Strategic Portfolio Management for Long-Term Investments: An Optimal Control Approach

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Abstract

In this thesis, a solution framework for the problem of strategic portfolio management for long-term investments is proposed which uses an optimal control approach. The aim of this work is to develop and apply methods of control engineering for solving the problem of multi-period portfolio optimization. The thesis introduces mathematical models in continuous and discrete time that describe the dynamics of assets. For these models, the corresponding feedback controllers are derived such that the objectives of the investors are optimized.

In continuous time, the basic modelling and optimization framework is introduced and the problem of portfolio optimization is discussed with respect to other well-known control solutions. For asset price models where the expected returns are an affine function of economic factors, the portfolio optimization problem is solved analytically. The conditions for solving this problem are also derived as well as the case where the economic factors are not directly observable. Since the analytical solutions depend on restrictive assumptions, such as no constraints on the control variables, a numerical method for solving the stochastic optimal control problems is derived and its convergence is proved. The continuous-time methods are applied in two short case studies to German and US market data.

In discrete time, models for the asset classes and the portfolio dynamics are introduced. The modelling of non-Gaussian asset distributions and the modelling of stochastic volatility (variance) is discussed. The problem of portfolio optimization in discrete time is stated and the conditions of optimality, i.e., the dynamic programming algorithm, are explained. Since for realistic assumptions, such as constraints, and large problem sizes, the solution of the dynamic programming algorithm is impractical and thus, two approximation methods are suggested. The first method is a suboptimal control strategy which uses the ideas from deterministic model predictive control. Model predictive control solves the problem of finding an optimal controller by consecutively solving the corresponding open-loop problem. The second method, called stochastic programming approximation, approximates the stochastic portfolio dynamics by a finite number of scenarios and solves the feedback problem for the approximated dynamics. The two approximation methods are used to derive solutions for portfolio optimizations for specific asset price models.
With model predictive control, a solution to the so-called linear Gaussian Factor model is derived and an extension to the model with time-varying and stochastic covariance matrices is given. For portfolio problems with transaction costs and liabilities, a stochastic programming solution is explained.

The methods for discrete-time portfolio optimizations are applied in three case studies to real-world asset data. In the first case study, the model predictive control method is applied to a problem of maximizing the expected returns subjected to a so-called coherent risk measure constraint. The results of the out-of-sample test with mostly US stock market data show that the realized portfolio returns comply with the risk constraint. In the second case study, the problem of constructing a balanced fund, which invests in stocks, bonds, and cash, is solved. The portfolio optimization in this case study uses the analytical model predictive control solution where a heuristic is used to periodically select factors for the expected return predictions. The results of the computed portfolios show that the method outperforms suitable benchmarks with respect to absolute and risk-adjusted returns. In the third case study, the asset allocation problem for a Swiss fund is considered which invests domestically and in the EU markets. The fund gives a performance guarantee and is assumed to be large, such that it faces considerable transaction costs. The problem resembles the situation that Swiss pension funds face for their asset allocation decisions. The case study shows that the computed strategy holds the portfolio values above the barrier of the performance guarantee. Furthermore, the optimization adjusts the risk aversion depending on the distance between actual portfolio value and current minimum performance guarantee.
Zusammenfassung


Programming Approximation bezeichnet wird, nähert man die stochastische Dynamik

**List of Symbols**

\( N \) normal distribution

\( (\mathcal{F}_t)_{t \geq 0} \) filtration

\( \epsilon^r \) white noise process of risky assets dynamics

\( \epsilon^x \) white noise process of the factor dynamics

\( \eta \) error covariance matrix of the Kalman Filter

\( \Gamma \) risk measure

\( \gamma \) coefficient of risk aversion

CVaR conditional Value-at-Risk

VaR Value-at-Risk

\( \Lambda \) correlation matrix

\( \mathcal{A} \) differential operator

\( \mathcal{F}(t) \) information at time \( t \) (filtration)

\( \mathcal{U} \) constraints for \( u \)

\( \mu_i \) expected returns of risky assets

\( \nu \) diffusion matrix of the factor process model

\( \Omega \) probability space

\( \omega \) sample

\( \bar{m} \) portfolio mean

\( \bar{V} \) portfolio variance

\( \psi \) diffusion of the factor process

\( \rho \) correlation matrix of Brownian motions

\( \Sigma \) covariance matrix of risky assets

\( \sigma_i \) standard deviation (volatility) of risky assets

\( \Theta \) drift of factor process
\(
\theta \quad \text{confidence level}
\)

\(\varphi\) \quad \text{quantile}

\(\xi\) \quad \text{observable variables} \((p, x)\)

\(\xi^r\) \quad \text{white noise process with unit variance}

\(A\) \quad \text{dynamics matrix of the factor process}

\(a\) \quad \text{constant of the factor process model}

\(B\) \quad \text{dynamics matrix of the unobservable factor process}

\(b\) \quad \text{constant of the unobservable factor process}

\(b\) \quad \text{constant of the unobservable factor process}

\(D\) \quad \text{drift of the state equation of} \(y\)

\(d^*\) \quad \text{sales of assets}

\(F\) \quad \text{factor loading matrix of the excess returns}

\(f\) \quad \text{constant of the factor model of excess returns}

\(F_0\) \quad \text{factor loading matrix of the risk-free asset}

\(f_0\) \quad \text{constant of the factor model of the risk-free asset}

\(G\) \quad \text{factor loading matrix of the risky assets}

\(g\) \quad \text{constant of the factor model of the risky assets}

\(H\) \quad \text{unobservable factor loading matrix of the risky assets}

\(J\) \quad \text{value function of the HJB equation}

\(K_i\) \quad \text{matrix function (riccati equations)}

\(L, M\) \quad \text{cost functions}

\(L_x, L_y\) \quad \text{portfolio loss}

\(m_z\) \quad \text{optimal estimate of} \(z\)

\(N_i\) \quad \text{number shares of asset} \(i\)

\(p(r)\) \quad \text{density of the returns}

\(p^*\) \quad \text{purchase of assets}

\(P_0\) \quad \text{price of the risk-free asset}

\(P_i\) \quad \text{prices of risky assets}

\(q\) \quad \text{in- or outflow of money from the portfolio}

\(R\) \quad \text{portfolio return}

\(r\) \quad \text{returns of the assets}

\(r_0\) \quad \text{risk-free interest rate}
S  diffusion of the state equation of $y$
$s$  scenario (number)
$T$  terminal date of the portfolio optimization
$U$  utility function
$u_i$  fraction of wealth invested in the $i$th asset
$V$  volatility matrix
$v_i$  volatility of risky asset $i$
$W$  wealth (value of the portfolio)
$w$  logarithm of wealth
$W_0$  initial wealth
$x$  factor levels
$x_0$  initial factor level
$y$  state variable vector
$z$  unobservable factor levels (Chapter 3)
$z_i$  amount of wealth invested in instrument $i$
$Z_P$  Brownian motion of the risky assets dynamics
$Z_x$  Brownian motion of the factor dynamics
$E$  expectation
$\mathbb{P}$  probability
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Introduction

In this chapter, a brief introduction to the topic of portfolio management is given. The standard single-period portfolio optimization problems are discussed and a critique is given for this type of models. The connection between the problem of long-term portfolio management and control engineering is explained and a discussion why stochastic optimal control is the appropriate framework for solving the problem of multi-period portfolio optimization is given. Furthermore, the goals and the structure of this thesis stated.

1.1 The portfolio management problem

The portfolio management problem is a well-known problem with many different solution ideas and methods. In general, the problem of portfolio construction is the problem of investing our own money in financial assets such that certain requirements regarding the expected profits and possible losses are matched.

1.1.1 Introduction and facts

In order to appraise the importance of portfolio management, Figure 1.1 shows the growth of the US mutual fund industry from 1990 to 2002. At the end of 2002, mutual funds had 6.5 trillion US $ in assets under management. Mutual funds are among the largest institutional investors that daily apply portfolio management and optimization. Many different types of funds exists, e.g., equity funds or bond funds, which invest in different asset classes or markets. The largest are clearly the equity funds, followed by the money market funds.

As shown in Figure 1.2, different fund types have quite different return patterns. Money market funds have the lowest returns but also the lowest fluctuations, whereas equity funds
have the highest returns with the highest return fluctuations. A portfolio is described by two main characteristics: return (expected profit) and risk (possible losses). Portfolios with high expected returns are in general more risky than portfolios with low expected returns (Sharpe, Alexander and Bailey 1998). Very often, the risk aversion defines in which assets investors tend to invest. But the risk return pattern is not the only variable that defines a portfolio. Other important variables are the investment horizon (duration of the investments), the asset classes, or the markets (domestic or international). In this
work, we consider portfolios of financial assets such as stocks, derivatives, bonds, indices of various markets, and cash.

Many different methods exist to solve the problem of portfolio construction. We only discuss portfolio construction methods based on mathematical methods, where we briefly discuss the best known methods of single-period models and give a short critique on this method below.

1.1.2 Asset classes

A financial investment, in contrast to a real investment which involves tangible assets (e.g. land, factories), is an allocation of money to financial contracts in order to make its value increase over time.

According to the statements above, an investment can be looked at as a sacrifice of current cash for future cash. The two main attributes that distinguish securities are time and risk, where risk also includes inflation. The interest rate or return is defined as the gain or loss of the investment divided by the initial value of the investment. An investment always contains some sort of risk. Therefore, the higher an investor considers the risk of a security, the higher the rate of return he demands, sometimes referred to as risk premium.

The first type of securities considered are treasury bills. They involve loaning on a short-term basis to the U.S. Treasury or any other national treasury. Treasury bills contain almost no risk. Other types of short-term interest rate products are money-market accounts.

The second type of securities are bonds. There are two major categories of bonds: government bonds and corporate bonds. Bonds, as treasury bills, involve lending money but on a fairly long-term basis. Bonds result in a cash payment on a regular basis, the coupon of a bond, up to its expiry, i.e., the maturity date, when the final cash payment is made. There is a market for bonds where they can be bought and sold. The price of a bond vary over its lifetime depending on the changes of interest rates. The bond rating system is a measure of a company’s credit risk. There are several rating agencies such as Moody’s, Standard & Poor, and Fitch.

Other type of securities are stocks. Stocks, also called shares, represent ownership in a part of a corporation. Stocks pay out part of their profits as dividends. The main reason for investing in stocks is the expected increase in the stock price.
The last category of securities is derivatives which are financial instruments that have a price that is derived from another tradable security, e.g., futures or options. In this work, besides currency forwards, we do not discuss the use of derivatives.

For more details on the different asset classes, the reader should refer to Reilly and Brown (1999), Sharpe et al. (1998), and Luenberger (1998).

1.1.3 Single-period portfolio management

The single-period portfolio management problem attempts to solve the problem of investment decisions for a fixed investment horizon without any possibilities to change the portfolio composition in between the investment period. Conceptually, the investor decides to allocate his capital among various assets and observes the final outcome at the end of the investment period.

Many single-period portfolio optimizations are given in the literature. The most famous and most popular model is the mean-variance framework suggested by Markowitz (1952, 1959). This model defines the risk as the variance of a portfolio, (see also the discussion in Section 2.2), and defines returns as expected returns. The problem can be efficiently solved as a quadratic program. Many other important single period models are: the mean deviation approach of Konno and Yamazaki (1991), the regret optimization approach of Dembo and King (1992), the min-max approach described in Young (1998), as well as the Conditional Value-at-Risk approach of Rockafellar and Uryasev (2000). They mostly differ in the modelling of return distributions and risk measures.

1.1.4 Critique of the single period model

Most financial planning systems today still rely on the classical mean-variance framework pioneered over 50 years ago by Markowitz. Despite its huge success, the single-period setting possesses some significant deficiencies. First, it is difficult to use it, in a long-term application where investors are able to re-balance their portfolio frequently. Second, for situations where investors face liabilities or goals at specific future dates, the investment decisions must be taken with regard to the dynamics and the time structure involved. The multi-period approach may also provide superior performance over the single-period approach, see Dantzig and Infanger (1993). Third, the definition of risk, such as variance or semi-variance, does not transfer any information regarding the chances of matching
the obligations or goals. Furthermore, variance or semi-variance are not coherent risk measures, see Section 2.3. Fourth, the mean-variance framework is extremely sensitive to the model inputs, i.e., mean values and covariances. Fifth, the mean-variance framework cannot easily handle issues such as taxes and transaction costs.

Nevertheless, economic growth theory recommends that a multi-period investor should maximize the expected logarithmic wealth at each time period, as suggested by Luenberger (1998, Chapter 15). However, it has been shown in Rudolf and Ziemba (2004) that a logarithmic utility may lead to a too high risk tolerance. In addition, the theory depends on various assumptions such as no transaction costs, identically and independently distributed asset returns, and neither liabilities nor in- or outflows to be time-dependent. When these assumptions are violated, a multi-period setting is the appropriate framework to handle such a problem. Multi-period models and the corresponding optimization methods (control methods) attempt to address the points raised above.

1.2 Portfolio management and control engineering

What is the connection between control engineering (optimal control) and portfolio management? What has the engineering science to offer for solving a problem that is firmly rooted in business and finance? To answer those questions let us look at the definition of control engineering from the quote in Wikipedia:

Control engineering is the engineering discipline that focuses on the mathematical modelling of systems of a diverse nature, analyzing their dynamic behavior, and using control theory to make a controller that will cause the systems to behave in a desired manner.

Using this definition, control engineering is the science that causes dynamic systems to behave in a desired manner. This definition fits the problem of multi-period portfolio management well, where we want to make asset allocation decisions such that the portfolio (system) behaves in a desired manner with regard to risk and return.

1.2.1 Portfolio management as dynamic optimization problem

The globalization of financial markets and the introduction of various new and complex products, such as options or other structured products, have significantly increased the volatility and risk for participants in the markets. Moreover, advances in communication
technology and computers have dramatically increased the reaction speed of financial markets to world events. This has occurred within any domestic market of OECD countries, as well as across markets internationally. The long-term nature of a portfolio amplifies the financial rewards for good decisions as well as the penalties for bad decisions.

The problem of multi-period (long-term) portfolio management is characterized by a long investment horizon and the possibility of rebalancing the portfolio frequently until the terminal date is reached. At every rebalancing period, new information is available to the investor, such as the asset prices, state of the economy, the results of his portfolio decisions, etc. The investor must react consistently to the new information such that the objectives and constraints of the portfolio are matched. Furthermore, the nature of asset prices are best captured by dynamic models and therefore, the portfolio is a dynamic system. Consequently, the problem of portfolio management is a dynamic optimization problem similar to a control problem. Given a strategy, e.g., a risk tolerance and a return expectation, the fund should react to the new state information similar to a feedback controller in a technical application.

1.2.2 Stochastic optimal control

Stochastic optimal control theory attempts to solve the kind of dynamic decision problems under uncertainty described above in an optimal fashion. The optimal control theory states the mathematical conditions of optimality given the dynamic model that describes the system and the objective function we want to optimize. When we meet the conditions of optimality, a feedback controller is found that reacts optimally with respect to the objective, the new information, the remaining investment horizon, and the assumed stochastic dynamics of our portfolio and assets. Therefore, we believe that stochastic optimal control is the correct framework to address the problem of long-term portfolio optimization.

1.3 Goals and structure of the thesis

In this section, we state the goals and give a brief outline of the contend of the thesis.
1.3 Goals and structure of the thesis

1.3.1 Goals of the thesis

The main motivation of this thesis is the consequent utilization of feedback methods to improve the solutions of multi-period portfolio optimizations. In the following list the goals of this work are stated:

- Consequent use of feedback from observed prices, economic factors, volatilities, etc., to solve dynamically the problem of portfolio optimization.
- Consistent use of dynamical models in continuous or discrete time to describe the stochastic asset dynamics and thus, make projections of future asset price evolutions.
- Correct consideration of multi-period investment problems with the corresponding investment horizons and rebalancing time periods.
- Derivation of optimization strategies for the most important time series models such as dynamic factor models or stochastic volatility (GARCH) models.
- Consistent use of modern risk measures and management methods for the portfolio characterizations.
- Testing of the portfolio optimization methods in realistic long-term out-of-sample tests.

1.3.2 Structure of the thesis

Chapter 1 is a brief introduction to this thesis and Chapter 2 discusses the topic of risk measures and decisions under uncertainty. In Chapter 3, continuous-time portfolio optimization methods are presented. In Chapter 4, discrete-time models and control are discussed. Chapter 5 applies the discrete-time methods in three case studies. The thesis concludes with Chapter 6.

Chapter 2: risk measurements and decisions under uncertainty

In this chapter, the problem of defining and measuring risk is explained. At first, utility functions and classical risk measures and their properties are presented.

In the second part, a discussion of the properties of coherent risk measures is given and the most important coherent risk measure (Conditional Value-at-Risk) is discussed.

The third part discusses the recent extensions of coherent risk measures to multi-period problems. After a short literature review, requirements for coherent multi-period risk measures are given.
Chapter 3: continuous-time methods

In this chapter, we use the continuous-time methods and stochastic optimal control theory to solve problems of long-term investments. At first, we introduce the asset price modelling framework and state the corresponding wealth dynamics. We limit our modelling framework to stochastic differential equation models with factor models for expected returns and volatilities. Furthermore, we derive the optimality conditions for portfolio optimizations where the asset dynamics are described by these kinds of models. The first part ends with an illustrative problem of portfolio optimization where the asset price dynamics are described by a stochastic volatility model.

In the second part, we introduce affine models for the expected returns of the risky assets and the interest rate for the bank account. The expected returns are assumed to be an affine function of the factor levels which themselves are governed by a linear stochastic process. The problem of long-term portfolio optimization over terminal wealth is solved for two utility functions. In both cases, the optimal control variable is computed by an affine function of the factor levels and time dependent gain matrices and vectors. The gain matrices are the solution of dynamic Riccati equations. Furthermore, the problem of partial information of the factor process is also discussed and a solution is derived. The second part concludes with a brief case study where the methods of this section are applied to German market data.

In the third part, we discuss numerical solutions for the continuous-time portfolio optimization problem. In particular, problems with constraints on the control vector are discussed. We derive the so-called successive approximation algorithm and prove its convergence. This part finishes with a case study, where we assume that the assets follow a stochastic differential equation with stochastic volatility and time-varying and stochastic expected returns.

The chapter concludes with a discussion of the weaknesses and strengths of continuous-time modelling and control. Especially, properties of optimal control strategies are discussed which maybe used in discrete-time to find optimal or suboptimal control strategies.

Chapter 4: discrete-time methods

In Chapter 4, we discuss discrete-time models and control strategies for long-term portfolio optimization problems. In the first part, we present discrete-time models for assets and
the corresponding portfolio optimization problems. Models for expected returns and models for stochastic volatility are discussed; however we restrict the modelling framework to models with a Markovian structure. Especially, the possibility to model multi-dimensional stochastic volatility models is presented. Moreover, other white noise processes than Gaussian white noise processes are also discussed. The conditions of optimality, which are described by the dynamic programming algorithm and their computational limitations are presented. Two approximation schemas are introduced: the Model Predictive Control method and the Stochastic Programming approximation. The Model Predictive Control method is a suboptimal control method that solves the optimal control problem by solving consecutively a series of open-loop control problems for the given time and state variables. The method is explained and conditions are given when the suboptimal strategy yields control decisions that are close to the true optimal control decisions. The Stochastic Programming approximation solves the optimal control problem by computing a so-called scenario approximation of the stochastic dynamics. Instead of approximating the dynamic programming by dividing the state space into a regular grid, we compute stochastic scenarios for the future state variables based on the current state values.

In the second part, we present the Model Predictive Control method applied to three specific asset return models. First, we assume a Gaussian factor model similar to the affine model presented in Chapter 2. Second, we assume that the asset returns are governed by a factor process for the expected returns and the white noise processes of the risky assets are described by a multivariate GARCH model. For this asset model, two Model Predictive Control strategies for two different objective functions are explained. The first solution is an analytical approach for the mean-variance utility function. The second solution uses a numerical approach to solve the problem where the utility function is defined as portfolio mean minus a weighted Conditional Value-at-Risk.

In the third part, the application of the Stochastic Programming approximation to problems of portfolio optimization with linear transaction costs is discussed. Two basic methods for solving the underlying stochastic program (optimization) are given. The first leads to a large scale linear program and the second to a nonlinear optimization which mimics the stochastic control strategies of Chapter 3.

The chapter ends with a discussion of advantages and disadvantages of discrete-time models and control strategies.
Chapter 5: case studies

In this chapter, we apply the discrete-time methods develop in Chapter 4 in three different case studies. In the first case study, which is based on US asset market data, the Model Predictive Control method with a coherent risk measure is used. The data set consists of nine stock market indices, a bond index, and a bank account. Three portfolios with two different investment horizons are computed. The theoretical limits for the portfolio risks are compared to the realized risk of the computed portfolio. The realized violations of the limits for portfolio risk agree well with theoretical predicted violations.

In the second case study, also based on US data, we use the Model Predictive Control method developed for the Gaussian factor model. In this case study, we simulate the situation of a balanced fund that invests in stocks, bonds, cash, and a commodity index. The long-term portfolio optimization is computed with a moving investment horizon of two-years and a Value-at-Risk constraint. The empirical results are encouraging, since the portfolios with different risk aversions all have a steady wealth evolution which mostly comply with the risk constraints.

In a third case study, we attempt to solve the problem of a Swiss based fund that gives a certain performance guarantee and faces considerable transaction costs. The fund invests domestically in stocks, bonds, and cash and internationally in stocks and bonds. The setup resembles the situation faced by Swiss pension funds. Due to the presence of transaction costs and the capital guarantee, the portfolio problem becomes path-dependent and therefore, we need to solve the control problem by the Stochastic Programming approximation. In an 8-year out-of-sample test, we find that the fund’s values are above the performance guarantee. Furthermore, the optimization adapts the risk aversion in function of the distance to the performance “barrier”. Also, the case study indicates that active asset management adds value even in the presence of transaction costs.

Chapter 6: summary and outlook

In the last chapter, the conclusion of the thesis is presented. We summarize the main ideas given in Chapter 3 and 4. We discuss the case studies presented in Chapter 5. In the outlook, we present open problems that were not solved in this work as well as open questions that were not addressed. Furthermore, we discuss possible further directions to improve the results given in this thesis.
Risk measures and decisions under uncertainty

Any decision under uncertainty must quantify its aims and its costs. As discussed in Section 1.1, any portfolio management problem consists of two crucial inputs: return (expected profits) and risk (possible losses). In order to quantify risk, we need to define risk in mathematical terms and discuss how the optimization changes under different risk definitions.

2.1 Introduction

In this section, we discuss the aspect of measuring risk in decision problems under uncertainty. Deterministic problems are characterized by known numbers, whereas stochastic problems are characterized by random variables. Naturally it is this stochastic nature that introduces the risk component to the system. When we control risk, we need to ask the question how risk should be measured. The quote taken from Gummerlock and Litterman (1998) exemplifies the problem of how to measure risk:

*Leaving aside risk for a moment, consider the measurement of human size. Everyone knows qualitatively what large and small mean, but life gets more difficult when we want to express size in a single number. Either height or weight can be useful, depending on the problem being addressed. Each metric is appropriate for a given problem, and neither serves all purposes. Indeed, if one asks for a definite answer to the question of which metric is the best measure of size, the answer is that neither height, nor weight, nor a linear combination of them is the best measure of size. The best measure of size is the one most appropriate to the purpose for which it is intended.*

As with human size, the same problem arises when measuring risk. Actually risk is too complex to characterize with one number. On the other hand, decision models call for a single measure of risk, which should be easy to understand. Generally a risk measure is
2.2 Utility functions and classical risk measures

In this part, we first discuss how traditionally utility functions were used to rank uncertain outcomes. Secondly, we introduce classical risk measures and discuss briefly their applicability.

2.2.1 Utility functions

Traditionally uncertain outcomes are ranked with so-called utility functions, which are a standard tool in economics to describe the preferences of individuals. Once the investor decides to allocate his capital among the alternatives, his future wealth is governed by the corresponding random cash flows of the investment opportunities. Utility functions provide a ranking to judge uncertain situations. For a risk averse investor, a utility function $U$ must fulfill certain properties.

- The utility function is an increasing continuous function: $U' > 0$
- The utility function must be concave: $U'' < 0$

The first property makes sure that an investor prefers always more wealth to less wealth. The second property captures the principle of risk aversion. Some commonly used utility functions for the uncertain wealth ($W$) include

- Exponential function ($\gamma > 0$) $U(W) = -e^{-\gamma W}$
- Logarithmic function $U(W) = \ln(W)$
- Power functions ($\gamma < 1$ and $\gamma \neq 0$) $U(W) = \frac{1}{\gamma} (W)^\gamma$
- Quadratic functions ($W \leq \frac{a}{2b}$) $U(W) = aW - bW^2$

Any of the utility functions shown above capture the principle of risk aversion. This is accomplished whenever the utility function is concave, see Luenberger (1998, Chapter 9).

2.2.2 Classical risk measures

From the early work on portfolio selection, the modern form of risk quantification find their origins in the article by Markowitz (1952), where the author defined risk as the
variance of a portfolio. Variance seems to be a natural candidate as a risk measure since it quantifies the deviation from the expectation as risk. Since then, variance or similar concepts such as lower semi-variance (Markowitz 1959) are the most used and popular risk measures. Using variance as risk measure for portfolio optimizations is equivalent to maximizing a quadratic utility or assuming that the uncertainty has a Gaussian distribution. The quadratic utility function has the undesirable property of satiation and of increasing absolute risk aversion which implies that the investment into risky assets decreases with increasing wealth. Furthermore, variance does not quantify the risk as a potential monetary loss and penalizes positive deviations from the expectation (which an investor would welcome) similar to negative deviations. For all these reasons, alternative down-side risk measure have been proposed and studied.

In recent years, the financial and banking industry has extensively investigated quantile-based downside risk measures. The most important these, the so-called Value-at-Risk (VaR), has been increasingly used as a risk management tool, see e.g., Jorion (1997), Duffie and Pan (1997), and Pearson and Smithson (2002). Value-at-Risk is defined as the largest possible loss of a portfolio for a given confidence level \( \theta \). However, it is not generally the largest possible loss, since in \( 1 - \theta \) percentage of the cases, the loss exceeds the Value-at-Risk number. The Value-at-Risk the at confidence level \( \theta \) is mathematically defined as

\[
\text{VaR}(u, \theta) = \arg \min_{\varphi \in \mathbb{R}} \left\{ \int_{f(r,u) \leq \varphi} p(r) dr \right\} \geq \theta,
\]

where \( f(u,r) \) is the portfolio loss function, \( \varphi \) denotes quantile, \( u \) the investments (positions) in the risky assets, \( r \) denotes the uncertain returns, and \( p(r) \) their distribution.

While VaR measures the worst losses which can be expected with certain probability (confidence), it does not address how large these losses can be expected when the bad (with small probability) events occur. Moreover, VaR for certain portfolios and return distributions cannot account for diversification.

\(^1\) A quantile of a distribution is the inverse cumulative distribution function for a given probability value
2.3 Coherent risk measures

The risk measures discussed so far (utility functions, variance, VaR) have all certain shortcomings for specific situations. Therefore, risk measures should be based on a set of axioms that make them acceptable under general situations.

2.3.1 Properties of coherent risk measures

The notion of a coherent risk measure was introduced in Artzner, Delbaen, Eber and Heath (1999), where the authors defined the mathematical properties of a risk measure in order to serve as a “monetary” loss measure. A coherent risk measure $\Gamma$ for a portfolio loss $L$ has the following properties:

1. Monotonicity: For all $L_x, L_y$, if $L_x(\omega) \geq L_y(\omega)$, for all $\omega \in \Omega$, then $\Gamma(L_x) \geq \Gamma(L_y)$.
2. Translation invariance: if $c$ is a constant and $r_f$ a strictly positive risk-free return, then for all $L_x$, $\Gamma(cr_f + L_x) = \Gamma(L_x) - c$.
3. Positive homogeneity: if $\lambda > 0$ then for all $L_x$, $\Gamma(\lambda L_x) = \lambda \Gamma(L_x)$.
4. Subadditivity: for all $L_x, L_y$, $\Gamma(L_x + L_y) \leq \Gamma(L_x) + \Gamma(L_y)$,

where $L_x$ and $L_y$ denote the loss (function) of two arbitrary portfolios, $\Omega$ the probability space, $\omega$ a sample, and $\Gamma$ the risk measure. Property 1 states that, when portfolio losses $L_x$ are larger than $L_y$ for every state of the world, then the risk measure should be larger. Property 2 states that the amount of capital invested in a risk-free instrument reduces the risk exposure. Property 3 ensures that holding the same portfolio multiple times, the risk measure increases by the same multiplier. The last property states that the risks of two arbitrary portfolios are always larger than or equal to the risk of the combined portfolios. It makes sure that $\Gamma$ properly accounts for diversification. Property 3 and 4 imply convexity.

Examples of risk measures and comments on their coherence properties:

- Variance: not coherent, because it violates property 1.
- Value-at-Risk: not coherent, because it violates property 4. With VaR it is impossible to show diversification effects, when we assume an arbitrary probability distribution.
- Maximum loss: coherent measure, can be used as worst case measure.
- CVaR: coherent measure with very useful properties.
Only for elliptical distributions, e.g., t-student or Gaussian, are variance or VaR coherent risk measures, as shown in McNeil, Frey and Embrechts (2005, Chapter 3).

Since properties of coherent risk measures in the sense of Artzner et al. seem to be too restrictive, Ziemba and Rockafellar (2000) proposed other properties of coherent risk measures. They replace property 3 (positive homogeneity) and property 4 (subadditivity) by

3a. Non-negativity: \( L_x > 0 \) then \( \Gamma(L_x) > 0 \) and \( \Gamma(0) = 0 \).

4a. Convexity: for all \( L_x, L_y \), \( \Gamma(\lambda_1 L_x + \lambda_2 L_y) \leq \lambda_1 \Gamma(L_x) + \lambda_2 \Gamma(L_y) \), \( \lambda_1 > 0, \lambda_2 > 0 \), and \( \lambda_1 + \lambda_2 = 1 \).

The two altered properties imply a partial substitute for positive homogeneity, since

\[
\Gamma(\lambda L_x) \leq \lambda \Gamma(L_x) \quad \text{if} \quad 0 < \lambda < 1
\]

\[
\Gamma(\lambda L_x) \geq \lambda \Gamma(L_x) \quad \text{if} \quad 1 \leq \lambda \leq \infty.
\]

The modified coherent risk measures do not measure risk as monetary loss measure (in currency units), but as some artificial number of losses. It allows penalizing losses above a given level stronger than a linear function. Furthermore, it allows us to use convex penalty function. From a computational point of view, both concepts of risk measures allow to efficiently implement portfolio optimizations since both definitions imply convexity.

### 2.3.2 Conditional Value-at-Risk

In this subsection, we introduce Conditional Value-at-Risk (CVaR) as a risk measure to optimize portfolios so as to reduce the risk of high losses. CVaR is probably the most important coherent risk measure since efficient computational techniques exist. By definition, with respect to a given confidence level \( \theta \), the VaR(\( \theta, u \)) of a portfolio is the minimum amount \( \varphi \) such that the loss will not exceed \( \varphi \) with confidence \( \theta \) for a given portfolio allocation \( u \). The CVaR(\( \theta \)), however, is the conditional expectation of a possible loss exceeding the level \( \varphi \). CVaR has been extensively studied by Rockafellar and Uryasev (2000, 2002). Recently, it has been proven that CVaR is a coherent risk measure which fulfills all of the four above stated properties, see Pflug (2000). Furthermore, Uryasev (2000) shows that minimizing CVaR with a finite number of scenarios can be solved by a linear program (LP) technique. Therefore, a computationally efficient optimization technique exist.
The portfolio losses are given by a function \( f(r,u) \), where \( r \) denotes returns of the portfolio instruments and \( u \) is the allocation of capital into the corresponding portfolio instruments. The probability of \( f(r,u) \) not exceeding a threshold \( \varphi \) is given by

\[
\Upsilon(u, \varphi) = \int_{f(r,u) \leq \varphi} p(r) \, dr,
\]

where \( p(r) \) is the probability density function of \( r \). Here we assume that the density of \( p(r) \) has no discontinuities and \( \Psi(u, \varphi) \) is everywhere continuous with respect to \( \theta \). The case when this assumption is violated is treated in Rockafellar and Uryasev (2002). VaR at confidence level \( \theta \) is defined as

\[
\text{VaR}(u, \theta) = \arg \min_{\varphi \in \mathbb{R}} \Upsilon(u, \varphi) \geq \theta. \tag{2.1}
\]

CVaR is defined as the conditional expectation of a loss being larger equal to VaR\((u, \theta)\). Mathematically it is given as

\[
\text{CVaR}(u, \theta) = \frac{1}{1 - \theta} \int_{f(r,u) \geq \text{VaR}(u, \theta)} f(r,u) p(r) \, dr. \tag{2.2}
\]

The probability of an event exceeding \( \text{VaR}(u, \theta) \) is \( 1 - \theta \) and thus the CVaR formula needs to be normalized by \( 1 - \theta \). Instead of working directly with (2.2), an alternative form is a key to efficiently compute CVaR and VaR

\[
C(u, \varphi, \theta) = \varphi + \frac{1}{1 - \theta} \left( \int_{r \in \mathbb{R}^n} \left[ f(r,u) - \varphi \right]^+ p(r) \, dr \right), \tag{2.3}
\]

where \([t]^+ = \max(t, 0)\). We use (2.3) in order to compute the CVaR. It can be shown that minimizing (2.3) with respect to \( \varphi \) yields the CVaR value and the minimizing argument is the VaR value. Formally we state

\[
\text{CVaR}(u, \theta) = \min_{\varphi \in \mathbb{R}} C(u, \varphi, \theta) \tag{2.4}
\]

\[
\text{VaR}(u, \theta) = \arg \min_{\varphi \in \mathbb{R}} C(u, \varphi, \theta). \tag{2.5}
\]

Furthermore, \( C(u, \varphi, \theta) \) is a convex functional with respect to \((\varphi, u)\) whenever \( f(r,u) \) is a convex function with respect to \( u \), see Rockafellar and Uryasev (2000).

### 2.4 Coherent multi-period risk measures and conclusions for this work

The extension form the one-period case to the multi-period case of risk measures is not straight forward. Currently two approaches exist in the literature. In Artzner, Delbaen,
Eber, Heath and Ku (2004) the authors extend their original definition of coherent risk measures to the multi-period case by defining a so-called risk measure process that fulfills the properties of coherent risk measures given in Section 2.3. Additionally, they add a consistency requirement that demands that judgments based on the risk measure are not contradictory over time. Their results differ from the work of Riedel (2004), who defines similar consistency requirements. The main differences of both works lies in their view of measuring risk at the final period versus measuring risk for all future time periods. Other works on multi-period coherent risk measures include Cvitanic and Karatzas (1999) and Balbas, Garrido and Mayoral (2002) which both define multi-period coherent risk measures for final period values similar to Riedel (2004).

2.4.1 Requirements used in this work

For this work, we define five main properties that any dynamic risk measure $\Gamma_t$ should possess in order to be dynamically coherent. We use the definition of Riedel (2004), since it allows use to use final values and the dynamic consistency requirement can be written as standard backward recursion from dynamic programming.

1. Monotonicity: For all $L_x, L_y$, if $L_x(\omega) \geq L_y(\omega)$, for all $\omega \in \Omega$, then $\Gamma_t(L_x) \geq \Gamma_t(L_y)$.

2. Translation invariance: if $c$ is a constant and $E[r_f]$ a strictly positive expected risk-free return, then for all $L_x$, $\Gamma_t(cE[r_f] + L_x) = \Gamma_t(L_x) - c$.

3. Non-negativity: $L_x > 0$ then $\Gamma(L_x) > 0$ and $\Gamma(0) = 0$.

4. Convexity: for all $L_x, L_y$, $\Gamma_t(\lambda_1 L_x + \lambda_2 L_y) \leq \lambda_1 \Gamma_t(L_x) + \lambda_2 \Gamma_t(L_y)$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_1 + \lambda_2 = 1$.

5. Dynamic consistency: $\Gamma_{t+1}(L_x) = \Gamma_{t+1}(L_y)$ implies $\Gamma_t(L_x) = \Gamma_t(L_y)$.

The first two properties are the same as for coherent single-period risk measures in the sense of Artzner et al. (1999). The next two properties are the weaker form of subadditivity and positive homogeneity, but still account for diversification. The last property states that a portfolio with uncertain results in, say, January 2010 and the knowledge that the portfolio is acceptable in January 2005, we also accept the portfolio in January 2003. The consistency requirement rules out that judgments based on the risk measure are contradictory over time (Riedel 2004). The fifth property allows us to use Bellman’s principle and is therefore suitable for optimization purposes. For example, in Riedel (2004) the following risk measure
\[ \Gamma_t = \mathbb{E} \left[ \sum_{\tau=t}^{T} \frac{-W(\tau)}{(1 + r_f)^{T-\tau}} \bigg| \mathcal{F}(t) \right] \]

is shown to be dynamically coherent. This measure is also positive homogenous and subadditive. Here \( W(\tau) \) denotes the portfolio wealth, \( r_f \) is a risk-free interest rate, \( \mathcal{F}(t) \) denotes the information at time \( t \). This risk measure can be written by the standard backward recursion and is obtained by

\[ \Gamma_\tau = \mathbb{E} \left[ -W(\tau) + \frac{\Gamma_{\tau+1}}{1 + r_f} \bigg| \mathcal{F}(t) \right], \quad \Gamma_T = \mathbb{E} \left[ \frac{-W(T)}{(1 + r_f)^{T-t}} \right]. \]

Whenever, a dynamic risk measure can be computed by this type of backward recursion, the the risk measure fulfills the dynamic consistency requirements.

### 2.4.2 Discussions of risk measures for different modelling frameworks

The application of the requirements of the dynamically coherent risk measures depends on the underlying modelling framework of asset prices and depends on whether we use final values or processes.

In the continuous-time case (Section 3.1), where we limit our modelling framework to stochastic differential equations driven by Brownian motions and where we limit our objective functions to the use of utility functions. In this modelling framework, any ranking of decisions which includes the variance of the decision is dynamically coherent. In this world, any probability distribution of a model is derived from the normal distribution. Even in the case of non linear models, e.g., square root processes, the distribution can be constructed by a transformation of normal variables. Also in the case of stochastic volatility models, the distributions are known as normal-variance mixtures, see McNeil et al. (2005, Chapter 3), which are elliptical distributions. Therefore, any ranking that uses the variance and can be written by a backward recursion is dynamically coherent and suitable for our purposes.

In the case of discrete-time methods presented in Section 4.2, the use of coherent risk measures depends on the probability distribution of the portfolio in the future. In the case that the portfolio is normally distributed, we use the variance at the terminal date as the risk measure. Alternatively, utility functions could be employed. In the case of non Gaussian white noise processes, we use a CVaR at terminal date. In the discussion of Artzner et al. (2004), CVaR is not a dynamically coherent risk measure since it is not
well defined on processes. In this work however, we take the view of Riedel (2004) and therefore, use risk measures that are only defined for final values and still can be seen as dynamically coherent. In Section 4.3, we introduce a risk measure for multi-period risk measurements that is dynamically coherent in the sense of Riedel (2004), similar to the example given in previous Subsection 2.4.1.
Continuous-time models and portfolio optimization

In general, the problem of long-term investments is a well-established research field since the work of Samuelson and Merton, see Samuelson (1969), Merton (1969), Merton (1971) and Merton (1973). Since then it is well understood that a short-term portfolio optimization can be very different from long-term portfolio optimization. In Merton (1973), where a continuous-time asset model with stochastic factors is described, the most important financial economic principles are established, using stochastic optimal control theory. Due to the very general model formulation, Merton did not give explicit results for portfolio choice problems. The paper highlights the difficulties in solving complex cases of asset dynamics with stochastic factors, because one has to solve a high-dimensional non-linear PDE.

Advances in numerical techniques and the growth of computing power led to the development of numerical solutions to multi-period portfolio optimization problem, where the optimization problem is solved by a discrete state approximation. Examples of this line of research are Brennan, Schwartz and Lagnado (1997) and Brennan and Schwartz (1999), Balduzzi and Lynch (1999), or Lynch (2001). The use of numerical dynamic programming is very often restricted to few factors, due to the fact that numerical procedures become slow and unstable (unreliable) in high-dimensions. Systematic research in field numerical approximation to stochastic dynamic programming show that even today’s problems with more than three state variables are extremely hard to solve, as shown in Peyrl (2003).

Closed-form solutions of the Merton model in continuous time have been discovered in the case of one stochastic the following authors: Kim and Omberg (1996) derive an analytical solution to the portfolio selection problem with utility defined over terminal wealth when the risk premium is stochastic and governed by an Ornstein-Uhlenbeck process. Canestrelli and Pontini (2000) studied examples of utility maximization of wealth at fi-
nite date, where one stochastic factor was taken into account. Haugh and Lo (2001) as well as Campbell, Rodriguez and Viceira (2004) present an analytical solution for the case in which the expected returns are governed by a continuous-time auto-regressive process, but with different investment objectives. Korn and Kraft (2001) solves portfolio problems with stochastic interest rates and utility maximization over terminal wealth. More complicated models with two or three factors in continuous-time, with utility maximization over terminal wealth, where the closed-form solution was derived, include Brennan and Yihong (2001) or Munk, Soerensen and Vinther (2004). An extensive study on solvable portfolio optimization problems with stochastic factors are given by Jun (2001) which includes the class of problems considered in Section 3.2. An extensive survey on stochastic control methods in finance can be found in Runggaldier (2003), where many problems of portfolio optimization are discussed. The paper of Schroder and Skiadas (2003) discuss the general conditions of optimality for portfolio strategies with arbitrary continuous price dynamics and convex trading constraints. The authors derive analytical solutions to class of parametric models, which includes many of the analytical solutions discussed in this chapter.

This chapter is organized as follows. In Section 3.1, we discuss the general modelling framework for asset price and introduce the portfolio dynamics. We show that the problem of portfolio optimization in continuous time is a stochastic optimal control problem and state the necessary conditions. Furthermore, for problems of maximizing the utility of terminal wealth we show the connection to risk-sensitive control methods. The section concludes with an example of portfolio optimization where we derive the analytical solution.

In Section 3.2, we solve the important problem of optimizing a linear portfolio. The mean returns are directly affected by financial and economic factors. These factors are modelled as stochastic Gaussian processes and the mean return of the assets are an affine function of the factor levels. The portfolio optimization problem with maximizing expected power and the exponential utility over terminal wealth is solved in closed form for \( m \) factors and \( n \) risky assets. Moreover, we also derive the optimization under partial information where we assume that not all of the factors are directly observable to the investor. The method is illustrated in a small case study with German data.
In Section 3.3, we discuss numerical methods to solve the Hamilton-Jacobi-Bellman equation. We derive the so-called successive approximation algorithm and prove the convergence. The implementation of the numerical method is discussed and a portfolio optimization example is solved with the numerical solver. A discussion on the applicability of the numerical methods concludes this section.

In Section 3.4, we discuss the general strength and weaknesses of continuous-time portfolio optimization. We point out the problems, e.g., difficulties to deal with constraints, no modelling of transaction costs, and discuss optimization strategies that follow from continuous-time results.

3.1 Portfolio optimization in continuous-time and optimal control

Following Merton’s approach, we introduce a modelling framework to describe asset price dynamics by stochastic differential equations (SDE). The asset price dynamics are embedded in a modelling structure that describes the dynamics of economics and financial factors which describe the state of the economy. The instantaneous expected returns and instantaneous volatility of the asset price dynamics are functions of the factors. The economic environment itself is modelled as a stochastic process and thus the drift and diffusion of the asset price dynamics might be stochastic in turn. An investor attempts to maximize his risk adjusted returns by maximizing a so-called utility function. The problem of portfolio optimization represents a stochastic optimal control problem. The general case of computing the optimal decisions about the asset allocation is found by solving the so-called Hamilton-Jacobi-Belman equation (HJB).

3.1.1 Asset price models

In order to model assets traded on an organized exchange, we make the following assumptions:
1) Trading is continuous.
2) There are no transaction costs, fees, or taxes.
3) The investor is a price taker and does not possess enough market power to influence prices.
4) Short selling of assets is allowed and the investor can borrow and save money at the same short-term interest rate.

We regard a market in which $n$ risk-bearing investments exist. The asset price processes $(P_1(t), P_2(t), \ldots, P_n(t))$ of the risk-bearing investments satisfy the stochastic differential equation

\[ \frac{dP_i(t)}{P_i(t)} = \mu_i(t, x(t)) dt + \sigma_i(t, x(t)) dZ_P(t) \]  
\[ P_i(0) = p_{i0} > 0. \]

Here $\mu_i(t, x(t)) \in \mathbb{R}$ is the relative expected instantaneous change in price of $P_i$ per unit time and $\sigma_i(t, x(t)) \sigma_i(t, x(t))^T$ is the instantaneous variance per unit time ($\sigma_i \in \mathbb{R}^{1 \times n}$ is the $i$-th row of the matrix $\sigma(t, x(t)) \in \mathbb{R}^{n \times n}$). The $n$-dimensional Brownian motion $Z_P(t)$ is defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$. The stochastic differential equation has the shape of a generalized geometric Brownian motion with possibly stochastic drift and diffusion. This model reflects important real stock or bond price characteristics (e.g., no negative prices are observed, variance (volatility) is stochastic, and expected returns are time-varying).

By adding a further, locally "risk-free" asset or rather a bank account with a short-term interest rate, with diffusion $\sigma_0 \equiv 0$, and instantaneous rate of return $\mu_0(t, x(t))$ referred to as $r_0(t, x(t))$, we obtain a risk free asset $P_0(t)$ given by the differential equation

\[ \frac{dP_0(t)}{P_0(t)} = r_0(t, x(t)) dt \]  
\[ P_0(0) = p_{00} > 0. \]

The process $x(t)$ may describe what often is called the exogenous factors, which have the economic interpretation as the "state of the economy". Therefore, we simple refer to the process $x(t)$ as the factor process.

\[ dx(t) = \Theta(t, x(t)) dt + \Psi(t, x(t)) dZ_x(t) \]  
\[ x(0) = x_0, \]
where \( x(t) \in \mathbb{R}^m, \Theta(t, x(t)) \in \mathbb{R}^m, \Psi(t, x(t)) \in \mathbb{R}^{m \times m}, \) and \( Z_x(t) \in \mathbb{R}^m. \) The \( m \)-dimensional Brownian motion \( Z_x(t) \) is defined on a fixed, filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\). The correlation matrix between \( Z_P(t) \) and \( Z_x(t) \) is \( \rho(t, x(t)) \in \mathbb{R}^{n \times m} \).

In the case when \( \mu_i(t, x(t)) \) and \( \sigma_i(t, x(t)) \) are constant and independent of \( x(t) \), the price process is a geometric Brownian motion. The asset price model which includes the factor dynamics describes an incomplete financial market, i.e., the investor faces more uncertainty (Brownian motion) than traded assets.

The process \( x(t) \) allows us to model variables that describe external influence (explanatory) factors of either macroeconomic, industry specific, or company specific nature. By carefully selecting external variables with some predictive capacity, we can model the time-varying features of the asset price dynamics. The framework is very broad and thus allows us to model different types of securities, such as stocks, bonds, options, etc. The setup of equations imposes a structure to model asset price dynamics. Moreover, we have made no assumptions so far about the SDE of the factors or on whether the factors directly observable or hidden.

### 3.1.2 Wealth dynamics and portfolio models in continuous-time

Based on the asset price dynamics stated in Section 3.1.1, we state the portfolio dynamics.

#### Self-financing portfolios and wealth dynamics

The wealth \( W(t) \) and share price \( P_i(t) \) are known at time \( t \). Further let \( N_i(t) \) be the number of shares of asset \( i \) purchased and let \( q(t) \) be the amount of in- or outflow (consumption) per time unit. In addition, we assume that no restrictions exist on short-selling of assets, as well as that we can borrow and save money at a single short-term interest rate. The wealth (portfolio value) \( W(t) \) at time \( t \) is thus computed by

\[
W(t) = \sum_{i=0}^{n} N_i(t) P_i(t) .
\]  

(3.4)

Differentiating (3.4), using Itô calculus we obtain

\[
dW(t) = \sum_{i=0}^{n} N_i(t) dP_i(t) + \sum_{i=0}^{n} dN_i(t) P_i(t) + \sum_{i=0}^{n} dN_i(t) dP_i(t) .
\]  

(3.5)

In equation (3.5) we interpret the leftmost term, \( \sum_{i=0}^{n} N_i(t) dP_i(t) \), as the income (or loss) from capital gains (or losses) resulting from stock price changes. The other terms,
\[ \sum_{i=0}^{n} dN_i(t)P_i(t) + \sum_{i=0}^{n} dN_i(t)dP_i(t) \], can be interpreted as income (or loss) from non-capital gains (or losses) such as from external sources or drains. In our model, we substitute \( \sum_{i=0}^{n} dN_i(t)P_i(t) + \sum_{i=0}^{n} dN_i(t)dP_i(t) = q(t) \) with the argumentation that those terms are equivalent to the consumption of wealth or external funding. The derivation and interpretation of the wealth dynamics are given by Merton (1992, Chapter 4). The differentiation of \( W(t) \) found in (3.5) then degenerates as follows:

\[
\begin{align*}
dW(t) &= \sum_{i=0}^{n} N_i(t)dP_i(t) + q(t)dt \\
&= \sum_{i=0}^{n} N_i(t)P_i(t)\mu_i(t, x(t)) dt \\
&\quad + \sum_{i=0}^{n} N_i(t)P_i(t)\sigma_i(t, x(t))dZ_{P_i}(t) + q(t)dt ,
\end{align*}
\]

where \( \sigma_0(t, x(t)) = 0 \) and \( \mu(t, x(t)) = r_0(t, x(t)) \). The initial value of \( W(t) \) can be computed using (3.4) at time \( t = 0 \) which yields \( W(0) = \sum_{i=0}^{n} N_i(0)p_{i0} = W_0 \). By defining a new variable \( u_i(t) = N_i(t)P_i(t)/W(t) \), the fraction of wealth invested in the \( i \)th asset at time \( t \), we can eliminate \( N_i(t) \) to obtain

\[
\begin{align*}
dW(t) &= \sum_{i=0}^{n} u_i(t)W(t)\mu_i(t, x(t)) dt + \sum_{i=0}^{n} u_i(t)W(t)\sigma_i(t, x(t))dZ_{P_i}(t) + q(t)dt .
\end{align*}
\]

By definition \( \sum_{i=0}^{n} u_i(t) = 1 \), we can rewrite equation (3.7) as

\[
\begin{align*}
dW(t) &= \sum_{i=1}^{n} u_i(t)(\mu_i(t, x(t)) - r_0(t, x(t)))W(t) dt + r(t, x(t))W(t) dt \\
&\quad + \sum_{i=1}^{m} u_i(t)W(t)\sigma_i(t, x(t))dZ_{P_i}(t) + q(t)dt ,
\end{align*}
\]

where \( u_0(t) = 1 - \sum_{i=1}^{n} u_i(t) \). For convenience we replace the sums with vector notation and rewrite the portfolio dynamics as

\[
\begin{align*}
dW(t) &= u(t)^T(\mu(t, x(t)) - \mathbf{1}r_0(t, x(t)))W(t) dt + r(t, x(t))W(t) dt \\
&\quad + W(t)u(t)^T\sigma(t, x(t))dZ_{P}(t) + q(t)dt ,
\end{align*}
\]

where \( u(t) = (u_1(t), \ldots, u_n(t))^T \in \mathbb{R}^n \), \( \mathbf{1} = (1, \ldots, 1)^T \in \mathbb{R}^n \),

\[ \mu(t, x(t)) = (\mu_1(t, x(t)), \ldots, \mu_n(t, x(t)))^T \in \mathbb{R}^n \], and initial condition \( W(0) = W_0 \).

### 3.1.3 Portfolio optimization and the Hamilton-Jacobi-Bellmann theory

In this section, we introduce the investment objectives and the resulting portfolio optimization problem. We show that the portfolio optimization problem is a stochastic con-
3.1 Portfolio optimization and optimal control

control problem which can be successfully treated by the Hamilton-Jacobi-Belmann theory. For the case of unconstrained control variables, we state the partial differential equation (PDE) that yields the solution to the portfolio optimization problem.

Investment objectives and optimization problem

We assume that the investor has a fixed endowment of capital at the beginning \((W_0)\), draws money from his wealth for consumption \((q(t) \leq 0)\), and invests with a fixed terminal date \(T\). We assume that there is no influx of capital from any other sources. The objective of the portfolio optimization problem is thus given by

\[
J = \max_{u(\cdot), q(\cdot) \leq 0} \left\{ \mathbb{E} \left[ \int_0^T U_1(q(t)) dt + U_2(W(T)) \right] \right\},
\]

where \(U_1\) and \(U_2\) are strictly concave utility functions that include the appropriate discount factors, i.e., the risk-free discount rate for the investment horizon. This objective maximizes both the utility of the consumption and the utility of the remaining terminal wealth \(W(T)\). A possible variation of this objective is to maximize consumption for an infinite horizon \(\mathbb{E}\{\int_0^\infty U_1(q(t)) dt\}\). In the case in which the investor chooses not to consume any of his wealth until the end of the investment period and therefore, maximizes the utility of wealth at time \(T\), the objective is \(\mathbb{E}\{U_2(W(T))\}\). The second case is more relevant in our view, since it characterizes more realistically the problem of a professionally managed investment funds, which does not distribute any money to shareholders.

Investment funds invest for a fixed investment horizon without any consumption from the portfolio.

We now have all the elements needed in order to mathematically state the portfolio optimization problem an investor faces. For the portfolio dynamics

\[
dW(t) = u(t)^T (\mu(t, x(t)) - \mathbb{1}_r(t, x(t))) W(t) dt + r_0(t, x(t)) W(t) dt \\
+ W(t) u(t)^T \sigma(t, x(t)) dZ_P(t) + q(t) dt
\]

\[
W(0) = W_0,
\]

and the factor dynamics that describe the state of the economy

\[
dx(t) = \Theta(t, x(t)) dt + \Psi(t, x(t)) dZ_x(t) \tag{3.12}
\]

\[
x(0) = x_0
\]

\[
dZ_P dZ_x = \rho(t, x(t)) dt,
\]
find the optimal control \( u^*(\cdot) \), \( q^*(\cdot) \leq 0 \) such that
\[
E \left[ \int_0^T U_1(q(t)) dt + U_2(W(T)) \right] \tag{3.13}
\]
is maximized. The optimization problem thus is a stochastic optimal control problem where the optimal asset allocation and the optimal consumption are determined simultaneously.

**Stochastic optimal control and portfolio optimization**

The problem of portfolio optimization can be successfully solved by the theory of stochastic optimal control where the HJB theory is instrumental for finding a solution. The HJB equation for this particular problem of portfolio optimization is obtained by

\[
\max_{u(\cdot), q(\cdot) \leq 0} \left[ J_t(t, W(t), x(t)) + U_1(q(t)) + \Theta(t, x(t))^T J_x(t, W(t), x(t)) + \frac{1}{2} \text{tr}\{\Psi(t, x(t)) \Psi^T(t, x(t)) J_{xx}(t, W(t), x(t))\} \right. \\
\left. + J_W(t, W(t), x(t)) \left( u(t)^T (u(t) - \frac{1}{2} r_0(t, x(t))) W(t) + r(t, x(t)) W(t) + q(t) \right) \right. \\
\left. + \frac{1}{2} W^2(t) J_{WW}(t, W(t), x(t))(u^T(t) \Psi(t, x(t))) + \frac{1}{2} W^2(t) J_{WW}(t, W(t), x(t)) u^T(t) \Sigma(t, x(t)) u(t) \right. \\
\left. + W(t) u^T(t) \sigma(t, x(t)) \rho(t, x(t)) \Psi^T(t, x(t)) J_{Wx}(t, W(t), x(t)) \right] = 0 \\
J(T, W(T), x(T)) = U_2(W(T)) \tag{3.14}
\]

where \( \Sigma(t, x) = \sigma(t, x) \sigma(t, x)^T \). This HJB equation can be derived by using the general HJB for stochastic control problems as stated Yong and Zhou (1999). The result given by (3.14) is also derived in Merton (1992) or Jun (2001).

In the next step, the first-order optimality conditions for \( u(t) \) and \( C(t) \) are computed and given by
\[
u^*(\cdot) = -\Sigma(t, x(t))^{-1} J_{W}^{-1}(t, W(t), x(t)) W^{-1}(t) \left[ (\mu(t, x(t)) - \frac{1}{2} r_0(t, x(t))) \right. \\
\left. \cdot J_W(t, W(t), x(t)) + \sigma(t, x(t)) \rho(t, x(t)) \Psi(t, x(t)) J_{Wx}(t, W(t), x(t)) \right] \tag{3.15}
\]
\[
U_{1q}(q^*(\cdot)) = J_W(t, W(t), x(t)) \tag{3.16}
\]

where \( U_{1q} \) denotes the derivative of \( U_1 \) with respect to \( q \). \( u^*(\cdot) \) denotes the optimal portfolio weight, and \( q^*(\cdot) \leq 0 \) denotes the optimal consumption. The portfolio weight \( u^*(\cdot) \) can be interpreted as follows:
• The first term,
\[-\Sigma(t, x(t))^{-1} J_{WW}^{-1}(t, W(t), x(t)) W^{-1}(t) (\mu(t, x(t)) - 1 r_0(t, x(t)))\]
\[-J_W(t, W(t), x(t)),\]
represents the mean-variance portfolio weight. It is commonly known as myopic demand, as it describes the portfolio weight chosen by an investor who has a single-period objective or a very short-term investment horizon. If \(\mu(t, x(t)), \Sigma(t, x(t))\) and \(r(t, x(t))\) are time-varying (and stochastic), the first term is not a static portfolio selection but a time-varying feedback controller.

• The second term, \(-\Sigma(t, x)^{-1} J_{WW}^{-1}(t, W, x) W^{-1}(t) \sigma(t, x) \rho(t, x) \Psi^T(t, x) J_{Wx}(t, W, x),\) accounts for changes in the mean and variance of the investment returns. For this reason Merton (1992) calls this term the inter-temporal hedging demand. Note that this term is zero in the case of uncorrelated Brownian motions of the factors and asset price dynamics. In this case the myopic demand governs the portfolio decisions.

The myopic demand is characterized by investing with a very short-term perspective that neglects that the risk-return pattern change over the investment horizon. The inter-temporal hedging demand corrects the myopic decision by adjusting the decision for the long-term changes in the risk-return pattern.

Substituting the expression for optimal consumption \(q^*(\cdot)\) and optimal portfolio choice \(u^*(\cdot)\) back into the HJB equation (3.14) we obtain

\[
J_t(\cdot) + U_1(q^*(\cdot)) + \Theta(t, x(t))^T J_x(\cdot) + \frac{1}{2} \text{tr} \{ J_{xx}(\cdot) \Psi(t, x(t)) \Psi^T(t, x(t)) \} + (r_0(t, x(t)) W(t) - q^*(\cdot)) J_W(\cdot) - \frac{1}{2} J_{WW}^{-1}(\cdot)(\mu(t, x(t)) - 1 r_0(t, x(t)))^T \Sigma^{-1}(t, x(t)) (\mu(t, x(t)) - 1 r_0(t, x(t))) - (\mu(t, x(t)) - 1 r_0(t, x(t)))^T \Sigma^{-1}(t, x(t)) \sigma(t, x(t)) \rho(t, x(t)) \Psi^T(t, x(t)) J_{Wx}(\cdot) - \frac{J_{Wx}(\cdot)}{2 J_{WW}(\cdot)} g(t, x(t)) \rho^T(t, x(t)) \rho(t, x(t)) \Psi^T(t, x(t)) J_{Wx}(\cdot) = 0 ,
\]
with terminal condition \( J(T, W(T), x(T)) = U_2(W(T)) \) and we suppressed the arguments of \( J(\cdot) \) for writing convenience. In order to compute the optimal feedback controller, we need to solve (3.17) which is a highly non-linear PDE. Only a few analytical solutions are known, such as the Merton Problem (Duffie 1996, Chapter 9) or the LQG Problem (Yong and Zhou 1999, Chapter 6). Very often one has to resort to numerical methods to solve the optimal control problem.
3.1.4 Remarks and observations for problems without consumption

In this part, we show the connection of portfolio optimization, as outlined in Section 3.1.3, with other optimal control problems. Especially, we consider the portfolio optimization situation without consumption, where the investor is only concerned with maximizing the utility at given terminal date in future.

Maximization of wealth at terminal date with power utility.

We consider a portfolio optimization without consumption \( q(t) = 0 \) and power utility for the remaining terminal wealth, i.e., \( U_2(W(T)) = \frac{1}{\gamma} W^\gamma(T) \). This utility has the property that any portfolio optimization is independent of the current wealth, since the Arrow-Pratt measure of relative risk aversion is constant, see Luenberger (1998, Chapter 9). In order to solve the portfolio optimization problem, we need to solve the associated HJB equation (3.14). Furthermore, we make no specific assumptions on the factor process.

Using the method of separation, we attempt to find a solution the HJB equation by the product from

\[
J(t, W(t), x(t)) = \frac{1}{\gamma} W^\gamma(t) l(t, x(t)),
\]

where \( l(t, x) \in \mathbb{R} \). This Ansatz separates the wealth from the factor levels by the product from. Using this transformation in (3.14) we obtain after some algebraic manipulations

\[
l_t(t, x(t)) + \max_u \left[ \Theta(t, x(t))^T l_x(t, x(t)) + \frac{1}{2} \text{tr} \{ \Psi(t, x(t)) \Psi^T(t, x(t)) l_{xx}(t, x(t)) \} ight.
\]

\[
\left. + \gamma \left( u(t)^T (\mu(t, x(t)) - 1 r_0(t, x(t))) + r_0(t, x(t)) \right) + \frac{1}{2} (\gamma - 1) u^T(t) \Sigma(t, x(t)) u(t) \right] l(t, x(t))
\]

\[
+ \gamma u^T(t) \sigma(t, x(t)) \rho(t, x(t)) \psi^T(t, x(t)) l_x(t, x(t)) = 0
\]

\[
l(T, x(T)) = 1.
\]

This type of HJB equation is commonly found in so-called risk-sensitive optimal control problems. The equivalent optimization problem that leads to (3.19) is given by

\[
\max_{u(\cdot)} \mathbb{E} \left[ e^{\gamma \int_0^T u^T(t)(\mu(t, x(t)) - \frac{1}{2} r_0(t, x(t))) + r_0(t, x(t)) + \frac{1}{2} (\gamma - 1) u^T(t) \Sigma(t, x(t)) u(t) du) \right]
\]

s.t.

\[
dx(t) = (\Theta(t, x(t)) + \gamma \Psi(t, x(t)) \rho^T(t, x(t)) \sigma(t, x(t)) u(t)) dt
\]

\[
+ \psi(t, x(t)) dZ_x(t).
\]

(3.20)
The optimization problem (3.20) is solved by finding a solution to (3.19). This HJB equation can be derived with the help of the theory of risk-sensitive control, see Whittle (1990). The risk sensitive optimal control problem (3.20) leads to the following remarks and observations:

- In the case of uncorrelated factors and asset dynamics ($\rho = 0$), the SDE in the risk sensitive control problem is a system which is not affected by the control variable. We can then solve the problem by simply maximizing the objective function which results in the myopic demand as outlined in Section 3.1.3. In this case, the handling of constraints for the decision variable is straight-forward.

- In the case that $\Theta(t, x(t)), \mu(t, x(t)), \text{and } r_0(t, x(t))$ are affine functions of the factor levels and $\rho(t, x(t)), \Psi(t, x(t)), \text{and } \sigma(t, x(t))$ are simply functions of the time, the risk sensitive problem is the well known linear-exponential-quadratic-Gaussian (LEQG) problem. The LEQG problem is introduced and solved by Jacobson (1973). In this case, we solve the portfolio optimization problem which we discuss in detail in Section 3.2.2.

- Other cases in which analytical solutions to the risk-sensitive control problem can be derived are given by Charalambous (1997). This could be used to find solutions to non-linear portfolio optimization problems.

- As shown in Pra, Menehini and Runggaldier (1996) or Jacobson (1973), the certainty equivalent to risk-sensitive control problems (without constraints on $u(t)$) are deterministic differential games. Therefore, we can state the dynamic differential game that is equivalent to the problem portfolio optimization under consideration.

This comparison of portfolio optimization with risk-sensitive control problems helps us to understand in which case the underlying portfolio problem is solvable. Furthermore, this puts the stochastic optimal control problem for portfolio optimizations into a wider optimal control context. The method of risk-sensitive control is successfully used to solve portfolio optimization problem, see Bielecki and Pliska (1999) or Fleming and Sheu (2005).

**Maximization of wealth at terminal date with exponential utility.**

Again, we consider a portfolio optimization without in- or outflows ($q(t) = 0$) and exponential utility for the terminal wealth, i.e., $U_2(W(T)) = -\frac{1}{\gamma} e^{-\gamma W(T)}$. This utility has
the property that any portfolio optimization depends on current wealth, since the Arrow-Pratt measure of absolute risk aversion is constant. Using the method of separation, we attempt to find a solution to the HJB equation by the product form

\[ J(t, W(t), x(t)) = -\frac{1}{\gamma} e^{-\gamma W(t)} l(t, x(t)), \]  

(3.21)

where \( l(t, x) \in \mathbb{R} \). We plug (3.21) in (3.14). We define \( \pi(t) = u(t)W(t) \) and \( \tau(t) = r_0(t)W(t) \) with the assumption that the interest rate is not a function of the factors and obtain after some manipulations

\[
l_t(t, x(t)) + \max_{\pi(t)} \left[ \Theta(t, x(t))^T l_x(t, x(t)) + \frac{1}{2} \text{tr}\{\Psi(t, x(t)) \Psi^T(t, x(t)) l_x(t, x(t))\} 
\right.
\]

\[
- \gamma \left( \pi(t)^T (\mu(t, x(t)) - 1r_0(t)) + \tau(t) \right) 
+ \frac{1}{2} \gamma \pi^T(t) \Sigma(t, x(t)) \pi(t) \right] l(t, x(t)) 
\]

\[
- \gamma \pi^T(t) \sigma(t, x(t)) \rho(t, x(t)) \Psi^T(t, x(t)) l_x(t, x(t)) \right] = 0
\]

(3.22)

By comparison of (3.22) with (3.19), we easily see that (3.22) can be written as risk-sensitive optimal control problem. The main difference to (3.19) is the fact that \( \pi(t) \) denotes absolute investments in the various assets and not anymore relative investments. The relative investments depend on the current value of the portfolio. Most of the observations and remarks given above also apply to this case. Especially, by choosing \( \Theta(t, x(t)) \) and \( \mu(t, x(t)) \) as affine function of the factor levels, and \( \rho(t, x(t)), \Psi(t, x(t)), \tau(t) \) and \( \sigma(t, x(t)) \) as function of time, this optimal control problem admits the same solution as the LEQP problem.

3.1.5 Example: Analytical portfolio optimization with stochastic volatility

The method to solve portfolio optimization problems, presented in Section 3.1.3, is illustrated in example where the volatility of risky assets is stochastic.

Asset model

We assume that the diffusion of the securities depends on a factor variable which describes the stochastic market volatility. The model of the risk bearing investments is given by

\[
\frac{dP_i(t)}{P_i(t)} = \mu_i(t)dt + \sqrt{\frac{1}{\nu(t)} \pi_i dZ_P}, \quad P_i(0) = p_{i0},
\]
where $\mu_i(t) \in \mathbb{R}$ is the expected return, $\sigma_i(t) \in \mathbb{R}^{1 \times n}$ is the dependence structure among the $n$ securities, $\sqrt{\frac{1}{v(t)}} \in \mathbb{R}$ is a factor describing the total market volatility, and $Z_P \in \mathbb{R}^{n \times 1}$ denotes the Brownian motion. The instantaneous covariance of the asset price dynamics is given by

$$\text{Cov} \left[ \frac{dP_i(t)}{P_i(t)} \right] = \frac{\sigma_i \sigma_i^T}{v(t)}.$$ 

The model for the inverse of total market volatility, also called precision, is given by

$$dv(t) = \kappa(t)(\theta(t) - v(t))dt + \nu(t)\sqrt{v(t)}dZ_v, \quad v(0) = v_0,$$

where $\kappa(t) \in \mathbb{R}$ denotes the mean reversion speed, $\theta(t) \in \mathbb{R}$ the long term mean value, $\nu(t) \in \mathbb{R}$ the diffusion parameter, and $Z_v \in \mathbb{R}$ denotes a Brownian motion. Furthermore, we assume that the Brownian motions are correlated, $dZ_PdZ_v = \rho dt \in \mathbb{R}^{n \times 1}$. This model is introduced in Chacko and Viceira (2005), as a solvable class of stochastic volatility models. This setup is mainly motivated by the observation that market volatility is time-varying and stochastic. We assume that the correlation structure between securities is stable, but the volatility of each individual security rises and falls with total market volatility. The state variable $v(t)$ can be interpreted as scaling factor of the covariance matrix.

We also assume that a risk-free bank account is available with interest rate $r_0(t)$ governed by the dynamics given in (3.2).

### Portfolio optimization problem with stochastic volatility

The portfolio optimization problem is maximizing the power utility function of the terminal wealth. Using (3.9), mathematically the problem is obtained by

$$\max_{u(t) \in \mathbb{R}^n} \mathbb{E}\left[ \frac{1}{\gamma} W^\gamma(T) \right]$$

where $W(t) = W(0) + \int_0^t u(s) \sigma(s) dZ_P(s) + \int_0^t \sqrt{\frac{1}{v(s)}} u(s) dZ_v(s)$, $W(0) = W_0$, $dZ_PdZ_v = \rho dt$, $dZ_PdZ_v = \rho dt$, $dZ_v = \kappa(t)(\theta(t) - v(t))dt + \nu(t)\sqrt{v(t)}dZ_v$, $v(0) = v_0$, $\gamma < 1$ denotes the coefficient of risk aversion. Using the results of (3.16) the first order optimality conditions are

$$u(\cdot) = -\Sigma^{-1}W^{-1}J_W^{-1}v[\mu - \frac{1}{2}r_0(t)] + \sigma \rho \nu J_{W_v}.$$
Using (3.17), the PDE that yields the solution to the portfolio optimization problem is given

\[ J_t + rWJ_W - \frac{1}{2} J_{WW}^2 v(\mu - 1r_0)^T \Sigma^{-1} (\mu - 1r_0) - \frac{1}{2} J_{WW}^2 v\rho^T \rho vJ_{Wv} - \frac{1}{2} J_{WW}^- v(\mu - 1r_0)^T \Sigma^{-1} \sigma \rho v + \kappa (\theta - v)J_v + \frac{1}{2} \nu^2 vJ_{vv} = 0. \] (3.23)

We conjecture the following functional form of the solution

\[ J = \frac{1}{\gamma} W^\gamma e^{a(t)+b(t)v(t)}, \]

with terminal conditions \( a(T) = 0 \) and \( b(T) = 0 \). The conjecture is put into (3.23) and yields two coupled scalar ODEs:

\[ \dot{a}(t) + r\gamma + \kappa \theta b(t) = 0 \] (3.24)

\[ a(T) = 0 \]

\[ \dot{b}(t) - \frac{\gamma}{2(\gamma - 1)} [(\mu - 1r)^T \Sigma^{-1} (\mu - 1r) + b^2(t)\nu^T \rho \nu + 2(\mu - 1r)^T \Sigma^{-1} \sigma \rho v + \frac{1}{2} \nu^2 b^2(t) - \kappa b(t) = 0. \] (3.25)

\[ b(T) = 0 \]

The controller can be computed by solving (3.25). By using the functional form of the solution, we calculate the explicit form of the control law, which is obtained by

\[ u(t, v) = \Sigma^{-1} \frac{v(t)}{1 - \gamma} [(\mu - 1r_0) + \sigma \rho v b(t)]. \]

Notice that the controller is linear in \( v(t) \) and the time varying function \( b(t) \) is governed by a Riccati equation. The function \( b(t) \) governs the inter-temporal hedging demand.

Since the model assumes constant expected return for the risky assets but time-varying and stochastic volatility, the controller scales the investments into the risky assets by the volatility. In periods of high volatility, the risk exposure is reduced and in other times it is increased. Furthermore, the inter-temporal hedging demand accounts for the correlation of the precision and the risky asset dynamics dynamics. In the case of negative correlations, i.e., volatility rises when stock market returns are falling, the inter-temporal hedging demand reduces the risk exposure \( (b(t) > 0) \), since the long-term risks are larger than the short-term risks.
3.2 Portfolio optimization with affine stochastic mean returns and interest rates

In this section we propose a multi-dimensional asset price model consisting of $m$ factors that influence the mean return of $n$ risk-bearing assets. The $m$ factors are modelled as Gaussian stochastic processes. The mean returns of the risk-bearing assets are an affine function of the factor levels. Additionally, the risk-free interest rate is also modelled as an affine function of the factor levels. The residuals of the factor dynamics can be correlated to residuals of the asset dynamics. The model introduced in this thesis resembles the models used Bielecki, Pliska and Sherris (2000) in continuous-time and by Campbell, Chan and Viceira (2003) in discrete time. Our model differs from Bielecki et al. (2000) in that the factors and the asset dynamics are correlated and an infinite-time objective is maximized and from Campbell et al. (2003) that their model is set in discrete time and they optimize an infinite-time Eppstein-Zin utility.

First, we solve the portfolio optimization (choice) problem with this model, when the investor maximizes a constant relative risk aversion (CRRA), i.e., power utility over terminal wealth. We derive a solution explicitly for the optimal portfolio weights and the value function via Riccati equations. Second, we propose the solution to the portfolio optimization problem with constant absolute risk aversion (CARA), i.e., exponential utility over terminal wealth, when risk-free interest rate is constant. At first, we solve the portfolio optimization problems under the assumption of full information of the factors. Secondly, we consider the case of partial information of the factor levels. We derive the filter equations and state the equivalent portfolio optimization problem with full information. In a short case study, the method for CRRA utility maximization under full information is applied to German data, with two assets and four factors.

3.2.1 Model setup

Given the general modelling setup in Section 3.1.1, we state the specific forms for the drift and the diffusion of the factor process, the expected returns, and the risk-free interest rate. In this case, we assume a general affine framework where most of the above mentioned parameters are modelled by affine functions. As stated in Section 3.1.4, we know that in this case, we are able to obtain analytical solutions to the problems of maximizing the terminal wealth of a portfolio.
Asset model assumptions

We assume that the drift terms of (3.1) and (3.2) are affine functions of the factor levels, as given by

$$\mu_i(t,x(t)) = G_i(t)x(t) + g_i(t),$$  \hspace{1cm} (3.26)
$$r_0(t,x(t)) = F_0(t)x(t) + f_0(t),$$  \hspace{1cm} (3.27)

where $x(t) = (x_1(t), x_1(t), \ldots, x_m(t))^T \in \mathbb{R}^m$, $G_i(t), F_0(t) \in \mathbb{R}^{1 \times m}$, and $g_i(t), f_0(t) \in \mathbb{R}$.

The diffusion of the risky asset is assumed to be independent of $x(t)$ and given by

$$\sigma_i(t,x(t)) = \sigma_i(t).$$ \hspace{1cm} (3.28)

The same affine assumption is made for the drift of the factors process (3.3) and we assume that the diffusion is only a function of time:

$$\theta(t,x(t)) = A(t)x(t) + a(t)$$
$$\Psi(t,x(t)) = \nu(t),$$ \hspace{1cm} (3.29)

where $A(t) \in \mathbb{R}^{m \times m}$, $a(t) \in \mathbb{R}^m$, and $\nu(t) \in \mathbb{R}^{m \times m}$.

Many authors of empirical studies have found evidence that macroeconomic and financial variables, such as long-term interest rates or the dividend-price ratio, are suitable return predictors. Among identified factors for US stock markets are the short-term interest rate (Fama and Schwert 1977, Glosten, Jagannathan and Runkle 1993), the dividend-price ratio (Campbell and Schiller 1988, Fama and French 1988), and the yield spread between long-term and short-term bonds (Fama and French 1989, Campbell and Schiller 1991). A systematic study to analyze the robustness and the economic significance of return predictors were undertaken by Pesaran and Timmermann (1995), where 1 month T-bill rates, 12 month T-bill rates, inflation rate, change in industrial production, and monetary growth rate were used as factors to explain the U.S. stock returns. Testing a simple allocation strategy, the authors concluded that investors could have exploited the predictability of returns during the volatile markets of the 1970’s. The authors found evidence on the predictability of U.K. stock returns in a more recent study (Pesaran and Timmermann 1998) and concluded that this could have been exploited by investors. In a similar manner, Patelis (1997) found evidence for the predictability of U.S. excess stock returns based on
five monetary policy factors as well as on interest rate spreads and one-month real interest rates. Ilmanen (1997) empirically showed that the excess returns of long-term T-bonds are predictable with factors such as term spread or momentum factors. Furthermore, Shen (2003) used the spread between long- and short-term interest rates and price-earnings ratio to predict future up- or down turns of the S&P 500 index. Combined with a simple switching strategy between the index and treasury bonds, he showed that a significant outperformance of the index by an investor was possible. Additional studies on return predictability are cited in the bibliographies of these cited papers.

These and other studies provide evidence that a dynamic asset allocation strategy provides significant portfolio improvements for investors. None of these studies however developed a systematic allocation strategy and relied on ad-hoc portfolio allocation methods.

**Wealth dynamics**

Assuming that an investor’s wealth only derives gains form his investments, his wealth dynamics (portfolio dynamics) can be computed with (3.9) and the specific forms for the drift of the assets as given in (3.26) and (3.27). Mathematically we obtain

\[
dW(t) = u(t)^T(F(t)x(t) + f(t))W(t)\,dt + (F_0(t)x(t) + f_0(t))W(t)\,dt \\
+ W(t)u(t)^T\sigma(t)dZ_P(t) + q(t)dt ,
\]

where \( u_i(t) \) denotes the fraction of wealth (portfolio value) invested in the i-th asset at time \( t \), and \( q(t) \) denotes in- or outflows of money. In most papers \( q(t) \) is assume to be consumption, which is an outflow of the portfolio. Moreover, we use the following abbreviations

\[
\begin{align*}
  u(t) &= (u_1(t), u_2(t), \ldots, u_n(t))^T \in \mathcal{U}, \\
  G(t) &= [G_1^T(t), G_2^T(t), \ldots, G_n^T(t)] \in \mathbb{R}^{n \times m}, \\
  F(t) &= G(t) - 1F_0(t) \in \mathbb{R}^{n \times m}, \\
  g(t) &= (g_1(t), \ldots, g_n(t))^T \in \mathbb{R}^{n \times 1}, \\
  f(t) &= f(t) - 1f_0(t) \in \mathbb{R}^{n \times 1}, \\
  \sigma &= [\sigma_1^T, \ldots, \sigma_n^T] \in \mathbb{R}^{n \times n}, \\
  1 &= (1, 1, \ldots, 1)^T \in \mathbb{R}^{n \times 1}.
\end{align*}
\]
The term $F(t)x(t) + f(t)$ denotes the excess mean return of the risky investments (i.e., the mean return above the risk-free interest rate) and the term $F_0(t)x(t) + f_0(t)$ denotes the risk-free interest rate.

### 3.2.2 Portfolio optimization problems

In this section, we introduce two portfolio optimization problems which depend on the modelling structure introduced in Section 3.2.1. Both portfolio optimization problems have in common, that an investor tries to maximize utility of wealth at a fixed, finite date. However, they differ in the utility function, the modelling of the risk-free asset, and the possibility of in- or outflow of money.

**Portfolio optimization with CRRA utility**

The first portfolio choice problem is maximizing the expected power utility defined over terminal wealth, i.e., $E\left[\frac{1}{\gamma} W(T)^\gamma\right]$. The mean returns of the risk-bearing assets and the interest rates of the risk-free bank account are affine functions of the factor process. Additionally, we assume that there is no in- or outflow of money from the portfolio dynamics, i.e., $q(t) = 0$. Furthermore, we assume that leveraging, short-selling, and borrowing at the risk-free rates are unrestricted, i.e., $\mathcal{U} = \mathbb{R}^n$. The factor dynamics and the risk-bearing asset dynamics are assumed to be correlated. Mathematically the problem statement is:

$$
\max_{u(\cdot) \in \mathbb{R}^n} E\left[\frac{1}{\gamma} W(T)^\gamma\right] \\
\text{s.t.} \\
\begin{align*}
    dW(t) &= W(t)[F_0(t)x(t) + f_0(t) + u^T(t)(F(t)x(t) + f(t))]dt + W(t)u^T(t)\sigma(t)dZ_P \\
    W(0) &= W_0 \\
    dx(t) &= (A(t)x(t) + a(t))dt + \nu(t)dZ_x \\
    x(0) &= x_0 \\
    dZ_PdZ_x &= \rho(t)dt ,
\end{align*}
$$

(3.31)

where $T$ denotes the time horizon and $\gamma < 1$ denotes the coefficient of risk aversion.

**Portfolio optimization with CARA utility**

The second portfolio choice problem is maximizing the exponential utility defined over terminal wealth, i.e., $E[-\frac{1}{\gamma} e^{-\gamma W(T)}]$. We make similar assumptions as for the problem of
portfolio optimization with power utility, with the difference that we assume the risk-free interest rate not to be a function of \( x(t) \), i.e., \( F_0(t) = 0 \) and that we allow for in- or outflows of money form the portfolio. The in- or outflow is modelled as an affine function of the factor levels, \( q(t) = H(t)x(t) + h(t) \). Possible applications are pension funds with known inflow (during the accumulation phase) and outflow (during the payout phase) or life-insurance portfolios.

Mathematically the problem statement is:

\[
\max_{u(\cdot) \in \mathbb{R}^n} \mathbb{E}\left[ -\frac{1}{\gamma} e^{-\gamma W(T)} \right] \\
\text{s.t.} \\
dW(t) = W(t)[f_0(t) + u^T(t)(F(t)x(t) + f(t))]dt \\
\quad + [H(t)x(t) + h(t)]dt + W(t)u^T(t)\sigma(t)dZ_P \\
W(0) = W_0 \\
x(t) = (A(t)x(t) + a(t))dt + \nu(t)dZ_x \\
x(0) = x_0 \\
dZ_PdZ_x = \rho(t)dt ,
\]

where \( T \) denotes the terminal date and \( \gamma > 0 \) denotes the coefficient of risk aversion.

### 3.2.3 Solutions to the portfolio optimization problems

The solution to the first portfolio optimization problem (3.31) is derived in detail. The solution for the second problem (3.32) is shown without detailed derivation, since the derivation closely resembles the derivation of first problem.

**Solution to the portfolio optimization with CRRA utility**

**Theorem 3.1.** The solution to the problem of portfolio optimization (3.31) is obtained by solving the following three ordinary differential equations (ODEs) and using an affine function for the control variable. The optimal feedback controller is calculated by

\[
u^*(\cdot) = \frac{\Sigma^{-1}(t)}{1-\gamma} \left( F(t)x(t) + f(t) + \sigma(t)\rho(t)\nu^T(t)(K_3(t)x(t) + k_2(t)) \right),
\]

where \( K_3(t) \in \mathbb{R}^{m \times m} \), \( k_2(t) \in \mathbb{R}^{m \times 1} \), and \( k_1(t) \in \mathbb{R} \) are given by three ODEs. The scalar \( k_1(t) \) is obtained by
\[ \dot{k}_1 + \frac{1}{2} \text{tr}\{\nu^T(k_2k_2^T + K_3)\nu\} + a^T k_2 + \gamma f_0 - \frac{\gamma}{2(\gamma - 1)} \left( f^T \Sigma^{-1} f + 2 f^T \Sigma^{-1} \sigma \rho \nu^T k_2 + k_2^T \nu \rho^T \rho \nu^T k_2 \right) = 0, \quad (3.34) \]

with terminal condition \( k_1(T) = 0 \). The vector \( k_2(t) \) is computed by

\[ \dot{k}_2 + \gamma F_0^T + K_3 \nu \nu^T k_2 + A^T k_2 + K_3 a - \frac{\gamma}{(\gamma - 1)} \left( F^T \Sigma^{-1} F + F^T \Sigma^{-1} \sigma \rho \nu^T K_3 \right. \]
\[ \left. + K_3 \nu \rho^T \rho \nu^T K_3 \right) = 0, \quad (3.35) \]

with terminal condition \( k_2(T) = 0 \). The matrix \( K_3(t) \) is obtained

\[ \dot{K}_3 + K_3 \nu \nu^T K_3 + K_3 A + A^T K_3 - \frac{\gamma}{(\gamma - 1)} \left( F^T \Sigma^{-1} F + F^T \Sigma^{-1} \sigma \rho \nu^T K_3 \right. \]
\[ \left. + K_3 \nu \rho^T \rho \nu^T K_3 \right) = 0, \quad (3.36) \]

with terminal condition \( K_3(T) = 0 \).

**Proof.** We use the HJB equation stated in (3.14) with the specific functions for this problem of portfolio optimization. In order to obtain the solution, we solve the following problem:

\[
J_t(\cdot) + \max_{u(\cdot) \in \mathbb{R}^n} \left[ W(t)(F_0^t(t)x(t) + f_0(t) + u^T(t)(F(t)x(t) + f(t)))J_W(\cdot) \right.
\]
\[ + (A(t)x(t) + a(t))^T J_x(\cdot) + \frac{1}{2} W^2(t) u^T(t) \Sigma(t) u(t) J_{WW}(\cdot) \right]
\[+ W(t) u^T(t) \sigma(t) \rho(t) \nu^T(t) J_{Wx}(\cdot) \right]
\[ + \frac{1}{2} \text{tr}\{J_{xx}(\cdot) \nu(t) \nu^T(t)\} \right] = 0 , \quad (3.37) \]

where \( \Sigma(t) = \sigma(t) \sigma^T(t) \) and terminal condition \( J(T, W(T), x(T)) = \frac{1}{\gamma} W(T) \gamma \). We have omitted the arguments of \( J(\cdot) = J(t, W(t), x(t)) \) for writing convenience. Using the first order conditions, we obtain the optimal portfolio control (composition) vector

\[
u^*(\cdot) = -\Sigma^{-1}(t) \frac{1}{W(t) J_{WW}(\cdot)} \left( J_W(\cdot)(F(t)x(t) + f(t)) \right.
\]
\[+ \sigma(t) \rho(t) \nu^T(t) J_{Wx}(\cdot) \right) \right). \quad (3.38) \]

By using the optimal portfolio vector (3.38) and inserting it back in (3.37), we obtain
\[ J_t(\cdot) + W(t)(F_0(t)x(t) + f_0(t))J_W(\cdot) \\
+ (A(t)x(t) + a(t))^T J_x(\cdot) + \frac{1}{2} \text{tr}\{J_{xx}(\cdot)\nu(t)\nu^T(t)\} \\
- \frac{1}{2} J_W^2(\cdot) (F(t)x(t) + f(t))^T \Sigma^{-1}(t) (F(t)x(t) + f(t)) \\
- (F(t)x(t) + f(t))^T \Sigma^{-1}(t) \sigma(t) \rho(t) \nu^T(t) \frac{J_W(\cdot)}{J_{WW}(\cdot)} J_{Wx}(\cdot) \\
- \frac{J_W^2(\cdot)}{2J_{WW}(\cdot)} \nu(t) \rho^T(t) \rho(t) \nu^T(t) J_{Wx}(\cdot) = 0, \tag{3.39} \]

with the terminal condition \( J(T, W(T), x(T)) = \frac{1}{\gamma} W(T)^\gamma \). The solution for \( J(t, W(t), x(t)) \) needs to be found in order to calculate the optimal feedback controller. Hence, we conjecture a functional form of the solution:

\[ J(t, W, x) = \frac{1}{\gamma} W(t)^{\gamma-1} l(t, x), \quad l(t, x) = e^{k_1(t)+k_2(t)x(t)} + \frac{1}{2} x^T(t) K_3(t)x(t), \tag{3.40} \]

with terminal conditions \( k_1(T) = 0, k_2(T) = 0, \) and \( K_3(T) = 0 \). From the observations made in Section 3.1.4, the separation of the wealth and the factors for CRRA utility is already known. The remaining functional form is clear from the comparison to the linear exponential Gaussian problems. Furthermore, one may notice that \( K_3(t) \) is a symmetric \( m \times m \) matrix, \( k_2(t) \) is a \( m \times 1 \) vector, and \( k_1(t) \) is a scalar. The partial derivatives of the value function \( J(t, W, x) \) are given by

\[
\begin{align*}
J_W(\cdot) &= W(t)^{\gamma-1} l(t, x) \\
J_{WW}(\cdot) &= (\gamma - 1)W^{\gamma-2}(t) l(t, x) \\
J_{Wx}(\cdot) &= W(t)^{\gamma-1} l(t, x)[k_2(t) + K_3(t)x(t)] \\
J_x(\cdot) &= \frac{1}{\gamma} W(t)^{\gamma-1} l(t, x)[k_2(t) + K_3(t)x(t)] \\
J_{xx}(\cdot) &= \frac{1}{\gamma} W(t)^{\gamma-1} l(t, x)[k_2(t)k_2^T(t) + K_3(t)x(t)k_2^T(t) \\
&+ k_2(t)x^T(t)K_3(t) + K_3(t)x(t)x^T(t)K_3(t) + K_3(t)] \\
J_t(\cdot) &= \frac{1}{\gamma} W(t)^{\gamma-1} l(t, x)[\dot{k}_1(t) + \dot{k}_2(t)x(t) + \frac{1}{2} x^T(t) \dot{K}_3(t)x(t)].
\end{align*}
\]

Replacing the derivatives of \( J(\cdot) \) in (3.39) and using the following rule for the trace operator

\[
\text{tr}\{J_{xx}(\cdot)\nu(t)\nu^T(t)\} = \text{tr}\{\nu^T(t)J_{xx}(\cdot)\nu(t)\} \\
= \frac{1}{\gamma} W(t)^{\gamma-1} l(t, x)[\text{tr}\{\nu^T(t)(k_2(t)k_2^T(t) + K_3(t))\nu(t)\} \\
+ 2k_2^T(t)\nu(t)\nu^T(t)K_3(t)x(t) + x^T(t)K_3(t)\nu(t)\nu^T(t)K_3(t)x(t)],
\]

where the terminal condition \( J(T, W(T), x(T)) = \frac{1}{\gamma} W(T)^\gamma \) and the solution for the optimal feedback controller is conjectured.
and after factoring $\frac{1}{7}W^\gamma(t)$ out and dividing by it, we arrive at

\[
\dot{k}_1 + \frac{1}{2}x^T \dot{K}_3 x + \gamma(F_0 x + f_0) + \frac{1}{2} \left[ \text{tr}\{\nu^T (k_2 k_2^T + K_3) \nu\} + 2k_2^T \nu \nu^T K_3 x + x^T K_3 \nu \nu^T K_3 x \right] + (Ax + a)^T (k_2 + K_3 x)
\]

\[
- \frac{\gamma}{2(\gamma - 1)} (x^T F^T \Sigma^{-1} F x + 2f^T \Sigma^{-1} F x + f^T \Sigma^{-1} f)
\]

\[
- \frac{\gamma}{(\gamma - 1)} \left( (F x + f^T) \Sigma^{-1} \sigma \nu^T (k_2 + K_3 x) \right) - \frac{1}{2} (k_2 + K_3 x)^T \nu \rho^T \nu^T (k_2 + K_3 x) = 0,
\]

where we suppressed $t$ in all functions for compactness. The last equation is quadratic in $x(t)$. We rearrange (3.41) as factors of $x(t)$ and obtain

\[
\dot{k}_1 + \frac{1}{2} \text{tr}\{\nu^T (k_2 k_2^T + K_3) \nu\} + a^T k_2 - \frac{\gamma}{2(\gamma - 1)} f^T \Sigma^{-1} f
\]

\[
- \frac{\gamma}{(\gamma - 1)} k_2^T \nu \rho^T \nu^T k_2 + \gamma f_0 - \gamma k_2^T \nu \rho^T \nu^T k_2
\]

\[
+ \left[ k_2^T + \gamma F_0 + k_2^T \nu \nu^T K_3 + k_2^T A + a^T K_3 - \frac{\gamma}{(\gamma - 1)} f^T \Sigma^{-1} F \right]
\]

\[
- \frac{\gamma}{(\gamma - 1)} k_2^T \nu \rho^T \sigma^T \Sigma^{-1} F - \frac{\gamma}{(\gamma - 1)} f^T \Sigma^{-1} \sigma \nu^T K_3 - \frac{\gamma}{(\gamma - 1)} k_2^T \nu \rho^T \nu^T K_3
\]

\[
+ \frac{1}{2} x^T \left[ \dot{K}_3 + K_3 \nu \nu^T K_3 + K_3 A + A^T K_3 - \frac{\gamma}{(\gamma - 1)} F^T \Sigma^{-1} F \right]
\]

\[
- \frac{\gamma}{(\gamma - 1)} \left( F^T \Sigma^{-1} \sigma \nu^T K_3 + K_3 \nu \rho^T \sigma^T \Sigma^{-1} F \right)
\]

\[
- \frac{\gamma}{(\gamma - 1)} K_3 \nu \rho^T \nu^T K_3 \right] x = 0.
\]

In order to achieve equality in (3.42), we set all factors of $x(t)$ to zero and obtain the system of the three ODES given above. The optimal feedback controller is calculated by using the value function conjecture (3.40) and plugging it into (3.38).

In order to compute the optimal controller, we need to calculate the solutions of (3.35) and (3.36), where parameters such as $F$ or $A$ are can be functions of time. Equation (3.36) is a well known Riccati type equation.

**Theorem 3.2.** In the case that all the parameters are time-invariant and if the following two conditions are met:

\[
\left[ \left( \frac{\gamma}{\gamma - 1} \nu \rho^T \sigma^T \Sigma^{-1} F - A \right), (\nu \nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \omega^T) \right] \text{ is controllable}
\]

\[
\left[ \left( \frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F \right), (\frac{\gamma}{\gamma - 1} \nu \rho^T \sigma^T \Sigma^{-1} F - A) \right] \text{ is observable},
\]

the Riccati equation (3.36) possesses a positive-definite and finite solution for $T \to \infty$. 

The matrix \((\nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \rho \nu^T)\) \(\frac{1}{2}\) is the full rank factorization of \((\nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \rho \nu^T)\) and the matrix \((\frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F)\) \(\frac{1}{2}\) is the full rank factorization of \((\frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F)\). Therefore, additionally the matrix \((\frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F)\) and the matrix \((\nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \rho \nu^T)\) must be positive-definite. We now give a proof for theorem 3.2.

**Proof.** Given an algebraic matrix Riccati equation

\[-H^T X - X H + X D X - G = 0 (3.43)\]

where all matrices are in \(\mathbb{R}^{n \times n}\) and \(D = B B^T, G = C^T C\) are full-rank factorizations of \(D\) and \(G\). If the matrices \(H, B, C\) satisfy the condition

\[(H, B)\] is controllable,

\[(C, H)\] is observable,

then \(X\) is the positive-definite and symmetric solution of the algebraic Riccati equation, see Xu and Lu (1995) or Anderson and Moore (1990). The conditions that (3.36) possesses a positive-definite solution for \(T \to \infty\) can be derived with conditions given for (3.43). Using condition for steady state, \(\dot{K}_3 = 0\), we can rewrite (3.43) and obtain

\[-K_3(\frac{\gamma}{\gamma - 1} \nu \rho^T \sigma^T \Sigma^T F - A) - (\frac{\gamma}{\gamma - 1} \nu \rho^T \sigma^T \Sigma^T F - A)^T K_3
+ K_3(\nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \rho \nu^T)K_3 - \frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F = 0 . (3.44)\]

By setting \(H = (\frac{\gamma}{\gamma - 1} \nu \rho^T \sigma^T \Sigma^T F - A), D = (\nu^T - \frac{\gamma}{\gamma - 1} \nu \rho^T \rho \nu^T), G = \frac{\gamma}{\gamma - 1} F^T \Sigma^{-1} F,\) we have derived the conditions for a positive-definite solution of (3.43). \(\square\)

If the matrix \((K_3 \nu^T + A^T - \frac{\gamma}{\gamma - 1}(\Sigma^{-1} \sigma \rho \nu + K_3 \nu \rho^T \rho \nu^T))\) is invertible and (3.36) possesses a positive-definite finite solution, then (3.35) possesses a finite solution for \(T \to \infty\).

The optimal controller can be split into two parts. The first being the myopic demand \(\frac{1}{1 - \gamma} \Sigma^{-1}(t)(F(t)x(t) + f(t))\) which is independent of the time horizon considered, similar to Merton Problem (Merton 1973). The second part is the inter-temporal hedging demand \(\frac{1}{1 - \gamma} \Sigma^{-1}(t)\rho(t) \nu^T(t)(K_3(t)x(t) + k_2(t))\), which depends on the time horizon and the correlation of the external variables with the asset price dynamics. The shorter the time horizon is, the smaller the inter-temporal hedging demand becomes because of the terminal conditions are \(K_3(T) = 0\) and \(k_2(T) = 0\). The time horizon \(T\) to some extent governs the time-varying gains and thus the feedback of the external variables. The portfolio
weights over time are therefore a function of the time horizon. The risk aversion controls how much wealth is invested in the risky assets. It also governs the portfolio composition, which is governed by $F(t), f(t), \gamma$, and all other variables that affect (3.35) and (3.36).

**Solution to the portfolio optimization with CARA utility**

In order to solve the problem of portfolio optimization with exponential utility (3.32), we need to find a solution to

$$J_t(\cdot) + (W(t) f_0(t) + H(t)x(t) + h(t))J_W(\cdot)$$

$$+ (A(t)x(t) + a(t))^T J_x(\cdot) + \frac{1}{2} \text{tr}\{J_{xx}(\cdot)\nu(t)\nu(t)^T\}$$

$$- \frac{1}{2} J_W^2(\cdot) \left( F(t)x(t) + f(t) \right)^T \Sigma^{-1}(t)(F(t)x(t) + f(t))$$

$$- (F(t)x(t) + f(t))^T \Sigma^{-1}(t) \sigma(t)\rho(t)\nu^T(t) \frac{J_W(\cdot)}{J_{WW}(\cdot)} J_{Wx}(\cdot)$$

$$- \frac{J_{Wx}(\cdot)}{2J_{WW}(\cdot)} \nu(t)\rho^T(t)\rho(t)\nu^T(t) J_{Wx}(\cdot) = 0,$$  (3.45)

with terminal condition $J(T, W(T), x(T)) = -\frac{1}{\gamma} e^{-\gamma W(T)}$. By using (3.17) we may compute (3.45). In order to solve (3.45) we conjecture the value function

$$J(t, W(t), x(t)) = -\frac{1}{\gamma} e^{k_1(t)+k_2(t)W(t)+k_3^T(t)x(t)+\frac{1}{2}x^T(t)K_4(t)x(t)}$$  (3.46)

with terminal conditions $k_1(T) = 0$, $k_2(T) = -\gamma$, $k_3(T) = 0$, and $K_4(T) = 0$. With this conjecture we state the solution as follows:

**Theorem 3.3.** The solution for the second affine portfolio optimization problem consists of an affine function of factor levels for the control variable and four ODEs. The optimal control law is given by

$$u(\cdot) = -\Sigma^{-1} \frac{1}{\gamma W(t)k_2(t)} \left[ F(t)x(t) + f(t) \right.$$

$$+ \sigma(t)\rho(t)\nu^T(t) \left( K_4(t)x(t) + k_3(t) \right) \right].$$  (3.47)

Note that $k_1(t)$ and $k_2(t)$ are scalars, $k_3(t)$ is a $m \times 1$ vector, and $K_4(t)$ is a $m \times m$ matrix. The scalar $k_1(t)$ is obtained by

$$k_1 + h k_2 + \frac{1}{2} \text{tr}\{\nu^T(k_3 k_3^T + K_4)\nu\} + a^T k_3$$

$$- \frac{1}{2} \left( f^T \Sigma^{-1} f + 2 f^T \Sigma^{-1} \sigma \rho^T k_3 + k_3^T \rho^T \nu^T k_3 \right) = 0,$$  (3.48)
with terminal condition \( k_2(T) = 0 \). The scalar \( k_2(t) \) is computed by

\[
\dot{k}_2 + f_0 k_2 = 0 ,
\]  

(3.49)

with terminal condition \( k_2(T) = -\gamma \). In the case that \( f_0(t) \) is a constant (time invariant) the solution to (3.49) is

\[
k_2(t) = -\gamma e^{f_0(T-t)} .
\]  

(3.50)

Otherwise, one has to solve the explicit form of the ODE (3.49). The vector \( k_3(t) \) is obtained by

\[
\dot{k}_3 + k_2 H + K_4 \nu \nu^T k_3 + A^T k_3 + K_4 a \\
- \left( F^T \Sigma^{-1} f + F^T \Sigma^{-1} \sigma \rho \nu k_3 + K_4 \nu \rho^T \sigma^T \Sigma^{-1} f + K_4 \nu \rho^T \rho \nu^T k_3 \right) = 0 ,
\]  

(3.51)

with terminal condition \( k_3(T) = 0 \). The matrix \( K_4(t) \) is calculated with

\[
\dot{K}_4 + K_4 \nu \nu^T K_4 + K_4 A + A^T K_4 \\
- \left( F^T \Sigma^{-1} F + F^T \Sigma^{-1} \sigma \rho \nu K_4 + K_4 \nu \rho^T \sigma^T \Sigma^{-1} F \\
+ K_4 \nu \rho^T \rho \nu^T K_4 \right) = 0 ,
\]  

(3.52)

where the terminal condition is \( K_4(T) = 0 \).

The proof of Theorem 3.3 is not stated here since (3.45) and (3.39) are very similar and the proof thus is almost identical to the proof for Theorem 3.1. Equation (3.52) is again a well known Riccati type equation.

**Theorem 3.4.** In the case that the all the parameters are time-invariant and if the following two conditions are met:

\[
\left[ (\nu \rho^T \sigma^T \Sigma^{-1} F - A) , (\nu \nu^T - \nu \rho^T \rho \nu^T)^{\frac{1}{2}} \right] \text{ is controllable}
\]  

(3.53)

\[
\left[ (F^T \Sigma^{-1} F)^{\frac{1}{2}} , (\nu \rho^T \sigma^T \Sigma^{-1} F - A) \right] \text{ is observable},
\]  

(3.54)

then the Riccati equation (3.52) possesses a positive-definite and finite solution for \( T \to \infty \), see Xu and Lu (1995).

Due to the similarities of Theorem 3.4 and Theorem 3.2 we omit the proof. The matrix \( (\nu \nu^T - \nu \rho^T \rho \nu^T)^{\frac{1}{2}} \) is the full rank factorization of \( (\nu \nu^T - \nu \rho^T \rho \nu^T) \) and matrix \( (F^T \Sigma^{-1} F)^{\frac{1}{2}} \) is the full rank factorization of \( (F^T \Sigma^{-1} F) \). Therefore, additionally the matrix \( (F^T \Sigma^{-1} F) \)
and the matrix $(\nu \nu^T - \nu \rho^T \rho \nu^T)$ must be positive-definite. If the matrix $(K_4 \nu \nu^T + A^T - \Sigma^{-1} \sigma \nu - K_4 \nu \rho^T \rho \nu^T)$ is invertible and a finite solution for $K_4$ exists, then (3.51) possesses a finite solution for $T \to \infty$.

The solution of the portfolio optimization in the case of the CARA utility and in the case of the CRRA utility are very similar with one clear distinction that in the case of the CRRA utility the optimal controller is independent of current wealth, whereas in the case of the CARA the controller is a function of current wealth. In the case of the CRRA utility, the investor cannot go bankrupt, because his wealth always remains positive. In the case of the CARA utility exists a finite probability that his wealth may become negative and thus the investor goes bankrupt.

### 3.2.4 Partial information case

In this subsection, we discuss the case when not all of the factor levels are directly observable. We assume that some factors that affect the expected returns of the risky asset are only observable through some kind of filtering (estimation). The presentation of this material given in this section closely follows Keel (2005) where all the results are derived.

#### Asset model with partial information

The $m_x$ observable factors $x(t)$ are modelled as affine Gaussian system given by

$$dx(t) = [A_x(t)x(t) + A_z(t)z(t) + a(t)]dt + \nu_x dZ_x(t)$$

$$x(0) = x_0,$$  \hspace{1cm} (3.55)

where $A_x(t) \in \mathbb{R}^{m_x \times m_x}$, $A_z(t) \in \mathbb{R}^{m_z \times m_z}$, $a(t) \in \mathbb{R}^{m_z}$, and $\nu_x \in \mathbb{R}^{m_x \times m_x}$. The $m_z$ unobservable factors $z(t)$ are also modelled as affine Gaussian system given by

$$dz(t) = [B_z(t)z(t) + B_x(t)x(t) + b(t)]dt + \nu_z dZ_z(t)$$

$$z(0) = z_0,$$  \hspace{1cm} (3.56)

where $B_z(t) \in \mathbb{R}^{m_z \times m_z}$, $B_x(t) \in \mathbb{R}^{m_x \times m_x}$, $b(t) \in \mathbb{R}^{m_x}$, and $\nu_z \in \mathbb{R}^{m_z \times m_z}$. The Brownian motions $Z_x(t)$ and $Z_z(t)$ are not generally assumed to be independent, but rather correlated with given correlation matrix $\rho_{xz}(t) \in \mathbb{R}^{m_x \times m_z}$. We assume that the drift terms of (3.1) and (3.2) are affine functions of the observable and unobservable factor levels, as given by
\[ \mu_t(t, x(t), z(t)) = G_t(t)x(t) + H_t(t)z(t) + g_t(t), \]
\[ r(t, x(t)) = F_0(t)x(t) + f_0(t), \]

where \( G_t(t), F_0(t) \in \mathbb{R}^{1 \times m_x}, H_t(t) \in \mathbb{R}^{1 \times m_z}, \) and \( g_t(t), f_0(t) \in \mathbb{R}. \) The unobservable factors \( z(t) \) affect the drift of the observable factors \( x(t) \) and the expected returns of the risky assets. Again we define \( G(t) = (G_1(t), G_2(t), ..., G_n(t))^T, \) \( g(t) = (g_1(t), g_2(t), ..., g_n(t))^T, \) and \( H(t) = (H_1(t), H_2(t), ..., H_n(t))^T. \)

**Filtering of the unobserved factors**

Since the investor is not able to observe the values of \( z(t) \) an estimation is needed. The estimation for the introduced asset price model is the standard Kalman-Bucy method.

Using the transformation \( p(t) = \ln(S(t)) \) on (3.1) we obtain
\[
dp_t = \left[ \mu_t(x(t), z(t)) - \frac{1}{2} \sigma_t(t) \sigma^T_t(t) \right] dt + \sigma_t(t) dZ_p(t) \\
p_0 = \ln(p_0),
\]
and \( p(t) = (p_1(t), p_2(t), ..., p_n(t))^T. \) In the vector \( \xi(t) = (p^T(t), x^T(t))^T \in \mathbb{R}^{n+m_x}, \) we summarize all directly observable variables. The estimation (filtering) problem consists of finding the optimal estimate \( m(t) \) given the stochastic dynamics
\[
d\xi(t) = \left[ A_\xi(t)\xi(t) + A_{\xi z}(t)z(t) + a_\xi(t) \right] dt + \nu_\xi(t) dZ_\xi(t) \\
\xi(0) = \xi_0 \\
dz(t) = \left[ B_\xi(t)z(t) + B_{\xi x}(t)x(t) + b(t) \right] dt + \nu_z dz_z(t) \\
z(0) = z_0,
\]

where \( Z_\xi(t) = (Z^p_\xi(t), Z^x_\xi(t))^T \in \mathbb{R}^{n+m_x}, A_\xi(t) \in \mathbb{R}^{n \times n+m_x}, A_{\xi z}(t) \in \mathbb{R}^{n \times m_z}, \) \( a_\xi(t) \in \mathbb{R}^{n \times m_x}, \) \( \nu_\xi(t) \in \mathbb{R}^{n \times n+m_x}, \) \( \Sigma(t) = \sigma(t) \sigma^T(t), \) and \( B_\xi(t) \in \mathbb{R}^{m_z \times n+m_x} \) are defined as
\[
A_\xi(t) = \begin{bmatrix} 0 & G(t) \\ 0 & A_x(t) \end{bmatrix}, \quad A_{\xi z}(t) = \begin{bmatrix} H(t) \\ A_z(t) \end{bmatrix}, \quad a_\xi(t) = \begin{bmatrix} g(t) - \frac{1}{2} \text{diag} \Sigma(t) \\ A_z(t) \end{bmatrix} \\
\nu_\xi(t) = \begin{bmatrix} \sigma(t) & 0 \\ 0 & \nu_x(t) \end{bmatrix}, \quad B_\xi(t) = \begin{bmatrix} 0 & B_z(t) \end{bmatrix}.
\]

In order to apply the standard results of the Kalman filter theory, we need to introduce independent Brownian motions. The Brownian motions \( Z_p(t), Z_x(t), \) and \( Z_z(t) \) are assumed to be correlated. The correlation matrix is given by
where $\mathbf{1}_i$, $i = 1, 2, 3$ denotes the identity matrix of appropriate dimensions, $\rho_{px}(t) \in \mathbb{R}^{n \times m_x}$, $\rho_{pz}(t) \in \mathbb{R}^{n \times m_z}$, $\rho_{xz}(t) \in \mathbb{R}^{m_x \times m_z}$, $\rho_{xz}(t) \in \mathbb{R}^{n \times m_z \times n + m_x}$, and $\rho_{xz}(t) \in \mathbb{R}^{n + m_x \times m_z}$ Since the correlation matrix is symmetric and positive definite, the matrix square root of $\rho(t)$ exists. We replace $Z(t) = (Z^T_x(t), Z^T_z(t), Z^T_{z\xi}(t)) \in \mathbb{R}^{n + m_x + m_z}$ with $Z(t) = \rho(t)Z(t)$, where $Z(t) \in \mathbb{R}^{n + m_x + m_z}$ are uncorrelated standard Brownian motions. We state based on the transformation of Brownian motions the filtering problem as

$$
\begin{align*}
    d\xi(t) &= [A_{\xi}(t)\xi(t) + A_{\xi}(t)z(t) + a_\xi(t)]dt + \nu_\xi(t)dZ(t), \quad \xi(0) = \xi_0, \\
    dz(t) &= [B_z(t)z(t) + B_\xi(t)\xi(t) + b(t)]dt + \nu_z dZ(t) + \nu_\xi dZ(t), \\
    z(0) &= z_0,
\end{align*}
$$

where $\nu_\xi(t) \in \mathbb{R}^{n + m_x + m_x}$, $\nu_z(t) \in \mathbb{R}^{m_x \times m_z}$, and $\nu_\xi(t) \in \mathbb{R}^{m_z \times m_z}$. The filtering problem is in a standard form where the Brownian motion of the measure and the process equations are independent. The relationship of the covariance matrix for $Z(t)$ and $Z(t)$ is

$$
\begin{align*}
    \begin{bmatrix}
        \nu_{\xi}\nu_{\xi}^T \\
        \nu_{\xi}\nu_{\xi}^T + \nu_{z}\nu_{z}^T
    \end{bmatrix}
    &=
    \begin{bmatrix}
        \nu_{\xi}\nu_{\xi}^T \\
        \nu_{z}\nu_{z}^T + \nu_{\xi}\nu_{\xi}^T
    \end{bmatrix}.
\end{align*}
$$

Using Lipster and Shiryaev (2001b, Theorem 10.3), we now state the Kalman-Bucy filter for the filtering problem given in (3.63). The dynamics of the optimal estimation of $z(t)$ are computed by

$$
\begin{align*}
    dm_{z}(t) &= [B_z(t)m_{z}(t) + B_\xi(t)\xi(t) + b(t)]dt \\
    &+ [C_1(t) + \eta(t)A_{\xi}(t)^T]C_2(t)[d\xi(t) - (A_{\xi}(t)\xi(t) + A_{\xi}(t)m_{z}(t) + a_\xi)dt] \\
    m_{z}(0) &= \mathbb{E}[z(0)|\xi(0)],
\end{align*}
$$

where $m_{z}(t)$ is the optimal estimation of $z(t)$. The error covariance matrix is denoted by $\eta(t)$ and is obtained by

$$
\begin{align*}
    \dot{\eta}(t) &= B_z(t)\eta(t) + \eta(t)B_z^T(t) + C_3(t) \\
    &- [C_1(t) + \eta(t)A_{\xi}(t)^T]C_2(t)[C_1(t) + \eta(t)A_{\xi}(t)^T]^T \\
    \eta(0) &= \mathbb{E}[(z(0) - m_z(0))(z(0) - m_z(0))^T],
\end{align*}
$$

where $\eta(t)$ is the optimal estimation error and $\eta(t)$ is the error covariance matrix. Theorem 10.3, we now state the Kalman-Bucy filter for the filtering problem given in (3.63). The dynamics of the optimal estimation of $z(t)$ are computed by

$$
\begin{align*}
    dm_{z}(t) &= [B_z(t)m_{z}(t) + B_\xi(t)\xi(t) + b(t)]dt \\
    &+ [C_1(t) + \eta(t)A_{\xi}(t)^T]C_2(t)[d\xi(t) - (A_{\xi}(t)\xi(t) + A_{\xi}(t)m_{z}(t) + a_\xi)dt] \\
    m_{z}(0) &= \mathbb{E}[z(0)|\xi(0)],
\end{align*}
$$

where $m_{z}(t)$ is the optimal estimation of $z(t)$. The error covariance matrix is denoted by $\eta(t)$ and is obtained by

$$
\begin{align*}
    \dot{\eta}(t) &= B_z(t)\eta(t) + \eta(t)B_z^T(t) + C_3(t) \\
    &- [C_1(t) + \eta(t)A_{\xi}(t)^T]C_2(t)[C_1(t) + \eta(t)A_{\xi}(t)^T]^T \\
    \eta(0) &= \mathbb{E}[(z(0) - m_z(0))(z(0) - m_z(0))^T],
\end{align*}
$$

where $\eta(t)$ is the optimal estimation error and $\eta(t)$ is the error covariance matrix.
where the matrices $C_1(t), C_2(t),$ and $C_3(t)$ are defined (see (3.64)) as follows:

$$C_1(t) = \Pi_{\xi} \Pi_{\xi}^T = \nu_z \rho_{\xi,\nu}^T \nu^T_{\xi} \in \mathbb{R}^{m_z \times n + m_z}$$

$$C_2(t) = \left[\Pi_{\xi} \Pi_{\xi}^T\right]^{-1} = \left[\nu \rho_{\xi,\nu} \nu^T_{\xi}\right]^{-1} \in \mathbb{R}^{n + m_z \times n + m_z} 
(3.67)$$

$$C_3(t) = \Pi_{\xi} \Pi_{\xi}^T + \nu_z \nu_{\xi}^T = \nu_z \nu_{\xi}^T \in \mathbb{R}^{m_z \times m_z}.$$  

The dynamics of $m(t)$ still contain $d\xi(t)$ which is a function of the unobservable variable $z(t)$. For this reason, we are not able to predict $m(t)$ into the future. By introducing a new Brownian motion $\tilde{Z}_{\xi}'(t)$, we eliminate the dependency on $d\xi(t)$. Using Lipster and Shiryaev (2001a, Thorem 12.5), we define

$$d\tilde{Z}_{\xi}'(t) = \Pi_{\xi}^{-1}(t)d\xi(t) = \Pi_{\xi}^{-1}(t)A_{\xi}(t)z(t)dt + \tilde{Z}_{\xi}(t). \tag{3.68}$$

Moreover, in Lipster and Shiryaev (2001a, Thorem 12.5) it is proven that $\tilde{Z}_{\xi}'(t)$ is a standard Brownian motion with the same filtration as $\tilde{Z}_{\xi}(t)$. Informally speaking, both Brownian motion generate the same information and thus, the same dynamics. Therefore, we can replace $\tilde{Z}_{\xi}(t)$ with $\tilde{Z}_{\xi}'(t)$ in (3.63) and obtain the completely observable system of equations

$$dm_z(t) = [B_{\xi}(t)m_z(t) + B_{\xi}(t)\xi(t) + b(t)]dt + [C_{1}(t) + n(t)A_{\xi}(t)^T]C_2(t)\Pi_{\xi'}d\tilde{Z}_{\xi}'(t)$$

$$d\xi(t) = [A_{\xi}(t)\xi(t) + A_{\xi}(t)z(t) + a(t)]dt + \Pi_{\xi}d\tilde{Z}_{\xi}'(t). \tag{3.69}$$

Decomposing $\xi(t)$, we obtain

$$dp(t) = \left[\mu(t, x(t), m_z(t)) - \frac{1}{2} \text{diag}\{\Sigma(t)\}\right]dt + \sigma(t)d\tilde{Z}_{\xi}'(t)$$

$$dx(t) = [A_{x}(t)x(t) + A_{z}(t)m_z(t) + a(t)]dt + \nu_x(t)d\tilde{Z}_{\xi}'(t), \tag{3.70}$$

where the relationships of $\sigma(t) \in \mathbb{R}^{n \times n + m_x}$ and $\nu_x(t) \in \mathbb{R}^{m_x \times n + m_x}$ with $\sigma(t)$ and $\nu_x(t)$ can be computed with the help of the covariance matrices

$$\left[\begin{array}{cc}
\sigma(t)\sigma^T(t) & \sigma(t)\nu_x^T(t) \\
\nu_x(t)\sigma^T(t) & \nu_x(t)\nu_x^T(t)
\end{array}\right] = \left[\begin{array}{cc}
\sigma(t)\sigma^T(t) & \sigma(t)\rho_{xx}(t)\nu_x^T(t) \\
\nu_x(t)\rho_{xx}(t)\sigma^T(t) & \nu_x(t)\nu_x^T(t)
\end{array}\right].$$

The dynamics of the risky asset with the transformed Brownian motion is given by

$$\frac{dP_i(t)}{P_i(t)} = \mu(t, x(t), m_z(t))dt + \sigma(t)d\tilde{Z}_{\xi}'(t). \tag{3.71}$$

It is important to notice that $P(t)$, with dynamics given in (3.71) and Brownian motion $\tilde{Z}_{\xi}'(t)$, has same values as when generated by the dynamics (3.1) with drift given by (3.57).
Equivalent problems of portfolio optimization

**Theorem 3.5.** The problem of portfolio optimization under partial information with unobservable factors $z(t)$ which is given by

$$\max_{u(\cdot) \in \mathbb{R}^n} E\left[\frac{1}{\gamma} W(T)^\gamma\right]$$

s.t. $$dW(t) = W(t)[F_0(t)x(t) + f_0(t) + u^T(t)(F(t)x(t) + H(t)z(t) + f(t))]dt$$

$$+ W(t)u^T(t)\sigma(t)dZ_P$$

$W(0) = W_0$

$$dx(t) = [A_x(t)x(t) + A_z(t)z(t) + a(t)]dt + \nu_x dZ_x(t)$$

$x(0) = x_0$

$$dz(t) = [B_x(t)z(t) + B_z(t)x(t) + b(t)]dt + \nu_z dZ_z(t)$$

$z(0) = z_0$, \hspace{1cm} (3.72)

can be replaced by the following problem of portfolio optimization with full information:

$$\max_{u(\cdot) \in \mathbb{R}^n} E\left[\frac{1}{\gamma} W(T)^\gamma\right]$$

s.t. $$dW(t) = W(t)[F_0(t)x(t) + f_0(t) + u^T(t)(F(t)x(t) + H(t)m_z(t) + f(t))]dt$$

$$+ W(t)u^T(t)\sigma(t)d\bar{Z}_P(t)$$

$W(0) = W_0$

$$dx(t) = [A_x(t)x(t) + A_z(t)m_z(t) + a(t)]dt + \nu_x d\bar{Z}_x(t)$$

$x(0) = x_0$

$$dm_z(t) = [B_z(t)m_z(t) + B_x(t)x(t) + b(t)]dt + [C_1(t) + \eta(t)A_z(t)T]C_2(t)\nu_z d\bar{Z}_x(t)$$

$m_z(0) = E[z(0)| (p(0), x(0))], \hspace{1cm} (3.73)$

where the error covariance matrix $\eta(t)$ is a function of time given by (3.66) and all other matrices are defined above.

**Proof.** We have shown that (3.63) can be replaced by the Kalman filter given in (3.69). The Brownian motions $\bar{Z}_x(t)$ and $\bar{Z}_x(t)$ possess the same filtration $\{\mathcal{F}_t\}_{t \geq 0}$ as proved in Lipster and Shiryaev (2001a, Thorem 12.5). The dynamics of the risky assets described
in (3.71) is equivalent to (3.1) with drift given by (3.57). Therefore, the wealth dynamics and the information generated by (3.72) and (3.73) are equivalent. The solution of portfolio optimization problem (3.72) can be derived by solving the problem of portfolio optimization stated in (3.73).

The full information problem stated in (3.73) can be reformulated in such a way that the problem fits the format of portfolio optimization problem (3.31). Using Theorem 3.1, we find the solution to the problem of portfolio optimization with partial information under the CRRA utility.

The solution procedure is not illustrated any further and can be found in Keel (2005). The same analysis can be performed for the problem of portfolio optimization under partial information with the CARA utility.

An important result in continuous-time optimal control of LQG problems is the separation of estimation and control. When the problem can be separated, the solution to the controller is not affected by the estimation. We can thus solve the partial information optimal control problem as if we would possess full information. In the case of portfolio optimization, the diffusion of the factor process of the equivalent problem is affected by the error covariance matrix of the Kalman filter. This alters the factor dynamics and the separation is not possible anymore. The main reason for the “non-separability” is the correlation of the Brownian motions for the risky-assets and the factors.

3.2.5 Application with German data

In a small scale example, we use the results derived in Section 3.2.3, in order to test a two asset model with German data with the CRRA utility. For the two data sets, we use as assets the stock market ($S$) and a short-term money market account as risk free investment. The factors for the stock market are a long-term interest rate ($l$), the short term money market interest rate ($s$), the dividend yield ($dy$), and the spread between short and long term government bond interest rates ($\delta$).

All the data sets were obtained from Thomson DATASTREAM (DS). The data set consists of the DS German total stock market index and the corresponding dividend yield data. The long term interest rate is obtained from the DS 10-year and 7 year German government bond index (constant maturities). The short rate is the 1 week Euro-Mark (London) interest rate. The interest rate spread is calculated between the DS 10 year and
the 2 year German government bond index. All German data sets started on January 1st 1980 and finished on January 20th 2004 and have a weekly frequency.

**Table 3.1.** Continuous-time parameter estimates for German data from 1980 to 2004 with t-statistics in parentheses

<table>
<thead>
<tr>
<th></th>
<th>constant</th>
<th>( \frac{dS_t}{S_t} )</th>
<th>( l )</th>
<th>( dy )</th>
<th>( s )</th>
<th>( \delta )</th>
</tr>
</thead>
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<tr>
<td></td>
<td></td>
<td>-0.31</td>
<td>-2.42</td>
<td>4.79</td>
<td>4.11</td>
<td>18.56</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-1.35)</td>
<td>(-0.49)</td>
<td>(0.67)</td>
<td>(0.96)</td>
<td>(1.94)</td>
</tr>
<tr>
<td>( dl )</td>
<td>0.01</td>
<td>-0.1316</td>
<td>-0.038</td>
<td>0.017</td>
<td>-0.153</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-2.39,99)</td>
<td>(-0.013)</td>
<td>(0.07)</td>
<td>(-0.35)</td>
<td></td>
</tr>
<tr>
<td>( ddy )</td>
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<td>0.18</td>
<td>-0.43</td>
<td>-0.146</td>
<td>-0.33</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.32)</td>
<td>(-296.05)</td>
<td>(-1.06)</td>
<td>(-1.34)</td>
<td></td>
</tr>
<tr>
<td>( s )</td>
<td>0.04</td>
<td>2.16</td>
<td>1.03</td>
<td>-3.1</td>
<td>-4.65</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(1.87)</td>
<td>(1.47)</td>
<td>(-99.1)</td>
<td>(-5.01)</td>
<td></td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.03</td>
<td>0.04</td>
<td>0.65</td>
<td>-0.58</td>
<td>-1.59</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>(2.83)</td>
<td>(0.13)</td>
<td>(-2.42)</td>
<td>(-122.31)</td>
<td></td>
</tr>
</tbody>
</table>

The estimation of the stock market model is done on a rolling basis. Starting with 10 years of data (January 1980 to January 1990), the parameters of the model are estimated and the optimal asset allocation is computed using the methods presented in Section 3.2.3. The portfolio composition is then tested on the next data point and the resulting portfolio wealth is computed. The next observation is added to the data set and the models are estimated again. In this way, the test moves through the data set from 1990 to 2004. The portfolio allocation decisions are always tested on out-of-sample data. The models are estimated using their discrete-time equivalent and based on the exact discretization. The continuous-time model was calculated using the methods presented in Kellerhals (2001) or Campbell et al. (2004).

**Table 3.2.** Cross correlation and standard deviation of residuals for German data from 1980 to 2004

<table>
<thead>
<tr>
<th></th>
<th>( \sigma )</th>
<th>( l )</th>
<th>( dy )</th>
<th>( s )</th>
<th>( \delta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta l )</td>
<td>0.186</td>
<td>-0.121</td>
<td>-0.892</td>
<td>-0.042</td>
<td>-0.069</td>
</tr>
<tr>
<td>( dl )</td>
<td>0.09</td>
<td>1</td>
<td>0.05</td>
<td>0.11</td>
<td>0.016</td>
</tr>
<tr>
<td>( dcp )</td>
<td>0.07</td>
<td>0.05</td>
<td>1</td>
<td>0.019</td>
<td>0.02</td>
</tr>
<tr>
<td>( s )</td>
<td>0.131</td>
<td>0.11</td>
<td>0.019</td>
<td>1</td>
<td>-0.07</td>
</tr>
<tr>
<td>( \delta )</td>
<td>0.089</td>
<td>0.016</td>
<td>0.02</td>
<td>-0.07</td>
<td>1</td>
</tr>
</tbody>
</table>
We test four strategies the data set. First, a myopic strategy, where the investor possesses a single period objective, i.e., $T = 0$ (Brennan et al. 1997). Second, a strategy where the time horizon is at every time in the simulation two years ahead, call this the two year constant time horizon strategy and a five year constant time horizon strategy. Third, a strategy where the assumption is that the terminal date is fixed at the 20th of January 2004. Under the last strategy, the portfolio allocation is computed with a horizon which is the remaining time until 20th January 2004. All four strategies were computed with risk aversion level $\gamma$ of $-2$, $-4$, $-6$, $-8$, and $-10$. The results for the myopic strategy are reported in tables found in Herzog, Dondi and Geering (2004). Table 3.1 and Table 3.2 report the parameter of the model for all German data. In Figure 3.1, the results of the four strategy for risk aversion level $\gamma = -4$ are shown.

From the results, two major conclusions can be drawn when we compare the various portfolios with the stock market index. First, the partial predictability of returns help to achieve a better performance than the stock index. The performance is better concerning returns as well as risk adjusted returns. This conclusion can be drawn from simply considering the results of the myopic strategy. The time frame used for the testing includes the bull market of the 1990’s and the bear market from 2000 to 2003. Often studies use only the data of the 1990’s to show that their strategies work. The results show that especially in the bear market the dynamic strategies achieve markedly better results than the index.

Second, the larger the time horizon the larger the demand for stocks. Since the dividend yield residuals and the stock market residuals are highly negative correlated, see Table 3.2, and the dividend yield itself is mean reverting, the stock market volatility grows with a small proportion with time and thus making investments in the stock market saver for those investors with a long horizon than those with short horizon. As other authors have noted, see Campbell et al. (2003) or Campbell and Viceira (2002), the demand for risky investments does not only depend on level of risk aversion, as the single period models predict, but also on the time horizon of the investor.

In Figure 3.2, the effect of different time horizons on the proportion of investments in the stock market for German data are shown. The stock demand is calculated by using the projected long-term values of $x(t)$, which we compute by solving for the steady state solution, i.e., $\bar{x} = -A^{-1}a$. For each of the three levels of risk aversion, the investment
Comparison of different time horizons for $\gamma = -4$ (Germany)

**Fig. 3.1.** Results for German data with $\gamma = -4$

...into stock market markedly increases with increasing time horizon. This explains why the demand for stocks for a long-term investor is larger than for a short-term investor.

**Fig. 3.2.** Influence of the time horizon on the demand for stocks for German data
3.3 Numerical solutions for portfolio optimizations

Other applications of this continuous-time framework of portfolio optimization in the case of investments in alternative assets is reported in Keel, Herzog and Geering (2004).

3.3 Numerical solutions for the portfolio optimization problem with general asset price SDEs

A necessary condition for an optimal solution of stochastic optimal control problems is the HJB equation, a second-order partial differential equation that is coupled with an optimization. Unfortunately, the HJB equation is difficult to solve analytically. Only for some special cases, with simple cost functionals and state equations, analytical solutions are known, e.g. the LQ regulator problem. In the following, we provide a successive approximation algorithm for a numerical solution of the HJB equation to tackle problems with no known analytical solution. The remaining material in this section follows closely the work presented in Peyrl (2003) and Peyrl, Herzog and Geering (2004b).

3.3.1 Successive approximation algorithm

Consider the $n$-dimensional stochastic process $y$ which is governed by the given stochastic differential equation (SDE)

$$dy = D(t, y, u)dt + S(t, y, u)dZ_y,$$

$$y(0) = y_0,$$  \hspace{1cm} (3.74)

where $dZ_y$ denotes $k$-dimensional uncorrelated standard Brownian motion defined on a fixed, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. The vector $u$ denotes the control variables contained in some compact, convex set $U \subset \mathbb{R}^m$, $D(t, y, u)$ and the diffusion $S(t, y, u)$ are given functions

$$D : [0, T] \times G \times U \rightarrow \mathbb{R}^n,$$

$$S : [0, T] \times G \times U \rightarrow \mathbb{R}^{n \times k},$$

for some open and bounded set $G \subset \mathbb{R}^n$. The value functional of our problem starting at arbitrary time $t \in (0, T)$ and state $y \in G$ with respect to a fixed control law $u$ is defined by

$$\mathcal{J}(t, y, u) = \mathbb{E} \left[ \int_t^T L(s, y, u)ds + M(\tau, y(\tau)) \right],$$  \hspace{1cm} (3.75)

where $\mathbb{E}$ denotes the expectation operator. The scalar functions $L$ and $M$ are defined as
The final time of our problem denoted by $\tau$ is the time when the solution $y(t)$ leaves the open set $Q = (0, T) \times G$: $\tau = \inf \{ s \geq t \mid (s, y(s)) \notin Q \}$. Our aim is to find the admissible feedback control law $u$ which maximizes the value of the functional $J(t, y, u)$ leading to the cost-to-go or value function $J(t, y)$:

$$J(t, y) = \max_{u(t, y) \in U} J(t, y, u).$$

This leads to the following version of the standard stochastic optimal control problem given by

$$\max_{u(t)} \mathbb{E} \left[ \int_0^\tau L(t, y, u) dt + M(\tau, y(\tau)) \right]$$

s.t.

$$dy(t) = D(t, y, u) dt + S(t, y, u) dZ_y$$

$$y(0) = y_0.$$  

In the following, we state the Hamilton-Jacobi-Bellman equation. The reader is referred to Yong and Zhou (1999) or Fleming and Rishel (1975) for a detailed derivation and proof for the HJB equation. By introducing the differential operator

$$A(t, y, u) = \frac{1}{2} \sum_{i,j=1}^n \Sigma_{ij} \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{i=1}^n D_i \frac{\partial}{\partial y_i},$$

where the matrix $\Sigma = (\Sigma_{ij})$ is given by $\Sigma(t, y, u) = S(t, y, u) S^T(t, y, u)$, the HJB equation can be written as follows

$$J_t + \max_{u \in U} \{ L(t, y, u) + A(t, y, u) J \} = 0, \text{ for } (t, y) \in Q,$$  \hspace{1cm} (3.76)

with the boundary condition

$$J(t, y) = M(t, y), \hspace{1cm} (t, y) \in \partial^*Q,$$  \hspace{1cm} (3.77)

where $\partial^*Q$ denotes a closed subset of the boundary $\partial Q$ such that $(\tau, y(\tau)) \in \partial^*Q$ with probability 1:

$$\partial^*Q = ([0, T] \times \partial G) \cup (\{T\} \times G).$$

The HJB equation (3.76) is a scalar linear second-order PDE which is coupled with an optimization over $u$. This fact makes solving the problem so difficult (apart from computational issues arising from problem sizes in higher dimensions). Note that the coefficients
of (3.76) are dependent on \( t, y, \) and \( u \). Solving the PDE and optimization problem at once would lead to unaffordable computational costs. Chang and Krishna (1986) propose a successive approximation algorithm which will be used in the following.

At first, we give the following lemma for the boundary value problem with a given (fixed) control strategy \( u \). With this lemma we compute the value of the functional given by (3.75) for an arbitrary control law (no optimization so far).

**Lemma 3.6 (without proof).** Let \( J^u \) be the solution of the boundary value problem corresponding to the arbitrary but fixed control law \( u \in U \):

\[
J^u_t + L(t, y, u) + A(t, y, u)J^u = 0, \quad (t, y) \in Q, \quad (3.78)
\]

with boundary condition

\[
J^u(t, y) = M(t, y), \quad (t, y) \in \partial^*Q. \quad (3.79)
\]

Then

\[
J^u(t, y) = F(t, y, u), \quad (3.80)
\]

where \( F(t, y, u) \) denotes the value functional. For the proof refer to Section V\S7 in (Fleming and Rishel 1975, pp. 129–130).

We present in the following the successive approximation algorithm. To begin with, assume \( J^k \) to be the solution of (3.78)–(3.79) corresponding to the arbitrary but fixed control law \( u^k \in U \)

\[
J^k_t + L(t, y, u^k) + A(t, y, u^k)J^k = 0, \quad (t, y) \in Q, \quad (3.81)
\]

\[
J^k(t, y) = M(t, y), \quad (t, y) \in \partial^*Q. \quad (3.82)
\]

Then the sequence of control laws is obtained by

\[
u^{k+1} = \arg \max_{u \in U} \{ L(t, y, u) + A(t, y, u)J^k \}. \quad (3.83)
\]

After computing the function \( J(t, y) \) we maximize over \( u \) in order to improve the fixed control law \( u^k \). Note that because of (3.81) and (3.83)

\[
L(t, y, u^{k+1}) + A(t, y, u^{k+1})J^k \geq L(t, y, u^k) + A(t, y, u^k)J^k. \quad (3.84)
\]
Now let $J^{k+1}$ be the solution of the boundary value problem (3.78)–(3.79) corresponding to the newly calculated control law $u^{k+1}$:

$$J^{k+1}_t + L(t, y, u^{k+1}) + A(t, y, u^{k+1})J^{k+1} = 0, (t, y) \in Q,$$  \hspace{1cm} (3.85)

$$J^{k+1}(t, y) = M(t, y), \hspace{1cm} (t, y) \in \partial^* Q.$$  \hspace{1cm} (3.86)

**Lemma 3.7.** Let the sequences of control laws $u^k$ and their affiliated value functionals $J^k$ be defined as above. Then the sequence $J^k$ satisfies

$$J^{k+1} \geq J^k.$$  \hspace{1cm} (3.87)

**Proof.** Define $P = J^{k+1} - J^k$ on $Q$ and compute $P_t + A(t, y, u^{k+1})P$:

$$P_t + A(t, y, u^{k+1})P = J^{k+1}_t + A(t, y, u^{k+1})J^{k+1} - J^k_t - A(t, y, u^{k+1})J^k.$$  \hspace{1cm} (3.88)

Adding and subtracting $L(t, y, u^{k+1})$ on the right-hand side and using (3.85) and (3.84) yields

$$P_t + A(t, y, u^{k+1})P = J^{k+1}_t + A(t, y, u^{k+1})J^{k+1} + L(t, y, u^{k+1})$$

$$- \left[ J^k_t + A(t, y, u^{k+1})J^k + L(t, y, u^{k+1}) \right].$$

Thus,

$$P_t + A(t, y, u^{k+1})P \leq - \left[ J^k_t + A(t, y, u^{k})J^k + L(t, y, u^{k}) \right].$$

The squared bracket term on the right-hand side of the previous equation is equal to zero due to (3.81). Hence,

$$P_t + A(t, y, u^{k+1})P \leq 0.$$  \hspace{1cm} (3.89)

Next, we compute the expectation of $P(\tau, y(\tau))$ by integration of its total derivative according to Itô’s rule subject to stochastic system dynamics and the last control law $u^{k+1}$:

$$E[P(\tau, y(\tau))] = P(t, y) + \mathbb{E} \left[ \int_t^\tau \{ P(s, y) + A(s, y, u^{k+1})P(s, y) \} ds \right].$$

By definition $P(\tau, y(\tau)) = J^{k+1}(\tau, y(\tau)) - J^k(\tau, y(\tau))$ vanishes on $\partial^* Q$ because of (3.82) and (3.86) and using (3.89), we obtain
3.3 Numerical solutions for portfolio optimizations

\[ P(t, y) = \mathbb{E}[P(\tau, y(\tau))] - \mathbb{E} \left[ \int_0^\tau \left\{ P_t(s, y) + \mathcal{A}(s, y, u^{k+1}) P(s, y) \right\} ds \right] \geq 0. \]

Since \( P(t, y) = J^{k+1} - J^k \), it follows that \( J^{k+1} \geq J^k \).

**Theorem 3.8.** Let the sequences of control laws \( u^k \) and their corresponding value functionals \( J^k \) be defined as above. Then they converge to the optimal feedback control law \( u(t, y) \) and the value function \( J(t, y) \) of the optimal control problem, i.e.,:

\[ \lim_{k \to \infty} u^k(t, y) = u(t, y) \text{ and } \lim_{k \to \infty} J^k(t, y) = J(t, y). \]

**Proof.** Due to Theorem 15 in (Friedman 1964, p. 80) (see also Theorem VI.6.1 in Fleming and Rishel (1975, pp. 208–209)) \( J^k, J^k_{x^i} \) converge uniformly on \( Q \) to \( J^*, J^*_{x^i} \), \( J^k_{x^i x^j} \) weakly to \( J^*_i, J^*_{x_i x_j} \) (for necessary properties of \( D, S, L \) and \( M \) refer to Yong and Zhou (1999)). Consider now the limit of (3.85) for \( k \to \infty \):

\[ \lim_{k \to \infty} \left[ J^{k+1}_t + L(t, y, u^{k+1}) + \mathcal{A}(t, y, u^{k+1}) J^{k+1} \right] = 0. \]

Since the limits of \( J^k \) and its derivatives exist, we may alter iteration indices:

\[ 0 = \lim_{k \to \infty} \left[ J^k_t + L(t, y, u^{k+1}) + \mathcal{A}(t, y, u^{k+1}) J^k \right]. \]

By using (3.83) and substituting \( \lim_{k \to \infty} J^k, \lim_{k \to \infty} J^k_t \) with their limits \( J^* \) resp. \( J^*_t \), we obtain

\[ 0 = \lim_{k \to \infty} \left[ J^k_t + \max_{u \in U} \left\{ L(t, y, u) + \mathcal{A}(t, y, u) J^k \right\} \right] \]
\[ = J^*_t + \max_{u \in U} \left\{ L(t, y, u) + \mathcal{A}(t, y, u) J^* \right\}, \quad (t, y) \in Q. \]

Since \( J^*(t, y) = M(t, y) \) on \( \partial^* Q \), \( J^*(t, y) \) solves (3.76) and (3.77) and hence is the solution of the optimal control problem. Consequently,

\[ u^*(t, y) = \arg \max_{u \in U} \left\{ L(t, y, u) + \mathcal{A}(t, y, u) J^* \right\} \]

is the optimal feedback control law.

We now state the successive approximation algorithm:

1. \( k = 0 \); choose an arbitrary initial control law \( u^0 \in U \).
2. Solve the boundary value problem for the fixed control law $u^k$, i.e., $J^k(t, y)$ solves

\[
\frac{\partial J^k}{\partial t} + L(t, y, u^k) + A(t, y, u^k)J^k = 0, \quad (t, y) \in Q, \\
J^k(t, y) = M(t, y), \quad (t, y) \in \partial^*Q.
\]

3. Compute the succeeding control law $u^{k+1}$, i.e., solve the optimization problem:

\[
u^{k+1} = \arg \max_{u \in U} \{ L(t, y, u) + A(t, y, u)J^k \}.
\]

4. $k = k + 1$; back to step 1.

To summarize, the algorithm is an iterative approach which separates the optimization from the boundary value problem. In other words, the difficult problem of finding a numerical solution of the HJB equation (3.76) has been separated into two easier problems which are solvable by standard numerical tools:

- Solving boundary value problem (3.81)–(3.82).
- Optimization of the nonlinear function with possible control variable constraints (3.83).

### 3.3.2 Computational implementation and limitations

The boundary value problem (3.81)–(3.82) is a scalar second-order PDE with nonlinear coefficients and thus, can be solved by standard methods for linear parabolic PDEs. With the mixed derivatives left out of consideration, (3.81) has the structure of the heat equation with advection and source terms. However, in contrast to the heat problem, the HJB-PDE has a terminal condition $J^k(T, y) = M(T, y)$ rather than an initial condition and therefore has to be integrated backwards in time. With the simple substitution $\bar{t} = T - t$, the problem is converted into a PDE which can be integrated forward in time (i.e. $\bar{t}$ runs from 0 to $T$):

\[
\frac{\partial J^k}{\partial \bar{t}} - \left[ L(\bar{t}, y, u^k) + A(\bar{t}, y, u^k)J^k \right] = 0, \quad (\bar{t}, y) \in Q, \\
J^k(\bar{t}, y) = M(\bar{t}, y), \quad (\bar{t}, y) \in \partial^*Q.
\]

Since available standard codes were not able to handle mixed derivatives and coefficients dependent on $t$, $x$ and $u$, we developed our own solvers. Although many different methods for solving PDEs have been developed so far, there does not exist one which is best-suited for all types of applications. We use finite difference schemes as they are both well-suited for simple (rectangular) shaped domains $Q$ and rather easy to implement. Our solver
employs an implicit scheme and uses upwind differences for the first order derivatives for
stability reasons. Second order and mixed derivatives are approximated by central space
differences. Although not required by (3.81), we extended the code to handle an additional
linear term of $J^k$ and Neumann boundary conditions on $\partial G$. This enables us to get better
results for some financial applications which can be significantly simplified by a special
transformation adding the linear term (for details see Section 3.3.3).

Optimization

According to Section 3.3.1, we are facing the following nonlinear optimization problem to
compute the succeeding control law:

$$u^{k+1} = \arg \max_{u \in U} \left\{ L(t, y, u) + A(t, y, u)J^k \right\}.$$ 

Since we approximate $J^k(t, y)$ on a finite grid, the optimization must be solved for every
grid point. This can be accomplished by standard optimization tools. For problems with
simple functions, it maybe possible to obtain an analytical solution for the optimal control
law.

Numerical Issues

Since the number of unknown grid points at which we approximate $J^k(t, y)$ grows by an
order of magnitude with dimension and grid resolution, we have to face issues of memory
limitations and computation time and accuracy.

The PDE solvers described above require the solution of large systems of linear equa-
tions. The coefficient matrix of these linear systems is banded and therefore strongly
encourages the use of sparse matrix techniques to save memory. Furthermore, applying
indirect solution methods for linear systems such as successive over-relaxation provides
higher accuracy and memory efficiency than direct methods for details see Press, Teukol-
sky, Vetterling and Flannery (2002). The method described in Section 3.3.1 is implemented
in a MATLAB code. MATLAB’s memory requirement for storing the coefficient matrix
corresponding to the implicit scheme is outlined examplarily in Peyrl (2003). While we
need only 2.4 MB for a two dimensional grid of 150 points in each space coordinate, the
matrix will allocate approximately 732 MB in the three dimensional case. Considering
the fact that today’s 32-bit architectures limit the virtual memory for variable storage to
1.5 GB (0.5 GB are needed by MATLAB), it is obvious that our solvers are restricted to rather coarse grids and low dimensions, showing that Bellman’s curse of dimensionality can not be overcome by the successive approximation. However, enhanced numerical methods such as alternating direction implicit (ADI) methods or domain decomposition algorithms could solve bigger problems. For further information on these topics the reader is referred to Strikwerda (1989) and Sun (1993).

### 3.3.3 Example and discussion of the numerical approach

We consider a portfolio optimization problem where an investor has the choice of investing in the stock market or to put his money in a bank account. We model the stock market as geometric Brownian motion with time-varying and stochastic mean returns and time-varying and stochastic diffusion term (volatility).

**Asset model**

Mathematically, the stock market model and the bank account model are given by

\[
\frac{dP_1(t)}{P_1(t)} = [Fx(t) + f]dt + \sqrt{v(t)}dZ_1, \quad (3.90)
\]

\[
\frac{dP_0(t)}{P_0(t)} = r_0dt, \quad (3.91)
\]

where \( P_1(t) \) is the stock market index, \( P_0(t) \) is the value of the bank account, \( r_0 \) is the risk-free interest rate, \( x(t) \) is a factor that directly affects the mean return, and \( v(t) \) is the square of the volatility. The model for stock market volatility is proposed by Stein and Stein (1991). The dynamics for the factor and the volatility are modelled as

\[
dx(t) = (a_1 + A_1x(t))dt + \nu dZ_2, \quad (3.92)
\]

\[
dv(t) = (a_2 + A_2v(t))dt + \sigma \sqrt{v(t)}dZ_3, \quad (3.93)
\]

where \( a_1, A_1, a_2, A_2, \nu, \) and \( \sigma \in \mathbb{R} \) are parameters describing the models. The volatility model is described in detail in Section 3.1.5, the factor model in Section 3.2. Furthermore, we assume that all three Brownian motions are correlated: \( dZ_1dZ_2 = \rho_{12}dt, \) \( dZ_1dZ_3 = \rho_{13}dt, \) and \( dZ_2dZ_3 = \rho_{23}dt. \)

**Portfolio Optimization**

The objective of the investor is maximizing the power utility of his wealth at a finite fixed time terminal date \( T: \max_{\gamma} \frac{1}{\gamma} W^\gamma(T). \) Thus, the portfolio optimization problem is
3.3 Numerical solutions for portfolio optimizations

\[
\max_{u \in [-1, 1]} \frac{1}{\gamma} W^\gamma(T)
\]

s.t.

\[
dW(t) = W(t)(r_0 + u(t)(Fx(t) + f - r_0))dt + W(t)u(t)\sqrt{v(t)}dZ_1
\]

\[
dx(t) = (a_1 + A_1x(t))dt + \nu dZ_2
\]

\[
dv(t) = (a_2 + A_2v(t))dt + \sigma \sqrt{v(t)}dZ_3,
\]

(3.94)

where \(\gamma < 1\) is coefficient of risk aversion, \(W(0) = W_0\), \(x(0) = x_0\), and \(v(0) = V_0\) are the initial conditions, and \(u(t) \in [-1, 1]\). We make the assumption that both of the processes \(x(t)\) and \(v(t)\) are measurable and we have both of the time series to estimate the model parameters. The assumption that volatility is directly observable is highly questionable, since volatility is normally obtained through the filtration of the stock prices. Instead of using the stock price process as the mean to observe the volatility, we may use a directly observable time-series such as the volatility index. Otherwise we need to resort to estimation techniques such as proposed by Chacko and Viceira (2003).

The HJB equation for portfolio problem (3.94) (suppressing \(t\) in all functions for compactness) using \(J = \frac{1}{\gamma} W^\gamma(t)H(t, x, v)\) is given by

\[
H_t + \max_{u \in [-1, 1]} \left\{ \gamma \left( r_0 + u(Fx + f - r_0) + \frac{1}{2} u^2 v(\gamma - 1) \right) H \\
+ (a_1 + A_1x + \gamma u \sqrt{v} \rho_{12} \nu) H_x + \frac{1}{2} \nu^2 H_{xx} \\
+ \frac{1}{2} \nu \sigma^2 H_{vv} + (a_2 + A_2v + \gamma uv \rho_{13} \sigma) H_v \\
+ \sqrt{v} \nu \rho_{23} \nu H_{vx} \right\} = 0,
\]

(3.95)

with terminal condition \(H(T, x, v) = 1\). This PDE (3.95) is a equation with two state variables \(x(t)\) and \(v(t)\). No analytical solution to this problem is known. The optimal control law in function of the unknown \(H\) is given by

\[
u^* = \frac{1}{(1 - \gamma)v} \left( (Fx + f - r_0) + \sigma \rho_{13} v \frac{H_v}{H} + \nu \sqrt{v} \rho_{12} \frac{H_x}{H} \right)
\]

(3.96)

and note that \(u(t) \in [-1, 1]\). If \(\nu^*\) violates the constraint, the value of \(\nu^*\) is set to the limits of the constraint. The controller is numerically computed based on the estimated parameters until 31/12/1997, see tables given Peyrl, Herzog and Geering (2004a). The time horizon is 1/1/2004, the HJB equation is solved with a monthly time step, and \(\gamma = -5\). In Figure 3.3, the optimal investment policy as function of \(x(t)\) and \(v(t)\) is shown.
The controller is the solution of (3.95). It is computed with the numerical procedure as outlined in Section 3.3.1. The optimal investment policy is almost a linear function of the expected returns and resembles a hyperbola with respect to the volatility. The investment in the risky stock market decreases when the volatility for each given expected return increases. Furthermore, the intertemporal hedging demand slightly increases the demand for stocks.

Simulation with Historical Data

The portfolio optimization problem (3.94) is applied with US data. We use the S&P 500 index as stock market index, the volatility index (VIX) as measurable time series for the volatility, and the difference between the E/P ratio of the S&P 500 and the 10 year Treasury Bond interest rate as factor that explains the expected mean returns. This factor is selected in accordance with the observations made by Shen (2003) and Oberuc (2004, Chapter 3 and 4). As short-term interest rate, we use the 1-month Treasury Notes. All five time series start on 1/1/1992 and end on 1/1/2004 and were obtained from Thomson DATASTREAM. We use the data from 1/1/1992 to 31/12/1997 to estimate the model parameters. The data from 1/1/1998 to 1/1/2004 is then used to evaluate the performance of our optimal portfolio controller. The parameters of model are again estimated based on the data from 1/1/1992 to 31/12/2000 in order to calculate the controller based on
more recent data. As similar procedure is used in Brennan et al. (1997) and Bielecki et al. (2000). The performance of the portfolio optimization therefore is always tested on the out-of-sample data.

![Graph showing portfolio value, S&P 500 index, bank account, and investment in stock market over time.]

**Fig. 3.4.** Results of the historical simulation with US data from 1998 to 2004

The result of the historical simulation is given in Figure 3.4 where the portfolio value, the S&P 500 index, the bank account, and the investment in the stock market are shown. Table 3.3 summarizes the statistics of the simulation. The portfolio outperforms both the stock market and the bank account. The investments in the stock market vary from -30% to 100%. The risk aversion used in this simulation is fairly high. Therefore, the portfolio exhibits a much lower volatility than the stock market. The portfolio manages to have higher returns than both assets and possesses a markedly higher Sharpe ratio than the stock market. The simulation shows that an investor could have exploited the partial predictability of the returns as well as the information on risk which is implied in the volatility index VIX.

<table>
<thead>
<tr>
<th>Table 3.3. Statistics of the historical simulation</th>
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<tbody>
<tr>
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<tr>
<td>return (%)</td>
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<tr>
<td>volatility (%)</td>
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<td>Sharpe ratio</td>
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</table>
Discussion of the numerical approach

The numerical approach allows us to solve portfolio optimization problems where the asset dynamics are arbitrary stochastic differential equations. Our numerical implementation however is limited to three state variables. This severely limits the applicability of the numerical scheme. In the case that we use the basic modelling approach presented in Section 3.1.1 and maximize power utility of terminal wealth, we are able to solve portfolio problems with an arbitrary number of risky assets. The economic factors affecting the assets are still restricted to three, since this is the limitation of our numerical HJB solver. One of the great strength of the numerical solution of portfolio optimization problems are the handling of control variable constraints. All analytical solutions presented in this work do not include control constraints. In general, it is very difficult to solve portfolio optimization problem with incomplete markets and control constraints. The successive approximation seems a suitable method to tackle numerically HJB problems of moderate size. The main problem remains solving the high-dimensional PDE and the curse of dimensionality.

3.4 Conclusion of continuous-time portfolio optimization

To conclude this chapter, we discuss the strength and weaknesses of the continuous-time modelling and portfolio optimization. Then we judge the continuous-time framework in respect to the applicability to real world long-term investment problems.

3.4.1 Weaknesses

In most portfolio optimization problems formulated in the continuous-time framework, it is very difficult to deal with constraints on the asset allocation or state constraints. Only special case such as no correlation of factor and asset dynamics or complete markets, the problem of portfolio optimization can be analytically solved with constraints, see Schroder and Skiadas (2003). In all other cases, one has to resort to numerical solutions.

Another important weakness of the modelling framework, used in Section 3.1, is the fact that all distribution of the risky assets are constructed by transformations of Brownian motions. This limits the distribution of risky assets to distributions that can be constructed on the basis of the Gaussian distribution.
Furthermore, it is difficult with continuous-time methods to construct multi-dimensional models of time-varying and stochastic volatility matrices. In the case of discrete-time models, we can resort to multi-dimensional GARCH models. Few continuous-time models for stochastic volatility in more than one dimension are stated in the literature. However, empirical analysis of asset prices suggests stochastic and time varying correlations and volatilities.

Moreover, continuous-time models are difficult to estimate. Especially, hidden factor models, such as non-observable factors that affect the mean or stochastic volatility models are inherently difficult to estimate, because they present filtering problems. This limits the applicability of these models for investment decisions.

3.4.2 Strengths

For many parametric portfolio optimizations, we are able to derive analytical solutions, see the examples given in Section 3.1.5 and 3.2. These results allow us to gain valuable insight into the properties of optimal portfolios. With the help of analytical solutions, we are able to analyze the effects of parameters of the optimization, e.g., the investment horizon or the risk aversion.

In contrast to numerical methods (or methods in discret-time), analytical solutions can be computed fast and efficiently. Instead of solving the HJB partial differential equation directly, we transform the problem into a system of ODEs which can then be more easily solved. This eases the computational burden dramatically.

Analytical solutions to continuous-time problems may be used as benchmark to judge the quality of suboptimal solutions and methods in discrete time. The insights from analytical portfolio optimizations help us to design optimal portfolio in other modelling frameworks or good suboptimal policies. Important observations are:

- Uncorrelated Brownian motions (residuals) of the factor and asset price process allow to use methods where the intertemporal hedging is not considered.
- The investment horizon effect saturates with longer horizons.
- The intertemporal hedging demand is strong for factors that affect the drift but less pronounced with factors that affect the volatility, see Chacko and Viceira (2005).
- In the case of the CRRA utility functions, the solution does not depend on current wealth.
These observations can be utilized for effective suboptimal strategies or other approximation methods, e.g., the horizon effect allows us to use a shorter but receding horizon strategy.
Discrete-time models and portfolio optimization

Portfolio optimizations with discrete-time models are a long established field of research. Many examples in discrete-time can be found in Bertsekas (1995, Chapter 4.3). Similar to the discussion at the beginning of Chapter 3, many models and solutions can be cited. In recent years, a growing number of real-world applications of portfolio management with discrete-time models have emerged. Insurance companies and pension funds pioneered these applications, which include the Russell-Yasuada investment system (Carino, Kent, Myers, Stacy, Sylvanus, Turner, Watanabe and Ziemba 1994), the Towers-Perrin System (Mulvey 1995), and the Siemens Austria Pension Fund (Ziemba 2003, Geyer, Herold, Kontriner and Ziemba 2004). In each of the applications, the investment decisions are linked to liability choices, and the system’s funds are maximized over time using multi-period optimization methods.

In this chapter, we discuss models and control strategies in discrete-time to solve the problem of long-term portfolio optimizations. In Section 4.1, we introduce the modelling framework, the wealth dynamics in discrete time and methods to compute control strategies. Two approximations for the dynamic programming method are discussed: Model Predictive Control method and Stochastic Programming Approximation.

In Section 4.2, the model predictive control method is applied to three different portfolio problems. First, we derive the strategy for a linear Gaussian factor model with different objective functions and constraints. Secondly, we derive an analytical method to solve the problem of mean-variance optimization for a model which is characterized by a stochastic volatility model for the covariance and a linear factor model for the expected returns. Third, we discuss a numerical procedure, where for portfolio dynamics with heavy-tailed white noise processes, a model predictive control method is derived which uses a coherent risk measure.
In Section 4.3, we present the applications of the stochastic programming approximation to a portfolio problem with transaction costs. Additionally, we introduce a dynamically coherent risk measure for asset liability situations. We discuss two ways to solve the underlying optimization task: a formulation that leads to a linear program (LP) and another that mimics the control strategies of Chapter 3.

In Section 4.4, the chapter concludes with a discussion of discrete-time modelling and control. The versatile modelling framework is discussed and compared with the continuous-time models. The control strategies developed in this chapter are discussed with respect to applicability and computational cost.

### 4.1 Discrete-time portfolio optimization and discrete-time optimal control

In this section, we discuss the basic modelling framework, the discrete-time wealth dynamics, and the conditions of optimality, i.e., the dynamic programming (DP) method. Since the DP method is difficult to apply to problems of realistic sizes, two approximation methods are discussed: the Model Predictive Control (MPC) method and the Stochastic Programming approximation (SPA).

The MPC method is a suboptimal control method which solves the control problem by repeatedly solving a series of open-loop control problems. The open-loop control problem always uses the most recent state value as the starting point and thus, introduces feedback into the system. Furthermore, we show that for utility functions that imply constant relative risk aversion, the MPC method yields good control decisions.

The SPA is used for problems where the optimization is path dependent or the objective function implies absolute relative risk aversion. We approximate the stochastic dynamics of the system by a multi-period sampling procedure (scenario approximation) which converges to the true distribution for an arbitrarily large number of scenarios.

#### 4.1.1 Asset price models

The returns of assets (or asset classes), in which we are able to invest, are described by

\[ r(t + 1) = \mu(t, x(t)) + \epsilon^r(t), \]  

(4.1)
where \( r(t) = (r_1(t), r_2(t), \ldots, r_n(t))^T \in \mathbb{R}^n \) is the vector of asset returns, \( \epsilon'(t) \in \mathbb{R}^n \) is a white noise process with \( E[\epsilon'(t)] = 0 \) and \( E(t)[\epsilon'(t)\epsilon'^T(t)] = \Sigma(t) \in \mathbb{R}^{n \times n} \) is the conditional covariance matrix of \( r(t) \), \( \mu(t, x(t)) \in \mathbb{R}^n \) is the expected return of \( r(t) \), and \( x(t) \in \mathbb{R}^m \) is the vector of factors. We assume that the conditional expectation and the conditional covariance can be time-varying and stochastic. The white noise process \( \epsilon'(t) \) is assumed to be strictly covariance stable, i.e., \( E[\epsilon'(t)\epsilon'^T(t)] < \infty \). The prices of the risky assets evolve according to

\[
P_i(t+1) = P_i(t)(1+r_i(t)), \quad P_i(0) = p_{i0} > 0,
\]

where \( P(t) = (P_1(t), P_2(t), \ldots, P_n(t)) \) denotes the prices of the risky assets. A locally risk-free bank account with interest rate \( r_0(t, x(t)) \) is given as

\[
P_0(t+1) = P_0(t)(1+r_0(t)), \quad P_0(0) = p_{00} > 0,
\]

where \( P_0(t) \) denotes the value of the bank account. The equations describing the price dynamics of the risky assets and the bank account are the discrete-time equivalents to (3.1) and (3.2), however with different assumptions about the white noise processes. The difference to the continuous-time models is that we do not explicitly specify the dependence of the volatility structure on the factors. The factor process affecting the expected returns of the risky assets and the interest rate of the bank account is described by

\[
x(t+1) = \Theta(t, x(t)) + \Psi(t, x(t))\epsilon^x(t), \quad (4.4)
\]

where \( \Theta(t, x(t)) \in \mathbb{R}^m, \Psi(t, x(t)) \in \mathbb{R}^{m \times k} \), and \( \epsilon^x(t) \in \mathbb{R}^k \) is a strictly covariance stable white noise process \( E[\epsilon^x(t)\epsilon^x^T(t)] < \infty \) with unity covariance. The factor process (4.4) is the discrete-time equivalent of (3.3).

In contrast to the continuous-time models, we have much more flexibility to model the uncertainty structure with discrete-time models. The white noise process of the risky asset dynamics \( \epsilon'(t) \) is not restricted to have a Gaussian distribution. Other distributions, which better reflect observed asset prices such as student t distribution, can be used. Realistic models for asset returns should reflect observed “stylized” facts of financial time series, such as the following: return data are not identically and independently distributed (i.i.d.) and possess low serial correlation; squared returns have high serial correlation; the standard deviation of return data, also called volatility, is observed to be time-varying and stochastic; volatility has the tendency to appear in clusters (persistence) and has
the tendency to increase more when equity returns are falling (leverage effect). Moreover, empirical data analysis of stock and bond prices suggests time variation of correlations. For this reasons, we choose to model the covariances by a multivariate GARCH\(^1\) model, which captures these stylized facts. The dynamics of the conditional covariance matrix \(\Sigma(t) = \sigma(t)\sigma^T(t)\) are described by a multi-variable GARCH process. The structure of the random process is given by

\[
\epsilon^r(t) = \sigma(t)\xi^r(t),
\]

where we assume that \(\xi^r(t)\) is strictly covariance-stable white noise process with \(E[\xi(t)] = 0\) and \(E[\xi^r(t)\xi^{rT}(t)] = I \in \mathbb{R}^{n \times n}\). The model of the covariance matrix is given by

\[
\Sigma(t) = V(t)\Lambda(t)V(t),
\]

where

\[
V(t) = \begin{pmatrix}
v_1(t) & 0 & \ldots & 0 \\
0 & v_2(t) & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & v_n(t)
\end{pmatrix}.
\]

The matrix \(V(t)\) is the diagonal matrix of the individual volatilities \(v_i(t)\) of the risky assets. The conditional covariance matrix \(\Sigma(t)\) must be a positive definite symmetric matrix and therefore \(v_i(t) > 0\) and \(\Lambda(t) > 0\). Any stochastic volatility model can be written as Markovian model, see Baillie and Bollerslev (1992) for GARCH models, and thus, we are able to write the dynamics of the returns with a Markovian structure. The GARCH structure is determined by the model we choose for the individual volatilities \(v_i(t)\) and the conditional correlation matrix \(\Lambda(t)\). This multi-dimensional GARCH approach replaces the factor dependency of the volatility matrix in the continuous-time models. We also assume that \(\epsilon^r(t)\) and \(\epsilon^x(t)\) are correlated.

4.1.2 Distribution of multi-period returns and prices

In the continuous-time case, the theory of stochastic differential equations and stochastic integrals allows in many cases to compute the long-term distributions from the differential descriptions of assets. In the discrete-time models, the long-term distribution is

\(^1\) Generalized AutoRegressive Conditional Heteroskedasticity
only known for few special cases such as Gaussian models. Only for self-similar distributions, e.g., the normal and the Poisson distribution (Johnson and Kotz 1995), we know the resulting distribution from the summation of individual distributions. The stochastic description of assets is often given in form of a difference equation and the resulting multi-period distributions of the assets are generally not known. Especially in the case of heavy-tailed distributions with finite second moment, the time aggregation cannot be expressed analytically. We have then to resort to simulations or a discrete approximation of the underlying distributions, the so-called scenario approach (Kall and Wallace 1994). In this case, the stochastic description is discrete in the realizations and we denote random variables by $\epsilon^s(t)$ where the superscript $s$ stands for the scenario number. In the case of linear Gaussian factor models, we are able to compute the multi-period distribution of the asset prices and wealth dynamics. In the case of heavy-tailed distributions or with GARCH models, no analytical time aggregation rules are known and thus we have to use the scenario approach.

### 4.1.3 Wealth dynamics in discrete-time

In contrast to the continuous-time models, we consider problems of portfolio optimization with (linear) transaction costs and without transaction costs. At first, we introduce the wealth dynamics without transaction costs and then with transaction costs.

#### Wealth dynamics without transaction costs

The investment into the bank account, denoted by $u_0(t)$ and the investments into the risky assets, denoted by $u(t)$, must sum up to 1, i.e., $u_0(t) + u^T(t)\mathbf{1} = 1$, where $\mathbf{1} = (1, 1, \ldots, 1)^T$, $u(t) = (u_1(t), \ldots, u_n(t))^T$, and $u_j(t)$ denotes the fraction of portfolio value (wealth) invested into the $j$-th risky investment. The portfolio return at time $t$ is given by

$$R(t) = u_0(t)r_0(t) + u(t)^T r(t) = r_0(t) + u(t)^T (r(t) - r_0(t)), \quad (4.8)$$

where $u_0(t) = 1 - \mathbf{1}^T u(t)$. The wealth dynamics are given by

$$W(t+1) = [1 + R(t)]W(t) + q(t)$$

$$= [1 + r_0(t) + u(t)^T (r(t) - r_0(t))] W(t) + q(t), \quad (4.9)$$
where \( W(t) \in \mathbb{R} \) denotes the wealth (portfolio value) at time \( t \) and \( q(t) \) possible in- or outflows due to non-capital gains or losses. Using (4.1) and (4.4) we obtain the discrete-time wealth dynamics as

\[
W(t+1) = W(t) \left[ 1 + r_0(t, x(t)) + u^T(t) (\mu(t, x(t)) - 1r_0(t, x(t))) \right] \\
+ q(t) + W(t)u^T(t)e^r(t) \\
x(t+1) = \Theta(t, x(t)) + \Psi(t, x(t))e^r(t) .
\] (4.10)

Here, similar to the continuous-time models, the dynamics of the factor and the dynamics of the wealth equation are coupled, but we only control the wealth equation. Often we work with the logarithm of the wealth (in the case \( q(t) = 0 \)) and corresponding wealth dynamics are given by

\[
w(t+1) = \ln \left( 1 + R(t) \right) + w(t) ,
\] (4.11)

where \( w(t) = \ln W(t) \). The difference \( w(t+1) - w(t) \) is interpreted as return of the portfolio, similar to the return definition based on the logarithmic difference of prices. The wealth dynamics can be used with an analytical description of the resulting distributions as well as with the scenario approach. The wealth dynamics with transaction costs are not discussed in this section, but they are explained in detail in Section 4.3.

4.1.4 Portfolio optimization and dynamic programming

Similar to the discussion presented in Section 3.1.3, we state that a multi-period problem of portfolio optimization for discrete-time models can be successfully solved by the method of dynamic programming.

We illustrate the dynamic programming (DP) technique in the case of portfolio optimizations without transaction costs. Mathematically, we state the problem of portfolio optimization as

\[
\max_{u(t), q(t) \leq 0} \left\{ E \left[ \sum_{t=0}^{T-1} U_1(q(t)) + U_2(W(T)) \right] \right\} \\
\text{s.t.} \\
W(t+1) = W(t) \left[ 1 + r_0(t, x(t)) + u^T(t) (\mu(t, x(t)) - 1r_0(t, x(t))) \right] \\
+ q(t) + W(t)u^T(t)e^r(t) \\
x(t+1) = \Theta(t, x(t)) + \Psi(t, x(t))e^r(t) .
\] (4.12)
where $U_1$ and $U_2$ are two strictly concave and monotone utility functions as defined in Section 2.2. The states of the system are the wealth $W(t)$ and the current value of the factors $x(t)$. In the case that the returns depend on economic factors and volatilities, the factor dynamics as well as the covariance of the white noise process can be written to accommodate this.

We state a general dynamic optimization problem (DOP) in discrete-time and show how to write the portfolio problem in this general form. Moreover, we state the dynamic programming (DP) algorithm to solve such DOP and discuss the difficulties in finding a solution.

In general, a dynamic optimization problem can be stated as

$$
J(t, y(t)) = \max_{u \in U} \left\{ \mathbb{E} \left[ \sum_{\tau=t}^{T-1} L(\tau, y, u) + M(T, y(T)) \right] \right\}
$$

s.t. $y(\tau+1) = D(\tau, y, u) + S(\tau, y, u) \epsilon(\tau)$,

(4.13)

This condition for optimality can be found in Bertsekas (1995, Chapter 1). We also assume that the expectation given in (4.14) exists. By setting $y(t) = (W(t), x(t))^T$, $\bar{u} = (u(t), q(t))^T$, $\epsilon(t) = (\epsilon^r(t), \epsilon^x(t))^T$, $L(t,y,u) = U_1(q(t))$, $M(t,y(T)) = U_2(W(T))$, and

$$
D(t,y,u) = \begin{bmatrix}
W(t) \left[ 1 + r_0(t,x(t)) + u^T(t)(\mu(t,x(t)) - 1r_0(t,x(t))) \right] + q(t) \\
\Theta(t,x(t))
\end{bmatrix}
$$

$$
S(t,y,u) = \begin{bmatrix}
W(t)u^T(t) & 0 \\
0 & \Psi(t,x(t))
\end{bmatrix}
$$

we convert (4.12) into the standard form of a DOP as given by (4.13) and may solve the problem by applying (4.14).
Ideally, we would like to use the DP algorithm to obtain a closed-form solution for $J(\cdot)$ or an optimal control strategy. In many cases, an analytical solution to the problem of portfolio optimization can be derived. Examples are given in Bertsekas (1995, Chapter 4.3). Similar to the analysis in Section 3.1.3, results are known for the case of independent asset returns and forecasts for the probability distribution of returns. We derive the DP solution to the discrete-time equivalent of the model given in Section 3.2 in Section A.2 of the Appendix. As in the case of continuous-time methods, we are only capable of deriving analytical solutions under restrictive assumptions, such as normal distribution of returns or no constraints on the asset allocation. Therefore, we need to resort to numerical approximations of the DP method. Since the numerical computation of DP solutions suffer from the curse of dimensionality, we need to approximate the DP algorithm, see Section 4.1.5 and 4.1.6, in order to solve models of realistic size and assumptions. The approximations are discussed for the general DP algorithm and the consequences for specific problems of portfolio optimizations are explained. The approximations are not discussed for portfolio problems, since they are applicable for any kind of DOP.

### 4.1.5 Model predictive control

For the problem of portfolio optimization with realistic assumptions for return distributions, e.g., non-normal distributions, and constraints on the asset allocation, the DP algorithm is very difficult to apply. Thus, we need suitable numerical methods or good suboptimal policies.

Model predictive control (MPC) is a technique developed for solving constrained optimal control problems for deterministic applications. Instead of directly solving the optimal control problem through the Hamilton-Jacobi-Bellman (HJB) equation or dynamic programming techniques (DP), MPC solves the problem in a receding horizon manner where a series of open-loop problems are consecutively solved. The constrained optimal control problem is solved, where the current control decision is obtained by calculating, at each sampling time, a finite horizon open-loop optimal control problem, using the actual state of the system as initial state. The optimization procedure yields an optimal control sequence and only the first control decision of the sequence is applied to the system. This procedure is repeated at each decision (sampling) time with a receding horizon. In deterministic problems, the MPC methods yields a closed-loop optimal control decision since
the closed-loop decision can be computed by solving the open-loop problem for every value of the state variables and every time step. An important advantage of this solution technique is its capability to deal with constraints on the control and state variables. For a detailed review of constrained MPC for deterministic control applications, refer to Mayne, Rawlings, Rao and Scokaert (2000) or Bemporad, Morari, Dua and Pistikoulos (2002) and the references therein.

### Stochastic Model Predictive Control

Most of the literature for MPC is concerned with solving control problems arising in technical applications. However, in recent years, the MPC approach has been extended to stochastic optimal control problems. The first line of research is the disturbance rejection problem with constraints in technical control applications, where the disturbances are modelled as stochastic processes. Examples of this line of research are found in van Hessel and Bosgra (2002b, 2002a), Li, Wendt and Wozny (2000), and Felt (2003). A second line of research in stochastic MPC is the application of MPC to inherently stochastic dynamic problems. Applications include stochastic resource allocation (Castanon and Wohletz 2002), multi-echelon supply chain networks (Seferlis and Giannelos 2004), sustainable development (Kouvaritakis, Cannon and Tsachouridis 2004), and option hedging (Meindl and Primbs 2004).

The solution of (4.13) is computed by the DP algorithm (4.14) which can be alternatively written as

\[
J(t, y(t)) = \max_{\pi(t) \in U} \left\{ \mathbb{E}[L(t, y(t), \pi(t))] + \max_{\pi(t+1) \in U} \left\{ \mathbb{E}[L(t+1, y(t+1), \pi(t+1))] + \ldots \\
+ \max_{\pi(T-1) \in U} \left\{ \mathbb{E}[L(T-1, y(T-1), \pi(T-1))] \right. \\
\left. + M(T, y(T)) | \mathcal{F}(T-1) \right\} \right. \\
\left. \mathcal{F}(T) \right\} \right. \\
\text{s.t. } y(\tau + 1) = D(\tau, y, \pi) + S(\tau, y, \pi) \epsilon(\tau), \quad (4.15)
\]

where \( \mathcal{F}(t) \) denotes the information (measurements) at time \( t \). The DP algorithm yields the feedback control decisions \( \pi(t, y(t)) \). We interchange iteratively the expectation and the maximization and using tower property for conditional expectations, \( (\mathbb{E}[\mathbb{E}[\epsilon(t)|\mathcal{F}(t)]|\mathcal{F}(t-1)]) = \mathbb{E}[\epsilon(t)|\mathcal{F}(t-1)] \), to obtain
\[
J(t, y(t)) = \max_{\mathcal{U}} \left\{ \mathbb{E} \left[ L(t, y(t), \bar{u}(t)) + L(t+1, y(t+1), \bar{u}(t+1)) + \ldots + L(T-1, y(T-1), \bar{u}(T-1)) + M(T, y(T)) \right| \mathcal{F}(t) \right\}
\]
\[\text{s.t. } y(\tau+1) = D(\tau, y, \bar{u}) + S(\tau, y, \bar{u}) \epsilon(\tau), \quad (4.16)\]

where \(J(t, y(t))\) is the objective value of the open-loop optimization. The open-loop optimization yields the control decision as a function of the current information but not as function of future measurements. By Jensen’s inequality \(\mathbb{E}\left[ \max \{ f(\cdot) \} \right] \geq \max \{ \mathbb{E}[f(\cdot)] \}\) it follows that
\[
J(t, y(t)) \leq J(t, y(t)), \quad (4.17)
\]
which shows that open-loop optimization is a sub-optimal control method. In deterministic applications however, both control methods have the same value of the objective function.

The model predictive control algorithm proceeds as given in Table 4.1. By resolving the open-loop optimization at every time step we introduce feedback into our system. The MPC method requires the solution of \(T = T - t + 1\) optimal control problems. We compute the control decisions along the trajectory of the system and thus, we avoid the curse of dimensionality, since the control decisions are not computed for states which we do not reach. Note that the MPC control policy and the DP control policy are the same at time \(T - 1\). The larger the control horizon and the more “stochastically” the system behaves, the larger the deviations become between the MPC and the DP solution.

**Table 4.1. Model Predictive Control algorithm**

1. Based on the information at time \(t\), determine (measure) \(y(t)\).
2. Compute the open-loop optimization problem given in (4.16) with information \(y(t)\).
   We either solve the open-loop optimization with the same horizon \((t + T)\), a so-called receding horizon policy, or we telescope towards a fixed finite date \((T)\). In this case, the horizon shrinks by one at every time step.
3. We apply only the first control decision decision, i.e., \(\bar{u}(t)\), of the sequence \(\bar{u}(t), \bar{u}(t+1), \ldots, \bar{u}(T-1)\) and we move one time step ahead.
4. The algorithm returns to step 1, until we have reached the final-time of our portfolio optimization.
4.1 Discrete-time portfolio optimization and discrete-time optimal control

As in any other suboptimal control method, we would like to be assured that the new information is advantageously used. We compare the MPC solution to the open-loop optimization in the case that we optimize the objective for a fixed finite date \((T)\).

**Theorem 4.1.** The value of the objective function \(J^*(t, y(t))\) corresponding to the MPC method satisfies

\[
J(t, y(t)) \leq J^*(t, y(t)),
\]

where \(J(t, y(t))\) is the value of the objective function corresponding to the open-loop optimal policy which is fixed at time \(t\).

**Proof.** We assume that we have solved the open-loop optimization problem given by (4.16) which yields the control decisions \(\tilde{u}(t|F(t)), \tilde{u}(t+1|F(t)), \ldots, \tilde{u}(T-1|F(t))\). The value of the objective function in this case is written as

\[
\tilde{J}(t, y(t)) = E\left[ L(t, y(t), \tilde{y}(t)) + E\left[ L(t+1, \tilde{y}(t+1), \tilde{u}(t+1)) + \cdots + M(T, \tilde{y}(T)) \right] \right] \quad \tilde{y}(\tau+1) = D(\tau, \tilde{y}, \tilde{u}) + S(\tau, \tilde{y}, \tilde{u}) \epsilon(\tau).
\]

At time \(t+1\) for the MPC method, we solve the open-loop optimization problem again under the information \(F(t+1)\). Note that \(\tilde{y}(t+1) = y(t+1)\), since the first control decision are the same for the MPC policy and the open-loop policy. For the MPC algorithm, we solve at time \(t+1\) the open-loop problem again:

\[
\tilde{J}'(t+1, y(t+1)) = \max_{\tilde{u}(t+1), \ldots, \tilde{u}(T-1) \in U} \left\{ E\left[ L(t+1, y(t+1), \tilde{u}(t+1)) + \cdots + M(T, y(T)) \right] \right\}
\]

\[
\text{s.t.} \quad y(\tau+1) = D(\tau, y, \tilde{u}) + S(\tau, y, \tilde{u}) \epsilon(\tau),
\]

which yields the new control sequence \(\tilde{u}'(t+1|F(t+1)), \ldots, \tilde{u}'(T-1|F(t+1))\). It is obvious when we compare (4.19) and (4.20) that \(\tilde{J}'(t+1, y(t+1)) \geq \tilde{J}(t+1, \tilde{y}(t+1))\). By resolving the open-loop optimal control problem at every time \(\tau > t\), we improve the remaining value of the objective function compared to the remaining value obtained at time \(\tau - 1\). Therefore, it is follows that \(J^*(t, y(t)) \geq J(t, y(t))\). The equality holds in the case that the resulting values of the states \(\tilde{y}(\tau) = y^*(\tau), \tau = t+1, \ldots, T-1\), which is only true for deterministic applications. \(\square\)
The open-loop optimization without any further measurements provides a lower performance bound of the MPC method. The MPC method performs at least as well as the open-loop method without any further information. In general, the following inequalities hold

\[ J(t, y(t)) \geq J^*(t, y(t)) \geq \mathbb{J}(t, y(t)) \]  

(4.21)

The MPC method solves at each time a suboptimal control problem and thus, the DP solution provides the true optimal control sequence. In the case of deterministic systems the inequalities become equalities. Similar observations to the MPC results presented above can be found in White and Harrington (1980) and Bertsekas (1995, Chapter 6.1).

**Application of stochastic MPC to portfolio optimization problems**

In this part, we discuss the application of the MPC method to portfolio optimization problems without transaction costs and objectives that imply constant relative risk aversion (CRRA). When using CRRA objectives, the current wealth of the portfolio does not influence the decision about the asset allocation. In this case, the decision is solely based on the trade-off between the risk and the return for the remaining investment horizon. Furthermore, we do not consider in- or outflows of funds to or from the portfolio \((q(t) = 0)\) and therefore, we optimize the expected utility defined for the terminal wealth. In general, we allow constraints for the asset allocation and probabilistic constraints for the wealth of the portfolio. Since we work with CRRA type objective functions and neglect transaction costs, we use the logarithmic form of the wealth dynamics given in (4.11). We assume that the utility functions possess either of the following properties:

\[ U(C + f(r, u)) = U(C) + U(f(r, u)) \]

\[ U(C \cdot f(r, u)) = U(C) \cdot U(f(r, u)) \]  

(4.22)

where \(C\) is an arbitrary constant, \(f(r, u) : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}\) denotes the function that converts the asset returns into the portfolio returns, \(r\) denotes the returns of the assets, and \(u\) denotes the vector of investments in the different assets. For utility functions fulfilling either property I or II, the optimal control decision is independent of \(w(t)\). In this case, the application of the MPC methods yields good results since dynamics of the returns are not affected by control decisions. In Section 4.2.3, we compare the DP solution to the MPC solution in a numerical case study. This comparison shows that for the specific case the difference is very small.
4.1 Discrete-time portfolio optimization and discrete-time optimal control

To solve this open-loop optimal control problem, we have two basic approaches. The first approach is the analytical approach which relies on knowing the distribution of $w(T)$ in function of $u(\tau)$. This is only possible in a few specific cases, such as Gaussian distributions for the asset returns which is discussed in Section 4.2.3. The second approach is the simulation approach which we use when we cannot give an analytical description of the distribution of $w(T)$. The optimization problem in this case can be written as

$$\max_{u(t), \ldots, u(T-1)\in U} \left\{ \frac{1}{N} \sum_{s=1}^{N} U(w^s(T)) \right\}$$

s.t. $w^s(T) = \sum_{\tau=t}^{T-1} \ln (1 + r_0^s(\tau) + u(\tau)^T (r^s(\tau) - 1 r_0^s(\tau))) + w(t), \ s = 1, \ldots, N,$

where the superscript $s$ denotes the scenario number and $N$ the number of scenarios. The scenario approximation is obtained by drawing samples from the underlying stochastic model of the returns. Also, the expectation $E[U(w(T))]$ is replaced by the sample mean $\frac{1}{N} \sum_{s=1}^{N} U(w^s(T))$.

**Proposition 4.2.** The simulated open-loop problem given by (4.23) is a concave (convex) optimization problem if $U(\cdot)$ is a concave function.

**Proof.** Since $U(\cdot)$ is concave with respect to $u(\tau)$ and thus $\sum_{s=1}^{N} U(w^s(T))$ is also concave, it suffices to prove that $\sum_{\tau=t}^{T-1} \ln (1 + r_0^s(\tau) + u(\tau)^T (r^s(\tau) - 1 r_0^s(\tau)))$ is concave with respect to $u(\tau)$. In Boyd and Vandenberghe (2004, 3.2.2) it is shown that for

$$g(u) = f(Au + b),$$

and $f : \mathbb{R}^n \to \mathbb{R}$ concave, $A \in \mathbb{R}^{n \times m}$, and $b \in \mathbb{R}^n$, that $g(u)$ is also concave. In our case we have $f(x) = \ln(x)$ which is concave, and $1 + r_0^s(\tau) + u(\tau)^T (r^s(\tau) - 1 r_0^s(\tau))$ is a linear function. Furthermore, the nonnegative weighted sums of concave functions are again concave.

The results of Proposition 4.2 is important, since it allows us to efficiently compute the open-loop problem even in the case when we have to resort to a simulation (scenario) procedure. For some open-loop problems, we show that the complexity of the optimization is independent of the number of scenarios. In this case, we may use a very large number of scenarios to get a good description of the uncertainty.
4.1.6 Stochastic programming approximation

Instead of solving the DP algorithm for the true stochastic dynamics, we approximate the stochastic dynamics by a finite number of scenarios. It is well known that an approximation of stochastic programming by discretizing the state variables leads to computational costs which increase exponentially with the dimension of the state space. When using a regular grid to discretize the state space, the optimization is computed at many points in the state space which are reached with low probability. A standard approach to overcome these drawbacks is to use a scenario approach and a sampling approximation of the true expectation.

Given the dynamic optimization problem stated in (4.13), the corresponding DP algorithm is given

$$J(\tau, y(\tau)) = \max_{u(\tau) \in U} \left\{ \mathbb{E} \left[ L(\tau, y(\tau), \bar{u}(\tau)) + J(\tau + 1, D(\tau, y, \bar{u}) + S(\tau, y, \bar{u}) \epsilon(\tau)) \right] \right\}$$

with terminal condition $J(T, y(T)) = M(T, y(T))$. In order to avoid mathematical (technical) issues, we assume that $\epsilon(t)$ takes a finite or countable number of values. The difficulties with solving this DP problem are the computation of the recursion under realistic assumptions. Instead of computing the exact optimal control policy, we solve the approximated problem where we replace the expectation by the sample mean and the true dynamics by finite number of representative scenarios. To calculate the sample mean and the scenarios, a number of samples have to be drawn from the underlying probability distribution. Given the information $\mathcal{F}(t)$ at time $t$, we replace the probability distribution of $\epsilon(t)$ by $\kappa(t)$ scenarios, denoted by $\epsilon^s(t)$. At time $t+1$ conditional on the scenario $\epsilon^s(t)$, we generate $\kappa(t + 1)$ scenarios for each previous scenario of $\epsilon^s(t)$, as shown in Figure 4.1. By utilizing a sufficiently large number of scenarios, we are able to approximate the random effects. The scenarios are defined as the set $S$ that represents a reasonable description of the future uncertainties. A scenario $s \in S$ describes a unique path through consecutive nodes of the scenario tree as depicted in Figure 4.1. In this way we generate a tree of scenarios that grows exponential with the time horizon and the number of scenarios is given by $N = \prod_{\tau=1}^{T} \kappa(\tau)$. By this procedure, we generate a grid of the state space along a highly probable evolution of the system. The optimization is then computed for the approximated dynamics and the sample expectation. For the approximated problem we still solve DP algorithm, but only for the approximated paths of the state dynamics.
Sampling approximations

In this subsection, we briefly explain the relationship between the expectation and its sample mean approximation. Given a set $\Omega$ with probability measure $P$ and the probability distribution is assumed to be unimodal. The function $L : \Omega \rightarrow \Xi$ is measurable with respect to $P$ and $\Xi$ is a subset of $\mathbb{R}$ or equal to $\mathbb{R}$. The expectation of $L$ is defined as $E[L] = \int_{\Omega} L(\omega) dP$. We assume that $E[|L|] < \infty$ and $E[L^2] < \infty$. A sampling approximation by drawing $N$ independent and identically distributed samples $\omega_1, \omega_2, \ldots, \omega_N$ from $\Omega$ is the sample mean given by

$$ \hat{E}[L] = \frac{1}{N} \sum_{i=1}^{N} L(\omega_i), \quad (4.25) $$

where $\hat{E}$ denotes the sample mean. Obviously, the sample mean is a random variable that depends on the sampling procedure from the underlying distribution. The variance of $\hat{E}[L]$ is

$$ \text{Var}[\hat{E}[L]] = \frac{\text{Var}[L]}{N}, \quad (4.26) $$

see Casella and Berger (2002, p. 214). Very often we do not know the variance of $L$ and have to use the sample variance given by

$$ \hat{\text{Var}}[L] = \frac{1}{N-1} \sum_{i=1}^{N} (L(\omega_i) - \hat{E}[L])^2, \quad (4.27) $$
where $\text{Var}$ denotes the sample variance. The variance of the sample mean error can thus be estimated by

$$\text{Var} \left[ \hat{E}[L] \right] \approx \frac{\text{Var}[L]}{N}. \quad (4.28)$$

This allows us to judge the quality of the sample mean from the samples. With the help of the Vysochanskii-Petunin inequality (Casella and Berger 2002, p.137) or the Chebychev inequality, we are able to state the confidence with which the bound $P(\left| E[L] - \hat{E}[L] \right| < \eta)$ holds. Using the Vysochanskii-Petunin inequality, since it gives 55% more accurate bound than the Chebychev inequality, the bound is given by

$$P \left( \left| E[L] - \hat{E}[L] \right| < \eta \right) \geq 1 - \frac{4\text{Var}[L]}{9\eta^2N}, \quad \eta \geq \sqrt{\frac{8\text{Var}[L]}{3N}}, \quad (4.29)$$

where $P$ denotes the probability.

**Theorem 4.3.** The sample mean converges in probability to the true expectation, i.e.,

$$\lim_{N \to \infty} P \left( \left| E[L] - \hat{E}[L] \right| < \eta \right) = 1, \quad \forall \eta > 0 \quad (4.30)$$

*Proof.* The theorem follows from (4.29) as $N \to \infty$. \qed

With the help of (4.29) and the approximation (4.28) we can determine the confidence of the sample mean or compute (iteratively) the number of samples needed to achieve a predetermined confidence.

**Stochastic programming approximation of dynamic programming**

When we replace the “true” stochastic dynamics of the state equation by the sample approximation, we need to compute sample mean of the objective function instead of the expectation of the objective function. The objective function at time $\tau$ and for one scenario $s$ with feedback mapping (policy) $\Pi^s(\tau) = [\pi^s(\tau), \pi^s(\tau+1), \ldots, \pi^s(T-1)]$ is calculated by

$$V^s(\tau, y^s(\tau), \Pi^s(\tau)) = \sum_{i=\tau}^{T-1} L(i, y^s(i), \pi^s(i)) + M(T, y^s(T)). \quad (4.31)$$

The feedback control decisions $\Pi^s(\tau)$ define a feedback mapping, since for each scenario $s$ and time $\tau$ a predetermined control decision based on a feedback rule $\pi^s(\tau) = u(\tau, y^s(\tau))$ is used. The sample mean of the objective function is given by

$$\hat{E} \left[ V^s(\tau, y^s(\tau), \Pi^s(\tau)) \right] = \frac{1}{N} \sum_{s=1}^{N} V^s(\tau, y^s(\tau), \Pi^s(\tau)). \quad (4.32)$$
Using (4.32), we define the sample approximation of the dynamic optimization problem as
\[
\hat{J}_s(\tau, y(\tau)) = \max_{\Pi(\tau) \in \mathcal{U}} \left\{ \mathbb{E}\left[ V_s(\tau, y^*\Pi(\tau)) \right] \right\} \\
\text{s.t. } y^*(\tau+1) = D(\tau, y^*, \Pi) + S(\tau, y^*, \Pi)e^*(\tau),
\] (4.33)

where \(\hat{J}_s(\tau, y(\tau))\) is the value of the objective function at time \(t\) under scenario \(s\). At the root of the scenario tree, \(\hat{J}_s(\cdot)\) is the same for all scenarios \((s = 1, \ldots, N)\).

**Theorem 4.4.** The sample approximation of the optimization problem given in (4.33) can be recursively computed by the following dynamic program:

\[
\hat{J}_s(\tau, y^*(\tau)) = \max_{u(\tau) \in \mathcal{U}} \left\{ \mathbb{E}\left[ L(\tau, y^*(\tau), u(\tau)) \right] \\
+ \hat{J}_s(\tau+1, D(\tau, y^*, u) + S(\tau, y^*, u)e^*(\tau)) \right\}
\] (4.34)

with terminal condition \(\hat{J}_s(T, y^*(T)) = M(T, y^*(T))\).

We give the proof of Theorem 4.4 in Section A.3, since it follows the standard arguments to derive the dynamic programming recursion. The sample approximation for specific state equations and objective functions, e.g., linear objectives and linear state equations, can be efficiently computed by stochastic programming techniques.

**Proposition 4.5.** The dynamic programming formulation of the approximated dynamic optimization problem can be written as multi-stage stochastic program.

**Proof.** By using (4.34) iteratively from the terminal condition we can write the approximated dynamic optimization given in (4.33) as

\[
\hat{J}_s(t, y(t)) = \max_{u(t) \in \mathcal{U}} \left\{ \mathbb{E}\left[ L(t, y^*(t), u(t)) \right] \\
+ \max_{u(t+1) \in \mathcal{U}} \left\{ \mathbb{E}\left[ L(t+1, y^*(t+1), u^*(t+1)) \right] + \ldots \right\} \\
+ \max_{u(T-1) \in \mathcal{U}} \left\{ \mathbb{E}\left[ L(T-1, y^*(T-1), u^*(T-1)) \right] + M(T, y^*(T)) \right\} \right\} \\
\text{s.t. } y^*(\tau+1) = D(\tau, y^*, u^*) + S(\tau, y^*, u^*)e^*(\tau)
\] (4.35)

which exactly defines a multi-stage stochastic program, see Louveaux and Birge (1997, Chapter 3).
Proposition 4.5 allows us to compute the approximation efficiently in the case that the multi-stage stochastic program possesses a suitable structure. Examples are linear or quadratic multi-stage stochastic programs. In the Table 4.2, we state the algorithm to compute the stochastic programming approximation of the dynamic programming method. By starting with a low number of scenarios, we ensure that the multi-stage stochastic

Table 4.2. Algorithm to compute the stochastic programming approximation

1. Based on the information at time $t$, determine $y(t)$. Set an accuracy parameter $\eta$, set $J = \infty$, and define the number of scenarios per period. Start with a relatively low number of scenarios.

2. Compute the scenario approximation of $\epsilon(\tau)$ for $\tau = t, t+1, \ldots, T-1$, based on the number of samples per time period.

3. Solve the multi-stage stochastic program as outlined in Proposition 4.5.

4. In the case that $|J^*(t, y(t)) - J| < \eta$ stop and go to step 5. Otherwise set $J = J^*(t, y(t))$ and increase the number of scenarios to refine the approximation. Go to step 2.

5. Apply only the first control decision $u(t)$ and disregard all future control decisions for the next time step. Until the optimization has reached its fixed end date (horizon) go to step 1, else stop.

program is solved rather quickly. The algorithm cycles between step 2 and step 4 until the desired accuracy has been reached. Alternatively, we could simply define the number of scenarios and solve the stochastic programming approximation. In this case however, we can not know in advance the accuracy of the approximation and have to “hope” that the number of scenarios is sufficiently large to achieve the desired accuracy. We may approximately determine the probability of being within the accuracy limit $\eta$ by using the sample variance defined in (4.28). The relation of the approximation algorithm and the true problem defined in (4.13) is given in Theorem 4.6.

**Theorem 4.6.** The sample approximation of the objective function $\hat{J}^*(\tau, y(\tau))$ defined in (4.33) converges with probability 1 to the true objective function $J(\tau, y(\tau))$ as defined in (4.13) for $N \to \infty$. Especially $\hat{J}^*(t, y(t))$ converges with probability 1 to $J(t, y(t))$ for $N \to \infty$. 
Proof. Given a predetermined feedback mapping \( \Pi^*(\tau) \) as defined before and using Theorem 4.3, the following holds:

\[
\lim_{N \to \infty} \mathbb{P}\left( \left| \mathbb{E}[V(\tau, y(\tau), \Pi(\tau))] - \hat{\mathbb{E}}[V(\tau, y^*(\tau), \Pi^*(\tau))] \right| < \eta \right) = 1, \quad \forall \eta > 0 \quad (4.36)
\]

where \( \tau = t, t+1, \ldots, T-1 \). We know that changing the expectation with the sample mean has a negligible effect with arbitrary large probability. Since \( \mathbb{E}[V(\tau, y(\tau), \Pi(\tau))] \) converges to \( \hat{\mathbb{E}}[V(\tau, y^*(\tau), \Pi^*(\tau))] \) with probability 1, and using

\[
\hat{J}^*(\tau, y^*(\tau)) = \max_{\Pi^*(\tau) \in \mathcal{U}} \left\{ \hat{\mathbb{E}}\left[ V^*(\tau, y^*(\tau), \Pi^*(\tau)) \right] \right\},
\]

it follows that

\[
\lim_{N \to \infty} \mathbb{P}\left( \left| \mathbb{E}[J(\tau, y(\tau))] - \hat{\mathbb{E}}[\hat{J}(\tau, y^*(\tau))] \right| < \eta \right) = 1, \quad \forall \eta > 0 . \quad (4.37)
\]

The results of Theorem 4.6 state that the approximated objective function converges in probability to the true value of the objective function. However, this does not imply that the control law computed by the approximation converges in probability to the true control law. The approximation by stochastic programming techniques determines only the control law for the scenarios used. For other values of the state variables, which are not covered by the scenario approximation, the control law is not defined. Since we only apply the first control decision for the fixed (measured) state \( y(t) \), the issue of feedback for uncovered state variables does not constitute a problem. By using the approximation procedure at every time step, the control decisions are always based on current information. A similar analysis for linear stochastic systems with quadratic performance criteria can be found in Batina, Stoorvogel and Weiland (2002, 2005), however without the explicit connection to stochastic programming.

The scenario approximation does not depend on the dimension of the state variables but on the number of scenarios used. The algorithm’s complexity is thus independent on the state space dimension and therefore, we avoid the curse of dimensionality. However, to obtain results with desired accuracy we need a sufficiently large number of scenarios. The number of scenarios \( \kappa(\tau) \) for each time step does not need to be constant. Instead of approximating \( \epsilon(\tau) \) for each time \( \tau \) by the same number of scenarios, we could use more scenarios for the first steps and fewer scenarios for the later time steps. This assumes, of course, that the random effects towards the end of horizon have less influence on the current control decision.
Portfolio optimizations using the stochastic programming approximation

For problems that do not have objective functions that imply constant relative risk aversion and/or problems of portfolio optimization with transaction costs, we cannot exploit any special structure to ease the computational burden. In these cases, the method of the stochastic programming approximation is applied, where the problem must be solved with the full tree structure. Problems of portfolio optimization with transaction costs are discussed in Section 4.3.

4.2 Portfolio optimizations using the MPC approach with different return models and objectives

In this section, we present the MPC method applied to three multi-period problems of portfolio optimization. First, we specify the asset model used for this method. In Subsection 4.2.3, we discuss the MPC approach for the discrete-time equivalent of the continuous-time models presented in Section 3.2. We discuss the case of constraints for $u(t)$ and probabilistic constraints for the wealth. We also compare the MPC solution to the DP solution, which we obtain for the unconstrained case of this model. In Subsection 4.2.4, we extend the framework of Subsection 4.2.3 to the case when the risky asset returns are described by non-Gaussian distributions. This includes heavy-tailed distributions, such as the student t distributions, or distributions that are the results of conditional heteroskedasticity, such as GARCH models. In this case, we assume a mean-variance objective for the portfolio optimization since this allows us to solve the open-loop optimal control problem analytically. However, for non-Gaussian distributions the variance is not a coherent risk measure and the mean-variance objective in this case might not be the most appropriate one. Therefore, in Subsection 4.2.6 we solve the problem of portfolio optimization with an objective that balances expected return and a coherent measure of risk. The open-loop optimization is based on the scenario approach, since the time aggregation of heavy-tailed distributions is not known analytically.
4.2 Portfolio optimizations using the MPC approach with different return models and objectives

4.2.1 Asset Model and Portfolio Model

Asset Model

We used the basic modelling structure presented in (4.1) and (4.4). We assume that the conditional expectation is time-varying and stochastic. The expected returns of the assets \( \mu(t) \) are thus modelled by

\[
\mu(t) = Gx(t) + g,
\]

(4.38)

where \( G \in \mathbb{R}^{n \times m} \) is the factor loading matrix, \( g \) is a constant, and \( x(t) \in \mathbb{R}^m \) is the vector of factor levels. The interest rate of the bank account, described by (4.3), is modelled by

\[
r_0(t) = F_0x(t) + f_0,
\]

(4.39)

where \( F_0 \in \mathbb{R}^{1 \times m} \) and \( f_0 \in \mathbb{R} \). We assume that the factors are driven by a linear stochastic process where

\[
\Theta(t, x(t)) = Ax(t) + a, \quad \Psi(t, x(t)) = \nu,
\]

(4.40)

where \( A \in \mathbb{R}^{m \times m}, a \in \mathbb{R}^m, \) and \( \nu \in \mathbb{R}^{m \times m} \). This model setup is similar to the model presented in Section 3.2.1. Empirical evidence for linear factor models is cited in Section 3.2.1.

Conditional Covariances

We use the modelling structure given by (4.6) and (4.7) to setup the multivariate conditional covariance structure. The GARCH structure is determined by the model we choose for the individual volatilities \( v_i(t) \) and the conditional correlation matrix \( \Lambda(t) \). For the individual conditional variances the following two models are proposed. First, the GARCH\((p,q)\) model given by

\[
v_i^2(t) = w_i + \sum_{j=1}^{q_i} \alpha_{ij} \epsilon_i^2(t-j) + \sum_{j=1}^{p_i} \beta_{ij} v_i^2(t-j),
\]

(4.41)

where \( w_i > 0, \alpha_{ij} > 0, \beta_{ij} > 0 \) in order to assure positive \( v_i^2(t) \), and initial condition \( v_i(0) \). The model is stationary if \( \sum_{j=1}^{q_i} \alpha_{ij} + \sum_{j=1}^{p_i} \beta_{ij} < 1, i = 1, \ldots, n \). The GARCH\((p,q)\) model correctly describes that volatility increases when large squared returns occur and that volatility shows persistence. Second, for the individual conditional variances the other model we use is the threshold GARCH (TARCH) model given by
\[ v_i^2(t) = \omega_i + \sum_{j=1}^{q_i} \alpha_{ij} \epsilon_i^r(t-j) + \sum_{j=1}^{p_i} \beta_{ij} v_j^2(t-j) + \sum_{j=1}^{\alpha_i} \gamma_{ij} \chi_{i(t-j)} \beta_{ij} \epsilon_i^r(t-j), \] (4.42)

where \( w_i > 0, \alpha_{ij} > 0, \beta_{ij} > 0, \gamma_{ij} > 0, \chi_{i(t-j)} = 1 \) if \( \epsilon_i^r(t-j) < 0 \) and zero otherwise, and initial condition \( v_i(0) \). The TARCH(o,p,q) model correctly describes the three main observed stylized facts. The volatility increases when large absolute returns occur, volatility shows persistence (clusters), and volatility rises strongly when negative returns occur. In Sadorsky (2004), an extensive comparison among various GARCH and stochastic volatility models concluded that for daily data for stocks and bonds, often the TARCH or GARCH model performed best as future volatility predictor. For this reason we limit the discussion of volatility models to these two models.

The conditional correlation, which describes the dependence structure of the asset returns, is modelled with the so-called dynamic conditional correlation (DCC) model. It was derived to improve the constant correlation model, since empirical data analysis of stock and bond prices suggest time variation of correlations. The so-called DCC(k,l) model is obtained by

\[
Q(t) = (1 - \sum_{j=1}^{k} \delta_j - \sum_{j=1}^{l} \eta_j) \bar{Q} + \sum_{j=1}^{k} \delta_j (\zeta(t-j)\zeta(t-j)^T) + \sum_{j=1}^{l} \eta_j Q(t-j)
\]

\[
\Lambda(t) = Q(t)^{-1}Q(t)Q(t)^{-1}, \quad Q(t)^* = \begin{pmatrix} \sqrt{q_{11}} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sqrt{q_{nn}} \end{pmatrix}, \quad (4.43)
\]

where \( \bar{Q} \in \mathbb{R}^{n \times n} \) denotes the unconditional correlation matrix of standardized residuals \( \zeta(t) = V(t)^{-1} \epsilon^r(t) \), \( \delta_j > 0, q_{ij} \) denotes the elements of the \( Q(t) \) matrix, and \( \eta_i > 0 \) to make sure that \( \Lambda(t) \) is a positive-definite matrix. Furthermore, we assume that \( Q^*(t) \) is invertible. The DCC model allows us to measure changes in the dependence structure of assets and helps us to optimally design a diversification strategy. With the assumption of constant correlations, we miss the important fact that correlations change over time and thus, we would not hedge accordingly. The constant conditional correlation model however greatly reduces the estimation effort, see Bollerslev (1990). The properties of the DCC model are discussed in Engle and Sheppard (2001) and Engle (2002).
4.2 Portfolio optimizations using the MPC approach with different return models and objectives

Portfolio model

For the portfolio optimization without transaction costs, we use the nonlinear wealth dynamics given in (4.11). We express the portfolio return as

\[ R(t) = r_0(t) + u^T(t)(r(t) - 1 r_0(t)) \]

\[ = F_0 x(t) + f_0 + u^T(t)(F x(t) + f) + u^T(t)e^r(t), \]

where \( F = G - 1F_0 \) and \( f = g - 1f_0 \). To simplify calculations, we replaced \( \ln(1 + R(t)) \) by the following Taylor series approximation (around the mean of \( R(t) \))

\[ \ln (1 + R(t)) \approx R(t) - \frac{1}{2} R(t)^2. \]

Also, we replace \( R(t)^2 \) by its conditional expectation, i.e., \( \text{Var}[R(t)] \). The same approximation has also been used by Campbell and Viceira (2002). When the noise terms of the assets \( e^r(t) \) are Gaussian, we can show that the wealth dynamics are exact, see Appendix A.1. The following wealth dynamics are obtained:

\[ w(t+1) = w(t) + F_0 x(t) + f_0 + u^T(t)(F x(t) + f) - \frac{1}{2} u^T(t)\Sigma(t)u(t) + u^T(t)e^r(t), \]

\[ x(t+1) = A x(t) + a + \xi^x(t). \]

In addition, we assume that the white noise processes for the asset returns and the factor dynamics are not independent. For this reason, we normalize the two processes and write \( e^r(t) = \nu \xi^x(t) \), with \( \nu \in \mathbb{R}^{m \times m} \), where the standard residuals are characterized by \( E[\xi^x(t)] = 0 \) and \( E[\xi^x(t)\xi^xT(t)] = I \in \mathbb{R}^{m \times m} \). The structure of the random process for the asset returns is \( e^r(t) = \sigma(t)\xi^r(t) \), where we assume that \( E[\xi(t)] = 0 \) and \( E[\xi^r(t)\xi^rT(t)] = I \in \mathbb{R}^{n \times n} \). In order to introduce dependence between the two white noise processes, we assume that the standard residuals of the two white noise processes are correlated, i.e., \( E[\xi^r(t)\xi^x(t)] = \rho \).

4.2.2 Portfolio mean and variance

For the system of equations that describe the portfolio dynamics given in (4.47), we calculate the mean and the variance. In the case that the two white noise variables \( e^x(t) \)
and \(\epsilon^\tau(t)\) are Gaussian, we know that the portfolio distribution conditioned on the future asset allocation \(u(\tau)\) is Gaussian. Then the first two moments completely describe the portfolio distribution. In the case of non-Gaussian distributions, we are still able to solve the mean-variance portfolio optimization problem.

The portfolio equations (4.47) can be written as

\[
\begin{pmatrix}
w(t+1) \\
x(t+1) \\
y(t+1)
\end{pmatrix}_{y(t+1)} = \begin{pmatrix} 1 & F_0 + u^T(t)F \\ 0 & A \end{pmatrix} \begin{pmatrix} w(t) \\
x(t) \\
y(t)
\end{pmatrix}_{y(t)} + \begin{pmatrix} f_0 + u^T(t)f - \frac{1}{2}u^T(t)\Sigma(t)u(t) \\
a \\
a_y
\end{pmatrix} + \begin{pmatrix} u^T(t)\epsilon^\tau(t) \\
\epsilon^\tau (t)
\end{pmatrix},
\]

or even more concisely as:

\[
y(t+1) = A_y y(t) + a_y + \epsilon(t+1).
\]

Note that

\[
E_\tau[\epsilon(t+1)\epsilon^T(t+1)] = \Omega(t) = \begin{bmatrix} u^T(t)\Sigma(t)u(t) & u^T(t)\sigma(t)\rho

\end{bmatrix},
\]

**Portfolio mean**

The conditional mean, \(\bar{m}(\tau) = E_\tau[y(\tau)]\), for a time \(\tau\) can be computed as

\[
\bar{m}(\tau+1|t) = A_y\bar{m}(\tau|t) + a_y
\]

and is obtained by

\[
\bar{m}^{(w)}(\tau+1|t) = \bar{m}^{(w)}(\tau|t) + F_0\bar{m}^{(x)}(\tau|t) + f_0

+ u^T(\tau)(F\bar{m}^{(x)}(\tau|t) + f) - \frac{1}{2}u^T(\tau)E_\tau[\Sigma(\tau)]u(\tau)

\]

\[
\bar{m}^{(x)}(\tau+1|t) = A\bar{m}^{(x)}(\tau|t) + a,
\]

where \(\bar{m}^{(w)}(\tau) = E_\tau[w(\tau)]\) and \(\bar{m}^{(x)}(\tau) = E_\tau[x(\tau)]\). We compute the mean at time \(t + T\) where \(T\) denotes the horizon. By iterating (4.50) \((T-1)\) times we obtain

\[
\bar{m}^{(w)}(t+T) = w(t) + f_0T + \sum_{i=0}^{T-1} \left\{ (F_0 + u^T(t+i)F) [A^ix(t)]

+ \sum_{j=0}^{i-1} A^{i-1-j}b_j + u^T(t+i)f - u^T(t+i)E_\tau[\Sigma(\tau+i)]u(t+i) \right\},
\]

where the fact that \(\bar{m}^{(x)}_{it} = x(t)\) and \(\bar{m}^{(w)}_{it} = w(t)\) is already used.
Portfolio variance

The dynamic evolution of the conditional covariance matrix is obtained by
\[
\overline{V}(\tau+1|t) = A_y \overline{V}(\tau|t) A_y^T + \mathbb{E}_t[\Omega(\tau)],
\]  
(4.52)
where \( \tau > t \) is a future time and \( \overline{V}(\tau|t) = \mathbb{E}_t[(y(\tau) - \mathbb{E}_t[y(\tau)])(y(\tau) - \mathbb{E}_t[y(\tau)])^T] \). The dynamics of covariance matrix are computed conditioned on the information at time \( t \).

The variance of the portfolio system can be decomposed into
\[
\overline{V}(\tau|t) = \begin{bmatrix} \overline{V}^{(w)}(\tau|t) & \overline{V}^{(wx)}(\tau|t) \\ \overline{V}^{(wx)^T}(\tau|t) & \overline{V}^{(x)}(\tau|t) \end{bmatrix},
\]
where the superscripted index in parentheses indicate the covariance between \( w \) and \( x \), and the covariance matrix of \( x \) and the variance of \( w \). We expand (4.52) by plugging in \( A_y, \overline{V}(\tau|t), \) and \( \mathbb{E}_t[\Omega(\tau)] \). This yields
\[
\begin{align*}
\overline{V}^{(w)}(\tau+1|t) &= \overline{V}^{(w)}(\tau|t) + (F_0 + u^T(\tau))F)(\overline{V}^{(wx)})^T(\tau|t) \\
&\quad + \overline{V}^{(wx)}(\tau|t)(F_0^T + F^T u(\tau)) + (F_0 + u^T(\tau)F)\overline{V}^{(x)}(\tau)(F_0^T + F^T u(\tau)) \\
&\quad + u^T(\tau)\mathbb{E}_t[\Sigma(\tau)]u(\tau) \\
\overline{V}^{(wx)}(\tau+1|t) &= \overline{V}^{(wx)}(\tau|t)A^T + (F_0 + u^T(\tau)F)\overline{V}^{(x)}(\tau|t)A^T + u^T(\tau)\mathbb{E}_t[\sigma(\tau)]v \\
\overline{V}^{(x)}(\tau+1|t) &= A\overline{V}^{(x)}(\tau|t)A^T + \nu v^T.
\end{align*}
\]
We can calculate the solutions of the difference equation by induction and get
\[
\begin{align*}
\overline{V}^{(x)}(t+\overline{T}|t) &= \sum_{i=0}^{T-1} A^i \nu v^T (A^T)^i \\
\overline{V}^{(wx)}(t+\overline{T}|t) &= \sum_{i=0}^{T-1} (F_0 + u^T(t+i)F)\overline{V}^{(x)}(t+i|t)A^T-i \\
&\quad + \sum_{i=0}^{T-1} u^T(t+i)\mathbb{E}_t[\sigma(t+i)]v A^T-1-i \\
\overline{V}^{(w)}(t+\overline{T}|t) &= \sum_{i=0}^{T-1} (F_0 + u^T(t+i)F)(\overline{V}^{(wx)}(t+i|t))^T \\
&\quad + (F_0 + u^T(t+i)F)\overline{V}^{(x)}(t+i|t)(F_0^T + F^T u(t+i)) \\
&\quad + u^T(t+i)\mathbb{E}_t[\Sigma(t+i)]u(t+i),
\end{align*}
\]
where \( \overline{T} \) denotes the investment horizon and we already used the initial value \( \overline{V}(t|t) = 0 \). In order to compute the variance of the wealth equation, we insert \( \overline{V}^{(wx)}(t+\overline{T}|t) \) and
\[ V^{(w)}(t+T|t) = V^{(w)}(t+T|t), \]

and write

\[ V^{(w)}(t+T|t) = \sum_{i=0}^{T-1} \left[ 2(F_0 + u^T(t+i)F) \cdot \left( \sum_{j=0}^{i-1} \left\{ A^{i-j} \nu^T \sigma^T(t+j)u(t+j) \right\} \right) 
+ A^{i-j} \sum_{l=0}^{j-1} (A^l \nu^T A^l)(F_0^T + F^T u(t+j)) \right] 
+ (F_0 + u^T(t+i)F) \sum_{j=0}^{i-1} (A^j \nu^T A^j)(F_0^T + F^T u(t+i)) 
+ u^T(t+i) \Sigma(t+i)u(t+i), \tag{4.53} \]

where \( \sigma^T(t+j) = E_t[\sigma^T(t+j)] \) and \( \Sigma(t+i) = E_t[\Sigma(t+i)] \) (conditional mean). In order to compute the variance of the portfolio we need to compute the expectation of the covariance matrix \( \bar{T} \) steps into the future, i.e., \( E_t[\Sigma(t+i)] \). One may notice that portfolio variance depends on multiplications of \( u(t+j) \) and \( u(t+i) \). This fact links the decision variables from one period with the decision variables from another period and thus, makes this problem a true multi-period decision problem. Therefore, the portfolio allocation is a strategic rather than a tactical asset allocation problem. The fact that links the different periods is the correlations of returns and factors.

### 4.2.3 Portfolio optimization for linear Gaussian factor models.

In this section, we make the additional assumption that the two white noise processes \( \epsilon^r(t) \) and \( \epsilon^x(t) \) have a Gaussian distribution and the joint covariance matrix is deterministic. Therefore

\[ E_t[\epsilon(t+1)\epsilon^T(t+1)] = \Omega(t) = \begin{bmatrix} u^T(t) \Sigma u(t) & u^T(t)\sigma \nu \\ \nu^T \rho \sigma^T u(t) & \nu \nu^T \end{bmatrix}, \tag{4.54} \]

and \( E_t[\Sigma(t+i)] = \Sigma \).

#### Portfolio distribution

The conditional density for wealth equations is

\[
\begin{align*}
    w(t+T) & \sim \mathcal{N}(\mu^{(w)}(t+T), V^{(w)}(t+T)) \\
    x(t+T) & \sim \mathcal{N}(\mu^{(x)}(t+T), V^{(x)}(t+T)),
\end{align*}
\tag{4.55}
\]

where the mean and variance of \( w(t+T) \) are computed by (4.51) and (4.53). The density is conditioned on the current value of the factors \( x(t) \) and the asset allocation decisions.
4.2 Portfolio optimizations using the MPC approach with different return models and objectives

Only for this special case we are able to derive the conditional distribution.

Probabilistic Constraints

MPC method is well suited for dealing with control as well as state constraints. We consider two kinds of probabilistic state constraints for the log-wealth.

**C1** State constraints for the log-wealth values are given as

\[ P(w(t+i) > L(t+i)) \geq p_{t+i} \quad i = 1, \ldots, T, \quad (4.57) \]

where \( L(t+i) \) denotes the constraint level at time \( t+i \) and \( p_{t+i} \) the minimum probability with which the constraint be satisfied. Given the mean and the variance of the wealth equation, we know the distribution of the wealth equation. Therefore, we are capable of computing the probability in (4.57). In a finance related context, the probabilistic constraint (4.57) is known as a Value-at-Risk (VaR) constraint. Mathematically, VaR is the \( p_{t+i} \)-quantile of the portfolio at time \( t+i \). For a given confidence level \( (p_{t+i}) \), we specify the minimum amount of the log-wealth \( (L(t+i)) \) which we want to attain.

**C2** Alternatively, we formulate another state constraint which is known as the Conditional Value-at-Risk (CVaR). CVaR is the expected wealth at a given time, conditional that the wealth is below the VaR level. With CVaR, we attempt to limit expected losses which are larger than the VaR. Mathematically, the CVaR constraints are given as

\[ E_t[w(t+i) \leq \varphi(p_{t+i})] \geq L(t+i), \quad i = 1, \ldots, T, \quad (4.58) \]

where \( \varphi(p_{t+i}) \) denotes the VaR with confidence \( p_{t+i} \) and \( L(t+i) \) denotes the value of the CVaR constraint.

Both of the state constraints (4.57) and (4.58) can explicitly be calculated since we know the future densities of the wealth.

**Proposition 4.7.** Given that the wealth is normally distributed, (4.57) can be computed as

\[ m^{(w)}(t+i) + \sqrt{V^{(w)}(t+i)\Phi^{-1}(1 - p_{t+i})} \geq L(t+i), \quad (4.59) \]

where \( \Phi^{-1}(\cdot) \) denotes the inverse cumulative distribution function of a normal distribution with zero mean and unit variance. The constraint **C1** given by (4.57) is equivalent to (4.59) and the constraint (4.59) is a convex constraint.
Proof. The constraint given by (4.57) can be equivalently written as
\[ P(w(t+i) \leq L(t+i)) \leq 1 - p_{t+i} \quad i = 0, \ldots, T-1. \] (4.60)

We insert (4.59) in (4.60) and obtain
\[
P(w(t+i) \leq L(t+i)) = P\left( \frac{w(t+i) - m^{(w)}(t+i)}{\sqrt{V^{(w)}(t+i)}} \leq \frac{L(t+i) - m^{(w)}(t+i)}{\sqrt{V^{(w)}(t+i)}} \right)
= P\left( \frac{w(t+i) - m^{(w)}(t+i)}{\sqrt{V^{(w)}(t+i)}} \leq \Phi^{-1}(1 - p_{t+i}) \right)
= \Phi\left( \Phi^{-1}(1 - p_{t+i}) \right) = 1 - p_{t+i}.
\]

Furthermore, we prove that (4.59) describes a convex set. First, we multiply (4.59) with
\(-1\) and get
\[
-m^{(w)}(t+i) - \sqrt{V^{(w)}(t+i)}\Phi^{-1}(1 - p_{t+i}) \leq -L(t+i)
\] (4.61)
and we show that the function \(l(t+i)\) (see above) is convex with respect to \(u(t+j), \ j = 0, \ldots, i-1\). The mean and variance of the wealth equation are linear-quadratic functions of \(u(t+j)\), see (4.51) and (4.53). Moreover, we know that the variance is convex w.r.t. \(u(t+j)\) and the mean is concave w.r.t. \(u(t+j)\). Furthermore, by inspecting (4.61) we notice that \(-\sqrt{V^{(w)}(t+i)}\Phi^{-1}(1 - p_{t+i}) > 0\), when \(p_{t+i} > 0.5\). The term \(\sqrt{V^{(w)}(t+i)}\) is a convex function w.r.t. \(u(t+j)\), since \(V^{(w)}(t+i)\) is convex and the square-root transformation preserves the convexity, see Boyd and Vandenberghe (2004, 3.2.4). The term \(-m^{(w)}(t+i)\) is also convex, since \(m^{(w)}(t+i)\) is concave. Thus, the function \(l(t+i)\) is convex because it is the sum of two convex functions. \(\square\)

**Proposition 4.8.** Given the normal distribution, we calculate the CVaR constraint (4.58) by
\[
m^{(w)}(t+i) - \sqrt{V^{(w)}(t+i)}\frac{\Phi^{-1}(1 - p_{t+i})}{1 - p_{t+i}} \geq L(t+i), \] (4.62)
where \(\Phi^{-1}(\cdot)\) denotes the inverse cumulative distribution function of a standard normal distribution and \(\phi(\cdot)\) is the standard normal density. The constraint \(C2\) given by (4.62) is equivalent to (4.58). Additionally, the constraint described by (4.62) is convex w.r.t. \(u(t+j)\).
From the arguments of proof of proposition 4.7, it is clear that (4.62) is again a convex function w.r.t. $u$. We write (4.58) as

$$\text{CVaR}(p_{t+i}) = \mathbb{E}[w(t+i) | w(t+i) \leq \varphi(p_{t+i})] = \frac{\mathbb{E}[w(t+i) | w(t+i) \leq q(1 - p_{t+i})]}{\mathbb{P}(w(t+i) \leq q(1 - p_{t+i}))},$$

(4.63)

where $q(1 - p_{t+i})$ denotes the $(1 - p_{t+i})$-quantile of the distribution and $\mathbb{P}(w(t+i) \leq q(1 - p_{t+i})) = 1 - p_{t+i}$. Moreover, note that

$$\text{CVaR}(p_{t+i}) = \overline{m}^{(w)}(t+i) - \frac{\sqrt{\overline{V}^{(w)}(t+i)}}{1 - p_{t+i}} \cdot \mathbb{E} \left[ \frac{w(t+i) - \overline{m}^{(w)}(t+i)}{\sqrt{\overline{V}^{(w)}(t+i)}} | \frac{w(t+i) - \overline{m}^{(w)}(t+i)}{\sqrt{\overline{V}^{(w)}(t+i)}} \leq q(1 - p_{t+i}) \right],$$

hence it suffices to compute the expected shortfall for the standard normal random variable $\frac{w(t+i) - \overline{m}^{(w)}(t+i)}{\sqrt{\overline{V}^{(w)}(t+i)}}$. We obtain

$$\text{CVaR}(p_{t+i}) = \overline{m}^{(w)}(t+i) - \sqrt{\overline{V}^{(w)}(t+i)} \frac{1}{1 - p_{t+i}} \int_{-\infty}^{\Phi^{-1}(1 - p_{t+i})} l\phi(l) \, dl$$

$$= \overline{m}^{(w)}(t+i) - \sqrt{\overline{V}^{(w)}(t+i)} \frac{1}{1 - p_{t+i}} \left[ -\phi(l) \right]_{-\infty}^{\Phi^{-1}(1 - p_{t+i})}$$

$$= \overline{m}^{(w)}(t+i) - \sqrt{\overline{V}^{(w)}(t+i)} \frac{\phi(\Phi^{-1}(1 - p_{t+i}))}{1 - p_{t+i}},$$

(4.64)

Furthermore, we demand that $\text{CVaR}(p_{t+i}) \geq \overline{L}_{t+i}$ which completes the proof that (4.58) is equivalent to (4.62). The formula (4.62) is again a convex function w.r.t. $u(t+j)$. We know that $\sqrt{\overline{V}^{(w)}(t+i)}$ and $-\overline{m}^{(w)}(t+i)$ are both convex functions. The expression $\frac{\phi(\Phi^{-1}(1 - p_{t+i}))}{1 - p_{t+i}}$ is a non-negative number. Multiplying (4.62) by $-1$, we get

$$\overline{m}^{(w)}(t+i) - \sqrt{\overline{V}^{(w)}(t+i)} \frac{\phi(\Phi^{-1}(1 - p_{t+i}))}{1 - p_{t+i}} \leq -\overline{L}(t+i).$$

(4.65)

From the arguments of proof of proposition 4.7, it is clear that $\overline{L}(t+i)$ is a convex function w.r.t. $u(t+j)$. □

We have shown that both kinds of state constraints are convex functions and therefore, are suitable for optimization purposes. Both constraints are a coherent measure of risk, since both constraints are convex functions and the portfolio distribution is a Gaussian distribution. Furthermore, they are both dynamically consistent in the sense of Property
5 presented in Section 2.4, when they are only used as risk measure for the terminal
wealth. They are dynamical coherent risk measures, since they fulfill the properties of
single-period risk measures and mean and variance can be written as stochastic difference
equations. This allows us to use Bellman’s principle in discrete-time.

Objective functions

After having discussed two state constraints, we discuss two kinds of objective functions
for future portfolio values.

O1 The first possibility is maximizing a risk-averse objective function, which balances
the expected return and the possible risk. In the case of a normal distribution, the
density is uniquely described by its conditional mean and variance. This corresponds
to maximizing a classical mean-variance objective. Mathematically, we obtain
\[
\max_{u(t+i)} \left\{ \frac{1}{12} \lambda V^{(w)}(t+T) \right\},
\]
where \( \lambda \leq 1 \) denotes the level of risk aversion. When we use this objective function, we
do not need any of the probabilistic constraints to describe a meaningful optimization
problem, where future gains (returns) and losses are balanced. The objective function
is linear quadratic in \( u(t+i) \). In the case \( \lambda = 1 \), the objective function may only be
used in connection with the state constraint \( C_1 \) or \( C_2 \), since otherwise we would not
balance the expected return and the possible risk.

O2 The second possibility is maximizing the probability of achieving a final predetermined
value. This objective function could be used in situations, where a guaranteed return
is promised to an investor. Applications are pension funds (defined benefit plans) or
life-insurance policies. Mathematically, this objective function is given as
\[
\max_{u(t+i)} \mathbb{P}(w(t+T) > O(t+T)),
\]
where \( O(t+T) \) denotes the final value we like to exceed.

Whereas objective \( O_1 \) is directly computable form (4.51) and (4.53), we need to analyze
objective \( O_2 \) more closely.

**Proposition 4.9.** The optimization of the objective function given in (4.67) can be solved
by the following optimization problem:
\[
\min_{s,u(t+j)} \ s \\
\text{s.t.} \quad -\overline{m}^{(w)}(t+T) - s\sqrt{\overline{V}^{(w)}(t+T)} \leq -O(t+T) \\
s \leq 0 \quad (4.68)
\]

The maximization of the objective function in (4.67) is equivalent to the optimization problem given in (4.68). Furthermore, the optimization problem (4.68) is a convex optimization.

**Proof.** We can rewrite (4.67) in terms of the portfolio mean and variance and obtain
\[
\mathbb{P}(w(t+T) > O(t+T)) = 1 - \mathbb{P}(w(t+T) \leq O(t+T)) \\
= 1 - \mathbb{P}\left(\frac{w(t+T) - \overline{m}^{(w)}(t+T)}{\sqrt{\overline{V}^{(w)}(t+T)}} \leq \frac{O(t+T) - \overline{m}^{(w)}(t+T)}{\sqrt{\overline{V}^{(w)}(t+T)}}\right) \\
= 1 - \Phi\left(\frac{O(t+T) - \overline{m}^{(w)}(t+T)}{\sqrt{\overline{V}^{(w)}(t+T)}}\right) \geq \alpha, \quad (4.69)
\]
where \(\Phi(\cdot)\) is standard normal cumulative distribution function and we require that \(\alpha \geq 0\). The fact that \(\mathbb{P}(w(t+T) > O(t+T)) > 0.5\) means that it must be possible that \(m_{t+\kappa}^{(w)} \geq O_{t+\kappa}\), otherwise it is not possible to exceed a probability of 50%. By further manipulations, we obtain
\[
\Phi\left(\frac{O(t+T) - \overline{m}^{(w)}(t+T)}{\sqrt{\overline{V}^{(w)}(t+T)}}\right) \leq 1 - \alpha \\
\frac{O(t+T) - \overline{m}^{(w)}(t+T)}{\sqrt{\overline{V}^{(w)}(t+T)}} \leq \Phi^{-1}(1-\alpha) = s \quad (4.70)
\]
where \(s \leq 0\), since \(\alpha \geq 0.5\). This analysis shows that we need to find the smallest \(s \leq 0\) such that \(-\overline{m}^{(w)}(t+T) - s\sqrt{\overline{V}^{(w)}(t+T)} \leq -O(t+T)\) is fulfilled. As shown in the case of the probabilistic constraints, the constraint in (4.68) is again convex, since \(s \leq 0\) and the functions \(-m_{t+\kappa}\) and \(\sqrt{V_{t+\kappa}}\) are convex w.r.t. \(u(t+j)\). \(\square\)

A similar argument about maximizing the probability to exceed a given level is also discussed in Kouvaritakis et al. (2004).

**Portfolio optimization problems**

We discuss several portfolio optimization problems. Based on the description of objectives and constraints, we combine them in order to obtain suitable portfolio optimization problems. We also discuss possible applications of the different optimization problems. Note
that for the first objective function, the control decisions are independent of the current value of the wealth. For this reason, future asset allocation decisions do not depend on the trajectory of the portfolio, but solely on the current trade-off between the satisfying the constraints and maximizing the objective. When we define the constraints relative to the current wealth, e.g., \( L(t+i) - w(t+i) \), the control decisions are again independent of the current value of the log-wealth.

\textbf{P1} The first optimization problem consists only of \( O_1 \) and is mathematically described as

\[
\max_{u(t+i)} \bar{m}^{(w)}(t+T) + \frac{1}{2} \lambda \bar{V}^{(w)}(t+T)
\]

s.t. \( Cu(t+i) \leq c \), \( (4.71) \)

where \( C \) and \( c \) can be used to impose linear constraints on the asset allocation variable \( u(t+i) \). This optimization problem is often encountered in the literature and has been solved in the case of no constraints on \( u(t+i) \). This problem is often called “Strategic Asset Allocation” problem as an expression describing a portfolio optimization problem with time-varying returns and objectives typical for long-term investments, see Brennan et al. (1997). When we remove the constraints on \( u(t+i) \), we solve the problem by dynamic programming techniques, which is given in Appendix A.2. The equivalent continuous-time problem (without any constraints) can be found in Section 3.2.

\textbf{P2} For the second optimization problem, we combine the objective of maximizing the return \( (O_1, \lambda = 0) \) with VaR constraints, i.e., \( C_1 \). The portfolio optimization problem is mathematically given as

\[
\max_{u(t+i)} \bar{m}^{(w)}(t+T) - \sqrt{\bar{V}^{(w)}(t+i)} \Phi^{-1}(1 - p_{t+i}) + L(t+i) \leq 0, \quad i = 1, \ldots, T
\]

\[
Cu(t+i) \leq c,
\]

\( (4.72) \)

where \( \bar{m}^{(w)}(t+i) \) and \( \bar{V}^{(w)}(t+i) \) can be computed with \( (4.51) \) and \( (4.53) \). This optimization problem can be applied to classical asset allocation for mutual funds, where the investor tries to maximize the future return while keeping the VaR under a certain limit \( (L(t+i)) \) for all future periods.

\textbf{P3} The third optimization problem, consists of \( O_1 \) and \( C_2 \). This example could be a pension fund portfolio, where the pension-fund has made a return promise to the
4.2 Portfolio optimizations using the MPC approach with different return models and objectives

An investor and wants to avoid possible extreme loss situation. For this reason, the CVaR constraints is combined with the maximization of probability that the portfolio value exceeds a predetermined value. Mathematically, the problem is described as

$$\min_{s, u(t+i)} s$$

s.t. 

$$-\bar{m}^{(w)}(t+T) - s\sqrt{V^{(w)}(t+T)} \leq -O(t+T)$$

$$-\bar{m}^{(w)}(t+i) + \sqrt{V^{(w)}(t+i)} \phi\left(\frac{\Phi^{-1}(1-p_{t+i})}{1-p_{t+i}}\right) + L(t+i) \leq 0,$$

$$Cu(t+i) \leq c, \quad i = 1, \ldots, T - 1$$

$$s \leq 0,$$

(4.73)

where the decision variables are $u(t+i)$ and $s$.

The dynamic portfolio optimization problems, which can be stated by combining the two objective function (O1 and O2) and the two possible state constraints (C1 and C2), are all convex optimization problems since all the functions are convex. The convexity of the optimization problem is important, since it allows us to solve problems of fairly large size in a computationally efficient manner. Furthermore, the optimal asset allocation is unique since convex optimization problem have only one unique solution.

When we apply the MPC algorithm in this manner, we always optimize the portfolio with a horizon of $T$ steps ahead. Alternatively, we could optimize the portfolio distribution for a given fixed calendar time, e.g., January 2034. Then the horizon of the optimization would shrink by one step at every cycle of the algorithm. The disadvantage of the fixed calendar time horizon is that we may have to compute a prohibitively large number of future asset allocation decisions initially. When the “true” fixed horizon is sufficiently far away, alternatively a smaller, but always constant horizon, could lead to the same asset allocation decision. Such a behavior is observed in continuous-time long-term unconstrained portfolio optimizations, see Section 3.4.

Comparison of unconstrained MPC solution with the DP solution

In Section A.2, the solution of the DP problem for mean-variance objective O1 is given. We compare numerically the two control methods for the data given in Section A.4. For this type of objective function, the DP solution is independent of the actual trajectory of
the log-wealth and only depends on the evolution of the factors. When the uncertainty of
the factor dynamics is small compared to the uncertainty of the asset dynamics ($\sigma \rho \gg \nu$),
the MPC solution yields good control decisions.

First, we compare the control decisions of both methods for different investment hori-
zons ($\kappa$). In Figure 4.2, the relative difference of the two control variables as function of
the investment horizon are shown. Since the DP method yields the “true” control deci-
sions, the difference to the MPC method is the error of the MPC method. The largest
relative error is approximately $1.4 \cdot 10^{-3}$ for the stock weight with a horizon of 76 months.
The relative error for the bond weight tends to be smaller. The relative error as function
of the current value of the state variables is shown in Figure 4.3 for the bond investments.
The largest relative error is again around $2 \cdot 10^{-3}$. In general the two methods give fairly
similar control decisions for $\sigma \rho \gg \nu$. For more complex situations, such as with $n = 10$
and $m = 30$, we get similar results. This short numerical comparison illustrates that the
MPC method yields satisfactory decisions even though it is a suboptimal procedure.

4.2.4 Portfolio optimization for factor models for expected returns and
non-Gaussian return distributions

In this part, we discuss the MPC method applied to the asset and portfolio model which
are introduced in Section 4.2.1. We assume that the white noise processes driving the
risky asset returns are non-Gaussian. We deal with two kinds of non-Gaussian white
noise assumptions. The first assumption is that volatilities and correlations are time-varying and stochastic. In this case, the covariance matrix is modelled by a multivariate GARCH model. Even if we assume that the multivariate GARCH model is driven by normal standard residuals, then the resulting white noise process is heavy-tailed. In this case, we need to compute the prediction of the covariance matrix \( E_t[\Sigma(t+i)], \quad i = 1, \ldots, T - 1 \) in order to compute the first two moments of the portfolio distribution.

The second assumption is that the white-noise process of the risky assets has a constant or deterministic covariance matrix, but the distribution is heavy-tailed, e.g., student t or Gaussian mixture distributions. In that case, we need no prediction of the covariance matrix.

**Covariance Matrix Prediction**

In this section, we briefly outline the prediction of the DCC, the GARCH, and the TARCH models. We explain the prediction equations based on the TARCH(1,1,1) and DCC(1,1) examples. The prediction of the conditional covariance matrix can be split into two parts, first the prediction of the individual variances \( v_i^2(t) \) and second the prediction of the conditional correlation matrix. The prediction is based on the available information till time \( t \) and calculated \( T \) steps ahead. For this reason, we derive the conditional expectation of the variances and correlations. We make an additional assumption that \( \xi^r \) is drawn from a symmetric distribution. Using (4.42) and setting \( p_i = q_i = o_i = 1 \), we compute the expectation of \( v(t+1) \) based on the conditional information until time \( t \) and obtain

\[
E_t[v_i^2(t+2)] = \omega_i + \alpha_i E_t[\epsilon_i^2(t+1)] + \beta_i E_t[v_i^2(t+1)] + \gamma_i E_t[\chi_i(t+1)\epsilon_i^2(t+1)] .
\]

Based on the conditional information until \( t \), we know \( E_t[\epsilon_i^2(t+1)] = v_i^2(t+1) \), \( E_t[v_i^2(t+1)] = v_i^2(t+1) \), and \( E_t[\chi_i(t+1)\epsilon_i^2(t+1)] = \frac{1}{2}v_i^2(t+1) \), because we made the assumption that \( \xi^r_i(t) \) possesses unit variance and is drawn from a symmetric distribution. Furthermore from (4.42) and the knowledge of all information at time \( t \), we know \( v_i(t+1) \). Using this fact, we obtain the following mathematical expression:

\[
E_t[v_i^2(t+2)] = \omega_i + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)v_i^2(t+1) . \quad (4.74)
\]

Computing the expectation of \( v_i^2(t+k) \), it results in the analogous discrete difference equation as (4.74), namely
\[ E_t[v_i^2(t+k)] = \omega_i + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)E_t[v_i^2(t+k-1)]. \] (4.75)

Combining (4.74) and (4.75) and iterating the result \((T-1)\) times, we obtain the prediction equation

\[ E_t[v_i^2(t+T)] = \omega_i \sum_{j=1}^{T-1} (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)^j + (\alpha_i + \beta_i + \frac{1}{2}\gamma_i)^{(T-1)}v_i^2(t+1). \] (4.76)

For \(\gamma_i = 0\), the TARCH(1,1,1) model is reduced to a GARCH(1,1) model, for which the prediction equation can be found in Baillie and Bollerslev (1992). In the case that we deal with a general GARCH\((p,q)\) model or a TARCH\((a,p,q)\), the prediction equations are given in Section A.5 of the Appendix.

In the case of dynamic conditional correlations, the prediction equations are difficult to derive, because the model is nonlinear: inspecting (4.43), we see that \(E_t[\zeta(T-1)\zeta(T-1)^T] = \Lambda(T-1) = Q(T-1)^{-1}Q(T-1)Q(T-1)^{-1}\) and thus, the model is nonlinear. In Engle and Sheppard (2001) the authors proposed to use the approximation \(E_t[Q(T-1)] \approx E_t[\Lambda(T-1)]\). This results in the following approximated prediction equation for the DCC(1,1) model

\[ E_t[\Lambda(T-1)] = \sum_{j=1}^{T-1} (1 - \delta - \eta)Q(\delta + \eta)^j + (\delta + \eta)^{(T-1)}\Lambda(t+1). \]

It possesses the same structure as the GARCH(1,1) model and therefore we can derive an equation that is similar to (4.76), but with \(\gamma = 0\). In the case of a general DCC\((k,l)\), we use the prediction equations derived for the GARCH\((p,q)\) model, to compute the DCC predictions. Having computed the prediction of the individual variances and the correlation, we now know the prediction of the conditional covariance matrix.

**Portfolio optimizations with a mean-variance objective**

The objective of the portfolio optimization is maximizing the mean-variance objective function of the portfolio \(\overline{T}\) time steps into the future. The objective is given by

\[ \max_{u(t+i) \in \mathcal{U}} \left\{ \overline{m}^{(w)}(t+T) + \frac{1}{2}\lambda\overline{V}^{(w)}(t+T) \right\}, \]

where \(\lambda < 1\) is the risk aversion coefficient, \(\overline{T}\) denotes the time horizon, and \(\mathcal{U}\) denotes a linear (convex) set of constraints. By inspecting the mean \(\overline{m}(t+T)\) and the variance \(\overline{V}(t+T)\), we note that both are linear-quadratic functions of \(u(t+i)\) for \(i = 0, \ldots, T-1\).
Thus, the optimization is a quadratic program (QP). The optimization problem is given by

$$
\max \left\{ d(t)^T U + \frac{1}{2} U^T D(t) U \right\}
$$

s.t. $CU \leq c,$ \hspace{1cm} (4.77)

where $d(t)$ and $D(t)$ can be assembled with (4.57), (4.58), and (4.77), $U = (u(t), u(t+1), ..., u(t+T-1))$, and $C$ and $c$ can be used to impose constraints on the asset allocation.

An example for $D(t)$ and $d(t)$ for $T = 2$ is given by

$$
D(t) = \begin{bmatrix}
(\lambda - 1)\Sigma(t+1) & \lambda \sigma(t+1) \rho \nu F^T \\
\lambda F \nu T \rho \sigma(t+1)^T & (\lambda - 1)\Sigma(t+2) + F H T T^T
\end{bmatrix}
$$

$$
d(t) = \begin{bmatrix}
2\lambda \sigma(t+1) \rho \nu F_0^T + F x(t) + f \\
h + F [Ax(t) + a] + 2\lambda F \nu \nu^T F_0
\end{bmatrix}.
$$

From the example above, we see that the decision for the first period is linked with the decision for the second period. In the case of uncorrelated factors and returns ($\rho = 0$) the two decision variables are independent and the optimization is reduced to multiple single period optimizations.

The strategy to implement this portfolio optimization algorithm is shown in Figure 4.4. The calculation of the asset allocation at time $t$ is as follows:

1. Parameter estimation of the factor and the GARCH models.
2. Computation of the covariance matrix prediction.
3. Calculation of the portfolio mean and variance $T$ steps ahead.
4. Solving the quadratic programm (4.77).
5. Application of $u(t)$ of the sequence $U$ and moving one step further.

It is not always necessary to update the model parameters, since the parameters may change slowly. However, when the parameters are always calculated anew, we base the portfolio optimization on the latest available information of the state variables and the model parameters.

4.2.5 Discussion of the advantages and disadvantages of the analytical MPC method

When we apply the analytical MPC method to models where distributions of the risky asset returns are non-Gaussian, the results are a bit ambiguous.
The mean variance-objective completely covers the risk-return trade-off in the case of Gaussian distributions or more generally in the case of elliptical distributions, see McNeil et al. (2005, Chapter 3). In the case of heavy-tailed and non-self-similar distributions, the variance is not a good risky measure, since it is not coherent. The mean-variance objective is though a well-known and well-understood portfolio objective. With variance as risk measure, we can account for diversification and it is convex, but it violates monotonicity property of coherent risk measures.

The main advantages are the computational efficiency due to the analytical calculations. Since the portfolio mean and variance as well as the covariance predictions can be analytically computed, the method is fast. Furthermore, the optimization is a quadratic program (QP) which can be solved with many variables. This method allows us to solve a problem with many assets and long investment horizons. In other cases, where we have to rely on numerical procedures, the size of the optimization problem becomes the limiting factor of the applicability of the model. In this case, problems with a few thousands variables can be solved with modern solvers and computers.
4.2.6 Portfolio optimization with coherent risk measures for terminal wealth

In this part, we address the problem of portfolio optimization with heavy-tailed distributions. In this case, as discussed above, the mean-variance objective is questionable. Instead, we use a coherent risk measures, as discussed in Section 2.3. We define a utility function as

\[ U(w(T)) = E[w(T)] + \lambda \Gamma(-w(T)), \]  

(4.78)

where \( \Gamma(\cdot) \) is coherent risk measure, \( \lambda < 0 \) is the coefficient of risk aversion, and \( w(T) \) is the log-wealth of the portfolio at final time \( T \). With this utility, we balance the expected return and the possible risk.

**Proposition 4.10.** The utility defined in (4.78) fulfills Property I of (4.22) and is a concave function of \( w(T) \).

**Proof.** The utility defined in (4.78) fulfills property I of (4.22) since both the expectation and the risk measure are translation invariant. The term \( \lambda \Gamma \) is concave by construction. \( \square \)

Thus any portfolio optimization using the utility definition of (4.78) is suitable for the MPC method, since the decisions about the asset allocation are independent of current value of the portfolio. Since we deal with portfolio optimizations of asset classes that exhibit heavy-tailed return distribution, we want to work with a risk measure that is suitable for those applications. The Conditional Value-at-Risk (CVaR) is ideally suited for those problems, since it measures the expected tail values and thus, we use this measure for the MPC application.

**Computational aspects of CVaR**

As shown in Section 2.3.2, CVaR can be efficiently computed by

\[ C(u, \varphi, \theta) = \varphi + \frac{1}{1 - \theta} \left( \int_{r \in \mathbb{R}^n} [f(r, u) - \varphi]^+ p(r) dr \right), \]  

(4.79)

where \([t]^+ = \max(t, 0)\), \( f(r, u) \) denotes the portfolio loss function, and \( p(r) \) denotes density of the asset returns. The integral given in (4.79) can be approximated in many ways. One possibility suggested by Uryasev (2000) is to sample the return distribution \( p(r) \) and generate a collection of return vectors \( r_1, r_2, \ldots, r_N \). Then, the corresponding approximation to (4.79) is
\[ \tilde{C}(u, \varphi, \theta) = \varphi + \frac{1}{1 - \theta} \sum_{k=1}^{N} \pi_k [f(r_k, u) - \varphi]^+, \tag{4.80} \]

where \( \pi_k \) is the probability of scenario \( r_k \). If the loss function \( f(u, r) \) is linear in \( u \), the function \( \tilde{C}(u, \varphi, \theta) \) is convex and piecewise linear. Moreover, we can replace \([f(r_k, u) - \varphi]^+\) by using dummy variables \( \alpha_k, k = 1, \ldots, N \), the function \( \tilde{C}(u, \varphi, \theta) \) is replaced by a linear function and a set of constraints given by

\[
\begin{align*}
\min_{u, \varphi} \left\{ \varphi + \frac{1}{1 - \theta} \left( \sum_{k=1}^{N} \pi_j \alpha_k \right) \right\} \\
\text{s.t. } \alpha_k \geq f(r_k, u) - \varphi, \\
\alpha_k \geq 0, \quad k = 1, \ldots, N. \tag{4.81}
\end{align*}
\]

As the number of scenarios increases, the piecewise linear approximation in (4.81) converges to the continuously differentiable functional \( C(u, \varphi, \theta) \). As noted in Alexander, Coleman and Li (2004), an alternative approximation to the piecewise linear approximation is a piecewise quadratic approximation to the continuously differentiable function \( C(u, \varphi, \theta) \). We therefore define,

\[
\overline{C}(u, \varphi, \theta) = \varphi + \frac{1}{1 - \theta} \sum_{k=1}^{N} \pi_k \beta(f(r_k, u) - \varphi), \tag{4.82}
\]

where \( \beta(y) \) is a continuously piecewise quadratic approximation of the piecewise linear function \( \max(y, 0) \). The function \( \beta(y) \) is defined as

\[
\beta(y) = \begin{cases} 
  y & \text{if } y \geq \epsilon \\
  \frac{(y + \eta)^2}{4\eta} & -\eta \leq y \leq \eta \\
  0 & \text{otherwise}
\end{cases}, \tag{4.83}
\]

where \( \eta \) is the parameter of the approximation. The main advantage of using (4.82) instead of (4.81) is that the optimization is independent of the number of scenarios. In the case of the piecewise linear approximation for each scenario an additional constraint is introduced. If the number of scenarios becomes very large, e.g., \( > 50000 \), the size of the optimization may become unmanageable. In the case of (4.82), the number of scenarios does not change the optimization problem size. The piecewise quadratic approximation can be solved by general convex optimization solvers.
4.2 Portfolio optimizations using the MPC approach with different return models and objectives

Long-term portfolio optimization with coherent risk measures for terminal wealth

As stated at the beginning of this section, we assume that the investor pursues a long-term goal. The investor wants to optimize the logarithm of the portfolio wealth with respect to the expected return and the risk. We compute the risk and the expected return of the final wealth at time \( t + T \), i.e., we optimize the portfolio at the final time which depends on the trajectory of the asset returns.

Due to the non-self-similarity of heavy-tailed distributions (with finite second moment), we have to resort to the scenario approximation method. Therefore, we discuss the portfolio optimization for the general case and not for a specific model. The optimization problem in the general case is written as

\[
\max_{\varphi, u(\tau)} \frac{1}{N} \left\{ \sum_{s=1}^{N} w^s(t + T) + \lambda \left( \varphi + \frac{1}{1-\theta} \sum_{s=1}^{N} \beta \left( -w^s(t + T) - \varphi \right) \right) \right\}
\]

s.t. \( w^s(t + T) = \sum_{\tau=t}^{t+T-1} \ln \left( 1 + r^s_0(\tau) \right) + u(\tau)^T (r^s(\tau) - 1) + w(t), \ s = 1, \ldots, N, \)

\[ u(\tau) \in U \]  

(4.84)

where \( \lambda < 0 \) is the coefficient of risk aversion, \( \beta \) is defined as in (4.83), \( u(\tau) \in \mathbb{R}^n \), and \( U \) defines a convex set.

**Proposition 4.11.** The optimization problem defined in (4.84) is a concave optimization problem of the size \( nT \) where \( n \) denotes the number of assets and \( T \) the time horizon.

**Proof.** The objective function of (4.84) is the sum of the sample expectation and the piecewise quadratic approximation of CVaR as given in (4.82). Therefore, the objective is a utility function as defined in (4.78) which is concave, see Proposition 4.10. Furthermore, by Proposition 4.2, the optimization problem is concave. When inserting the equality constraint into the objective, we have \( nT \) investment periods.

In Section 5.1, the MPC method with a CVaR risk constraint for a linear factor model with heavy-tailed noise is applied to a long-term portfolio optimization problem. In the case study, we show that risk-limits imposed on the portfolio were not violated in an
out-of-sample study with 10 assets and 777 weeks testing period. This suggests that the
MPC method with coherent risk-measures works quite well in real-world situations.

4.3 Portfolio optimization with linear transaction costs and
constraints

In this section, we discuss the problem of portfolio optimization with linear transaction
costs. As described in Section 4.1.3, the wealth dynamics with transaction costs change
considerably, since we need to keep track of absolute investments in every asset. In the
case of no transaction costs, it suffices to keep track of the total wealth and the percentage
of the wealth invested into the different assets.

The term transaction costs includes not only brokerage and banking fees, but also
market impact costs as well as taxes and other levies. These non-fee transaction costs are
caused by several factors, especially thin or volatile markets, large size trades, or hedging
costs for international transactions. Today, brokerage fees as well as other exchange related
levies in OECD countries have been dramatically reduced for major asset classes such as
government bonds or large-cap stocks. Therefore, fees can be neglected for optimization
purposes as long as there is not enough power to influence prices. The situation is how-
ever different for thinly traded stocks, international markets which often have significant
brokerage fees, and alternative (non-traditional) investments such as hedge funds or real
estate investments. In these cases, we are not able to neglect the presence of transaction
costs without a significant impact on the optimal decisions.

4.3.1 Transaction costs and path dependent optimizations

In the case of transaction costs, we limit our description of the wealth dynamics to linear
transaction costs and the scenario approach to describe multi-period asset prices.

Many different formulations of multi-period investment problems can be found in the
literature, see Ziemba and Mulvey (1998) or Louveaux and Birge (1997). We adopt the
basic model formulation presented in Mulvey and Simsek (2002) and Mulvey and Shetty
(2004). The portfolio optimization horizon consists of $T$ time steps represented by $t =
\{1, 2, \ldots, T\}$. At every time step $t$, the investors are able to make a decision regarding
their investments and faces the inflow of funds and outflows due to obligations. The
investment classes belong to the set \( I = \{1, 2, \ldots, n\} \). By utilizing a sufficiently large number of scenarios, we are able to represent all of the random effects the investor faces. The scenarios are defined as the set \( S \) that represents a reasonable description of the future uncertainties. A scenario \( s \in S \) describes a unique path through consecutive nodes of the scenario tree as depicted in Figure 4.1.

Let \( z^s_i(t) \) be the amount of wealth invested in instrument \( i \) at the beginning of the time step \( t \) under scenario \( s \). The units we use are investor’s home currency (e.g., Swiss Francs). Foreign assets, hedged as well as un-hedged, are also denoted in the portfolio’s home currency. At time \( t \) the total wealth of the portfolio is

\[
W^s(t) = \sum_{i=1}^{n} z^s_i(t), \quad \forall s \in S, \quad (4.85)
\]

where \( W^s(t) \) denotes the total wealth under scenario \( s \). Given the returns of each investment class, the asset values at the end of the time period are

\[
z^s_i(t)(1 + r^s_i(t)) = \tilde{z}^s_i(t), \quad \forall s \in S, \quad \forall i \in I, \quad (4.86)
\]

where \( r^s_i(t) \) is the return of investment class \( i \) at time \( t \) under scenario \( s \). The returns are obtained from the scenario generation system. Therefore, \( \tilde{z}^s_i(t) \) is the \( i \)-th asset value at the end of the time period, where \( d^s_i(t) \geq 0 \) denotes amount of asset \( i \) sold at time \( t \) under scenario \( s \), and \( p^s_i(t) \geq 0 \) denotes the purchase of asset \( i \) at time \( t \) under scenario \( s \). The asset balance equation for each asset is

\[
z^s_i(t) = \tilde{z}^s_i(t-1) + p^s_i(t)(1-\delta_i) - d^s_i(t), \quad \forall s \in S, \quad \forall i \in I \setminus \{1\}, \quad (4.87)
\]

where \( \delta_i \) is the proportional (linear) transaction cost of asset \( i \). We make the assumption that the transaction costs are not a function of time, but depend only on the investment class involved. Note that the transaction costs appear only for the purchase variables in (4.87) and for the sales variables below in (4.88). In the case that we purchase an asset and spend 100 Swiss Francs, only for 100\((1-\delta_i)\) Francs we acquire asset \( i \), i.e., when we face a cost of 1% we only get 99 Francs of the asset, but from the bank account 100 Francs are subtracted. In the case that we sell 100 Francs of asset \( i \), only 100\((1-\delta_i)\) Francs are deposited in the bank account, e.g., we sell the asset for 100 Francs but only get 99 Francs credited in the bank account. In this way, the transaction costs are correctly considered.
We treat the cash component of our investments as a special asset. The balance equation for cash is

\[ z_1^s(t) = \tilde{z}_1^s(t-1) + \sum_{i=2}^{n} d_i^s(t)(1-\delta_i) - \sum_{i=2}^{n} p_i^s(t) + q^s(t), \quad \forall s \in S, \quad (4.88) \]

where \( z_1^s(t) \) is the cash account at time \( t \) under scenario \( s \) and \( q^s(t) \) is the in- or outflow of funds due to non-capital gains or losses at time \( t \) under scenario \( s \), respectively. The cash account equals the interest rate earned from the cash account’s value of the last period, plus all money earned from sales of assets, minus all money used for the purchase of assets, plus the in- or outflows.

All of the variables in Equations (4.85) - (4.88) are dependent on the actual scenario \( s \). These equations could be decomposed into subproblems for each scenario where we anticipate which scenario will evolve. To model reality, we must however, impose non-anticipativity constraints. All scenarios which inherit the same past up to a certain time period must evoke the same decisions in that time period, otherwise the non-anticipativity requirement would be violated. So \( z_i^s(t) = z_i^{s'}(t) \) when \( s \) and \( s' \) have same past until time \( t \).

In the following system of equations, we summarize the wealth dynamics under transaction costs:

\[ W^s(t) = \sum_{i=1}^{n} z_i^s(t), \quad \forall s \in S, \]

\[ z_i^s(t)(1 + r_i^s(t)) = \tilde{z}_i^s(t), \quad \forall s \in S, \quad \forall i \in I, \]

\[ z_i^s(t) = \tilde{z}_i^s(t-1) + p_i^s(t)(1-\delta_i) - d_i^s(t), \quad \forall s \in S, \quad \forall i \in I \setminus \{1\}, \]

\[ z_i^s(t) = \tilde{z}_i^s(t-1) + \sum_{i=2}^{n} d_i^s(t)(1-\delta_i) - \sum_{i=2}^{n} p_i^s(t) + q^s(t), \quad \forall s \in S. \quad (4.89) \]

In this setup, we keep track of each possible investment opportunity for all times and scenarios. This increases the number of state variables from one wealth equation to \( n \) asset dynamics. Additionally, we have to keep track of all the state equations that describe the evolution of the returns \( r_i^s(t) \) such as the models given in Section 4.1.1. The optimization is path dependent, since the decision variables \( p_i^s(t) \) and \( d_i^s(t) \) must be chosen for each time \( t \) and each scenario \( s \). In the case of no transaction costs, the proportional investments do not depend on the actual scenario. The system of equations (4.89) can be rewritten in the standard form for dynamic optimization problems, but is not followed here.
4.3 Portfolio optimization with linear transaction costs and constraints

4.3.2 Stochastic programming solution to optimization problem under transaction costs

In this section, we discuss two concepts to solve the stochastic programming approximation. The first concept results in a large-scale linear programming problem (LP). The second concept uses the ideas from continuous-time optimizations and uses the fraction of wealth invested into the different assets as control variable.

Stochastic programming approach with linear objective functions

The stochastic programming approach finds the optimal sales variables \( p_t^i(t+k) \) and the optimal purchase variables \( d_t^i(t+k) \) given the current time and scenario including the non-anticipativity constraints. By inspecting (4.89), we see that these equations are linear. Furthermore, when we use a linear objective function then the optimization problem becomes a large-scale linear programming (LP) problem.

We introduce a objective function which leads to an LP problem. The objective function is well suited for problems with assets and liabilities. Liabilities can be explicit payments promised at future dates as well as capital guarantees (promises) to investors or investment goals. Standard coherent risk measures, such as CVaR, can be used in ALM situations, when applied to the portfolio’s net wealth, e.g., the sum of the assets minus all the remaining liabilities. CVaR penalizes linearly all events which are below the VaR limit for a given confidence level. The VaR limit therefore depends on the confidence level chosen and the shape of the distribution. The VaR limit maybe a negative number, i.e., a negative net wealth result. In the situation with assets and liabilities, e.g., pension funds, we do not only want to penalize scenarios that are smaller than a given VaR value, but all scenarios where the net wealth is non-positive. For these reasons, we define risk as a penalty function for the net wealth. We want to penalize small “non-achievement” of liabilities differently from large “non-achievement”. Therefore, the penalty function should have an increasing slope with increasing “non-achievement”. The risk of the portfolio is measured as one-sided downside risk based on non-achievement of the liabilities or goals.

As penalty function \( P_f(\cdot) \) we choose the expectation of a piecewise linear function of the net wealth as shown in Figure 4.5. The penalty function is convex, but not a coherent risk measure in the sense of Artzner et al. However, it fulfills the modified properties of Ziemba and Rockafellar (2000). Furthermore, we can formulate a backward recursion to
compute the risk measure and thus, it is dynamically coherent in the sense of Section 2.4. The same approach is discussed in detail by Dondi, Herzog, Schumann and Geering (2005) where this method is applied to the management of a Swiss pension fund. Furthermore, in the case of “American” capital guarantees, a suitable risk measure that is linear is given by

\[ \Gamma = \mathbb{E} \left[ \sum_{\tau=t}^{T} P_f (W(\tau) - G(\tau)) \mid \mathcal{F}(t) \right], \]  

(4.90)

where \( P_f \) denotes the piecewise linear penalty function, \( G(\tau) \) denotes the American capital guarantee at time \( \tau \), \( W(\tau) \) the portfolio value at time \( \tau \), and \( \mathcal{F}(t) \) the information at time \( t \). This risk measure is convex and fulfills the property of a dynamic risk measure.

Many specialized algorithms exist to solve this special form of an LP, such as the L-shaped algorithm, see van Slyke and Wets (1969), Birge, Donohue, Holmes and Svintsitski (1994), and Birge and Holmes (1992). Other approaches, are known as the progressive hedging algorithm, see Rockafeller and Wets (1991), Berger, Mulvey and Ruszczynski (1994), and Barro and Canestrelli (2004).

The advantage of the stochastic programming approach over the dynamic stochastic control approach is that the decisions are computed similarly to a feedback control system. The fraction of wealth invested in the different asset classes differs for different scenarios, which sets this apart from the fixed-mix approach. The main disadvantage is that for systems with many future time periods, the number of scenarios \( S = \prod_{\tau=t}^{T} \kappa(\tau) \), where \( \kappa(\tau) \) is the number of branches at each time step \( \tau \), increase exponentially.
Dynamic stochastic control approach to stochastic programming

The dynamic control approach works by using control policies that are independent of the current and future scenarios. Such a policy is the fixed-mix strategy, where we require that a fixed proportion of the wealth is invested in a certain asset (Ziemba 2003). Fixed-mix strategies are motivated by stochastic control solutions but also have another strong theoretical motivation. Recently, Dempster, Evstigneev and Schenk-Hopp (2002) have proven, under general assumptions, that any fixed-mix investment policy leads to exponential portfolio growth with probability one in the long-run, where asset return dynamics follow stationary stochastic process. The result even holds when the investor faces small transaction costs. Alternatively, we could also allow that the mix is time-varying, but still independent of the scenarios. This results in a dynamic mix (or time-varying mix) strategy. The fraction of wealth at time $t$ invested in the $i$-th asset is

$$ u_i(t) = \frac{z^s_i(t)}{W^s(t)}, \ s \in S. \quad (4.91) $$

The mix strategies reduce the number of decision variables to a large extent, but they introduce nonlinearity into the problem. Theoretical results from continuous-time finance support this approach, see Merton (1992) and the results of Chapter 3. By allowing dynamic mix strategies, i.e., the mix is not constant throughout the planning horizon, we can also introduce feedback into the system. This feedback depends on the time dependencies of the asset returns. The main disadvantage of the stochastic control strategies is that they introduce nonlinearities into the optimization problem regardless of the functional form of the objective function. To analyze (4.91), we use system dynamics (4.89) and we obtain

$$ \ddot{z}_j^s(t-1) + p_j^s(t)(1 - \delta_j) - d_j^s(t) = u_j(t) \left( \sum_{i=1}^{n} \{ \dot{z}_i^s(t-1) \} \right) + q^s(t) $$

$$ - \sum_{i=2}^{n} \{ \delta_i(p_i^s(t) + d_i^s(t)) \} , \ j \in I \setminus \{1\}, \ s \in S. \quad (4.92) $$

The asset sales and purchases under each scenario and at any time can be computed in this manner. The wealth at time $t$ before re-balancing minus the net transaction costs is divided among the $n$ assets according to the mix rule $u_j(t)$. The various nonlinearities in Equation (4.92) are visible such as the term.
\[ u_j(t) \left( \sum_{i=2}^{n} \delta_i(p^s_i(t) + d^s_i(t)) \right). \] The stochastic control approach results in a non-convex nonlinear optimization, which requires specialized global solution algorithms, see Maranas, Androulakis, Floudas, Berger and Mulvey (1997) and Konno and Wijayanayake (2002), or other global optimization techniques, such as genetic algorithms.

Many authors have investigated stochastic control problems with transaction costs. Generally, the findings can be described in terms of an allowable zone around the targeted fraction of wealth invested in certain assets (Konno and Wijayanayake 2002, Konno and Wijayanayake 2001, Shreve and Soner 1991, Dumas and Luciano 1991, Davis and Norman 1990, Constantinides 1986). These no-trade zones can be included in the optimization by requiring

\[ u_i^{\text{min}}(\tau) \leq \frac{z^s_i(\tau)}{W^s(\tau)} \leq u_i^{\text{max}}(\tau), \quad s \in S. \] (4.93)

Note that \( u_i^{\text{min}}(\tau) \) and \( u_i^{\text{max}}(\tau) \) are also decision variables of the optimization. In this way, the optimization is not forced to trade under every scenario and thus saves transaction costs. In this way, the stochastic control solutions are mimicked since the investment allocation is only rebalanced when one of the constraints given in (4.93) are violated.

**Long-term portfolio optimization with transaction costs**

In the case of problems of portfolio optimizations with long-term goals and transaction costs, we use the approach presented in Section 4.1.6. The basic approximated optimization problem with a linear penalty function is given by

\[
\max_{P^s(\tau), P^d(\tau)} \left\{ \sum_{s=1}^{N} [W^s(T) + \lambda \sum_{\tau=t}^{T} P_f(\tau, W(\tau) - G(\tau)|F(t))] \right\} \\
W^s(\tau) = \sum_{i=1}^{n} z^s_i(\tau), \quad \forall s \in S, \\
z^s_i(\tau)(1 + r^s_i(\tau)) = \tilde{z}^s_i(\tau), \quad \forall s \in S, \quad \forall i \in I, \\
z^s_i(\tau) = \tilde{z}^s_i(\tau - 1) + p^s_i(\tau)(1 - \delta_i) - d^s_i(\tau), \quad \forall s \in S, \quad \forall i \in I \setminus \{1\}, \\
z^s_1(\tau) = \tilde{z}^s_1(\tau - 1) + \sum_{i=2}^{n} d^s_i(\tau)(1 - \delta_i) - \sum_{i=2}^{n} p^s_i(\tau) + q^s(\tau), \quad \forall s \in S, \] (4.94)

where \( \tau = t, t+1, \ldots, T, \lambda < 0 \) denotes coefficient of risk aversion, and initial conditions \( z_i(t), i = 1, \ldots, n \) are given. Furthermore, we may impose constraints for short positions or leveraging and we may limit the position in a specific asset (or asset class) by imposing the following constraints
\[ u_i^{(\text{min})} \leq \frac{z_i^*(\tau)}{W^*(\tau)} \leq u_i^{(\text{max})}, \]

where \( u_i^{(\text{min})} \) is the minimum fraction of wealth invested in asset \( i \) and \( u_i^{(\text{max})} \) is the maximum fraction of wealth invested in asset \( i \). Similar constraints can be introduced to enforce minimum and maximum investments into certain sets of assets, e.g., international investments or stocks. These kind of constraints can be formulated as linear constraints and thus, the optimization given in (4.94) is still a large-scale LP.

The portfolio optimization is solved by employing the algorithm outlined in Section 4.1.6. At every time step, we generate the scenario approximation of the return dynamics, based on the current information at time \( t \). We then solve the optimization problem given by (4.94) with all of the corresponding constraints. We check the accuracy and refine the scenario tree. When we reach the predetermined accuracy, we terminate the optimization and tree generation and apply the investment decisions \( p_i(t) \) and \( d_i(t) \). Then the procedure moves one time step ahead.

### 4.4 Discussion of applicability and limitations of the discrete-time models

We discuss the discrete-time problem of portfolio optimization with respect to the modelling and optimization aspects.

#### 4.4.1 Modelling aspects

The discrete-time models allow for much more modelling flexibility than their continuous-time counterparts. Especially, the modelling of time-varying and possibly stochastic variances in more than one dimension is, from practical point of view, only feasible with discrete-time models. Even though continuous-time equivalent models for the individual volatility models of the GARCH class exist, the extension to multi-dimensional problems is not easily accomplished in continuous-time. Furthermore, no equivalent continuous-time models to the DCC model exist or to general multi-variable GARCH models. In addition, the parameters of stochastic volatility models in continuous-time are inherently difficult to estimate, since their dynamics have to be filtered from the observations of prices. In contrasts, the GARCH models can be efficiently estimated and predicted. Financial observations are discrete in nature and therefore, the discrete-time models are the correct
modelling framework, whereas continuous-time models are an approximation of the reality. This feature is totally different from the modelling of physical processes, where the continuous-time models describe the real physics and the discrete-time models are an approximation.

Discrete-time models are easier to use with various investment classes. Certain asset classes are difficult to model in continuous-time, such as dividend paying stocks, coupon paying bonds, asset classes with fixed holding periods, etc. Sometimes only discrete-time models allow us to realistically include those assets in a portfolio optimization.

Discrete-time models are also more flexible in the white noise modelling. In continuous-time, the basic modelling elements are either Brownian motions or Poisson processes. In discrete-time, we can choose any distribution, as long as the expectation is finite and it is strictly covariance stable. This leaves us many options, such as the normal distribution, white noise processes with GARCH effects or other heavy-tailed distributions. Moreover, we may also use a bootstrap approach where the observed standard residuals of the fitting procedures are used to simulate the white noise process. In this approach, we do not need to make any distributional assumptions and thus, keep all of the observed facts of the standard residuals, such as non-zero skewness, excess kurtosis, and tail dependency. For bootstrap methods the reader may refer to Davison and Hinkley (1999).

The discrete-time models help us to build models for investment decisions that reflect much better the observed stylized facts and replicate more closely the observations from financial time-series. Important aspects of the problem of long-term portfolio optimization can be much better modelled, such as realistic description of risk or predictions for variances and covariances. This increases the applicability of discrete-time models for real world applications to a large extent.

### 4.4.2 Optimization aspects

The optimization of discrete-time models is much more computationally demanding than their continuous-time counterparts. Since most of the models discussed have a Markovian structure, the problem of portfolio optimization can always be solved by applying the dynamic programming (DP) method. However, except for a few special cases, such as the linear Gaussian factor model without constraints, we do not find any analytical solutions. Thus, we have to resort to numerical procedures. Unfortunately, the DP algorithm suffers
from the curse of dimensionality and each of the individual optimization problems are difficult to solve under realistic assumptions. Therefore, we present two approximation methods to deal with this problem.

The first method, the so-called stochastic Model Predictive Control approach consecutively solves a series of open-loop optimization problems under various constraints. The open loop optimization is repeated at every time-step and based on the latest available information. This is a suboptimal procedure, but when we solve asset allocation problems without transaction costs and utility functions that imply constant relative risk aversion, the MPC method yields satisfactory decisions with acceptable computing costs with respect to the DP method. For the special case of Gaussian linear factor models, the MPC method allows fast computations with a wide array of state constraints and objectives. In addition, this method allows us to include a realistic description of risk by using coherent risk measures. Moreover, this approach is also applicable to situations where the open-loop optimization needs to be computed by a so-called scenario approach.

In the case that we deal with transaction costs or investment objectives that are path dependent, we can approximate the DP problem by the so-called stochastic programming approach. Similar to the MPC approach, we solve the problem at every time step based on the scenario projection in the future. This yields a decision rule similar to a feedback control law. This method allows us to included state as well as asset allocation constraints and can be efficiently solved for specific models. The disadvantage of the scenario approximation approach is the exponential explosion of scenarios with time. This very easily creates models that are too large to solve. An alternative is the stochastic control approach which allows to compute a fixed-mix or dynamic mix that does not depend on the scenarios. However, then the optimizations become nonlinear and non-convex.

To summarize, discrete-time optimal control approximation can be computationally demanding, but by exploiting modelling structures and properties of dynamics and objectives, we are often able to solve the approximation efficiently. In the discrete-time framework, we can use realistic constraints and objectives which to improve the applicability of these methods.
Case studies

In this chapter, we apply the discrete-time asset allocation method presented in Chapter 4. The first two case studies are asset allocation problems in US asset markets. The first case study uses 9 stock market indices, 1 bond market index, and a money market account. In the first case study with US data, we use the MPC method with a coherent risk measure. We use a rather high trading frequency of a weekly period with two different horizons.

The second case study uses the analytical MPC solution presented in Section 4.2 with a VaR constraint. We use again US data, where three different bond indices, a stock index, and a commodity index are the investment universe. The trading frequency is monthly and we test the method over a period of 15 years.

The third case study is calculated from the perspective of a Swiss fund that invests domestically in stocks, bond, and cash, as well as in the EU stock and bond markets. The fund gives a performance guarantee and faces transaction costs and thus, we use the optimization ideas presented in Section 4.3. The control strategy is test in a 8-year out-of-sample test with a quarterly trading frequency.

5.1 MPC with coherent risk measure applied to US data

In this case study, we compare the asset allocation and performance results that are generated by the same MPC method with two different investment horizons. We model the asset returns with the models described in Section 4.2.1, i.e., a linear factor model to explain the expected returns and a multi-variate GARCH model to explain volatility and correlations. In this case study, we want to illustrate the risk management capabilities of the MPC method with CVaR as risk measure. By using a bootstrap approach together with the dynamic models, the portfolio management adapts to the newest data and thus, effectively limits the risk exposure.
5.1.1 Implementation

The implementation of the portfolio optimization method depends crucially on two inputs. First, we need to estimate all the relevant parameters of the dynamic factor and dynamic covariance models. Second, we need a suitable method of generating scenarios of future trajectories of the asset returns. The scenarios are the essential inputs for the portfolio optimization procedure.

Asset model estimation

Portfolio management depends primarily on the quality of the estimation of expected returns and future risks. The expected return are computed with help of a factor model. The factor model is a predictive regression of factors which have a measurable impact on the expected returns. By inspecting (4.1) with (4.38), and (4.4) with (4.40), we notice that they form a system of two coupled linear regressions. The first regression is the predictive regression that will explain the returns and the second regression allows us to estimate the dynamics of the factor levels. The factor model and the factor loading matrix are estimated with a predetermined number of past data using an ordinary least squares approach. This is asymptotically correct despite the presence of heteroskedasticity in the innovations of the asset returns, see Hamilton (1994, Chapter 8). Alternatively, we could have included the estimation of the factor loading matrix in the estimation of the multivariable GARCH models by specifically estimating the mean terms of the individual time series. We choose the first method, because it is computationally much faster.

The GARCH structure of the dynamic covariance is estimated using a pseudo-maximum-likelihood method. At first, the parameters for the individual variance models, either GARCH(p,q) or TARCH(p,q,o), are estimated and then the DCC(k,l) model is estimated from the standard residuals of the individual variance models. This two-stage estimation determines the efficiency of the separation in individual variances and the correlation matrix. For proofs and further explanations consult Engle and Sheppard (2001). The maximum-likelihood estimation is computed using a numerical optimization algorithm.

Scenario generation and optimization for the numerical approach

The portfolio optimization depends on two inputs, first the newest set of model parameters and second, on the scenarios generated by the multivariate asset model. For the scenario
generation, we need specifications for the distributions of the standard residuals $\xi^r(t)$ and $\xi^x(t)$. We make no specification for the distribution and compute the empirically observed standard residuals, which we obtain from the model estimations. Then, we use these residuals in a bootstrap method to generate scenarios. Here we work with 10'000 simulations, since we use the standard residuals of 10 years, i.e., 520 data points. By resampling the standard residuals more than 10 times, the simulations are stable and we get almost the same decisions independent of the simulations. For using a bootstrap approach the reader may refer to Davison and Hinkley (1999).

The scenario generation is a Monte-Carlo (MC) simulation based on: model parameters, latest observed factor levels, variances, correlations, and the way we generate the standard residuals. A scenario denotes a trajectory of the asset returns computed into the future. The MC simulation generates the scenarios for the two white noise processes. It uses (4.41) or (4.42) to generate the individual variances, (4.43) to simulate the correlations, in order to compute $\epsilon^r(t+i)$. The simulations of $\epsilon^x(t+i)$ are easier, since we just need to simulate $\epsilon^x_t$. After having computed a sufficient number of future white noise processes, we have obtained the scenarios in order to solve (4.84).

### 5.1.2 Data description

The case study starts on 1/1/1989 and ends on 1/11/2003. For both methods, we use nine US stock market indices and a government bond index as risky assets. The stock market indices are nine direct sub-indices of the S&P 500 which are obtained from Thomson DATASTREAM. In Table 5.1 are all indices shown and $r$ denotes the mean returns, $\sigma$ the volatility, and $SR$ the Sharpe ratio.

All time series start on 01/01/1973 and end on 28/11/2003 with a weekly sampling frequency. The bond index is the 10-year constant maturity Treasury bond index. For the bank account, we use the 1-month Treasury note interest rate. For both studies, we use the 10-year Treasury bond interest rate, the earnings-price (EP) ratio of the index, past 150 weeks average return of the index, and the past one week return of the index as factors for each stock market index. Thus, every index return is modeled by four factors. The interest rate and the EP ratio are fundamental values, which should influence the expected returns due to their impact on the underlying economics of the industry. The other two factors should rather reflect short-term trends, which are motivated by the behavioral

<table>
<thead>
<tr>
<th>Time series</th>
<th>$r$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Basic Industries.</td>
<td>6.5</td>
<td>19.7</td>
<td>0.08</td>
</tr>
<tr>
<td>Cyclical Consumer Goods</td>
<td>4.7</td>
<td>19.7</td>
<td>-0.02</td>
</tr>
<tr>
<td>Cyclical Services</td>
<td>9.8</td>
<td>18.7</td>
<td>0.26</td>
</tr>
<tr>
<td>Financial Services</td>
<td>12.9</td>
<td>19.0</td>
<td>0.41</td>
</tr>
<tr>
<td>Information Technology</td>
<td>12.5</td>
<td>29.3</td>
<td>0.25</td>
</tr>
<tr>
<td>Noncyclical Consumer Goods</td>
<td>12.2</td>
<td>16.5</td>
<td>0.43</td>
</tr>
<tr>
<td>Noncyclical Services</td>
<td>4.6</td>
<td>18.8</td>
<td>-0.02</td>
</tr>
<tr>
<td>Resources</td>
<td>6.7</td>
<td>18.9</td>
<td>0.09</td>
</tr>
<tr>
<td>Utilities</td>
<td>2.8</td>
<td>15.2</td>
<td>-0.14</td>
</tr>
<tr>
<td>Treasury bond (10-year)</td>
<td>8.7</td>
<td>5.8</td>
<td>0.63</td>
</tr>
<tr>
<td>S&amp;P 500</td>
<td>9.6</td>
<td>16.0</td>
<td>0.29</td>
</tr>
</tbody>
</table>

aspects of asset prices. For the 10-year treasury index we use the 10-year, the 1-year, and the 3-month Treasury bond interest rates, as well as the average 150 weeks’ return and the past weeks’ return of the index. The three interest rates are the fundamental factors and the two return factors are again behaviorally motivated. In total we use 32 factors to model the returns of the 10 risky assets. The reader may consult Oberuc (2004, Chapter 3 and 4) for a good reference on factor models.

5.1.3 Results for case study with a myopic horizon

We evaluate our strategy in an out-of-sample test with a duration of 777 weeks. In order to judge the risk measures, we slightly change the optimization proposed in (4.84). We maximize the expected returns subjected to a CVaR limit. As shown in Alexander et al. (2004), for every mean-CVaR utility, we can find an equivalent optimization problem, where we maximize the mean subjected to a CVaR constraint. In this part, we compute three portfolios with $\bar{T} = 1$, which only differ in their CVaR-limits, namely -2%, -4%, and -8% (portfolio loss of initial wealth) at confidence level $\theta = 99\%$. The summary statistics for returns, volatilities, and Sharpe ratios of the risky assets are given in Table 5.2, where $r$ denotes the return, $\sigma$ denotes the volatility, and $SR$ the Sharpe ratio. We compute the Sharpe ratio despite that we do not use variance as a risk measure, since in practice it is a widely monitored ratio.

In Figure 5.1, the evolution of a 1$ investment in the portfolios with CVaR limits -4% and -8% and three indices are shown. The portfolios outperform the S&P 500 index.
Table 5.2. Summary statistics for the portfolios with $T = 1$ from Dec. 1982 to Nov. 2003.

<table>
<thead>
<tr>
<th>Time series</th>
<th>$\mu$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio CVaR -2%</td>
<td>10.5</td>
<td>5.1</td>
<td>1.07</td>
</tr>
<tr>
<td>Portfolio CVaR -4%</td>
<td>11.2</td>
<td>9.4</td>
<td>0.65</td>
</tr>
<tr>
<td>Portfolio CVaR -8%</td>
<td>9.9</td>
<td>16.0</td>
<td>0.30</td>
</tr>
</tbody>
</table>

because of the strong market downturn after 2000. Before 2000, the portfolios do not increase as strongly as the stock market, but are much better diversified. All three portfolios out-perform the S&P 500 index, but the Financial Services, the Information Technology and the Noncyclical Consumer Goods indices possess higher returns than any of the three portfolios. However, the portfolios with the two lower CVaR-limits have better Sharpe ratios than any of the stock market indices and even possess a higher Sharpe ratio than the government bond index. The portfolio with the lowest CVaR constraint possesses the lowest returns because the asset allocation is far less diversified than in the other two portfolios and the return prediction quality of the factor model is moderate.

The result of the portfolio optimization yields the optimal asset allocation as well as the next VaR-level ($\xi$). We compute how often empirically the portfolio returns exceed the VaR-level and compute the losses conditioned on the violations of the VaR-level. The results are given in Table 5.3, where VV stands for VaR-level violations, ECVaR for the
empirically computed CVaR. The theoretical confidence level for violating the VaR-level is 1%. The occurrence of empirical VaR violations is higher in the case of the portfolio with CVaR -2% and portfolio with CVaR -4% but lower for the portfolio with CVaR -8% than the theoretical value. The CVaR-limits imposed on the portfolios hold in the out-of-sample tests. The CVaR limit also successfully limits the maximum portfolio loss which is in the range of the CVaR limit. The constraint also effectively limits the extreme events. The maximum losses for all three portfolios are much lower than those of the stock market indices, which are between -12% and -28%.

In Figure 5.2, the asset allocation of the portfolio with -4% CVaR-limit into the three main asset classes (stocks, bonds, and cash) is shown. Inspecting Figure 5.2, one may notice the good market-timing capabilities of this portfolio construction. During the stock market downturn from mid 2000 to early 2003, the portfolio changes its investments

![Image of Figure 5.2](image_url)

**Fig. 5.2.** Asset allocation of the portfolio with CVaR-limit -4%.
from mostly stock market investments into the bond market. Thereby, the portfolio keeps its value and even increases its net worth, because the gains in the bond market are larger than the losses of the stock market investments. In early 2003, the capital is mostly reinvested into the stock market and thus, the portfolio participates in the stock market upturn. Similar market-timings can be observed in other market cycles, such as during the crisis in 1998.

5.1.4 Results for case study with a 10 weeks horizon

We compute four portfolios with $T = 10$ (weeks), which only differ in their CVaR-limits, namely -5%, -10%, -15%, and -25% of portfolio loss in percentage of initial wealth at a confidence level of $\theta = 99\%$. The summary statistics for returns, volatilities, and Sharpe ratios of of the portfolios are given in Table 5.4. The first two portfolios beat all stock market indices with respect to the Sharpe ratio. The average returns of the indices (Financial Services, Information Technology, and Noncyclical Consumer Goods) are higher than the return of the portfolio with CvaR limit of -10%. The portfolio with CVaR-limit of -25% (portfolio loss) produces a relatively weak performance, because it diversifies too little and bets too much on the asset with the highest expected return. In Figure 5.3, the evolution of a 1$ investment in either portfolios with CVaR-limit - 10% and -15% or the S&P 500 index is shown. In order to compute how often the four portfolios have violated the risk constraints, we compute the 10 week cumulative returns and compare them to the VaR-value for that corresponding period. Then, we compute how often the VaR-level is violated and calculate the conditional losses. The results are given in Table 5.5. In two of the four portfolios, the empirical VaR violations are lower than the theoretical value of 1% and in the two others it is somewhat higher. The empirical CVaR for all portfolios is lower than the theoretical level, which shows again, that the risk management aspect

<table>
<thead>
<tr>
<th>Time series</th>
<th>$r$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio CVaR -5%</td>
<td>10.7</td>
<td>6.3</td>
<td>0.9</td>
</tr>
<tr>
<td>Portfolio CVaR -10%</td>
<td>11.9</td>
<td>11.3</td>
<td>0.61</td>
</tr>
<tr>
<td>Portfolio CVaR -15%</td>
<td>10.2</td>
<td>14.2</td>
<td>0.37</td>
</tr>
<tr>
<td>Portfolio CVaR -25%</td>
<td>7.2</td>
<td>17.9</td>
<td>0.13</td>
</tr>
</tbody>
</table>
Fig. 5.3. Results of the historical simulation of the portfolios with CVaR limits -10% and -15% with $T = 10$

of our portfolio construction works well. The CVaR-limit constrains, in real terms, the possible portfolio losses, as Figure 5.4 shows. We manage to effectively limit the downside risk and the corresponding losses. Furthermore, this case study also indicates that the dynamic models are important to closely track the expected risks and returns of the assets. In Figure 5.5, the asset allocation of the portfolio with $-10\%$ CVaR-limit into the three main asset classes (stocks, bonds, and cash) is shown. During the stock market downturn from mid 2000 to early 2003, the portfolio shifts its capital from mostly stock market investments into the bond market. We notice the good timing capabilities, which are the reason why this portfolio performs very well in the crises from mid 2000 to early 2003.

**Table 5.5.** Empirical VaR and other risk statistics for case study 2

<table>
<thead>
<tr>
<th>Portfolio</th>
<th>VV (%)</th>
<th>ECVaR (%)</th>
<th>Max. loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>CVaR -5%</td>
<td>0.4</td>
<td>-4.8</td>
<td>-5.3</td>
</tr>
<tr>
<td>CVaR -10%</td>
<td>1.41</td>
<td>-9.4</td>
<td>-10.5</td>
</tr>
<tr>
<td>CVaR -15%</td>
<td>1.16</td>
<td>-14.8</td>
<td>-16.1</td>
</tr>
<tr>
<td>CVaR -25%</td>
<td>0.26</td>
<td>-23.2</td>
<td>-23.6</td>
</tr>
</tbody>
</table>
5.1.5 Comparison and conclusion

When we compare the two results, the portfolio optimization with the longer horizons outperforms (at its best) the portfolio with the myopic horizon. The longer horizon allows the investor to take more risk in the short-term. By resolving the problem at every-time step, we adjust new information and the controller reacts accordingly. The MPC method, combined with the constant updating of the model parameters, allows the method to closely track any changes in the relevant optimization parameters, such as expected return, volatilities, or correlations. The main drawbacks are the high trading frequency to achieve this result and the strong changes in the portfolio compositions. Sometimes the risk exposure to a single stock market index must be drastically reduced not to violate the risk limits. In the case studies, we demonstrate the feasibility of this method on a relatively large portfolio decision problem. Furthermore, we show in two out-of-sample studies that the risk constraints effectively limit the future portfolio losses and thereby, we can shape the risk exposure of the portfolios. The results also indicate that the obtained portfolios possess superior risk and return characteristics in comparison to the indices used in the case study.
5.2 Strategic asset allocation in US asset markets

In the second case study based on US asset market data, we want to simulate the situation of a balanced fund (strategy fund). Balanced funds invest their portfolio value among stocks, bonds, and cash. Sometimes, to a small degree, alternative investments are allowed as well as investments outside their native market. In this case study, we assume that the fund only invests in domestic US assets.

5.2.1 Data set and data analysis

The data set consists of 6 indices which starts on 1/1/1982 and ends on December 31/12/2004. The indices are the S&P 500 total return index (S&P 500 corrected for dividend payments), 5- and 10-year constant maturity US Treasury bond index (total return), the Goldman Sachs Commodity index (total return), and the Moody’s BAA 10-year corporate bond index (total return). The case study starts on 1/1/1990 and ends on 31/12/2004 with a monthly frequency, however the total time series data starts on 1/1/1982. In order to model the returns of the five risky assets, we compute the statistics for the return data set from 1/1/1982 to 31/12/1989. This is the data at the beginning of the historical back-test. The statistics also include two tests for the normality. Both tests, the Jarque-Bera test (J-B) and the Lilliefors test (LF), see Alexander (2001, Chapter
10.1), are used with usual confidence of 1% and 5%. In order to test the return data for normality, we have removed the return of October 1987 since this event seems to be an outlier, as argued in Johansen and Sornette (2001). The statistics given in Table 5.6 are

Table 5.6. Summary statistics of the 5 indices from Jan. 1982 to Dec. 1989 and results of the normality test. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

<table>
<thead>
<tr>
<th>Time series</th>
<th>r (%)</th>
<th>σ (%)</th>
<th>SR</th>
<th>krt</th>
<th>skew</th>
<th>J-B (1%)</th>
<th>J-B (5%)</th>
<th>LF (1%)</th>
<th>LF(5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>14.7</td>
<td>15.8</td>
<td>0.38</td>
<td>-1.5</td>
<td>-0.3</td>
<td>0(1)</td>
<td>0(1)</td>
<td>0(0)</td>
<td>0(0)</td>
</tr>
<tr>
<td>Goldman Sachs</td>
<td>11.4</td>
<td>12.6</td>
<td>0.22</td>
<td>0.4</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5-year Treasury Bond</td>
<td>11.5</td>
<td>12.6</td>
<td>0.38</td>
<td>0.26</td>
<td>0.47</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>10-year Treasury Bond</td>
<td>13.4</td>
<td>12.3</td>
<td>0.48</td>
<td>-0.44</td>
<td>0.26</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10-year BAA Moody</td>
<td>14.0</td>
<td>8.1</td>
<td>0.65</td>
<td>0.02</td>
<td>0.031</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1-month Treasury notes</td>
<td>8.7</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

mean return ($r$), volatility ($\sigma$), Sharpe ratio ($SR$), excess kurtosis ($krt$), skewness ($skw$), and the results of the normality tests. The results in brackets for the S&P 500 index are results of the test when the data from Oct. 1987 is included. It is noteworthy that for most of the indices, the normality assumption is not rejected, even for the stock market. In Figure 5.6, the quantile-quantile (QQ) plots for the return data of the five risky assets and the interest rate of the money market account are shown. Similar to the normality test results given in Table 5.6, the deviation from normality is small. Only the interest rate data significantly deviate from normality for high levels. Given the evidence between 1/1/1982 and 31/12/1989, we do not reject that the data is generated by a normal distribution and thus, we use the MPC method explained in Section 4.2.3.

In order to justify the use of a portfolio allocation method based on the normal distribution assumption, we need to test for normality given the new information while moving through the historical back-test. For this reason, every year we test the data of the five risky asset returns and the interest rate for normality based on the last 8 years of data. The results of the rolling normality tests are given in Table A.1. Except for 3 years for the S&P 500 index and for 3 years for the 1-month Treasury note interest rate, the normality is not rejected at least by one of the four tests. For the Treasury bond indices and the commodity index, the normality assumption is almost never rejected and for the Moody index it is not rejected in half of the test. Therefore, we think that the MPC methods based on the assumption of a normal distribution is justified to be used.
The factors to explain the expected returns of the five risky assets are given in Table 5.7. Also, the results of the normality test for standard residuals of (4.4) is given for the time period between 1982 and 1989. Only for the Federal fund rate we have not tested for normality, since it is not a stochastic process.

Again based on the analysis, we do not reject the hypothesis of normally distribution except for the difference of the 1-month Treasury notes interest rates. The short-term interest rates follow very closely the Federal fund rate and therefore, the rejection is not very surprising. Similar to the analysis of the risky asset returns, we test the factors for normality on a rolling basis. Only for factors no. 3, 5, 8, 11, 17, and 22 we reject the assumption of normality for some of the tested years.

In order to compare the results of the portfolios, we show the summary statistics of the six asset classes in Table 5.8. For the asset data from 1990 to 2004 the QQ plots are shown in Figure A.1 which can be found in Section A.6.1.
### Table 5.7
Factors for case study 2 with US data and result of the normality tests for the standard residuals. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

<table>
<thead>
<tr>
<th>Factor no.</th>
<th>1-month TB int. rates</th>
<th>J-B (1%)</th>
<th>J-B (5%)</th>
<th>LF (1%)</th>
<th>LF (5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10-year TB int. rates</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>spread 2-year 10-year TB int. rates</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>spread 5-year 10-year TB int. rates</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>spread Moody BAA - 10-year TB int. rates</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>Federal Fund rate</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>US $-JPY FX</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>US $-Euro/DEM FX</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>Oil (Brent) prices</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>Gold prices</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>S&amp;P 500 EP-ratio</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>S&amp;P 500 DDY</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>3-months momentum. S&amp;P 500</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>3-months momentum. GSCI</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>3-months momentum. 5 year TB index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>3-months momentum. 10 year TB index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>3-months momentum. Moody BAA index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>12-months momentum. S&amp;P 500</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>12-months momentum. GSCI</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>12-months momentum. 5 year TB index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>12-months momentum. 10 year TB index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>12-months momentum. Moody BAA index</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>12-months momentum. GSCI</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

### Table 5.8
Summary statistics of the 5 indices from 1/1/1990 to 31/12/2004 and result of the normality tests for the standard residuals. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

<table>
<thead>
<tr>
<th>Time series</th>
<th>$r$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
<th>$skew$</th>
<th>$J-B$ (1%)</th>
<th>$J-B$ (5%)</th>
<th>$LF$ (1%)</th>
<th>$LF$ (5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P 500</td>
<td>10.8</td>
<td>14.3</td>
<td>0.45</td>
<td>0.52</td>
<td>-0.36</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Goldman Sachs</td>
<td>7.8</td>
<td>18.3</td>
<td>0.19</td>
<td>0.5</td>
<td>0.12</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5-year Treasury Bond</td>
<td>7.5</td>
<td>8.4</td>
<td>0.39</td>
<td>0.22</td>
<td>-0.02</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10-year Treasury Bond</td>
<td>7.9</td>
<td>7.4</td>
<td>0.48</td>
<td>0.66</td>
<td>-0.36</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>10-year BAA Moody</td>
<td>8.87</td>
<td>4.9</td>
<td>0.93</td>
<td>1.27</td>
<td>-0.43</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>1-month Treasury notes</td>
<td>4.1</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

### 5.2.2 Implementation
The implementation of the out-of-sample test for the portfolio allocation method consists of three main steps: the factor selection, the parameter estimation, and the computation of the asset allocation.
Factor selection

The factor selection determines which of the factors best explain the expected returns of the risky assets. The factor selection is only used for the regression problem given by (4.1) with (4.38). Often this selection is predetermined using literature recommendations or economic logic. However, when we use a very large set of factors, it is difficult to decide which factors explain the expected returns best. In order to solve this problem, we employ a heuristic given in Table 5.9. When we want to choose the best possible subset of factors, we face a combinatorial number of possible subsets. For this reason, we use a “greedy” strategy that reduces the number of factors by the factor with the lowest impact. Given \( m \) factors, the heuristic creates only \( m \) subsets and we choose the best subset by a so-called information criterion. The information criterion is a trade-off between the number of regressors and the regression quality. For the factor selection we either use the modified Akaike’s (AIC) or the Schwartz Bayesian (SBIC) criterion. Both mainly differ in their weights for the number of factors, where the SBIC weights the number of factors stronger than AIC. This factor selection procedure is used for all risky assets independently. Different risky assets are regressed by different sets of factors. The factor selection is recomputed every 12 month, i.e., the factors are selected at the beginning of every year. This heuristic and the two information criteria are discussed in detail in Illien (2005).

Parameter estimation and dependence

When we have chosen the factors, we still need to estimate the parameters of (4.4) with (4.40). When we use a large set of factors, it is important that we do not introduce statistical dependencies that are very questionable. For this reason, we estimate \( A \) and

<table>
<thead>
<tr>
<th>Table 5.9. Steps of the factor selection heuristics</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Estimate the parameters of the linear regression using all available factors.</td>
</tr>
<tr>
<td>2. Remove the factor that contributes as little as possible to the ( R^2 ) of the regression. The ( R^2 ) indicates how much of the data variation is explained by the regression. Estimate the parameters again with the reduced factor set.</td>
</tr>
<tr>
<td>3. Continue removing the factor with lowest ( R^2 ) impact until only one factor is left.</td>
</tr>
<tr>
<td>4. Choose the number factors and the regression model that has the highest information criteria, such as the Akaike criterion.</td>
</tr>
</tbody>
</table>
with a method of maximum likelihood (ML), starting with an unrestricted model. We then use an iterative procedure, where one insignificant factor, usually the one with highest p-value\(^1\) is removed and the ML estimates for the remaining parameters are recalculated until all insignificant parameters at 5% (or 1%) significance level have been removed. The same procedure is also proposed in Koivu, Pennanen and Ranne (2005). The parameter estimation for the returns and the factor dynamics are computed in every step of the out-of-sample test with 8 years of past data.

**Asset allocation strategy**

In Figure 5.7, the overall implementation of the asset allocation strategy is shown. The figure is taken from Illien (2005). As discussed, periodically the factors are selected which

\[\nu\]

\(^1\) The p-value is the probability of rejection of the estimated value.
Capacity Utilization, are omitted from the table. A complete list of all factors are given in Table A.2 in the Appendix A.6.1. Given the factors selection, we first estimate all relevant parameters and then compute the asset allocation. The asset allocation decisions are then used in an out-of-sample test and we record the portfolio performances. In this way, the algorithm moves one step forward until we have to select the factors again.

5.2.3 Results of out-of-sample test

In the out-of-sample test from 1/1/1990 until 1/12/2004 with a monthly frequency where we use the MPC method as described in Section 4.2.3. The portfolio optimization problem is maximizing the expected return while limiting the VaR of the portfolio with a confidence level of 99%. This corresponds to the portfolio optimization problem $P_2$.

We compute the VaR limit based on the previous value of the portfolio. In this manner, the optimization does not depend on current value of portfolio since the risk limit is defined relative to the current portfolio value. We define three relative limits, namely -2.0%, -5.0%, and -10% of portfolio loss with respect to the current portfolio value. Furthermore, we assume that the MPC Strategy is computed with a two-year horizon. The VaR limit is not only defined at the end of the investment horizon but also for intermediate periods. In this way, we try to limit the risk exposure not only for the terminal time but also for the time in between. Moreover, we impose the following maximum limits for investments (in percentage of the portfolio value): 60% stock market, 20% commodity index, 100% for both Treasury bond indices, and 80% for the Moody’s index. We do not allow any short selling or leveraging.

In order to compare the portfolios, we calculate two benchmarks. The composition of the two benchmarks are given in Table 5.10 which mainly differ in their stock market weights and the weights for the bond indices. The performance results of the portfolio test and the benchmarks are given in Table 5.11, where $r$ denotes the average return, $\sigma$ denotes the volatility, $SR$ denotes the Sharpe-ratio, EVaR denotes the empirical one

<table>
<thead>
<tr>
<th></th>
<th>S&amp;P 500</th>
<th>GS Comm</th>
<th>5-year TB</th>
<th>10-year TB</th>
<th>10-year Moody BAA</th>
<th>Money Market</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark 1</td>
<td>45%</td>
<td>5%</td>
<td>5%</td>
<td>30%</td>
<td>5%</td>
<td>10%</td>
</tr>
<tr>
<td>Benchmark 2</td>
<td>30%</td>
<td>10%</td>
<td>10%</td>
<td>30%</td>
<td>10%</td>
<td>10%</td>
</tr>
</tbody>
</table>
month VaR with 99% confidence level, and VaR viol. denotes how often the theoretical VaR limit is violated. The results are quite promising, since the portfolios have higher

Table 5.11. Portfolio results and statistics of the out-of-sample test.

<table>
<thead>
<tr>
<th>Time series</th>
<th>$r$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
<th>max. loss</th>
<th>EVaR</th>
<th>VaR viol.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Portfolio 1 (-2%)</td>
<td>8.17</td>
<td>5.0</td>
<td>0.77</td>
<td>-4.4%</td>
<td>-3.1%</td>
<td>3.5%</td>
</tr>
<tr>
<td>Portfolio 2 (-5%)</td>
<td>10.6</td>
<td>7.5</td>
<td>0.84</td>
<td>-6.8%</td>
<td>-4.6%</td>
<td>2.3%</td>
</tr>
<tr>
<td>Portfolio 3 (-10%)</td>
<td>10.7</td>
<td>8.0</td>
<td>0.79</td>
<td>-6.8%</td>
<td>-5.1%</td>
<td>1.1%</td>
</tr>
<tr>
<td>Benchmark 1</td>
<td>8.5</td>
<td>7.1</td>
<td>0.61</td>
<td>-4.6%</td>
<td>-4.1%</td>
<td>-</td>
</tr>
<tr>
<td>Benchmark 2</td>
<td>8.0</td>
<td>5.7</td>
<td>0.66</td>
<td>-3.4%</td>
<td>-3.0%</td>
<td>-</td>
</tr>
</tbody>
</table>

Sharpe ratios than both of the benchmarks. The returns of portfolio 3 are little bit lower than the average return of the S&P 500 in this period. However, the S&P 500 index outperforms the three portfolios considerably until its peak in 2000, as Figure 5.9 shows. The Portfolio 2 is omitted from the figure since its evolution resembles the evolution of Portfolio 3. The Sharpe ratios of the three portfolios all beat the benchmarks and most of the indices. Portfolios 2 and 3 both outperform the two benchmarks with respect to average return as well as risk adjusted returns. Very often, the constraints for the investments limit the asset allocation into the two most risky asset (S&P 500 and CSCI indices) and not the VaR constraints of -5% and -10%. Thus, the Portfolios 2 and 3

Fig. 5.8. Results of the out-of-sample test and comparison to other assets and benchmarks
have often similar investment decisions and performance results. However, the theoretical confidence level of 1% violations of the VaR limit does not hold in the out-of-sample test. All three portfolios have higher empirical violations which indicates that the distribution of returns and factors have some divergence from the normality assumption. When we test the returns of the portfolios a-posteriori for normality, we reject the assumption of normal distributed returns for both tests and confidence levels for Portfolio 1. For Portfolio 2 and 3 the J-B test does not reject the assumption of normal distributed returns for both confidence levels. This coincides well with the observations of the empirical VaR violations. The empirical VaR of the first portfolio is about 50% higher than the theoretical limit imposed by the optimization. The empirical VaR of portfolio 2 and 3 is below the theoretical limit, which indicates that the risk management aspect works quite well.
5.2.4 Conclusion

The result of the case study where we use the MPC method based on the assumption of a normal distribution are quite encouraging. The sophisticated statistical modelling, i.e., the factor selection heuristic and the significant parameter estimation, yields good results. The implementation of the historical test attempts to exploit as much information as possible with this kind of model and tries to avoid statistical artifacts. The extensive testing for normality on a rolling basis justifies using this method through the historical test.

The portfolios with relative VaR limit of -5% and -10% hold the risk limits which we theoretical try to achieve with the portfolio optimization. However, the risk limits do not hold for the first portfolio. The main reason is probably the deviation from normality of the stock returns. The portfolios have a quite steady wealth evolution through the test with only one period (Sept. 1998 until Feb 2000) where a small drawdown happens. During the long bull market from 1995 to 2000 the stock market considerably outperforms all of the portfolios and benchmarks, but this case study was designed to replicate a balanced fund.

From this case study and others, see Herzog et al. (2004), we conclude that the MPC method is a suitable suboptimal control procedure to optimize long-term portfolio decisions. With careful statistical modelling, we believe that these optimization methods achieve suitable dynamic portfolio constructions.

5.3 Asset and liability management with transaction costs for a Swiss fund

In the third case study, we take the view of an investment fund that resides in Switzerland which invests domestically and abroad. The fund is assumed to be large and we cannot neglect the market impact of its trading activities which results in trading costs. The problem of portfolio optimization is solved with the framework presented in Section 4.3. Moreover, we assume that the fund gives a capital (performance) guarantee which introduces a liability. The situation resembles the situation of a Swiss pension fund and thus, we impose similar restrictions on the case study. Designing funds with performance
guarantees is also discussed in Dempster, Germano, Medova, Rietbergen, Sandrini and Scrowston (2004).

5.3.1 Data sets and data analysis

The data sets consists of the Datastream (DS) Swiss total stock market index, the DS Swiss government benchmark bond index, the DS European Union (EU) total stock market, and the DS EU government benchmark bond index. For the money market account, we use the 3-month Libor (SNB) interest rate. The data set starts on 1/1/1988 and finishes on 1/1/2005 with quarterly frequency. The two international indices are used twice in our case study. One time, the indices are simply recalculated in Swiss Francs (CHF) and the other time, the currency risk is eliminated by completely hedging the currency risk. The risk-return profile is thus changed, since hedging introduces costs that reduce the performances but eliminates the currency risk. The hedging costs are computed on the basis of the 3-month forward rates between the Swiss Franc and Euro. Before 1999, we use the forward rates between the Swiss Franc and German Mark as approximation for the Euro. Also before 1999, the EU stock and bond indices are calculated in German marks. In Figure 5.10, the histogram for the Swiss stock market, the Swiss bond market, the EU stock market in CHF, and the EU stock market hedged are shown. The figure shows the histograms and the best fits of a normal distribution. Except for the bond market index, we clearly reject the assumption that the stock market data is normally distributed. When we fit a multivariate student-t distribution on a rolling basis to the stock market data, we get degrees of freedom between 6.7 and 9.6 which indicate a very clear deviation from normality. This result is supported by various tests for normality which all reject the assumption of normality at 5% confidence. For these reasons, we have to model the returns of the stock market data by a non-normal distribution. The results of the normality test for the out-of-sample period are given in Table 5.15. We get similar results for the time period from 1/1/1988 to 1/6/1996.

5.3.2 Implementation

By using the methods discussed in Section 4.3, one crucial step of the implementation is the generation of scenarios. The scenarios describe the future stochastic evolution of the assets and the scenarios must reasonably well approximate the underlying stochastic model.
Scenario generation

The problem of approximating stochastic models is a well-known and active field of research. The most common techniques to generate scenarios for multistage stochastic programs are discussed in Dupacova, Consigli and Wallace (2000). Among the most important methods of scenario generation are moment matching (Hoyland, Kaut and Wallace 2003), importance sampling (Dempster and Thompson 1999), or discretizations via integration quadratures (Pennanen and Koivu 2005, Pennanen 2005).

We use the method of discretization via integration quadratures because we believe that this method is superior to Monte-Carlo methods especially for high dimensional problems. Furthermore, numerical test validate the stability of the optimization results, as shown in Pennanen and Koivu (2005). The method is used by approximating the white noise process at every stage of the dynamic model, i.e., $\epsilon'(t)$. In the case that we know the time aggregation rules for distributions (normal distributions), we approximate the distribution of the asset model at pre-specified future points. For the generation of low discrepancy sequences, which are the essential for the discretization, we use either the Sobol sequence, see Bratley and Fox (1988) or the original paper Sobol and Levitan (1976),
or the Niederreiter sequence, see Bratley, Fox and Niederreiter (1994) and Niederreiter (1988).

We discuss the implementation of scenario approximation for two distributions: the normal and student-t distribution. A multivariate $n$ dimensional normal distribution with mean vector $\mu$ and covariance matrix $\Sigma = \sigma \sigma^T$ can be constructed as

$$X_i \sim \mathcal{N}(0,1), \quad X = (X_1, X_2, \ldots, X_n)^T$$

$$Y = \mu + \sigma X, \quad \Rightarrow Y \sim \mathcal{N}(\mu, \Sigma),$$

where $\mathcal{N}$ denotes the normal distribution. The scenario generation algorithm for the normal distribution is given in Table 5.12, where we use the basic idea of the method of inverses. The algorithm ensures that first and second moments are matched. The normal-

<table>
<thead>
<tr>
<th>Table 5.12.</th>
<th>Scenario generation for a multivariate normal distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Use the Sobol or Niederreiter sequence to generate $s$ uniformly unit cube $[0,1]^n$ distributions, denoted by $S \in [0,1]^{n \times s}$.</td>
<td></td>
</tr>
<tr>
<td>2. Use the inverse standard normal distribution to transform the uniformly distributed random variables ($S$) to standard normal distributed random variables ($\xi(s)$), i.e., $\xi(s) = \Phi^{-1}(S(s))$.</td>
<td></td>
</tr>
<tr>
<td>3. Compute the covariance matrix $\bar{\Sigma}$ of $\xi$ and calculate the normalized sequence of standard random variables $\xi(s) = \bar{\Sigma}^{-1} \xi$. In this way, we make sure that $\xi(s)$ possesses unit variance.</td>
<td></td>
</tr>
<tr>
<td>4. Calculate the scenarios by $\epsilon(s) = \mu + \sigma \xi(s)$.</td>
<td></td>
</tr>
</tbody>
</table>

The student-t distribution possesses the following parameters: mean vector $\mu$, degree of freedom $\nu$, and diffusion matrix $\Sigma$. The algorithm to compute a multivariate $t$ student distribution is based on the following construction (Glasserman 2004, Chapter 9):

$$X \sim \mathcal{N}(\mu, \Sigma), \quad z \sim \chi^2_{\nu}$$

$$Y = \sqrt{\nu} \frac{X}{\sqrt{z}},$$

where $\mathcal{N}$ denotes the normal distribution and $\chi^2_{\nu}$ the standard chi-square distribution with degree of freedom $\nu$. The covariance of student-t distribution is given by $\frac{\nu}{\nu-2} \Sigma$ and exists only for $\nu > 2$. The scenario generation algorithm for the student $t$ distribution is given in Table 5.13. With similar algorithms for scenario generation, any kind of normal (variance) mixture distributions (McNeil et al. 2005, Chapter 3) can be approximated as long as the
Table 5.13. Scenario generation for a multivariate student-t distribution

1. Use the Sobol or Niederreiter sequence to generate $s$ uniformly unit cube $[0, 1]^{n+1}$ distributions, denoted by $S \in [0, 1]^{n+1 \times s}$.

2. Use the inverse standard normal distribution to transform the uniformly distributed random variables $(S_i, i = 1, ..., n)$ to standard normal distributed random variables $(\xi_i, i = 1, ..., n)$, i.e., $\xi_i(s) = \Phi^{-1}(S_i(s)), i = 1, ..., n$.

Furthermore, we use the inverse of the chi-square distribution to transform $S_{n+1}$ to chi-square distributed random variables which we denote by $\eta$.

3. Compute the covariance matrix $\Sigma$ of $\xi$ and calculate the normalized sequence of standard random variables $\xi(s) = \Sigma^{-1/2} \Phi^{-1}(s)$. In this way, we make sure that $\xi(s)$ possesses unit variance.

4. Calculate the scenarios by $\epsilon(s) = \sqrt{\nu(\mu + \sigma \xi(s))} \eta(s)$.

The method of inverses is applicable. Not only elliptical distributions can be approximated, but also distributions with non-zero skewness such as the generalized hyperbolic or the skewed student t distribution.

Factor selection, parameter estimation, and asset allocation strategy

In this case study, we use the same method for factor selection as described in Table 5.9. In Table A.3 we report the factors that are selected at least once. The factors that were never selected, such as the dividend yield, are omitted from the table. The estimation of parameters for the risky return is computed on a rolling basis where always the last 8 years of data are used. The factor selection determines which time series of factors are used to predict the expected returns of the four risky assets. This implementation of model estimation and factor selection is the same as discussed in Section 5.2.2.

We assume that the fund faces different transaction costs for domestic and international assets. The costs (due to market impact) for the Swiss stock market are assumed to be 1.5% and the bond market 0.5%. For the European stock market, the transaction costs are assumed to be 2.0% and for the bond market 1%. The transaction cost for European assets are independent of the hedging, since we calculate the hedging cost as part of the realized returns in Swiss Francs.

The asset allocation decisions are calculated with the optimization algorithm proposed in (4.94). We assume that the fund possess a two-year moving investment horizon and we use a tree structure with 50, 20, and 5 branches which results in 5000 scenarios which we denote by 5000 (50,20,5). The algorithm to approximate (locally) the DP algorithm given in Section 4.1.6 is used by first computing 500 (10,10,5), 1000 (20,10,5), 3000 (20,15,10),
4000 (40,20,5), and 5000 (50,20,5) scenarios. The relative error between using 4000 and
5000 scenarios was smaller than 1% (measured by the objective function value obtained
with 5000 scenarios). This test was done at the first time step of the out-of-sample test
and repeated every 12 quarters. In all tests, the difference was smaller than the 1%.

For the first branching we use a one quarter time step, for the second branching we
use 2 quarters, and for the third branching we use 5 quarters. The constraints for the
optimization are similar to the constraints which Swiss pension funds face. In Table 5.14
the maximum limits for investments in the different asset classes are given. As mentioned
before, we assume that the fund gives the same capital guarantee of 4% as a Swiss pension
fund (until 2002). When we solve the optimization problem at every time step, the current
asset allocation is taken as the initial asset allocation. In this way, the transaction costs
for every rebalancing of the portfolio are correctly considered.

<table>
<thead>
<tr>
<th>Table 5.14. Investment constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swiss stock market</td>
</tr>
<tr>
<td>Swiss bond market</td>
</tr>
<tr>
<td>EU stock market in CHF</td>
</tr>
<tr>
<td>EU bond market in CHF</td>
</tr>
<tr>
<td>EU stock market in CHF (hedged)</td>
</tr>
<tr>
<td>EU bond market in CHF (hedged)</td>
</tr>
<tr>
<td>All international assets</td>
</tr>
<tr>
<td>All stock market investments</td>
</tr>
</tbody>
</table>

The risk measure used for this case study is an expected-shortfall measure, where
we compute the expected shortfall of the portfolio wealth minus the capital guarantees.
Therefore, we use the risk measure given in (4.90), but with only one linear function. The
expected shortfall is not only used at the terminal date of the optimization, but at all
time steps in between. In our strategy, we compute the expected shortfall for one quarter,
3 quarters, and 8 quarters in advance.

The optimization results in an LP with 6902 variables and 69050 constraints. The case
study is computed in MATLAB and we use a C implementation of the LP solver CLP
which is connected to MATLAB. The solution of the LP takes about 20 seconds on a
Pentium 4 with 2.6 GHz and 768 MB memory under Windows 2000. All the necessary
steps to setup the optimization problem take about 40 minutes, which limits the extent of
the historical out-of-sample test. Therefore, only two portfolios are computed with high and low transaction costs.

### 5.3.3 Results of the out-of-sample test

The out-of-sample test starts on 1/6/1996 and ends on 1/1/2005 with quarterly frequency. The statistics of the out-of-sample test for the portfolios and the assets are shown in Table 5.15. The portfolio and the asset evolution throughout the historical out-of-sample test are shown in Figure 5.11. The graph shows that the portfolio has a relatively steady evolution throughout the historical back-test with only one longer drawdown period of 5 quarters from third quarter 2000 until the first quarter 2002. Furthermore, the largest losses happens in third quarter of 1998 where the portfolio loss is 16.3%. The initial investments mostly into EU bond market and the subsequent portfolio gains allows the system to invest more into the stock market between 1996 and 1998. These changes in the asset allocation are shown in Figure 5.12 as percentage of portfolio value. The large loss in the third quarter of 1998 leads to a dramatic increase of the money market investments and a sharp decrease of the investments into the Swiss stock market. The capacity to incur loss is reduced at this moment and consequently the risky investments are reduced. A similar behavior can be seen during the drawdown from 2000 to 2002, where the fund invests mostly into Swiss bonds and non-hedged EU bonds. Large changes in the asset allocation happen usually after significant changes in the portfolio value or after significant changes in the risk-return perception of the assets.

#### Table 5.15. Summary statistics of the 4 indices from 1/6/1996 to 1/1/2005 and results of the normality tests for the standard residuals. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

<table>
<thead>
<tr>
<th>Time series</th>
<th>$r$ (%)</th>
<th>$\sigma$ (%)</th>
<th>$SR$</th>
<th>$kurt$</th>
<th>$skew$</th>
<th>J-B (1%)</th>
<th>J-B (5%)</th>
<th>LF (1%)</th>
<th>LF(5%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Swiss stock market</td>
<td>7.0</td>
<td>24.6</td>
<td>0.22</td>
<td>1.5</td>
<td>-0.9</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>Swiss bond market</td>
<td>5.5</td>
<td>4.5</td>
<td>0.88</td>
<td>-0.9</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EU stock market in CHF</td>
<td>9.1</td>
<td>26.2</td>
<td>0.28</td>
<td>0.5</td>
<td>-0.8</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>EU bond market in CHF</td>
<td>9.2</td>
<td>7.4</td>
<td>1.04</td>
<td>-0.7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>EU stock market hedged</td>
<td>10.0</td>
<td>22.1</td>
<td>0.39</td>
<td>0.0</td>
<td>-0.5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>EU bond market hedged</td>
<td>9.4</td>
<td>6.1</td>
<td>1.29</td>
<td>-0.3</td>
<td>-0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3-month Libor (SNB)</td>
<td>1.41</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Portfolio (4% guarantee)</td>
<td>7.2</td>
<td>9.5</td>
<td>0.6</td>
<td>3.4</td>
<td>-1.4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
The exchange rate risk of the international investments may be hedged. Most of the stock investments are hedged as shown in Figure 5.12. The risk-return trade-off for stocks seems to be more acceptable with hedging than without hedging. Moreover, most of the investment into stocks occurs before 2001 and in this period the Swiss Franc is constantly gaining in value and thus reducing returns from the international investments. The Swiss Franc loses value after mid 2002 when most of the international bond investment occurs. Therefore, most of the bond investment takes place without hedging. By introducing the international assets twice as either hedged or not hedged, we use the standard portfolio optimization to make the hedging decision in parallel with the portfolio construction.

The capability of holding the portfolio above the capital guarantee barrier is demonstrated in this out-of-sample test. However, a historical back-test starting after 2000 would have much more difficulties to remain above the barriers, since the two least-risky assets (Swiss money market and bond market) did not yield returns above 4%. For this reason, many pension funds came into a situation of severe financial stress.

Despite the relatively high transaction costs, the performance is satisfactory with an average return of 7.1%. The performance is similar to the Swiss stock market but markedly higher than that of the Swiss bond market or money market accounts. The standard devi-
5.3 ALM with transaction for a Swiss fund

Fig. 5.12. Asset allocation of the Swiss fund in percentage of wealth

Allocation is moderate, much smaller than any of the stock market investment possibilities but larger than the bond markets. Hence, the Sharpe ratio lies between the one for the stock and the bond markets. When we solve the same case study with much lower transaction cost of 15 basis points, the return is increased to 9.36% and the volatility is decreased to 8.9%. However, the changes in investments are much more volatile and the portfolio optimization reacts more strongly to small changes in the risk-return prediction. These observations can be seen in Figure A.2 and Figure A.3.

5.3.4 Conclusion

The case study with transaction costs and international assets demonstrates the applicability of the ideas presented in Section 4.3. In the case that we deal with transaction costs or investment objectives that are path dependent, we can approximate the DP problem by the so-called stochastic programming approach. We solve the problem at every time step based on the scenario projection. This yields a decision rule similar to feedback control law. This method allows to included state as well as asset allocation constraints and can be efficiently solved for specific models. The disadvantage of the scenario approximation
approach is the exponential explosion of scenarios with time. This very easily creates models that are too large to solve.

Furthermore, the case study also shows that it is possible to construct funds that give a certain performance guarantee. The performance guarantee together with the transaction costs makes the portfolio optimization path depended and therefore, we need to solve the stochastic programming approximation. In this case study, the risk aversion varies throughout the historical back-test and depends on the distance to the barrier. The optimization always reduces the risk exposure when the portfolio wealth moves closer to the barrier and increases the risk exposure when the portfolio moves away from the barrier. In this way, we introduce a feedback from the portfolio results to our current portfolio decisions and adapt the risk aversion to our loss incurring capacity.
Summary and outlook

In this work, the problem of strategic portfolio management for long-term (multi-period) investment is successfully addressed for continuous-time and discrete-time models. Many different concepts from control engineering are linked with topics from financial engineering and applied to the problem of portfolio optimization. The conclusions and the outlook of this thesis are presented in this chapter.

6.1 Summary

In the first part of this work, we give a brief introduction to the topic of portfolio management and explain the connection to control engineering. In the second chapter, the problem of defining and measuring risk is discussed. A list of requirements for risk measures, used in this thesis, is given and we explain which risk measure constitutes a coherent risk measure for which modeling framework.

The third chapter discusses the application of stochastic optimal control theory to continuous-time problems of portfolio optimization. For a class of affine portfolio models, the optimal control strategy for two utility functions are derived under the assumption of no constraints on the asset allocation. Furthermore, the situation where not all factors are directly measurable is solved. Valuable insights, especially how the risk aversion and the time horizon govern the investment decisions, can be drawn from this model. An application of the affine models with German asset market data is given where we conclude that a longer investment horizon increases the demand for risky assets. In order to address the problem of constraints on the optimal control vector, the last part of this chapter discusses a numerical solution procedure for the Hamilton-Jacobi-Belman equation. We derive and prove the convergence of the so-called successive approximation algorithm. The results of the numerical algorithm are applied in a case study with US data.
In the fourth chapter, we discuss stochastic control strategies for the problem of portfolio optimization with discrete-time models. Two approximations of the dynamic programming method are derived and explained. The first method, the model predictive control method, solves the problem by repeatedly solving the corresponding open-loop optimal control problem. The model predictive control method is applied to three specific portfolio models, which differ mostly in their objective functions and their modelling of the white noise terms. For a linear Gaussian factor model, similar to the results of the affine portfolio models of Chapter 3, we are able to derive the open-loop optimization problem for two different objective functions and Value-at-Risk and Conditional Value-at-Risk risk constraints. For other portfolio distributions, we derive convex optimization problems that allow us to solve the open-loop control problem efficiently. The second approximation method, the stochastic programming method, solves locally a discretized feedback control problem. We discretize the original state dynamics by drawing samples from its distribution. A tree of so-called scenarios is computed that represents the stochastic dynamic of the portfolio. We apply the first control decision of the scenario approximation and repeat the procedure again at next time step. In this way, a feedback solution is computed that does not suffer from the curse of dimensionality.

In the fifth chapter, we apply the discrete-time models in three different case studies, two with data from the US and one with European data. In the first case study, we apply the model predictive control method where the underlying distributions are heavy-tailed and where we optimize a coherent risk measure. In the 777 weeks out-of-sample test, we demonstrated that the empirical risk limits agree well with theoretical risk limits of the optimization. In a second case study, we apply the analytical solutions of the model predictive control method for linear Gaussian factor models with US asset market data. The empirical results, obtained with the model predictive control method and a carefully way to select the factors, are encouraging since the portfolios comply mostly with the risk requirements and possess a suitable risk-return characteristic. In the last case study, the problem of a Swiss fund that invests domestically and in the EU markets, which gives a performance guarantee, is considered. With the stochastic programming approximation, we show that the portfolio results exceed the performance guarantee and that the optimization adapts its risk tolerance relative to the performance barrier.
6.2 Outlook

Certain aspects of feedback control applied to the problem of portfolio optimization are not discussed or solved in this thesis. In this part, we list open question and unsolved problems that would help to improve the main goals of this work.

- Numerical results for the time-aggregations of heavy-tailed distributions would help to compute multi-period distributions.
- A theoretical error bound for the model predictive control method could be found (especially for problems with constraints on $u(t)$). Then it would be better known when it is justified to use the model predictive control approximation.
- For the linear Gaussian factor model with objective $O_1$ and no risk-constraints (e.g., Value-at-Risk), computation of a feedback map (for every state value and time) could be solved. The solution of this multi-parametric problem could be found by using the solution presented by Bemporad et al. (2002) for deterministic model predictive control problems, since the open-loop problem is reduced to solving a quadratic programm.
- Extension of the model predictive control method for problems with partial information. The continuous-time results for partial information indicate that similar results could be obtained for the linear Gaussian factor model.
- Using copulas and univariate distributions to improve the statistical modelling, in particular the scenario modelling of unconditional models.
- The combination of time-series modelling with models from (macro) economics could alleviate the problem of long-term return predictions.
- Testing of the stochastic programming approach that mimics the control solutions for problems with transaction costs. This approach could be used to link the strategic asset allocation with the tactical asset allocation since it leaves decision room for tactical asset allocation.

Some items of the list are uncovered issues from the thesis. Other points could help to improve issues of modelling and optimization for the problem of strategic portfolio management.
Appendix

A.1 Derivation of the exact portfolio dynamics for linear Gaussian factor models

The wealth dynamics of the linear factor model in continuous-time is given by
\[
dW(t) = u(t)^T (F(t)x(t) + f(t))W(t) \, dt + (F_0(t)x(t) + f_0(t)) \, W(t) \, dt \\
+ W(t)u(t)^T \sigma(t)dZ_P(t) .
\] (A.1)

This is the continuous-time equivalent of discrete-time wealth dynamics for the linear Gaussian factor model. We use the transformation \( w(t) = \ln \left( W(t) \right) \) and obtain by using Itô’s lemma:
\[
dw(t) = \left[ F_0(t)x(t) + f_0(t) + u(t)^T (F(t)x(t) + f(t)) - \frac{1}{2} u(t)^T \Sigma(t) u(t) \right] dt \\
+ u(t)^T \sigma(t) dZ_P(t) ,
\] (A.2)

where \( \Sigma(t) = \sigma(t) \sigma^T(t) \). By using the Euler discretization with \( dw(t) \approx w(t+1) - w(t) \), \( dt \approx \delta t \), \( dZ_P(t) \approx \sqrt{\delta t} \xi^r(t) \), and \( \xi^r(t) \sim N(0, I) \), we obtain
\[
w(t+1) = w(t) + \left[ F_0(t)x(t) + f_0(t) + u(t)^T (F(t)x(t) + f(t)) - \frac{1}{2} u(t)^T \Sigma(t) u(t) \right] \delta t \\
+ u(t)^T \sigma(t) \sqrt{\delta t} \xi^r(t) ,
\] (A.3)

which has the same functional form as (4.47).

A.2 Dynamic programming solution of the linear Gaussian factor model

We consider the stochastic system given in (4.47) and the objective function \( O_1 \). Since the log-wealth has a normal distribution (for \( u(t) \) as function of \( t \), the objective function
**O1** can be written as $E^\frac{1}{\lambda}e^{\lambda w(T)}$, since this is the moment generating function. For the unconstrained case, we solve the following equivalent problem of portfolio optimization:

$$
\max_{u(t)} E^\frac{1}{\lambda}e^{\lambda w(T)}
$$

st.

$$
w(t+1) = w(t) + F_0 x(t) + f_0 + u^T(t)(Fx(t) + f) - \frac{1}{2}u^T(t)\Sigma(t)u(t) + u^T(t)M_1 \epsilon(t),
$$

$$
x(t+1) = Ax(t) + a + M_2 \epsilon(t),
$$

(A.4)

where $M_1 = [\psi \Phi, \psi \rho] \in \mathbb{R}^{n \times n+m}$, $M_2 = [0, \nu] \in \mathbb{R}^{m \times n+m}$, $\Phi \Phi^T = I - \rho \rho^T$, and $\epsilon(t) = (\epsilon^r(t), \epsilon^x(t))^T$. The dynamic programming algorithm to solve problem (A.4) is given by

$$
J(T, x(T), w(T)) = \frac{1}{\lambda}e^{\lambda w(T)}
$$

$$
J(t, x(t), w(t)) = \max_{u_t} \left\{ E \left[ J(t+1, x(t+1), w(t+1)) \right] \right\}.
$$

(A.5)

Based on the results from continuous-time (Section 3.2.3), we use the following Ansatz to solve the dynamic programming problem:

$$
J(t, x(t), w(t)) = \frac{\alpha_t}{\lambda}e^{\lambda(w(t)+\beta(t)x(t)+\frac{1}{2}x^T(t)\Delta(t)x(t))},
$$

(A.6)

where $\alpha(t) \in \mathbb{R}$, $\beta(t) \in \mathbb{R}^m$, and $\Delta(t) \in \mathbb{R}^{m \times m}$. The terminal conditions are $\alpha(T) = 1$, $\beta(T) = 0$ and $\Delta(T) = 0$. In order to derive a solution for (A.5) with the Ansatz (A.6), we need to compute the expectation involved in the DP algorithm. Similar to the lemma stated in Jacobson (1973), we state the following proposition:

**Proposition A.1.**

The solution of (A.5) consists of three coupled difference equations and an affine function for the control variable. The control variable is computed by

$$
u(t, x(t)) = H^{-1} \cdot \left( Fx(t) + f + K_1(\Delta(t+1)(Ax(t) + a) + \beta(t+1)) \right)
$$

$$
H = \left( (1 - \lambda)\Sigma + \lambda \rho \sigma (I - (I - \nu^T \Delta(t+1)\nu)^{-1})\rho^T \sigma^T \right)
$$

$$
K_1 = \sigma \rho (I - \nu^T \Delta(t+1)\nu)^{-1}\nu^T
$$

$$
K_2 = \nu (I - \nu^T \Delta(t+1)\nu)^{-1}\nu^T.
$$

(A.7)

The Riccati equations to compute the control law are:

First,
\[ \Delta(t) = A^T \Delta(t+1)A + A^T \Delta(t+1)K_2 \Delta(t+1)A + \lambda \left( F^T H^{-1} F \right) + A^T \Delta(t+1)K_1^T H^{-1} K_1 \Delta(t+1)A + F^T H^{-1} K_1 \Delta(t+1)A + A^T \Delta(t+1)K_1^T H^{-1} F \),

with terminal condition \( \Delta(T) = 0 \). Note that \( \Delta(t) \) is positive definite and symmetric matrix. Second,

\[ 
\beta(t) = \beta^T(t+1)A + a^T \Delta(t+1)A + \beta^T(t+1)K_2 \Delta(t+1)A + a^T \Delta(t+1)K_2 \Delta(t+1)A + \lambda \left( F_0 + f^T H^{-1} F + \beta^T(t+1)K_1^T H^{-1} F + a^T \Delta(t+1)K_1^T H^{-1} \Delta(t+1)A + f^T H^{-1} K_1 \Delta(t+1)A \right),
\]

with terminal condition \( \beta(T) = 0 \). The last difference equation for \( \alpha(t) \) is given by

\[ 
\alpha(t) = \frac{\alpha(t+1)}{\sqrt{1 - \nu^2 \Delta(t+1)\nu}} \exp \left( \frac{1}{2} a^T \Delta(t+1)a + \beta^T(t+1)a + \frac{1}{2} \beta^T(t+1)K_2 \beta(t+1) + \frac{1}{2} \lambda \beta^T(t+1)K_1^T H^{-1} K_1 \beta(t+1) \right. 
\]

\[ 
+ \frac{1}{2} \lambda a^T \Delta(t+1)K_2^2 \Delta(t+1)a + \beta^T(t+1)K_2 \Delta(t+1)a + \frac{1}{2} \lambda f^T H^{-1} f 
\]

\[ 
+ \frac{1}{2} \lambda \beta^T(t+1)K_1^T H^{-1} K_1 \beta(t+1) + \frac{1}{2} \lambda a^T \Delta(t+1)K_1^T H^{-1} K_1 \Delta(t+1)a 
\]

\[ 
+ \lambda f^T H^{-1} K_1 \Delta(t+1)a + \lambda f_0 \right),
\]

with terminal condition \( \alpha(T) = 1 \).

### A.3 Dynamic programming recursion for the sample approximation

The proof of Theorem 4.4 is given as follows:

**Proof.** We denote by \( \Pi^*(\tau) = [\pi^*(\tau), \pi^*(\tau+1), \ldots, \pi^*(T-1)] \) and insert (4.31) into (4.33) which yields

\[ 
\hat{J}^*(\tau, \gamma(\tau)) = \max_{\Pi^*(\tau)} \left\{ \hat{E} \left[ \sum_{i=\tau}^{T-1} L(i, y^*(i), \pi^*(i)) + M(T, \pi^*(T)) \right] \right\}
\]

\[ 
= \max_{\pi^*(\tau), \Pi^*(\tau+1)} \left\{ \hat{E} \left[ L(\tau, y^*(\tau), \pi^*(\tau)) + \sum_{i=\tau+1}^{T-1} L(i, y^*(i), \pi^*(i)) + M(T, \pi^*(T)) \right] \right\}.
\]
Since $L(\tau, y^*(\tau), \pi^*(\tau))$ is independent of the future decisions $\Pi^*(t+1)$ and the scenarios for $\epsilon^*(i), i > \tau$ are independent of $\pi^*(\tau)$, we move the maximization operator inside the bracket and obtain

$$\hat{J}^s(\tau, y(\tau)) = \max_{\pi^*(\tau)} \left\{ \hat{E}\left[ L(\tau, y^*(\tau), \pi^*(\tau)) \right] 
+ \max_{\Pi^*(t+1)} \left\{ \hat{E}\left[ \sum_{i=\tau+1}^{T-1} L(i, y^*(i), \pi^*(i)) + M(T, \pi^*(T)) \right] \right\} \right\} = \max_{\pi^*(\tau)} \left\{ \hat{E}\left[ L(\tau, y^*(\tau), \pi^*(\tau)) + \hat{J}^s(\tau+1, y^*(\tau+1)) \right] \right\}. $$

Using the fact that $y^*(\tau+1) = D(\tau, y^*, \pi^*) + S(\tau, y^*, \pi^*)\epsilon^*(t)$ and basic idea of Bellman’s principle

$$\hat{J}^s(\tau+1, y(\tau+1)) = \max_{\Pi^*(t+1)} \left\{ \hat{E}\left[ \sum_{i=\tau+1}^{T-1} L(i, y^*(i), \pi^*(i)) + M(T, \pi^*(T)) \right] \right\}, $$

we obtain the following result:

$$\hat{J}^s(\tau, y(\tau)) = \max_{\pi^*(\tau)} \left\{ \hat{E}\left[ L(\tau, y^*(\tau), \pi^*(\tau)) \right] 
+ \hat{J}^s(\tau+1, D(\tau, y^*, \pi^*) + S(\tau, y^*, \pi^*)\epsilon^*(\tau)) \right\} \right\} = \max_{u^*(\tau) \in \mathcal{U}} \left\{ \hat{E}\left[ L(\tau, y^*(\tau), u^*(\tau)) \right] 
+ \hat{J}^s(\tau+1, D(\tau, y^*, u^*) + S(\tau, y^*, u^*)\epsilon^*(\tau)) \right\}, $$

where we converted the maximization over $\pi^*(\tau)$ to a maximization over $u^*(\tau)$, using the fact that for any function $f$ of $x$ and $u$ it is true that

$$\max_{\pi \in \mathcal{Q}} \{ f(x, \pi(x)) \} = \max_{u \in \mathcal{U}} \{ f(x, u) \}, $$

where $\mathcal{Q}$ is the set of all functions $\pi(x)$ such that $\pi(x) \in \mathcal{U}$, $\forall x$. This statement can be found in Bertsekas (1995, Chapter 2).

\section*{A.4 Parameters for numerical comparison of the MPC and the DP method}

The parameters for this example are representative for Swiss market data from 1984 until 2004 for the Swiss bond and stock market. The parameters are obtained by fitting the model to the Swiss market data set with a monthly frequency using the “seemingly unrelated regression” method (Hamilton 1994). The model parameters are as follows:
\[ F = \begin{bmatrix} -1.04 & 13.12 \\ 2.01 & -1.75 \end{bmatrix}, \quad f = \begin{bmatrix} -0.103 \\ -0.019 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0.178 & 0.005 \\ 0.005 & 0.048 \end{bmatrix}, \]
\[ \rho = \begin{bmatrix} 0.028 & -0.803 \\ -0.969 & -0.045 \end{bmatrix}, \quad A = \begin{bmatrix} 0.9904 & 0.011 \\ 0.006 & 0.9566 \end{bmatrix}, \]
\[ a = \begin{bmatrix} 1.42 \cdot 10^{-3} \\ 4.65 \cdot 10^{-3} \end{bmatrix}, \quad \nu = \begin{bmatrix} 1.86 \cdot 10^{-3} & 0.07 \cdot 10^{-3} \\ 0.07 \cdot 10^{-3} & 1.02 \cdot 10^{-3} \end{bmatrix}, \]
\[ F_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad f_0 = 0.015. \]  

(A.12)

All parameters are expressed in values per annum (p.a.).

**A.5 Prediction of the GARCH(p,q) and the TARCH(o,p,q) models**

The expected conditional variance in the case of the GARCH(p,q) model is given by

\[
E_0[\sigma_t^2(t+T)] = \omega_0 + \sum_{j=0}^{p-1} \delta_{ij} \sigma_t^2(t-j) + \sum_{j=0}^{m-1} \rho_{ij} \sigma_t^2(t-j) \quad (A.13)
\]

where \( m_i = \max\{p_i, q_i\}, \) \( \alpha_{ij} = 0 \) for \( j > q_i, \beta_{ij} = 0 \) for \( i > p_i, \) and

\[
\omega_0 = \bar{\varepsilon}_1^T \Gamma_i \left( I + \Gamma_i + \ldots + \Gamma_i^{T-1} \right) \bar{\varepsilon}_1 \omega_i \\
\delta_{ij} = -\bar{\varepsilon}_1^T \Gamma_i \bar{\varepsilon}_{m_i+j+1} \\
\rho_{ij} = \bar{\varepsilon}_1^T \Gamma_i \left( \bar{\varepsilon}_{1+j} + \bar{\varepsilon}_{m_i+j+1} \right) \\
\rho_{ij} = \bar{\varepsilon}_1^T \Gamma_i \bar{\varepsilon}_{1+j}
\]

where

\[
\Gamma_i = \begin{pmatrix} \alpha_{i1} + \beta_{i1} & \ldots & \alpha_{im_i} + \beta_{im_i} - \beta_{i1} & \ldots & -\beta_{ip_i} \\ 1 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\ 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\ \end{pmatrix}
\]

and \( \bar{\varepsilon}_j \) denotes a vector of zeros except for unity in the \( j \)-th element. The derivation and proof of the GARCH(p,q) prediction equations can be found in Baillie and Bollerslev (1992).
Proposition A.2. The conditional variances in the case of the TARCH(o,p,q) model is given by

\[
E_t[v_i^2(t+T)] = \omega_i + \sum_{j=0}^{m_2i-1} \tau_{ij}^T \chi_{it-j}^2 \varepsilon_i^2(t-j) \\
+ \sum_{j=0}^{m_1i-1} \delta_{ij}^2 \varepsilon_i^2(t-j) + \sum_{j=0}^{m_1i-1} \rho_{ij}^T \varepsilon_i^2(t-j),
\]

where \(m_2i = \max\{p_i, o_i\}\), \(m_1i = \max\{p_i, q_i, o_i\}\), \(\alpha_{ij} = 0\) for \(j > q_i\), \(\beta_{ij} = 0\) for \(i > p_i\), \(\delta_{ij} = 0\) for \(j > o_i\), and

\[
\omega_i = \varepsilon_i^T (I + T_i + \ldots + T_i^{T-1}) \varepsilon_i, \quad i = 1, \ldots, n \\
\tau_{ij} = \varepsilon_i^T T_i \varepsilon_{m_1i+m_2i+j+1}, \quad j = 1, \ldots, o_i - 1, \quad i = 1, \ldots, n \\
\rho_{ij} = -\varepsilon_i^T T_i \varepsilon_{m_1i+m_2i+j+1}, \quad j = 0, 1, \ldots, o_i - 1, \quad i = 1, \ldots, n \\
\delta_{ij} = -\varepsilon_i^T T_i \varepsilon_{m_1i+m_2i+j+1}, \quad j = o_i, \ldots, m_2i - 1, \quad i = 1, \ldots, n \\
\delta_{ij} = -\varepsilon_i^T T_i \varepsilon_{m_1i+m_2i+j+1}, \quad j = 0, 1, \ldots, m_2i - 1, \quad i = 1, \ldots, n
\]

where

\[
T_i = \begin{pmatrix}
\alpha_{i1} + \beta_{i1} + \frac{1}{2} \gamma_{i1} & \ldots & \alpha_{im_1} + \beta_{im_1} + \frac{1}{2} \gamma_{m_1} - \beta_{i1} - \frac{1}{2} \gamma_{i1} & \ldots & -\beta_{m_2i} - \frac{1}{2} \gamma_{m_2i} & \gamma_{i1} & \ldots & \gamma_{io} \\
1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\
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\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 1 \\
\end{pmatrix},
\]

and \(\varepsilon_j\) denotes a vector of zeros except for unity in the \(j\)-th element.

Proof. The TARCH model is given by

\[
v^2(t) = \omega + \sum_{j=1}^{q} \alpha_j \varepsilon^2(t-j) + \sum_{j=1}^{p} \beta_j v^2(t-j) + \sum_{j=1}^{o} \gamma_j \chi_{t-j} \varepsilon^2(t-j).
\]
The innovations are given by \( e^r(t) = v(t)\xi^r(t) \) where \( \xi(t) \) is a identically and independently distributed random variable with mean zero and unity variance. Moreover, we assume that the distribution of \( \xi(t) \) is symmetric. The expectation is \( \mathbb{E}_{t-1}[\chi_t e^r(t)] = \frac{1}{2}v^2(t) \) because \( v(t) > 0 \) and \( \mathbb{E}_{t-1}[\chi_t \xi^2(t)] = \frac{1}{2} \). We now introduce two new random variables, which are defined as

\[
\begin{align*}
\zeta_1(t) &= e^r(t) - v^2(t), \quad \mathbb{E}_t[\zeta_1(t + T)] = 0, \quad T > 0 \\
\zeta_2(t) &= \chi_t e^r(t) - \frac{1}{2}v^2(t), \quad \mathbb{E}_t[\zeta_2(t + T)] = 0, \quad T > 0
\end{align*}
\]

The equation of the T ARCH model is now rewritten with the two new variables as

\[
\epsilon^2(t) = \zeta_1(t) + \omega + \sum_{j=1}^{m_1} (\alpha_j + \beta_j + \frac{1}{2}\gamma_j)\epsilon^2(t-j) \\
- \sum_{j=1}^{m_2} (\beta_j + \frac{1}{2}\gamma_j)\zeta_1(t-j) + \sum_{j=1}^{o} \gamma_j\zeta_2(t-j),
\]

where \( m_2 = \max\{p, o\}, \ m_1 = \max\{p, q, o\}, \ \alpha_j = 0 \) for \( j > q, \ \beta_j = 0 \) for \( i > p, \) and \( \delta_j = 0 \) for \( j > o. \) In matrix notion, (A.16) is given as

\[
\Xi_{t+1} = T\Xi_t + (\tau_1 + \tau_{m_1+1})v_t + (\tau_{m_1+m_2+1})z_t + \tau_1\omega,
\]

where \( T \) is given by (A.15), \( \tau_j \) denotes a vector of zeros expected for unity in the \( j \)-th element, and \( \Xi(t) = (\epsilon^2(t), \epsilon^2(t-1), \ldots, \epsilon^2(t-m_1+1), \zeta_1(t), \ldots, \zeta_1(t-m_2+1), \zeta_2(t), \ldots, \zeta_2(t-o+1))^T. \) We compute \( \Xi(t+T) \) by iterating (A.17) \( T \)-times and get

\[
\Xi(t+T) = T^T \Xi(t) + \sum_{j=0}^{T-1} \left( T^j \left[ (\tau_1 + \tau_{m_1+1})\zeta_1(t+k-j) + (\tau_{m_1+m_2+1})\zeta_2(t+T-j) + (\tau_1)\omega \right] \right).
\]

The prediction equation for \( v^2(t+T) \) can be derived by using \( v^2(t+T) = \tau_1^T \Xi(t+T), \)

\[
\mathbb{E}_t[\zeta_2(t+T)] = 0, \quad \mathbb{E}_t[\zeta_1(t+T)] = 0,
\]

\[
\Xi(t) = \sum_{j=0}^{m_1-1} \tau_{j+1}\epsilon^2(t-j) + \sum_{j=0}^{m_2-1} \tau_{m_1+j}\zeta_1(t-j) + \sum_{j=0}^{o-1} \tau_{m_1+m_2+1+j}\zeta_2(t-j).
\]

Therefore, it follows that

\[
\mathbb{E}_t[v^2(t+T)] = \omega T + \sum_{j=0}^{o-1} \tau_j T \chi_{t-j} T v^2(t-j) + \sum_{j=0}^{m_2-1} \delta_j T v^2(t-j) + \sum_{j=0}^{m_1-1} \rho_j T \epsilon^2(t-j),
\]
where \( m_2 = \max\{p, o\} \), \( m_1 = \max\{p, q, o\} \), \( \alpha_j = 0 \) for \( j > q \), \( \beta_j = 0 \) for \( i > p \), \( \delta_j = 0 \) for \( j > o \), and

\[
\omega_T = \bar{e}_1^T (I + T + \ldots + T^{T-1}) \bar{e}_1 \omega , \\
\tau_j = \bar{e}_1^T T \bar{e}_{m_1+m_2+j+1} j = 1, \ldots, o - 1 , \\
\rho_j = -\bar{e}_1^T T \bar{e}_{m_1+1+j} + \bar{e}_{m_1+m_2+j+1} j = 0, 1, \ldots, o - 1 , \\
\rho_j = -\bar{e}_1^T T \bar{e}_{m_1+1+j} j = 0, \ldots, m_2 - 1 , \\
\delta_j = -\bar{e}_1^T T \bar{e}_{m_1+1+j} j = 0, 1, \ldots, m_2 - 1 , \\
\delta_j = -\bar{e}_1^T T \bar{e}_{1+j} j = m_2, \ldots, m_1 - 1 .
\]
A.6 Additional results for the case studies

A.6.1 Additional data and results for case study 2 with US data

In Table A.1, the results of the rolling normality tests for case study two with US data are shown. For each set of return data, the two normality tests are computed with eight-years of past data. The results justify the use of optimization methods based on the normal distributions. Only for the S&P 500 data in the years 1992, 1993, and 1994 all four tests reject the normal distribution assumption. For the 1-month Treasury note interest rate in the years 1998, 2000, and 2001 all four tests reject the normal distribution assumption. For all other data sets and years at least one test does not reject the normality assumption. Most of the times, the results are quite stable. For the data from Jan. 1990 to Dec. 2004 the QQ plot is shown in Figure A.1. Similar to the results of the normality tests shown in Table A.1, the deviation from the normal distribution is minor. In Table A.2, all the factors

![QQ plots of the asset return data from Jan. 1980 to Dec. 1988.](image)

used in case study 2 (Section 5.2) are given. Only the first 22 factors are selected at least once. The factors 23 to 41 where never selected. Furthermore in Table A.2, a * behind the factor number indicates factors that might be unsuitable for a out-of-sample test, since their values are subjected to revision up to 9 months after their initial publication. In this case, the record factor value was not available at the time and after the revision might indicate any future development. This problem arises whenever GDP components are used as factors.
Table A.1. Results of the rolling normality tests. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

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J-B denotes the Jarque-Bera test, LF denotes Lilliefors, TB denotes Treasury bonds, Mdy denotes the Moody BAA bond index, TN denotes Treasury notes, and GS CI denotes the Goldman Sachs Commodity index

A.6.2 Additional data and results for case study with Swiss data

The portfolio with low transaction costs and the asset evolution throughout the historical back test are shown in Figure A.2. The asset allocation in the case of low transaction costs are given in Figure A.3. The asset allocation is much more volatile than in the case with higher transaction costs. In Table A.3 are all the selected factors of the third case study reported.
Table A.2. All factors for case study 2 with US data and result of the normality tests for the standard residuals. A 0 indicates that we do not reject the normality assumption and a 1 indicates the rejection of normality.

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The * denotes time series that are subjected to revisions after their initial publishing.
Fig. A.2. Results of the out-of-sample test for the Swiss case study with low transaction costs

Fig. A.3. Asset allocation for the Swiss case study with low transaction costs
Table A.3. All factors for Case Study 3 with Swiss and EU data

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<td>10-year EU Benchmark Government bond (GB) interest rate</td>
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<td>10-year GB rate - 3-month LIBOR Switzerland</td>
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<td>10-year GB rate - 3-month LIBOR EU</td>
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<td>FX spot rate CHF/Euro (DM)</td>
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<td>8</td>
<td>3-months momentum. Swiss stock market</td>
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<td>12</td>
<td>FX 3-month forward rate CHF/Euro (DM)</td>
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References


References


References


Curriculum Vitae

Personal Data

Name: Florian Herzog

Date of birth: July 18th, 1975 in Munich, Germany

Citizenship: Germany

Education

2002-2005 Doctoral studies and research assistant at the Measurement and Control Laboratory, Swiss Federal Institute of Technology (ETH) Zurich. Research topic: application of optimal control methods to problems of portfolio management

2002 Diploma (Dipl. Ing. ETH) ETH Zurich

2001 Master of Science (M.Sc.) Georgia Institute of Technology (Georgia Tech)

1996-2002 Studies in mechanical engineering at the ETH Zurich with specializations in control engineering and operations research. Electives in management studies

2000-2001 Graduate studies at the Georgia Tech in engineering with specializations in optimization and control engineering.


1982-1986 Primary School Grundschule an der Oselstrasse, Munich, Germany

Professional Experience

2001 SIG Pack Systems, Berringen: Development of assembly line controllers. Named as inventor of the corresponding European patent

1997 Hypovereinsbank, Munich: Internship risk management