Viscoelastic truss metamaterials as time-dependent generalized continua

Journal Article

Author(s):
Glaesener, Raphaël N.; Bastek, Jan-Hendrik; Telgen, Bastian; Gonon, Frederick; Kannan, Vignesh; Spöttling, Ben; Steiner, Stephan; Kochmann, Dennis M.

Publication date:
2021-11

Permanent link:
https://doi.org/10.3929/ethz-b-000498808

Rights / license:
Creative Commons Attribution 4.0 International

Originally published in:
Viscoelastic truss metamaterials as time-dependent generalized continua

Raphaël N. Glaesener, Jan-Hendrik Bastek, Frederick Gonon, Vignesh Kannan, Bastian Telgen, Ben Spöttling, Stephan Steiner, Dennis M. Kochmann

Mechanics & Materials Lab, Department of Mechanical and Process Engineering, ETH Zürich, 8092 Zürich, Switzerland

ABSTRACT

Mechanical metamaterials provide tailorable functionality based on a careful combination of base material and structural architecture. Truss-based metamaterials, e.g., exploit structural topology and beam geometry to achieve beneficial mechanical and physical properties from stiffness and wave dispersion to strength and toughness. While the focus to date has been primarily on static metamaterial properties or elastic wave motion, 3D-printed polymeric base materials naturally come with significant viscoelasticity, making the effective truss response time- and rate-dependent. Here, we report a theoretical-numerical-experimental study which (i) deploys a linear viscoelastic corotational beam description (capturing finite rotations at small strains), (ii) implements the latter in a finite element framework, (iii) calibrates a generalized Maxwell model based on viscoelastic experiments on 3D-printed polymer samples, (iv) validates the theory and implementation through experimental truss benchmark tests, and (v) introduces a generalized continuum formulation for the efficient simulation of viscoelastic truss metamaterials containing large numbers of structural members. We show that the viscoelastic beam approach, calibrated via tension tests on individual strut samples, performs well when applied to complex truss lattices undergoing time-dependent stress relaxation — as verified by the effective mechanical response and full-field deformation maps. The resulting variational generalized continuum framework uses on-the-fly periodic homogenization based on a representative unit cell and is extended to dynamics by including inertial effects. By comparison to discrete numerical simulations we demonstrate the accuracy of the continuum approach, which is promising for modeling and optimizing 3D-printed truss metamaterials for engineering applications from shock-absorbing structures to rate-dependent architected materials and soft robotics.

1. Introduction

Architectural design plus base material determine the effective mechanical properties of metamaterials based on, e.g., trusses (Wadley et al., 2008; Meza et al., 2015), plates and shells (Zheng et al., 2018; Bonatti and Mohr, 2019), or spinodal structures (Vidyasagar et al., 2018; Kumar et al., 2020). The available design space has enabled metamaterials with optimized, application-dependent static (Surjadi et al., 2019) and dynamic properties, the latter including shock mitigating by impact energy absorption (Schaedler et al., 2014; Frenzel et al., 2016) and wave guiding (Zelhofer and Kochmann, 2017). As additive manufacturing is the primary means of fabrication, versatile 3D printing materials have been used as base materials for metamaterials, which requires detailed understanding of their mechanical behavior to accurately design, model, and optimize the engineered metamaterial
response. While for additively manufactured trusses made of metallic (Zadi-Maada et al., 2018) and ceramic base materials (Zocca et al., 2015) the assumption of linear elasticity is oftentimes sufficient, compliant polymers require rate-dependent models to describe the complex time-dependent behavior and history dependence (Gibson et al., 2010; Wang et al., 2017). Although being a technical complication for modeling, the viscoelastic relaxation, creep, frequency-dependent harmonic or generally time-dependent response of such architectures facilitates fascinating new features and applications (Gomez et al., 2019) ranging from rate-dependent buckling patterns and bistable beams (Dykstra et al., 2019; Janbaz et al., 2020) to advanced vibration damping (Wang and J., 2018) and frequency control and dispersion tuning in soft phononic crystals (Frazier and Hussein, 2015; Parnell and De Pascalis, 2019). Seizing the opportunities provided by viscoelastic metamaterials requires accurate and efficient models for the design space exploration and property optimization.

While the classical theory of viscoelasticity was laid out long time ago (see e.g. Schapery (1969), Taylor et al. (1970), Christensen (1982), Lakes (1999)), the modeling of viscoelastic structural members is a more recent topic, including, e.g., finite element (FE) models for viscoelastic plates (Marques and Creus, 1994; Yi and Hilton, 1994), beams (Hilton, 2009), and sandwich structures (Galucio et al., 2004). Rate-dependent trusses were studied, among others, by Ghayesh et al. (2016) who studied the viscoelastic response of a single Euler–Bernoulli beam, whereas Hamed (2012) focused on a 2D Timoshenko beam, and Bottoni et al. (2008) modeled linear viscoelastic thin-walled beams. Recently, Ananthapadmanabhan and Saravanaan (2020) reported numerical techniques to calculate the response of nonlinear viscoelastic truss networks (yet ignoring bending strains). In addition, Lestringant et al. (2020) and Lestringant and Kochmann (2020) introduced a general formulation of slider, geometrically exact beams made of viscoelastic base materials, which has not yet been applied to trusses.

With the objective of providing a simple and accurate truss description and an efficient finite element treatment, we here extend the corotational beam element of Crisfield (1990) to linear viscoelastic beams, which accounts for rate dependence in the axial, flexural and torsional stress–strain relations (based on a general one-dimensional linear viscoelastic constitutive law) and accounts for finite rotations. Implementation of the model in a FE framework allows us to simulate the linear viscoelastic response of complex truss networks in two and three dimensions (2D and 3D, respectively). Validation is realized by comparison to relaxation experiments on 3D-printed 2D hexagonal trusses, whose generalized Maxwell model parameters are extracted from tensile tests. The comparison between simulations and experiments on the strained hexagon lattice shows excellent agreement, which confirms the applicability of our viscoelastic corotational beam formulation to polymeric trusses.

Even though we present an efficient truss description, the cost of solving boundary problems involving thousands to millions of struts inside a truss becomes prohibitive, so multiscale techniques become beneficial (Kochmann et al., 2019). Here, we incorporate the linear viscoelastic beam into a generalized continuum formulation, which was previously introduced for linear elastic beam networks (Glaesener et al., 2019, 2020). The discrete truss is replaced by a continuous body, whose effective mechanical constitutive behavior is obtained from on-the-fly numerical homogenization of a representative unit cell, and whose deformation is solved by a finite element calculation on the macroscale (comparable to FE²). This two-scale model captures finite strains on the macroscale (accommodated by finite rotations of truss members on the microscale), and it applies to stretching- and bending-dominated truss topologies. Unlike Vigliotti et al. (2014) and Pal et al. (2016), who pursued similar approaches in 2D but with rotational degrees of freedom condensed out on the microscale, we introduce both translational and rotational degrees of freedom on the macroscale and pass those to the representative unit cell. We compare simulation results of discrete numerical calculations to those of the two-scale generalized continuum approximation for several 3D relaxation and vibration benchmarks, using a selection of bending- and stretching-dominated truss topologies, which demonstrate the homogenization scheme’s accuracy and efficiency as well as its applicability. In addition, our results highlight the exciting opportunities for exploiting viscoelastic base materials for the design of time-dependent metamaterial properties.

The remainder is structured as follows. Section 2 lays out the linear viscoelastic beam theory, introduces its FE implementation through corotational beam elements, and summarizes the formulation of the two-scale generalized continuum representation of viscoelastic trusses. Section 3 validates the viscoelastic truss model by comparing simulated results to experimental relaxation tests on 2D hexagonal trusses. 3D quasistatic stress relaxation and dynamic vibration examples in Section 4 demonstrate good agreement between fully resolved discrete numerical calculations and efficient FE simulations based on the two-scale representation, before Section 5 concludes our study.

2. Linear viscoelastic corotational beams

We consider slender beams made of a linear viscoelastic material in 3D, which we describe by an extension of the corotational beam formulation originally introduced by Crisfield (1990) for linear elastic Euler–Bernoulli beams. Owing to the linear constitutive relations and the resulting applicability of Boltzmann's superposition principle, the formulation and numerical implementation of our viscoelastic beams follows that of their linear elastic counterparts and, particularly, admits the decoupling of stretching, flexural and torsional stress and strain components.

2.1. Viscoelastic beam theory

We assume our corotational beam to be sufficiently slender, so it experiences a linear combination of axial strains due to stretching and bending, on the one hand, and shear strains due to torsion, on the other hand, while we neglect all further strain components. Let the x-axis be locally aligned with the neutral axis of the beam (Fig. 1(a)), so that

\[ \varepsilon_{xx}(x,t) = \varepsilon_{ax}(x,t) - y \kappa_x(x,t) + z \kappa_y(x,t) \] (1)
Owing to the linearity of the constitutive relations, we may reduce the number of internal variables to \( n \) per cross-section, viz. \( \sigma_{xx}^k, \sigma_{yy}^k, \sigma_{zz}^k, \tau_{xy}^k, \tau_{xz}^k, \tau_{yz}^k \) with \( k = 1, \ldots, n \), by defining

\[
\varepsilon_p^k(x,t) = \varepsilon_{ax}^k(x,t) - \varepsilon_{ap}^k(x,t), \quad \omega_p^k(x,t) = \gamma_p^k(x,t) - \gamma_{ap}^k(x,t). 
\]

Owing to the linearity of the constitutive relations, we may reduce the number of internal variables to 4n per cross-section, viz. \( (\varepsilon_{ax}^k, \varepsilon_{ay}^k, \varepsilon_{az}^k, \tau_{xy}^k, \tau_{xz}^k, \tau_{yz}^k) \) with \( k = 1, \ldots, n \), by defining

\[
\varepsilon_p^k(x,t) = \varepsilon_{ax}^k(x,t) - y k_{ax}^k(x,t) + z k_{az}^k(x,t) \quad \text{and} \quad \gamma_p^k(x,t) = \sqrt{y^2 + z^2}.
\]
Insertion into (6) yields the kinetic relations for the reduced set of internal variables for each of the $n$ Maxwell elements, which decouple into

$$
\tau_k e_{ax,p}^k(x,t) = \epsilon_{ax}(x,t) - \epsilon_{ax,p}^k(x,t), \quad \tau_k k_{z,p}^k(x,t) = k_y(x,t) - k_{z,p}^k(x,t), \quad \tau_k \dot{k} z_{p}^{k}(x,t) = \kappa_z(x,t) - \kappa_{z,p}^k(x,t)
$$

and

$$
o_{k,p} \dot{I}_p^k(x,t) = \theta'(x,t) - \theta_{p}^k(x,t).
$$

As we deal with general, non-monotonic time histories, it is convenient to introduce a time-incremental setting with constant time steps of size $\Delta t = \tau^p - \tau^t$, so that $\tau^p = a \Delta t$ and $\tau^t = 0$. Applying a backward-Euler finite-difference scheme leads to

$$
e^{k}_{ax}(x,\tau^{p+1}) = e^{k}_{ax}(x,\tau^{p}) + \frac{1}{1 + \tau^k/\Delta t} \left[ e^{k}_{ax}(x,\tau^{p+1}) - e^{k}_{ax}(x,\tau^{p}) \right],
$$

$$
\kappa^{k}_{z,p}(x,\tau^{p+1}) = \kappa^{k}_{z,p}(x,\tau^{p}) + \frac{1}{1 + \tau^k/\Delta t} \left[ \kappa^{k}_{z,p}(x,\tau^{p+1}) - \kappa^{k}_{z,p}(x,\tau^{p}) \right], \quad \xi \in \{y, z\},
$$

$$
I^k_p(x,\tau^{p+1}) = I^k_p(x,\tau^{p}) + \frac{1}{1 + \omega_{k}/\Delta t} \left[ \theta^{k}(\tau^{p+1}) - \theta_{p}^k(x,\tau^{p}) \right].
$$

Inserting (7) into (5), applying the finite difference discretization and applying (10) leads to an explicit solution for the stress at the new time $\tau^{p+1}$:

$$
\sigma_{xx}(x,\tau^{p+1}) = \left( E_{\infty} + \sum_{k=1}^{n} \frac{E_k}{1 + \Delta t/\tau_k} u'_{x}(x,\tau^{p+1}) - \sum_{k=1}^{n} \frac{E_k}{1 + \Delta t/\tau_k} e^{k}_{ax,p}(x,\tau^{p}) \right) - y \left( E_{\infty} + \sum_{k=1}^{n} \frac{E_k}{1 + \Delta t/\tau_k} u''_{x}(x,\tau^{p+1}) - \sum_{k=1}^{n} \frac{E_k}{1 + \Delta t/\tau_k} \kappa^{k}_{z,p}(x,\tau^{p}) \right)
$$

and analogously

$$
\sigma_{yy}(x,\tau^{p+1}) = \frac{1}{2} \sqrt{y^2 + z^2} \left[ G_{\infty} + \sum_{k=1}^{n} \frac{G_k}{1 + \Delta t/\omega_k} \theta'(x,\tau^{p+1}) - \sum_{k=1}^{n} \frac{G_k}{1 + \Delta t/\omega_k} I^k_p(x,\tau^{p}) \right].
$$

The normal force $N$, the two bending moments $M_y$ and $M_z$ as well as the torsional moment $M_z$ follow from integration over the cross-section as

$$
N = \int_A \sigma_{xx} y \, dy \, dz, \quad M_y = \int_A \sigma_{xx} z \, dy \, dz, \quad M_z = - \int_A \sigma_{xx} y \, dy \, dz, \quad M_x = \int_A \sigma_{yy} \sqrt{y^2 + z^2} \, dy \, dz.
$$

The above model can be cast into a variational framework by exploiting variational constitutive updates (Ortiz and Stainier, 1999; Lestringant et al., 2020). Integration of the effective energy density of Ortiz and Stainier (1999) over the cross-section leads to an incremental energy density per beam length, which depends on the internal variables from the previous time step only:

$$
W_b(x,\tau^{p+1}) = \frac{E_{\infty}}{2} \left[ A_b \left( u''_{x}(x,\tau^{p+1}) \right)^2 + I_y \left( u''_{y}(x,\tau^{p+1}) \right)^2 + I_z \left( u''_{z}(x,\tau^{p+1}) \right)^2 + 2 I_{yz} \left( u''_{y}(x,\tau^{p+1}) u''_{z}(x,\tau^{p+1}) \right) \right]
$$

$$
+ \sum_{k=1}^{n} \frac{E_k}{2(1 + \Delta t/\tau_k)} \left[ A_b \left( u''_{x}(x,\tau^{p+1}) - e^{k}_{ax,p}(x,\tau^{p}) \right)^2 + I_y \left( u''_{y}(x,\tau^{p+1}) - k^{k}_{z,p}(x,\tau^{p}) \right)^2 \right]
$$

$$
+ I_z \left( u''_{z}(x,\tau^{p+1}) + k^{k}_{y,p}(x,\tau^{p}) \right)^2 + 2 I_{yz} \left( u''_{y}(x,\tau^{p+1}) - k^{k}_{z,p}(x,\tau^{p}) \right) \left( u''_{y}(x,\tau^{p+1}) + k^{k}_{p,z}(x,\tau^{p}) \right) \right] + \frac{G_{\infty}}{2} \left[ \theta'(x,\tau^{p+1}) - I^k_p(x,\tau^{p}) \right]^2
$$

with the area and area moments of inertia defined by

$$
A_k = \int_A \, dy \, dz, \quad I_y = \int_A y^2 \, dy \, dz, \quad I_z = \int_A z^2 \, dy \, dz, \quad I_{yz} = \int_A yz \, dy \, dz, \quad I_{xyz} = \int_A (y^2 + z^2) \, dy \, dz.
$$

This completes the description of our linear viscoelastic corotational beam. Without finite rotations, the above framework admits closed-form solutions for monotonic loading and various boundary and initial conditions. Under finite rotations, the framework still applies (Crisfield, 1990; Crisfield and Cole, 1990) but the $y$-axis refers to the current, deformed center-line of the beam, so that the resulting nonlinear governing equations require a numerical formulation for solving initial boundary value problems.

### 2.2 Finite element setting

Following the corotational beam formulation of Crisfield (1990) and its numerical implementation by Philpot and Kochmann (2019), we represent a beam as a discrete set of segments separated by nodes, whose deformed and undeformed locations in space are given by, respectively, position vectors $X \in \mathbb{R}^d$ and $x \in \mathbb{R}^d$ in $d$ dimensions. The deformed and undeformed nodal orientations are
defined, respectively, by rotation vectors \( \theta \in \mathbb{R}^{2d-3} \) and \( \Theta \in \mathbb{R}^{2d-3} \). Altogether, the degrees of freedom of a node \( i \) are abbreviated by the vector of nodal degrees of freedom, \( \mathbf{x}_i = (x^i, \theta)^T \in \mathbb{R}^{3d-i} \). For a beam segment with nodes at positions \( x^1 \) and \( x^2 \) in the current configuration, the deformed orientation of the beam is defined by \( I = (x^2 - x^1)/l \) with the deformed segment length \( l = |x^2 - x^1| \) (whereas \( L = |x^2 - x^1| \) denotes the undeformed segment length). To account for finite rotations of the beam that do not contribute to the beam's deformation, we define by \( \psi^i \) the rotations of node \( i \) relative to the current beam orientation \( I \), defined separately for each segment (for details see, e.g., Philpot and Kochmann (2019)).

Let \( \psi^y \) and \( \psi^z \) denote the angles of rotation at node \( i \) about the \( y \)- and \( z \)-axes, respectively, with respect to the current orientation \( I \) of a segment, and let \( \psi^x \) denote the twist angle at node \( i \). The FE interpolation is based on linear ansatz functions for the axial displacement \( u_i(x) \) and twist \( \theta(x) \), while third-order Hermite polynomial ansatz functions are used for the transverse displacements \( u_{ij}(x) \) and \( u_{ij}(x) \) of the beam's center-line (which includes the exact solution for a beam loaded at its end points). Following a Rayleigh–Ritz approach, inserting those ansatz functions (with the essential boundary conditions enforced at nodes) into (14) and integrating over the beam segment yields the nodal reactions by differentiating the resulting solution with respect to the nodal degrees of freedom. This includes the axial forces

\[
N^1(t^{e+1}) = -N^2(t^{e+1}) = E_{\text{eff}} A_k \frac{u^y_i(t^{e+1}) - u^y_i(t^{e+1})}{L} - \sum_{k=1}^n \bar{E}_k A_k \epsilon_{\text{ax},p}^k (t^a)
\]

with moduli

\[
E_{\text{eff}} = E_\infty + \sum_{k=1}^n \bar{E}_k \quad \text{and} \quad \bar{E}_k = \frac{E_k}{1 + \tau_k / r_k}.
\]

where we point out that the axial strain is constant along the beam segment, so one internal variable \( \epsilon_{\text{ax},p}^k \) per segment is sufficient (and the same applies to the torsional internal variables \( \epsilon_{\text{t},p}^k \)). Assuming that the coordinate axes align with the principal axes of the beam’s cross-section (so \( I_{yz} = 0 \)), the nodal bending and torsional moments are obtained as

\[
\begin{pmatrix}
M^x_1(t^{e+1}) \\
M^x_2(t^{e+1}) \\
M^z_1(t^{e+1}) \\
M^z_2(t^{e+1})
\end{pmatrix} = \frac{1}{L} \begin{pmatrix}
G_{\epsilon I_p} & 0 & 0 & -G_{\epsilon I_p} & 0 & 0 \\
0 & 4E_{\text{eff}}I_y & 0 & 0 & 2E_{\text{eff}}I_y & 0 \\
0 & 0 & 4E_{\text{eff}}I_z & 0 & 0 & 2E_{\text{eff}}I_z \\
0 & 2E_{\text{eff}}I_y & 0 & 0 & 4E_{\text{eff}}I_y & 0 \\
0 & 0 & 2E_{\text{eff}}I_z & 0 & 0 & 4E_{\text{eff}}I_z
\end{pmatrix} \begin{pmatrix}
\epsilon_{\text{ax},p}^1(t^{e+1}) \\
\epsilon_{\text{ax},p}^2(t^{e+1}) \\
\epsilon_{\text{ax},p}^3(t^{e+1}) \\
\epsilon_{\text{ax},p}^4(t^{e+1}) \\
\epsilon_{\text{ax},p}^5(t^{e+1}) \\
\epsilon_{\text{ax},p}^6(t^{e+1})
\end{pmatrix}
\]

with moduli

\[
G_{\epsilon I} = G_\infty + \sum_{k=1}^n \bar{G}_k \quad \text{and} \quad \bar{G}_k = \frac{G_k}{1 + \tau_k / r_k}.
\]

For the linear viscoelastic bending contribution, one set of internal variables \( \{k_{\text{t},p}^{ij}, k_{\text{t},p}^{ij}\} \) is required per node \( i \) to capture the history dependence. All nodal reactions must be rotated from the deformed frame to the undeformed reference frame within a Lagrangian FE setup (see Crisfield (1990)).

The above linear viscoelastic corotational beam element is implemented within a massively-parallel in-house computational mechanics code, so that we may subsequently use it either for discrete numerical calculations or within the generalized continuum framework described in the following. For quasistatic simulations, a Newton–Raphson solver is used, whereas dynamic problems are solved by a Newmark-\( \beta \) implicit time integration scheme (Newmark, 1959) with the lumped mass matrix (Iura and Atluri, 1995) for the linear beam element. A validation example is presented in Appendix A, which simulates a viscoelastic cantilever beam utilizing both the above linear viscoelastic corotational beam formulation and a fully-resolved FE description based on solid elements for comparison.

### 2.3. Modeling viscoelastic trusses as generalized continua

We deploy the above linear viscoelastic beam description in our recently introduced generalized continuum framework (Glaesener et al., 2019, 2020), previously used only for static, linear elastic beam networks. The objective is to considerably reduce the computational costs associated with fully resolved discrete calculations involving large numbers of beams. Our strategy is to replace the periodic truss by an effective continuum whose response is approximated by 3D solid finite elements with the effective constitutive behavior on the macroscale obtained from a nested FE calculation on a representative unit cell (RUC) on the microscale — assuming a separation of scales. Efficiency is gained by our FE\(^2\)-type formulation (Glaesener et al., 2019), which uses periodic homogenization on the microscale, while allowing for consistent analytical tangents to be used in implicit macroscale FE simulations in 3D (Glaesener et al., 2020). Fig. 2 illustrates the general procedure, which we briefly summarize here.
On the macroscale, the deformation of a body \( \Omega \in \mathbb{R}^d \) in \( d \) dimensions is described by a deformation mapping \( \varphi(x,t) : \Omega \times [0, T] \rightarrow \mathbb{R}^d \) and a rotation field \( \theta(x,t) : \Omega \times [0, T] \rightarrow S^2 \), where \( \theta(x,t) \) is a rotation tensor. The deformation gradient tensor is \( F = \nabla \varphi \), while the curvature tensor is \( \kappa = \nabla \theta \) (with gradients defined with respect to the undeformed configuration). The finite-strain formulation is required to capture finite rotations. Each required constitutive model evaluation on the macroscale at a quadrature point \( X_q \) passes the kinematic quantities \( \Lambda(X_q) = (\varphi(X_q), F(X_q), \theta(X_q), \kappa(X_q)) \) down to a RUC, which consists of the smallest periodic unit representing the effective truss response (either a single unit cell or, if buckling gains importance, an array of unit cells, sufficiently large to capture the relevant buckling modes; see Glaesener et al. (2019)). In a nested RUC calculation, \( \Lambda(X_q) \) is applied on average to the RUC while enforcing periodic boundary conditions. Owing to the variational structure of our model, the resulting RUC energy is computed as the volume-averaged RUC energy, whose derivatives provide all required stress (and couple stress) measures on the macroscale along with their consistent tangent matrices. The interested reader is referred to Glaesener et al. (2019, 2020) for further information and implementation details (for linear elastic trusses). The extension to linear viscoelasticity implements internal variables in each beam segment within the RUC and hence stores those variables independently for each quadrature point on the macroscale. Otherwise, the numerical setup for linear viscoelastic trusses is analogous to that of linear elastic trusses. As an explicit form for the updated internal variables is available and no numerical condensation is required, the numerical efficiency is marginally affected by viscoelasticity, while computational storage is increased due to the requirement for storing internal variables.

As a further extension of our generalized continuum approach, we consider dynamic effects, which requires the introduction of effective inertia within the macroscale FE problem. We introduce the relative density (fill fraction) of the RUC as

\[
\rho^* = \frac{\sum_{b \in \text{RUC}} A_{bi} L_i}{V_{\text{RUC}}},
\]

where \( A_{bi} \) is the \( i \)th beam’s cross-sectional area, \( L_i \) its undeformed length, and \( \rho \) the density of the beam’s base material. \( V_{\text{RUC}} \) is the RUC volume, and we ensure that beams lying on the RUC boundary are uniquely accounted for. The total mass of a finite element \( e \) of volume \( V_e \) follows as \( m_e = \rho^* V_e \), which in turn defines the translational inertia in the lumped mass matrix of the macroscale element as

\[
M_{\text{lumped,trans}} = \frac{m_e}{n_e} I
\]

where \( n_e \) represents the number of nodes of the macroscale element. The rotational inertia of the effective lumped mass matrix of a macroscale finite element is established by matching the kinetic energy \( E_e \) of element \( e \) with that of the underlying RUC, \( E_{\text{RUC}} \), each normalized by its respective volume \( (E_e/V_e = E_{\text{RUC}}/V_{\text{RUC}}) \), which yields

\[
\frac{1}{2} \mathbf{\omega} \cdot M_{\text{rot}} \mathbf{\omega} = \frac{V_e}{2V_{\text{RUC}}} \sum_{i \in \text{RUC}} \omega_{bi} \cdot I_{bi} \omega_{bi},
\]

where \( I_{bi} \) is the (diagonal) mass moment of inertia tensor of the \( i \)th beam in the RUC and \( \omega_{bi} = (\dot{\varphi}_{b1}, \dot{\varphi}_{b2}, \dot{\varphi}_{b3}) \) is the angular velocity vector associated with its two end points 1 and 2. Analogously, \( M_{\text{rot}} \) is the effective (diagonal) lumped mass matrix of the macroscale finite element associated with rotations only, with \( \mathbf{\omega} = (\dot{\varphi}_{1}, \ldots, \dot{\varphi}_{n_e}) \) containing the angular velocity vectors of all \( n_e \) nodes of the element. In 3D, rotations are accounted for about all three axes, whereas in 2D only a single (in-plane) rotational degree of freedom must be considered. The thus-obtained lumped mass matrix of an individual beam and of the macroscale finite element (including both translational and rotational contributions) is provided in Appendix B along with an illustration of the principle of matching kinetic energies between macro- and microscales.

We acknowledge that this approximation of matching the effective inertia in the continuum representation is only suitable in the long-wavelength limit, i.e., for relatively slow dynamic processes that engage the inertia of the overall truss rather than exciting resonances of individual truss members (e.g., we do not claim to capture wave dispersion with this approach). Yet, it will prove effective in computing the dynamic and vibrational response of large truss structures at low frequencies.
Fig. 3. Snapshots of experiments 1 (left) and experiment 2 (right), each comparing the respective undeformed to the deformed truss at the maximum applied average strain.

The generalized continuum model is implemented inside an in-house 3D FE framework and uses 8-node brick elements with tri-linear interpolation along with the nonlinear Newton–Raphson solver from the Portable, Extensible Toolkit for Scientific Computation (PETSc) (Abhyankar et al., 2018; Balay et al., 1997, 2019, 2020). Dynamic simulations use implicit Newmark-\(\beta\) time integration of the FE equations of motion.

3. Model calibration and validation by experiments

Before assessing the continuum approximation, we demonstrate the applicability and accuracy of the viscoelastic beam formulation of Sections 2.1 and 2.2 by simulating a discrete, 3D-printed polymeric truss. We choose the representative example of a 2D hexagonal truss lattice made of a thermoplastic elastomer by selective laser sintering (SLS) on a Sintratec S1 printer. The hexagonal topology combines stretching and bending deformation, making it a valuable benchmark. (We deliberately choose a 2D example that admits detailed extraction of the truss deformation, rather than using a 3D truss with the added complication of imaging a cellular 3D structure). Fig. 3 illustrates samples clamped at their top and bottom and stretched under displacement control to total vertical strains of 8.1% (experiment 1) and 14.3% (experiment 2), leading to viscoelastic stress relaxation over time.

To calibrate the Prony series of the base material, we performed separate uniaxial extension experiments on dog-bone samples and fitted the simulated response force vs. time of the generalized Maxwell model to the experimental force–time history, leading to the Prony series detailed in Appendix C along with all material parameters.

The undeformed truss in Fig. 3 measured 94mm \(\times\) 55.4mm with individual beams being \(L = 8\) mm long with a measured beam width of \(W = 0.66\) mm and hence a slenderness ratio of \(\lambda = W/L = 0.0825\). The truss had a constant out-of-plane thickness of \(T = 5.0\) mm, so that deformation is approximately 2D (and out-of-plane displacements can be neglected in simulations). The sample was mounted on a tensile testing machine (Instron Model 5943, 1kN force capacity) using custom clamps, while a 450N load cell (mounted to the bottom clamp) measured the total vertical force applied to the truss. A high-speed camera (Photron AX200) was used to image the truss during deformation at a frame rate of 50 frames per second. Images were processed to track the center-lines of individual beams and the positions of nodes, using a custom-built image processing code. We define the (average) vertical strain of the truss as the relative displacement of the top- and bottom-most nodes on the vertical center-line of the sample normalized by the initial height of the sample (see Fig. 3). Note that, before applying any tensile strain, the starting configuration was already pre-stretched due to gravity (and this pre-stretched truss is used as the initial configuration in simulations).

Two sets of experiments were performed with maximum tensile strains of \(\varepsilon = 8.1\%\) and \(\varepsilon = 14.3\%\); strain histories are shown in Fig. 4. Each experiment starts with a constant uniaxial displacement rate of 100 mm/min applied to the clamped top edge of the truss (section \(\mathcal{a} - \mathcal{b}\) in Fig. 4), followed by the strain being held constant (\(\mathcal{b} - \mathcal{c}\)), until the structure is brought back to its original configuration (\(\mathcal{c} - \mathcal{d}\)) at a rate of 10 mm/min in experiment 1 and 100 mm/min in experiment 2 (all circled labels refer to the points marked in Fig. 4).

A visual inspection of the deformed trusses at the respective maximum strains, shown in Fig. 5, indicates a remarkable match between experimental observations and simulation results. The total vertical forces measured during the strain paths of Fig. 4 are summarized in Fig. 6. After an initial nonlinear rise of the tensile force, typical viscoelastic relaxation behavior is observed (\(\mathcal{b} - \mathcal{c}\)), before unloading sets in. Simulations use the linear viscoelastic corotational beam element (with 10 elements per strut) and the Prony series of Table C.3, and they demonstrate an overall convincing agreement with experiments. The error between numerical
Fig. 4. Applied average strain vs. time up to maximum strains of $\varepsilon = 8.1\%$ and $\varepsilon = 14.3\%$ in experiments 1 and 2, respectively. The tensile loading rate is 100 mm/min for both experiments, unloading rates differ (10 mm/min in experiment 1 and 100 mm/min in experiment 2). Simulations use the exact same strain histories to mimic experiments 1 and 2.

Fig. 5. Relaxation test of a 2D hexagonal truss loaded up to $\varepsilon = 8.1\%$ (left) and $\varepsilon = 14.3\%$ (right) in experiments 1 and 2, respectively. For both experiments, the initial truss configuration before loading (point $\mathbb{a}$) is compared to the deformed configuration at the indicated times (point $\mathbb{b}$). Gray trusses represent the simulation results, while black lines were extracted from experiments. Optical images of the two samples are shown in Fig. 3.

We evaluate the accuracy of the simulations for experiments 1 and 2 by the normalized root-mean-square error (NRMSE)

$$\text{NRMSE} = \sqrt{\frac{\sum_{i=0}^{N}(\hat{F}_i - F_i)^2}{\hat{F}_{\text{max}} - \hat{F}_{\text{min}}}}$$

where $\hat{F}_i$ are experimentally measured forces (for all $N$ time steps) and $F_i$ are the ones obtained from the discrete model; $T$ denotes the total number of data points. Normalization is performed over the total range of forces in the experiment (difference between

and experimental data, which is almost constant during the relaxation phase, is attributed to imperfections in the printed material and to nodal effects (Portela et al., 2018), which are not considered in simulations (nodes are rigid in our corotational setup). In addition, due to experimental constraints we lack knowledge of the Prony series for faster relaxation times (as explained in Appendix C).
i) Force-time response in experiment 1 (strain $\varepsilon = 8.1\%$)

Fig. 6. Force vs. time of the 2D hexagonal trusses of experiments (i) 1 and (ii) 2. The viscoelastic stress relaxation of the 3D-printed base material leads to the typical relaxation behavior between $\mathbf{b}$ and $\mathbf{c}$. Simulations based on the viscoelastic corotational beam element match experimental data to within a normalized root-mean-square error (NRMSE) of (i) $\sim 4.0\%$ and (ii) $\sim 15.48\%$, measured between $\mathbf{a}$ and $\mathbf{d}$.

maximum and minimum force values: $F_{\text{max}} - F_{\text{min}}$. The NRMSE between $\mathbf{a}$ and $\mathbf{d}$ evaluates to $\sim 4.0\%$ and $\sim 15.48\%$ in experiments 1 and 2, respectively. The error for experiment 1 is remarkably low, considering the moderate strains and resulting nonlinear deformation as well as the uncertainty in printing-induced imperfections. As expected, the error grows with increasing applied strain (cf. experiment 1 vs. experiment 2), which we attribute to increasing nodal effects and the decreasing legitimacy of the linearized-kinematics beam model — even though the deformed trusses in Fig. 5 show convincing agreement even at 14.3% strain. When stretching the structure up to 14.3% in experiment 2, especially those beams connected to the clamps undergo large strains that are likely to violate the small-strain assumption. The identically shaped but stiffer numerical force response leads to the conclusion that the elastic modulus $E_\infty$ is slightly lower in experiments, whereas the contributions of the $n$ dashpots in the Maxwell model match measurements well. A detailed analysis of the local strain and curvature values within individual struts in both experiments is provided in Appendix D, which reveals that local strain measures are significantly higher in experiment 2, which invalidates the assumption of linearized kinematics, hence leading to the observed larger error for experiment 2.

We point out that, like in any FE discretization, simulated results converge with $h$-refinement, as demonstrated in Fig. 7, presenting the total force vs. time for the hexagonal truss relaxation in experiment 1 for six different mesh resolutions (ranging from a single element per strut to 10 beam elements per strut). With increasing refinement, bending deformation is increasingly concentrated at nodes, which leads to larger rotations and hence higher bending moments at the nodes. As a consequence, we observe convergence from below. As we deem 10 elements per strut to show a sufficient level of convergence, we used this resolution in our comparison to experiments in Fig. 6. Overall, we see convincing agreement between simulations and experimental measurements — both in the full-field deformation maps of Fig. 5 and the force–time histories in Fig. 6, based on which we use the viscoelastic truss framework in the following to simulate more complex examples.
Fig. 7. Magnified view of the force–time relaxation curve in experiment 1 from Fig. 6(i), comparing the results of FE calculations with different levels of beam refinement to the experimental data (shown as mean plus standard deviation, the latter being indicated by the region shaded in red). The legend reports the normalized root-mean-square error (NRMSE) during stress-relaxation between $b$ and $c$.

Table 1

<table>
<thead>
<tr>
<th>Topology</th>
<th>Density $V_f/V_{UC}$</th>
<th>Slenderness ratio $\lambda$</th>
<th>Beam diameter $D$</th>
<th>Beam length $L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>0.833%</td>
<td>0.05945</td>
<td>0.05945 $L_{UC}$</td>
<td>$L_{UC}$</td>
</tr>
<tr>
<td>Octahedron</td>
<td>0.833%</td>
<td>0.05</td>
<td>0.03535 $L_{UC}$</td>
<td>$L_{UC}/\sqrt{2}$</td>
</tr>
<tr>
<td>Octet</td>
<td>0.833%</td>
<td>0.03535</td>
<td>0.02499 $L_{UC}$</td>
<td>$L_{UC}/\sqrt{2}$</td>
</tr>
<tr>
<td>Bitruncated Octahedron</td>
<td>0.833%</td>
<td>0.1</td>
<td>0.03535 $L_{UC}$</td>
<td>$\sqrt{2}/4L_{UC}$</td>
</tr>
</tbody>
</table>

4. 3D simulations and the generalized continuum approximation

Having verified the applicability of the viscoelastic beam model for 3D-printed trusses, we proceed to assess the validity of the generalized continuum approximation introduced in Section 2.3 as a reduced-order model for efficient truss simulations. In the following, we refer to the FE-type framework resulting from viscoelastic corotational beams used in the homogenized RUC employed at every quadrature point of the macroscale FE problem as two-scale model. As benchmarks, we illustrate the quasistatic relaxation of a 3D truss cube after torsional twisting as well as the dynamic response of the same truss undergoing damped free vibrations.

All struts in the trusses are described as viscoelastic corotational beams with a circular cross-section of diameter $D$, which gives the cross-sectional area $A_b = \pi D^2$, area moment of inertia $I = \pi D^4/4$, and polar moment of inertia $I_p = 2I$. To cover a range of stretching- and bending dominated trusses, we analyze the response of four different truss topologies based on cubic, octet, octahedral, and bitruncated octahedral unit cells (Porcela et al., 2018). For a fair comparison, we ensure a constant effective density, while individual struts are ensured to remain below a maximum slenderness ratio of $\lambda = D/L \leq 0.1$. All struts within each topology possess the same length $L$ and diameter $D$.

For each of the four topologies, we simulate the truss response both via discrete FE calculations (resolving all struts by corotational beam elements as in Section 3) and using the two-scale model within an efficient macroscale FE calculation based on a mesh of brick elements. The highly slender beams in the octahedral topology cause the effective homogenized body to behave in an approximately incompressible manner (Philpot and Kochmann, 2019), which causes convergence challenges and calls for specialized element types on the macroscale. We previously showed (Glaesener et al., 2020) that the two-scale approximation indeed captures instabilities and localizations well when applied to linear elastic trusses. To avoid the complexity of dynamically snapping structures, we here limit our benchmarks to relatively small strains at the macroscale which, however, yields representative examples that verified the applicability of the dynamic, viscoelastic continuum representation. For all benchmarks, we continue to use the previously calibrated Prony series of Table C.3.

4.1. Torsional relaxation test

We consider a cube filled by $40 \times 40 \times 40$ truss unit cells (each of side length $L_{UC} = 10$mm), which is undergoing torsional deformation. While the bottom of the cube is clamped to a rigid substrate, the top is free to move vertically but forced to undergo a
prescribed in-plane twisting motion. A visual comparison of the fully resolved discrete octet structure and the generalized continuum approximation used in the two-scale model is shown in Fig. 8.

After twisting the top at a constant rate of $72^\circ$/min up to a maximum angle of $3^\circ$ (section (1–2)), the top is held fixed (2–3) for 5s, before being rotated back to its initial state at a rate of $72^\circ$/min (3–4). Fig. 9 compares the average applied torque vs. time curves for both a fully resolved FE simulation of the discrete truss and the continuum approximation based on a mesh of $10 \times 10 \times 10$ brick elements and the smallest possible RUC $\Omega_RUC$ for all calculations, shown within the plots of Fig. 9.

Analogous to the results of the 2D relaxation experiment in Section 3, the viscoelastic base material leads to the relaxation of internal stresses and, as a result, to a decrease in the applied torque while keeping the twist angle constant. The torque-vs.-time curves for the different topologies in Fig. 9 show convincing agreement between the fully-resolved discrete calculations and the continuum approximation. We note that a proper representation of the cubic topology (Fig. 9d) within the two-scale approach requires an enlarged RUC, assembled from $2 \times 2 \times 2$ cubic unit cells. We attribute the too stiff response obtained form only a single cubic unit cell ($1 \times 1 \times 1$) to the suppression of bifurcation modes that arise in the soft, bending-dominated cubic topology at relatively low strains — comparable to what Glaesener et al. (2019) showed for hexagonal and triangular linear elastic structures in 2D. Interestingly, we observe an almost identical response of the stretching-dominated octet topology (Fig. 9c) and the octahedron (Fig. 9b), which is due to their identical effective stiffness matrices when both trusses have the same relative density (see Table E.4).

The error between the discrete truss calculation and the homogenized FE approximation strongly depends on the mesh size of the macroscale FE problem, as shown in Fig. 10. With $h$-refinement of the macroscale brick mesh, we see convergence from above of the error in the computed net torque to the solution of the discrete calculation. We note that, unlike in linear elasticity, the linear viscoelastic model’s dependence on the previous time step’s internal variables results in an accumulation of errors over time, so that an inappropriate FE mesh resolution may lead to significant errors over time.

4.2. Damped vibrations

Let us reconsider the cube-twist example of Section 4.1, now with the additional consideration of inertia. Instead of slowly twisting the cube after relaxation back in a displacement-controlled fashion, the truss is now released to dynamically return to its initial configuration under damped vibrations — with significant damping stemming from the viscoelastic base material (we ensure no numerical damping from the Newmark-$\beta$ scheme). Specifically, we twist a $30 \times 30 \times 30$ unit-cell truss at a rate of $89^\circ$/min up to $3.7^\circ$, then allow the structure to relax until $t = 5s$, and then remove all constraints at the top face instantaneously, leading to vibrations. The evolution of the average twist angle of the top face over time is shown in Fig. 11 for each of the four truss topologies.

As in the quasistatic examples, the results obtained from the homogenized continuum approximation show convincing agreement with those from the considerably more expensive fully-resolved truss calculations, confirming that inertial effects within the RUC at the microscale are successfully passed to the macroscopic FE problem. The cube topology again necessitates the use of a $2 \times 2 \times 2$ unit cell for an accurate agreement, so we use the same RUCs shown in Fig. 9. Also analogous to Section 2.3, the octet and octahedron show identical oscillations due to the same stiffness matrix and almost identical homogenized lumped mass matrices (only those terms associated with torsional inertia differ, yet those play only a minor role compared to the other inertial terms, see Appendix E).
Fig. 9. Simulated torque vs. time response, comparing the exact structural response of a cube-shaped truss made of 40 × 40 × 40 unit cells to that obtained from the two-scale model used in a 10 × 10 × 10 FE mesh. In both cases, the cube is initially twisted by 3° (a→b), then held fixed while stress relaxation occurs due to the viscoelastic base material (b→c), before being returned to the initial configuration (c→d). Results are presented for four types of truss topologies (bitruncated octahedron, octahedron, octet, and cube), all using the same linear viscoelastic base material based on the Prony series of Table C.3. All trusses have the same relative density (cf. Table 1).

Note that the underlying Prony series was identified from experiments without consideration of relaxation times τ < 0.3 s, which plays a significant role in high-frequency oscillations, so a re-calibration of the Maxwell model may be required to accurately account for such behavior if needed.

5. Conclusions

We have presented a framework of linear viscoelastic slender beams along with their numerical implementation as 3D corotational beam elements (undergoing finite rotations while considering linearized axial, flexural and torsional strains). By formulating the viscoelastic corotational beam in terms of reduced internal variables (one per each of the aforementioned strain contributions), the model becomes computationally tractable. A generalized Maxwell model was calibrated experimentally for the thermoplastic base material of 3D-printed truss by identifying the parameters of the associated Prony series from quasistatic uniaxial relaxation experiments. We have validated the model by comparing the simulated total force–time history and full-field deformation maps of 2D hexagonal truss lattices under tension to those of 3D-printed trusses. The comparison showed convincing qualitative and quantitative agreement with a normalized root-mean-square error as small as ∼ 4.0% for experiments with moderate strains.
Fig. 10. Relative error of the net torque applied to the twisted cube truss after relaxation (computed at point $c$ in Fig. 9), where the torque of an exact discrete truss calculation (having $40 \times 40 \times 40$ unit cells) is compared to the approximate torque obtained from the two-scale approximation with $h$ elements per side. Included in the log–log plot is a linear regression fit for each topology. Results in Fig. 9 correspond to $1/h = 1/10$.

Subsequently, we introduced the viscoelastic beam formulation into a generalized continuum description (previously available only for linear elastic trusses), which admits a computationally inexpensive finite element treatment based on a separation of scales between strut characteristics on the microscale and the macroscale body filled by the truss. This two-scale formulation was validated by comparison to discrete, fully-resolved truss calculations, showing agreement at small deformation for both the quasistatic relaxation behavior (relative errors smaller than 1.5%) as well as for damped vibrations (where the macroscale kinetic energy was matched with the homogenized kinetic energy of the truss unit cell).

Of course, our approach can be generalized and extended in various directions. While we focused on linear viscoelasticity, nonlinearity may have to be considered when the material shows inherent nonlinear constitutive behavior or when struts experience large local strains (Marques and Creus, 2012), as observed in the presented experiments, which show increasing differences from simulations with increasing levels of applied strain. Base materials typically also include viscoplastic effects at significant strain levels, which can in principle be included in an analogous reduced-order beam model, yet one may no longer assume linear strain variations across the cross-section, which leads to added complexity. Moreover, the kinetic energy of the generalized continuous body in the two-scale setup has been approximated by means of the long-wavelength kinetic energy of undeformed beams, which fails if, e.g., local resonance or wave dispersion effects must be accounted for or in case of large deformation or snapping instabilities. Despite those limitations, the presented model has provides an accurate and efficient reduced-order representation of complex viscoelastic trusses and truss-based architected materials.

CRediT authorship contribution statement

Raphaël N. Glaesener: Methodology, Software, Validation, Formal analysis, Investigation, Writing – original draft, Visualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

R.N.G. thanks Dr. Manuel Weberndorfer for supporting the numerical implementation.

Appendix A. Comparison of the linear viscoelastic corotational beam description with fully-resolved FE simulations

To assess the accuracy of the presented linear viscoelastic corotational beam, we compare results obtained from the latter to those of a fully resolved FE calculation based on 3D solid elements. In both simulations, we use the experimentally obtained Prony series (Table C.3) to describe the constitutive behavior of a cantilever beam of length $L = 10$mm and circular cross-section with diameter $D = 1$mm. Clamped at one end, the beam’s free end is exposed to essential boundary conditions to uniformly (i) stretch, (ii) bend, and (iii) twist the beam in a time-dependent fashion, analogous to the relaxation tests performed on the hexagonal trusses. For each case, we measure the NRSME between our simulations (using the presented linear viscoelastic corotational beam element) and fully-resolved FE simulations based on a mesh of 6501 linear tetrahedral solid elements in the commercial software Abaqus (Smith,
Fig. 11. Simulated rotation vs. time response of a twisted truss cube made of $30 \times 30 \times 30$ unit cells of four different truss topologies (bitruncated octahedron, octahedron, octet, cube) in comparison to the results of the two-scale model used in a $10 \times 10 \times 10$ brick mesh. The cube is initially twisted up to $3.7^\circ$ at a rate of $89^\circ$/min and held at that twist angle for 2.5 s to allow the material to relax, and then released to return to the initial state through damped vibrations. All trusses have the same linear viscoelastic base material model with the parameters of Table C.3 and the same effective density according to Table 1. For comparison, dashed lines illustrate the quasistatic relaxation response of each structure without considering inertial effects.

To estimate the validity limits of the linearized kinematics formulation underlying the corotational beam formulation, Abaqus simulations were performed using both updated-Lagrangian (UL) and total-Lagrangian (TL) settings — the former providing higher accuracy in case of increased deformation levels. As a worst-case scenario, we use a single corotational beam element within the discrete calculations, expecting the error to further decrease with refinement. Note that the use of Abaqus requires a representation of the Prony series in terms of the instantaneous elastic shear modulus $G_0$, viz.

$$G(t) = G_\infty + \sum_{k=1}^{n} g_k G_0 e^{-t/\omega_k} \quad \text{with} \quad G_0 = G_\infty + \sum_{k=1}^{n} G_k,$$

(A.1)

(and analogously for the bulk modulus $K(t)$ with parameters $k_i$), which admits the extraction of all dimensionless Prony series contributions $g_i$ (and $k_i$), as summarized in Table A.2.
The mass matrix of the individual beam is used to compute the kinetic energy of the RUC, which in turn is matched with the kinetic energy of the macroscale finite element to provide the latter’s (lumped) mass matrix. Fig. B.14 schematically illustrates the concept of obtaining the effective rotational inertia in 2D. An average angular velocity \( \omega \) is applied to every node of the macroscopic (constant-strain) finite element as well as to each node in the RUC, whose matching kinetic energies yields the effective mass matrix of the macroscale element. Uniform \( \omega \)-values applied to all nodes of an element result in \( \omega \) also being active at each quadrature point (and \( \nabla \omega = 0 \)), which in turn results in the same \( \omega \) being applied to every node in the RUC. In 3D the kinetic energy matching procedure is repeated for each component of the angular velocity vector, so that three independent components of the lumped mass matrix are obtained. (In all our examples in 3D, the symmetry of the RUC leads to identical values about each of the three axes.)

The translational components of the macroscale finite element simply follow from mass lumping onto the nodes of the macroscale element. Overall, the lumped mass matrix of the macroscale finite element for each node is then obtained as (to be duplicated for the two nodes of a corotational beam element)

\[
M_{\text{lumped, 3D}} = \frac{\rho A_b L}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{A_b}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L^2}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{L^2}{12}
\end{pmatrix},
\]

where \( L \) and \( A_b \) are the beam’s length and cross-sectional area, and \( \rho \) denotes the base material’s mass density. For simplicity, this mass matrix is constant (i.e., we do not take into account changes to the above mass matrix due to finite rotations of the beam, which can be accounted for in an extension).

The mass matrix of the individual beam is used to compute the kinetic energy of the RUC, which in turn is matched with the kinetic energy of the macroscale finite element to provide the latter’s (lumped) mass matrix. Fig. B.14 schematically illustrates the concept of obtaining the effective rotational inertia in 2D. An average angular velocity \( \omega \) is applied to every node of the macroscopic (constant-strain) finite element as well as to each node in the RUC, whose matching kinetic energies yields the effective mass matrix of the macroscale element. Uniform \( \omega \)-values applied to all nodes of an element result in \( \omega \) also being active at each quadrature point (and \( \nabla \omega = 0 \)), which in turn results in the same \( \omega \) being applied to every node in the RUC. In 3D the kinetic energy matching procedure is repeated for each component of the angular velocity vector, so that three independent components of the lumped mass matrix are obtained. (In all our examples in 3D, the symmetry of the RUC leads to identical values about each of the three axes.)

The translational components of the macroscale finite element simply follow from mass lumping onto the nodes of the macroscale element. Overall, the lumped mass matrix of the macroscale finite element for each node is then obtained as (to be duplicated for each of the two nodes of a corotational beam element)

\[
M_{\text{lumped, 3D}} = \frac{\rho A_b L}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{A_b}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L^2}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{L^2}{12}
\end{pmatrix},
\]

where \( L \) and \( A_b \) are the beam’s length and cross-sectional area, and \( \rho \) denotes the base material’s mass density. For simplicity, this mass matrix is constant (i.e., we do not take into account changes to the above mass matrix due to finite rotations of the beam, which can be accounted for in an extension).

The mass matrix of the individual beam is used to compute the kinetic energy of the RUC, which in turn is matched with the kinetic energy of the macroscale finite element to provide the latter’s (lumped) mass matrix. Fig. B.14 schematically illustrates the concept of obtaining the effective rotational inertia in 2D. An average angular velocity \( \omega \) is applied to every node of the macroscopic (constant-strain) finite element as well as to each node in the RUC, whose matching kinetic energies yields the effective mass matrix of the macroscale element. Uniform \( \omega \)-values applied to all nodes of an element result in \( \omega \) also being active at each quadrature point (and \( \nabla \omega = 0 \)), which in turn results in the same \( \omega \) being applied to every node in the RUC. In 3D the kinetic energy matching procedure is repeated for each component of the angular velocity vector, so that three independent components of the lumped mass matrix are obtained. (In all our examples in 3D, the symmetry of the RUC leads to identical values about each of the three axes.)

The translational components of the macroscale finite element simply follow from mass lumping onto the nodes of the macroscale element. Overall, the lumped mass matrix of the macroscale finite element for each node is then obtained as (to be duplicated for each of the two nodes of a corotational beam element)

\[
M_{\text{lumped, 3D}} = \frac{\rho A_b L}{2} \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{A_b}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{L^2}{12} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{L^2}{12}
\end{pmatrix},
\]
Fig. A.12. Net normal force $N$, transverse force $Q$, and torsional moment $M$ vs. time of for a cantilever beam of $L = 10 \text{mm}$ and diameter $D = 1 \text{mm}$. All plots compare the response of the presented corotational beam element to a detailed FEM simulation using 6501 linear tetrahedral solid elements in Abaqus (Smith, 2009), based on updated-Lagrangian (UL) and total-Lagrangian (TL) schemes. The deformation is ramped up linearly over 2.5s, then kept constant for 5.0s, until the beam is returned to its initial configuration (analogous to the hexagonal truss study). Tables indicate the NRSME between results from the corotational beam description and the 3D FE simulation. (c) includes a visualization of the fully resolved beam modeled using Abaqus.
Fig. A.13. NRSME between the corotational beam formulation and the 3D FE model for an increasing slenderness ratio $\lambda$. The applied deformation is kept constant for all simulations. In Abaqus simulations, the average element size is adjusted such as to prevent $h$-refinement when thickening the beam. The NRSME between the normal force for tension, the transverse force for bending, and the torsional moment for twisting is measured over the entire course of the simulation, shown in Fig. A.12. FE results were obtained within updated-Lagrangian (UL) and total-Lagrangian (TL) settings.

Fig. B.14. Macroscale finite element with the same angular velocity $\omega$ applied at each node, whose kinetic energy is matched with that of a beam-based RUC, whose nodes are deformed by the same angular velocity.

every node)

$$M_{\text{lumped, 3D}}^c = \begin{pmatrix}
\frac{1}{n_e} \rho^* V_e & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{n_e} \rho^* V_e & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{n_e} \rho^* V_e & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{2}{n_e} \frac{V_e}{V_{\text{RUC}}} \sum_{k} n_k \rho_{A_k} L_k^3 I_k^3 4 \pi & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{2}{n_e} \frac{V_e}{V_{\text{RUC}}} \sum_{k} n_k \rho_{A_k} L_k^3 I_k^3 24 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{2}{n_e} \frac{V_e}{V_{\text{RUC}}} \sum_{k} n_k \rho_{A_k} L_k^3 I_k^3 24 & 0
\end{pmatrix}, \quad (B.2)$$

with $\rho^*$ from (20), $V_e$ the volume of the macroscale element, and the rotational terms obtained from (22) for each rotational component.

Appendix C. Mechanical response of the TPE base-material

Trusses in all experiments were produced by powder-based selective laser sintering (SLS), using the rubbery Sintratec TPE material with a measured mass density of $\rho = 0.95$ g/cm$^3$. For this base material, we extracted the Prony series parameters of Eq. (4) (i.e., modulus $E_\infty$ and the $n$ moduli $E_k$ and relaxation times $\tau_k$) by fitting experimentally measured stress–strain curves to the simulated response obtained from the generalized Maxwell model. To keep model calibration separate from validation, we calibrate the Prony series based on data from tensile relaxation tests on dog-bone samples (not trusses), which were manufactured using the same print parameters and geometric settings as for all trusses (see Section 3) to eliminate artifacts of the printing process. The gauge length and width of the dogbone samples were 30mm and 6mm, respectively. Before each experiment, we introduced two
white markers within the gauge section of the sample’s face and imaged their motion using a high-speed camera (Photron AX200). The effective strain was obtained from the relative spacing between the markers (this eliminates errors due to deformation outside the gauge section). We tested five samples along each of three different orientations within the print bed plane, whose data show negligible differences in material response along those three orientations, so we may assume isotropy.

We performed uniaxial tensile tests by clamping the specimens to the testing machine (Instron Model 5943) and applying a constant displacement rate of 100 mm/min, followed by holding the displacement constant for 10 min. The rate is selected sufficiently fast to retain most of the viscous energy, while minimizing the strain overshoot at the end of the displacement ramp, shown in Fig. C.15. The overshoot, which is ignored during parameter fitting, appears due to limitations in the testing machine’s control loop and results in an increase in stiffness at that point. As expected, the viscoelastic TPE base material undergoes stress relaxation when subjected to a constant average strain.

Since experiments could not provide a reliable force–time history for the fast initial loading (0 < t < 1.6 s) of the extension experiments, we neglect the measured force–time history for the time range 0 < t < 1.6s and instead use the data from 1.6s onward to estimate the time-dependent material response of the sample (of course, this may incur errors for the fastest relaxation time(s), so a re-calibration may be required if faster responses are to be simulated). Using the resulting stress–strain curve, an n = 4-element Prony-series is calibrated by fixing the four relaxation times $\tau_1$ through $\tau_4$ and fitting the respective elastic constants $E_\infty$ and $E_1$ through $E_4$, resulting in the relaxation function $\varepsilon(t)$ whose parameters are summarized in Table C.3.

We note that a complete model calibration (including the torsional shear stresses) would require a separate set of shear or torsion experiments to identify the Prony series parameters associated with the torsional behavior (parameters $G_\infty$, $G_1$ through $G_n$ and $\omega_i$ through $\omega_n$). As we do not have access to such experimental data, we assumed a constant Poisson’s ratio of $\nu = 0.3$, so that the torsional moduli $G_\infty$ and $G_1$ through $G_n$ can be obtained directly from the uniaxial moduli $E_\infty$ and $E_1$ through $E_n$. We admit that this approximation may require refinement, yet torsional effects play only a minor role in all our experiments (as confirmed by the shown agreement between simulated and experimental truss data).

Appendix D. Analysis of local beam deformation

To point out the limitations of the structural representation of trusses by the viscoelastic corotational beam model, we compute the local stretching strains and curvature values within individual struts — both for experiment 1 (up to a stretch of 8.1%) and experiment 2 (up to 14.3%). Fig. D.16 illustrates the axial strain due to stretching and the maximum curvature in each beam at point $b^\circ$, corresponding to the beginning of the relaxation process (see Fig. 5). As before, all beams are subdivided into 10 segments and have a slenderness ratio of $\lambda = 0.1$. While both values remain small in experiment 1, we observe an increase in the maximum...
Fig. D.16. Stretching strains and curvature values within individual struts of the hexagonal truss, as obtained from discrete viscoelastic beam simulations of experiment 1 (left) and experiment 2 (right).

**Table E.4**

Effective elastic moduli of the octahedron and octet lattices with the same relative density of 0.833\% (all moduli are normalized by the base material’s Young’s modulus $E_\infty$), computed by using the homogenized continuum framework.

<table>
<thead>
<tr>
<th>Modulus</th>
<th>Octahedron</th>
<th>Octet</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{111}/F_m$</td>
<td>$1.391 \cdot 10^{-3}$</td>
<td>$1.389 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$C_{222}/F_m$</td>
<td>$1.391 \cdot 10^{-3}$</td>
<td>$1.389 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$C_{333}/F_m$</td>
<td>$0.693 \cdot 10^{-3}$</td>
<td>$0.693 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$C_{113}/F_m$</td>
<td>$0.693 \cdot 10^{-3}$</td>
<td>$0.693 \cdot 10^{-3}$</td>
</tr>
<tr>
<td>$C_{121}/F_m$</td>
<td>$0.695 \cdot 10^{-3}$</td>
<td>$0.695 \cdot 10^{-3}$</td>
</tr>
</tbody>
</table>

Axial strains by a factor of $\sim 10$ and in the maximum curvature by a factor of $\sim 3.5$ in experiment 2 (even though the applied average vertical strain increases only by a factor of $\sim 1.7$). This strong increase in beam elongation and bending highlights the emergence of nonlinearity, which is why the Euler–Bernoulli-type beam model loses accuracy when simulating experiment 2. It is for this reason that we observe a strong increase in the NMRSE from experiment 1 to experiment 2. More accurate results can be obtained by (i) introducing finite kinematics in the beam formulation and (ii) extending the presented theory in Section 2.1 to a nonlinear viscoelastic constitutive response. (Both go beyond the scope of this study.)

**Appendix E. Effective incremental elastic moduli and homogenized mass matrix of the octahedron and octet lattices**

As discussed in Section 4, the effective response of the octahedron and octet lattices show only marginal differences, when considering slender struts and low relative densities as those chosen here. To underline the similarities in their mechanical behavior, Table E.4 summarizes their effective elastic moduli, computed from the generalized continuum model.

The homogenized lumped mass matrix of the two topologies also shows noticeable similarities, when the trusses have the same relative densities — with the exception of those components associated with torsional inertia. The latter for the octet is about half that of the octahedron. However, since torsional inertia of individual beams plays only a minor role in all examples reported here where axial and bending modes dominate, its effect on the damped vibrations in Fig. 11 is negligible. For comparison, the two diagonal lumped mass matrices (using the geometric and material parameters of all truss simulations, cf. Table 1) are

$$M_{\text{octahedron\ lumped,3D}} = \text{diag} (0.0267013, 0.0267013, 0.0267013, 0.0004170, 0.1112550, 0.1112550),$$
$$M_{\text{octet\ lumped,3D}} = \text{diag} (0.0267012, 0.0267012, 0.0267012, 0.0002085, 0.1112550, 0.1112550).$$

(E.1)


