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Author(s):
Mei, Wenjun; Chen, Ge; Dörfler, Florian

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Structural Balance and Interpersonal Appraisals Dynamics: Beyond All-to-All and Two-Faction Networks ⋆

Wenjun Mei * **Ge Chen** ** **Florian Dörfler** ***

* *Automatic Control Laboratory, ETH Zurich, 8006 Zurich, Switzerland (e-mail: wmei@ethz.ch)*
** *Academy of Mathematics and Systems Science, Chinese Academy of Science, Beijing 100190, China (e-mail: chenge@amss.ac.cn)*
*** *Automatic Control Laboratory, ETH Zurich, 8006 Zurich, Switzerland (e-mail: dorfler@ethz.ch)*

Abstract: Structural balance theory describes stable configurations of topologies of signed interpersonal appraisal networks. Various mathematical models have been proposed to explain how initially unbalanced appraisal networks evolve to structural balance. However, the existing models either diverge in finite time, or could get stuck in jammed states, or converge to only non-all-to-all graphs starting from certain sets of initial conditions. It remains an open problem how non-all-to-all structural balance emerges via local dynamics of interpersonal appraisals. In this paper, we first compare two well-justified definitions of structural balance, i.e., the triad-wise structural balance and the two-faction structural balance, and establish the conditions with clear graph-theoretic interpretations, under which these two definitions of structural balance are equivalent. Secondly, we propose a simple model of gossip-like appraisal dynamics in which the appraisal network, starting from any initial condition, almost surely achieves structural balance in finite time, while its topology remains unchanged. Our model is based on three widely adopted sociological mechanisms: the symmetry mechanism, the influence mechanism and the homophily mechanism. Our main theoretical contribution is manifold: First, we show that the equilibrium set of our gossip-like appraisal dynamics corresponds to the set of all the possible triad-wise structural balance configurations of the appraisal networks. Second, we prove that, for any initial condition, the appraisal network almost surely achieves triad-wise structural balance in finite time. Third, we provide a sufficient condition, under which the appraisal networks almost-surely achieve two-faction structural balance in finite time.

Keywords: Structural balance, Network formation, Graph theory, Multi-agent systems

1. INTRODUCTION

Adversarial or antagonistic relations are prevalent in various biological, engineering, social, and economic systems. Such relations are often modelled as links with negative weights in signed networks. *Structural balance* (also referred to as *social balance*) theory, first proposed in the seminal works by Heider (Heider, 1944, 1946), characterizes the stable configurations of interpersonal appraisal networks with both friendly and antagonistic relations. An appraisal network satisfies structural balance if each individual obeys the famous Heider's axioms: Friends' friends are friends; Friends' enemies are enemies; Enemies' friends are enemies; Enemies enemies are friends." Empirical studies (Leskovec et al., 2010; King, 1964; Taylor, 1970) indicate that social balance is a type of stable configurations frequently observed in real social networks. Dynamic social balance theory, aiming to explain how an initially unbalanced network evolves to a balanced state, has recently attracted much interest. However, in existing models (Antal et al., 2006; van de Rijt, 2011; Malekzadeh et al., 2011; Kułakowski et al., 2005; Marvel et al., 2011; Traag et al., 2013; Wongkaew et al., 2015; Mei et al., 2019), the appraisal network either diverges in finite time, or converges to an all-to-all graph satisfying social balance, or gets stuck in unbalanced equilibria. It is still an open problem what models lead to the convergence of appraisal networks to social balance with arbitrary network topology.

In this paper, we first clarify the meaning of structure balance in non-all-to-all graphs by presenting two well-justified definitions: the *triad-wise structural balance* and the *two-faction structural balance*, and establish their relations. We then propose a novel gossip-like appraisal dynamics that almost-surely achieve triad-wise structural balance in finite time. We also provide a sufficient condition that the proposed appraisal dynamics achieve two-faction structural balance in finite time. We postpone all the proofs to a later journal submission.

2. NOTATION AND TWO DEFINITIONS OF NON-ALL-TO-ALL STRUCTURAL BALANCE

Let ϕ be the empty set and $\mathbb{N}$ be the set of natural numbers. Let $V = \{1, 2, \dots, n\}$. Two type of graphs are involved in this paper. A *directed and unweighted signed graph* with n nodes is denoted by $\vec{\mathcal{G}} = (V, \vec{E}^+, \vec{E}^-)$, where V is the nodes

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set and $\vec{E}^{+} \subseteq V \times V$ ($\vec{E}^{-} \subseteq V \times V$ resp.) is the set of all the positive (negative resp.) directed links in $\vec{\mathcal{G}}$. We only consider unweighted graphs and thereby the term "unweighted" is omitted in the rest of this paper. An *undirected unsigned graph* is denoted by $|\mathcal{G}| = (V, E)$, where E is the set of all the undirected links. A *directed path* (*path* resp.) in $\vec{\mathcal{G}}$ ($|\mathcal{G}|$ resp.) from node i_1 to node i_m is an ordered sequence of nodes $i_1, \dots, i_m$ such that $(i_k, i_{k+1}) \in \vec{E}^{+} \cup \vec{E}^{-}$ ($\{i_k, i_{k+1}\} \in E$ resp.) for any $k = 1, \dots, m-1$. This directed path (path resp.) has length $m-1$. A graph $\vec{\mathcal{G}}$ ($|\mathcal{G}|$ resp.) is *strongly connected* (*connected* resp.) if, for any $i, j \in V$, there exists at least one directed path (path resp.) from i to j.

Any matrix $X \in \{-1, 0, 1\}^{n\times n}$ defines a directed signed graph $\vec{\mathcal{G}}(X) = (V, \vec{E}^{+}(X), \vec{E}^{-}(X))$, where, $\vec{E}^{+}(X) = \{(i,j) \in V \times V \,|\, X_{ij} = 1\}$ and $\vec{E}^{-}(X) = \{(i,j) \in V \times V \,|\, X_{ij} = -1\}$. This matrix X also defines an undirected unsigned graph $|\mathcal{G}|(X) = (V, E(X))$, where $E(X) = \big\{\{i,j\} \,|\, i, j \in V,\ X_{ij} \neq 0, \text{ or } X_{ji} \neq 0\big\}$. An ordered sequence $(i_1, \dots, i_m)$ of non-repeating nodes is a simple cycle with length m in $\vec{\mathcal{G}}$ ($|\mathcal{G}|$ resp.) if $(i_k, i_{k+1}) \in \vec{E}^{+} \cup \vec{E}^{-}$ ($\{i_k, i_{k+1}\} \in E$ resp.) for any $k \in \{1, \dots, m\}$. Here we take i_{m+1} as i_1. On $\vec{\mathcal{G}}$, a simple cycle is *positive* if it contains either zero or an even number of negative links. A triad is a simple cycle with length 3 and a pair of nodes linking to each other is a simple cycle with length 2. Given a graph $\vec{\mathcal{G}}$ ($|\mathcal{G}|$ resp.) and a subset of nodes $\tilde{V} \subseteq V$, denoted by $\vec{\mathcal{G}}_{\tilde{V}}$ ($|\mathcal{G}|_{\tilde{V}}$ resp.) the subgraph involving all the nodes in $\tilde{V}$. For any $i \in V$ in $\vec{\mathcal{G}}(X)$, define $N_i = \{j \in V \,|\, X_{ij} \neq 0\} \cup \{i\}$ and define $\vec{\mathcal{G}}_{N_i}(X)$ as node i's *ego-network*.

Given a group of n individuals, their interpersonal appraisals are characterized by the *appraisal matrix* $X = (X_{ij})_{n\times n}$. We assume that $X_{ij} \in \{-1, 0, 1\}$ for any $i, j \in \{1, \dots, n\}$. Here $X_{ij} = 1$ ($X_{ij} = -1$ resp.) means that individual i is friendly (antagonistic resp.) to individual j, while $X_{ij} = 0$ means that either i does not know j or i holds neutral attitude towards j. An appraisal matrix X defines a directed and signed graph $\vec{\mathcal{G}}(X)$, referred to as the *appraisal network*. We use the terms "graph" and "network" interchangeably. The appraisals network $\vec{\mathcal{G}}(X)$ is *bilateral* if "$X_{ij} \neq 0 \Leftrightarrow X_{ji} \neq 0$". We do not consider the self loops, i.e., X_{ii} is taken to be 0 for any $i \in V$.

For non-all-to-all appraisal networks, one could intuitively think of two definitions of structural balance. The first one is a straightforward generalization of the aforementioned Heider' four axioms: "Friends' friends are not enemies; Friends' enemies are not friends; Enemies' friends are not friends; Enemies' enemies are not enemies." Here "not enemies" means either being friends, or not knowing each other, or neutral relations. This definition is equivalent to say that any existing triad in the appraisal network is positive and is formalized as follows.

Definition 1. (Triad-wise structural balance). An appraisal network $\vec{\mathcal{G}}(X)$ satisfies triad-wise structural balance, or, equivalently, is triad-wise balanced, if the appraisal matrix X satisfies the following properties:

P1: (Symmetric appraisals) $X_{ij}X_{ji} > 0$ for any $i \neq j$;
P2: (Positive triads) $X_{ij}X_{jk}X_{ki} > 0$, $\forall\,(i,j,k)$ in $\vec{\mathcal{G}}(X)$.

Note that, by the definition above, any appraisal network satisfying triad-wise structural balance is bilateral.

The second definition is widely used in the opinion dynamics over signed graphs (Altafini, 2013; Liu et al., 2017; Shi et al., 2019). It requires that the appraisal network can be partitioned into two unfriendly factions, whereas inside each faction the social links are non-negative. The formal definition is as follows.

Definition 2. (Two-faction structural balance). An appraisal network $\vec{\mathcal{G}}(X)$ satisfies two-faction structural balance, or, equivalently, is two-faction balanced, if either $\vec{\mathcal{G}}(X)$ has no negative link or its nodes set V can be partitioned into two disjoint sets V_1 and V_2 such that $X_{ij} \geqslant 0$, for any $i, j \in V_1$ or any $i, j \in V_2$, and $X_{ij} \leqslant 0$, for any $i \in V_1$, $j \in V_2$, or any $i \in V_2$, $j \in V_1$.

The following lemma was first proposed in (Cartwright and Harary, 1956) and provides a necessary and sufficient condition for two-faction structural balance in strongly connected graphs.

Lemma 3. (Cycle-wise structural balance). Given an appraisal matrix $X \in \{-1, 0, 1\}^{n\times n}$ such that $\vec{\mathcal{G}}(X)$ is bilateral and strongly connected, $\vec{\mathcal{G}}(X)$ satisfies two-faction structural balance if and only if every simple cycle in $\vec{\mathcal{G}}(X)$ is positive.

Triad-wise structural balance is about a local feature of signed appraisal networks, while two-faction structural balance characterizes some global structure. It is well known that, in all-to-all appraisal networks, these two definitions are equivalent (Cartwright and Harary, 1956). In non-all-to-all bilateral appraisal networks, their connections are more sophisticated. Before presenting the main theorem of this section, we first propose some important graph-theoretic definitions.

Definition 4. (Maximal cyclic subgraph). Consider an undirected unsigned graph $|\mathcal{G}| = (V, E)$. A subgraph $|\mathcal{G}|_{\tilde{V}}$ with $\tilde{V} \subseteq V$ is a maximal cyclic subgraph if it satisfies the following two conditions:

(1) There exists a simple cycle $(i_1, \dots, i_m)$ in $|\mathcal{G}|$ such that $\tilde{V} = \{i_1, \dots, i_m\}$, i.e., this simple cycle passes through every node in $|\mathcal{G}|_{\tilde{V}}$;
(2) For any $j \in V \backslash \tilde{V}$ (if any), there does not exist any simple cycle in $|\mathcal{G}|$ that passes through every node in $\tilde{V} \cup \{j\}$.

In undirected unsigned graphs, any simple cycle can be geometrically arranged as a "circle", and any triad whose nodes are chosen from such a simple cycle geometrically forms a triangle. From such a geometric perspective, the definition given below characterizes an important feature of simple cycles.

Definition 5. (Triad-connected cycle). In an undirected unsigned graph $|\mathcal{G}|$, a simple cycle $(i_1, \dots, i_m)$ is triad-connected if the subgraph $|\mathcal{G}|_{\{i_1,\dots,i_m\}}$ contains a subset Δ of triads such that, if the simple cycle $(i_1, \dots, i_m)$ is geometrically arranged as a circle, then the triangles formed by all the triads in Δ fill the interior of the circle with no geometric overlap.

Below we present the main theorem on the relations between triad-wise structural balance and two-faction structural balance.

Theorem 6. (Relations between two structural balances). For any $X \in \{-1, 0, 1\}^{n\times n}$ such that $\vec{\mathcal{G}}(X)$ is bilateral, the following statements hold:

(i) If $\vec{\mathcal{G}}(X)$ is two-faction balanced, then $\vec{\mathcal{G}}(X)$ is triad-wise balanced;

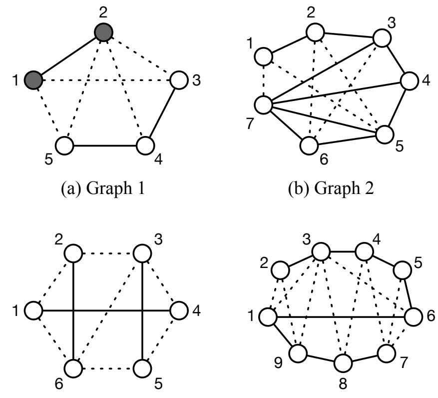

Fig. 1. Examples of graphs on which the triad-wise structural balance is or is not equivalent to the two-faction structural balance. In all these directed signed graphs, the links are bilateral and sign-symmetric, and the arrows of links are thereby omitted. Solid (dashed resp.) lines represent positive (negative resp.) bilateral links. In panel (a), $(1,\dots,5)$ is a simple cycle. Its interior is covered by the triads set $\{(1,2,5),\,(2,4,5),\,(2,3,4)\}$ with no geometric overlap and the link $(1,3)$ satisfies the condition in Theorem 6(iii)2). As one could easily see, Graph 1 satisfies both triad-wise structural balance and two-faction structural balance. The grey nodes form one faction and the other nodes form another faction. In panel (b), the corresponding undirected unsigned graph $|\mathcal{G}|$ itself is maximal cyclic and its interior can be covered by the set of all the triads, but not without overlap. As a result, Graph 2 satisfies triad-wise structural balance but its nodes cannot be partitioned into two factions. In panel (c), the simple cycle $(1,2,\dots,6)$ in the corresponding undirected unsigned graph is triad-connected but the link $\{1,4\}$ violates condition (iii)2) in Theorem 6. As a result, Graph 3 is triad-wise balanced but not two-faction balanced. In panel (d), the simple cycle $(1,2,\dots,9)$ in the corresponding undirected unsigned graph is triad-connected and every link is in at least one triad. However, since links $\{3,6\}$ and $\{1,6\}$ violate condition (iii)2) in Theorem 6, Graph 4 is triad-wise balanced but not two-faction balanced.

(ii) If $\vec{\mathcal{G}}(X)$ is triad-wise balanced, then, for any $i \in V$, node i's ego-network $\vec{\mathcal{G}}_{N_i}(X)$ satisfies two-faction structural balance;

(iii) Suppose $\vec{\mathcal{G}}(X)$ is strongly connected. $\vec{\mathcal{G}}(X)$ is two-faction balanced if $\vec{\mathcal{G}}(X)$ is triad-wise balanced and, for any maximal cyclic subgraph in $|\mathcal{G}|(X)$ with $m > 3$ nodes,
   1) it contains a simple cycle, denoted by $(i_1,\dots,i_m)$, that passes through all the nodes in this subgraph and is triad-connected;
   2) for any $j,k \in \{1,\dots,m\}$ with $\{j,k\} \in E$ and $k \geqslant j+2$, at least one of the simple cycles $(i_j, i_{j+1},\dots,i_k)$ or $(i_1,\dots,i_j,i_k,\dots,i_m)$ is triad-connected.

Statement (i) in Theorem 6 is straightforward to see. Statement (ii) in Theorem 6 has a clear sociological interpretation. According to Heider's structural balance theory (Heider, 1946), imbalance of interpersonal relations sensed by individuals leads to cognitive dissonances that the individuals strive to resolve. It is intuitive to assume that individuals have access to information from their ego-networks. As long as the entire appraisal network satisfies triad-wise structural balance, all the individuals' ego-networks satisfy both triad-wise structural balance and two-faction structural balance. Individuals should not feel any cognitive dissonance and thereby have no motivation to further adjust their appraisals of others. In this sense, in non-all-to-all appraisal networks, triad-wise structural balance characterizes a class of steady configurations of the interpersonal appraisals. In Figure 1(a) we provide one example where the graph $|\mathcal{G}|$ satisfies the conditions in Theorem 6(iii) and the two definitions of structural balance are equivalent. We also provide some examples in Figure 1(b)-(d) where $|\mathcal{G}|$ violates some of the conditions in Theorem 6(iii) and, as the consequence, there exists $\vec{\mathcal{G}}(X)$ with $|\mathcal{G}|(X) = |\mathcal{G}|$ such that $\vec{\mathcal{G}}(X)$ satisfies triad-wise structural balance but not two-faction balance.

3. SYMMETRY-INFLUENCE-HOMOPHILY DYNAMIC STRUCTURAL BALANCE MODEL

In this section we propose and analyze a discrete-time gossip-like model that characterizes how appraisal networks evolves to structural balance configurations (either triad-wise or two-faction) while their zero-patterns remain unchanged. This model is built on three sociologically intuitive mechanisms: 1) the symmetry mechanism (Emerson, 1976), i.e., individuals tend to be friendly (unfriendly resp.) to those who are friendly (unfriendly resp.) to themselves; 2) the influence mechanism (Friedkin and Johnsen, 2011), i.e., any individual i's appraisal of individual j is influenced by individual i's friends' and enemies' appraisals of individual j; 3) the homophily mechanism (Lazarsfeld and Merton, 1954), i.e., individual i tends to be friendly to individual j if they have similar appraisals of others. Such a model is referred to as the *symmetry-influence-homophily* (SIH) dynamics and is formally defined as follows.

Definition 7. (SIH dynamics). Given any initial appraisal matrix $X(0) \in \{-1,0,1\}^{n\times n}$ such that $\vec{\mathcal{G}}(X(0))$ is bilateral, the evolution of $X(t)$ is defined by the following stochastic process. At each time $t \in \mathbb{N}$, randomly pick a link (i,j) with $X_{ij}(t) \neq 0$,

(1) if there does not exists $k \in V\backslash\{i,j\}$ such that $X_{ik}(t)X_{jk}(t) \neq 0$, then update $X_{ij}(t)$ according to the symmetry mechanism, i.e.,
$$X_{ij}(t+1) = X_{ji}(t);$$
(2) if there exists $k \in V\backslash\{i,j\}$ such that $X_{ik}(t)X_{jk}(t) \neq 0$, i.e., if i and j have a common neighbor, then randomly pick such a common neighbor k and let
$$X_{ij}(t+1) = \begin{cases} X_{ji}(t), & \text{with probability } p_1,\\ X_{ik}(t)X_{kj}(t) & \text{with probability } p_2,\\ X_{ik}(t)X_{jk}(t) & \text{with probability } p_3,\end{cases}$$
for some $p_1 > 0$, $p_2 > 0$, and $p_3 > 0$ with $p_1 + p_2 + p_3 = 1$. These three equations above correspond to the symmetry mechanism, the influence mechanism, and the homophily mechanism respectively.

All the other links remain unchanged from t to $t+1$.

For SIH dynamics, X^* is an equilibrium if X^* is not changed under any possible update given in Definition 7. The equilibria of the SIH dynamics are characterized below.

Proposition 8. (Equilibrium set). For any appraisal matrix $X \in \{-1, 0, 1\}^{n \times n}$, X is an equilibrium of the SIH dynamics if and only if $\vec{\mathcal{G}}\ (X)$ is triad-wise balanced.

Now we present the main theorem on the almost-sure convergence of the SIH dynamics to triad-wise structural balance.

Theorem 9. (Convergence to triad-wise structural balance). Consider the SIH dynamics given by Definition 7. For any initial condition $X(0) \in \{-1, 0, 1\}^{n \times n}$ such that $\vec{\mathcal{G}}\ (X(0))$ is bilateral, the trajectory $X(t)$ almost surely reaches an equilibrium, i.e., a triad-wise balanced configuration, in finite time.

Regarding the convergence to two-faction structural balance, we have the following result based on Theorem 6(iii).

Proposition 10. (Convergence to two-faction balance). Consider the SIH dynamics given by Definition 7 with any initial condition $X(0) \in \{-1, 0, 1\}^{n \times n}$ such that $\vec{\mathcal{G}}\ (X(0))$ is bilateral. The appraisal network $\vec{\mathcal{G}}\ (X(t))$ almost surely reaches in finite time a two-faction balanced configuration, which is an equilibrium of the SIH dynamics, if and only if $|\mathcal{G}|(X(0))$ satisfies that, for any maximal cyclic subgraph with $m > 3$ nodes,

(1) it contains a simple cycle $(i_1, \dots, i_m)$ that passes through all its nodes and it is triad-connected;
(2) for any $j, k \in \{1, \dots, m\}$ with $X_{jk}(0) \neq 0$ and $k \geqslant j + 2$, either the simple cycle $(i_j, i_{j+1}, \dots, i_k)$ or $(i_1, \dots, i_j, i_k, \dots, i_m)$ is triad-connected.

4. CONCLUSION

This paper studies how an interpersonal appraisal network converges to a non-all-to-all structural balance configuration. We first introduce two well-justified definitions of non-all-to-all structural balance: the triad-wise structural balance and the two-faction structural balance, and establish the graph-theoretic conditions for their equivalence. We then propose a discrete-time gossip-like dynamics model of interpersonal appraisals referred to as the Symmetry-Influence-Homophily (SIH) dynamics. We conduct a comprehensive analysis of the dynamical behavior of this model. We prove that the set of equilibria of the SIH dynamics is equal to the set of all the possible triad-wise structural balance configurations. Moreover, we prove that for any initial condition, the appraisal networks achieves a triad-wise structural balance configuration almost surely in finite time. We also provide a sufficient condition for the almost-sure convergence to two-faction structural balance in finite time. Future research directions include the investigation on the conditions for the convergence of the SIH dynamics to all-friendly appraisal networks, a special class of structural balance configurations. It is also of research value to consider extending the applicability of the SIH dynamics to weighted signed graphs.

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