{2\Gamma(s)}{(\pi |D|/N)^s} \displaystyle\sum_{Q \in \mathcal{Q}_{\beta,D}/\Gamma_0(N)} E_{z_Q}^{\mathrm{ell}}(z, 2s), & \text{if } D < 0, \end{cases}
$$

for $z \in \mathbb{H} \setminus H_{\beta,D}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Here, $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

It is remarkable that averaged versions of all three types of Eisenstein series arise as the theta lift of a single type of Poincaré series, being distinguished by the sign of D. This is due to the fact that the averaged Eisenstein series on the right-hand side can essentially be written as

$$
y^{2s} \sum_{Q \in \mathcal{Q}_{\beta,D}} |Q(z,1)|^{-2s}.
$$

These can be seen as non-holomorphic weight 0 analogs of the holomorphic cusp forms $f_{k,\beta,D}(z) = \sum_{Q \in \mathcal{Q}_{\beta,D}} Q(z,1)^{-k}$ of weight $2k \geq 4$ defined by Zagier in [Zag75a] for positive discriminants D, and their natural generalizations for negative discriminants, the meromorphic cusp forms $f_{k,\beta,D}$ introduced by Bengoechea in [Ben15]. For more details on these non-holomorphic modular forms associated to discriminants and their relation to Eisenstein series we refer to the PhD thesis of the third named author (see [Völ18]).

We further note that Matthes in [Mat99], Theorem 1.1, uses a similar Poincaré series to realize averaged versions of the hyperbolic kernel function $\sum_{M \in \Gamma_0(N)} \cosh(d_{\mathrm{hyp}}(Mz, w))^{-s}$ as a theta lift. The relation between his and our result is explained by the identities given in [JPS16], Proposition 11 and 15.

Theorem 1.2 also yields a new and unified proof of the meromorphic continuation of all three types of (averaged) Eisenstein series for the group $\Gamma_0(N)$. Since the continuations of the Eisenstein series are well documented in the literature, we do not focus on this aspect here. Instead, we will employ Borcherds' theory of automorphic products developed in [Bor98] to establish explicit Kronecker limit type formulas for the averaged Eisenstein series given in Theorem 1.2.

Finally, we remark that the realization of individual hyperbolic and elliptic Eisenstein series for $\Gamma_0(N)$ as regularized theta lifts is given by the third named author in [Völ18].

These realizations also provide a conceptual approach to all three types of Eisenstein series and can be used to define and study generalized hyperbolic and elliptic Eisenstein series on orthogonal groups.

1.3. Kronecker limit formulas

Using the functional equation of $E_\infty^{\mathrm{par}}(z, s)$ for $N = 1$, the classical Kronecker limit formula (1.1) is equivalent to the more attractive looking Laurent expansion

$$E_\infty^{\mathrm{par}}(z, s) = 1 + \log\left(|\Delta(z)|^{1/6} \operatorname{Im}(z)\right) \cdot s + O(s^2)$$

at $s = 0$. Here $\Delta(z)$ is the unique normalized cusp form of weight 12 for $\mathrm{SL}_2(\mathbb{Z})$. In this article, we establish Kronecker limit type formulas for the averaged Eisenstein series appearing in Theorem 1.2 at $s = 0$, namely, we prove that the corresponding Laurent expansions at $s = 0$ are of the form

$$a_0 + \mathcal{K}(z) \cdot s + O(s^2),$$

where $a_0 \in \mathbb{C}$ is a constant, and $\mathcal{K}(z)\colon \mathbb{H} \to \mathbb{C}$ is some $\Gamma_0(N)$-invariant function, which, for brevity, we call Kronecker limit function. In particular, we show that $a_0 = 0$ in the hyperbolic and elliptic case. The explicit computation of the Kronecker limit function $\mathcal{K}(z)$ consists of the following three main steps:

(1) Firstly, we explicitly determine the functions $P_{\beta,D}(\tau, 0)$. They turn out to be of rather different nature for different signs of D. For $D < 0$, the Poincaré series is in general a properly non-holomorphic harmonic Maass form which is determined by its principal part, for $D = 0$ it is a holomorphic modular form which can explicitly be written as a linear combination of unary theta functions, and for $D > 0$ it is a cusp form which is characterized by the fact that the Petersson inner product with a cusp form f of weight $1/2$ for ρ essentially gives the (β, D)-th Fourier coefficient of f. We refer to Theorem 3.10 for the details.
(2) Next, we show that the functions

$$\Phi(z, P_{\beta,D}(\,\cdot\,, s))|_{s=0} \quad \text{and} \quad \Phi(z, P_{\beta,D}(\,\cdot\,, 0))$$

essentially agree (some care is necessary for $D = 0$), see Proposition 4.1. Since $\Gamma(s)$ and $\zeta^*(2s)$ have a simple pole at $s = 0$, Theorem 1.2 then implies that the Kronecker limit function $\mathcal{K}(z)$ is basically given by the theta lift $\Phi(z, P_{\beta,D}(\,\cdot\,, 0))$.
(3) By the theory of automorphic products developed in [Bor98] and [BO10], it is known that the theta lift $\Phi(z, P_{\beta,D}(\,\cdot\,, 0))$ is essentially given by the logarithm of the absolute value of the Borcherds product associated to $P_{\beta,D}(\tau, 0)$. Our explicit description of the functions $P_{\beta,D}(\tau, 0)$ enables us to determine the required Borcherds products, which in turn gives the Kronecker limit functions $\mathcal{K}(z)$.

In the following, we present the three Kronecker limit type formulas we obtained via the above process, depending on the sign of D. However, since the Kronecker limit functions for the averaged elliptic and hyperbolic Eisenstein series look quite technical for arbitrary squarefree integers N, we only state simplified versions in these cases in the introduction, restricting to special values of N. For the general theorems we refer to Section 5.

For $D = 0$ the parabolic Kronecker limit function generalizes the classical Kronecker limit formula for $N = 1$ seen above:

Theorem 1.3. *At $s = 0$ we have the Laurent expansion*

$$\sum_{p \in C(\Gamma_0(N))} E_p^{\mathrm{par}}(z, s) = 1 + \frac{1}{\sigma_0(N)} \sum_{c | N} \log\left(|\Delta(cz)|^{1/6} \operatorname{Im}(z)\right) \cdot s + O(s^2).$$

Here $C(\Gamma_0(N))$ denotes the set of cusps of $\Gamma_0(N)$, and $\sigma_0(N) = \sum_{d|N} 1$ is the number of positive divisors of N. In fact, this is the Kronecker limit formula of the parabolic Eisenstein series for the generalized Fricke group $\Gamma_0^*(N)$, which is the extension of $\Gamma_0(N)$ by all Atkin-Lehner involutions, compare [JST16].

In the hyperbolic case, we will see that the Kronecker limit function for the averaged hyperbolic Eisenstein series vanishes for trivial reasons if $N = 1$ or $N = p$ is a prime, or if D is not a square. Thus it is reasonable to assume that N is the product of at least two different primes, and that D is a square, in order to obtain an interesting statement. In the following theorem we deal with the simplest non-trivial situation.

Theorem 1.4. *Let $\beta = n$ and $D = n^2$ for some positive integer n, and let $N = pq$ be the product of two different primes. Then the averaged hyperbolic Eisenstein series admits a Laurent expansion at $s = 0$ of the form*

$$\sum_{Q \in \mathcal{Q}_{n,n^2}/\Gamma_0(N)} E_{c_Q}^{\mathrm{hyp}}(z, s) = \mathcal{K}(z) \cdot s + O(s^2),$$

where the Kronecker limit function $\mathcal{K}(z)$ is given by

$$\mathcal{K}(z) = \begin{cases} \dfrac{n}{4(p-1)(q-1)} \log\left|\dfrac{\Delta(z)\Delta(Nz)}{\Delta(pz)\Delta(qz)}\right|, & \textit{if } (n, N) = 1, \\ 0, & \textit{if } (n, N) > 1. \end{cases}$$

For general squarefree N the Kronecker limit function is given by the logarithm of the absolute value of a more complicated quotient of Delta functions, which is, however, still of weight 0. We refer to Theorem 5.5 for the general statement.

In order to present a Kronecker limit formula in the elliptic case, we recall that there are finitely many N such that the modular curve $\Gamma_0(N)\backslash\mathbb{H}$ has genus 0, namely

$$N = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 16, 18, 25.$$

For such N, p a cusp of $\Gamma_0(N)$ and $w \in \mathbb{H}$ let $j_{N,p,w}(z) \in M_0^!(\Gamma_0(N))$ be the unique normalized Hauptmodul, which is holomorphic up to a simple pole at p and which vanishes exactly at the point w, modulo $\Gamma_0(N)$. Here normalized means that the leading coefficient in the Fourier expansion at ∞ is 1, i.e., we have $j_{N,\infty,w}(z) = e(-z) + O(1)$ if $p = \infty$ and $j_{N,p,w}(z) = 1 + O(e(z))$ if $p \neq \infty$. In this situation we obtain the following result.

Theorem 1.5. *Let $D < 0$, and let N be a squarefree positive integer such that the group $\Gamma_0(N)$ has genus 0. Then the averaged elliptic Eisenstein series has a Laurent expansion at $s = 0$ of the form*

$$\sum_{Q \in \mathcal{Q}_{\beta,D}/\Gamma_0(N)} E_{z_Q}^{\mathrm{ell}}(z,s) = -\frac{1}{\sigma_0(N)} \sum_{Q \in \mathcal{Q}_{\beta,D}/\Gamma_0(N)} \sum_{c|N} \log\left(\left| j_{N,1/c,z_Q}(z) \right|^{2/|\Gamma_0(N)_Q|} \right) \cdot s$$
$$+ O(s^2).$$

Note that for $N = 1$ we have $j_{1,\infty,w}(z) = j(z) - j(w)$ for $w \in \mathbb{H}$ where $j(z)$ is the usual modular j-function. In this case Theorem 1.5 is simply an averaged version of the Kronecker limit formula (1.2). For general squarefree N, the Kronecker limit function is given by the logarithm of the absolute value of a holomorphic function $\Psi_{\beta,D}$ on $\mathbb{H}$ which transforms like a modular form of weight 0 for some unitary character of possibly infinite order, and which is determined by the orders of its roots at the Heegner points z_Q for $Q \in \mathcal{Q}_{\beta,D}$ and its orders at the cusps of $\Gamma_0(N)$. See Theorem 5.7 for the details.

1.4. Outline of the work

We now describe the organization of our work. In Section 2 we introduce the basic concepts used in this presentation, such as the Grassmannian model of the upper half-plane and its underlying lattice, vector valued harmonic Maass forms and unary theta functions, and Borcherds' theory of regularized theta lifts and automorphic products.

Afterwards, we study a vector valued version of Selberg's Poincaré series in Section 3. We use spectral theory to establish the meromorphic continuation of the corresponding Kloosterman zeta functions involved in the Fourier expansion of the Poincaré series, which enables us to continue our Poincaré series to the whole complex plane. In particular, we can explicitly describe the Poincaré series at the particular point $s = 0$.

In Section 4 we prove the existence and continuation of the regularized theta lift of the above Poincaré series, showing that the evaluation of the continuation of the lift at $s = 0$ essentially agrees with the lift of the evaluation of the Poincaré series at this point. Moreover, we determine the regularized theta lift of the Poincaré series via unfolding against the Poincaré series, seeing that it actually can be written as an averaged sum

of hyperbolic, parabolic or elliptic Eisenstein series, depending on the parameter of the Poincaré series.

Finally, in Section 5 we use this identity between the theta lift of our Poincaré series and the different non-holomorphic Eisenstein series to obtain Kronecker limit type formulas for all three types of Eisenstein series. As these are of rather different nature we study them separately, starting with the classical parabolic case. The general hyperbolic and elliptic Kronecker limit formulas are then given in Theorem 5.5 and Theorem 5.7. We end both sections with some comments on these formulas.

1.5. Acknowledgments

2. Preliminaries

2.1. The modular curve $\Gamma_0(N)\backslash\mathbb{H}$

The group $\mathrm{SL}_2(\mathbb{R})$ acts on the complex upper-half plane $\mathbb{H}$ by fractional linear transformations. In the present work we focus on the congruence subgroup $\Gamma_0(N)$, consisting of integer matrices with determinant 1 and lower left entry being divisible by N. In particular, the group $\Gamma_0(N)$ is a Fuchsian group of the first kind. In order to simplify the presentation we assume that N is squarefree. In this case the cusps of $\Gamma_0(N)$ can be represented by the fractions $1/c$ where c runs through the positive divisors of N, and the width of the cusp $1/c$ is given by N/c. We denote the set of cusps of $\Gamma_0(N)$ by $C(\Gamma_0(N))$. We also note that for N being squarefree the index of $\mathrm{SL}_2(\mathbb{Z})$ in $\Gamma_0(N)$ is simply given by $\sigma_1(N)$.

2.2. The Grassmannian model of the upper half-plane

We consider the rational quadratic space V of signature $(2,1)$ given by the trace zero matrices in $\mathrm{Mat}_2(\mathbb{Q})$ together with the quadratic form $Q(X) = -N\det(X)$. The associated bilinear form is $(X,Y) = N\operatorname{tr}(XY)$. The group $\mathrm{SL}_2(\mathbb{Q})$ acts on V by $g.X = gXg^{-1}$. We let $\mathfrak{D}$ be the Grassmannian of 2-dimensional subspaces in $V(\mathbb{R}) = V\otimes\mathbb{R}$ on which the quadratic form Q is positive definite, and we identify $\mathfrak{D}$ with the complex upper half-plane $\mathbb{H}$ by associating to $z = x+iy \in \mathbb{H}$ the orthogonal complement $X(z)^\perp$ of the negative line $\mathbb{R}X(z)$ generated by

$$X(z) = \frac{1}{\sqrt{N}y} \begin{pmatrix} -x & |z|^2 \\ -1 & x \end{pmatrix}.$$

Since $g.X(z) = X(gz)$ for $g \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$ the above identification of $\mathfrak{D}$ and $\mathbb{H}$ is $\mathrm{SL}_2(\mathbb{R})$-equivariant.

In V we consider the lattice

$$L = \left\{ \begin{pmatrix} b & c/N \\ -a & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

Its dual lattice is

$$L' = \left\{ \begin{pmatrix} b/2N & c/N \\ -a & -b/2N \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}.$$

We see that L'/L is isomorphic to $\mathbb{Z}/2N\mathbb{Z}$, equipped with the quadratic form $x \mapsto x^2/4N \bmod \mathbb{Z}$. Note that the group $\Gamma_0(N)$ acts on L and fixes the classes of L'/L.

For fixed $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$ we let

$$L_{\beta,m} = \{X \in L + \beta : Q(X) = m\}.$$

If $m \neq 0$ the group $\Gamma_0(N)$ acts on $L_{\beta,m}$ with finitely many orbits, and elements $X \in L_{\beta,m}$ with $X = \begin{pmatrix} b/2N & c/N \\ -a & -b/2N \end{pmatrix}$ correspond to integral binary quadratic forms

$$Q_X(x,y) = aNx^2 + bxy + cy^2$$

of discriminant $4Nm$ with $b \equiv \beta \bmod 2N$. This identification is compatible with the corresponding actions of $\Gamma_0(N)$ in the sense that $Q_X.g = Q_{g^{-1}.X}$ for $g \in \Gamma_0(N)$. In particular, we have a bijection between $\Gamma_0(N) \backslash L_{\beta,m}$ and $\mathcal{Q}_{\beta,4Nm}/\Gamma_0(N)$. Moreover, if $m > 0$ or $m < 0$ we associate to $X \in L_{\beta,m}$ the Heegner geodesic $c_X = c_{Q_X}$ or the Heegner point $z_X = z_{Q_X}$, respectively. Clearly, this identification is again compatible with the corresponding actions of $\Gamma_0(N)$.

The set $\mathrm{Iso}(V)$ of isotropic lines in V can be identified with $\mathbb{P}^1(\mathbb{Q})$ via the map

$$\mathbb{P}^1(\mathbb{Q}) \to \mathrm{Iso}(V), \quad (\alpha : \beta) \mapsto \begin{pmatrix} -\alpha\beta & \alpha^2 \\ -\beta^2 & \alpha\beta \end{pmatrix}, \tag{2.1}$$

and this identification respects the corresponding actions of $\Gamma_0(N)$. In particular, it gives rise to a bijection between $\Gamma_0(N) \backslash \mathrm{Iso}(V)$ and the set of cusps of $\Gamma_0(N)$. We write p_X for the cusp associated to the isotropic line $\mathbb{Q}X$. Further, we choose a generator $X_\ell \in \ell$ with $\ell \cap L = \mathbb{Z}X_\ell$ for every line $\ell \in \mathrm{Iso}(V)$. Clearly, X_ℓ is unique up to a sign.

2.3. Harmonic Maass forms

We let $\mathrm{Mp}_2(\mathbb{R})$ be the metaplectic group consisting of all pairs (M, ϕ) with $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\phi \colon \mathbb{H} \to \mathbb{C}$ holomorphic such that $\phi^2(\tau) = c\tau + d$. Further, we denote the preimage of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map $(M, \phi) \mapsto M$ by $\mathrm{Mp}_2(\mathbb{Z})$. For $\beta \in L'/L$ we let $\mathfrak{e}_\beta$ be the standard basis vectors of the group ring $\mathbb{C}[L'/L]$, and we let $\langle \cdot, \cdot \rangle$ be the inner product on $\mathbb{C}[L'/L]$ which satisfies $\langle \mathfrak{e}_\beta, \mathfrak{e}_\gamma \rangle = \delta_{\beta, \gamma}$ and is antilinear in the second variable. The associated Weil representation ρ is defined on the generators $(T, 1)$ and $(S, \sqrt{\tau})$ of $\mathrm{Mp}_2(\mathbb{Z})$ by

$$\rho(T, 1)\mathfrak{e}_\beta = e(Q(\beta))\mathfrak{e}_\beta, \qquad \rho(S, \sqrt{\tau})\mathfrak{e}_\beta = \frac{e(-1/8)}{\sqrt{|L'/L|}} \sum_{\gamma \in L'/L} e(-(\beta, \gamma))\mathfrak{e}_\gamma.$$

Here $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $e(z) = e^{2\pi i z}$ for $z \in \mathbb{C}$, as usual. The dual Weil representation is denoted by $\bar{\rho}$.

Recall from [BF04] that a smooth function $f \colon \mathbb{H} \to \mathbb{C}[L'/L]$ is called a harmonic Maass form of weight $k \in \mathbb{Z} + 1/2$ for ρ if it is annihilated by the weight k hyperbolic Laplace operator Δ_k given by

$$\Delta_k = -v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} \right) + ikv \left(\frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right), \quad (\tau = u + iv), \tag{2.2}$$

if it is invariant under the weight k slash operator given by

$$f \big|_{k, \rho} (M, \phi) = \phi(\tau)^{-2k} \rho^{-1}(M, \phi) f(M\tau)$$

for all $(M, \phi) \in \mathrm{Mp}_2(\mathbb{Z})$, and if it grows at most linearly exponentially at ∞. We denote the space of harmonic Maass forms by $H_{k, \rho}$, and we let $M_{k, \rho}^!$ be the subspace of weakly holomorphic modular forms, consisting of the forms in $H_{k, \rho}$ which are holomorphic on $\mathbb{H}$.

The antilinear differential operator $\xi_k f = 2iv^k \overline{\frac{\partial}{\partial \bar{\tau}} f(\tau)}$ defines a surjective map $\xi_k \colon H_{k, \rho} \to M_{2-k, \bar{\rho}}^!$ with kernel $M_{k, \rho}^!$. We let $H_{k, \rho}^+$ be the space of harmonic Maass forms which map to cusp forms under ξ_k. A form $f \in H_{k, \rho}^+$ has a Fourier expansion of the form

$$f(\tau) = \sum_{\beta \in L'/L} \left(\sum_{\substack{m \in \mathbb{Z} + Q(\beta) \\ m \gg -\infty}} c_f^+(\beta, m) e(m\tau) \right.$$

$$\left. + \sum_{\substack{m \in \mathbb{Z} + Q(\beta) \\ m < 0}} c_f^-(\beta, m) \Gamma(1 - k, 4\pi |m| v) e(m\tau) \right) \mathfrak{e}_\beta, \tag{2.3}$$

where $\Gamma(s, x) = \int_x^\infty t^{s-1} e^{-t} dt$ denotes the incomplete gamma function. The finite Fourier polynomial $\sum_{\beta \in L'/L} \sum_{m \leq 0} c_f^+(\beta, m) e(m\tau) \mathfrak{e}_\beta$ is called the principal part of f.

The regularized inner product of $f \in H_{k,\rho}^+$ and $g \in M_{k,\rho}$ is defined by

$$(f, g)^{\mathrm{reg}} = \lim_{T \to \infty} \int_{\mathcal{F}_T} \langle f(\tau), g(\tau) \rangle v^k \frac{du\, dv}{v^2}, \tag{2.4}$$

where $\mathcal{F}_T = \{\tau \in \mathbb{H} : |u| \leq \frac{1}{2}, |\tau| \geq 1, v \leq T\}$ is a truncated fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$. If the integral converges without the regularization (e.g., if f and g are cusp forms), this is just the usual Petersson inner product (f, g).

2.4. Unary theta functions

We let K be the one-dimensional positive definite sublattice of L generated by the vector $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Its dual lattice K' is generated by $\frac{1}{2N}\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. We see that $K'/K \cong L'/L$, so modular forms for the Weil representation of K are the same as modular forms for ρ. Thus the unary theta function associated to the lattice K given by

$$\theta(\tau) = \sum_{\beta \in K'/K} \sum_{X \in K+\beta} e(Q(X)\tau)\mathfrak{e}_\beta = \sum_{\beta(2N)} \sum_{\substack{n \in \mathbb{Z} \\ n \equiv \beta(2N)}} e(n^2\tau/4N)\mathfrak{e}_\beta$$

is a modular form of weight $1/2$ for ρ by Theorem 4.1 in [Bor98].

The orthogonal group $O(L'/L)$ acts on vector valued functions $f = \sum_\beta f_\beta \mathfrak{e}_\beta$ modular of weight $1/2$ for ρ by $f^w = \sum_\beta f_\beta \mathfrak{e}_{w(\beta)}$. The elements of $O(L'/L)$ are all involutions, so-called Atkin-Lehner involutions, and as N is squarefree they correspond to the positive divisors c of N. More precisely, the automorphism w_c corresponding to $c \mid N$ is defined by the equations

$$w_c(\beta) \equiv -\beta \ (2c) \quad \text{and} \quad w_c(\beta) \equiv \beta \ (2N/c) \tag{2.5}$$

for $\beta \in L'/L \cong \mathbb{Z}/2N\mathbb{Z}$ (compare [EZ85], Theorem 5.2). We also note that $f^{w_c} = f^{w_{N/c}}$ and $(f^{w_c}, g) = (f, g^{w_c})$ for f, g modular of weight $1/2$ for ρ.

Using a dimension formula for $M_{1/2,\rho}$ by Skoruppa, it is easy to show that the unary theta functions θ^{w_c}, with c running through the positive divisors of N modulo the relation $c \sim N/c$, form a basis of $M_{1/2,\rho}$ (compare [BS17], Lemma 2.1).

2.5. Borcherds products

The Siegel theta function associated to the lattice L is defined by

$$\Theta(\tau, z) = v^{1/2} \sum_{\beta \in L'/L} \sum_{X \in L+\beta} e(Q(X_z)\tau + Q(X_{z^\perp})\bar{\tau})\mathfrak{e}_\beta,$$

where X_z denotes the orthogonal projection of X onto the positive definite subspace $X(z)^\perp$, and $X_{z^\perp}$ the projection of X onto the negative line $\mathbb{R}X(z)$. By [Bor98], Theorem

4.1, the Siegel theta function is $\Gamma_0(N)$-invariant in z, and transforms like a modular form of weight $1/2$ for ρ in τ.

Given a function $f\colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $1/2$ with respect to the Weil representation ρ, Borcherds' regularized theta lift of f is defined by

$$\Phi(z,f) = \mathrm{CT}_{t=0} \left[\lim_{T\to\infty} \int_{\mathcal{F}_T} \langle f(\tau), \Theta(\tau,z)\rangle \, v^{1/2-t} \frac{du\,dv}{v^2} \right], \tag{2.6}$$

where $\mathrm{CT}_{t=0}\, F(t)$ denotes the constant term of the Laurent expansion of the analytic continuation of $F(t)$ at $t = 0$. The lift was studied by Borcherds in [Bor98] for weakly holomorphic forms $f \in M_{1/2,\rho}^{!}$, and generalized by Bruinier and Ono [BO10] to harmonic Maass forms $f \in H_{1/2,\rho}^{+}$. It turns out that for such f, the lift $\Phi(z,f)$ defines a $\Gamma_0(N)$-invariant real analytic function with logarithmic singularities at certain Heegner points, which are determined by the principal part of f.

The so-called Weyl vector $\rho_{f,1/c}$ associated to a harmonic Maass form $f \in H_{1/2,\rho}^{+}$ and the cusp $1/c$ with $c \mid N$ is defined by

$$\rho_{f,1/c} = \frac{\sqrt{N}}{8\pi} (f, \theta^{w_c})^{\mathrm{reg}}. \tag{2.7}$$

Note that θ^{w_c} can also be seen as the unary theta function associated to the one-dimensional positive definite sublattice of L corresponding to the cusp $1/c$ (compare [BO10], Section 4.1).

Theorem 2.1 *([BO10], Theorem 6.1). Let $f \in H_{1/2,\rho}^{+}$ be a harmonic Maass form with real coefficients $c_f^+(\beta, m)$ for all $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. Moreover, assume that $c_f^+(\beta, m) \in \mathbb{Z}$ for $m \leq 0$. The infinite product*

$$\Psi(z,f) = e(\rho_{f,\infty} z) \prod_{n=1}^{\infty} \left(1 - e(nz)\right)^{c_f^+(n,n^2/4N)} \tag{2.8}$$

converges for $\mathrm{Im}(z)$ sufficiently large and has a meromorphic continuation to all of $\mathbb{H}$ with the following properties:

(1) It is a meromorphic modular form of weight $c_f^+(0,0)$ for $\Gamma_0(N)$ with a unitary character which may have infinite order.

(2) The orders of $\Psi(z,f)$ in $\mathbb{H}$ are determined by the Heegner divisor

$$\frac{1}{2} \sum_{\beta\in L'/L} \sum_{\substack{m\in\mathbb{Z}+Q(\gamma)\\ m<0}} c_f^+(\beta, m) \sum_{X\in L_{\beta,m}} (z_X).$$

(3) *The order of $\Psi(z, f)$ at the cusp $1/c$ for $c \mid N$ is given by the corresponding Weyl vector, i.e.,*

$$\mathrm{ord}_{1/c}(\Psi(z, f)) = \rho_{f, 1/c}.$$

(4) *The regularized theta lift of f is given by*

$$\Phi(z, f) = -c_f^+(0, 0)(\log(4\pi N) + \Gamma'(1)) - 4 \log \left| \Psi(z, f) \, \mathrm{Im}(z)^{c_f^+(0,0)/2} \right|.$$

Here we quickly recall that given a meromorphic modular form f of weight k for $\Gamma_0(N)$ and a cusp p of $\Gamma_0(N)$ the order of f at p is defined as follows: Let $\sigma_p \in \mathrm{SL}_2(\mathbb{R})$ be a scaling matrix for the cusp p. Then the function $f \big|_k \sigma_p$ has a Fourier expansion of the form $\sum_{n \in \mathbb{Z}} c_f(n; \sigma_p) e(nz)$. Now the order of f at p is the smallest integer n such that $c_f(n; \sigma_p) \neq 0$.

3. Selberg's Poincaré series

In the following we define a vector valued version of the non-holomorphic Poincaré series introduced by Selberg in his famous work [Sel65]. We will establish the meromorphic continuation of this Poincaré series via its Fourier series and finally evaluate it at the special value $s = 0$. For more details on this technical process we refer the reader to [Völ18], Chapter 6.

Given $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, we define the non-holomorphic Poincaré series of index (β, m) as

$$P_{\beta, m}(\tau, s) = \frac{1}{2} \sum_{(M, \phi) \in \langle (T, 1) \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} v^s e(m\tau) \mathfrak{e}_\beta \Big|_{1/2, \rho} (M, \phi) \tag{3.1}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. The sum is absolutely convergent and defines an analytic function in s, which is by construction modular of weight $1/2$ with respect to ρ. However, $P_{\beta, m}(\tau, s)$ is not an eigenfunction of the hyperbolic Laplace operator, but it satisfies the differential equation

$$\Delta_{1/2} P_{\beta, m}(\tau, s) = s \left(\frac{1}{2} - s \right) P_{\beta, m}(\tau, s) + 4\pi m s \, P_{\beta, m}(\tau, s + 1). \tag{3.2}$$

In order to study the meromorphic continuation of $P_{\beta, m}(\tau, s)$ we also need to investigate the Kloosterman zeta functions appearing in its Fourier expansion. In fact, in [Sel65], [GS83] and [Pri00] these Poincaré series were studied for the purpose of finding good growth estimates for the Kloosterman zeta functions. However, the case of Δ_k having continuous spectrum was mostly neglected.

Given $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, $n \in \mathbb{Z} + Q(\gamma)$, we define the Kloosterman zeta function by

$$Z(s; \beta, m, \gamma, n) = \sum_{c \neq 0} |c|^{1/2 - 2s} H_c(\beta, m, \gamma, n) \tag{3.3}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. Here $H_c(\beta, m, \gamma, n)$ is the generalized Kloosterman sum of weight $k = 1/2$ defined for example in [Bru02, eq. (1.38)], i.e.,

$$H_c(\beta, m, \gamma, n) = \frac{e(-\operatorname{sign}(c)k/4)}{|c|} \sum_{\substack{d \in (\mathbb{Z}/c\mathbb{Z})^* \\ M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})}} \langle \mathfrak{e}_\beta, \rho(\tilde{M}) \mathfrak{e}_\gamma \rangle\, e\left(\frac{ma + nd}{c}\right). \tag{3.4}$$

The coefficients $\langle \mathfrak{e}_\beta, \rho(\tilde{M}) \mathfrak{e}_\gamma \rangle$ are universally bounded because ρ factors through a double cover of the finite group $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Thus $H_c(\beta, m, \gamma, n)$ is bounded by some constant only depending on the underlying lattice L, and the sum in (3.3) defines a holomorphic function in s for $\mathrm{Re}(s) > 3/4$. We further remark that using Shintani's formula for the Weil representation (see Proposition 1.6 in [Shi75]) one can easily check that

$$H_c(\beta, m, \gamma, n) = H_c(\gamma, n, \beta, m)$$

for all $c \neq 0$, $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, $n \in \mathbb{Z} + Q(\gamma)$. Thus the same is true for the Kloosterman zeta function, i.e., we also have $Z(s; \beta, m, \gamma, n) = Z(s; \gamma, n, \beta, m)$.

Proposition 3.1. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. The Poincaré series $P_{\beta,m}(\tau, s)$ has a Fourier expansion of the form*

$$P_{\beta,m}(\tau, s) = v^s e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) + b(0,0; v, s)\mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n \neq 0}} b(\gamma, n; v, s) e(nu)\mathfrak{e}_\gamma,$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. The Fourier coefficients $b(\gamma, n; v, s)$ are given by

$$b(0,0; v, s) = \frac{2^{3/2 - 2s} \pi v^{1/2 - s}}{\Gamma(s)} \sum_{j=0}^{\infty} \frac{1}{j!} \left(-\frac{\pi m}{v}\right)^j \frac{\Gamma(2s - 1/2 + j)}{\Gamma(s + 1/2 + j)}\, Z(s + j; \beta, m, 0, 0),$$

if $n = 0$, and by

$$b(\gamma, n; v, s) = 2^{1/2} \pi^{s + 1/2} |n|^{s - 1/2} \sum_{j=0}^{\infty} \frac{(-4\pi^2 |n| m)^j}{j!} Z(s + j; \beta, m, \gamma, n)$$

$$\times \begin{cases} \dfrac{(4\pi |n| v)^{-1/4 - j/2}}{\Gamma(s + 1/2 + j)}\, W_{1/4 + j/2,\, s - 1/4 + j/2}(4\pi |n| v), & \text{if } n > 0, \\[2ex] \dfrac{(4\pi |n| v)^{-1/4 - j/2}}{\Gamma(s)}\, W_{-1/4 - j/2,\, s - 1/4 + j/2}(4\pi |n| v), & \text{if } n < 0, \end{cases}$$

if $n \neq 0$.

Proof. The proof proceeds by standard calculations, so we omit it. We refer the reader to [Pri99], Section 5, for some hints on the computation in the scalar valued case. □

3.1. Eisenstein series of weight $1/2$

Since N is squarefree, the only element $\beta \in L'/L$ with $Q(\beta) = 0 \bmod \mathbb{Z}$ is $\beta = 0$. For $m = \beta = 0$ the Poincaré series $P_{0,0}(\tau, s)$ is an Eisenstein series, which we also denote by $E_0(\tau, s)$. It satisfies the Laplace equation

$$\Delta_{1/2} E_0(\tau, s) = s \left(\frac{1}{2} - s \right) E_0(\tau, s). \tag{3.5}$$

Note that the Fourier expansion given in Proposition 3.1 greatly simplifies for $m = 0$ since the summands for $j > 0$ vanish. Further, one can rewrite the Kloosterman zeta functions $Z(s; 0, 0, \gamma, n)$ appearing in the Fourier expansion of $E_0(\tau, s)$ in terms of Dirichlet L-functions as in [BK03], Theorem 3.3. More precisely, following the arguments in [BK01], Section 4, or [BK03], Section 3, (the calculations in [Ara90], pp. 146–147, are helpful in the case $\gamma = n = 0$), we find

$$Z(s; 0, 0, \gamma, n) = \begin{cases} \dfrac{\sqrt{2}}{\sqrt{N}} \dfrac{\zeta(4s-1)}{\zeta(4s)} \displaystyle\prod_{p|N} \dfrac{1 + p^{1-2s}}{1 + p^{-2s}}, & \text{if } n = 0, \\[3ex] \dfrac{\sqrt{2}}{\sqrt{N}} \dfrac{L(\chi_{D_0}, 2s)}{\zeta(4s)} \sigma_{\gamma, -n}(2s + 1/2), & \text{if } n \neq 0, \end{cases} \tag{3.6}$$

for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$, $\gamma \in L'/L$ and $n \in \mathbb{Z} + Q(\gamma)$. Here D_0 is the unique fundamental discriminant such that $D_0 f^2 = 4N \, \mathrm{ord}(\gamma)^2 n$ for some $f \in \mathbb{N}$, $L(\chi_{D_0}, s)$ is the Dirichlet L-function associated to the Kronecker symbol $\chi_{D_0} = \left(\frac{D_0}{\cdot} \right)$, and $\sigma_{\gamma, -n}(s)$ is a generalized divisor sum defined by

$$\sigma_{\gamma, -n}(2s + 1/2) = \prod_{\substack{p | D_0 f^2 \\ (p, 2N) = 1}} \left(\sigma_{1-4s} \left(p^{\nu_p(f)} \right) - \chi_{D_0}(p) p^{-2s} \sigma_{1-4s} \left(p^{\nu_p(f)-1} \right) \right)$$

$$\times \prod_{p | 2N} \frac{1 - \chi_{D_0}(p) p^{-2s}}{1 - p^{-4s}} L^{(p)}_{\gamma, -n}(p^{-2s}),$$

where $\sigma_s(m) = \sum_{d|m} d^s$ is the usual divisor sum, $L^{(p)}_{\gamma, -n}(X)$ is the polynomial

$$L^{(p)}_{\gamma, -n}(X) = N_{\gamma, -n}(p^{w_p}) X^{w_p} + (1 - X) \sum_{\nu=0}^{w_p - 1} N_{\gamma, -n}(p^\nu) X^\nu \in \mathbb{Z}[X],$$

with $w_p = 1 + 2v_p(2 \, \mathrm{ord}(\gamma) n)$ and

$$N_{\gamma,-n}(a) = \#\{x \in \mathbb{Z}/a\mathbb{Z} : N(x - \gamma/2N)^2 - n \equiv 0 \bmod a\}.$$

Here we understand γ as an element of $\mathbb{Z}/2N\mathbb{Z}$. From the above formula for the Kloosterman zeta function $Z(s; 0, 0, \gamma, n)$ we can deduce the following lemma.

Lemma 3.2. *For $\gamma \in L'/L$ and $n \in \mathbb{Z} + Q(\gamma)$ the Kloosterman zeta function*

$$Z(s; 0, 0, \gamma, n) = Z(s; \gamma, n, 0, 0)$$

has a meromorphic continuation in s to $\mathbb{C}$, which is holomorphic at $s = 0$. Further, there is a possible simple pole at $s = 1/2$ which appears if and only if $4Nn$ is zero or a square.

Proof. Part (ii) of Theorem 6 in [BK01] implies that if $N_{\gamma,-n}(p^{w_p}) \neq 0$ then $\chi_{D_0}(p) = 1$. Thus $\sigma_{\gamma,-n}(2s + 1/2)$ is holomorphic on $\mathbb{C}$ up to possible simple poles on the purely imaginary axis. The analytic properties of the other terms appearing in $Z(s; 0, 0, \gamma, n)$ are well known and yield the desired analytic continuation. $\quad\square$

Further, we obtain a very explicit Fourier expansion of the Eisenstein series, namely

$$E_0(\tau, s) = 2v^s \mathfrak{e}_0 + v^{1/2-s} \frac{2}{\sqrt{N}} \frac{\zeta^*(4s-1)}{\zeta^*(4s)} \prod_{p|N} \frac{1 + p^{1-2s}}{1 + p^{-2s}} \mathfrak{e}_0 \tag{3.7}$$

$$+ \frac{2\pi^{s+1/2}}{\sqrt{N}} \sum_{\gamma \in L'/L} \sum_{n \neq 0} |n|^{s-1/2} \frac{L(\chi_{D_0}, 2s)}{\zeta(4s)} \sigma_{\gamma,-n}(2s + 1/2) e(nu) \mathfrak{e}_\gamma$$

$$\times \begin{cases} \dfrac{(4\pi|n|v)^{-1/4}}{\Gamma(s+1/2)} \, W_{1/4, s-1/4}(4\pi|n|v), & \text{if } n > 0, \\[3mm] \dfrac{(4\pi|n|v)^{-1/4}}{\Gamma(s)} \, W_{-1/4, s-1/4}(4\pi|n|v), & \text{if } n < 0, \end{cases}$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. Here $\zeta^*(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$.

Proposition 3.3. *The function*

$$N^s \prod_{p|N}(1 + p^{-2s}) \, \zeta^*(4s) E_0(\tau, s) \tag{3.8}$$

has a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic up to simple poles at $s = 0$ and $s = 1/2$. Further, it is invariant under $s \mapsto 1/2 - s$. In particular, $E_0(\tau, s)$ has a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic at $s = 0$ and has a simple pole at $s = 1/2$. Its Fourier expansion is given by (3.7) for all s where it is holomorphic.

Proof. The generalized divisor sum $\sigma_{\gamma,-n}(2s + 1/2)$ is holomorphic on $\mathbb{C}$ up to possible simple poles on the imaginary axis. The analytic properties of the remaining terms in

the Fourier expansion of $E_0(\tau, s)$ are well known, and imply that the Fourier expansion (3.7) converges absolutely and locally uniformly in s whenever the coefficients are holomorphic. This gives a meromorphic continuation of (3.8) to $\mathbb{C}$ which is holomorphic up to possible simple poles at $s = 0, s = 1/4$ and $s = 1/2$, coming from $\zeta^*(4s), \zeta^*(4s - 1)$, and $L(\chi_{D_0}, 2s)$. It is easy to see that the pole at $s = 1/4$ does in fact not appear.

If we consider the function in (3.8) with s replaced by $1/2 - s$ and take the difference with (3.8), then this function has Laplace eigenvalue $s(1/2 - s)$ and vanishes at ∞. In particular, for $\mathrm{Re}(s) > 1/2$ it defines a square integrable function with a negative Laplace eigenvalue, and hence has to vanish identically, compare [Roe67], Satz 10.2. This shows the invariance of (3.8) under $s \mapsto 1/2 - s$. $\square$

We remark that the meromorphic continuation of the Eisenstein series also follows by the fundamental work of Selberg and Roelcke, see [Sel56] and [Roe66,Roe67], but the methods used above yield a more explicit continuation which is convenient for our applications.

3.2. Continuation of Kloosterman zeta functions

In order to establish the meromorphic continuation of the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ for arbitrary $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, $n \in \mathbb{Z} + Q(\gamma)$ with $m, n \neq 0$ it turns out to be useful to work with a slightly modified non-holomorphic Poincaré series: Given $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$ we define

$$P^*_{\beta,m}(\tau, s) = \frac{1}{2} \sum_{(M,\phi) \in \langle (T,1) \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} v^s e^{-2\pi|m|v} e(mu) \mathfrak{e}_\beta \Big|_{1/2, \rho} (M, \phi) \qquad (3.9)$$

for $\tau = u + iv \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 3/4$. For $m \geq 0$ the functions $P_{\beta,m}(\tau, s)$ and $P^*_{\beta,m}(\tau, s)$ agree, but for $m < 0$ they are fundamentally different since $P^*_{\beta,m}(\tau, s)$ is square-integrable for $\mathrm{Re}(s) > 3/4$ and $P_{\beta,m}(\tau, s)$ is not. However, for $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$ with $m < 0$ we do have the identity

$$v^{1/2} \, \overline{P^*_{\beta,m}(\tau, \bar{s} - 1/2)} = \frac{1}{2} \sum_{(M,\phi) \in \langle (T,1) \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})} v^s e(-m\tau) \mathfrak{e}_\beta \Big|_{-1/2, \bar{\rho}} (M, \phi),$$

where the function on the right-hand side is Selberg's non-holomorphic Poincaré series of weight $-1/2$ for the dual Weil representation $\bar{\rho}$ and index $(\beta, -m)$.

We start by establishing the meromorphic continuation of the modified Poincaré series $P^*_{\beta,m}(\tau, s)$ for $m \neq 0$, using the spectral theory of vector valued modular functions developed in the extensive works [Roe66,Roe67] of Roelcke. Adapting his results to the setting of this work, a combination of Satz 7.2 and Satz 12.3 in [Roe67] yields the following spectral theorem: Given a real analytic function $f \colon \mathbb{H} \to \mathbb{C}[L'/L]$ modular of weight $1/2$ with respect to the Weil representation ρ and satisfying $(f, f) < \infty$, the function f admits a spectral expansion of the form

$$f(\tau) = \sum_{j=0}^{\infty} (f, \psi_j) \psi_j(\tau) + \frac{1}{64\pi} \int_{-\infty}^{\infty} (f, E_0(\,\cdot\,, 1/4 + it)) E_0(\tau, 1/4 + it) dt. \quad (3.10)$$

Here $(\psi_j)_{j \geq 0}$ is an orthonormal system of square-integrable eigenfunctions of the Laplace operator $\Delta_{1/2}$, modular of weight $1/2$ with respect to ρ, and $E_0(\tau, s) = P_{0,0}(\tau, s)$ is the unique vector valued Eisenstein series of weight $1/2$ for the given lattice L. The fudge factor $1/64\pi$ comes from our normalization of the Eisenstein series.

Proposition 3.4. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$ with $m \neq 0$. Then $P_{\beta,m}^*(\tau, s)$ has a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic at $s = 0$.*

Proof. The following proof is a translation of Selberg's work in [Sel65] to our situation. Some caution is necessary since in our case the Laplace operator $\Delta_{1/2}$ has a continuous spectrum. For $m \neq 0$ the Poincaré series $P_{\beta,m}^*(\tau, s)$ is square-integrable, and thus admits a spectral expansion as in (3.10), i.e.,

$$P_{\beta,m}^*(\tau, s) = \sum_{j=0}^{\infty} a_j(s) \psi_j(\tau) + \frac{1}{64\pi} \int_{-\infty}^{\infty} a_\infty(s, t) E_0(\tau, 1/4 + it) \, dt, \quad (3.11)$$

with spectral coefficients given by

$$a_j(s) = 2(4\pi|m|)^{3/4-s} \, \overline{c_{\psi_j}(\beta, m)} \begin{cases} \dfrac{\Gamma(s - 1/4 + it_j)\Gamma(s - 1/4 - it_j)}{\Gamma(s)}, & \text{if } m > 0, \\[2mm] \dfrac{\Gamma(s - 1/4 + it_j)\Gamma(s - 1/4 - it_j)}{\Gamma(s + 1/2)}, & \text{if } m < 0, \end{cases}$$

and

$$a_\infty(s, t)$$
$$= \frac{4^{1+it}\pi}{(4\pi|m|)^{s-1/4+it}} \, Z(1/4 - it; 0, 0, \beta, m) \begin{cases} \dfrac{\Gamma(s - 1/4 + it)\,\Gamma(s - 1/4 - it)}{\Gamma(s)\,\Gamma(3/4 - it)}, & \text{if } m > 0, \\[2mm] \dfrac{\Gamma(s - 1/4 + it)\,\Gamma(s - 1/4 - it)}{\Gamma(s + 1/2)\,\Gamma(1/4 - it)}, & \text{if } m < 0. \end{cases}$$

Here ψ_j is an eigenfunction with eigenvalue $\lambda_j = 1/16 + t_j^2$ where $t_j > 0$ or $t_j \in [-i/4, 0]$. Further, ψ_j has a Fourier expansion of the form

$$\psi_j(\tau) = c_j(0,0) v^{1/4 - it_j} \mathfrak{e}_0 + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n \neq 0}} c_j(\gamma, n) v^{-1/4} W_{\text{sign}(n)/4, it_j}(4\pi|n|v) e(nu) \mathfrak{e}_\gamma.$$

It is not difficult to see that the part in the spectral expansion in (3.11) coming from the discrete spectrum of $\Delta_{1/2}$ has a meromorphic continuation in s to all of $\mathbb{C}$. Furthermore, the integral in (3.11) defines a holomorphic function in s away from the vertical lines

$\mathrm{Re}(s) = 1/4 - n$ for n any non-positive integer. Following the proof of Theorem 4.2 in [JKP10] we can use the residue theorem to step by step establish the meromorphic continuation of the Eisenstein part in (3.11), at first to $\mathrm{Re}(s) \geq 1/4$, then to $\mathrm{Re}(s) > -3/4$, and so on. However, in each step we pick up some residue term which mainly consists of a product a Kloosterman zeta function $Z(t; 0, 0, \beta, m)$ and an Eisenstein series $E_0(\tau, t)$. In particular, for $-3/4 < \mathrm{Re}(s) < 1/4$ the continuation of $P^*_{\beta,m}(\tau, s)$ is given by the expression

$$\sum_{j=0}^{\infty} a_j(s)\psi_j(\tau) + \frac{1}{64\pi} \int_{-\infty}^{\infty} a_\infty(s,t)E_0(\tau, 1/4 + it)dt \tag{3.12}$$

$$+ \frac{(2\pi|m|)^{1/2-2s}\,\pi}{8}\, Z(1/2 - s; 0, 0, \beta, m)\, E_0(\tau, s) \begin{cases} \dfrac{\Gamma(2s - 1/2)}{\Gamma(s)\,\Gamma(1 - s)}, & \text{if } m > 0, \\ \dfrac{\Gamma(2s - 1/2)}{\Gamma(s + 1/2)\,\Gamma(1/2 - s)}, & \text{if } m < 0, \end{cases}$$

$$+ \frac{2^{1/2-2s}\,\pi}{8}\, \frac{\Gamma(2s - 1/2)}{\Gamma(s)\,\Gamma(s + 1/2)}\, Z(s; 0, 0, \beta, m)\, E_0(\tau, 1/2 - s).$$

Here the term in the second line of the equation comes from establishing the continuation onto the vertical line $\mathrm{Re}(s) = 1/4$, and the term in the third line from continuing further to the strip $-3/4 < \mathrm{Re}(s) < 1/4$.

The first sum is holomorphic at $s = 0$ up to a possible pole coming from the eigenvalue $\lambda_j = 0$ corresponding to $t_j = -i/4$. However, if $m > 0$ this pole is compensated for by the Gamma function $\Gamma(s)$ in the denominator of $a_j(s)$, and if $m < 0$ the coefficient $a_j(s)$ vanishes identically as the corresponding eigenfunction ψ_j is a holomorphic modular form in this case which does not have any negative Fourier coefficients. Moreover, the integral is clearly holomorphic at $s = 0$, and thus it remains to consider the two additional terms coming from crossing the vertical line $\mathrm{Re}(s) = 1/4$. We recall that the Eisenstein series $E_0(\tau, s)$ is holomorphic at $s = 0$, and has a simple pole at $s = 1/2$. Further, the Kloosterman zeta function $Z(s; 0, 0, \beta, m)$ is holomorphic at $s = 0$ for all m and at $s = 1/2$ if $m < 0$, but may have a simple pole at $s = 1/2$ if $m > 0$. However, all these (possible) poles are compensated for by some Gamma function. Therefore the continuation of $P^*_{\beta,m}(\tau, s)$ is holomorphic at $s = 0$. $\quad\square$

Next we use the previous proposition to conclude the meromorphic continuation of the Kloosterman zeta functions $Z(s; \beta, m, \gamma, n)$ as in [Sel65] and [Pri00].

Lemma 3.5. *For $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, $n \in \mathbb{Z} + Q(\gamma)$ the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ has a meromorphic continuation in s to all of $\mathbb{C}$. If $m \geq 0$ or $n \geq 0$ this meromorphic continuation is holomorphic at $s = 0$, and otherwise, i.e., if $m, n < 0$, it has at most a simple pole at $s = 0$.*

Proof. For $m = 0$ or $n = 0$ the statement is given in Lemma 3.2. Let $m, n \neq 0$. We consider the inner product $(P^*_{\beta,m}(\,\cdot\,, s), P^*_{\gamma,n}(\,\cdot\,, \overline{s}+\ell))$ for fixed $\ell \in \mathbb{N}$ and $\mathrm{Re}(s) > 3/4$. Unfolding against $P^*_{\gamma,n}(\tau, \overline{s}+\ell)$ and using the Fourier expansion of $P^*_{\beta,m}(\tau, s)$ one obtains an expression of the form

$$(P^*_{\beta,m}(\,\cdot\,, s), P^*_{\gamma,n}(\,\cdot\,, \overline{s}+\ell)) = \sum_{j=0}^{\ell-1} A^{(j)}_\ell(s; m, n) Z(s+j; \beta, m, \gamma, n) + R_\ell(s; \beta, m, \gamma, n). \tag{3.13}$$

Here the functions $A^{(j)}_\ell(s; m, n)$ for $j = 0, \ldots, \ell-1$ and the function $R_\ell(s; \beta, m, \gamma, n)$ are holomorphic in s for $\mathrm{Re}(s) > 1/4 - \ell/2$. Although, in contrast to Selberg's original work, the meromorphic continuation of $P^*_{\beta,m}(\tau, s)$ is for $\mathrm{Re}(s) \leq 1/4$ no longer square-integrable because of the additional Eisenstein terms appearing while crossing the vertical lines $\mathrm{Re}(s) = 1/4, -3/4, \ldots$ (compare equation (3.12) from the previous proof), it turns out that the above inner product still defines a meromorphic function in s for $\mathrm{Re}(s) > 3/4 - \ell$. Therefore we inductively obtain the meromorphic continuation of $Z(s; \beta, m, \gamma, n)$ by considering (3.13) for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/4 - \ell/2$ and $\ell = 1, 3, 5, \ldots$.

For $\ell = 1$ equation (3.13) takes the special form

$$Z(s; \beta, m, \gamma, n) = A^{(0)}_1(s; m, n)^{-1} \left((P^*_{\beta,m}(\tau, s), P^*_{\gamma,n}(\tau, \overline{s}+2)) - R_1(s; \beta, m, \gamma, n) \right),$$

where the difference on the right is holomorphic at $s = 0$ by the above considerations and since $P^*_{\beta,m}(\tau, s)$ is holomorphic at $s = 0$ by Proposition 3.4. However, $A^{(0)}_1(s; m, n)^{-1}$ is only holomorphic at $s = 0$ if $n > 0$, and has a simple pole at $s = 0$ if $n < 0$. We thus find that the continuation of $Z(s; \beta, m, \gamma, n)$ is holomorphic at $s = 0$ if $n > 0$, or if $n < 0$ and $m > 0$ since $Z(s; \beta, m, \gamma, n) = Z(s; \gamma, n, \beta, m)$, but if $m, n < 0$ we might indeed encounter a simple pole at $s = 0$. □

In order to study the Fourier coefficients of $P_{\beta,m}(\tau, s)$ for $m \neq 0$ we also need to study the asymptotic behaviour of the Kloosterman zeta function $Z(s; \beta, m, \gamma, n)$ as $n \to \pm\infty$. Following the work of Pribitkin in [Pri00] and adapting it to the present situation we obtain the following lemma.

Lemma 3.6. *Let $\beta, \gamma \in L'/L$, $m \in \mathbb{Z} + Q(\beta)$ with $m \neq 0$ and $\Omega \subseteq \mathbb{C}$ compact such that $Z(s; \beta, m, \gamma, n)$ is holomorphic for all $s \in \Omega$ and $n \in \mathbb{Z} + Q(\gamma)$. Then*

$$|Z(s; \beta, m, \gamma, n)| = O(|n|^\ell) \qquad as \ n \to \pm\infty \tag{3.14}$$

for some $\ell > 0$, uniformly in s for $s \in \Omega$. Moreover, for $m < 0$ and Ω being a sufficiently small neighbourhood of $s = 0$ we have that

$$\left| \frac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)} \right| = O(|n|^\ell) \qquad as \ n \to -\infty \tag{3.15}$$

for some $\ell > 0$, uniformly in s for $s \in \Omega$. In both cases the implied constant depends on β, m, γ and Ω.

We quickly comment on the necessary adaptations of Pribitkin's work: Firstly, we need to translate his results to our vector valued setting. This is a tedious, but straightforward task. Secondly, Pribitkin only proves the above estimate for a multiplier system not allowing the existence of Eisenstein series, i.e., he assumes that the Laplace operator $\Delta_{1/2}$ has only discrete spectrum. This is not true in our case. However, this assumption is solely used in the proof of part (c) of Lemma 3 in his work, in order to derive good estimates for the Petersson norms $\|P^*_{\beta,m}(\tau, s)\|$ as $m \to \infty$. Since m is fixed in our setting, we can simply estimate this single Petersson norm by taking the maximum over all $s \in \Omega$.

3.3. Continuation to $s = 0$

We now establish the meromorphic continuation of the Poincaré series $P_{\beta,m}(\tau, s)$ for $m < 0$. Since $P_{\beta,m}(\tau, s)$ is not square-integrable in this case, we cannot use spectral theory for the meromorphic continuation as in the case $m > 0$. Instead we use the Fourier expansion of $P_{\beta,m}(\tau, s)$ given in Proposition 3.1. We also remark that in addition to proving the meromorphic continuation for $m < 0$, we obtain that for $m > 0$ the known meromorphic continuation of $P_{\beta,m}(\tau, s) = P^*_{\beta,m}(\tau, s)$ (compare Proposition 3.4) is given by its Fourier expansion.

Theorem 3.7. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. Then $P_{\beta,m}(\tau, s)$ has a meromorphic continuation in s to $\mathbb{C}$ which is given by its Fourier expansion.*

Proof. We let $m \neq 0$ since for $m = 0$ this has already been shown in Proposition 3.3. Further, we fix $\gamma \in L'/L$, $v > 0$ and $\Omega \subseteq \mathbb{C}$ compact such that $Z(s + j; \beta, m, \gamma, n)$ is holomorphic for all $s \in \Omega$, $j \in \mathbb{N}_0$ and $n \in \mathbb{Z} + Q(\gamma)$. Recall that apart from the harmless term $v^s e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$ the Fourier coefficients of $P_{\beta,m}(\tau, s)$ are all of the form

$$b(\gamma, n; v, s) = \sum_{j=0}^{\infty} c_j(n; v, s) Z(s + j; \beta, m, \gamma, n).$$

We choose $\ell \in \mathbb{N}$ such that $\mathrm{Re}(s) + \ell > 3/4$ for all $s \in \Omega$, and split the above sum over j into a finite sum $j = 0, 1, \ldots, \ell - 1$ and an infinite sum starting with $j = \ell$. The finitely many terms of the first sum can now be estimated using Lemma 3.6 and the formula $W_{\kappa,\mu}(x) = O(x^\kappa e^{-x/2})$ as $x \to \infty$. In the second (infinite) sum the Kloosterman zeta functions can be bounded by a universal constant since the choice of ℓ guarantees that $\mathrm{Re}(s + j) > 3/4$ for all $s \in \Omega$.

If $n = 0$ then the remaining (infinite) sum can be written as part of a confluent hypergeometric series $_1F_1(a, b; x)$, which is meromorphic in a and b. Thus we may assume

that $n \neq 0$. In this case it remains to study the asymptotic behaviour of the Whittaker functions $W_{\pm 1/4 \pm j/2, s-1/4+j/2}(4\pi |n|v)$ in j and n. Recall that

$$W_{\kappa,\mu}(x) = e^{-x/2} x^{\mu+1/2} U(1/2 + \mu - \kappa, 1 + 2\mu, x),$$

where $U(a, b, x)$ is Tricomi's confluent hypergeometric function. If $\mathrm{Re}(a) > 0$ we can use the integral representation of $U(a, b, x)$ to obtain

$$|\Gamma(a)U(a,b,x)| \leq \begin{cases} 2^{\mathrm{Re}(b-a)-1}(x^{1-\mathrm{Re}(b)}\Gamma(\mathrm{Re}(b)-1)+1/\mathrm{Re}(a)), & \text{if } \mathrm{Re}(b-a) > 1, \\ x^{-\mathrm{Re}(a)}\Gamma(\mathrm{Re}(a)), & \text{if } \mathrm{Re}(b-a) \leq 1. \end{cases} \tag{3.16}$$

If $n < 0$ then the condition $\mathrm{Re}(a) > 0$ translates to $\mathrm{Re}(s) + 1/2 + j > 0$, which is true for all $j \geq \ell$. Hence we may use the above estimate in this case. Putting everything together, we find that

$$b(\gamma, n; v, s) = O(e^{-\varepsilon|n|}) \qquad \text{as } n \to -\infty \tag{3.17}$$

for some $\varepsilon > 0$, uniformly in s for $s \in \Omega$. The implied constant depends on β, m, γ and Ω. If $n > 0$ we repeatedly use the recurrence relation

$$U(a, b, z) = zU(a+1, b+1, z) - (b-a-1)U(a+1, b, z)$$

from [AS84, eq. 13.4.18], until $\mathrm{Re}(a+k) = \mathrm{Re}(s) + k$ is positive for all $s \in \Omega$. In each of these finitely many pieces appearing during this process, we can now use the estimate (3.16), yielding the same bound as in (3.17) for the coefficients $b(\gamma, n, v, s)$ with n positive.

Altogether we have shown that the Fourier series given in Proposition 3.1 defines a meromorphic function in s on all of $\mathbb{C}$. This function is by construction the analytic continuation of $P_{\beta,m}(\tau, s)$ in s. $\quad\square$

Remark 3.8. Let $\beta, \gamma \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. Going carefully through the proof of Theorem 3.7 we find that, for all $\varepsilon > 0$,

$$b(\gamma, n; v, s) = O(e^{-2\pi|n|v+\varepsilon|n|}) \qquad \text{as } |n| \to \infty,$$

locally uniformly in s and uniformly in v for $v > 1$. Moreover, if $n = \gamma = 0$ we obtain

$$b(0, 0; v, s) = O(v^{1/2-s}) \qquad \text{as } v \to \infty,$$

locally uniformly in s.

Lemma 3.9. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. The meromorphic continuation of $P_{\beta,m}(\tau, s)$ is holomorphic at $s = 0$ with a Fourier expansion of the form*

$$P_{\beta,m}(\tau, 0) = e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}) + \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n>0}} \tilde{b}(\gamma, n) e(n\tau) \mathfrak{e}_\gamma$$

$$+ \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n<0}} \tilde{b}(\gamma, n) \Gamma(1/2, 4\pi|n|v) e(n\tau) \mathfrak{e}_\gamma,$$

where

$$\tilde{b}(\gamma, n) = \begin{cases} 2^{1/2} n^{-1/2} \sum_{j=0}^{\infty} \dfrac{(-16\pi^2 mn)^j}{(2j)!} \, Z(j; \beta, m, \gamma, n), & \text{if } n > 0, \\[4mm] (2\pi)^{1/2} |n|^{-1/2} \lim_{s \to 0} \dfrac{Z(s; \beta, m, \gamma, n)}{\Gamma(s)}, & \text{if } n < 0. \end{cases}$$

In particular, the coefficients $\tilde{b}(\gamma, n)$ are all real, and if $m \geq 0$ then $\tilde{b}(\gamma, n) = 0$ for all $\gamma \in L'/L$ and $n \in \mathbb{Z} + Q(\gamma)$ with $n < 0$. We define

$$P_{\beta,m}(\tau) = P_{\beta,m}(\tau, 0).$$

Proof. First of all we note that by Lemma 3.5 the Kloosterman zeta functions $Z(s + j; \beta, m, \gamma, n)$ with $j \in \mathbb{N}_0$ showing up in the Fourier expansion are all holomorphic at $s = 0$ except for the case $j = 0$ and $m, n < 0$ in which we may have a simple pole at that point. However, for $n < 0$ the factor $1/\Gamma(s)$ in the Fourier coefficient $b(\gamma, n; v, s)$ compensates for this possible pole if $m < 0$, and implies the vanishing of $b(\gamma, n; v, s)$ at $s = 0$ if $m \geq 0$. In particular, the Fourier coefficients of $P_{\beta,m}(\tau, s)$ are holomorphic at $s = 0$. Using the formulas

$$W_{-1/4,-1/4}(x) = x^{1/4} e^{x/2} \Gamma(1/2, x) \qquad \text{and} \qquad W_{\mu,\mu-1/2}(x) = x^\mu e^{-x/2}$$

for $x > 0$ (see [EMOT55], formulas (2), (21) and (36) in Section 6.9) we obtain the claimed Fourier expansion. Moreover, the Fourier coefficients $\tilde{b}(\gamma, n)$ are indeed real since $\overline{Z(s; \beta, m, \gamma, n)} = Z(\overline{s}; \beta, m, \gamma, n)$. $\quad\square$

Theorem 3.10. *The function $P_{\beta,m}(\tau) \in H^+_{1/2,\rho}$ can be characterized as follows:*

(1) *For $m > 0$ the function $P_{\beta,m}$ is the unique cusp form of weight $1/2$ for ρ characterized by the inner product formula*

$$(f, P_{\beta,m}) = -8\pi \sqrt{m} \, c_f(\beta, m) \tag{3.18}$$

for each $f = \sum_\gamma \sum_n c_f(\gamma, n) e(n\tau) \mathfrak{e}_\gamma \in S_{1/2,\rho}$.

(2) For $m = \beta = 0$ we have

$$P_{0,0}(\tau) = \frac{2}{\sigma_0(N)} \sum_{c|N} \theta^{w_c} \in M_{1/2,\rho},$$

where $\sigma_0(N) = \sum_{c|N} 1$. This is the unique holomorphic modular form of weight $1/2$ for ρ with constant term $2\mathfrak{e}_0$ which is orthogonal to cusp forms in $S_{1/2,\rho}$.

(3) For $m < 0$, the function $P_{\beta,m}$ is the unique harmonic Maass form of weight $1/2$ for ρ which has principal part $(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})e(m\tau)$, which is orthogonal to cusp forms with respect to the regularized inner product, and which maps to a cusp form under the differential operator $\xi_{1/2}$.

Proof. Applying $\Delta_{1/2}$ to the Fourier expansion of $P_{\beta,m}$ given in Lemma 3.9, or evaluating the differential equation (3.2) at $s = 0$, we see that $P_{\beta,m}$ is harmonic. Moreover, $P_{\beta,m}$ is by analytic continuation modular of weight $1/2$ for ρ. Looking at its Fourier expansion we thus find that $P_{\beta,m}$ is an element of $S_{1/2,\rho}$, $M_{1/2,\rho}$ or $H_{1/2,\rho}^+$ if $m > 0$, $m = 0$ or $m < 0$, respectively.

If $m = \beta = 0$ then $P_{0,0} \in M_{1/2,\rho}$ can be written as a linear combination of the unary theta series θ^{w_c} for $c \mid N$ introduced in Section 2.4, i.e., $P_{0,0} = \sum_{c|N} \lambda_c \theta^{w_c}$. On the other hand, $P_{0,0}$ is invariant under all Atkin-Lehner involutions and orthogonal to cusp forms, which can be checked for the defining series of $P_{0,0}(\tau, s)$ for $\mathrm{Re}(s) \gg 0$, and follows for $P_{0,0}(\tau)$ by analytic continuation. In particular, all λ_c agree. Comparing constant coefficients, we obtain $\lambda_c = 2/\sigma_0(N)$.

Let now $m \neq 0$. The usual unfolding argument shows that for $f \in S_{1/2,\rho}$ and $\mathrm{Re}(s) \gg 0$ we have $(f, P_{\beta,m}(\cdot, s))^{\mathrm{reg}} = 0$ if $m < 0$, and

$$(f, P_{\beta,m}(\,\cdot\,, s))^{\mathrm{reg}} = 2(4\pi m)^{-s+1/2} \Gamma(s - 1/2) c_f(\beta, m)$$

if $m > 0$. One can check that the regularized integral on the left-hand side has a meromorphic continuation to $s = 0$, and that its evaluation at $s = 0$ agrees with $(f, P_{\beta,m})^{\mathrm{reg}}$. Hence $P_{\beta,m}$ is orthogonal to cusp forms if $m < 0$, and satisfies the claimed formula if $m > 0$.

It remains to note that for $m < 0$ the function $P_{\beta,m}$ is uniquely determined by the given conditions. Let $f \in H_{1/2,\rho}^+$ with $c_f^+(\gamma, n) = 0$ for all $\gamma \in L'/L$ and $n < 0$. Then

$$(\xi_{1/2}f, \xi_{1/2}f) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z}+Q(\gamma) \\ n < 0}} c_f^+(\gamma, n) c_{\xi_{1/2}f}(\gamma, -n) = 0$$

by Proposition 3.5 in [BF04], and thus $\xi_{1/2}f = 0$, implying $f \in M_{1/2,\rho}$. Hence $P_{\beta,m}$ is uniquely determined by its principal part $(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})e(m\tau)$ upto an element $f \in M_{1/2,\rho}$. Using that $c_{P_{\beta,m}}^+(0,0) = 0$, and that $P_{\beta,m}$ is orthogonal to cusp forms, we thus obtain the claimed uniqueness. $\quad\square$

Example 3.11. Let $N = 1$. Then we have $P_{0,0} = 2\theta$, and $P_{\beta,m} = 0$ for $m > 0$ since $S_{1/2,\rho} = \{0\}$ for $N = 1$. Further, for $m < 0$ the function $P_{\beta,m}$ is the unique weakly holomorphic modular form of weight $1/2$ with principal part $2e(m\tau)\mathfrak{e}_\beta$. Under the isomorphism $f_0(\tau)\mathfrak{e}_0 + f_1(\tau)\mathfrak{e}_1 \mapsto f_0(4\tau) + f_1(4\tau)$ between $M^!_{1/2,\rho}$ and the space of scalar valued weakly holomorphic modular forms of weight $1/2$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition, $P_{\beta,m}$ is identified with Borcherds' basis element $2f_{4m} = 2q^{4m} + O(q)$, whose Fourier coefficients of positive index are given by twisted traces of the j-function (compare [Zag02], Section 5).

4. Eisenstein series as theta lifts

In the current section we realize averaged versions of parabolic, hyperbolic and elliptic Eisenstein series as regularized theta lifts of Selberg's Poincaré series $P_{\beta,m}(\tau, s)$. We start by investigating the analytic properties of this lift. For $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$ with $m < 0$ we define $H_{\beta,m}$ as the set of all Heegner points, i.e.

$$H_{\beta,m} = \{z_X \in \mathbb{H} \colon X \in L_{\beta,m}\}.$$

If $m \geq 0$ we simply put $H_{\beta,m} = \emptyset$.

Proposition 4.1. *Let* $\beta \in L'/L$ *and* $m \in \mathbb{Z} + Q(\beta)$.

(1) For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$ the regularized theta lift

$$\Phi(z, P_{\beta,m}(\,\cdot\,,s)) = \mathrm{CT}_{t=0}\left[\lim_{T\to\infty}\int_{\mathcal{F}_T} \langle P_{\beta,m}(\tau,s), \Theta(\tau,z)\rangle \, \mathrm{Im}(\tau)^{1/2-t}\frac{du\,dv}{v^2}\right]$$

defines a real analytic function in $z \in \mathbb{H} \setminus H_{\beta,m}$ and a holomorphic function in s, which can be meromorphically continued to all $s \in \mathbb{C}$.

(2) For $m \neq 0$, it is holomorphic at $s = 0$, and we have

$$\Phi(z, P_{\beta,m}(\,\cdot\,,0)) = \Phi(z, P_{\beta,m}(\,\cdot\,,s))\big|_{s=0}.$$

(3) For $m = \beta = 0$, it has a simple pole at $s = 0$ with residue $-\frac{2}{s}$, and we have

$$\Phi(z, P_{0,0}(\,\cdot\,,0)) = \left(\Phi(z, P_{0,0}(\,\cdot\,,s)) + \frac{2}{s}\right)\bigg|_{s=0}.$$

Proof. The integral of $P_{\beta,m}(\tau, s)$ against $\Theta(\tau, z)$ over the compact set $\mathcal{F}_1$, the fundamental domain truncated at $y = 1$, converges and is real analytic for all $z \in \mathbb{H}$, and holomorphic in s whenever $P_{\beta,m}(\tau, s)$ is. Thus it suffices to consider

$$\varphi(z, s, t) = \int\limits_1^\infty \int\limits_{-1/2}^{1/2} \langle P_{\beta,m}(\tau, s), \Theta(\tau, z) \rangle\, v^{1/2-t} \frac{du\, dv}{v^2}.$$

Plugging in the Fourier expansion of $P_{\beta,m}(\tau, s)$ and the defining series for $\Theta(\tau, z)$, and carrying out the integral over u, we find

$$\varphi(z, s, t) = 2 \int\limits_1^\infty \sum_{X \in L_{\beta,m}} v^{s-1-t} \exp(-4\pi v Q(X_z))dv + \int\limits_1^\infty b(0,0;v,s) v^{-1-t} dv$$

$$+ \int\limits_1^\infty \sum_{\substack{X \in L' \\ X \neq 0}} b(X, Q(X); v, s) v^{-1-t} \exp(-2\pi v Q_z(X)) dv, \tag{4.1}$$

where $Q_z(X) = Q(X_z) - Q(X_{z^\perp})$ is the positive definite majorant of Q associated to z. If $m = \beta = 0$, we further split off the summand for $X = 0$ in the first integral, giving the additional term

$$2 \int\limits_1^\infty v^{s-1-t} dv = -\frac{2}{s-t} \tag{4.2}$$

for $\mathrm{Re}(t) > \mathrm{Re}(s)$. For $\mathrm{Re}(s) > 0$, the right-hand side has a continuation to $t = 0$, and taking the constant term at $t = 0$ yields a simple pole of $\Phi(z, P_{0,0}(\cdot, s))$ at $s = 0$ with residue $-\frac{2}{s}$.

The simple estimate $b(0,0;v,s) = O(v^{1/2-s})$ shows that for $\mathrm{Re}(s) > 1/2$ the second integral in (4.1) is holomorphic at $t = 0$. We plug in $t = 0$, insert the explicit formula for $b(0,0;v,s)$ and evaluate the integrals over the powers of v, yielding

$$\frac{2^{3/2-2s}\pi}{\Gamma(s)} \sum_{j=0}^\infty \frac{(-\pi m)^j}{j!} \frac{\Gamma(2s-1/2+j)}{\Gamma(s+1/2+j)}\, Z(s+j;\beta,m,0,0) \frac{1}{s+j-1/2}. \tag{4.3}$$

By Lemma 3.5 and since the Kloosterman zeta functions $Z(s+j;\beta,m,0,0)$ are universally bounded for $\mathrm{Re}(s) > 3/4 - j$ the remaining expression has a meromorphic continuation in s to $\mathbb{C}$, which is holomorphic and indeed vanishing at $s = 0$.

Next we recall that for $n \neq 0$ the Fourier coefficients $b(\gamma, n; v, s)$ are rapidly decreasing in n and v by Remark 3.8. Hence it can be shown as in the proof of Proposition 2.8 in [Bru02] that for $z \in \mathbb{H} \setminus H_{\beta,m}$ each of the remaining integrals in (4.1) has a continuation to $t = 0$, which is real analytic in $z \in \mathbb{H} \setminus H_{\beta,m}$ and has a continuation to all $s \in \mathbb{C}$ for which $P_{\beta,m}(\tau, s)$ is holomorphic. In particular, we can just plug in $t = s = 0$ in these integrals.

In order to prove the equations in part (2) and (3) of the proposition, we have to go through the same proof again, replacing $P_{\beta,m}(\tau, s)$ by $P_{\beta,m}(\tau, 0)$. Using similar arguments as before one can show that all the integrals appearing during this process apart from

the one corresponding to (4.2) have holomorphic continuation to $t = 0$. This proves part (2). For $m = \beta = 0$, the integral corresponding to (4.2) is now given by

$$2 \int_1^\infty v^{-1-t} = \frac{2}{t} \tag{4.4}$$

for $\mathrm{Re}(t) > 0$, with vanishing constant term at $t = 0$. Comparing the constant terms of (4.2) and (4.4) at $t = 0$, we obtain part (3) of the proposition. □

Recall that given $X \in L_{\beta,m}$ with $m \neq 0$ there is an associated Heegner geodesic c_X or Heegner point z_X if $m > 0$ or $m < 0$, respectively. Moreover, given $X \in L_{0,0}$ we find $\lambda \in \mathbb{Z}$ such that $X = \lambda X_\ell$ where X_ℓ is a generator of the isotropic line $\ell = \mathbb{Q}X$ satisfying $\ell \cap L = \mathbb{Z}X_\ell$, and there is an associated cusp p_X corresponding to the line ℓ.

Lemma 4.2. *Let $X \in L_{\beta,m}$ with $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$, and let $z \in \mathbb{H}$. For $m \neq 0$ we have*

$$Q(X_z) = \begin{cases} m \cosh^2(d_{\mathrm{hyp}}(z, c_X)), & \text{if } m > 0, \\ |m| \sinh^2(d_{\mathrm{hyp}}(z, z_X)), & \text{if } m < 0, \end{cases}$$

and for $m = \beta = 0$ and $X \neq 0$ we have

$$Q(X_z) = \frac{\lambda^2}{4N} \mathrm{Im}(\sigma_{p_X}^{-1} z)^{-2},$$

were $\lambda \in \mathbb{Z}$ with $X = \lambda X_\ell$ for a generator X_ℓ of $\mathbb{Q}X \cap L$, and $\sigma_{p_X} \in \mathrm{SL}_2(\mathbb{R})$ is a scaling matrix for the cusp p_X.

Proof. The two formulas for $m \neq 0$ are well known and follow by a direct calculation. Let $m = \beta = 0$, and let c be a positive divisor of N. Then the cusp $1/c$ of $\Gamma_0(N)$ corresponds to the isotropic line $\mathbb{Q}X_c$ with generator $X_c = \begin{pmatrix} -1 & 1/c \\ -c & 1 \end{pmatrix} \in L$, and a simple calculation shows that

$$Q((X_c)_z) = \frac{1}{4N} \mathrm{Im}(\sigma_{1/c}^{-1} z)^{-2} \quad \text{with} \quad \sigma_{1/c} = \sqrt{\frac{c}{N}} \begin{pmatrix} N/c & 1 \\ N & 1+c \end{pmatrix}.$$

Here $\sigma_{1/c}$ is a scaling matrix for the cusp $1/c$. Since $Q((\lambda X)_z) = \lambda^2 Q(X_z)$ for $\lambda \in \mathbb{Z}$, and since the identification in (2.1) is compatible with the action of $\Gamma_0(N)$, this proves the claimed formula. □

Theorem 4.3. *Let $\beta \in L'/L$ and $m \in \mathbb{Z} + Q(\beta)$. Then*

$$\Phi(z, P_{\beta,m}(\,\cdot\,, s)) = \begin{cases} \dfrac{2\Gamma(s)}{(4\pi m)^s} \displaystyle\sum_{X \in \Gamma_0(N)\backslash L_{\beta,m}} E_{cX}^{\mathrm{hyp}}(z, 2s), & \text{if } m > 0, \\[2.5em] 4N^s \zeta^*(2s) \displaystyle\sum_{p \in C(\Gamma_0(N))} E_p^{\mathrm{par}}(z, 2s), & \text{if } m = 0, \\[2.5em] \dfrac{2\Gamma(s)}{(4\pi|m|)^s} \displaystyle\sum_{X \in \Gamma_0(N)\backslash L_{\beta,m}} E_{zX}^{\mathrm{ell}}(z, 2s), & \text{if } m < 0, \end{cases}$$

for $z \in \mathbb{H} \setminus H_{\beta,m}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$. Here, $\zeta^*(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function.

Proof. Splitting the sum over $(M, \phi) \in \langle (T, 1) \rangle \backslash \mathrm{Mp}_2(\mathbb{Z})$ defining the Poincaré series into matrices M with lower left entry $c = 0$ and $c \neq 0$, we find

$$P_{\beta,m}(\tau, s) = v^s e(m\tau)\left(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta}\right) + \frac{1}{2} \sum_{\substack{(M,\phi) \in \langle (T,1) \rangle \backslash \mathrm{Mp}_2(\mathbb{Z}) \\ M \neq \pm 1}} v^s e(m\tau)\mathfrak{e}_\beta \Big|_{1/2,\rho} (M, \phi) \quad (4.5)$$

As the infinite sum behaves nicely at $i\infty$, we may plug in $t = 0$ and take the limit in the corresponding part of the regularized integral, yielding

$$\int_{\mathcal{F}}^{\mathrm{reg}} \left\langle \frac{1}{2} \sum_{M \neq \pm 1} v^s e(m\tau)\mathfrak{e}_\beta \Big|_{1/2,\rho} \tilde{M}, \Theta(\tau, z) \right\rangle v^{1/2} \frac{du\,dv}{v^2} = 2 \int_{\mathcal{G}} v^{s+1/2} e(m\tau)\overline{\Theta_\beta(\tau, z)} \, \frac{du\,dv}{v^2},$$

where $\mathcal{G} = \{\tau \in \mathbb{H}\colon |\mathrm{Re}(\tau)| \leq 1/2, |\tau| < 1\}$ and $\Theta = \sum_{\gamma \in L'/L} \Theta_\gamma \mathfrak{e}_\gamma$. The asymptotic behaviour $\Theta_\beta(\tau, z) = O(v^{-1})$ as $v \to 0$, uniformly in u (see [Bru02], Lemma 2.13), shows that the unfolding is justified for $\mathrm{Re}(s) > 3/2$. If we split the regularized integral corresponding to the first part of (4.5) at $v = 1$, we see that $\Phi(z, P_{\beta,m}(\,\cdot\,, s))$ equals

$$2 \int_0^1 \int_{-1/2}^{1/2} v^{s+1/2} e(m\tau)\overline{\Theta_\beta(\tau, z)} \, \frac{du\,dv}{v^2} + \mathrm{CT}_{t=0} \, 2 \int_1^\infty \int_{-1/2}^{1/2} v^{s+1/2-t} e(m\tau)\overline{\Theta_\beta(\tau, z)} \, \frac{du\,dv}{v^2}.$$

Plugging in the definition of $\Theta_\beta(\tau, z)$ and evaluating the integral over u this becomes

$$2 \int_0^1 v^{s-1} \sum_{X \in L_{\beta,m}} \exp(-4\pi v Q(X_z)) dv + \mathrm{CT}_{t=0} \, 2 \int_1^\infty v^{s-1-t} \sum_{X \in L_{\beta,m}} \exp(-4\pi v Q(X_z)) dv.$$

For $m = \beta = 0$ a short calculation shows that the integrals over the summands for $X = 0$ in the two sums above cancel out. Thus we may assume $X \neq 0$ in both sums. Now for

$z \notin H_{\beta,m}$ and $X \in L_{\beta,m} \setminus \{0\}$ we have $Q(X_z) \neq 0$, so we can simply plug in $t = 0$. Substituting $v' = 4\pi v Q(X_z)$ we obtain

$$\Phi(z, P_{\beta,m}(\,\cdot\,,s)) = \frac{2\,\Gamma(s)}{(4\pi)^s} \sum_{X \in L_{\beta,m}\setminus\{0\}} Q(X_z)^{-s}. \tag{4.6}$$

Next we split the sum in (4.6) into a sum over classes $[X]$ in $\Gamma_0(N)\backslash L_{\beta,m} \setminus \{0\}$ and a sum over matrices M in $\Gamma_0(N)_X\backslash\Gamma_0(N)$ where $\Gamma_0(N)_X$ is the stabilizer of X in $\Gamma_0(N)$. Since $Q((MX)_z) = Q(X_{M^{-1}z})$, we find

$$\Phi(z, P_{\beta,m}(\,\cdot\,,s)) = \frac{2\,\Gamma(s)}{(4\pi)^s} \sum_{X \in \Gamma_0(N)\backslash(L_{\beta,m}\setminus\{0\})} \sum_{M \in \Gamma_0(N)_X\backslash\Gamma_0(N)} Q(X_{Mz})^{-s}.$$

Note that $\Gamma_0(N)_X = \Gamma_0(N)_{c_X}$ if $m > 0$, and $\Gamma_0(N)_X = \Gamma_0(N)_{z_X}$ if $m < 0$. Hence for $m \neq 0$ the statement of the theorem follows from Lemma 4.2 and using holomorphic continuation of both sides to $\mathrm{Re}(s) > 1/2$. Let now $m = \beta = 0$. Given $X \in L_{0,0} \setminus \{0\}$ we write $X = \lambda X_\ell$ with $\lambda \in \mathbb{Z}$ and X_ℓ a generator of $\mathbb{Q}X \cap L$. Note that $\Gamma_0(N)_X = \Gamma_0(N)_{X_\ell} = \Gamma_0(N)_p$ with p being the cusp associated to the isotropic line $\mathbb{Q}X$. Applying Lemma 4.2 we obtain

$$\Phi(z, P_{\beta,m}(\,\cdot\,,s)) = \frac{2\,\Gamma(s)}{(4\pi)^s} \sum_{\substack{\lambda \in \mathbb{Z} \\ \lambda \neq 0}} \sum_{\ell \in \Gamma_0(N)\backslash \mathrm{Iso}(V)} \sum_{M \in \Gamma_0(N)_{X_\ell}\backslash\Gamma_0(N)} Q((\lambda X_\ell)_{Mz})^{-s}$$

$$= \frac{4N^s\Gamma(s)\zeta(2s)}{\pi^s} \sum_{p \in C(\Gamma_0(N))} \sum_{M \in \Gamma_0(N)_p\backslash\Gamma_0(N)} \mathrm{Im}(\sigma_p^{-1}Mz)^{2s}.$$

Here σ_p is a scaling matrix for the cusp p. As before the claimed statement follows by holomorphic continuation. $\square$

We conclude this section with a simple corollary on the meromorphic continuation of Eisenstein series which follows directly from Proposition 4.1. Finally, we remark a possible generalization of averaged hyperbolic and elliptic Eisenstein series.

Corollary 4.4. *The averaged Eisenstein series appearing in Theorem 4.3 have a meromorphic continuation in s to $\mathbb{C}$ which is holomorphic at $s = 0$. Furthermore, the averaged hyperbolic and elliptic Eisenstein series vanish at $s = 0$.*

Remark 4.5. Up to equation (4.6) we did not use the special shape of the lattice L. In fact, given an even lattice (L, Q) of arbitrary signature (b^+, b^-) and some pair (β, m) with $\beta \in L'/L$, $m \in \mathbb{Z}+Q(\beta)$ one could use the right-hand side of (4.6) to define an associated averaged Eisenstein series, which we call hyperbolic, parabolic or elliptic if $m > 0$, $m = 0$ or $m < 0$, respectively. This generalized Eisenstein series lives on the Grassmannian of b^+-dimensional positive definite subspaces of $L\otimes\mathbb{R}$, and it is invariant under the action of

$O_d(L)$, which is the subgroup of the special orthogonal group of L fixing the classes of L in its discriminant group L'/L. Moreover, using the same techniques as above an analog of equation (4.6) shows that these generalized Eisenstein series can again be realized as the Borcherds theta lift of Selberg's Poincaré series for the lattice L of index (β, m).

5. Kronecker limit formulas

We now compute Kronecker limit formulas for the averaged parabolic, hyperbolic and elliptic Eisenstein series given in Theorem 4.3. Since $\Gamma(s)$ and $\zeta^*(2s)$ have simple poles at $s = 0$, Theorem 4.3 tells us that the linear coefficient of the Laurent expansion of the averaged Eisenstein series at $s = 0$ is essentially given by the value of the regularized theta lift $\Phi(z, P_{\beta,m}(\,\cdot\,, s))$ at $s = 0$. By Proposition 4.1 we need to compute the regularized theta lift of the harmonic Maass form $P_{\beta,m}(\tau) = P_{\beta,m}(\tau, 0)$, which by Theorem 2.1 is given by the logarithm of the absolute value of the Borcherds product $\Psi(z, P_{\beta,m})$ associated to $P_{\beta,m}(\tau)$. In the following, we compute these Borcherds products explicitly.

5.1. The parabolic case

We start with the classical, parabolic case. Let $m = \beta = 0$. In Theorem 3.10 we have seen that

$$P_{0,0}(\tau) = \frac{2}{\sigma_0(N)} \sum_{c|N} \theta^{w_c}(\tau)$$

is a holomorphic modular form of weight $1/2$ for ρ. In order to compute the Borcherds product associated to $P_{0,0}$ we first compute the Borcherds product of the unary theta functions θ^{w_c}.

Lemma 5.1. *For $c \mid N$ the Borcherds product associated to θ^{w_c} is given by*

$$\Psi(z, \theta^{w_c}) = \eta(cz)\eta\left(\tfrac{N}{c}z\right),$$

where $\eta(z) = e(z/24) \prod_{n \geq 1}(1 - e(nz))$ is the Dedekind eta function.

Proof. The Fourier coefficients of the unary theta function θ^{w_c} relevant for its Borcherds product are given by

$$c_{\theta^{w_c}}(n, n^2/4N) = \begin{cases} 1, & \text{if } n \equiv 0 \bmod c \text{ or } n \equiv 0 \bmod N/c, \text{ and } n \not\equiv 0 \bmod N, \\ 2, & \text{if } n \equiv 0 \bmod N, \\ 0, & \text{else}, \end{cases} \tag{5.1}$$

for $n \neq 0$. Further, the Weyl vector associated to θ^{w_c} and the cusp ∞ computes to

$$\rho_{\theta^{w_c}, \infty} = \frac{\sqrt{N}}{8\pi} (\theta^{w_c}, \theta^{w_N})^{\text{reg}} = \frac{c + N/c}{24}$$

by Example 5.6 in [BS17]. Therefore, using Theorem 2.1 we obtain the product expansion

$$\Psi(z, \theta^{w_c}) = e((c + N/c)z/24) \prod_{n=1}^{\infty} (1 - e(ncz)) \prod_{n=1}^{\infty} \left(1 - e\left(n\frac{N}{c}z\right)\right).$$

This proves the claimed formula. □

Theorem 5.2. *At $s = 0$ we have the Laurent expansion*

$$\sum_{p \in C(\Gamma_0(N))} E_p^{\text{par}}(z, s) = 1 + \frac{1}{\sigma_0(N)} \sum_{c|N} \log\left(|\Delta(cz)|^{1/6} \operatorname{Im}(z)\right) \cdot s + O(s^2).$$

Proof. For brevity, we write $E(z, s)$ for the sum of the parabolic Eisenstein series on the left-hand side. By Proposition 4.1 we have the identity

$$\Phi(z, P_{0,0}) = \left(\Phi(z, P_{0,0}(\cdot, s)) + \frac{2}{s}\right)\Bigg|_{s=0}$$

of regularized theta lifts. Applying Theorem 2.1 with $c_{P_{0,0}}^+(0,0) = 2$ on the left-hand side and Theorem 4.3 on the right-hand side, where we also replace s by $s/2$, we obtain

$$-2(\log(4\pi N) + \Gamma'(1)) - 4\log|\Psi(z, P_{0,0})y| = \left(4N^{s/2}\zeta^*(s)E(z, s) + \frac{4}{s}\right)\Bigg|_{s=0}. \quad (5.2)$$

Using the expansion $\zeta^*(s) = -\frac{1}{s} - \frac{1}{2}(\Gamma'(1) + \log(4\pi)) + O(s)$ and the fact that the right-hand side of (5.2) is holomorphic at $s = 0$, we see that the constant term in the Laurent expansion of $E(z, s)$ at $s = 0$ equals 1. Now a short calculation shows that the right-hand side of (5.2) is given by

$$-2(\log(4\pi N) + \Gamma'(1)) - 4 \operatorname{FT}_{s=0} E(z, s),$$

where $\operatorname{FT}_{s=0} E(z, s)$ denotes the first term in the Laurent expansion of $E(z, s)$ at $s = 0$, i.e., the coefficient of the linear term. So by (5.2) we find that $E(z, s)$ has the Laurent expansion

$$E(z, s) = 1 + \log|\Psi(z, P_{0,0})y| \cdot s + O(s^2)$$

at $s = 0$. In order to determine the remaining Borcherds product, we write $P_{0,0}$ as a linear combination of theta functions θ^{w_c} for $c \mid N$ as in Theorem 3.10 and apply Lemma 5.1. This yields

$$\Psi(z, P_{0,0}) = \prod_{c|N} \left(\eta(cz)\eta\left(\frac{N}{c}z\right)\right)^{2/\sigma_0(N)} = \prod_{c|N} \eta(cz)^{4/\sigma_0(N)}.$$

Using $\Delta(z) = \eta(z)^{24}$ and taking everything together we obtain the stated formula.   □

Remark 5.3. For $N = 1$ we recover the classical Kronecker limit formula

$$E^{\mathrm{par}}_\infty(z, s) = 1 + \log\left(|\Delta(z)|^{1/6}\operatorname{Im}(z)\right) \cdot s + O(s^2).$$

Moreover, for the extension $\Gamma_0^*(N)$ of $\Gamma_0(N)$ by all Atkin-Lehner involutions, having only one cusp at ∞, the sum of all parabolic Eisenstein series for $\Gamma_0(N)$ is actually the unique parabolic Eisenstein series for $\Gamma_0^*(N)$, i.e.,

$$E^{\mathrm{par},\Gamma_0^*(N)}_\infty(z, s) = \sum_{M \in \Gamma_0^*(N)_\infty \backslash \Gamma_0^*(N)} \operatorname{Im}(Mz)^s = \sum_{p \in C(\Gamma_0(N))} E^{\mathrm{par}}_p(z, s).$$

Therefore, the above theorem actually states the Kronecker limit formula for the group $\Gamma_0^*(N)$, given for example in Section 1.5 of [JST16].

5.2. The hyperbolic case

Let $m > 0$ and $\beta \in L'/L$ with $m \in \mathbb{Z} + Q(\beta)$. Recall from Theorem 3.10 that in this case the Poincaré series $P_{\beta,m}(\tau) = P_{\beta,m}(\tau, 0)$ is the holomorphic Poincaré series of weight $1/2$ for ρ, i.e., the unique cusp form satisfying the Petersson coefficient formula (3.18).

We will first construct an orthogonal basis for the space $S_{1/2,\rho}$ consisting of linear combinations of unary theta functions, whose associated Borcherds products were computed in Lemma 5.1. This will then enable us to compute the Borcherds product associated to the Poincaré series $P_{\beta,m}(\tau)$.

Note that we can identify the orthogonal group $O(L'/L)$ with the group $D(N)$ given by the set of positive divisors of N together with the operation $c \star d = cd/(c,d)^2$ via the isomorphism $c \mapsto w_c$, where $w_c \in O(L'/L)$ is defined by (2.5). The characters of $D(N)$ are given by the functions $c \mapsto \mu((c,d))$ for $d \mid N$, where $\mu(n)$ denotes the Moebius function. We let $E(N)$ be the set of all $d \mid N$ for which $\mu(d) = 1$, and for $d \in E(N)$ we define

$$\theta_d = \sum_{c \mid N} \mu((c,d))\theta^{w_c}. \tag{5.3}$$

Lemma 5.4. *The functions θ_d for $d \in E(N) \backslash \{1\}$ form an orthogonal basis of $S_{1/2,\rho}$ with norms*

$$(\theta_d, \theta_d) = \frac{2\pi\sigma_0(N)\sigma_1(N/d)\varphi(d)}{3\sqrt{N}}.$$

Here $\sigma_k(N) = \sum_{d \mid N} d^k$, and φ denotes Euler's totient function.

Proof. Since $(f^{w_c}, g) = (f, g^{w_c})$ one easily computes

$$(\theta_d, \theta_e) = \sum_{b,c|N} \mu((c,d))\mu((b,e))(\theta^{w_c}, \theta^{w_b}) = \sum_{c|N} \mu((c, d \star e)) \cdot \sum_{a|N} \mu((a,e))(\theta^{w_a}, \theta) \quad (5.4)$$

for $d, e \in E(N)$. Thus, using the orthogonality relations for characters of finite groups, we see that the functions θ_d for $d \in E(N)$ are orthogonal. Further, using formula (5.1) we easily find that the $(d, d^2/4N)$'th Fourier coefficient of θ_d is non-zero, i.e., $\theta_d \not\equiv 0$. Hence, the θ_d are linearly independent, and since $M_{1/2,\rho}$ is isomorphic to the space $J^*_{1,N}$ of skew-holomorphic Jacobi forms of weight 1 and index N (compare [EZ85], Theorem 5.7), the dimension formula $\dim(J^*_{1,N}) = \frac{1}{2}\sigma_0(N)$ from [SZ88, page 130] shows that the θ_d for $d \in E(N)$ form a basis of $M_{1/2,\rho}$. Taking into account the constant coefficient of θ_d one finds that θ_d is a cusp form unless $d = 1$. Hence the θ_d for $d \in E(N) \setminus \{1\}$ form an orthogonal basis of $S_{1/2,\rho}$.

It remains to compute their norms. Recall that we have $(\theta^{w_c}, \theta) = \frac{\pi}{3\sqrt{N}}(c + N/c)$ for $c \mid N$ by Example 5.6 in [BS17]. Thus (5.4) yields

$$(\theta_d, \theta_d) = \sigma_0(N) \cdot \frac{2\pi}{3\sqrt{N}} \sum_{c|N} \mu((c,d))c$$

for $d \in E(N)$. Since

$$\sum_{c|N} \mu((c,d))c = \prod_{p|d}(1-p) \prod_{p|\frac{N}{d}}(1+p) = \mu(d)\sigma_1(N/d)\varphi(d)$$

and $\mu(d) = 1$, we also obtain the stated formula for the norms. $\square$

We are now ready to state a general Kronecker limit type formula for the averaged sum of hyperbolic Eisenstein series given in Theorem 4.3.

Theorem 5.5. *Let $m > 0$ and $\beta \in L'/L$ with $m \in \mathbb{Z} + Q(\beta)$. At $s = 0$ we have the Laurent expansion*

$$\sum_{X \in \Gamma_0(N) \backslash L_{\beta,m}} E^{\mathrm{hyp}}_{cX}(z, s) = \sum_{d \in E(N) \setminus \{1\}} C_{\beta,m}(d) \log \left(\prod_{c|N} |\eta(cz)|^{\mu((c,d))} \right) \cdot s + O(s^2),$$

where $E(N)$ is the set of all positive divisors of N having an even number of prime factors, and the constants $C_{\beta,m}(d)$ are given as follows:

(1) If $4Nm$ is not a square, or if $4Nm = n^2$ is a square with $N/(n, N)$ being 1 or a prime, then the sum on the right-hand side vanishes, i.e., the sum of hyperbolic Eisenstein series has a double root at $s = 0$.

(2) *If $4Nm = n^2$ with $n \in \mathbb{Z}_{>0}$, and such that $N/(n, N)$ has at least two different prime factors, then the constants $C_{\beta,m}(d)$ are given by*

$$
C_{\beta,m}(d) = \begin{cases} \dfrac{24n\,\mu((f,d))}{\sigma_0(N/(n,N))\sigma_1(N/d)\varphi(d)}, & \text{if } (n,d) = 1, \\[2mm] 0, & \text{if } (n,d) > 1. \end{cases}
$$

Here we chose $f \mid N$ such that $w_f(n) = \beta$. In particular, the sum on the right-hand side of the above Laurent expansion does not vanish in this case.

Proof. Combining Theorem 4.3, Proposition 4.1 and Theorem 2.1 we obtain that the Laurent expansion of the left-hand side at $s = 0$ is given by

$$
\sum_{X \in \Gamma_0(N) \backslash L_{\beta,m}} E_{c_X}^{\mathrm{hyp}}(z,s) = \frac{1}{4}\Phi(z, P_{\beta,m}) \cdot s + O(s^2) = -\log|\Psi(z, P_{\beta,m})| \cdot s + O(s^2).
$$

In order to compute the Borcherds product $\Psi(z, P_{\beta,m})$, we write $P_{\beta,m}$ in terms of the orthogonal basis given in Lemma 5.4, and use Lemma 5.1 to obtain

$$
-\log|\Psi(z, P_{\beta,m})| = -2 \sum_{d \in E(N) \backslash \{1\}} \frac{(P_{\beta,m}, \theta_d)}{(\theta_d, \theta_d)} \log\left(\prod_{c|N} |\eta(cz)|^{\mu((c,d))}\right). \tag{5.5}
$$

If $4Nm$ is not a square we have $(P_{\beta,m}, \theta_d) = 0$ for all $d \in E(N) \setminus \{1\}$ by the coefficient formula (3.18) since the Fourier coefficients of the theta functions θ_d are supported on indices of the form $n^2/4N$ for $n \in \mathbb{Z}$.

Let now $4Nm = n^2$ with $n \in \mathbb{Z}_{>0}$. Then we find $f \mid N$ such that $w_f(n) = \beta$ as $\beta^2 \equiv n^2 \bmod 2N$. Moreover, we have $P_{\beta,m} = P_{n,n^2/4N}^{w_f}$ and $\theta_d^{w_f} = \theta_{d \star f} = \mu((f,d))\theta_d$. Using again formula (3.18) we thus find

$$
(P_{\beta,m}, \theta_d) = \mu((f,d)) \sum_{c|N} \mu((c,d)) \left(P_{n,n^2/4N}, \theta^{w_c}\right)
$$

$$
= -\frac{4\pi n}{\sqrt{N}} \mu((f,d)) \sum_{c|N} \mu((c,d)) c_{\theta^{w_c}}(n, n^2/4N).
$$

The latter Fourier coefficients are given in (5.1), yielding

$$
\sum_{c|N} \mu((c,d)) c_{\theta^{w_c}}(n, n^2/4N) = 2 \sum_{c|(n,N)} \mu((c,d)) = 2\sigma_0((n,N/d)) \sum_{c|(n,d)} \mu((c,d)).
$$

Now the remaining sum over the values of the character $c \mapsto \mu((c,d))$ of the group $D((n,d))$ is 1 if $(n,d) = 1$, and vanishes otherwise. Note that in the first case, i.e., if $(n,d) = 1$, then $\sigma_0((n,N/d)) = \sigma_0((n,N))$. Therefore, the formula in item (2) follows if we take into account the norm (θ_d, θ_d) computed in Lemma 5.4. Moreover, we see that the

coefficients $C_{\beta,m}(d)$ can only vanish simultaneously if $(n,d) > 1$ for all $d \in E(N) \setminus \{1\}$, i.e., if $N/(n,N)$ is 1 or a prime, proving the second part of item (1).

Finally, we note that since an eta quotient is uniquely determined by its exponents, one can check that the eta quotients $\prod_{c|N} |\eta(cz)|^{\mu((c,d))}$ with $d \in E(N) \setminus \{1\}$ appearing in (5.5) are linearly independent (in the multiplicative sense). Hence the first term in the Laurent expansion of the averaged hyperbolic Eisenstein series indeed vanishes if and only if $C_{\beta,m}(d) = 0$ for all $d \in E(N) \setminus \{1\}$. $\square$

Remark 5.6. The proof of the previous theorem actually shows that for $m > 0$ the Poincaré series $P_{\beta,m}$ vanishes exactly if $4Nm$ is not a square, i.e., if the geodesics c_X with $X \in L_{\beta,m}$ are closed in the modular curve $\Gamma_0(N) \backslash \mathbb{H}$, or if $4Nm = n^2$ is a square with $N/(n,N)$ being 1 or a prime. In the following we explain the vanishing of the Poincaré series in the latter case:

If $N = 1$ or $N = p$ is a prime we trivially have $P_{\beta,m} = 0$ as $S_{1/2,\rho} = \{0\}$ by Lemma 5.4. We claim that if $(n,N) > 1$, the Poincaré series $P_{\beta,m}$ of level N can actually be written as a certain lift of a Poincaré series of lower level N/q with $q = (n,N)$. In order to distinguish lattices, we write (L_N, Q_N) for the lattice of level $4N$, and $(L_{N/q}, Q_{N/q})$ for the lattice of level $4N/q$. Similarly, we will put superscripts at some places to emphasize the underlying lattice. We now define an operator

$$V_q \colon M_{1/2,\rho^{N/q}} \to M_{1/2,\rho^N}$$

acting on the Fourier expansion of $f = \sum_{\gamma \in L'_{N/q}/L_{N/q}} \sum_{n \geq 0} c_f(\gamma,n)e(n\tau)\mathfrak{e}_\gamma \in M_{1/2,\rho^{N/q}}$ via

$$f|V_q = \sum_{\substack{\gamma \in L'_N/L_N \\ }} \sum_{\substack{n \in \mathbb{Z}+Q_N(\gamma) \\ n \geq 0}} \left(\sum_{a|(n-Q_{(\gamma)},\gamma,q)} c_f(\gamma/a, qn/a^2) \right) e(n\tau)\mathfrak{e}_\gamma.$$

Here we identify L'_N/L_N with $\mathbb{Z}/2N\mathbb{Z}$, as usual. This operator is a direct translation of the corresponding V_q operator acting on skew-holomorphic Jacobi forms of weight 1 and index N/q (compare [EZ85], §4). One can check that

$$\left(f|V_q \right)^{w_c^N} = f^{w_c^{N/q}}|V_q \tag{5.6}$$

for $f \in M_{1/2,\rho^{N/q}}$ and $c \mid \frac{N}{q}$. Furthermore, comparing Fourier expansions we find

$$\theta^{N/q}|V_q = \sum_{a|q} \left(\theta^N \right)^{w_a^N}, \tag{5.7}$$

and combining (5.6) and (5.7) we obtain $\theta_d^{N/q}|V_q = \theta_d^N$ for $d \in E(N/q)$. Conversely, computing inner products with $\theta_e^{N/q}$ for $e \in E(N/q)$ we get

$$\theta_d^N | V_q^* = \begin{cases} \frac{\sigma_0(q)\sigma_1(q)}{\sqrt{q}} \theta_d^{N/q}, & \text{if } (d, q) = 1, \\ 0, & \text{if } (d, q) > 1, \end{cases} \qquad (5.8)$$

for $d \in E(N)$. Here V_q^* denotes the adjoint operator of V_q. Finally, we can use (5.8) and the fact that $(\beta, N) = (n, N) = q$ to prove

$$P_{\beta, n^2/4N}^N = \frac{q}{\sigma_1(q)} P_{\beta/q, (n/q)^2/(4N/q)}^{N/q} | V_q.$$

Thus, if N/q is 1 or a prime, the Poincaré series on the right-hand side vanishes identically since the corresponding space of cusp forms is trivial, and hence the Poincaré series on the left-hand side vanishes as well.

5.3. The elliptic case

Let $m < 0$ and $\beta \in L'/L$ with $m \in \mathbb{Z} + Q(\beta)$. Theorem 3.10 tells us that the Poincaré series $P_{\beta, m}$ is the unique harmonic Maass form of weight $1/2$ for ρ which is orthogonal to cusp forms and has principal part $e(mz)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$. For general squarefree N, the function $P_{\beta, m}$ is a proper harmonic Maass form, i.e., not weakly holomorphic, and the associated Borcherds product can only be described in terms of its roots and poles on $\mathbb{H}$ and at the cusps. Note that the Fourier coefficients of the holomorphic part of $P_{\beta, m}$ are real by Lemma 3.9, so the corresponding Borcherds product is well-defined.

Theorem 5.7. *Let $m < 0$ and $\beta \in L'/L$ with $m \in \mathbb{Z} + Q(\beta)$. At $s = 0$ we have the Laurent expansion*

$$\sum_{X \in \Gamma_0(N) \backslash L_{\beta, m}} E_{z_X}^{\mathrm{ell}}(z, s) = -\log |\Psi_{\beta, m}(z)| \cdot s + O(s^2),$$

where $\Psi_{\beta, m} \colon \mathbb{H} \to \mathbb{C}$ is a weakly holomorphic modular form of weight 0, level $\Gamma_0(N)$ and some unitary character of possibly infinite order. Further, $\Psi_{\beta, m}$ is uniquely determined by the following properties:

(1) The roots of $\Psi_{\beta, m}$ in $\mathbb{H}$ are located at the Heegner points z_X for $X \in L_{\beta, m}$, with order 2 if $\beta \equiv -\beta \bmod L$ and order 1 otherwise.
(2) The order of $\Psi_{\beta, m}$ at the cusp $1/c$ for $c \mid N$ is given by

$$\mathrm{ord}_{1/c}(\Psi_{\beta, m}) = -\frac{H_N(\beta, m)}{\sigma_0(N)}$$

   where

$$H_N(\beta, m) = \sum_{X \in \Gamma_0(N) \backslash L_{\beta, m}} \frac{2}{|\Gamma_0(N)_X|}$$

is the Hurwitz class number.

(3) The leading coefficient in the Fourier expansion of $\Psi_{\beta,m}$ at ∞ is 1.

Proof. As in the proof of Theorem 5.5, a combination of Theorem 4.3, Proposition 4.1 and Theorem 2.1 yields that the Laurent expansion of the averaged sum of elliptic Eisenstein series at $s = 0$ is of the form

$$\sum_{X \in \Gamma_0(N) \backslash L_{\beta,m}} E_{z_X}^{\text{ell}}(z,s) = -\log|\Psi_{\beta,m}| \cdot s + O(s^2)$$

with $\Psi_{\beta,m}(z) = \Psi(z, P_{\beta,m})$ a meromorphic modular form of weight $c_{P_{\beta,m}}(0,0) = 0$, level $\Gamma_0(N)$ and some unitary character. Its product expansion implies property (3). Moreover, as the principal part of $P_{\beta,m}$ is given by $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$ we have

$$\frac{1}{2} \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} + Q(\gamma) \\ n < 0}} c_{P_{\beta,m}}^+(\gamma,n) \sum_{X \in L_{\gamma,n}} (z_X) = \frac{1}{2} \sum_{X \in L_{\beta,m}} [(z_X) + (z_{-X})] = \sum_{X \in L_{\beta,m}} (z_X).$$

Thus part (2) of Theorem 2.1 yields property (1). In order to determine the order of $\Psi_{\beta,m}$ at the cusp $1/c$ using part (3) of Theorem 2.1 we need to compute the Weyl vector associated to $P_{\beta,m}$ and the cusp $1/c$, i.e.,

$$\rho_{P_{\beta,m},1/c} = \frac{\sqrt{N}}{8\pi}(P_{\beta,m}, \theta^{w_c})^{\text{reg}}.$$

Since $\theta^{w_c} - \frac{1}{\sigma_0(N)} \sum_{d|N} \theta^{w_d}$ is a cusp form, and $P_{\beta,m}$ is orthogonal to cusp forms by Theorem 3.10, we can write

$$\rho_{P_{\beta,m},1/c} = \frac{\sqrt{N}}{8\pi\sigma_0(N)} \left(P_{\beta,m}, \sum_{d|N} \theta^{w_d} \right)^{\text{reg}}. \tag{5.9}$$

In order to compute the latter inner product, we introduce Zagier's non-holomorphic Eisenstein series $E_{3/2}$ of weight $3/2$, which was first studied by Zagier for level $N = 1$ in [Zag75b], and later generalized to arbitrary N by Bruinier and Funke, see [BF06], Remark 4.6 (i). It is a harmonic Maass form of weight $3/2$ for the dual representation $\bar\rho$ with holomorphic part

$$E_{3/2}^+(\tau) = \sum_{\gamma \in L'/L} \sum_{\substack{n \in \mathbb{Z} - Q(\gamma) \\ n \geq 0}} H_N(\gamma, -n) e(n\tau) \mathfrak{e}_\gamma,$$

where $H_N(\gamma, -n)$ is the Hurwitz class number, which is defined in the theorem for $n < 0$, and for $n = 0$ we set $H_N(0,0) = -\sigma_1(N)/6$. Furthermore, the Eisenstein series $E_{3/2}$ is orthogonal to cusp forms with respect to the regularized inner product, and its image under the differential operator $\xi_{3/2}$ is given by

$$\xi_{3/2}E_{3/2}(\tau) = -\frac{\sqrt{N}}{4\pi}\sum_{d|N}\theta^{w_d}(\tau), \tag{5.10}$$

which can be checked using the Fourier expansion of $E_{3/2}$ given in [BF06]. Therefore, by equation (5.9) and Stokes' theorem (applied as in [BF04], Proposition 3.5) we find

$$\begin{aligned}
\rho_{P_{\beta,m},1/c} &= -\frac{1}{2\sigma_0(N)}(P_{\beta,m},\xi_{3/2}E_{3/2})^{\mathrm{reg}} \\
&= \frac{1}{2\sigma_0(N)}(E_{3/2},\xi_{1/2}P_{\beta,m})^{\mathrm{reg}} \\
&\quad - \frac{1}{2\sigma_0(N)}\sum_{\gamma\in L'/L}\sum_{\substack{n\in\mathbb{Z}+Q(\gamma)\\ n\leq 0}}c_{P_{\beta,m}}^+(\gamma,n)c_{E_{3/2}}^+(\gamma,-n).
\end{aligned}$$

The remaining inner product vanishes since $\xi_{1/2}P_{\beta,m}$ is a cusp form by Theorem 3.10, and as the principal part of $P_{\beta,m}$ is given by $e(m\tau)(\mathfrak{e}_\beta + \mathfrak{e}_{-\beta})$, we are left with

$$\rho_{P_{\beta,m},1/c} = -\frac{1}{2\sigma_0(N)}\left(c_{E_{3/2}}^+(\beta,-m) + c_{E_{3/2}}^+(-\beta,-m)\right) = -\frac{H_N(\beta,m)}{\sigma_0(N)}.$$

Here we also used that $H_N(\beta,m) = H_N(-\beta,m)$.

Finally, in order to see that $\Psi_{\beta,m}$ is indeed uniquely determined by the properties given in the theorem, we let $g\colon\mathbb{H}\to\mathbb{C}$ be another weakly holomorphic modular form of weight 0, level $\Gamma_0(N)$ and some unitary character, which satisfies the properties (1) – (3) of the theorem. Then the function $h(z) = \log|\Psi_{\beta,m}(z)/g(z)|$ defines a harmonic function on the modular curve $\Gamma_0(N)\backslash(\mathbb{H}\cup\mathbb{P}^1(\mathbb{Q}))$ without singularities. Hence, the maximum principle for harmonic functions gives that h is constant, showing that $\Psi_{\beta,m}$ and g agree up to a constant multiple, which needs to be one as both functions are normalized. □

Remark 5.8. We emphasize that the character of the Borcherds product $\Psi_{\beta,m}$ may have infinite order. Since $P_{\beta,m}$ is orthogonal to cusp forms, Theorem 6.2 and the subsequent remark from [BO10] show that the character has finite order if and only if the Fourier coefficients $c_{P_{\beta,m}}^+(n, n^2/4N)$ for $n \geq 1$ are all rational. However, since these coefficients are expected to be transcendental if $P_{\beta,m}$ is not weakly holomorphic (see the conjecture in the introduction of [BO10]) the character will typically have infinite order.

Remark 5.9. Let N be a squarefree positive integer such that the group $\Gamma_0(N)$ has genus 0. Then Theorem 1.5 from the introduction follows from Theorem 5.7 by noting that for $X \in L_{\beta,m}$ with $m < 0$ and $c \mid N$ the normalized Hauptmodul $j_{N,1/c,z_X}(z)$ has order -1 at the cusp $1/c$ and order $|\Gamma_0(N)_X|/2$ at the root z_X.

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