Reversion Porisms for Lines

Master Thesis

Author(s):
Schiltknecht, Marco

Publication date:
2021

Permanent link:
https://doi.org/10.3929/ethz-b-000499919

ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Reversion Porisms for Lines

Master's Thesis

M. Schiltknecht

25 June 2021

Advisor: Prof. Dr. N. Hungerbühler
Department of Mathematics, ETH Zürich

Abstract

We show that the Butterfly Porism about cyclic quadrilaterals also applies to a degenerate case of a conic. Moreover, we prove that the collinearity used in this theorem is a necessary condition and we give an alternative proof for Pappus's Hexagon Theorem. The main result is a reversion porism for polygons with an arbitrary number of vertices on two distinct lines. We define a polar line for a degenerate case of a conic and present a conjugated reversion porism. Lastly, we consider reversions on more than two lines.

Contents

Introduction

The Butterfly Porism about cyclic quadrilaterals states that three collinear reversions can be replaced by one reversion, whose corresponding point lies on the same line as the three given points. This problem goes back at least forty years to the work of Dixon Jones [2] from 1980, where it was shown for a circle using Pascal's Theorem, which is a generalization of Pappus's Hexagon Theorem for conics. The Butterfly Porism was rediscovered by Jerzy Kocik [3] in the year 2013 using Möbius transformations and two years later, Ivan Izmestiev [1] showed that this problem could also be solved with cross-ratios.

Roadmap: In the first chapter, we define the projective plane and reversions on two lines. We will prove the Butterfly Porism in the second chapter using Izmestiev's approach. An alternative proof for Pappus's Hexagon Theorem and the main result are part of the third chapter. We will use the main result to show that every finite composition of reversions can be turned into the identity by adding at most four points. In Chapter IV, we define the polar line and the process of conjugation and in the last chapter, we take a look at reversions on more than two lines.

Acknowledgements: I would like to thank my advisor Prof. Dr. Norbert Hungerbühler for allowing me to write my master's thesis under his supervision and for the helpful remarks during the semester.

Chapter I

Reversions Between Two Lines

1 Definition of a Reversion

We refer the reader to [5] for an introduction to projective geometry. We denote the projective plane, which is essentially the Euclidean plane along with a set of points at infinity, by

$$\mathbb{P}^2 := (\mathbb{R}^3 \backslash \{0\}) / \sim,$$

where two points $(x_1, x_2, x_3), (y_1, y_2, y_3) \in \mathbb{R}^3$, both different from the origin, are equivalent if $(x_1, x_2, x_3) = (\lambda y_1, \lambda y_2, \lambda y_3)$ for a $\lambda \in \mathbb{R}\backslash\{0\}$. We write $(x_1 : x_2 : x_3)$ for the equivalence class of the vector (x_1, x_2, x_3). We can think about the Euclidean plane as points of the form $(x_1 : x_2 : 1)$, whereas $(x_1 : x_2 : 0)$ corresponds to a point at infinity. This notation allows for the use of matrix operations and vector products on elements of the projective plane. Recall that an element in $\mathbb{P}^2$ can represent both a point and a line in the projective plane. Using the standard scalar product, we can denote the set of points on a line $g \in \mathbb{P}^2$ by

$$L_g := \{p \in \mathbb{P}^2 \mid g \cdot p = 0\}$$

and the set of lines incident to a point $p \in \mathbb{P}^2$ can be expressed as

$$L_p := \{g \in \mathbb{P}^2 \mid g \cdot p = 0\}.$$

Definition 1.1 Let $g, h \in \mathbb{P}^2$ be two lines and let $p \in \mathbb{P}^2\backslash K$ be a point, where $K := L_g \cup L_h$. A *reversion* through p is a function

$$\varphi_p^K \colon K \to K$$
$$x \mapsto \varphi_p^K(x)$$

that maps a point x from one line to a point $\varphi_p^K(x)$ on the other line such that $x, p, \varphi_p^K(x)$ are collinear.

Remarks 1.2 First a few remarks to Definition 1.1.

a) The set K is a degenerate case of a conic.

b) Note that $\varphi_p^K \circ \varphi_p^K = \mathrm{id}|_K$.

c) A reversion φ_p^K splits up into two maps, namely

$$\varphi_p^K(x) = \begin{cases} \varphi_p^h(x) & \text{for } x \in L_g, \\ \varphi_p^g(x) & \text{for } x \in L_h, \end{cases}$$

where $\varphi_p^h\colon L_g \to L_h$ and $\varphi_p^g\colon L_h \to L_g$.

d) The map φ_p^h is a bijection. To see this, let x be a point on the line h and let l be the line that connects x and p. Note that l is different from g, since p is not incident to g. Therefore there is exactly one point in L_g that gets mapped to x by φ_p^h, which is given by the unique intersection point of l and g. By symmetry, φ_p^g is also a bijection.

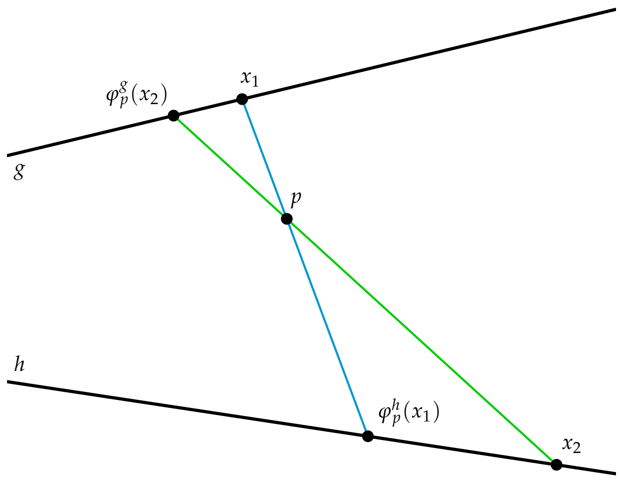

Figure 1.1: Reversion through p.

We can quantify Definition 1.1 using the cross product. To this end, let $g, h \in \mathbb{P}^2$ be the two straight lines. If we want to map a point x from g to h, then we can write

$$\varphi_p^h(x) = h \times (p \times x).$$

Recall that the cross product can be written in matrix form the following way:

$$p \times x = \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Therefore we can write the reversion map from g to h as $\varphi_p^h(x) = M_p^h x$, where

$$M_p^h := \begin{pmatrix} 0 & -h_3 & h_2 \\ h_3 & 0 & -h_1 \\ -h_2 & h_1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -p_3 & p_2 \\ p_3 & 0 & -p_1 \\ -p_2 & p_1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} -h_2p_2 - h_3p_3 & h_2p_1 & h_3p_1 \\ h_1p_2 & -h_1p_1 - h_3p_3 & h_3p_2 \\ h_1p_3 & h_2p_3 & -h_1p_1 - h_2p_2 \end{pmatrix}.$$

One thing to note here is that we did not use g in this matrix representation. We now compute the eigenspaces of the matrix M_p^h. Clearly we have $M_p^h p = h \times (p \times p) = 0$ and a point $x \in \mathbb{P}^2$ that is incident to h satisfies

$$\varphi_p^h(x) = h \times (p \times x) = (h \cdot x)p - (h \cdot p)x = -(h \cdot p)x.$$

Hence the kernel of M_p^h is given by the span of p and the eigenspace associated with the eigenvalue $-(h \cdot p)$ is the plane that has h as its normal vector. Since p is not incident to h and φ_p^h fixes the line h, we see that

$$M_p^h \colon \mathbb{P}^2 \backslash \{p\} \to L_h$$

is a projection.

Suppose for a moment that p is incident to h. Then all eigenvalues are zero and the kernel of M_p^h is again given by the line h, because M_p^h is not the zero matrix. So for $h \cdot p = 0$, we have

$$M_p^h \colon \mathbb{P}^2 \backslash L_h \to \{p\},$$

which is a constant map.

2 Basic Properties

We now look at three important lemmas, which we will often use later on. The first lemma basically states that two reversions are the same if and only if their underlying points are equal to each other. By the second lemma, a composition of an even number of reversions fixes either one, two or all points on each of the two lines. Hence if such a composition fixes at least three points on both lines, then it is equal to the identity on both lines, which we usually denote as a restriction of the map

$$\begin{aligned} \mathrm{id} \colon \mathbb{P}^2 &\to \mathbb{P}^2 \\ x &\mapsto x. \end{aligned}$$

However, it is also possible that an even number of reversions equals the identity on just one of the two lines. This brings us to the third lemma, which ends this section. It says that a composition of an odd number of reversions can always be extended by one point such that it becomes the identity on one line.

Lemma 2.1 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and let $p, q \in L_l \backslash K$, where $K := L_g \cup L_h$. If there is a point $x \in K \backslash (L_l \cup \{g \times h\})$ that satisfies $\varphi_q^K \circ \varphi_p^K(x) = x$, then $p = q$. In particular, we then have $\varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$.*

Proof Assume that $p \neq q$. Let l_1 be the line through x and p. We have $\varphi_p^K(x) \neq x$, since x is not the intersection point of g and h. Let l_2 be the line through $\varphi_p^K(x)$ and q. Now l_1 and l_2 are distinct lines, since x is not incident to l. Thus x does not sit on l_2 and therefore $\varphi_q^K \circ \varphi_p^K(x)$ cannot equal x. This is a contradiction. □

Lemma 2.2 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n} \in \mathbb{P}^2 \backslash K$ be points, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $\varphi := \varphi_{p_{2n}}^K \circ \cdots \circ \varphi_{p_1}^K$. If φ fixes three pairwise distinct points on g, then $\varphi|_{L_g} = \mathrm{id}|_{L_g}$.*

Proof We can write

$$\varphi|_{L_g} = \varphi_{p_{2n}}^g \circ \varphi_{p_{2n-1}}^h \circ \cdots \circ \varphi_{p_2}^g \circ \varphi_{p_1}^h,$$

so consider the matrix

$$M := M_{p_{2n}}^g M_{p_{2n-1}}^h \cdots M_{p_2}^g M_{p_1}^h.$$

We know that $Mp_1 = 0$, therefore one eigenvalue of M is zero, say $\lambda_3 = 0$. By the equivalence relation defined on $\mathbb{P}^2$, we see that all three pairwise distinct fixed points of $\varphi|_{L_g}$ are eigenvectors of M with respect to some non-zero eigenvalues. The only situation, where this becomes possible, is when the other two eigenvalues satisfy $\lambda_1 = \lambda_2 \in \mathbb{R} \backslash \{0\}$. Hence the corresponding eigenspace is the plane that has g as its normal vector, since this plane is spanned by the three fixed points. □

Remark 2.3 The map $\varphi|_{L_g}$ from Lemma 2.2 fixes either one, two or all points on the line g, because the intersection point of g and h is always fixed.

Lemma 2.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $\varphi := \varphi_{p_{2n-1}}^K \circ \cdots \circ \varphi_{p_1}^K$. There exists a unique point $p_g \in \mathbb{P}^2 \backslash K$ such that*

$$\varphi|_{L_g} = \varphi_{p_g}^K|_{L_g}.$$

Proof Let $A, B \in L_g \backslash L_h$ be two distinct points and define $A' := \varphi(A)$ and $B' := \varphi(B)$. We define $p_g \in \mathbb{P}^2$ to be the intersection point of the two distinct lines $\overline{AA'}$ and $\overline{BB'}$. Note that both of these two lines are not incident to the intersection point of g and h. Since $A \neq B$ and $A' \neq B'$, we have $p_g \in \mathbb{P}^2 \backslash K$. Now we see that $\varphi^K_{p_g} \circ \varphi$ fixes three pairwise distinct points on g, namely A, B and the intersection point of g and h. By Lemma 2.2, we have

$$\varphi^K_{p_g} \circ \varphi|_{L_g} = \mathrm{id}|_{L_g},$$

which is equivalent to

$$\varphi|_{L_g} = \varphi^K_{p_g}|_{L_g}.$$

To see that the first equality implies the second, consider

$$\varphi(x) = \varphi^K_{p_g} \circ \varphi^K_{p_g} \circ \varphi(x) = \varphi^K_{p_g}(x),$$

where $x \in L_g$. Uniqueness of the point p_g follows from Lemma 2.1. □

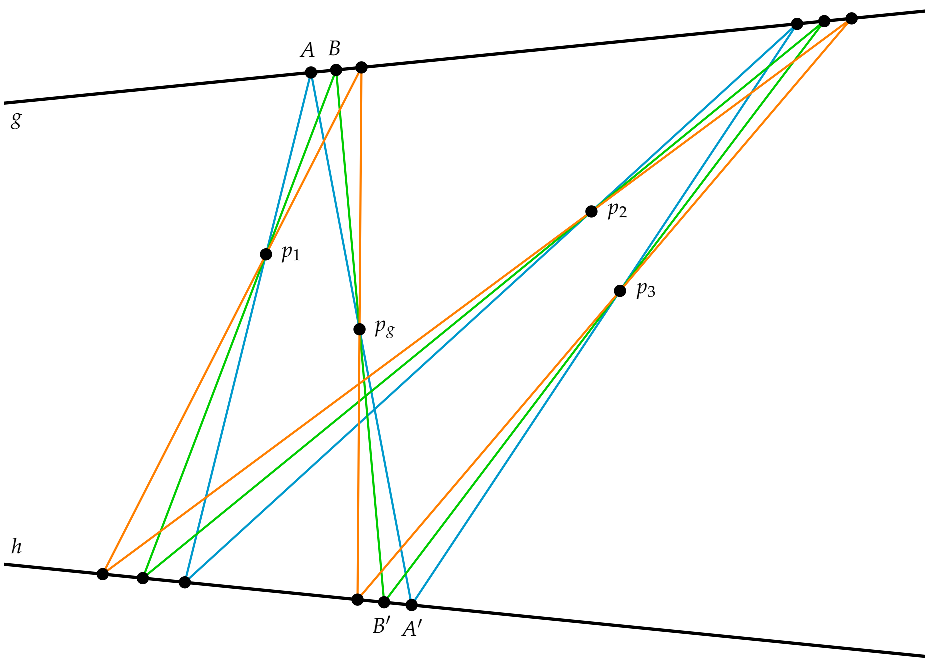

Figure 2.1: Construction of the point p_g.

Remarks 2.5 Let φ be defined as in Lemma 2.4.

a) There is also a point $p_h \in \mathbb{P}^2 \backslash K$ such that

$$\varphi|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

b) The points p_g and p_h also satisfy

$$\varphi^{-1}|_{L_h} = \varphi^K_{p_g}|_{L_h} \qquad \text{and} \qquad \varphi^{-1}|_{L_g} = \varphi^K_{p_h}|_{L_g},$$

where $\varphi^{-1} = \varphi^K_{p_1} \circ \cdots \circ \varphi^K_{p_{2n-1}}$ is the process of passing through the given points in reverse order.

c) If $p_g = p_h$, then $\varphi = \varphi^K_{p_{2n}}$, where $p_{2n} := p_g = p_h$. In this case, we have

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

d) We have

$$\varphi \circ \varphi = \mathrm{id}|_K \qquad \Longleftrightarrow \qquad p_g = p_h.$$

To see that the first equation induces the second, note that $\varphi = \varphi^{-1}$ implies

$$\varphi^K_{p_g}|_{L_g} = \varphi|_{L_g} = \varphi^{-1}|_{L_g} = \varphi^K_{p_h}|_{L_g}.$$

By Lemma 2.1, we then have $p_g = p_h$.

3 Polygonal Chains

In this section, we define polygonal chains, which we will often use to give geometrical versions of the theorems we prove.

Definition 3.1 A finite sequence $A_1 \dots A_n$ of points $A_1, \dots, A_n \in \mathbb{P}^2$, where each pair of consecutive points is joined by a straight line segment, is called *polygonal chain*.

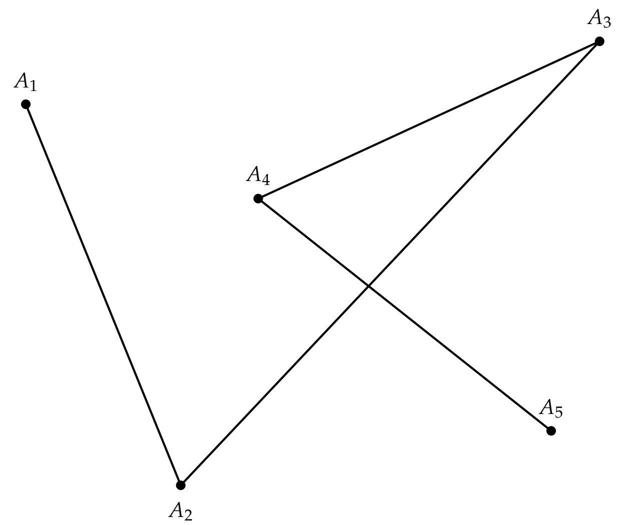

Figure 3.1: An example of a polygonal chain $A_1 \dots A_5$.

Remarks 3.2 A few remarks to Definition 3.1.

a) A polygonal chain $A_1 \dots A_n$ *closes*, if $A_1 = A_n$.

b) Let $g, h \in \mathbb{P}^2$ be lines and $p_1, \dots, p_{n-1} \in \mathbb{P}^2 \backslash K$ be points, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$. Now let $A_1 \dots A_n$ be a polygonal chain, where $A_1 \in K$. We say that $A_1 \dots A_n$ *passes through* $p_1, \dots, p_{n-1}$ with respect to K, if $A_{k+1} = \varphi^K_{p_k}(A_k)$ for all $k \in \{1, \dots, n-1\}$.

Chapter II

The Butterfly Porism

4 Cross-Ratio

As shown by Izmestiev in [1], the cross-ratio is a useful tool to work with reversions and it can be used to prove the Butterfly Porism. Hence we give a small introduction to the topic as well as a few key properties of the cross-ratio here. This section closely follows the book *Perspectives on Projective Geometry* [5] by Jürgen Richter-Gebert (2011).

Definition 4.1 Let $l \in \mathbb{P}^2$ and $a, b, c, d \in L_l$ be such that $|\{a, b, c, d\}| \geq 3$. The *cross-ratio* of a, b, c, d is defined as

$$(a, b; c, d) := \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]},$$

where $o \in \mathbb{P}^2 \backslash L_l$ and $[o, a, c] := o \cdot (a \times c)$.

Remarks 4.2 Let $a, b, c, d \in L_l$ with $|\{a, b, c, d\}| \geq 3$ for some $l \in \mathbb{P}^2$.

a) We can interpret a, b, c, d as collinear points, but also as concurrent lines.

b) Note that $[o, a, c] = \det N$, where

$$N := \begin{pmatrix} o_1 & a_1 & c_1 \\ o_2 & a_2 & c_2 \\ o_3 & a_3 & c_3 \end{pmatrix}$$

is the matrix that has o, a and c as its column vectors.

c) For the cross-ratio, we allow operations like

$$\frac{1}{0} = \infty \qquad \text{and} \qquad \frac{1}{\infty} = 0.$$

Hence we have $(a, b; c, d) \in \mathbb{R} \cup \{\infty\}$. Since we require a, b, c, d to be at least three pairwise distinct elements of $\mathbb{P}^2$, we do not get the expression $\frac{0}{0}$.

d) Let $M \in \mathbb{R}^{3\times 3}$ be an invertible matrix. By b), we have

$$\frac{[Mo, Ma, Mc][Mo, Mb, Md]}{[Mo, Ma, Md][Mo, Mb, Mc]} = \frac{(\det M)^2[o, a, c][o, b, d]}{(\det M)^2[o, a, d][o, b, c]} = \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]}.$$

e) The cross-ratio satisfies

$$(a, b; c, d) = (b, a; d, c) = (c, d; a, b) = (d, c; b, a)$$

and

$$(a, b; c, d) = \frac{1}{(a, b; d, c)}.$$

f) To verify that $(a, b; c, d)$ does not depend on the choice of $o \in \mathbb{P}^2\backslash L_l$, let $o' \in \mathbb{P}^2\backslash L_l$. By d), we can assume without loss of generality that $l = (1 : 0 : 0)$. Then we have $o = (1 : o_2 : o_3)$ and $o' = (1 : o'_2 : o'_3)$. Consider the invertible matrix

$$M := \begin{pmatrix} 1 & 0 & 0 \\ o'_2 - o_2 & 1 & 0 \\ o'_3 - o_3 & 0 & 1 \end{pmatrix}.$$

The matrix M satisfies $Mo = o'$ and $Mx = x$ for every $x \in L_l$. Hence

$$\frac{[o', a, c][o', b, d]}{[o', a, d][o', b, c]} = \frac{[Mo, Ma, Mc][Mo, Mb, Md]}{[Mo, Ma, Md][Mo, Mb, Mc]} = \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]}.$$

g) It remains to check that the cross-ratio is well-defined with respect to the equivalence relation defined on $\mathbb{P}^2$. To see this, consider

$$\frac{[\lambda o, \alpha a, \gamma c][\lambda o, \beta b, \delta d]}{[\lambda o, \alpha a, \delta d][\lambda o, \beta b, \gamma c]} = \frac{\lambda^2\alpha\beta\gamma\delta[o, a, c][o, b, d]}{\lambda^2\alpha\beta\gamma\delta[o, a, d][o, b, c]} = \frac{[o, a, c][o, b, d]}{[o, a, d][o, b, c]},$$

where $\lambda, \alpha, \beta, \gamma, \delta \in \mathbb{R}\backslash\{0\}$.

Lemma 4.3 *Let $l \in \mathbb{P}^2$ and $p_1, p_2, p_3, p_4 \in L_l$ be such that $|\{p_1, p_2, p_3, p_4\}| \geq 3$. For $o \in \mathbb{P}^2\backslash L_l$, we define $l_k := o \times p_k$ for all $k \in \{1, 2, 3, 4\}$. We have*

$$(l_1, l_2; l_3, l_4) = (p_1, p_2; p_3, p_4).$$

Proof We can assume without loss of generality that p_1, p_2, p_3, p_4 are points and l_1, l_2, l_3, l_4 are lines. Now o is the intersection point of l_1, l_2, l_3, l_4 and does not lie on l. Moreover, the line o is not incident to the point o, since $o \cdot o \neq 0$. Thus we have

$$(l_1, l_2; l_3, l_4) = \frac{[o, l_1, l_3][o, l_2, l_4]}{[o, l_1, l_4][o, l_2, l_3]}.$$

Note that

$$[o, l_1, l_3] = o \cdot ((o \times p_1) \times (o \times p_3)) = (o \cdot o)[o, p_1, p_3].$$

Hence we have

$$(l_1, l_2; l_3, l_4) = \frac{(o \cdot o)^2 [o, p_1, p_3][o, p_2, p_4]}{(o \cdot o)^2 [o, p_1, p_4][o, p_2, p_3]} = (p_1, p_2; p_3, p_4),$$

which concludes the proof. □

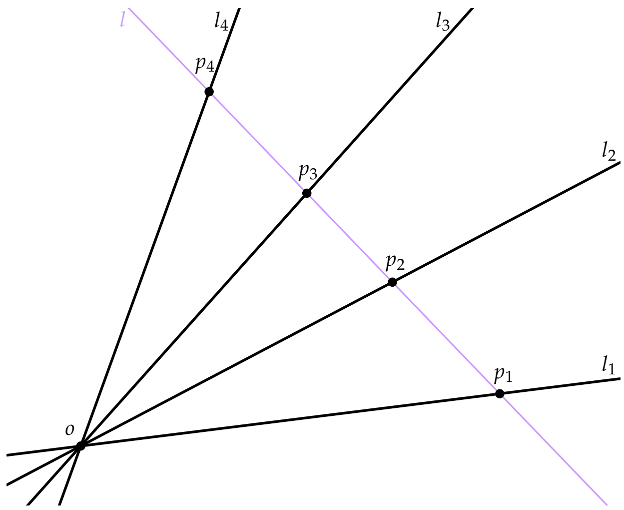

Figure 4.1: We have $(l_1, l_2; l_3, l_4) = (p_1, p_2; p_3, p_4)$ by Lemma 4.3.

Lemma 4.4 *Let $a, b, c, d, d' \in L_l$ for some $l \in \mathbb{P}^2$. If a, b, c are pairwise distinct and $(a, b; c, d) = (a, b; c, d')$, then d is equivalent to d'.*

Proof Let $o \in \mathbb{P}^2 \backslash L_l$. Since a, b, c are pairwise distinct, we see that the equation $(a, b; c, d) = (a, b; c, d')$ is equivalent to

$$\begin{aligned} 0 &= [o, a, d][o, b, d'] - [o, a, d'][o, b, d] \\ &= (o \times b) \cdot \big([o, a, d]d' - [o, a, d']d\big) \\ &= (o \times b) \cdot \big((o \times a) \times (d' \times d)\big) \\ &= (o \times b) \cdot \big([o, d', d]a - [a, d', d]o\big) \\ &= [o, a, b][o, d, d'], \end{aligned}$$

which implies that $d \times d' = 0$, i.e. d is equivalent to d'. □

Lemma 4.5 *Let $l \in \mathbb{P}^2$ and $a, b, c, d \in L_l$ be such that a, b, c are three pairwise distinct points. If $c \neq d$ and*

$$(a, b; c, d) = (a, b; d, c),$$

then $(a, b; c, d) = -1$.

Proof We must have $(a, b; c, d) \in \{-1, 1\}$ by Remark 4.2 e). Since $(a, b; c, c) = 1$ and $c \neq d$, we can use Lemma 4.4 to conclude. □

Remark 4.6 If four collinear points or concurrent lines $a, b, c, d \in \mathbb{P}^2$ satisfy $(a, b; c, d) = -1$, then they are called *harmonic* (see [5, Definition 5.1.]).

5 The Butterfly Porism

In this section, we will prove the Butterfly Porism for a degenerate case of a conic using Izmestiev's approach. This result was already shown for a circle by Dixon Jones in the year 1980 (see [2]). It says that three collinear reversions can be replaced by one. But first, we look at another theorem, which states that if a composition of four reversions is the identity on both lines, then the four underlying points must be collinear. This then implies that the collinearity condition in the Butterfly Porism is necessary.

Theorem 5.1 *Let $g, h \in \mathbb{P}^2$ be two distinct lines and let $p, q, r, s \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$. If $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$, then the points p, q, r, s are collinear.*

Proof By Lemma 2.1, we can assume without loss of generality that $p \neq q$, $q \neq r$, $r \neq s$ and $p \neq s$. Assume towards contradiction that p, q, r, s are not collinear. Let l_1 be the line through p and q, and l_2 be the line through r and s. Since p, q, r, s are not collinear, the lines l_1, l_2 are distinct.

Suppose that one line, say l_1, has two distinct intersection points with the set K. Let x be one of those two intersection points such that x is not incident to l_2. Now $\varphi_q^K \circ \varphi_p^K(x) = x$, but we have $\varphi_s^K \circ \varphi_r^K(x) \neq x$ again by Lemma 2.1, which is a contradiction to $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$.

Suppose now that both lines l_1 and l_2 have only one intersection point with K, i.e. the lines g, h, l_1, l_2 are concurrent. We define l_3 to be the line through q and r. The line through p and s shall be denoted by l_4. Since $l_1 \neq l_2$, we see that l_3 is not incident to the intersection point of g and h. Hence l_3 has two intersection points with K and is different from l_4. Thus we can repeat the same argument as above for l_3 and l_4 instead of l_1 and l_2. □

Lemma 5.2 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p, q, r \in L_l \backslash K$, where $K := L_g \cup L_h$. Let $p_g \in \mathbb{P}^2 \backslash K$ be the point that satisfies*

$$\varphi|_{L_g} = \varphi_{p_g}^K|_{L_g},$$

where $\varphi := \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K$. Then we have $p_g \in L_l \backslash K$.

Proof Suppose g, h, l are non-concurrent. Let $A \in L_g \backslash (L_h \cup L_l)$ and define $A' := \varphi(A)$. By Lemma 2.2, we have $p_g = l \times (A \times A')$.

Now let g, h, l be concurrent lines and assume that $p_g \notin L_l$. We can define $B := g \times (r \times p_g) \in L_g \backslash L_h$. Note that $\varphi(B) = \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K(B) = \varphi_{p_g}^K(B)$, which implies $\varphi_q^K \circ \varphi_p^K(B) = B$. By Lemma 2.1, we have $p = q$. Hence we see that $r = p_g$, which is a contradiction. □

Lemma 5.3 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct, concurrent lines and $p, q \in L_l \backslash K$ be two distinct points, where $K := L_g \cup L_h$. Let $p_g, p_h \in \mathbb{P}^2 \backslash K$ be the points that satisfy*

$$\varphi|_{L_g} = \varphi_{p_g}^K|_{L_g} \qquad \textit{and} \qquad \varphi|_{L_h} = \varphi_{p_h}^K|_{L_h},$$

where $\varphi := \varphi_p^K \circ \varphi_q^K \circ \varphi_p^K$. Then $p_g = p_h \neq q$.

Proof We have $r := p_g = p_h$ by Remark 2.5 d). Assume towards contradiction that $q = r$. Then we have

$$\varphi_q^K \circ \varphi_p^K = \varphi_p^K \circ \varphi_q^K.$$

We can assume without loss of generality that $o := g \times h = (0 : 0 : 1)$ and $g = (1 : 0 : 0)$. Then $h = (h_1 : 1 : 0)$ and $p = (1 : p_2 : p_3)$. We can write $q = ao + bp = (b : bp_2 : a + bp_3)$, where $a, b \in \mathbb{R} \backslash \{0\}$. For $A := (0 : 1 : 0)$, we have $A \in L_g \backslash L_h$. Now we define $A' := M_q^g M_p^h(A)$ and $A'' := M_p^g M_q^h(A)$. We can calculate that

$$\begin{aligned}
A' &= \begin{pmatrix} 0 & 0 & 0 \\ bp_2 & -b & 0 \\ a + bp_3 & 0 & -b \end{pmatrix} \begin{pmatrix} -p_2 & 1 & 0 \\ h_1 p_2 & -h_1 & 0 \\ h_1 p_3 & p_3 & -(h_1 + p_2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ bp_2 & -b & 0 \\ a + bp_3 & 0 & -b \end{pmatrix} \begin{pmatrix} 1 \\ -h_1 \\ p_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ b(h_1 + p_2) \\ a \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
A'' &= \begin{pmatrix} 0 & 0 & 0 \\ p_2 & -1 & 0 \\ p_3 & 0 & -1 \end{pmatrix} \begin{pmatrix} -bp_2 & b & 0 \\ bh_1 p_2 & -bh_1 & 0 \\ h_1(a + bp_3) & a + bp_3 & -b(h_1 + p_2) \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ p_2 & -1 & 0 \\ p_3 & 0 & -1 \end{pmatrix} \begin{pmatrix} b \\ -bh_1 \\ a + bp_3 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ b(h_1 + p_2) \\ -a \end{pmatrix}.
\end{aligned}$$

Now we have $A' \times A'' = -2ab(h_1 + p_2)g \neq 0$, since $h \cdot p \neq 0$. This is a contradiction to $\varphi_q^K \circ \varphi_p^K = \varphi_p^K \circ \varphi_q^K$. □

Proposition 5.4 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p, q, r \in L_l \backslash K$, where $K := L_g \cup L_h$. Let $p_g \in \mathbb{P}^2 \backslash K$ be the point that satisfies*

$$\varphi|_{L_g} = \varphi_{p_g}^K|_{L_g},$$

where $\varphi := \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K$.

(i) If g, h, l are non-concurrent, then

$$(G, H; p, q) = (G, H; p_g, r),$$

where $G := g \times l$ and $H := h \times l$.

(ii) If g, h, l are concurrent and $p \neq r$, then

$$(o, r; p, q) = (o, p; r, p_g),$$

where $o := g \times h$.

(iii) If g, h, l are concurrent, $p \neq q$ and $p = r$, then

$$(o, p; q, p_g) = -1,$$

where $o := g \times h$.

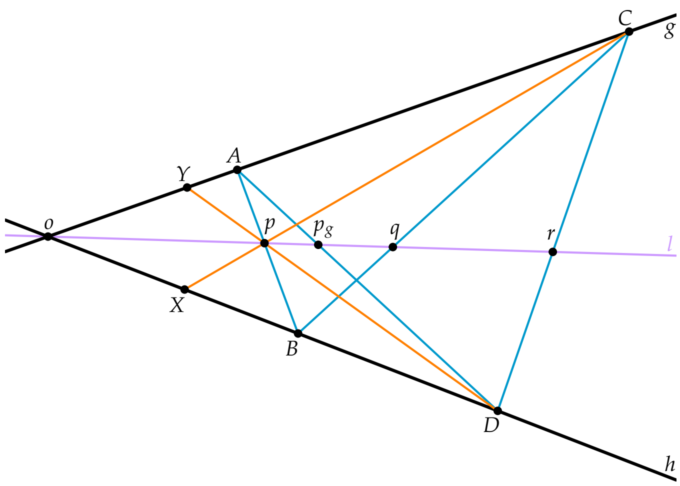

Figure 5.1: The proof for Proposition 5.4 (ii).

Proof Note that we have $p_g \in L_l \backslash K$ by Lemma 5.2. Let $A \in L_g \backslash (L_h \cup L_l)$ and define $B := \varphi_p^K(A)$, $C := \varphi_q^K(B)$ and $D := \varphi_r^K(C)$. We use Lemma 4.3 on all three cases.

(i) We have

$$\begin{aligned}(G, H; p, q) &= (B \times G, h; B \times p, B \times q)\\ &= (G, g \times h; A, C)\\ &= (D \times G, h; D \times A, D \times C)\\ &= (G, H; p_g, r).\end{aligned}$$

(ii) For $X := \varphi_p^K(C)$ and $Y := \varphi_p^K(D)$, we get

$$\begin{aligned}(o, r; p, q) &= (g, C \times r; C \times p, C \times q)\\ &= (o, D; X, B)\\ &= (l, p \times D; p \times X, p \times B)\\ &= (o, Y; C, A)\\ &= (h, D \times Y; D \times C, D \times A)\\ &= (o, p; r, p_g).\end{aligned}$$

(iii) Define $k := A \times D$ and $x := k \times (B \times C)$. Then

$$\begin{aligned}(o, p; q, p_g) &= (g, C \times p; C \times q, C \times p_g)\\ &= (k \times g, k \times (C \times p); k \times (C \times q), k \times (C \times p_g))\\ &= (A, D; x, p_g)\\ &= (B \times A, h; B \times x, B \times p_g)\\ &= (p, o; q, p_g).\end{aligned}$$

The result now follows from Remark 4.2 e), Lemma 4.5 and Lemma 5.3. □

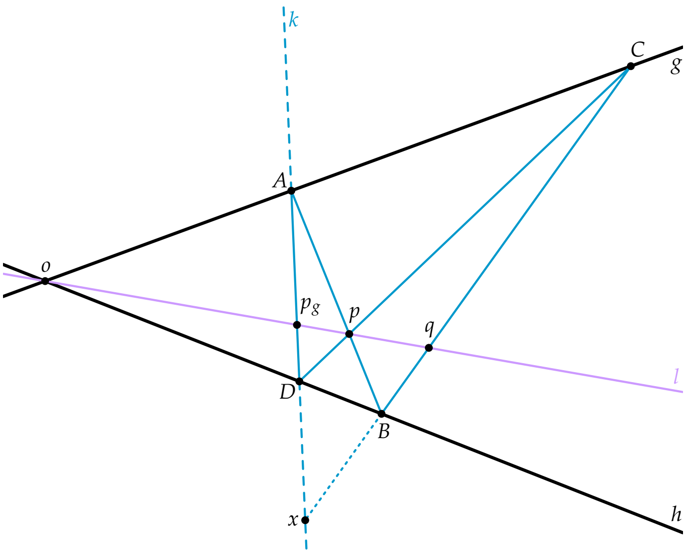

Figure 5.2: The proof for Proposition 5.4 (iii).

Theorem 5.5 (Butterfly Theorem) *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p, q, r \in \mathbb{P}^2 \setminus K$, where $K := L_g \cup L_h$. Define $\varphi := \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K$ and $p_g, p_h \in \mathbb{P}^2 \setminus K$ such that*

$$\varphi|_{L_g} = \varphi_{p_g}^K|_{L_g} \qquad \text{and} \qquad \varphi|_{L_h} = \varphi_{p_h}^K|_{L_h}.$$

We have $p_g = p_h$ if and only if the points p, q, r are collinear.

Proof If $p_g = p_h$, then we define $s := p_g$. By Remark 2.5 c), we have

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K.$$

The collinearity of p, q, r now follows from Theorem 5.1.

Suppose that p, q, r are collinear. We can assume without loss of generality that $p \neq q$. By exchanging g and h, we can see that Proposition 5.4 also holds for the point p_h. The result now follows from Lemma 4.4. Note that we used Remark 4.2 e) twice here. □

Remark 5.6 For three collinear points $p, q, r \in \mathbb{P}^2 \setminus K$, where $K := L_g \cup L_h$ and $g, h \in \mathbb{P}^2$ are two distinct lines, we have

$$\varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \varphi_s^K$$

by Theorem 5.5. Therefore we can write three collinear reversions as one. Note that $s \in \mathbb{P}^2 \setminus K$ lies on the same line as p, q, r by Lemma 5.2.

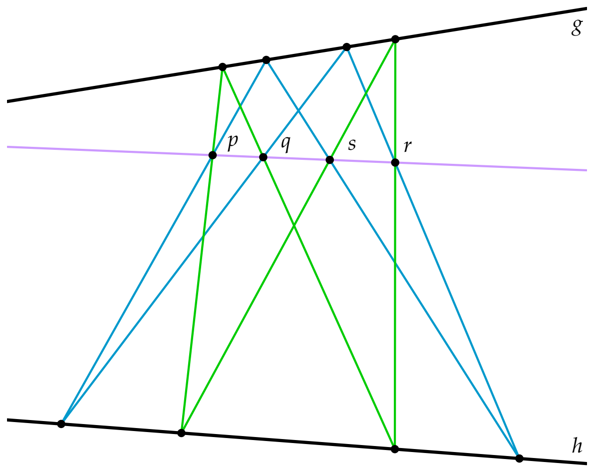

Figure 5.3: If the three points p, q, r are collinear, then there is a fourth point s on one line with p, q, r such that $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$. The blue quadrilateral has its starting point on g and the green one starts on h.

Porism 5.7 (Butterfly Porism) *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p, q, r \in L_l \setminus K$, where $K := L_g \cup L_h$. Then there exists a unique point $s \in L_l \setminus K$ such that the polygonal chain $A_1A_2A_3A_4A_5$ that passes through p, q, r, s with respect to K closes for every starting point $A_1 \in K$.*

Proof This is a consequence of Theorem 5.5. $\square$

Corollary 5.8 *Let $g, h, l \in \mathbb{P}^2$ be three pairwise distinct lines and $p_1, \ldots, p_{2n} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. If there is a point $x \in K \backslash (L_l \cup \{g \times h\})$ that satisfies $\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1}(x) = x$, then*

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

Proof By Remark 5.6, we can keep reducing the number of collinear reversions by two until we are left with exactly two reversions. Hence we can assume without loss of generality that $n = 1$, which is the case covered by Lemma 2.1. $\square$

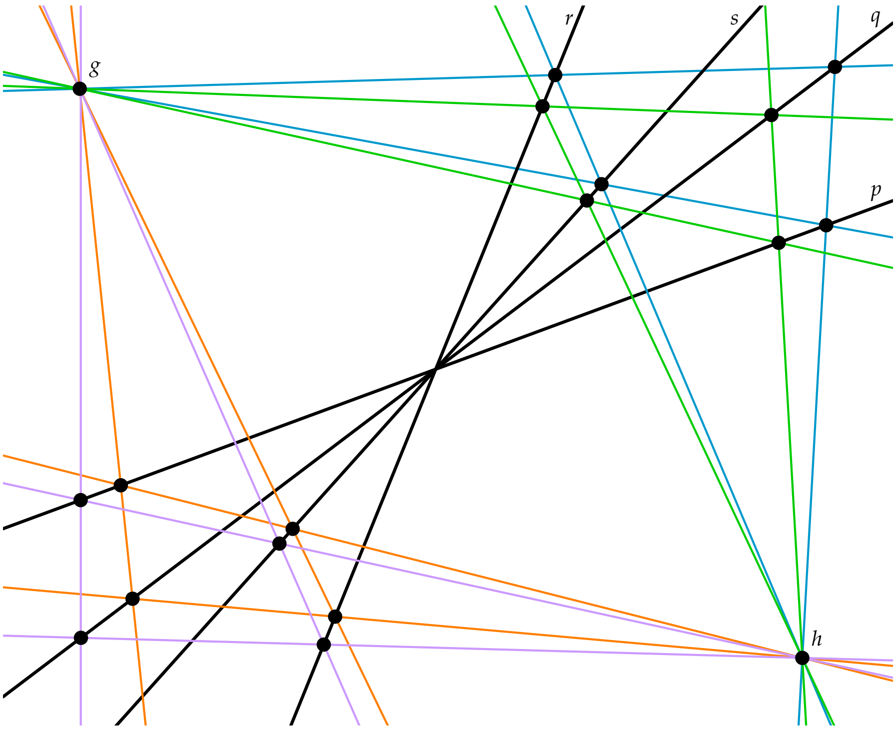

Figure 5.4: The dual version of Theorem 5.5, i.e. points and lines are interchanged.

Porism 5.9 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p_1, \ldots, p_{2n} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. If there is a starting point $A_1 \in K \backslash (L_l \cup \{g \times h\})$ such that the polygonal chain $A_1 \ldots A_{2n+1}$ that passes through $p_1, \ldots, p_{2n}$ with respect to K closes, then the same polygonal chain closes for every starting point $A_1 \in K$.*

Proof This is a consequence of Corollary 5.8. $\square$

Remark 5.10 Corollary 5.8 implies the Scissors Theorem [4, Satz (S)].

Theorem 5.11 (Izmestiev) *Let $g, h, l \in \mathbb{P}^2$ be three pairwise distinct lines and let $p, q, r, s \in L_l \backslash K$, where $K := L_g \cup L_h$.*

(i) If g, h, l are non-concurrent, then

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K \qquad \Longleftrightarrow \qquad (G, H; p, q) = (G, H; s, r),$$

where $G := g \times l$ and $H := h \times l$.

(ii) If g, h, l are concurrent and $p \neq r$, then

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K \qquad \Longleftrightarrow \qquad (o, r; p, q) = (o, p; r, s),$$

where $o := g \times h$.

(iii) If g, h, l are concurrent, $p \neq q$ and $p = r$, then

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K \qquad \Longleftrightarrow \qquad (o, p; q, s) = -1,$$

where $o := g \times h$.

Proof The result follows immediately from Lemma 4.4, Proposition 5.4 and Theorem 5.5. □

6 Collinear Reversions

We have seen in Theorem 5.11 that the equation $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$, where $p, q, r, s \in L_l \backslash K$ and $K := L_g \cup L_h$ for three pairwise distinct lines $g, h, l \in \mathbb{P}^2$, does not depend on g and h, but on the points $o := g \times h$, $G := g \times l$ and $H := h \times l$. We specify the consequences of that theorem in this section.

Theorem 6.1 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p, q, r, s \in L_l \backslash K$, where $K := L_g \cup L_h$. We define $K' := L_{g'} \cup L_{h'}$, where*

$$g' := o' \times (g \times l) \qquad \text{and} \qquad h' := o' \times (h \times l)$$

for $o' \in \mathbb{P}^2 \backslash L_l$. If $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$, then $\varphi_s^{K'} \circ \varphi_r^{K'} \circ \varphi_q^{K'} \circ \varphi_p^{K'} = \mathrm{id}|_{K'}$.

Proof This result follows immediately from Theorem 5.11 (i). □

Corollary 6.2 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p_1, \ldots, p_{2n} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. We define $K' := L_{g'} \cup L_{h'}$, where*

$$g' := o' \times (g \times l) \qquad \text{and} \qquad h' := o' \times (h \times l)$$

for some point $o' \in \mathbb{P}^2 \backslash L_l$. If $\varphi_{p_{2n}}^K \circ \cdots \circ \varphi_{p_1}^K = \mathrm{id}|_K$, then $\varphi_{p_{2n}}^{K'} \circ \cdots \circ \varphi_{p_1}^{K'} = \mathrm{id}|_{K'}$.

Proof We prove by induction on n. The base case $n = 1$ follows immediately from Lemma 2.1. So let $n > 1$. We can use Remark 5.6 to write

$$\varphi^K_{p_{2n}} \circ \varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n-2}} = \varphi^K_p$$

for some $p \in L_l \backslash K$. By Theorem 6.1, we have

$$\varphi^{K'}_{p_{2n}} \circ \varphi^{K'}_{p_{2n-1}} \circ \varphi^{K'}_{p_{2n-2}} = \varphi^{K'}_p.$$

We can now use the induction hypothesis to get

$$\varphi^{K'}_p \circ \varphi^{K'}_{p_{2n-3}} \circ \cdots \circ \varphi^{K'}_{p_1} = \mathrm{id}|_{K'}.$$

Putting the last two equations together yields

$$\varphi^{K'}_{p_{2n}} \circ \cdots \circ \varphi^{K'}_{p_1} = \mathrm{id}|_{K'},$$

which concludes the proof. □

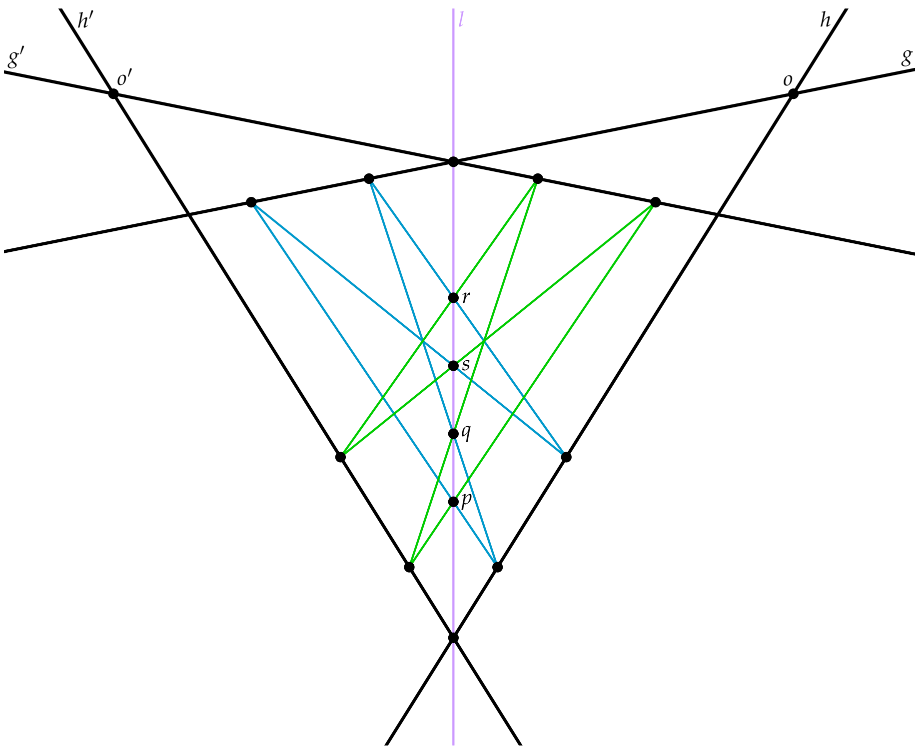

Figure 6.1: Visualization of Theorem 6.1. Here K' is the reflection of K at l.

Theorem 6.3 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct, concurrent lines and $p, q, r, s \in L_l \backslash K$, where $K := L_g \cup L_h$. We define $o := g \times h$ and $K' := L_{g'} \cup L_{h'}$, where $g', h' \in L_o \backslash \{l\}$. If $\varphi^K_s \circ \varphi^K_r \circ \varphi^K_q \circ \varphi^K_p = \mathrm{id}|_K$, then $\varphi^{K'}_s \circ \varphi^{K'}_r \circ \varphi^{K'}_q \circ \varphi^{K'}_p = \mathrm{id}|_{K'}$.*

Proof We can assume without loss of generality that $g' \neq h'$. The result follows immediately from Theorem 5.11. $\square$

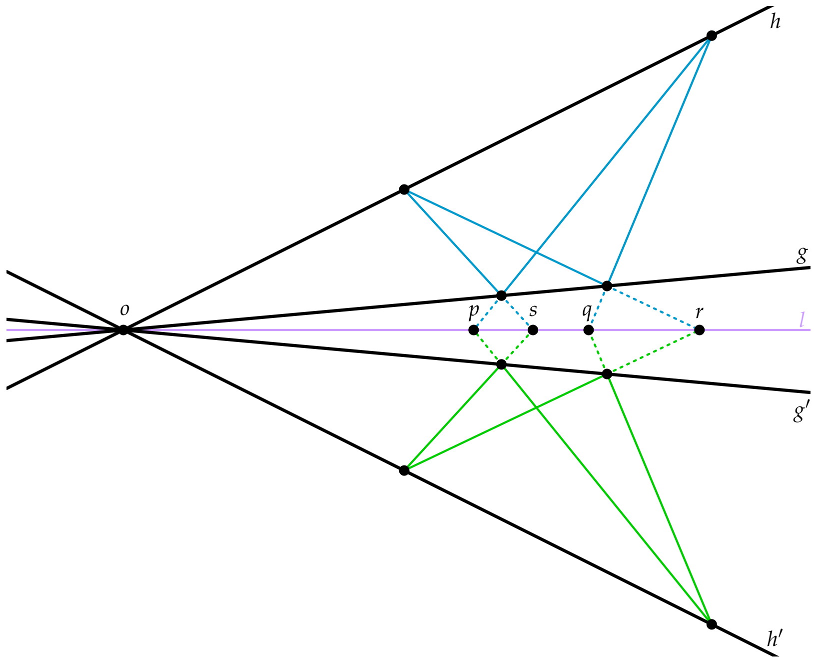

Figure 6.2: Visualization of Theorem 6.3. Here K' is the reflection of K at l.

Corollary 6.4 *Let $g, h, l \in \mathbb{P}^2$ be three pairwise distinct, concurrent lines and $p_1, \dots, p_{2n} \in L_l \setminus K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. We define $o := g \times h$ and $K' := L_{g'} \cup L_{h'}$, where $g', h' \in L_o \setminus \{l\}$. If $\varphi^K_{p_{2n}} \circ \dots \circ \varphi^K_{p_1} = \mathrm{id}|_K$, then $\varphi^{K'}_{p_{2n}} \circ \dots \circ \varphi^{K'}_{p_1} = \mathrm{id}|_{K'}$.*

Proof Again we can assume that $g' \neq h'$. We prove by induction on n, analogously to the proof of Corollary 6.2. The base case $n = 1$ follows immediately from Lemma 2.1. So let $n > 1$. We can use Remark 5.6 to write

$$\varphi^K_{p_{2n}} \circ \varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n-2}} = \varphi^K_p$$

for some $p \in L_l \setminus K$. By Theorem 6.3, we have

$$\varphi^{K'}_{p_{2n}} \circ \varphi^{K'}_{p_{2n-1}} \circ \varphi^{K'}_{p_{2n-2}} = \varphi^{K'}_p.$$

We can now use the induction hypothesis to get

$$\varphi^{K'}_p \circ \varphi^{K'}_{p_{2n-3}} \circ \dots \circ \varphi^{K'}_{p_1} = \mathrm{id}|_{K'}.$$

Putting the last two equations together yields

$$\varphi^{K'}_{p_{2n}} \circ \dots \circ \varphi^{K'}_{p_1} = \mathrm{id}|_{K'},$$

which concludes the proof. $\square$

Porism 6.5 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p_1, \ldots, p_{2n} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Moreover, let $K' := L_{g'} \cup L_{h'}$ such that $K \cap L_l = K' \cap L_l$, where $g', h' \in \mathbb{P}^2$ are lines. If the polygonal chain $A_1 \ldots A_{2n+1}$ that passes through $p_1, \ldots, p_{2n}$ with respect to K closes for every starting point $A_1 \in K$, then the polygonal chain $B_1 \ldots B_{2n+1}$ that passes through $p_1, \ldots, p_{2n}$ with respect to K' also closes for every starting point $B_1 \in K'$.*

Proof This is a consequence of Corollary 6.2 and Corollary 6.4. □

Corollary 6.6 *Let $l \in \mathbb{P}^2$ be a line and $p, q, r, s \in L_l$ be points such that $p \neq q$, $q \neq r$, $r \neq s$ and $s \neq p$. Moreover, let $x \in L_l \backslash \{p, q, r, s\}$. There is a unique point $y \in L_l \backslash \{p, q, r, s\}$ such that for every $o \in \mathbb{P}^2 \backslash L_l$, we have*

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K,$$

where $g := o \times x$, $h := o \times y$ and $K := L_g \cup L_h$.

Proof Let $g' \in L_x \backslash \{l\}$ and $A \in \mathbb{P}^2 \backslash (L_{g'} \cup L_l)$. We define

$$B := \varphi_p^{g'}(A) \qquad \text{and} \qquad D := \varphi_s^{g'}(A).$$

Since $A \notin L_{g'} \cup L_l$ and $p \neq s$, we see that $B, D \in L_{g'} \backslash \{x\}$ are two distinct points. Furthermore, we define

$$C := (q \times B) \times (r \times D).$$

We have $C \notin L_{g'} \cup L_l$, because $q \neq r$ and $B \neq D$. Note that $p \neq q$ implies $A \neq C$, hence the line $h' := A \times C$ is well defined. Lastly, we define $y := l \times h'$. Now $p \neq q$ and $r \neq s$ implies $y \notin \{p, q, r, s\}$. By construction, we have

$$\varphi_s^{K'} \circ \varphi_r^{K'} \circ \varphi_q^{K'} \circ \varphi_p^{K'}(A) = A,$$

where $K' := L_{g'} \cup L_{h'}$. Thus we can apply Corollary 5.8 to get

$$\varphi_s^{K'} \circ \varphi_r^{K'} \circ \varphi_q^{K'} \circ \varphi_p^{K'} = \mathrm{id}|_{K'}.$$

By Theorem 6.1, we see that for every $o \in \mathbb{P}^2 \backslash L_l$, we have

$$\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K,$$

where $g := o \times x$, $h := o \times y$ and $K := L_g \cup L_h$.

To show uniqueness, let $y' := L_l \backslash \{p, q, r, s, y\}$ and define $h'' := y' \times A$. Since

$$\varphi_q^{h''} \circ \varphi_p^{g'}(A) \neq \varphi_r^{h''} \circ \varphi_s^{g'}(A),$$

we see that

$$\varphi_s^{K''} \circ \varphi_r^{K''} \circ \varphi_q^{K''} \circ \varphi_p^{K''} \neq \mathrm{id}|_{K''},$$

where $K'' := L_{g'} \cup L_{h''}$. □

Chapter III

Main Theorem

7 Pappus's Hexagon Theorem

In this section, we give an alternative proof for Pappus's Hexagon Theorem [5, Theorem 1.1], which we will use to prove the main result.

Lemma 7.1 *Let $g, h, l \in \mathbb{P}^2$ be pairwise distinct lines and $p, q, r \in L_l \backslash K$, where $K := L_g \cup L_h$. Define $o := g \times h$ and let p_g be the point that satisfies*

$$\varphi_r^K \circ \varphi_q^K \circ \varphi_p^K|_{L_g} = \varphi_{p_g}^K|_{L_g}.$$

Then we have

$$(g, h; o \times p, o \times q) = (g, h; o \times p_g, o \times r).$$

Proof If g, h, l are concurrent, then the result follows from Lemma 5.2. If g, h, l are non-concurrent, then we can use Proposition 5.4 (i) to conclude. □

Theorem 7.2 *Let $g, h \in \mathbb{P}^2$ be two distinct lines and $p, q, r \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$. Define $o := g \times h$ and let $p', q', r' \in \mathbb{P}^2 \backslash K$ be points such that the triples*

$$o, p, p', \qquad o, q, q' \quad \text{and} \quad o, r, r'$$

are all respectively collinear. We can write

$$\varphi_r^K \circ \varphi_q^K \circ \varphi_p^K|_{L_g} = \varphi_{p_g}^K|_{L_g} \qquad \text{and} \qquad \varphi_{r'}^K \circ \varphi_{q'}^K \circ \varphi_{p'}^K|_{L_g} = \varphi_{p'_g}^K|_{L_g}.$$

The triple o, p_g, p'_g is also collinear.

Proof By Lemma 7.1, we have

$$\begin{aligned}(g, h; o \times p'_g, o \times r') &= (g, h; o \times p', o \times q') \\ &= (g, h; o \times p, o \times q) \\ &= (g, h; o \times p_g, o \times r).\end{aligned}$$

We use Lemma 4.4 to conclude. □

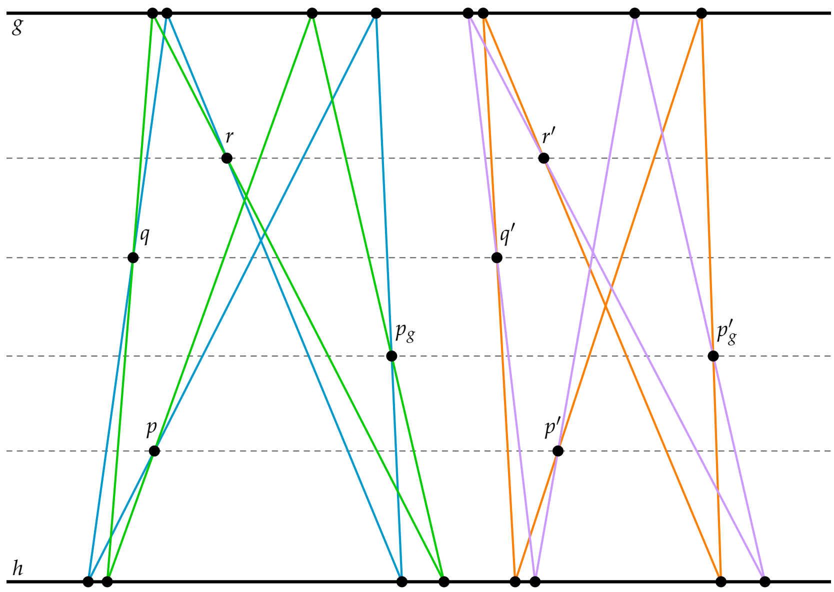

Figure 7.1: Visualization of Theorem 7.2, where the point o is at infinity.

Corollary 7.3 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $o := g \times h$ and let $p'_1, \ldots, p'_{2n-1} \in \mathbb{P}^2 \backslash K$ be points such that the triple o, p_k, p'_k is collinear for all $k \in \{1, \ldots, 2n-1\}$. We can write*

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \textit{and} \qquad \varphi^K_{p'_{2n-1}} \circ \cdots \circ \varphi^K_{p'_1}|_{L_g} = \varphi^K_{p'_g}|_{L_g}.$$

The triple o, p_g, p'_g is also collinear.

Proof We prove by induction on n. Case $n = 1$ is trivial, so let $n > 1$. We can write

$$\varphi^K_{p_{2n-3}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{q_g}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p'_{2n-3}} \circ \cdots \circ \varphi^K_{p'_1}|_{L_g} = \varphi^K_{q'_g}|_{L_g}.$$

By induction hypothesis, the points o, q_g, q'_g are collinear. Now we have

$$\varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n-2}} \circ \varphi^K_{q_g}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p'_{2n-1}} \circ \varphi^K_{p'_{2n-2}} \circ \varphi^K_{q'_g}|_{L_g} = \varphi^K_{p'_g}|_{L_g}.$$

The result follows from Theorem 7.2. □

Corollary 7.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $\varphi := \varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}$ and $o := g \times h$. Moreover, let $p_g, p_h \in \mathbb{P}^2 \backslash K$ be the points that satisfy*

$$\varphi|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \textit{and} \qquad \varphi|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

The triple o, p_g, p_h is collinear.

Proof Define $l := o \times p_g$ and let $p'_1, \ldots, p'_{2n-1} \in \mathbb{P}^2 \backslash K$ be collinear points such that the triple o, p_k, p'_k is also collinear for all $k \in \{1, \ldots, 2n-1\}$. The points p'_g and p'_h that are defined by

$$\varphi^K_{p'_{2n-1}} \circ \cdots \circ \varphi^K_{p'_1}|_{L_g} = \varphi^K_{p'_g}|_{L_g} \quad \text{and} \quad \varphi^K_{p'_{2n-1}} \circ \cdots \circ \varphi^K_{p'_1}|_{L_h} = \varphi^K_{p'_h}|_{L_h}$$

satisfy $p'_g = p'_h$, which follows from Remark 5.6. Moreover, the point p'_g lies on l by Corollary 7.3 and thus p_h lies on l as well, which is again a consequence of Corollary 7.3. □

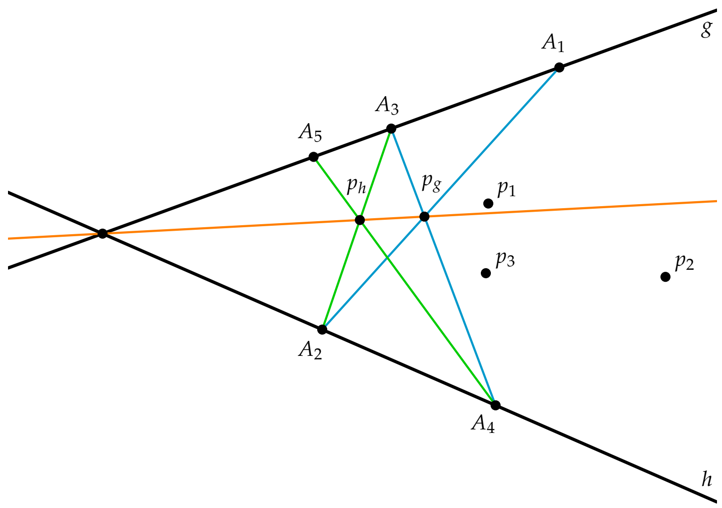

Figure 7.2: Another way to construct p_g and p_h is to take a starting point $A_1 \in L_g \backslash L_h$ and define $A_{k+1} := \varphi(A_k)$ for all $k \in \{1, 2, 3, 4\}$, where $\varphi := \varphi^K_{p_3} \circ \varphi^K_{p_2} \circ \varphi^K_{p_1}$ and $K := L_g \cup L_h$. Now p_g and p_h are the intersection points of the line pairs $(\overline{A_1A_2}, \overline{A_3A_4})$ and $(\overline{A_2A_3}, \overline{A_4A_5})$ respectively. We can see that the line through p_g and p_h is incident to the intersection point of g and h, as predicted in Corollary 7.4.

Corollary 7.5 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $\varphi := \varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}$. If there is a point $x \in K \backslash \{g \times h\}$ that satisfies $\varphi \circ \varphi(x) = x$, then $\varphi \circ \varphi = \mathrm{id}|_K$.*

Proof Let p_g and p_h be the points corresponding to φ. We can assume without loss of generality that $x \in L_g \backslash L_h$. We now have

$$x = \varphi \circ \varphi(x) = \varphi^K_{p_h} \circ \varphi^K_{p_g}(x).$$

Let $l \in \mathbb{P}^2$ be the line through p_g and the intersection point of g and h. Note that p_h is incident to l, by Corollary 7.4. The result now follows from Lemma 2.1 and Remark 2.5 d). □

Porism 7.6 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. If a polygonal chain $A_1 \ldots A_{4n-1}$ that passes through $p_1, \ldots, p_{2n-1}$ twice with respect to K closes for a starting point $A_1 \in K \backslash \{g \times h\}$, then the same polygonal chain closes for every starting point $A_1 \in K$.*

Proof This is a consequence of Corollary 7.5. □

Theorem 7.7 (Pappus's Hexagon Theorem) *Let $g, h \in \mathbb{P}^2$ be two lines. Furthermore, let $A_1, A_2, A_3 \in L_g$ and $B_1, B_2, B_3 \in L_h$. We define p, q and r to be the intersection points of the line pairs $(\overline{A_1B_2}, \overline{A_2B_1})$, $(\overline{A_1B_3}, \overline{A_3B_1})$ and $(\overline{A_2B_3}, \overline{A_3B_2})$ respectively. If existent, the points p, q, r are collinear.*

Proof We can assume without loss of generality that g and h are distinct and we define $K := L_g \cup L_h$. We can also assume that $A_1, A_2, A_3 \in L_g \backslash L_h$ and $B_1, B_2, B_3 \in L_h \backslash L_g$ are six pairwise distinct points. Let p_g and p_h be the points that satisfy

$$\varphi|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \text{and} \qquad \varphi|_{L_h} = \varphi^K_{p_h}|_{L_h},$$

where $\varphi := \varphi^K_r \circ \varphi^K_q \circ \varphi^K_p$. By construction, we have $\varphi \circ \varphi(A_2) = A_2$, hence Corollary 7.5 implies that

$$\varphi \circ \varphi = \mathrm{id}|_K.$$

We have $p_g = p_h$ by Remark 2.5 d). Lastly, Theorem 5.5 implies that p, q, r are collinear. □

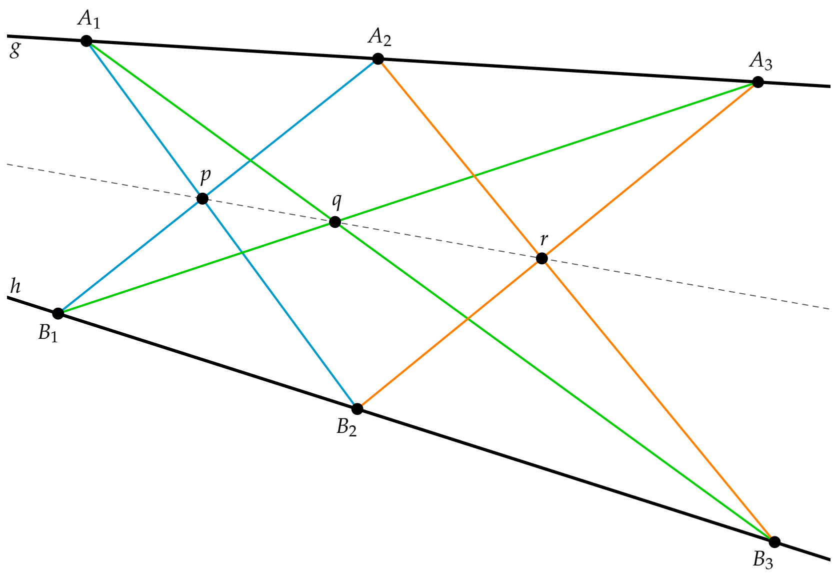

Figure 7.3: Visualization of Pappus's Hexagon Theorem. The line through p, q, r is called *Pappus line*. Other proofs of Theorem 7.7 can be found in [5].

8 Main Theorem

We have already seen compositions of reversions that equal the identity on one or even both lines, but we often had to make strong assumptions to get the identity on both lines. Now we want to see, whether arbitrary compositions of reversions can be turned into the identity on both lines by adding a few points. Of course, if we have a composition of n reversions, we could just add the same points in reverse order to get the identity on both lines. But this can also be done by adding a number of points, which does not depend on n. A first step towards this claim are the following results.

Proposition 8.1 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \dots, p_{2n-2} \in \mathbb{P}^2 \backslash K$ be points, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} \neq \mathrm{id}|_{L_g} \qquad \text{and} \qquad \varphi|_{L_h} \neq \mathrm{id}|_{L_h}.$$

Then there exist two points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

Proof Let $A \in L_g$ be a point that is not fixed by φ and define $A' := \varphi(A)$. Furthermore, let $B, C \in L_h$ be two distinct points that are also not fixed by φ. Again we define $B' := \varphi(B)$ and $C' := \varphi(C)$.

Let S and T be the intersection points of the line pairs $(\overline{AB'}, \overline{A'B})$ and $(\overline{AC'}, \overline{A'C})$ respectively. Note that $A, A', B, B' \notin L_g \cap L_h$. This implies that $\overline{AB'}$, $\overline{A'B}$, g and h are pairwise distinct lines. Hence S is different from A and sits on $\overline{AB'}$. Since $B' \neq C'$, we see that $S \neq T$. By symmetry, we also have $A, A' \notin \{S, T\}$. If we assume that A lies on $\overline{ST}$, then this implies that A, B', C' are collinear. This is a contradiction. The same argument can be used to show that A' is not incident to $\overline{ST}$ either.

Now let $A'' \in L_h \backslash \{B, B', C, C'\}$ be a point that does not lie on $\overline{ST}$ and is not the intersection point of g and h. We define p_{2n-1} and p_{2n} as intersection points of the line pairs $(\overline{ST}, \overline{A'A''})$ and $(\overline{ST}, \overline{AA''})$ respectively. Since A, A', A'' are all not incident to $\overline{ST}$, we see that $p_{2n-1}, p_{2n} \notin K$. We can now define $B'' := \varphi_{p_{2n-1}}(B')$ and $C'' := \varphi_{p_{2n-1}}(C')$.

By construction, p_{2n-1} is the intersection point of the line pair $(\overline{A'A''}, \overline{B'B''})$ and also of the line pair $(\overline{A'A''}, \overline{C'C''})$.

Let U and V be the intersection points of the line pairs $(\overline{AA''}, \overline{BB''})$ and $(\overline{AA''}, \overline{CC''})$ respectively. Consider the hexagon $A'A''AB'B''B$ as well as $A'A''AC'C''C$. Using Theorem 7.7, we see that the triple p_{2n-1}, U, S is collinear and so is the triple p_{2n-1}, V, T.

Assume that $p_{2n-1} = S$. Since $A' \neq S$, this implies $\overline{A'B} = \overline{A'A''}$ and in particular $B = A''$, which is a contradiction. Analogously, one can verify that $p_{2n-1} \neq T$. Since p_{2n-1} lies on $\overline{ST}$ by definition, we see that the two Pappus lines of the considered hexagons coincide and are both equal to $\overline{ST}$.

Since U, V, p_{2n} are all incident to the two distinct lines $\overline{ST}$ and $\overline{AA''}$ at the same time, we can conclude that $U = V = p_{2n}$.

Hence we have

$$\varphi^K_{p_{2n}}(A'') = A,$$
$$\varphi^K_{p_{2n}}(B'') = B,$$
$$\varphi^K_{p_{2n}}(C'') = C,$$

and $\varphi^K_{p_{2n-1}}(A') = A''$. We can write

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

Since p_{2n} is the intersection point of $\overline{BB''}$ and $\overline{CC''}$, we see that $p_h = p_{2n}$. By Corollary 7.4, we know that p_g is the intersection point of $\overline{AA''}$ and the line through p_h and the intersection point of g and h. Hence $p_g = p_{2n} = p_h$ and therefore

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1} = \varphi^K_{p_{2n}},$$

which is equivalent to $\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K$. □

Remark 8.2 Consider the line $\overline{ST}$, as it was constructed in Proposition 8.1. We have seen that there are two points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ on $\overline{ST}$ such that $\varphi = \varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n}}$. Suppose there are two other points p'_{2n-1} and p'_{2n} that satisfy $\varphi = \varphi^K_{p'_{2n-1}} \circ \varphi^K_{p'_{2n}}$. Then we must have

$$\varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n}} \circ \varphi^K_{p'_{2n}} \circ \varphi^K_{p'_{2n-1}} = \mathrm{id}|_K.$$

By Theorem 5.1, the points $p_{2n-1}, p_{2n}, p'_{2n-1}, p'_{2n}$ are collinear. Hence the two points p'_{2n-1}, p'_{2n} also lie on $\overline{ST}$, which implies that the line $\overline{ST}$ does not depend on the choice of the points A, B, C. Since φ can be written as the composition of two reversions, whose corresponding points lie on $\overline{ST}$, we see that the fixed points of φ are the intersection point of g and h, but also the points in K that sit on the line $\overline{ST}$. By Lemma 2.1, φ has no other fixed points.

Theorem 8.3 (Main Theorem) *Let $g, h \in \mathbb{P}^2$ be lines and $p_1, \ldots, p_{2n-2} \in \mathbb{P}^2 \backslash K$ be points, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} \neq \mathrm{id}|_{L_g} \qquad \text{and} \qquad \varphi|_{L_h} \neq \mathrm{id}|_{L_h}.$$

Moreover, let $A \in L_g$ and $A'' \in L_h$ be points that are not fixed by φ and define $A' := \varphi(A)$. Then there exist two unique points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that $\varphi^K_{p_{2n-1}}(A') = A''$, $\varphi^K_{p_{2n}}(A'') = A$ and

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

Proof Let $B, C \in L_h$ be two distinct points that are not fixed by φ and satisfy $A'' \notin \{B, B', C, C'\}$, where $B' := \varphi(B)$ and $C' := \varphi(C)$. Since A'' does not lie on $\overline{ST}$ by Remark 8.2, we can use the same proof as in Proposition 8.1 to show existence. Uniqueness follows also from Remark 8.2. $\square$

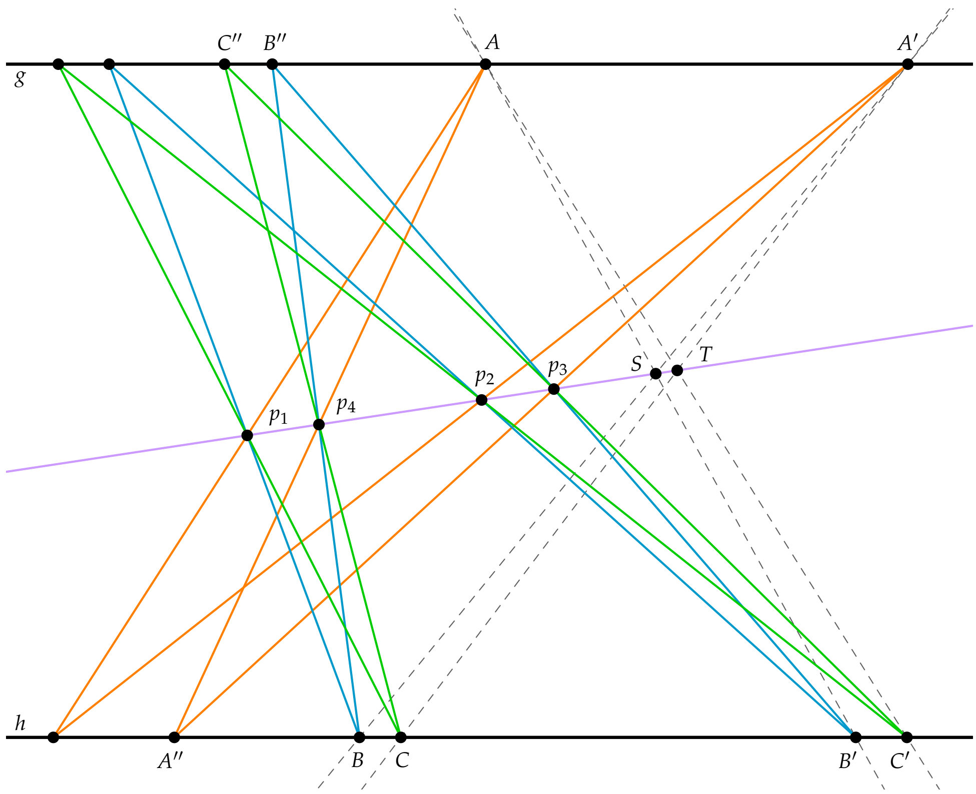

Figure 8.1: Theorem 8.3 for $n = 2$. The lines $\overline{A'A''}$ and $\overline{AA''}$ uniquely define the locations of p_3 and p_4. Note that $\overline{ST}$ passes through p_1 and p_2 by Theorem 5.1.

Porism 8.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-2} \in \mathbb{P}^2 \backslash K$ be points, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} \neq \mathrm{id}|_{L_g} \quad \text{and} \quad \varphi|_{L_h} \neq \mathrm{id}|_{L_h}.$$

Let $A_1 \ldots A_{2n} A_1$ be a closed polygonal chain such that $A_1 \in L_g$ and $A_{2n} \in L_h$ are both not fixed by φ and the polygonal chain $A_1 \ldots A_{2n-1}$ passes through $p_1, \ldots, p_{2n-2}$ with respect to K. Then there are two unique points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that every polygonal chain that passes through $p_1, \ldots, p_{2n}$ with respect to K closes for every starting point in K, including $A_1 \ldots A_{2n} A_1$.

Proof This is a consequence of Theorem 8.3. $\square$

Corollary 8.5 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$, be points such that $\varphi := \varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} \neq \mathrm{id}|_K$. Then there exists a line $l \in \mathbb{P}^2$ such that the fixed points of φ are given by the set $(K \cap L_l) \cup \{g \times h\}$.*

Proof If $\varphi|_{L_g} \neq \mathrm{id}|_{L_g}$ and $\varphi|_{L_h} \neq \mathrm{id}|_{L_h}$, then the result follows from Remark 8.2. Suppose that $\varphi|_{L_g} = \mathrm{id}|_{L_g}$. We claim that l is equal to g in this case. Assume that there is a point $x \in L_h \backslash L_g$ that is fixed by φ. Consider the points p_g and p_h that are defined by

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

Now we see that $p_g \neq p_h$ and $p_g = p_{2n}$. Therefore

$$x = \varphi(x) = \varphi^K_{p_g} \circ \varphi^K_{p_h}(x).$$

By Lemma 2.1 and Corollary 7.4, the map $\varphi^K_{p_g} \circ \varphi^K_{p_h}$ can only fix the intersection point of g and h. Hence we get a contradiction. □

Definition 8.6 Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$, be points such that $\varphi := \varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} \neq \mathrm{id}|_K$. Furthermore, let $l \in \mathbb{P}^2$ be the line such that the fixed points of φ are given by the set $(K \cap L_l) \cup \{g \times h\}$. We call the elements of $K \cap L_l$ *non-trivial fixed points* of φ.

9 Identity on One Line

In Lemma 2.4, we have seen that a composition of an odd number of reversions can always be turned into the identity on one line by adding a point. For an even number of reversions, we can simply add an arbitrary point and then use Lemma 2.4 to get the identity on one line. But similarly to Theorem 8.3, we now want to consider the situation, where not only the reversion points are given, but also a polygonal chain.

Theorem 9.1 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-2} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, be points such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} \neq \mathrm{id}|_{L_g}.$$

Let $A \in L_g$ be a point that is not fixed by φ, $A' := \varphi(A)$ and $A'' \in L_h \backslash L_g$. Finally, let $p_{2n-1} \in L_{A' \times A''} \backslash K$ and $p_{2n} \in L_{A \times A''} \backslash K$ be points such that the triple x, p_{2n-1}, p_{2n} is collinear, where $x \in L_g$ is the non-trivial fixed point of φ. Then we have

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \mathrm{id}|_{L_g}.$$

Proof Let $o \in \mathbb{P}^2$ be the intersection point of g and h. If $x \in L_g \backslash L_h$, then the map $\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g}$ fixes three pairwise distinct points, namely o, x and A. The result then follows from Lemma 2.2. Hence we can assume without loss of generality that $x = o$. We prove by induction on n, so let $n = 2$. Note that the triple o, p_1, p_2 is collinear in this case. Define the points p'_1, p'_2, p'_3 such that $p'_1 = p'_2 = p_1$ and $p'_3 = p_3$. Now we can write

$$\varphi^K_{p_3} \circ \varphi^K_{p_2} \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \quad \text{and} \quad \varphi^K_{p'_3} \circ \varphi^K_{p'_2} \circ \varphi^K_{p'_1}|_{L_g} = \varphi^K_{p'_g}|_{L_g}.$$

Let $l \in \mathbb{P}^2$ be the line trough x, p_3 and p_4. By Theorem 7.2, we see that o, p_g and $p'_g = p_3$ are collinear. Hence p_g is the intersection point of l and $\overline{AA''}$, which is p_4.

Now let $n > 2$. We can write

$$\varphi^K_{p_3} \circ \varphi^K_{p_2} \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{q_g}|_{L_g}.$$

Since we still have

$$\varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_4} \circ \varphi^K_{q_g}|_{L_g} = \varphi|_{L_g} \neq \mathrm{id}|_{L_g},$$

we can use the induction hypothesis to get

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_4} \circ \varphi^K_{q_g}|_{L_g} = \mathrm{id}|_{L_g},$$

which concludes the proof. □

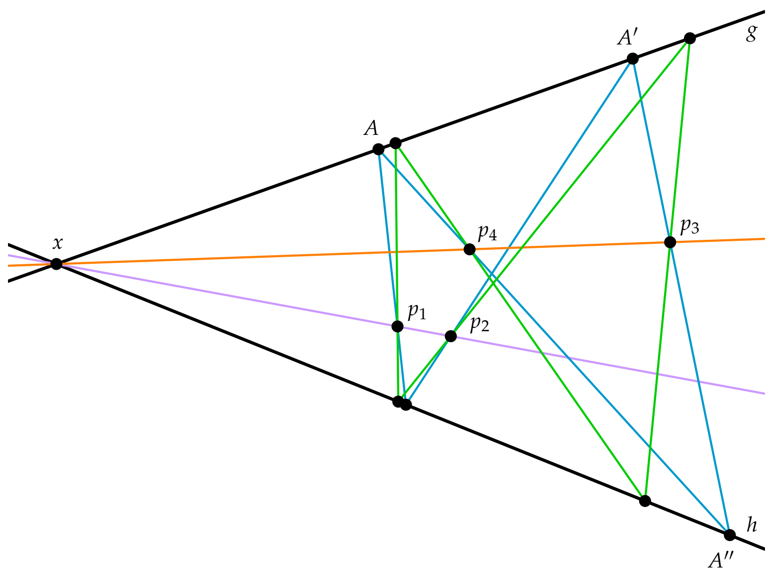

Figure 9.1: Visualization of Theorem 9.1 for $n = 2$. Here the non-trivial fixed point x is equal to the intersection point of g and h. Note that the map $\varphi^K_{p_4} \circ \varphi^K_{p_3} \circ \varphi^K_{p_2} \circ \varphi^K_{p_1}$, where $K := L_g \cup L_h$, fixes all points on g, but not on h, since the points p_1, p_2, p_3, p_4 are not collinear.

Porism 9.2 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-2} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, be points such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} \neq \mathrm{id}|_{L_g}.$$

Let $A_1 \ldots A_{2n} A_1$ be a closed polygonal chain such that $A_1 \in L_g$ is not fixed by φ, $A_{2n} \in L_h \backslash L_g$ and the polygonal chain $A_1 \ldots A_{2n-1}$ passes through $p_1, \ldots, p_{2n-2}$ with respect to K. Finally, let $p_{2n-1} \in L_{A_{2n-1} \times A_{2n}} \backslash K$ and $p_{2n} \in L_{A_1 \times A_{2n}} \backslash K$ be points such that the triple x, p_{2n-1}, p_{2n} is collinear, where $x \in L_g$ is the non-trivial fixed point of φ. Then every polygonal chain that passes through $p_1, \ldots, p_{2n}$ with respect to K closes for every starting point on g, including $A_1 \ldots A_{2n} A_1$.

Proof This is a consequence of Theorem 9.1. □

10 Arbitrary Compositions of Reversions

Finally, we prove the claim about arbitrary compositions of reversions mentioned in Section 8.

Lemma 10.1 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-3} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$. There exist three points $p_{2n-2}, p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that*

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

Proof We can write

$$\varphi^K_{p_{2n-3}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p_{2n-3}} \circ \cdots \circ \varphi^K_{p_1}|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

Let $p_{2n-2} \notin \{p_g, p_h\}$. Then

$$\varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} \neq \mathrm{id}|_{L_g} \qquad \text{and} \qquad \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}|_{L_h} \neq \mathrm{id}|_{L_h}.$$

Hence we can use Proposition 8.1 to conclude. □

Corollary 10.2 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-2} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 2}$, be such that $\varphi := \varphi^K_{p_{2n-2}} \circ \cdots \circ \varphi^K_{p_1}$ satisfies*

$$\varphi|_{L_g} = \mathrm{id}|_{L_g} \qquad \text{and} \qquad \varphi|_{L_h} \neq \mathrm{id}|_{L_h}.$$

For all points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$, we have

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} \neq \mathrm{id}|_K.$$

Proof Assume towards contradiction that there are two points $p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that $\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K$. Then

$$\varphi = \varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n}}.$$

Since $\varphi|_{L_g} = \mathrm{id}|_{L_g}$, we have $p_{2n-1} = p_{2n}$ by Lemma 2.1. Thus $\varphi = \mathrm{id}|_K$, which is a contradiction. □

Corollary 10.3 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \ldots, p_{2n-4} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{Z}_{\geq 3}$. There exist four points $p_{2n-3}, p_{2n-2}, p_{2n-1}, p_{2n} \in \mathbb{P}^2 \backslash K$ such that*

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \mathrm{id}|_K.$$

Proof The point p_{2n-3} can be chosen arbitrarily. The result then immediately follows from Lemma 10.1. □

Porism 10.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and define $K := L_g \cup L_h$. Every finite set of points in $\mathbb{P}^2 \backslash K$ can be extended by at most four points such that every polygonal chain that passes through the extended set of points with respect to K closes for every starting point on K.*

Proof This is a consequence of Lemma 10.1 and Corollary 10.3. □

Chapter IV

The Polar Line

11 Conjugated Points

Roughly, the polar line of a point p with respect to some non-degenerate conic is a line that connects the tangent points, if two tangents can be drawn from p to the conic. Interestingly, there is also a way to define the polar line for a degenerate case of a conic, more specifically for two distinct lines. Using the polar line, we can define the conjugation of points. We will show in this section that if a composition of collinear reversions equals the identity on both lines, then the conjugated points also yield a composition of reversions that equals the identity on both lines.

Definition 11.1 Let $g, h \in \mathbb{P}^2$ be distinct lines, $p \in \mathbb{P}^2 \backslash \{g \times h\}$ and $K := L_g \cup L_h$. The *polar line* of p with respect to K is defined as

$$l_p := (h \cdot p)g + (g \cdot p)h.$$

Remarks 11.2 Let $g, h \in \mathbb{P}^2$ be distinct lines and let l_p be the polar line of $p \in \mathbb{P}^2 \backslash \{g \times h\}$ with respect to $K := L_g \cup L_h$.

a) The lines g, h and l_p are concurrent by definition.

b) We have $p \in K \backslash \{g \times h\}$ if and only if $l_p \in \{g, h\}$.

c) The line l_p is incident to p if and only if $p \in K \backslash \{g \times h\}$, because

$$l_p \cdot p = 2(g \cdot p)(h \cdot p).$$

Definition 11.3 Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p \in L_l \backslash K$, where $K := L_g \cup L_h$. The point $\bar{p} := l_p \times l \in L_l \backslash K$ is called the *conjugated* point of p with respect to l and K, where l_p is the polar line of p with respect to K.

Lemma 11.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p, q \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$. Moreover, let l_p and l_q be the polar lines of p and q with respect to K. If $l_p \cdot q = 0$, then $l_q \cdot p = 0$. In particular, the process of conjugation is an involution.*

Proof If q lies on l_p, then q is the conjugated point of p with respect to $l := p \times q$ and K. Thus we have

$$l_q \cdot p = (h \cdot q)(g \cdot p) + (g \cdot q)(h \cdot p) = l_p \cdot q = 0.$$

Hence the conjugated point $\bar{q} := l_q \times l$ is indeed equal to p. □

Corollary 11.5 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p, q \in \mathbb{P}^2 \backslash K$ be two points such that o, p, q are collinear, where $o := g \times h$ and $K := L_g \cup L_h$. Then the polar lines l_p and l_q of p and q with respect to K are equivalent.*

Proof Let $r \in L_{l_p} \backslash \{o\}$. Since the polar line l_r of r with respect to K is incident to o, we see that $l_r \cdot q = 0$ if and only if $l_r \cdot p = 0$. As seen in Lemma 11.4, we have $l_r \cdot p = l_p \cdot r = 0$. Hence $l_q \cdot r = l_r \cdot q = 0$. □

Lemma 11.6 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p, q \in L_l \backslash K$, where $K := L_g \cup L_h$. We define $G := g \times l$ and $H := h \times l$. Moreover, let l_p and l_q be the polar lines of p and q with respect to K. Now we have*

$$(g, h; l_p, l_q) = (G, H; p, q).$$

Proof We define $o := g \times h$. First note that

$$[o, g, l_p] = ((h \cdot p)g + (g \cdot p)h) \cdot (o \times g) = (g \cdot p)[o, g, h]$$

and

$$[o, g, o \times p] = (o \times p) \cdot (o \times g) = (o \cdot o)(g \cdot p) - (g \cdot o)(o \cdot p) = (o \cdot o)(g \cdot p).$$

Hence

$$(g, h; l_p, l_q) = \frac{[o, g, l_p][o, h, l_q]}{[o, g, l_q][o, h, l_p]} = \frac{(g \cdot p)(h \cdot q)}{(g \cdot q)(h \cdot p)}$$

and by Lemma 4.3, we also have

$$(G, H; p, q) = (g, h; o \times p, o \times q) = \frac{[o, g, o \times p][o, h, o \times q]}{[o, g, o \times q][o, h, o \times p]} = \frac{(g \cdot p)(h \cdot q)}{(g \cdot q)(h \cdot p)},$$

which concludes the proof. □

Theorem 11.7 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p, q, r, s \in L_l \backslash K$, where $K := L_g \cup L_h$. If $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$, then*

$$\varphi_{\bar{s}}^K \circ \varphi_{\bar{r}}^K \circ \varphi_{\bar{q}}^K \circ \varphi_{\bar{p}}^K = \mathrm{id}|_K,$$

where $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ are the conjugated points of p, q, r, s with respect to l and K.

Proof We define $o := g \times h$, $G := g \times l$ and $H := h \times l$. Using a combination of Lemma 4.3, Theorem 5.11 and Lemma 11.6, we get

$$\begin{aligned}(G, H; \bar{p}, \bar{q}) &= (g, h; l_p, l_q) \\ &= (G, H; p, q) \\ &= (G, H; s, r) \\ &= (g, h; l_s, l_r) \\ &= (G, H; \bar{s}, \bar{r}).\end{aligned}$$

The result now follows from Theorem 5.11. □

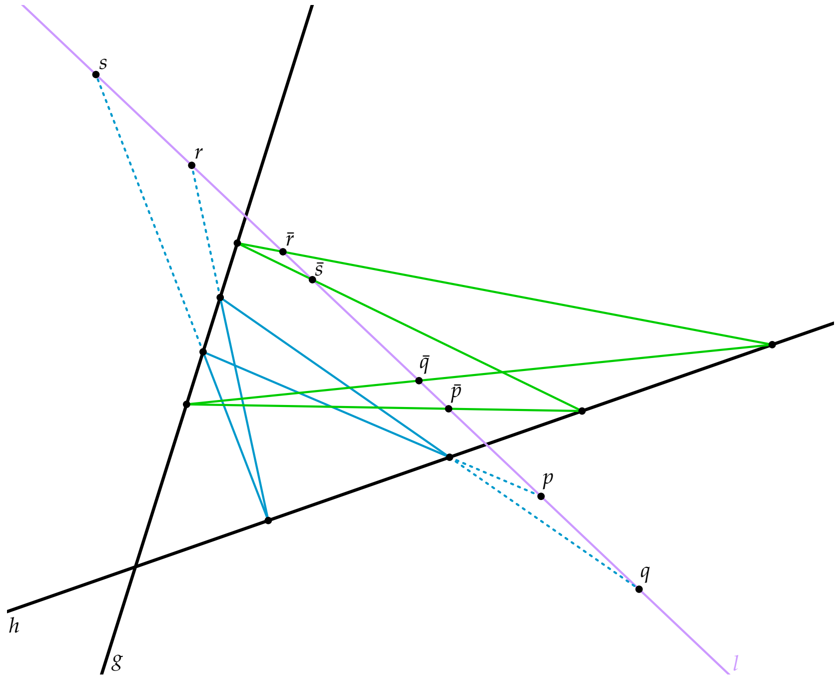

Figure 11.1: Visualization of Theorem 11.7. The blue quadrilateral corresponds to the points p, q, r, s and starts on g. The green quadrilateral corresponds to the conjugated points $\bar{p}, \bar{q}, \bar{r}, \bar{s}$ and starts on h. By Corollary 5.8, we can conclude from this picture that $\varphi_s^K \circ \varphi_r^K \circ \varphi_q^K \circ \varphi_p^K = \varphi_{\bar{s}}^K \circ \varphi_{\bar{r}}^K \circ \varphi_{\bar{q}}^K \circ \varphi_{\bar{p}}^K = \mathrm{id}|_K$, where $K := L_g \cup L_h$.

Corollary 11.8 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p_1, \ldots, p_{2n} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. If $\varphi_{p_{2n}}^K \circ \cdots \circ \varphi_{p_1}^K = \mathrm{id}|_K$, then*

$$\varphi_{\overline{p_{2n}}}^K \circ \cdots \circ \varphi_{\overline{p_1}}^K = \mathrm{id}|_K,$$

where $\overline{p_1}, \ldots, \overline{p_{2n}}$ are the conjugated points of $p_1, \ldots, p_{2n}$ with respect to l and K.

Proof We prove by induction on n, similarly to Corollary 6.2. The base case $n = 1$ follows immediately from Lemma 2.1. So let $n > 1$. We can use Remark 5.6 to write

$$\varphi^K_{p_{2n}} \circ \varphi^K_{p_{2n-1}} \circ \varphi^K_{p_{2n-2}} = \varphi^K_p$$

for some point $p \in L_l \backslash K$. By Theorem 11.7, we have

$$\varphi^K_{\overline{p_{2n}}} \circ \varphi^K_{\overline{p_{2n-1}}} \circ \varphi^K_{\overline{p_{2n-2}}} = \varphi^K_{\overline{p}}.$$

We can now use the induction hypothesis to get

$$\varphi^K_{\overline{p}} \circ \varphi^K_{\overline{p_{2n-3}}} \circ \cdots \circ \varphi^K_{\overline{p_1}} = \mathrm{id}|_K.$$

Putting the last two equations together yields

$$\varphi^K_{\overline{p_{2n}}} \circ \cdots \circ \varphi^K_{\overline{p_1}} = \mathrm{id}|_K,$$

which concludes the proof. □

Porism 11.9 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines, $p_1, \ldots, p_{2n} \in L_l \backslash K$ be points and $\overline{p_1}, \ldots, \overline{p_{2n}} \in L_l \backslash K$ be the conjugated points of $p_1, \ldots, p_{2n}$ with respect to l and K, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. If the polygonal chain $A_1 \ldots A_{2n+1}$ that passes through $p_1, \ldots, p_{2n}$ with respect to K closes for every starting point $A_1 \in K$, then the polygonal chain $B_1 \ldots B_{2n+1}$ that passes through $\overline{p_1}, \ldots, \overline{p_{2n}}$ with respect to K also closes for every starting point $B_1 \in K$.*

Proof This is a consequence of Corollary 11.8. □

Corollary 11.10 *Let $g, h, l \in \mathbb{P}^2$ be non-concurrent lines and $p_1, \ldots, p_{2n-1} \in L_l \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Furthermore, let $\overline{p_1}, \ldots, \overline{p_{2n-1}}$ be the conjugated points of $p_1, \ldots, p_{2n-1}$ with respect to l and K. Then the two points $p_g, \overline{p_g} \in \mathbb{P}^2 \backslash K$ that satisfy*

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}|_{L_g} = \varphi^K_{p_g}|_{L_g} \qquad \textit{and} \qquad \varphi^K_{\overline{p_{2n-1}}} \circ \cdots \circ \varphi^K_{\overline{p_1}}|_{L_g} = \varphi^K_{\overline{p_g}}|_{L_g}$$

are conjugated to each other with respect to l and K.

Proof Since the points $p_1, \ldots, p_{2n-1}$ and $\overline{p_1}, \ldots, \overline{p_{2n-1}}$ are collinear, we can use Remark 5.6 to consecutively replace three reversions by one until we are left with just one reversion, i.e. we can write

$$\varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1} = \varphi^K_{p_g} \qquad \text{and} \qquad \varphi^K_{\overline{p_{2n-1}}} \circ \cdots \circ \varphi^K_{\overline{p_1}} = \varphi^K_{p'_g}.$$

Now we can use Corollary 11.8 to get

$$\varphi^K_{\overline{p_g}} = \varphi^K_{\overline{p_{2n-1}}} \circ \cdots \circ \varphi^K_{\overline{p_1}} = \varphi^K_{p'_g}.$$

By Lemma 2.1, we have $\overline{p_g} = p'_g$. □

12 Construction of the Polar Line

So far we only have a mathematical expression for the polar line, using the standard scalar product. Now we want to show a construction of the polar line, for which only a ruler is used.

Lemma 12.1 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$. Define $o := g \times h$ and let $l \in L_o$. Now we have*

$$(g, h; o \times p, l) = -1 \qquad \Longleftrightarrow \qquad l = l_p,$$

where l_p is the polar line of p with respect to K

Proof Let $q \in L_{l_p} \backslash \{o\}$. Then we have $l_q \cdot p = 0$ by Lemma 11.4, where l_q is the polar line of q with respect to K. Define $G := g \times (p \times q)$ and $H := h \times (p \times q)$. Note that $G \neq H$ by Remark 11.2 c). Using Lemma 4.3 and Lemma 11.6, we get

$$\begin{aligned}(g, h; o \times p, l_p) &= (g, h; l_q, l_p) \\ &= (G, H; p, q) \\ &= (g, h; l_p, l_q) \\ &= (g, h; l_p, o \times p).\end{aligned}$$

By Lemma 4.5, we have $(g, h; o \times p, l_p) = -1$.

If $(g, h; o \times p, l) = -1$, then $l = l_p$ by Lemma 4.4. □

Remark 12.2 By Lemma 12.1, we see that the polar line could also be defined in terms of harmonic lines.

Theorem 12.3 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p, q \in \mathbb{P}^2 \backslash K$ be distinct points, where $K := L_g \cup L_h$. The polar line of p with respect to K is incident to q, i.e. $l_p \cdot q = 0$, if and only if*

$$\varphi_q^K \circ \varphi_p^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K.$$

Proof Let $o := g \times h$ and $l := p \times q$. We denote the polar line of q with respect to K by l_q.

Suppose that $l_p \cdot q = 0$. Then we have $l \cdot o \neq 0$ by Remark 11.2 c). We define $G := g \times l$ and $H := h \times l$. Lemma 11.4 implies that $l_q \cdot p = 0$. Using Lemma 4.3 and Lemma 11.6, we get

$$(G, H; p, q) = (g, h; l_p, l_q) = (G, H; q, p).$$

The result now follows from Theorem 5.11 (i).

For the other direction, suppose that $\varphi_q^K \circ \varphi_p^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$. By Lemma 5.3, we have $l \cdot o \neq 0$. Again we can define $G := g \times l$ and $H := h \times l$. Theorem 5.11 (i) implies that $(G, H; p, q) = (G, H; q, p)$. Thus we get

$$(g, h; o \times p, o \times q) = (G, H; p, q) = -1$$

by Lemma 4.5. The result now follows from Lemma 12.1. □

Corollary 12.4 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p \in \mathbb{P}^2 \backslash K$ be a point, where $K := L_g \cup L_h$. Moreover, let $A, B \in L_g \backslash L_h$ be two distinct points with images $A' := \varphi_p^K(A)$ and $B' := \varphi_p^K(B)$. The intersection point q of the two lines $\overline{AB'}$ and $\overline{A'B}$ is not equal to $o := g \times h$ and sits on the polar line l_p of p with respect to K. In particular, the line through o and q is precisely the polar line l_p.*

Proof Since $A, B \in L_g \backslash L_h$ and $A', B' \in L_h \backslash L_g$ are four pairwise distinct points, we see that $q \in \mathbb{P}^2 \backslash K$ and $p \neq q$. Note that A and B cannot both be incident to the line $p \times q$, but they are both fixed by $\varphi_q^K \circ \varphi_p^K \circ \varphi_q^K \circ \varphi_p^K$. Hence Corollary 5.8 implies that $\varphi_q^K \circ \varphi_p^K \circ \varphi_q^K \circ \varphi_p^K = \mathrm{id}|_K$. By Theorem 12.3, the point q lies on l_p. □

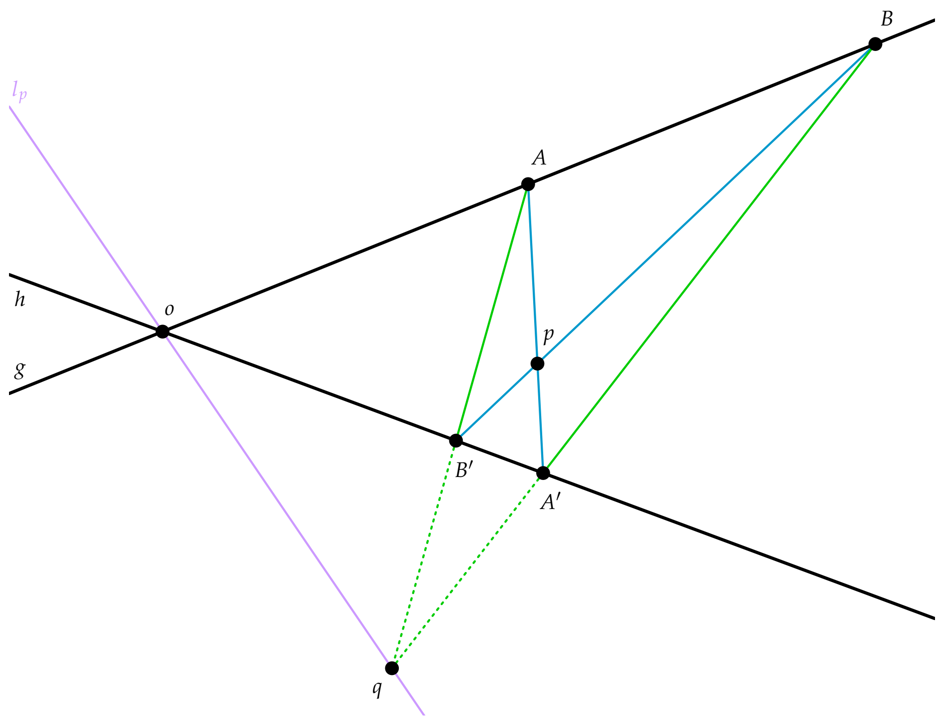

Figure 12.1: A way to construct the polar line of p.

Corollary 12.5 *Let $g, h \in \mathbb{P}^2$ be distinct lines and $p_1, \dots, p_{2n-1} \in \mathbb{P}^2 \backslash K$, where $K := L_g \cup L_h$ and $n \in \mathbb{N}$. Define $p_g \in \mathbb{P}^2 \backslash K$ to be the point that satisfies*

$$\varphi|_{L_g} = \varphi^K_{p_g}|_{L_g},$$

where $\varphi := \varphi^K_{p_{2n-1}} \circ \cdots \circ \varphi^K_{p_1}$. Moreover, let l_{p_g} be the polar line of p_g with respect to K. For $p_{2n} \in L_{l_{p_g}} \backslash \{g \times h\}$, we have $\varphi^K_{p_{2n}} \circ \varphi \circ \varphi^K_{p_{2n}} \circ \varphi = \mathrm{id}|_K$, which is equivalent to

$$\varphi^K_{p_{2n}} \circ \cdots \circ \varphi^K_{p_1} = \varphi^K_{p_1} \circ \cdots \circ \varphi^K_{p_{2n}}.$$

Proof By Corollary 7.4 and Corollary 11.5, we have $l_{p_g} = l_{p_h}$, where l_{p_h} is the polar line of p_h with respect to K and $p_h \in \mathbb{P}^2 \backslash K$ is the point that satisfies

$$\varphi|_{L_h} = \varphi^K_{p_h}|_{L_h}.$$

Theorem 12.3 implies that

$$\varphi^K_{p_{2n}} \circ \varphi \circ \varphi^K_{p_{2n}} \circ \varphi|_{L_g} = \varphi^K_{p_{2n}} \circ \varphi^K_{p_g} \circ \varphi^K_{p_{2n}} \circ \varphi^K_{p_g}|_{L_g} = \mathrm{id}|_{L_g}$$

and

$$\varphi^K_{p_{2n}} \circ \varphi \circ \varphi^K_{p_{2n}} \circ \varphi|_{L_h} = \varphi^K_{p_{2n}} \circ \varphi^K_{p_h} \circ \varphi^K_{p_{2n}} \circ \varphi^K_{p_h}|_{L_h} = \mathrm{id}|_{L_h},$$

which concludes the proof. □

13 Invertible Matrix

Let $g, h \in \mathbb{P}^2$ be two lines and $p \in \mathbb{P}^2 \backslash K$ be a point, where $K := L_g \cup L_h$. As seen in Section 1, we are able to split a reversion into two parts, i.e.

$$\varphi^K_p(x) = \begin{cases} \varphi^h_p(x) & \text{for } x \in L_g, \\ \varphi^g_p(x) & \text{for } x \in L_h. \end{cases}$$

We have seen that both maps φ^h_p and φ^g_p can be represented by matrices M^h_p and M^g_p respectively. But there is also a way to represent φ^K_p by a single matrix M^K_p such that $\varphi^K_p(x) = M^K_p x$ for all $x \in K$.

Consider the matrix

$$M^K_p := (h \cdot p) p g^T + (g \cdot p) p h^T - (g \cdot p)(h \cdot p)\mathbb{1},$$

where $\mathbb{1} \in \mathbb{R}^{3\times 3}$ is the identity matrix. First we are going to compute the eigenspaces of M^K_p. For $x \in L_{l_p}$, where l_p is the polar line of p with respect to K, we have

$$\begin{aligned} M^K_p x &= (h \cdot p)(g \cdot x)p + (g \cdot p)(h \cdot x)p - (g \cdot p)(h \cdot p)x \\ &= (l_p \cdot x)p - (g \cdot p)(h \cdot p)x \\ &= -(g \cdot p)(h \cdot p)x \end{aligned}$$

and for p we have $M_p^K p = (g \cdot p)(h \cdot p)p$. Thus the eigenspace of M_p^K corresponding to the eigenvalue $(g \cdot p)(h \cdot p)$ is given by the span of p and the eigenspace associated with the eigenvalue $-(g \cdot p)(h \cdot p)$ is the plane that has l_p as its normal vector. Since the determinant of a matrix is the product of its eigenvalues, we see that M_p^K is invertible if and only if $p \notin K$. Indeed, a short calculation shows that $(M_p^K)^2 = (g \cdot p)^2(h \cdot p)^2\mathbb{1}$. It remains to show that $\varphi_p^K(x) = M_p^K x$ for all $x \in K$. Without loss of generality, let $x \in L_g$. Then we have

$$\begin{aligned} h \cdot (M_p^K x) &= h \cdot ((g \cdot p)(h \cdot x)p - (g \cdot p)(h \cdot p)x) \\ &= (g \cdot p)(h \cdot x)(h \cdot p) - (g \cdot p)(h \cdot p)(h \cdot x) \\ &= 0, \end{aligned}$$

which shows that $M_p^K x \in L_h$ and

$$\begin{aligned} (M_p^K x) \cdot (x \times p) &= ((g \cdot p)(h \cdot x)p - (g \cdot p)(h \cdot p)x) \cdot (x \times p) \\ &= (g \cdot p)(h \cdot x)(p \cdot (x \times p)) - (g \cdot p)(h \cdot p)(x \cdot (x \times p)) \\ &= 0, \end{aligned}$$

which proves that $M_p^K x$ is incident to the line through x and p, i.e. x, p and $M_p^K x$ are collinear as required. Hence

$$\begin{aligned} M_p^K \colon \mathbb{P}^2 &\to \mathbb{P}^2 \\ x &\mapsto M_p^K x \end{aligned}$$

is a bijective map that is equal to φ_p^K if restricted to the set K.

Chapter V

Reversions Between Multiple Lines

14 Restricted Reversions

In this chapter, we want to consider reversions on any finite number of lines and not just two. If there are more than two lines involved, we need to specify which line a point should be mapped onto. One way to do this is to restrict a reversion to one line in the following way.

Definition 14.1 Let $g_1, \dots, g_n \in \mathbb{P}^2$ be lines, where $n \in \mathbb{N}$. Let $i, j \in \{1, \dots, n\}$ and $p \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{g_j})$. A *restricted reversion* through p from g_i to g_j is a map

$$\begin{aligned} \varphi_p^{g_j} : L_{g_i} &\to L_{g_j} \\ x &\mapsto \varphi_p^{g_j}(x) \end{aligned}$$

such that the three points $x, p, \varphi_p^{g_j}(x)$ are collinear. To indicate the domain of a restricted reversion $\varphi_p^{g_j}$, we also write $\varphi_p^{g_j}|_{L_{g_i}}$.

Remark 14.2 Note that a restricted reversion is a bijection from one line to another. The reversions that we defined in Chapter I were bijections from a set of two lines onto itself.

Lemma 14.3 *Let $g_1, \dots, g_n \in \mathbb{P}^2$ be lines and let*

$$\varphi := \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1} : L_{g_i} \to L_{h_m}$$

be a composition of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m\}$ and $h_m = g_i$. If φ fixes three pairwise distinct points on g_i, then $\varphi = \varphi|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}}$.

Proof Note that $p_1 \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_1})$ and $p_k \in \mathbb{P}^2 \backslash (L_{h_{k-1}} \cup L_{h_k})$ for all $k \in \{2, \dots, m\}$ by Definition 14.1. Since $\varphi|_{L_{g_i}}$ can be written as a matrix, we can use the same proof as for Lemma 2.2. □

Corollary 14.4 *Let $g_1, \dots, g_n \in \mathbb{P}^2$ be concurrent lines and let*

$$\varphi := \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1} \colon L_{g_i} \to L_{h_m}$$

be a composition of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m\}$ and $h_m \neq g_i$. Then there exists a unique point $p_{g_i} \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_m})$ such that

$$\varphi|_{L_{g_i}} = \varphi_{p_{g_i}}^{h_m}|_{L_{g_i}}.$$

Proof Let $A, B \in L_{g_i} \backslash \{o\}$ be two distinct points, where $o \in \mathbb{P}^2$ is the intersection point of $g_1, \dots, g_n$. Define $A' := \varphi(A)$ and $B' := \varphi(B)$. Now p_{g_i} is the intersection point of the line pair $(\overline{AA'}, \overline{BB'})$. Analogously to Lemma 2.4, we have $p_{g_i} \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_m})$. The result now follows from Lemma 14.3. □

Remark 14.5 As seen in the proof of Lemma 2.4, the equation

$$\varphi|_{L_{g_i}} = \varphi_{p_{g_i}}^{h_m}|_{L_{g_i}} \qquad \text{is equivalent to} \qquad \varphi_{p_{g_i}}^{g_i} \circ \varphi|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}}.$$

Hence we can also define $p_{m+1} := p_{g_i}$ to get

$$\varphi_{p_{m+1}}^{g_i} \circ \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1}|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}}.$$

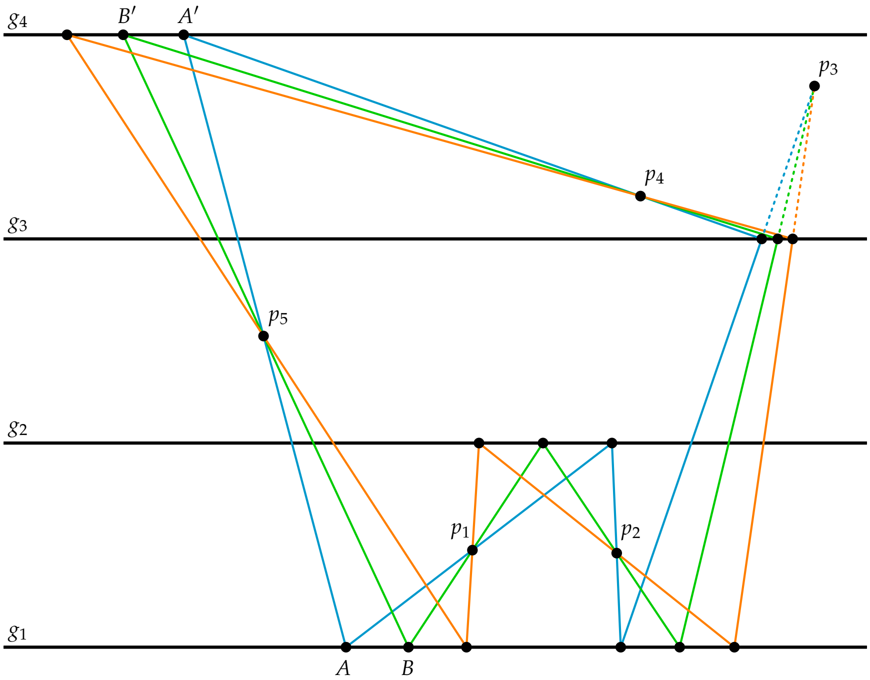

Figure 14.1: Corollary 14.4 for four parallel lines that meet in a point at infinity. Indeed we have $\varphi_{p_5}^{g_1} \circ \varphi_{p_4}^{g_4} \circ \varphi_{p_3}^{g_3} \circ \varphi_{p_2}^{g_1} \circ \varphi_{p_1}^{g_2}|_{L_{g_1}} = \mathrm{id}|_{L_{g_1}}$, since A, B and the intersection point of g_1, g_2, g_3, g_4 are fixed.

Porism 14.6 *Let $g_1, \dots, g_n \in \mathbb{P}^2$ be concurrent lines and let*

$$\varphi := \varphi_{p_m}^{h_m} \circ \dots \circ \varphi_{p_1}^{h_1} \colon L_{g_i} \to L_{h_m}$$

be a composition of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m\}$ and $h_m \neq g_i$. Then there exists a unique point $p_{m+1} \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_m})$ such that the polygonal chain $A_1 \dots A_{m+2}$ that passes through $p_1, \dots, p_{m+1}$ according to $\varphi_{p_{m+1}}^{g_i} \circ \varphi$ closes for every starting point $A_1 \in L_{g_i}$.

Proof This is a consequence of Corollary 14.4. □

Corollary 14.7 *Let $g_1, \dots, g_n \in \mathbb{P}^2$ be non-concurrent lines and let*

$$\varphi := \varphi_{p_{m-2}}^{h_{m-2}} \circ \dots \circ \varphi_{p_1}^{h_1} \colon L_{g_i} \to L_{h_{m-2}}$$

be a composition of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m-2\}$ and $h_{m-2} \neq g_i$. Let $j \in \{1, \dots, n\}$ be such that g_i, g_j and h_{m-2} are three non-concurrent lines and let $A := g_i \times g_j$. We define $A' := \varphi(A)$ and $l := A \times A'$. For any $p_{m-1} \in L_l \backslash (L_{g_j} \cup L_{h_{m-2}})$, there is a point $p_m \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{g_j})$ such that

$$\varphi_{p_m}^{g_i} \circ \varphi_{p_{m-1}}^{g_j} \circ \varphi_{p_{m-2}}^{h_{m-2}} \circ \dots \circ \varphi_{p_1}^{h_1}|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}}.$$

Proof Let $B, C \in L_{g_i} \backslash \{A\}$ be two distinct points and define

$$B'' := \varphi_{p_{m-1}}^{g_j} \circ \varphi(B) \qquad \text{and} \qquad C'' := \varphi_{p_{m-1}}^{g_j} \circ \varphi(C).$$

Lastly, we define $p_m \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{g_j})$ as the intersection point of the line pair $(\overline{BB''}, \overline{CC''})$. Now we see that

$$\varphi_{p_m}^{g_i} \circ \varphi_{p_{m-1}}^{g_j} \circ \varphi_{p_{m-2}}^{h_{m-2}} \circ \dots \circ \varphi_{p_1}^{h_1}|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}},$$

since the map $\varphi_{p_m}^{g_i} \circ \varphi_{p_{m-1}}^{g_j} \circ \varphi$ fixes A, B, C and thus all points on g_i by Lemma 14.3. □

Porism 14.8 *Let $g_1, \dots, g_n \in \mathbb{P}^2$ be non-concurrent lines and let*

$$\varphi := \varphi_{p_{m-2}}^{h_{m-2}} \circ \dots \circ \varphi_{p_1}^{h_1} \colon L_{g_i} \to L_{h_{m-2}}$$

be a composition of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m-2\}$ and $h_{m-2} \neq g_i$. Let $j \in \{1, \dots, n\}$ be such that g_i, g_j and h_{m-2} are three non-concurrent lines and let $A := g_i \times g_j$. We define $A' := \varphi(A)$ and $l := A \times A'$. For any $p_{m-1} \in L_l \backslash (L_{g_j} \cup L_{h_{m-2}})$, there is a point $p_m \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{g_j})$ such that the polygonal chain $A_1 \dots A_{m+1}$ that passes through $p_1, \dots, p_m$ according to $\varphi_{p_m}^{g_i} \circ \varphi_{p_{m-1}}^{g_j} \circ \varphi$ closes for every starting point $A_1 \in L_{g_i}$.

Proof This is a consequence of Corollary 14.7. □

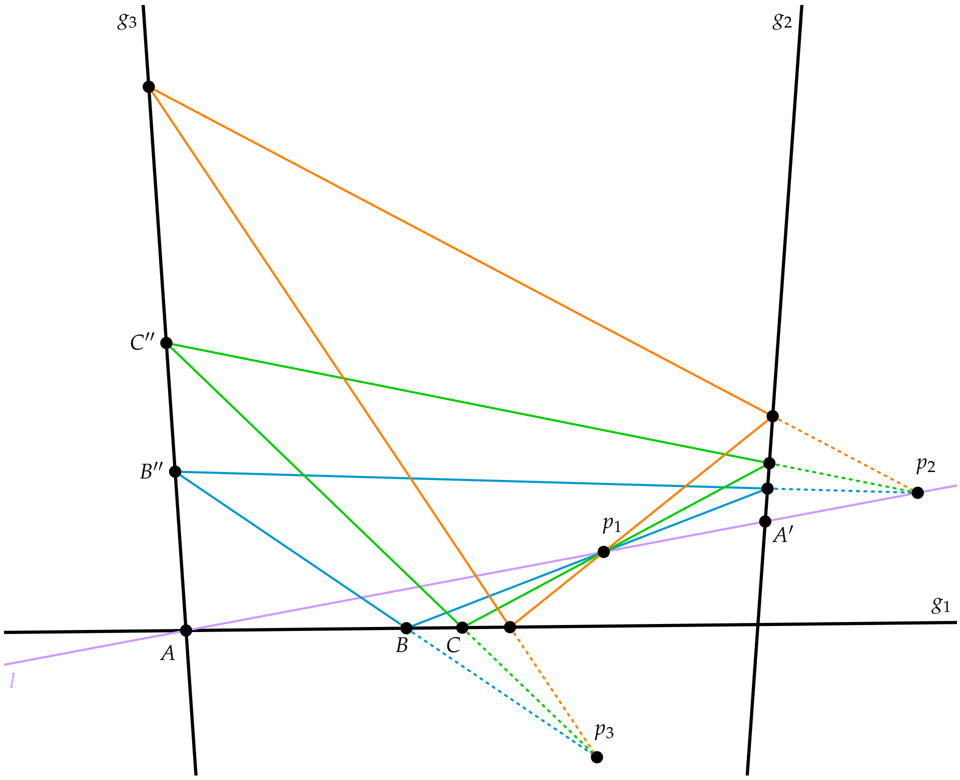

Figure 14.2: Visualization of Corollary 14.7 for $n = 3$. The composition of restricted reversions $\varphi_{p_3}^{g_1} \circ \varphi_{p_2}^{g_3} \circ \varphi_{p_1}^{g_2}|_{L_{g_1}}$ fixes A, B, C and hence all points on the line g_1.

Lemma 14.9 *Let $o \in \mathbb{P}^2$ be a point and $g_1, g_2, g_3 \in L_o$ be lines such that $g_1 \neq g_3$ and let*

$$\varphi := \varphi_q^{g_3} \circ \varphi_p^{g_2} : L_{g_1} \to L_{g_3}$$

be a composition of restricted reversions. Moreover, let $l \in \mathbb{P}^2 \backslash \{g_1, g_2, g_3\}$ be a line such that $p, q \in L_l$. We define $p_{g_1} \in \mathbb{P}^2 \backslash (L_{g_1} \cup L_{g_3})$ to be the point that satisfies

$$\varphi|_{L_{g_1}} = \varphi_{p_{g_1}}^{g_3}|_{L_{g_1}}.$$

Then we have $p_{g_1} \in L_l \backslash (L_{g_1} \cup L_{g_3})$, i.e. the points p, q, p_{g_1} are collinear.

Proof We can assume without loss of generality that $p \neq q$ and g_1, g_2, g_3 are pairwise distinct.

Suppose $l \cdot o \neq 0$. Let $A \in L_{g_1} \backslash (L_l \cup \{o\})$. By Lemma 14.3, we have $p_{g_1} = l \times (A \times A')$, where $A' := \varphi(A)$.

Now let $l \cdot o = 0$ and assume that p, q, p_{g_1} are not collinear. We define $B := g_1 \times (p \times p_{g_1}) \in L_{g_1} \backslash L_l$. Then we have $\varphi^{g_3}_{p_{g_1}}(B) = \varphi^{g_3}_{q} \circ \varphi^{g_2}_{p}(B)$, which is a contradiction, since q, $\varphi^{g_2}_{p}(B)$ and $\varphi^{g_3}_{p_{g_1}}(B)$ are not collinear. $\square$

Lemma 14.10 *Let $g_1, \ldots, g_n \in \mathbb{P}^2$ be concurrent lines and let*

$$\varphi := \varphi^{h_m}_{p_m} \circ \cdots \circ \varphi^{h_1}_{p_1} \colon L_{g_i} \to L_{h_m}$$

be a composition of restricted reversions, where $i \in \{1, \ldots, n\}$, $h_k \in \{g_1, \ldots, g_n\}$ for all $k \in \{1, \ldots, m\}$ and $h_m \neq g_i$. If $p_1, \ldots, p_m \in L_l$ for some line $l \in \mathbb{P}^2 \backslash \{h_1, \ldots, h_m, g_i\}$, then the point $p_{g_i} \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_m})$ that satisfies

$$\varphi|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}}$$

is also incident to l.

Proof We prove by induction on m. The case $m = 1$ is trivial and the case $m = 2$ was covered in Lemma 14.9. So let $m > 2$.

If $h_{m-1} \neq g_i$, then we can write

$$\varphi^{h_{m-1}}_{p_{m-1}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}} = \varphi^{h_{m-1}}_{q_{g_i}}|_{L_{g_i}},$$

where $q_{g_i} \in L_l \backslash (L_{g_i} \cup L_{h_{m-1}})$ by induction hypothesis. Now we have

$$\varphi^{h_m}_{p_m} \circ \varphi^{h_{m-1}}_{q_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}},$$

for which we can also use the induction hypothesis.

If $h_{m-1} = g_i$ and $h_{m-2} \neq g_i$, then we can write

$$\varphi^{h_{m-2}}_{p_{m-2}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}} = \varphi^{h_{m-2}}_{r_{g_i}}|_{L_{g_i}},$$

where $r_{g_i} \in L_l \backslash (L_{g_i} \cup L_{h_{m-2}})$. If $h_{m-2} = h_m$, then we can use Lemma 5.2 on

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}} \circ \varphi^{h_m}_{r_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}}$$

to conclude. Otherwise, if $h_{m-2} \notin \{g_i, h_m\}$, then we can write

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}}|_{L_{h_{m-2}}} = \varphi^{h_m}_{s_{h_{m-2}}}|_{L_{h_{m-2}}},$$

where $s_{h_{m-2}} \in L_l \backslash (L_{h_{m-2}} \cup L_{h_m})$, and use the induction hypothesis on

$$\varphi^{h_m}_{s_{h_{m-2}}} \circ \varphi^{h_{m-2}}_{r_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}}.$$

If $h_{m-1} = g_i$ and $h_{m-2} = g_i$, then we have

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}} \circ \varphi^{g_i}_{p_{m-2}} \circ \varphi^{h_{m-3}}_{p_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}} = \varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-2}} \circ \varphi^{h_{m-3}}_{p_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}},$$

for which we can use the induction hypothesis right away. $\square$

Lemma 14.11 *Let $o \in \mathbb{P}^2$ be a point, $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines and*

$$\varphi := \varphi_q^{g_3} \circ \varphi_p^{g_2} \colon L_{g_1} \to L_{g_3}$$

be a composition of restricted reversions. We define $p_{g_1} \in \mathbb{P}^2 \backslash (L_{g_1} \cup L_{g_3})$ to be the point that satisfies

$$\varphi|_{L_{g_1}} = \varphi_{p_{g_1}}^{g_3}|_{L_{g_1}}.$$

Then we have

$$(g_1, g_2; o \times p, o \times q) = (g_1, g_3; o \times p_{g_1}, o \times q).$$

Proof By Lemma 14.9, we can assume that $p \neq q$ and that $l := p \times q$ is not incident to o. Let $A \in L_{g_1} \backslash (L_l \cup \{o\})$ and define

$$B := \varphi_p^{g_2}(A), \qquad C := \varphi_q^{g_3}(B) \quad \text{and} \quad X := \varphi_q^{g_1}(C).$$

Furthermore, we define $G_k := g_k \times l$ for all $k \in \{1, 2, 3\}$. By Lemma 4.3, we have

$$\begin{aligned}
(g_1, g_2; o \times p, o \times q) &= (G_1, G_2; p, q) \\
&= (B \times G_1, g_2; B \times p, B \times q) \\
&= (G_1, o; A, X) \\
&= (C \times G_1, g_3; C \times A, C \times X) \\
&= (G_1, G_3; p_{g_1}, q) \\
&= (g_1, g_3; o \times p_{g_1}, o \times q)
\end{aligned}$$

as claimed. □

Lemma 14.12 *Let $o \in \mathbb{P}^2$ be a point and $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines. Moreover, let*

$$\varphi := \varphi_q^{g_3} \circ \varphi_p^{g_2} \colon L_{g_1} \to L_{g_3} \qquad \textit{and} \qquad \varphi' := \varphi_{q'}^{g_3} \circ \varphi_{p'}^{g_2} \colon L_{g_1} \to L_{g_3}$$

be compositions of restricted reversions. The underlying points are chosen in such a way that the two triples

$$o, p, p' \qquad \textit{and} \qquad o, q, q'$$

are both respectively collinear. Let $p_{g_1}, p'_{g_1} \in \mathbb{P}^2 \backslash (L_{g_1} \cup L_{g_3})$ be the points that satisfy

$$\varphi|_{L_{g_1}} = \varphi_{p_{g_1}}^{g_3}|_{L_{g_1}} \qquad \textit{and} \qquad \varphi'|_{L_{g_1}} = \varphi_{p'_{g_1}}^{g_3}|_{L_{g_1}}.$$

The points o, p_{g_1}, p'_{g_1} are collinear.

Proof Again we can assume that $p \neq q$. If $l := p \times q$ is incident to o, then the result follows directly from Lemma 14.9. If $l \cdot o \neq 0$, then we can use Lemma 4.4 and Lemma 14.11 to conclude. □

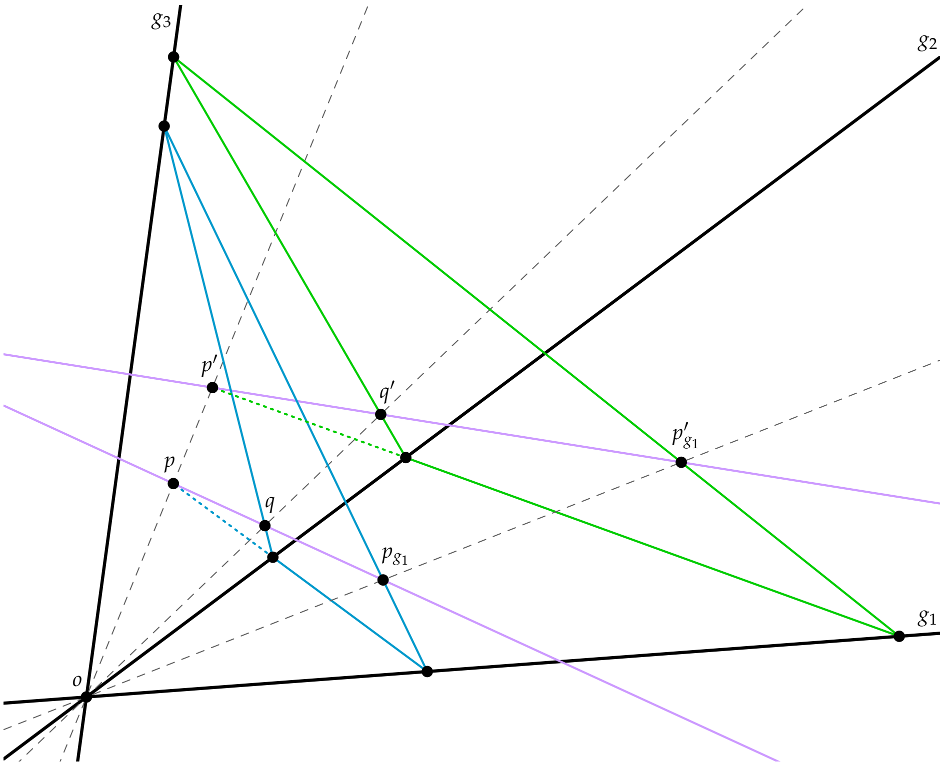

Figure 14.3: The triples p, q, p_{g_1} and p', q', p'_{g_1} are both respectively collinear by Lemma 14.9 and the triple o, p_{g_1}, p'_{g_1} is collinear by Lemma 14.12.

Corollary 14.13 *Let $o \in \mathbb{P}^2$ be a point and $g_1, \dots, g_n \in L_o$ be lines. Moreover, let*

$$\varphi := \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1} \colon L_{g_i} \to L_{h_m} \qquad \text{and} \qquad \varphi' := \varphi_{p'_m}^{h_m} \circ \cdots \circ \varphi_{p'_1}^{h_1} \colon L_{g_i} \to L_{h_m}$$

be compositions of restricted reversions, where $i \in \{1, \dots, n\}$, $h_k \in \{g_1, \dots, g_n\}$ for all $k \in \{1, \dots, m\}$ and $h_m \neq g_i$. Moreover, the underlying points are chosen in a way that the triple o, p_k, p'_k is collinear for all $k \in \{1, \dots, m\}$. Let $p_{g_i}, p'_{g_i} \in \mathbb{P}^2 \backslash (L_{g_i} \cup L_{h_m})$ be the points that satisfy

$$\varphi|_{L_{g_i}} = \varphi_{p_{g_i}}^{h_m}|_{L_{g_i}} \qquad \text{and} \qquad \varphi'|_{L_{g_i}} = \varphi_{p'_{g_i}}^{h_m}|_{L_{g_i}}.$$

Then the triple o, p_{g_i}, p'_{g_i} is also collinear.

Proof We prove by induction on $m \in \mathbb{N}$, similarly to the proof of Lemma 14.10. The base case $m = 1$ is trivial and the case $m = 2$ was covered in Lemma 14.12. So let $m > 2$.

If $h_{m-1} \neq g_i$, then we can write

$$\varphi_{p_{m-1}}^{h_{m-1}} \circ \cdots \circ \varphi_{p_1}^{h_1}|_{L_{g_i}} = \varphi_{q_{g_i}}^{h_{m-1}}|_{L_{g_i}} \qquad \text{and} \qquad \varphi_{p'_{m-1}}^{h_{m-1}} \circ \cdots \circ \varphi_{p'_1}^{h_1}|_{L_{g_i}} = \varphi_{q'_{g_i}}^{h_{m-1}}|_{L_{g_i}},$$

where $q_{g_i}, q'_{g_i} \in \mathbb{P}^2 \setminus (L_{g_i} \cup L_{h_{m-1}})$. The triple o, q_{g_i}, q'_{g_i} is collinear by induction hypothesis. Now we have

$$\varphi^{h_m}_{p_m} \circ \varphi^{h_{m-1}}_{q_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}} \qquad \text{and} \qquad \varphi^{h_m}_{p'_m} \circ \varphi^{h_{m-1}}_{q'_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p'_{g_i}}|_{L_{g_i}},$$

for which we can also use the induction hypothesis.

If $h_{m-1} = g_i$ and $h_{m-2} \neq g_i$, then we can write

$$\varphi^{h_{m-2}}_{p_{m-2}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}} = \varphi^{h_{m-2}}_{r_{g_i}}|_{L_{g_i}} \qquad \text{and} \qquad \varphi^{h_{m-2}}_{p'_{m-2}} \circ \cdots \circ \varphi^{h_1}_{p'_1}|_{L_{g_i}} = \varphi^{h_{m-2}}_{r'_{g_i}}|_{L_{g_i}},$$

where $r_{g_i}, r'_{g_i} \in \mathbb{P}^2 \setminus (L_{g_i} \cup L_{h_{m-2}})$. The triple o, r_{g_i}, r'_{g_i} is collinear by induction hypothesis. If $h_{m-2} = h_m$, then we can use Theorem 7.2 on

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}} \circ \varphi^{h_m}_{r_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}} \qquad \text{and} \qquad \varphi^{h_m}_{p'_m} \circ \varphi^{g_i}_{p'_{m-1}} \circ \varphi^{h_m}_{r'_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p'_{g_i}}|_{L_{g_i}}$$

to conclude. Otherwise, if $h_{m-2} \notin \{g_i, h_m\}$, then we can write

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}}|_{L_{h_{m-2}}} = \varphi^{h_m}_{s_{h_{m-2}}}|_{L_{h_{m-2}}} \qquad \text{and} \qquad \varphi^{h_m}_{p'_m} \circ \varphi^{g_i}_{p'_{m-1}}|_{L_{h_{m-2}}} = \varphi^{h_m}_{s'_{h_{m-2}}}|_{L_{h_{m-2}}},$$

where $s_{h_{m-2}}, s'_{h_{m-2}} \in \mathbb{P}^2 \setminus (L_{h_{m-2}} \cup L_{h_m})$. The triple $o, s_{h_{m-2}}, s'_{h_{m-2}}$ is also collinear. Hence we can use the induction hypothesis on

$$\varphi^{h_m}_{s_{h_{m-2}}} \circ \varphi^{h_{m-2}}_{r_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p_{g_i}}|_{L_{g_i}} \qquad \text{and} \qquad \varphi^{h_m}_{s'_{h_{m-2}}} \circ \varphi^{h_{m-2}}_{r'_{g_i}}|_{L_{g_i}} = \varphi^{h_m}_{p'_{g_i}}|_{L_{g_i}}.$$

If $h_{m-1} = g_i$ and $h_{m-2} = g_i$, then we have

$$\varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-1}} \circ \varphi^{g_i}_{p_{m-2}} \circ \varphi^{h_{m-3}}_{p_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}} = \varphi^{h_m}_{p_m} \circ \varphi^{g_i}_{p_{m-2}} \circ \varphi^{h_{m-3}}_{p_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p_1}|_{L_{g_i}}$$

and

$$\varphi^{h_m}_{p'_m} \circ \varphi^{g_i}_{p'_{m-1}} \circ \varphi^{g_i}_{p'_{m-2}} \circ \varphi^{h_{m-3}}_{p'_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p'_1}|_{L_{g_i}} = \varphi^{h_m}_{p'_m} \circ \varphi^{g_i}_{p'_{m-2}} \circ \varphi^{h_{m-3}}_{p'_{m-3}} \circ \cdots \circ \varphi^{h_1}_{p'_1}|_{L_{g_i}},$$

for which we can use the induction hypothesis right away. □

Lemma 14.14 *Let $o \in \mathbb{P}^2$ be a point and $g_2, g_3 \in L_o$ be distinct lines. Moreover, let $p, q, r \in \mathbb{P}^2$ be collinear points such that $p \notin L_{g_2}$, $q \notin L_{g_2} \cup L_{g_3}$, $r \notin L_{g_3}$ and $p \neq r$. Then there exists a unique line $g_1 \in L_o \setminus (L_p \cup L_r)$ such that*

$$\varphi := \varphi^{g_1}_r \circ \varphi^{g_3}_q \circ \varphi^{g_2}_p \colon L_{g_1} \to L_{g_1}$$

is a composition of restricted reversions that satisfies

$$\varphi|_{L_{g_1}} = \mathrm{id}|_{L_{g_1}}.$$

Proof Let $l := p \times r$ and $A' \in L_{g_2} \backslash (L_l \cup \{o\})$. We define $A'' := \varphi_q^{g_3}(A')$ and $g_1 := o \times A$, where

$$A := (p \times A') \times (r \times A'').$$

Note that $g_1 \in L_o \backslash (L_p \cup L_r)$ is well-defined, because of the way p, q, r are chosen. We have $g_1 = g_3$ if and only if $p = q$. Since this case is trivial, we can assume without loss of generality that $g_1 \neq g_3$. Let $p_{g_1} \in \mathbb{P}^2 \backslash (L_{g_1} \cup L_{g_3})$ be the point that satisfies

$$\varphi_q^{g_3} \circ \varphi_p^{g_2}|_{L_{g_1}} = \varphi_{p_{g_1}}^{g_3}|_{L_{g_1}}.$$

By Lemma 14.9, we see that p_{g_1} is incident to the lines $\overline{AA''}$ and l, which are distinct since $A'' \notin L_l$. Hence $r = p_{g_1}$ and we have

$$\varphi|_{L_{g_1}} = \mathrm{id}|_{L_{g_1}}.$$

Uniqueness follows from the construction of g_1. □

Corollary 14.15 *Let $o \in \mathbb{P}^2$ be a point and $g_2, \ldots, g_n \in L_o$ be lines. Moreover, let $p_1, \ldots, p_{m+1} \in \mathbb{P}^2$ be collinear points such that*

$$\varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_2}^{h_2} \colon L_{h_1} \to L_{h_m}$$

is a composition of restricted reversions, where $h_k \in \{g_2, \ldots, g_n\}$ for all $k \in \{1, \ldots, m\}$, $h_1 \neq h_m$, $p_1 \notin L_{h_1}$, $p_{m+1} \notin L_{h_m}$ and $p_1 \neq p_{m+1}$. Then there exists a unique line $g_1 \in L_o \backslash (L_{p_1} \cup L_{p_{m+1}})$ such that

$$\varphi := \varphi_{p_{m+1}}^{g_1} \circ \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1} \colon L_{g_1} \to L_{g_1}$$

is also a composition of restricted reversions that satisfies

$$\varphi|_{L_{g_1}} = \mathrm{id}|_{L_{g_1}}.$$

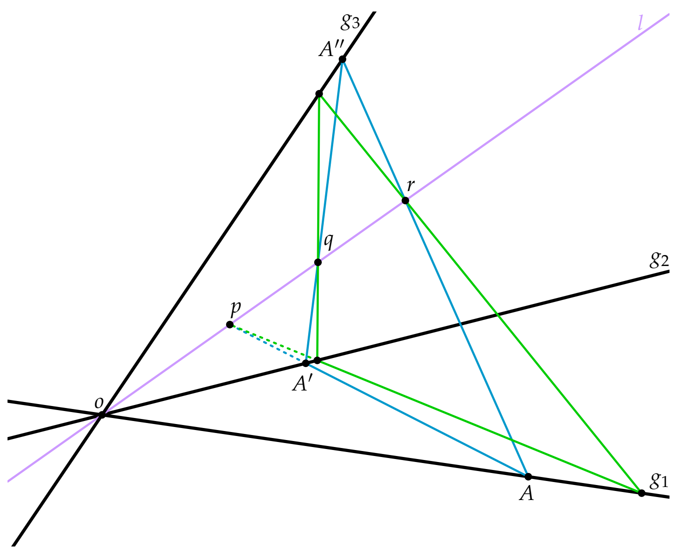

Figure 14.4: Visualization of Lemma 14.14.

Proof Let $p_{h_1} \in \mathbb{P}^2 \backslash (L_{h_1} \cup L_{h_m})$ be the point that satisfies

$$\varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_2}^{h_2}|_{L_{h_1}} = \varphi_{p_{h_1}}^{h_m}|_{L_{h_1}}.$$

The points p_1, p_{m+1}, p_{h_1} are collinear by Lemma 14.10. We can use Lemma 14.14 to conclude. □

Porism 14.16 *Let $o \in \mathbb{P}^2$ be a point and $g_2, \ldots, g_n \in L_o$ be lines. Moreover, let $p_1, \ldots, p_{m+1} \in \mathbb{P}^2$ be collinear points such that*

$$\varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_2}^{h_2} \colon L_{h_1} \to L_{h_m}$$

is a composition of restricted reversions, where $h_k \in \{g_2, \ldots, g_n\}$ for all $k \in \{1, \ldots, m\}$, $h_1 \neq h_m$, $p_1 \notin L_{h_1}$, $p_{m+1} \notin L_{h_m}$ and $p_1 \neq p_{m+1}$. Then there exists a unique line $g_1 \in L_o \backslash (L_{p_1} \cup L_{p_{m+1}})$ such that

$$\varphi := \varphi_{p_{m+1}}^{g_1} \circ \varphi_{p_m}^{h_m} \circ \cdots \circ \varphi_{p_1}^{h_1} \colon L_{g_1} \to L_{g_1}$$

is also a composition of restricted reversions and the polygonal chain $A_1 \ldots A_{m+2}$ that passes through $p_1, \ldots, p_{m+1}$ according to φ closes for every starting point $A_1 \in L_{g_1}$.

Proof This is a consequence of Corollary 14.15. □

15 Permuting Reversions

Another way to define reversions on multiple lines $g_1, \ldots, g_n$ is to take a permutation $\sigma \in S_n$ that maps the points from line g_k to $g_{\sigma(k)}$ for all $k \in \{1, \ldots, n\}$, where S_n is the symmetric group. With this definition, reversions are bijections from a set of n lines onto itself. Now we can try to find compositions of reversions that equal the identity on all n lines. To make things easier, we will consider compositions of these reversions, where every reversion follows the same permutation σ. In this case, it makes sense to just consider the permutation $\sigma = (1 \ \ldots \ n)$ that contains only one cycle and no fixed points, otherwise we could just consider a smaller amount of lines. Note that $(1 \ \ldots \ n)$ is written in cycle notation, i.e. $\sigma(k) = k + 1$ for all $k \in \{1, \ldots, n-1\}$ and $\sigma(n) = 1$.

Definition 15.1 Let $g_1, \ldots, g_n \in \mathbb{P}^2$ be concurrent lines, where $n \in \mathbb{N}$. Let $p \in \mathbb{P}^2 \backslash K$ be a point, where $K := \bigcup_{k=1}^{n} L_{g_k}$. A *permuting reversion* is a map

$$\varphi_p^\sigma \colon K \to K$$

that sends $x \in L_{g_k}$ to $\varphi_p^\sigma(x) \in L_{g_{\sigma(k)}}$ such that the triple $x, p, \varphi_p^\sigma(x)$ is collinear for all $k \in \{1, \ldots, n\}$, where σ is an element of the symmetric group S_n that operates on the set of lines $\{g_1, \ldots, g_n\}$.

Remarks 15.2 Let φ_p^σ and φ_q^σ be permuting reversions.

a) Note that φ_p^σ is not well-defined, when the lines $g_1, \dots, g_n$ are not concurrent. To see this, assume that g_1, g_2, g_3 are not concurrent and let $\sigma = (1\ 2\ 3)$. Consider the intersection point x of g_1 and g_2. Now we have
$$\varphi_p^\sigma|_{L_{g_1}}(x) = x \qquad \text{and} \qquad \varphi_p^\sigma|_{L_{g_2}}(x) \in L_{g_3}.$$
Since $x \notin L_{g_3}$, we see that φ_p^σ is not a function anymore.

b) The inverse of φ_p^σ is given by
$$(\varphi_p^\sigma)^{-1} = \varphi_p^{\sigma^{-1}},$$
where $\sigma^{-1} \in S_n$ is the inverse of σ.

c) Let $g_k \in \{g_1, \dots, g_n\}$. The equation
$$\varphi_p^\sigma|_{L_{g_k}} = \varphi_q^\sigma|_{L_{g_k}}$$
is equivalent to
$$\varphi_q^{\sigma^{-1}} \circ \varphi_p^\sigma|_{L_{g_k}} = \mathrm{id}|_{L_{g_k}}$$
and hence still implies $p = q$, by Lemma 2.1.

Proposition 15.3 *Let $o \in \mathbb{P}^2$ be a point and $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines. We define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. Moreover, let $l \in \mathbb{P}^2 \backslash L_o$ and $p, q, r \in L_l \backslash K$, where $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$. We have*
$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K$$
if and only if
$$(g_k, g_{\sigma(k)}; o \times p, o \times q) = (g_k, g_{\sigma^2(k)}; o \times r, o \times q)$$
for all $k \in \{1, 2, 3\}$.

Proof If $\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K$, then the result follows from Lemma 14.11.

Let $p_{g_k} \in L_l \backslash (L_{g_k} \cup L_{g_{\sigma^2(k)}})$ be the point that satisfies
$$\varphi_q^{g_{\sigma^2(k)}} \circ \varphi_p^{g_{\sigma(k)}}|_{L_{g_k}} = \varphi_{p_{g_k}}^{g_{\sigma^2(k)}}|_{L_{g_k}}$$
and suppose we have
$$(g_k, g_{\sigma(k)}; o \times p, o \times q) = (g_k, g_{\sigma^2(k)}; o \times r, o \times q)$$
for all $k \in \{1, 2, 3\}$. Permuting the lines g_1, g_2, g_3 in Lemma 14.11 yields
$$\begin{aligned}(g_k, g_{\sigma^2(k)}; o \times p_{g_k}, o \times q) &= (g_k, g_{\sigma(k)}; o \times p, o \times q) \\ &= (g_k, g_{\sigma^2(k)}; o \times r, o \times q)\end{aligned}$$
for all $k \in \{1, 2, 3\}$. Since $p_{g_1}, p_{g_2}, p_{g_3}, r \in L_l$ and $l \cdot o \neq 0$, we can use Lemma 4.4 to get $r = p_{g_1} = p_{g_2} = p_{g_3}$. Note that $r \in L_l \backslash K$, because $p_{g_k} \in L_l \backslash (L_{g_k} \cup L_{g_{\sigma^2(k)}})$ for all $k \in \{1, 2, 3\}$. □

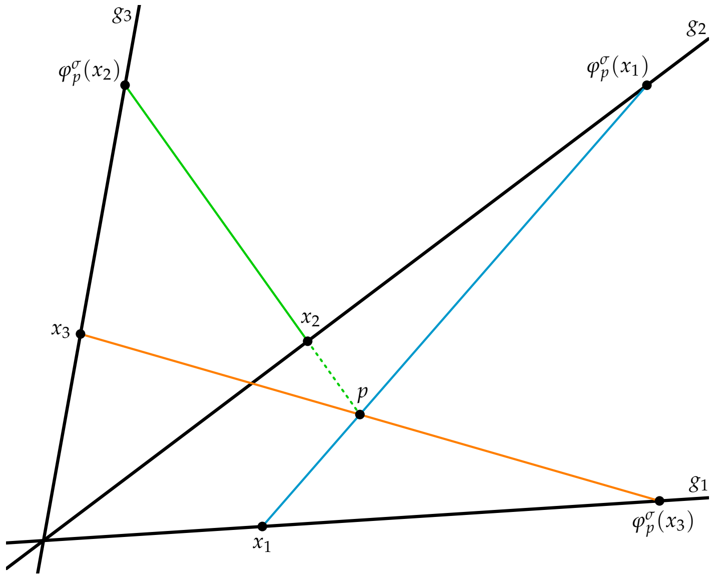

Figure 15.1: A permuting reversion φ_p^σ, where $\sigma := (1\ 2\ 3)$.

Remark 15.4 Let $o \in \mathbb{P}^2$ be a point and $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines. We define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. Moreover, let $l \in \mathbb{P}^2 \backslash L_o$ be a line and $p, q, r \in L_l \backslash K$ be points such that

$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K,$$

where $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$. Note that

$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \varphi_p^\sigma \circ \varphi_r^\sigma \circ \varphi_q^\sigma = \varphi_q^\sigma \circ \varphi_p^\sigma \circ \varphi_r^\sigma = \mathrm{id}|_K.$$

Hence we can also permute the points p, q, r in Lemma 14.11 to get even more equations. In particular, we now have

$$\begin{aligned} \lambda_k &:= (g_k, g_{\sigma(k)}; o \times p, o \times q) \\ &= (g_k, g_{\sigma^2(k)}; o \times r, o \times q) \\ &= (g_{\sigma(k)}, g_{\sigma^2(k)}; o \times r, o \times p), \end{aligned}$$

where $k \in \{1, 2, 3\}$. Using Definition 4.1, we have

$$\lambda_1 \lambda_2 \lambda_3 = (g_1, g_2; o \times p, o \times q)(g_2, g_3; o \times p, o \times q)(g_3, g_1; o \times p, o \times q) = 1.$$

Corollary 15.5 *Let $l \in \mathbb{P}^2$ be a line and $G_1, G_2, G_3 \in L_l$ be pairwise distinct points. Moreover, let $p, q, r \in L_l \backslash \{G_1, G_2, G_3\}$ such that*

$$(G_k, G_{\sigma(k)}; p, q) = (G_k, G_{\sigma^2(k)}; r, q),$$

where $k \in \{1, 2, 3\}$ and $\sigma := (1\ 2\ 3)$. We define $g_k := o \times G_k$ for all $k \in \{1, 2, 3\}$ and $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$, where $o \in \mathbb{P}^2 \backslash L_l$, and we allow σ to act on $\{g_1, g_2, g_3\}$. Now we have

$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K.$$

Proof The result follows directly from Proposition 15.3. □

Proposition 15.6 *Let $o \in \mathbb{P}^2$ be a point and $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines. For $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$, let $p, q \in \mathbb{P}^2 \backslash K$ be two distinct points such that o, p and q are collinear and define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. Then there is no point $r \in \mathbb{P}^2 \backslash K$ that satisfies*

$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K.$$

Proof Assume towards contradiction that there is such a point r. Define $j := g_1$, $k := g_2$ and $l := g_3$. We can assume without loss of generality that $o = (0 : 0 : 1)$ and that $j = (1 : 0 : 0)$, which can be achieved by a rotation around the axis given by the vector o. Furthermore, we can assume that $k = (k_1 : 1 : 0)$, $l = (l_1 : 1 : 0)$ and $p = (1 : p_2 : p_3)$, since $j \notin \{k, l\}$ and $j \cdot p \neq 0$. Now we can write

$$q = ao + bp,$$

where $a, b \in \mathbb{R} \backslash \{0\}$. We have $r = p_{g_1}$, where p_{g_1} is defined by

$$\varphi_q^\sigma \circ \varphi_p^\sigma|_{L_{g_1}} = \varphi_{p_{g_1}}^{\sigma^{-1}}|_{L_{g_1}}.$$

Hence we can construct r in the following way. Let $A := (0 : 1 : 0)$, then $A \in L_{g_1} \backslash \{o\}$. We define

$$A' := M_q^l M_p^k(A).$$

By Lemma 14.9, we have

$$r = (o \times p) \times (A \times A').$$

The point $B = (-1 : k_1 : 0)$ satisfies $B \in L_{g_2} \backslash \{o\}$. Define

$$C := M_r^k M_q^j M_p^l(B).$$

It suffices to show that $B \neq C$. Thus we have to make sure that $B \times C$ is not the zero vector. Using Mathematica, we get

$$B \times C = -ab(k \cdot p)\left(k_1(k_1 - l_1 + p_2) + l_1^2 + l_1 p_2 + p_2^2\right) k.$$

It remains to show that the polynomial

$$z(z - x + y) + x^2 + xy + y^2$$

is different from zero, where $x := l_1$, $y := p_2$ and $z := k_1$. If $z = 0$, then $l \cdot p \neq 0$ implies that the polynomial is strictly positive, because

$$2(x^2 + xy + y^2) = x^2 + y^2 + (x + y)^2 \geq 0$$

and x, y cannot simultaneously be zero. If $z \neq 0$, we can assume without loss of generality that $z = 1$, since the polynomial is homogeneous. We then define

$$f(x,y) := 1 - x + y + x^2 + xy + y^2.$$

Now we have

$$\frac{\partial f}{\partial x}(x,y) = -1 + 2x + y, \qquad \frac{\partial f}{\partial y}(x,y) = 1 + x + 2y$$

and the Hessian matrix of f is given by

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x,y) & \frac{\partial^2 f}{\partial x \partial y}(x,y) \\ \frac{\partial^2 f}{\partial y \partial x}(x,y) & \frac{\partial^2 f}{\partial y^2}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Since the Hessian matrix of f is positive definite and does not depend on x, y, we see that f is convex. Hence the global minimum of f is given by $(x_0, y_0) := (1, -1)$, because

$$\frac{\partial f}{\partial x}(x_0,y_0) = \frac{\partial f}{\partial y}(x_0,y_0) = 0.$$

We also have $f(x_0, y_0) = 0$. Lastly, $l \cdot p \neq 0$ implies that $x + y \neq 0$. Hence $B \times C$ is not the zero vector. □

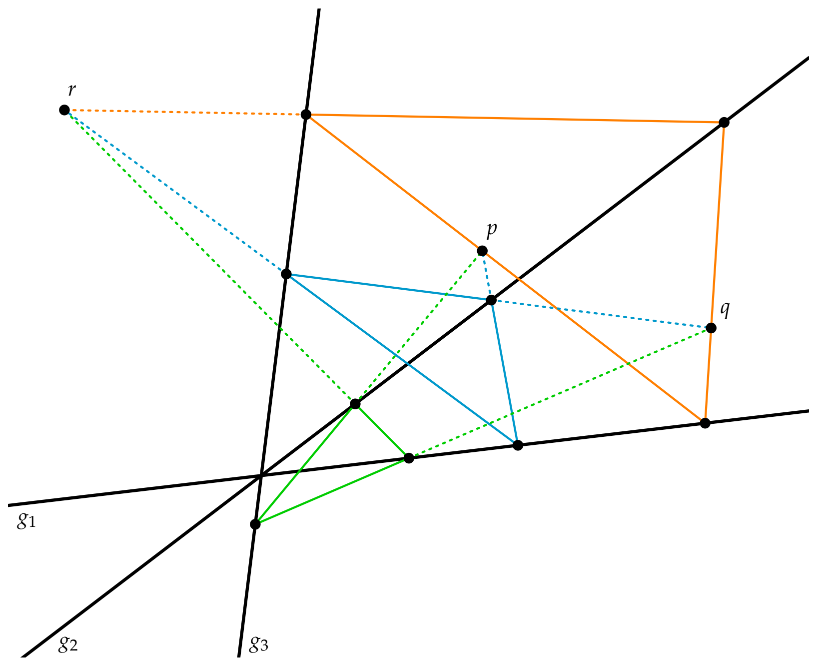

Figure 15.2: An example of three points p, q, r that satisfy $\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K$, where $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$ and $\sigma := (1\ 2\ 3)$. The blue triangle starts on g_1, the green one on g_2 and the orange one on g_3. The three points p, q, r must be collinear by Lemma 14.9.

Theorem 15.7 *Let $o \in \mathbb{P}^2$ be a point, $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines and $p \in \mathbb{P}^2 \backslash K$, where $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$. We define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. There is a line $l \in L_o \backslash \{g_1, g_2, g_3, o \times p\}$ such that for all $q \in (L_l \backslash \{o\}) \cup \{p\}$, there is a point $r \in \mathbb{P}^2 \backslash K$ with*

$$\varphi_r^\sigma \circ \varphi_q^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K.$$

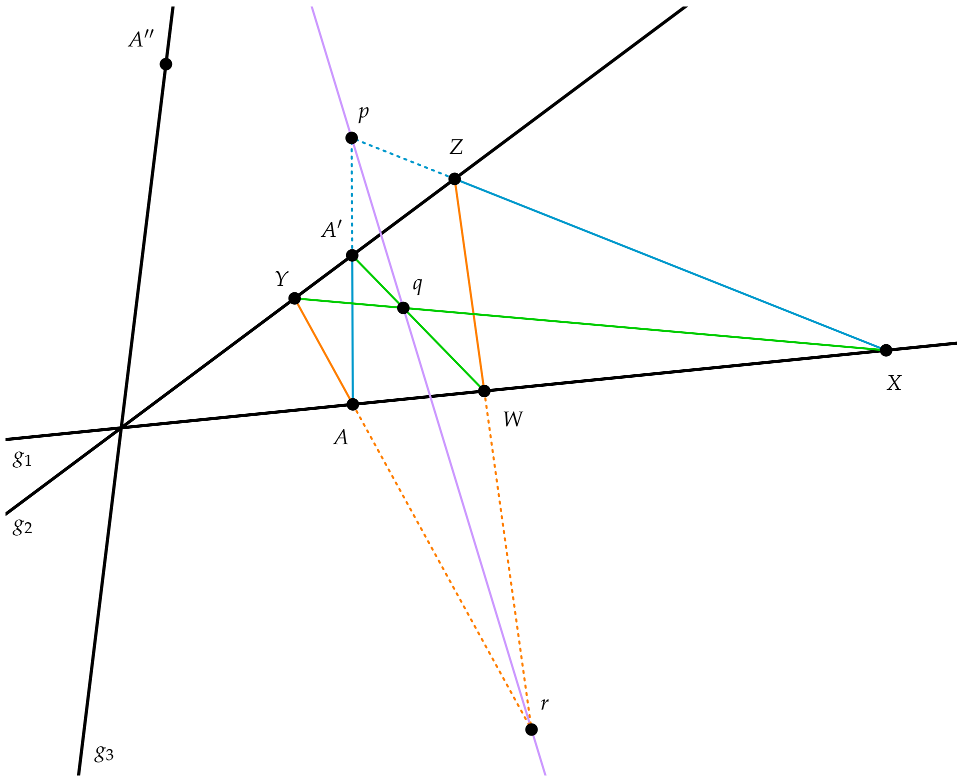

Figure 15.3: Here we see the hexagon $AA'WZXY$ and also the locations of all the points used in the proof of Theorem 15.7.

Proof Clearly, we have $\varphi_p^\sigma \circ \varphi_p^\sigma \circ \varphi_p^\sigma = \mathrm{id}|_K$.

Let $A \in L_{g_1} \backslash \{o\}$ and $A'' \in L_{g_3} \backslash \{o\}$ such that p, A and A'' are not collinear and let $A' := \varphi_p^\sigma(A)$. Define $X := \varphi_p^\sigma(A'')$ and let Y be the intersection point of g_2 and $\overline{AA''}$. Note that $A \neq X$ and $A' \neq Y$, hence the lines $\overline{XY}$ and $\overline{A'A''}$ are distinct. Now let q be the intersection point of $\overline{XY}$ and $\overline{A'A''}$. Since A, A', A'' are not collinear, $A \neq X$ and $A' \neq Y$, we see that $p \neq q$ and $q \in \mathbb{P}^2 \backslash K$. Let $Z := \varphi_p^\sigma(X)$ and $W := \varphi_q^\sigma(A'')$. Since $A \neq X$, we have $Y \neq Z$. Hence the lines $\overline{ZW}$ and $\overline{AA''}$ are distinct. We define r to be the intersection point of $\overline{ZW}$ and $\overline{AA''}$. The non-collinearity of A, A', A'' implies that p, q, r are pairwise distinct.

Consider the hexagon $AA'WZXY$. By Theorem 7.7, we see that p, q, r are collinear. Assume that $r \in K$. Then $r \in \{A, Y, A''\}$. If $r \in \{A, A''\}$, then A, A', A'' must be collinear, which is a contradiction. If $r = Y$, then we have $p = X$, since p is now incident to $\overline{XY}$ and $\overline{XA''}$, which are two distinct lines. This is again a contradiction. Thus $r \in \mathbb{P}^2 \backslash K$.

For $i \in \{1, 2, 3\}$, let $p_{g_i} \in \mathbb{P}^2$ be the point that satisfies

$$\varphi^{\sigma}_{p_{g_i}} \circ \varphi^{\sigma}_{q} \circ \varphi^{\sigma}_{p}|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}}.$$

By Lemma 14.9, the points $p_{g_1}, p_{g_2}, p_{g_3}$ are all incident to the line through p and q. Moreover, p_{g_1} is incident to $\overline{AA''}$, p_{g_2} lies on $\overline{ZW}$ and p_{g_3} sits on $\overline{A''Y}$. Hence $r = p_{g_1} = p_{g_2} = p_{g_3}$.

Since $p \neq q$ and $\varphi^{\sigma}_{r} \circ \varphi^{\sigma}_{q} \circ \varphi^{\sigma}_{p} = \mathrm{id}|_K$, we know from Proposition 15.6 that the line $l := o \times q$ is not incident to p. Now let $q' \in L_l \backslash \{o\}$. Again we consider the three points p'_{g_i} that satisfy

$$\varphi^{\sigma}_{p'_{g_i}} \circ \varphi^{\sigma}_{q'} \circ \varphi^{\sigma}_{p}|_{L_{g_i}} = \mathrm{id}|_{L_{g_i}},$$

where $i \in \{1, 2, 3\}$. By Lemma 14.9 and Lemma 14.12, the points p'_{g_1}, p'_{g_2} and p'_{g_3} simultaneously lie on the two distinct lines $o \times r$ and $p \times q'$. Therefore $p'_{g_1} = p'_{g_2} = p'_{g_3}$. □

Corollary 15.8 *Let $o \in \mathbb{P}^2$ be a point, $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines and define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. Let $h_1 \in L_o \backslash \{g_1, g_2, g_3\}$ be a line and define $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$. There exist two distinct lines $h_2, h_3 \in L_o \backslash \{g_1, g_2, g_3, h_1\}$ such that for all $l \in \mathbb{P}^2 \backslash L_o$, we have*

$$\varphi^{\sigma}_{l \times h_3} \circ \varphi^{\sigma}_{l \times h_2} \circ \varphi^{\sigma}_{l \times h_1} = \mathrm{id}|_K.$$

Proof Let $l \in \mathbb{P}^2 \backslash L_o$ and define $p := l \times h_1$. By Theorem 15.7, there is a line $h_2 \in L_o \backslash \{g_1, g_2, g_3, h_1\}$ such that for $q := l \times h_2$, there is a point $r \in L_l \backslash K$ with

$$\varphi^{\sigma}_{r} \circ \varphi^{\sigma}_{q} \circ \varphi^{\sigma}_{p} = \mathrm{id}|_K.$$

We define $h_3 := o \times r$. Since p, q, r are pairwise distinct, we see that h_1, h_2, h_3 are also pairwise distinct. □

Porism 15.9 *Let $o \in \mathbb{P}^2$ be a point, $g_1, g_2, g_3 \in L_o$ be pairwise distinct lines and define $\sigma := (1\ 2\ 3)$ to be the permutation that acts on $\{g_1, g_2, g_3\}$. Let $h_1 \in L_o \backslash \{g_1, g_2, g_3\}$ be a line and define $K := L_{g_1} \cup L_{g_2} \cup L_{g_3}$. There exist two distinct lines $h_2, h_3 \in L_o \backslash \{g_1, g_2, g_3, h_1\}$ such that for all $l \in \mathbb{P}^2 \backslash L_o$, the polygonal chain $A_1A_2A_3A_4$ that passes through $l \times h_1, l \times h_2, l \times h_3$ according to σ closes for every starting point $A_1 \in K$.*

Proof This is a consequence of Corollary 15.8. □

Conjecture 15.10 *Let $o \in \mathbb{P}^2$ be a point, $g_1, \ldots, g_n \in L_o$ be pairwise distinct lines and define $\sigma := (1 \ \ldots \ n)$ to be the permutation that acts on $\{g_1, \ldots, g_n\}$, where $n \in \mathbb{Z}_{\geq 3}$. Let $h_1 \in L_o \backslash \{g_1, \ldots, g_n\}$ be a line and define $K := \bigcup_{k=1}^{n} L_{g_k}$. There exist pairwise distinct lines $h_2, \ldots, h_n \in L_o \backslash \{g_1, \ldots, g_n, h_1\}$ such that for all $l \in \mathbb{P}^2 \backslash L_o$, we have*

$$\varphi^{\sigma}_{l \times h_n} \circ \cdots \circ \varphi^{\sigma}_{l \times h_1} = \mathrm{id}|_K.$$

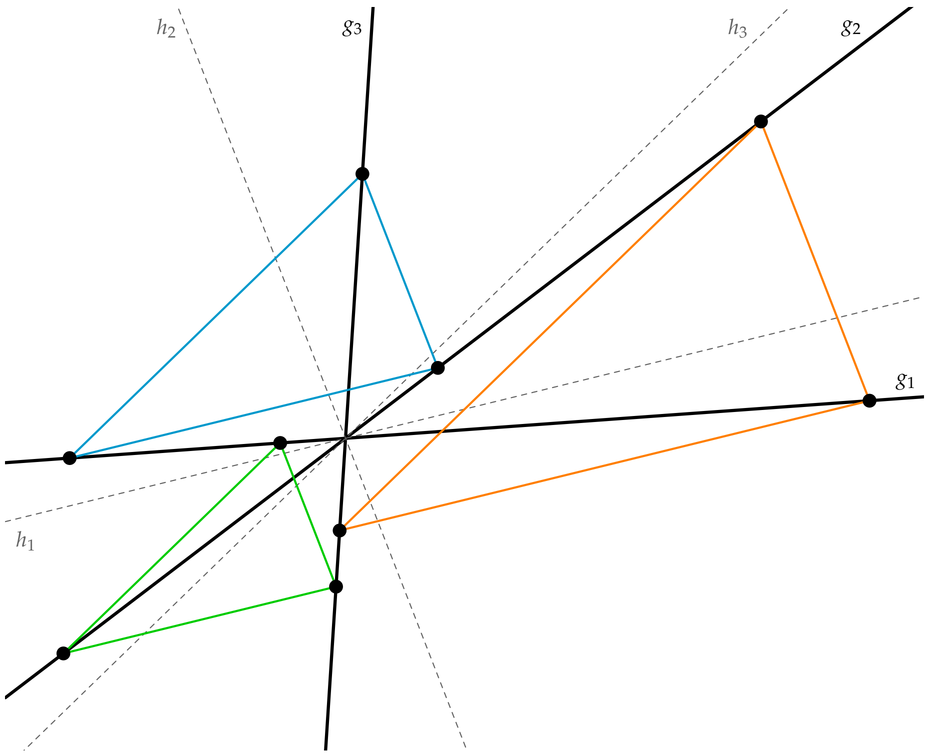

Figure 15.4: Visualization of Corollary 15.8, where l is the line at infinity.

Bibliography

[1] Ivan Izmestiev. A porism for cyclic quadrilaterals, butterfly theorems, and hyperbolic geometry. *The American Mathematical Monthly*, 2015.

[2] Dixon Jones. Quadrangles, butterflies, pascal's hexagon, and projective fixed points. *The American Mathematical Monthly*, 1980.

[3] Jerzy Kocik. A porism concerning cyclic quadrilaterals. *Geometry*, 2013.

[4] Rolf Lingenberg. *Grundlagen der Geometrie*. Durchges. ed. Mannheim U.a: Bibliographisches Institut, 1978.

[5] Jürgen Richter-Gebert. *Perspectives on Projective Geometry*. Springer, 2011.

Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Declaration of originality

The signed declaration of originality is a component of every semester paper, Bachelor's thesis, Master's thesis and any other degree paper undertaken during the course of studies, including the respective electronic versions.

Lecturers may also require a declaration of originality for other written papers compiled for their courses.

I hereby confirm that I am the sole author of the written work here enclosed and that I have compiled it in my own words. Parts excepted are corrections of form and content by the supervisor.

Title of work (in block letters):

Reversion Porisms for Lines

Authored by (in block letters):
For papers written by groups the names of all authors are required.

Name(s):	**First name(s):**
Schiltknecht	Marco

With my signature I confirm that
- I have committed none of the forms of plagiarism described in the 'Citation etiquette' information sheet.
- I have documented all methods, data and processes truthfully.
- I have not manipulated any data.
- I have mentioned all persons who were significant facilitators of the work.

I am aware that the work may be screened electronically for plagiarism.

Place, date	**Signature(s)**
Hettlingen, 25.06.21	[signature]

For papers written by groups the names of all authors are required. Their signatures collectively guarantee the entire content of the written paper.