On the Nonstationary Navier-Stokes Equations in Space Dimensions 3 and 4

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ON THE NONSTATIONARY NAVIER-STOKES EQUATIONS IN SPACE DIMENSIONS 3 AND 4

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presented by

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2021

To my parents.

赵客缦胡缨，吴钩霜雪明。

银鞍照白马，飒沓如流星。

— 李白，《侠客行》

Abstract

This thesis first presents partial regularity theory for the nonstationary Navier-Stokes equations in space dimensions 3 and 4. In particular, we show that there exist global-in-time partially regular weak solutions to the nonstationary Navier-Stokes equations whose singular sets have finite 2-dimensional parabolic Hausdorff measure in space dimension $n = 4$.

Secondly, in space dimensions $n \geq 3$, motivated by studying possible sigularities of the Navier-Stokes equations, we prove singularity formation in a linear toy model of the axi-symmetric Navier-Stokes equations. As a by-product, we construct time-independent supercritical drifts in $L^{n-\lambda}(\mathbb{R}^n)$ with arbitrarily small $\lambda > 0$ such that the Harnack inequality and the Hölder continuity fail in both elliptic and parabolic equations associated to these drifts.

Zusammenfassung

Diese Arbeit präsentiert erstens partielle Regularitätstheorie für die nicht-stationären Navier-Stokes Gleichungen in den Dimensionen 3 und 4. Insbesondere zeigen wir, dass zeitlich globale partiell reguläre schwache Lösungen für die nicht-stationären Navier-Stokes Gleichungen existieren, für welche die Menge der Singularitäten endliches 2-dimensionales Hausdorff-Mass in der Raumdimension $n = 4$ hat.

Zweitens, motiviert durch das Studium von möglichen Singularitäten der Navier-Stokes Gleichungen, beweisen wir in den Dimensionen $n \geq 3$ Singularitätsbildung in einem linearen Toy Modell der achsensymmetrischen Navier-Stokes Gleichungen. Als ein Nebenprodukt konstruieren wir zeitunabhängige superkritische Drifts in $L^{n-\lambda}(\mathbb{R}^n)$ mit beliebig kleinem $\lambda > 0$, so dass die Harnack-Ungleichung und die Hölder-Stetigkeit in den zu diesen Drifts assoziierten elliptischen und parabolischen Gleichungen nicht gelten.

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Contents

Chapter 1

Introduction

The nonstationary Navier-Stokes equations governing the motion of an incompressible viscous fluid in $\mathbb{R}^n \times [0,T]$ with $T \in (0,\infty]$ are given by

$$\begin{aligned} \partial_t u - \nu\Delta u + (u\cdot\nabla)u + \nabla p &= f, \\ \operatorname{div} u &= 0 \end{aligned} \tag{1.0.1}$$

with initial condition $u(x,0) = u_0(x), u_0 \in L^2(\mathbb{R}^n)$. Here, the constant $\nu > 0$ is the viscosity of the fluid, which is a measure of its resistance to deformation at a given rate. The function $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is the velocity field and the function $p : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is the pressure of the fluid. The function $f : \mathbb{R}^n \times [0,T] \to \mathbb{R}^n$ is a force which acts on the fluid. The function $u_0 : \mathbb{R}^n \to \mathbb{R}^n$ is the initial velocity field. By solving these equations, we mean solving the pair (u,p) for given u_0, f, ν and sufficiently large T in a suitable sense.

The Navier-Stokes equations (1.0.1) not only are a physical model of fundamental importance in continuum mechanics, they also are a fascinating nonlinear system of partial differential equations, whose rigorous study is a classical and large field in mathematics which is still very active. In particular, the existence and the uniqueness of smooth solutions in the space-time domain $\mathbb{R}^3 \times [0,\infty)$ or $\mathbb{R}^3/\mathbb{Z}^3 \times [0,\infty)$ for arbitrary smooth initial data are one of the millennium problems posed by Clay Mathematical Institute [14].

1.1 Technical setting

We set $\nu = 1$ without losing any generality from the perspective of studying the existence and the regularity. Indeed, if (u, p) is a solution to (1.0.1) with (u_0, f) and viscosity 1, then (u_λ, p_λ) defined by

$$u_\lambda(x,t) := \nu\lambda u(\lambda x, \nu\lambda^2 t), \tag{1.1.1}$$

$$p_\lambda(x,t) := \nu^2\lambda^2 p(\lambda x, \nu\lambda^2 t) \tag{1.1.2}$$

is a solution to (1.0.1) with $(u_{0,\lambda}, f_\lambda)$ defined by

$$\begin{aligned} u_{0,\lambda}(x,t) &:= \nu\lambda u_0(\lambda x), \\ f_\lambda(x,t) &:= \nu^2\lambda^3 f(\lambda x, \nu\lambda^2 t) \end{aligned}$$

and viscosity $\nu > 0$. When we set $\nu = 1$ and vary λ, (1.1.1) is still a solution to the Navier-Stokes equations for scaled initial data and force. This is called *scaling invariance* of the Navier-Stokes equations.

It suffices to consider weakly divergence-free forces $f \in L^q(\mathbb{R}^n)$ for some $q > 1$. Indeed, by Helmholtz-Weyl decomposition, for any external force $f \in L^q(\mathbb{R}^n)$, $q > 1$, we have a decomposition $f = f_s + f_{p'}$ such that $\operatorname{div} f_s = 0$ and $f_{p'} = \nabla p'$ for some $p' \in W^{1,q}(\mathbb{R}^n)$. We can insert the component $f_{p'}$ into the pressure term ∇p. Therefore, the main object of our study is the following system of equations with divergence-free force f,

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla)u + \nabla p = f, \\ \operatorname{div} u = 0, \\ u(x,0) = u_0(x). \end{aligned} \tag{1.1.3}$$

1.2 Partial regularity theory

The existence and the uniqueness of solutions of (1.1.3) in $\mathbb{R}^2 \times [0,\infty)$ was known to Leray [18] in 1933. Later, the case of 2D domains with boundaries was settled by Ladyzhenskaya [21] in 1959. Leray [26] and Hopf [17] proved the existence of weak solutions of (1.1.3) in dimensions $n \geq 2$ in the whole space and on bounded open domains with smooth boundary in 1934 and 1950, respectively. These weak solutions, called *Leray-Hopf weak solutions*, are distributional solutions to (1.1.3) in the natural energy space $L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)$ which satisfy the following global energy inequality

$$\frac{1}{2}\|u(\cdot,t)\|^2_{L^2(\mathbb{R}^n)} + \int_0^t \int_{\mathbb{R}^n} |\nabla u|^2 dxdt \leq \frac{1}{2}\|u_0\|^2_{L^2(\mathbb{R}^n)} \tag{1.2.1}$$

for almost every $t \in [0,T]$. Here, we use the notation $D := \mathbb{R}^n \times [0,T]$ and the definition of distributional solutions to (1.1.3) is given in Definition 2.1.1. The Leray-Hopf weak solutions can be obtained from a Galerkin approximation. The inequality (1.2.1) follows with the help of weak convergence results, instead of testing (1.1.3) with u. Indeed, we only have $(u\cdot\nabla)u \in L^{(n+2)/(n+1)}$ because of the embedding $L_t^\infty L_x^2 \cap L_t^2 H_x^1 \subset L_{t,x}^{2+4/n}$, and the product of this term with $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ is not necessarily integrable.

Although the existence and the regularity of solutions in dimensions $n \geq 3$ are still open, remarkable progress has been made in dimension $n = 3$ since the pioneering work by Leray in 1930s. Leray's argument yields that the $\frac{1}{2}$-dimensional Hausdorff measure of the singular times vanishes. Here, the set of singular times refers to the set $\mathcal{T} \subset [0,\infty)$ such that the weak solution is continuous from $[0,\infty)\backslash\mathcal{T}$ to $H^1(\mathbb{R}^3)$. Another important step was made by Scheffer [42, 43, 44] and Caffarelli, Kohn and Nirenberg [5]. In [44] Scheffer pioneered the partial regularity theory by introducing the notion of suitable weak solutions and proving their existence in dimension $n = 3$ when $f = 0$. Moreover, he proved that the singular sets of these suitable weak solutions have finite $\frac{5}{3}$-dimensional Hausdorff measure in space-time. Caffarelli, Kohn and Nirenberg made remarkable improvements and generalizations, still in dimension $n = 3$, by proving local partial regularity results for a general force and by proving that the 1-dimensional parabolic Hausdorff measure of the singular sets of suitable weak solutions is zero.

Theorem 1.2.1 (Caffarelli, Kohn and Nirenberg, [5], Theorem A′)**.** *In dimension $n = 3$, suppose for some $q > \frac{5}{2}$,*

$$f \in L^2(D) \cap L^q_{loc}(D), \quad \operatorname{div} f = 0.$$

Then the Navier-Stokes equations (1.1.3) *with initial data $u_0 \in L^2(\mathbb{R}^3)$ has a suitable weak solution on D whose singular set S satisfies $\mathcal{P}^1(S) = 0$.*

Here, $\mathcal{P}$ is the parabolic Hausdorff measure defined in Definition 2.2.13. The definition of suitable weak solutions is given below.

Definition 1.2.2 (Suitable weak solution)**.** A pair of functions (u, p) is called a suitable weak solution of the Navier Stokes equations (1.1.3) on $D := \mathbb{R}^n \times [0, T]$ if it is a Leray-Hopf weak solution of (1.1.3) on D and satisfies the following local energy inequality

$$\begin{aligned}&\int_{B_{r_0}(x_0)\times\{t\}} |u|^2 \phi dx + 2 \iint_{Q_{r_0}(x_0,t_0)} |\nabla u|^2 \phi dx ds \\ &\le \iint_{Q_{r_0}(x_0,t_0)} \Big(|u|^2(\partial_t \phi + \Delta \phi) + (|u|^2 + 2p) u \cdot \nabla \phi + 2 f \cdot u\Big) dx ds.\end{aligned} \tag{1.2.2}$$

for any point $(x_0, t_0) \in D$, any $Q_{r_0}(x_0, t_0) \subset D$ with $r_0 > 0$, almost every $t \in [t_0 - r_0^2, t_0)$, any $r_1 \in (0, r_0)$, any scalar function $0 \le \phi \in C^\infty(Q_{r_0}(x_0, t_0))$ with $\phi = 0$ in $Q_{r_0}(x_0, t_0) \backslash Q_{r_1}(x_0, t_0)$. Here, we use the notation for parabolic cylinder

$$Q_{r_0}(x_0,t_0) := \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \,|\, t_0 - r_0^2 < t < t_0, |x - x_0| < r_0\}.$$

Remark 1.2.3. For a general function in $L_t^\infty L_x^2 \cap L_t^2 H_x^1$, the first integral in (1.2.2) may not be well-defined for all t. However, for Leray-Hopf weak solutions, one can redefine the solution in those time slices with zero measure in t and make this solution weakly continuous as a mapping from the time interval to L_x^2.

Remark 1.2.4. It is unknown if Leray-Hopf weak solutions satisfy the local energy inequality, since $u\phi$ is not an admissible test function, thus, they might not be suitable weak solutions. Caffarelli, Kohn and Nirenberg [5] proved their existence in a general setting by retarded mollification, and then showing that the sequence of the approximation solutions $\{u_k\}_{k\in\mathbb{N}}$ of the mollified equations is relatively compact in the $L_{t,x}^3$-topology. Because we have $L_t^\infty L_x^2 \cap L_t^2 H_x^1 \subset L_{t,x}^{2+4/n}$, the compactness in $L_{t,x}^3$ in dimension $n = 3$ can be obtained from boundedness of $\{u_k\}_{k\in\mathbb{N}}$ in $L_{t,x}^{10/3}$ and compactness in $L_{t,x}^2$. In contrast, in dimension $n = 4$ we only have that the approximation sequence is relatively compact in $L_{t,x}^\alpha$ for $\alpha < 3$, which is not enough for the local energy inequality to hold in the limit. In dimensions $n \geq 5$, this is even more difficult, because Leray-Hopf weak solutions may not belong to $L_{t,x}^3$ and the local energy inequality (1.2.2) may not even make sense.

Also partial regularity becomes very difficult for higher dimensions $n \geq 4$, as mentioned in Remark 1.2.4. In dimension $n = 4$, Scheffer [43] showed that there exist weak solutions whose singular sets have finite 3-dimensional Hausdorff measure in space-time. Since the local energy inequality is the most important ingredient of Theorem 1.2.1, the following natural question arises, mentioned by Dong and Du in Remark 1.1 of [11]: Do there exist partially regular weak solutions of the Navier-Stokes equations in 4D which satisfy a local energy inequality in a certain sense?

This problem has not been answered for quite a long time; however, there have been many results in this direction. Dong and Du [11] showed that the 2-dimensional Hausdorff measure of the singular sets of local-in-time regular weak solutions at the first blow-up time is zero. Under the assumption on the existence of the suitable weak solutions, Dong, Gu [12] and Wang, Wu [53] independently proved that the 2-dimensional parabolic Hausdorff measure of the singular sets is zero. A similar study of partial regularity has also been carried out for the magneto-hydrodynamic equations by Choe and Yang [8]. Biryuk, Craig and Ibrahim [4] discussed the difficulty of validating the local energy inequality in higher dimensions $n \geq 4$. Taniuchi [51] proved the local energy inequality in the dimensions $3 \leq n \leq 10$, given some conditional regularity on distributional solutions.

The first contribution of this thesis is to answer Dong and Du' question, by constructing weak solutions of the Navier-Stokes equations in dimension $n = 4$ satisfying the local energy inequalities (2.1.19) and (2.1.20) below and showing that these solutions are global-in-time partially regular with singular sets of locally finite 2-dimensional parabolic Hausdorff measure, i.e. we prove the following result

Theorem 1.2.5 (Wu, [55]). *In dimension $n = 4$, given any (possibly large) $T > 0$ and a weakly solenoidal force $f \in L^q_{loc} \cap L^{3/2}(D)$ for some $q > 3$, there exists a weak solution set (u, p, λ, ω) defined in Definition 2.1.11 for the nonstationary Navier-Stokes equations* (1.1.3) *in D which satisfies the local energy inequalities* (2.1.19) *and* (2.1.20). *Moreover, (u, p) is a weak solution of the Navier-Stokes equations with $u \in L^\infty_t L^2_x \cap L^2_t H^1_x(D)$ and $p \in L^{3/2}(D)$, and the singular set S of u as defined in Definition 2.2.12 satifies $\mathcal{P}^2(S) < \infty$.*

Theorem 1.2.5 improves Scheffer's result in [43] by refining the estimate of the Hausdorff dimension of singular sets from 3 to 2 and by allowing general forces. We remark that the local energy inequalities

(2.1.19) and (2.1.20) are slightly weaker than the local energy inequality (1.2.2). Nevertheless, they suffice to give all the partial regularity criteria which Caffarelli, Kohn and Nirenberg have obtained for dimension $n = 3$.

1.3 Towards the singularity formation

The partial regularity theory tells us that the singular sets of solutions to the Navier-Stokes equations in dimensions $n = 3, 4$ cannot be too large. However, the ultimate question posed by the Clay Mathematical Institute is still open, i.e. we do not know if the singular sets are empty for all smooth initial data with sufficient decay at infinity. Not being able to answer the question in general, many mathematicians turned to study various special cases. An important example is the axi-symmetric Navier-Stokes equations in $\mathbb{R}^3 \times [0, \infty)$. In cylindrical coordinate system of $\mathbb{R}^3$, a velocity field $u = (u_r, u_\theta, u_z)$ is given by

$$u = u_r \mathbf{e}_r(r, \theta, z) + u_\theta \mathbf{e}_\theta(r, \theta, z) + u_z \mathbf{e}_z(r, \theta, z).$$

Assume (u, p) is a solution to the Navier-Stokes equations which is axi-symmetric with respect to z axis, then we have

$$\begin{aligned} \partial_t u_r + u \cdot \nabla u_r - r^{-1} u_\theta^2 + \partial_r p &= \Delta u_r - r^{-2} u_r, \\ \partial_t u_\theta + u \cdot \nabla u_\theta - r^{-1} u_r u_\theta &= \Delta u_\theta - r^{-2} u_\theta, \\ \partial_t u_z + u \cdot \nabla u_z + \partial_z p &= \Delta u_z, \\ \operatorname{div} u &= 0. \end{aligned} \tag{1.3.1}$$

The question of global regularity in the axi-symmetric case also is still open. However, as a special and somewhat simpler case which still captures the main properties and difficulties of the original equations, most remarkably the natural scaling invariance, the axi-symmetric equation (1.3.1) is intensively studied in [22], [52], [7], [19], [6], [45], [25],

[54] etc. An important observation is that the scalar quantity $\chi = ru_\theta$ satisfies the following parabolic equation

$$\partial_t \chi - \Delta \chi + (u \cdot \nabla)\chi + 2r^{-1}\partial_r \chi = 0. \tag{1.3.2}$$

The parabolic equation (1.3.2) with nonzero divergence-free drift u has been shown to be very useful for excluding singularities of the first type for the axi-symmetric Navier-Stokes equations ([7], [19], [6], [45], etc.).

Theorem 1.3.1 (Chen, Strain, Yau and Tsai, [7], Theorem 1.1)**.** *Let (u, p) be a weak solution of the axi-symmetric Navier-Stokes equations (1.3.1) in $L^3_{x,t} \times L^{3/2}_{x,t}(Q_1(0,1))$ such that $u \in L^\infty(Q_1(0,1) \backslash Q_r(0,1))$ for any $r \in (0,1)$, then u cannnot develop a first-type singularity at $(0,1)$, i.e. a singularities at $(0,1)$ for which there holds*

$$|u(r,z,\theta,t)| \le \frac{C}{\sqrt{r^2 + 1 - t}}. \tag{1.3.3}$$

The bound (1.3.3) is invariant under the natural scaling of the Navier-Stokes equations. Roughly speaking, Theorem 1.3.1 is a result of scaling invariance and the following result of a linear toy model.

Theorem 1.3.2 (Chen, Strain, Yau and Tsai, [7], Theorem 3.1)**.** *Let $v \in L^\infty_t L^2_x \cap L^2_t H^1_x$ be a weak solution of the following parabolic equation in $Q_1(0,1)$,*

$$\partial_t v - \Delta v + (u \cdot \nabla)v + 2r^{-1}\partial_r v = 0. \tag{1.3.4}$$

Assume $\operatorname{div} u = 0$ *and*

$$|u(r,z,\theta,t)| \le \frac{C}{r}, \tag{1.3.5}$$

then $\|v\|_{C^\alpha(Q_{1/2}(0,1))} \le C\|v\|_{L^\infty(Q_1(0,1))}$ for some $\alpha \in (0,1)$.

Theorem 1.3.2 can be proved by De Giorgi-Nash-Moser iteration. To show Theorem 1.3.1, note that χ is locally bounded, for example, using some kernel estimates proved by Nash [37]. Then by Theorem 1.3.2, χ is locally Hölder continuous and one can carry out some scaling argument in [7] to show u is regular.

The global regularity of the axi-symmetric Navier-Stokes equations in $\mathbb{R}^3 \times [0, \infty)$ would be solved if v is Hölder continuous in the toy model (1.3.4) for general drifts $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$ without the pointwise bound (1.3.5). This motivates the study of the regularity requirement on the drift u for obtaining the Hölder continuity of the solution v, as in [46], [48] and etc. However, as our second contribution here, we show that the Hölder continuity of v in (1.3.4) may fail for certain supercritical drifts in the space $L_t^\infty L_x^{3-} \cap L_t^{4-} H_x^1$.

Theorem 1.3.3. *For any $\lambda \in (0, 2)$, any $\alpha \in (0, \frac{\lambda}{3-\lambda})$ and any $q \in \left[1, \frac{4+2\alpha}{1+2\alpha}\right)$, there exists $T > 0$ and a time-dependent drift $u \in L_t^\infty L_x^{3-\lambda} \cap L_t^q H_x^1$ satisfying*

$$|u(r, z, \theta, t)| \leq \min \left\{ \frac{C(\lambda)}{|x|^{1+\alpha}}, \frac{C(\lambda)}{(T-t)^{(1+\alpha)/(2+\alpha)}} \right\}$$

and such that $\operatorname{div} u = 0$*, with the following property. The parabolic equation* (1.3.4) *in $B \times [0, T]$, where B is the unit ball of $\mathbb{R}^3$, has a bounded weak solution which is not continuous at the origin at time $T > 0$. In particular, v/r blows up with rate r^{-1}.*

Note that the quantity v/r corresponds to u_θ in the axi-symmetric Navier-Stokes equations (1.3.1). Theorem 1.3.3 shows that one cannot just rely on classical techniques in regularity theory to exclude singularities in the axi-symmetric Navier-Stokes equations in space dimensions $n \geq 3$. Actually, this theorem shows that the toy model (1.3.4) of the axi-symmetric Navier-Stokes equations can develop a finite-time singularity at a linear level. Although the dynamics in the linearized equation is significantly simpler than the original nonlinear equation, we still hope this can shed some light and lead to further progress on the Navier-Stokes equations.

Chapter 2

Partial regularity theory

This chapter presents partial regularity theory for the Navier-Stokes equations in dimensions $n = 3, 4$. The partial regularity theory in dimension $n = 3$ is pioneered by Scheffer [42, 43, 44] and Caffarelli, Kohn and Nirenberg [5]. The case $n = 4$ is proved by Wu [55]. The presentation follows [55] to discuss the case $n = 4$ in details. Along with the presentation, we shall point out how the argument can be easily adapted to the case $n = 3$.

The key idea in [5] in dimension $n = 3$ is to show the existence of certain weak solutions satisfying local energy inequalities and then prove the partial regularity from these local estimates. Obtaining similar estimates is harder in dimension $n = 4$. We introduce a new notion of generalized solutions which is called weak solution sets. This notion couples some defect measures which appear in local energy inequalities. We prove the existence of weak solution sets in Section 2.1, then we show that the weak solutions in these weak solution sets are partially regular in Section 2.2.

2.1 The existence of weak solution sets

To construct weak solutions for the Navier-Stokes equations in the space-time domain $D := \mathbb{R}^n \times [0, T]$ with $n = 3, 4$, we consider the regularized Navier-Stokes equations, namely

$$\begin{aligned} \partial_t u_k - \Delta u_k + [(\chi_k * u_k) \cdot \nabla] u_k + \nabla p_k &= f \\ \operatorname{div} u_k &= 0 \\ u(\cdot, 0) &= u_0, \end{aligned} \tag{2.1.1}$$

where $\{\chi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ are standard mollifiers. This regularization was used by Hopf in [17] to show the existence of weak solutions. The first step is to use a Galerkin method to construct stronger solutions for the regularized equations with uniform energy estimates.

2.1.1 Solving regularized equations and weak compactness of measures. Before we prove the existence of weak solutions of the regularized Navier-Stokes equations (2.1.1), we recall the following definition of distributional solutions of the Navier-Stokes equations (1.1.3), which also avoids any possible ambiguity of realizing the initial data u_0.

Definition 2.1.1. A pair of functions $(u, p) \in L_t^2 H_{x,\mathrm{loc}}^1(D) \times L_{\mathrm{loc}}^{1+2/n}(D)$ are distributional solutions of (1.1.3) if u is weakly divergence-free and for any $\varphi \in C_c^\infty(\mathbb{R}^n \times [0,T])$ and any $t \in [0,T]$, we have

$$
\begin{aligned}
&-\int_0^t \int_{\mathbb{R}^n} u^i \partial_t \varphi_i dx dt + \int_0^t \int_{\mathbb{R}^n} \partial_j u^i \partial_j \varphi_i dx dt + \int_0^t \int_{\mathbb{R}^n} u^j \partial_j u^i \varphi_i dx dt \\
&-\int_0^t \int_{\mathbb{R}^n} p \operatorname{div} \varphi dx dt - \int_0^t \int_{\mathbb{R}^n} f_i \varphi_i dx dt = \int_{\mathbb{R}^n} \big(u_0^i \varphi_i(0) - u^i(t)\varphi_i(t)\big) dx.
\end{aligned}
\tag{2.1.2}
$$

Remark 2.1.2. If we restrict the test functions to divergence-free functions, then we have a weak formulation of (1.1.3) without p. If we test with $\nabla\phi$ where ϕ is a scalar function, then we obtain the following well-known elliptic equation for the pressure

$$
-\Delta p = \partial_i \partial_j (u^i u^j). \tag{2.1.3}
$$

Distributional solutions for the regularized Navier-Stokes equations (2.1.1) can be defined in a similar way. We now show that there exist weak solutions for (2.1.1) with uniform energy bounds and satisfying local energy inequality.

Lemma 2.1.3. *For $n = 3, 4$, let $\{\chi_k\}_{k\in\mathbb{N}}$ be a sequence of standard mollifiers and $f \in L^{3/2}(D)$, then we have a sequence $\{(u_k, p_k)\}_{k\in\mathbb{N}} \subset L_t^\infty L_x^2 \cap L_t^2 H_x^1(D) \times L^{3/2}(D)$ such that (u_k, p_k) is a distributional solution to the regularized nonstationary Navier-Stokes equations* (2.1.1). *Moreover,*

(i) $\{u_k\}_{k\in\mathbb{N}}$ *is uniformly bounded in* $L_t^\infty L_x^2 \cap L_t^2 \dot{H}_x^1(D)$,

(ii) $\{p_k\}_{k\in\mathbb{N}}$ *is uniformly bounded in* $L^{3/2}(D)$,

(iii) $\{\partial_t u_k\}_{k\in\mathbb{N}}$ *is uniformly bounded in* $L_t^1 \mathbb{H}_{x,loc}^*(D)$,

where $\mathbb{H}(\mathbb{R}^n) := \{\varphi \in H^1(\mathbb{R}^n) | \operatorname{div} \varphi = 0\}$ *and* $\mathbb{H}^*(\mathbb{R}^n)$ *is the dual space of* $\mathbb{H}(\mathbb{R}^n)$. *Thereby we can pass to the weak limit,*

$$\begin{aligned} u_k \to u &\quad \text{weakly in } L_t^2 H_x^1, \\ u_k \to u &\quad \text{weakly} - * \text{ in } L_t^\infty L_x^2, \\ p_k \to p &\quad \text{weakly in } L^{3/2}. \end{aligned} \tag{2.1.4}$$

This sequence satisfies the local energy inequality, i.e. for any bounded smooth function ϕ with bounded derivatives,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(|u_k(t)|^2\phi(t) - |u_0|^2\phi(0)\right)dx + \int_0^t \int_{\mathbb{R}^n} |\nabla u_k|^2 \phi dxdt \\ \le &\int_0^t \int_{\mathbb{R}^n} |u_k|^2 |\partial_t \phi + \Delta\phi| dxdt + \int_0^t \int_{\mathbb{R}^n} |u_k|^2 (\tilde{u}_k \cdot \nabla)\phi dxdt \\ &+ \int_0^t \int_{\mathbb{R}^n} 2p_k(u_k \cdot \nabla)\phi dx + \int_0^t \int_{\mathbb{R}^n} f \cdot u_k \phi dxdt, \end{aligned} \tag{2.1.5}$$

where $\tilde{u}_k := \chi_k * u_k$.

Remark 2.1.4. We do not need ϕ to have compact support. $\chi_k * u_k$ is bounded and smooth for every k. For any bounded function ϕ with bounded derivatives, $u_k\phi$ is in $L_t^\infty L_x^2 \cap L_t^2 H_x^1(D) \subset L^3(D)$; thus, $u_k\phi$ is an admissible test function.

Proof. We focus on the case $n = 4$. The case $n = 3$ can be proved by the same argument with minor modifications.

The existence of $u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)$ can be proved by a standard Galerkin method. We refer to Theorem 4.4 and Theorem 14.1 in [41] for an exposition. The existence of $p \in L^{3/2}(D)$ is obtained by L^p-theory of elliptic operators and Calderon-Zygmund theory.

For the rest, note that in the regularized equations u_k and $u_k\phi$ are admissible test functions. Testing with u_k and $u_k\phi$ yields the uniform boundedness of $\{u_k\}_{k\in\mathbb{N}}$ and $\{p_k\}_{k\in\mathbb{N}}$ and the local energy inequality (2.1.5).

For the uniform boundedness of $\{\partial_t u_k\}_{k\in\mathbb{N}}$, we remark that the weak formulation (2.1.2) for u_k is equivalent to

$$\forall\, \xi \in \mathbb{H}_x,\ \ \langle \partial_t u_k, \xi\rangle_{\mathbb{H}_x^* \times \mathbb{H}_x} = -\int_{\mathbb{R}^4} \Big(\partial_j u_k^i \partial_j \xi_i + u_k^j \xi_i \partial_j u_k^i - f_i \xi_i\Big) dx$$

for almost every $t \in [0, T]$.

For every $\xi \in C_c^\infty(\Omega)$, $\Omega \subset\subset \mathbb{R}^4$ and almost every $t \in [0, T]$, we can estimate

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^4} \Big(\partial_j u_k^i \partial_j \xi_i + u_k^j \xi_i \partial_j u_k^i - f_i \xi_i\Big) dx\Big| \\
\leq& \|\nabla u_k\|_{L_x^2} \|\nabla \xi\|_{L_x^2} + \|\xi\|_{L_x^3} \|f\|_{L_x^{3/2}} + \|\nabla u_k\|_{L_x^2} \|\xi\|_{L_x^4} \|u_k\|_{L_x^4} \\
\leq& C\big(\|u_k\|_{\dot{H}_x^1}^2 + \|u_k\|_{\dot{H}_x^1} + \|f\|_{L_x^{3/2}}\big) \|\xi\|_{H_x^1}.
\end{aligned}$$

Then integrating in time yields that $\{\partial_t u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L_t^1 \mathbb{H}_{x,\mathrm{loc}}^*(D)$. □

Next, we prove that certain measures in the limit are relatively compact in dimension $n = 4$, which yields a crucial requirement in our parabolic concentration-compactness framework. This is equivalent to tightness of measures. Recall that a collection of measures $\{m_i\}_{i\in\Lambda}$ on $\mathbb{R}^n$ is called **tight** if for any $\epsilon > 0$, there exists a compact set $E_\epsilon \subset \mathbb{R}^n$

such that $m_i(\mathbb{R}^n \backslash E_\epsilon) < \epsilon$ for any $i \in \Lambda$. The case $n = 3$ is much easier and a stronger compactness holds.

Lemma 2.1.5. *Let the assumptions be as in Lemma 2.1.3 and $n = 3, 4$, then $\{|\nabla u_k|^2 dxdt\}_{k\in\mathbb{N}}$, $\{|u_k|^2 dxdt\}_{k\in\mathbb{N}}$ and $\{|u_k|^3 dxdt\}_{k\in\mathbb{N}}$ are tight in the sense of measures.*

Proof. We focus on the case $n = 4$. The case $n = 3$ can be proved by the same argument with minor modifications.

We define a cut-off function $\xi \in C^\infty(\mathbb{R}^4)$ with bounded derivatives by

$$0 \le \xi \le 1, \quad \xi|_{B_\rho} = 0, \quad \xi|_{\mathbb{R}^4\backslash B_{2\rho}} = 1, \quad |\nabla \xi| \le C\rho^{-1}, \quad |\nabla^2 \xi| \le C\rho^{-2}.$$

and let $\rho > 0$ to be determined. From Remark 2.1.4, we know $u_k\xi$ is an admissible test function. Testing the regularized Navier-Stokes equations (2.1.1) with $u_k\xi$ yields

$$\begin{aligned}
&\sup_t \int_{\mathbb{R}^4} |u_k(t)|^2 \xi dx - \int_{\mathbb{R}^4} |u_0|^2 \xi dx + \iint_D |\nabla u_k|^2 \xi dxdt \\
\le& \iint_D |u_k|^2 |\Delta \xi| dxdt + \iint_D |u_k|^2 |\tilde{u}_k| |\nabla \xi| dxdt \\
&+ \iint_D 2|p_k u_k| |\nabla \xi| dxdt + \iint_D |f \cdot u_k| \xi dxdt.
\end{aligned}$$

The bounds for ξ gives

$$\begin{aligned}
&\sup_t \int_{\mathbb{R}^4\backslash B_{2\rho}} |u_k(t)|^2 dx - \int_{\mathbb{R}^4\backslash B_\rho} |u_0|^2 dx + \iint_{D^c_{2\rho}} |\nabla u_k|^2 dxdt \\
\le& C\rho^{-2} \iint_{D_{\rho,2\rho}} |u_k|^2 dxdt + C\rho^{-1} \iint_{D_{\rho,2\rho}} |u_k|^2 |\tilde{u}_k| dxdt \\
&+ 2C\rho^{-1} \iint_{D_{\rho,2\rho}} |p_k u_k| dxdt + \iint_{D_{\rho,2\rho}} |f||u_k| dxdt \\
\le& C\rho^{-2/3} T^{1/3} + C\|f\|_{L^{3/2}(D_{\rho,2\rho})},
\end{aligned}$$

where $D_{2\rho}^c := (\mathbb{R}^4 \backslash B_{2\rho}) \times [0,T]$, and $D_{\rho,2\rho} := (B_{2\rho} \backslash B_\rho) \times [0,T]$. The second inequality follows from Hölder inequality and the fact that $\{u_k\}_{k\in\mathbb{N}}$ is uniformly bounded in $L^3(D)$. Finally, letting ρ be arbitrarily large concludes the tightness of $\{|\nabla u_k|^2 dxdt\}_{k\in\mathbb{N}}$. This also implies that $\{|u_k(t)|^2 dx\}_{k\in\mathbb{N}}$ is tight uniformly in t, which leads to the tightness of $\{|u_k|^2 dxdt\}_{k\in\mathbb{N}}$.

For the tightness of the measures $\{|u_k|^3 dxdt\}_{k\in\mathbb{N}}$, we use Sobolev inequality and the same cutoff function,

$$
\begin{aligned}
&\iint_D |u_k\xi|^3 dxdt \\
\leq &\|u_k\xi\|_{L_t^\infty L_x^2} \iint_D |\nabla(u_k\xi)|^2 dxdt \\
\leq &2\|u_k\xi\|_{L_t^\infty L_x^2} \Big(\iint_D |\nabla u_k|^2 |\xi|^2 dxdt + \iint_D |u_k|^2 |\nabla\xi|^2 dxdt \Big) \\
\leq &2\|u_k\xi\|_{L_t^\infty L_x^2} \Big[\iint_D |\nabla u_k|^2 |\xi|^2 dxdt + C\rho^{-2/3} T^{1/3} \Big(\iint_D |u_k|^3 dxdt \Big)^{2/3} \Big].
\end{aligned}
$$

Given the tightness of $\{|\nabla u_k|^2 dxdt\}_{k\in\mathbb{N}}$ and uniform boundedness of u_k in the natural energy space, arbitrarily large ρ yields the tightness of $\{|u_k|^3 dxdt\}_{k\in\mathbb{N}}$. □

With the tightness of the measures, we obtain convergence of $\{u_k\}_{k\in\mathbb{N}}$ in $L^2(D)$.

Lemma 2.1.6. *Let the assumptions be as in Lemma 2.1.3 and $n = 3, 4$. The sequence $\{u_k\}_{k\in\mathbb{N}}$ is relatively compact in $L^2(D)$. Consequently, the weak limit (u, p) are distributional solutions of the Navier-Stokes equations* (1.1.3).

Lemma 2.1.6 is a direct consequence of the bounds in Lemma 2.1.3 and the following compactness result.

Lemma 2.1.7 (Corollary 6, Simon, [49])**.** *Let X, Y, B be Banach spaces* *and assume that we have the embeddings*

$$X \xhookrightarrow{compact} B \hookrightarrow Y.$$

If a sequence $\{u_k\}_{k\in\mathbb{N}}$ *is bounded in* $L^\alpha(0,T,B) \cap L^1_{loc}(0,T,X), \alpha \in (1,+\infty]$ *and* $\partial_t u_n$ *is bounded in* $L^1_{loc}(0,T,Y)$*, then there exists a subsequence of* $\{u_k\}_{k\in\mathbb{N}}$ *which converges strongly in* $L^\beta(0,T,B)$ *for any* $\beta \in [1,\alpha)$*.*

Proof of Lemma 2.1.6. Note that, to get a compact Sobolev embedding, we first restrict to $B_l \subset \mathbb{R}^n$ with $l \in \mathbb{N}^*$, then letting $\alpha = +\infty$ and

$$X = H^1(B_l), \quad B = L^2(B_l), \quad Y = \mathbb{H}^*(B_l)$$

give the strong convergence of a subsequence of $\{u_k\}_{k\in\mathbb{N}}$ in $L^2(B_l \times [0,T])$. By enlarging r to infinity, a diagonal argument gives a subsequence which converges in $L^2(\Omega \times [0,T])$ for any compact subset Ω of $\mathbb{R}^4$. Note that Lemma 2.1.5 yields the tightness of $\{|u_k|^2 dxdt\}_{k\in\mathbb{N}}$, so it is easy to show the subsequence converges in $L^2(D)$.

With the strong convergence of u_k in $L^2(D)$ and the weak convergence criteria in (2.1.4), it is easy to verify the weak limit (u,p) solves (1.1.3) in the distributional sense. □

2.1.2 Parabolic concentration-compactness. To obtain local energy inequalities for the weak limit (u,p), one would like to pass to the limit $k \to \infty$ in the local energy inequalities (2.1.5) for the approximation solutions. As we mentioned in Remark 1.2.4, the case $n = 4$ is difficult, because the approximation sequence of solutions may not be relatively compact in $L^3_{t,x}$. In similar critical variational problems, concentration phenomena may occur. This motivates to look for an analogue of Lions's [28] concentration-compactness principle in parabolic setting.

Note that concentration-compactness in elliptic setting may not be applicable to the parabolic setting, since it is hopeless to get $\{\nabla u_k(t)\}_{k\in\mathbb{N}}$ is bounded in L^2 for almost every t, even for a subsequence. A relevant example in [29] by Lopes Filho and Nussenzveig Lopes shows that a bounded sequence in L^1 might blow up at almost every point up to any subsequence.

Lemma 2.1.8. *Let $n = 4$. Given a bounded sequence $\{u_k\}_{k\in\mathbb{N}} \subset L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)$, let u be given by the limit in (2.1.4). Suppose u_k converges to u in $L^1_{loc}(D)$. Assume that $\mu_k = |\nabla u_k|^2 dxdt \to \mu$, $\nu_k = |u_k|^3 dxdt \to \nu$ weakly in the sense of measures, where μ and ν are bounded nonnegative measures on $\mathbb{R}^4 \times [0,T]$. Then there exist nonnegative finite measures ω and λ on $\mathbb{R}^4 \times [0,T]$, such that for any $\varphi \in C_c^\infty(D)$,*

$$\iint \varphi d\mu = \iint \varphi |\nabla u|^2 dxdt + \iint \varphi d\lambda, \tag{2.1.6}$$

$$\iint \varphi d\nu = \iint \varphi |u|^3 dxdt + \iint \varphi d\omega. \tag{2.1.7}$$

Moreover, $\omega \ll \lambda$, and we have for any open subdomain Q of D,

$$\iint_Q d\omega \le C \liminf_{k\to\infty} \|u_k - u\|_{L_t^\infty L_x^2(Q)} \iint_Q d\lambda. \tag{2.1.8}$$

In particular, the Radon–Nikodym derivative satisfies

$$\frac{d\omega}{d\lambda} \le C \lim_{r\to 0} \liminf_{k\to\infty} \|u_k - u\|_{L_t^\infty L_x^2(Q_r^*(x_0,t_0))}, \tag{2.1.9}$$

where $Q_r^*(x_0,t_0) := B_r(x_0) \times (t_0 - \frac{r^2}{2}, t_0 + \frac{r^2}{2})$.

Remark 2.1.9. We remark that this lemma only requires $\{u_k\}_{k\in\mathbb{N}}$ to be bounded in $L_t^\infty L_x^2 \cap L_t^2 H_x^1(D)$. $\{u_k\}_{k\in\mathbb{N}}$ does not necessarily solve certain equations. Indeed, we only need $u_k \to u$ converges in $L^1_{\mathrm{loc}}(D)$.

Remark 2.1.10. Although we only state this result for space dimension 4, one can easily see a trivial generalization to higher dimensions.

Proof. Let $v_k = u_k - u \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$, then

$$v_k \to 0 \quad \text{strongly in } L_{t,x}^2, \text{ locally in space,} \tag{2.1.10}$$
$$v_k \to 0 \quad \text{weakly in } L_t^2 \dot{H}_x^1, \tag{2.1.11}$$
$$v_k \to 0 \quad \text{weakly} - * \text{ in } L_t^\infty L_x^2. \tag{2.1.12}$$

Define $\omega_k := |v_k|^3 dxdt$. It is easy to check $\{\omega_k\}_{k\in\mathbb{N}}$ is tight. Indeed, for any compact subset $\Omega \subset D$, denote $\Omega^c := (\mathbb{R}^4 \backslash \Omega) \times [0,T]$, then

$$\|v_k\|_{L^3(\Omega^c)} \leq \|u_k\|_{L^3(\Omega^c)} + \|u\|_{L^3(\Omega^c)}.$$

Because of the weak convergence of $\{\nu_k\}_{k\in\mathbb{N}}$, we know that $\{\nu_k\}_{k\in\mathbb{N}}$ is tight and thus $\|v_k\|_{L^3(\Omega^c)}$ is arbitrarily small given Ω large enough. Thus we can extract a weakly convergent subsequence with a limit denoted by ω. For any $\varphi \in C_c^\infty(D)$, we have

$$\begin{aligned}\iint \varphi d\nu &= \lim_{k\to\infty} \iint \varphi d\nu_k = \lim_{k\to\infty} \iint \varphi |u_k|^3 dxdt \\ &= \iint \varphi |u|^3 dxdt + \lim_{k\to\infty} \iint \varphi |v_k|^3 dxdt \\ &= \iint \varphi |u|^3 dxdt + \iint \varphi d\omega.\end{aligned}$$

The third equality follows from the fact that $u_k \to u$ in L^α locally in space for $\alpha \in [1,3)$, then all the interaction terms vanish. Let $\lambda_k := |\nabla v_k|^2 dxdt \to \lambda$ weakly in the sense of measures. A similar argument verifies (2.1.7), and the interaction term vanishes there since $u_k \to u$ weakly in $L_t^2 \dot{H}_x^1(D)$.

Now we prove (2.1.8). For any $\varphi \in C_c^\infty(D)$, we have

$$\begin{aligned}\iint_D |\varphi|^3 d\omega &= \lim_{k\to\infty} \iint_D |\varphi|^3 d\omega_k = \lim_{k\to\infty} \iint_D |v_k\varphi|^3 dxdt \\ &\le \liminf_{k\to\infty} \sup_{0<t<T} \|v_k\varphi\|_{L^2_x} \int_0^T \|v_k\varphi\|^2_{L^4_x} dt \\ &\le C \liminf_{k\to\infty} \sup_{0<t<T} \|v_k\varphi\|_{L^2_x} \iint_D |\nabla(v_k\varphi)|^2 dxdt \qquad (2.1.13) \\ &\le C \liminf_{k\to\infty} \sup_{0<t<T} \|v_k\varphi\|_{L^2_x} \iint_D |\varphi|^2 |\nabla v_k|^2 dxdt \\ &\le C \liminf_{k\to\infty} \sup_{0<t<T} \|v_k\varphi\|_{L^2_x} \iint_D |\varphi|^2 d\lambda.\end{aligned}$$

The first inequality follows from the interpolation between L^2 and L^4. The second inequality follows from Sobolev embedding. For the third inequality, note that the terms converge to zero when at least one derivative hits φ. Using smooth functions to approximate the indicator function of Q yields the inequality (2.1.8).

Therefore, ω is absolutely continuous with respect to λ, and by Radon–Nikodym theorem, we have

$$\frac{d\omega}{d\lambda} \in L^1(D;\lambda)$$

with

$$\frac{d\omega}{d\lambda}(x_0,t_0) \le C \lim_{r\to 0} \liminf_{k\to\infty} \|v_k\|_{L_t^\infty L_x^2(Q_r^*(x_0,t_0))}$$

for any $(x_0,t_0) \in \mathbb{R}^4 \times (0,T)$. □

Using the concentration-compactness theorem in parabolic setting Lemma 2.1.8 and the tightness results in Lemma 2.1.5, we now can define the notion of weak solution sets involving concentration measures.

Definition 2.1.11. The quadruple (u,p,λ,ω) is a *weak solution set* of the Navier-Stokes equations (1.1.3) if

(i) u and p are obtained as weak limits of the weak solutions $\{(u_k, p_k)\}_{k\in\mathbb{N}}$ of the regularized Navier-Stokes equations (2.1.1), as in Lemma 2.1.3.

(ii) λ and ω are obtained as weak limits of the measures in Lemma 2.1.8.

One can see that every weak solution set comes with a sequence of approximation solutions. However, this is in a sense necessary because a single L^p function is not able to represent concentration of any form. As we shall see, this is effective for analytical purposes in certain critical cases.

2.1.3 Local energy inequalities. In this subsection, we show two energy inequalities with purely local nature for weak solution sets in dimension $n = 4$. Although these inequalities are weaker than the local energy inequality (1.2.2) in a sense, they suffice to establish partial regularity of the distributional solutions (u, p). For technical reasons only, in (2.1.19) and (2.1.20), we present two distinct forms of these estimates. As a by-product, we also prove the local energy inequality (1.2.2) in dimension $n = 3$.

From the elliptic equation (2.1.3) for the pressure p, one may guess p has the same regularity as $|u|^2$, so the pressure term in the local energy inequalities (2.1.5) may also exhibit concentration of mass in dimension $n = 4$. As a preparation for our main goal in this section, we show the concentration in $|up|dxdt$ is localizable and comparable to the concentration in $|u|^3dxdt$.

Lemma 2.1.12. *Let $n = 4$. Suppose $\{(u_k, p_k)\}_{k\in\mathbb{N}}$ are the solutions of the regularized equations (2.1.1) and (u, p, λ, ω) is the corresponding weak solution set, then*

$$\limsup_{k\to\infty} \iint_D \zeta \big|u_k(p_k - \gamma) - u(p - \gamma)\big| dxdt \lesssim \iint_D \zeta d\omega$$

for any $\zeta \in C_c^\infty(D)$ and any $\gamma \in \mathbb{R}$ with $\zeta \geq 0$.

Proof. To prove this result, we need an interpolation inequality. For any $\alpha \in (3, +\infty), \beta \in (2, 3), \vartheta \in (0, 1)$ with $\frac{1}{\alpha} + \frac{2}{\beta} = 1$ and $\frac{1}{\beta} = \frac{\vartheta}{2} + \frac{1-\vartheta}{3}$, we have for any $w \in L_t^\infty L_x^2 \cap L_t^2 H_x^1$,

$$
\begin{aligned}
&\int \|w(t)\|_{L_x^\beta}^3 dt \\
=&\int \|w(t)\|_{L_x^\beta} \cdot \|w(t)\|_{L_x^\beta}^2 dt \\
\leq&\|w\|_{L_t^\alpha L_x^\beta} \|w\|_{L_{t,x}^\beta}^2 \\
\leq&\|w\|_{L_t^\alpha L_x^\beta} \|w\|_{L_{t,x}^2}^{2\vartheta} \|w\|_{L_{t,x}^3}^{2(1-\vartheta)} \\
=&\|w\|_{L_{t,x}^2}^{2\vartheta} \|w\|_{L_{t,x}^3}^{2(1-\vartheta)} \Big(\int \|w(t)\|_{L_x^\beta}^\alpha dt \Big)^{1/\alpha} \\
\leq&\|w\|_{L_{t,x}^2}^{2\vartheta} \|w\|_{L_{t,x}^3}^{2(1-\vartheta)} \Big(\int \|w(t)\|_{L_x^2}^{\alpha(4-\beta)/\beta} \|w(t)\|_{L_x^4}^{\alpha(2\beta-4)/\beta} dt \Big)^{1/\alpha} \\
\leq&\|w\|_{L_{t,x}^2}^{2\vartheta} \|w\|_{L_{t,x}^3}^{2(1-\vartheta)} \|w\|_{L_t^\infty L_x^2}^{(4-\beta)/\beta} \|\nabla w\|_{L_{t,x}^2}^{2/\alpha}.
\end{aligned}
\tag{2.1.14}
$$

The first inequality follows from Hölder inequality. The second and the third inequalities follow from Lebesgue interpolation inequality. The fourth ineuqality follows from the Sobolev inequality.

Now we analyze the concentration phenomena of the measures involving the pressure p_k. Note the following Poisson equation

$$-\Delta p_k = \partial_i \partial_j (\tilde{u}_k^i u_k^j),$$

where $\tilde{u}_k := \chi_k * u_k$. From Remark 2.1.2, we know this equation holds in the sense of distributions for almost every t, then we localize this equation with an arbitrary Lipschitz function $\xi \in C^{0,1}(\mathbb{R}^4)$, i.e.

$$\begin{aligned}&-\Delta(p_k\xi)\\ =&\xi\partial_i\partial_j(\tilde{u}_k^i u_k^j) - \operatorname{div}(p_k\nabla\xi) - \nabla p_k\cdot\nabla\xi\\ =&\partial_i\partial_j(\xi\tilde{u}_k^i u_k^j) - \operatorname{div}(\tilde{u}_k u_k^j\partial_j\xi + p_k\nabla\xi) - \partial_j(\tilde{u}_k^i u_k^j)\partial_i\xi - \nabla p_k\cdot\nabla\xi.\end{aligned}$$

Next, we decompose the pressure $p_k\xi = p_k^1 + p_k^2 + p_k^3$ with

$$\begin{aligned}-\Delta p_k^1 &= \partial_i\partial_j(\xi\tilde{u}_k^i u_k^j),\\ -\Delta p_k^2 &= -\operatorname{div}(\tilde{u}_k u_k^j\partial_j\xi + p_k\nabla\xi),\\ -\Delta p_k^3 &= -\partial_j(\tilde{u}_k^i u_k^j)\partial_i\xi - \nabla p_k\cdot\nabla\xi.\end{aligned}$$

and $p\xi$ in a similar way. Intuitively, the concentration takes place in the component p_k^1, since at least one differentiation hits the cutoff function ξ in other components.

Now we do rigorous estimates term by term. $p_k^1\xi$ can be obtained by the Riesz transformation. Calderon-Zygmund theory yields

$$\|p_k^1(t) - p^1(t)\|_{L_x^{3/2}} \lesssim \|\xi\tilde{u}_k^i(t)u_k^j(t) - \xi\tilde{u}^i(t)u^j(t)\|_{L_x^{3/2}}. \qquad (2.1.15)$$

By writing the right hand side of (2.1.17) into a telescoping sum and using Vitali's convergence theorem, we have

$$\limsup_{k\to\infty}\iint_D |p_k^1 - p^1|^{3/2}dxdt \lesssim \iint_D |\xi|^{3/2}d\omega.$$

Here, we also use the fact that $\omega_k \to \omega$ weakly as $k\to\infty$.

Also, for almost every t, $u_k(t)\in L^3(\mathbb{R}^4)$, then $p_k^2\xi$ can be obtained by convolution with singular kernels. Calderon-Zygmund theory yields

$$\begin{aligned}
&\|p_k^2 - p^2\|_{L^{3/2}_{t,x}(D)} \\
=&\big\|(-\Delta)^{-1}\big[-\operatorname{div}\big((u_k u_k^j - u u^j)\partial_j \xi + (p_k - p)\nabla\xi\big)\big]\big\|_{L^{3/2}_{t,x}(D)} \\
\lesssim&\big\|(u_k u_k^j - u u^j)\partial_j \xi + (p_k - p)\nabla\xi\big\|_{L^{3/2}_t L^{12/11}_x(D)} \\
\lesssim&\|\nabla\xi\|_{L^\infty}\big(\|u_k u_k^j - u u^j\|_{L^{3/2}_t L^{12/11}_x(D)} + \|p_k - p\|_{L^{3/2}_t L^{12/11}_x(D)}\big) \\
\lesssim&\|\nabla\xi\|_{L^\infty}\big(\|u_k - u\|_{L^3_t L^{24/11}_x(D)} + \|p_k - p\|_{L^{3/2}_t L^{12/11}_x(D)}\big).
\end{aligned}$$

Similarly, for $p_k^3\xi$ we have

$$\begin{aligned}
&\|p_k^3 - p^3\|_{L^{3/2}_{t,x}(D)} \\
\lesssim&\|\nabla\xi\|_{L^\infty}\big(\|u_k - u\|_{L^3_t L^{24/11}_x(D)} + \|p_k - p\|_{L^{3/2}_t L^{12/11}_x(D)}\big).
\end{aligned}$$

Let $w = u_k - u$ and $\beta = \frac{24}{11}$, the interpolation inequality (2.1.14) yields

$$\limsup_{k\to\infty} \|u_k - u\|_{L^3_t L^{24/11}_x(D)} = 0 \tag{2.1.16}$$

and Calderon-Zygmund theory yields that for $l = 1$ or 2,

$$\limsup_{k\to\infty} \|p_k^l - p^l\|_{L^3_t L^{12/11}_x(D)} = 0. \tag{2.1.17}$$

Now we combine the estimates for p_k^1, p_k^2 and p_k^3. From (2.1.16) and (2.1.17), we know that p_k^2 and p_k^3 have no contribution to the concentration, then

$$\begin{aligned}
\limsup_{k\to\infty} \|(p_k - p)\xi\|^{3/2}_{L^{3/2}_{t,x}(D)} &\le \limsup_{k\to\infty} \sum_{l=1}^{3} \|p_k^l - p^l\|^{3/2}_{L^{3/2}_{t,x}(D)} \\
&\le \iint_{D_r} |\xi|^{3/2} d\omega.
\end{aligned} \tag{2.1.18}$$

Therefore, we can choose $\xi = \zeta^{2/3}$ and bound the concentration of the measure as follows,

$$\begin{aligned}
&\limsup_{k\to\infty}\iint_D \zeta\big|u_k(p_k-\gamma)-u(p-\gamma)\big|dxdt\\
\le&\limsup_{k\to\infty}\iint_D \zeta|u_k||p_k-p|dxdt+\limsup_{k\to\infty}\iint_D \zeta|u_k-u||p-\gamma|dxdt\\
\le&\limsup_{k\to\infty}\iint_D \zeta|u_k-u||p_k-p|dxdt+\limsup_{k\to\infty}\iint_D \zeta|u||p_k-p|dxdt\\
\lesssim&\iint_D \zeta d\omega.
\end{aligned}$$

Due to Vitali's convergence theorem, the second term in the second line and the second term in the third line converge to zero. Note that ζ is nonnegative and smooth. By Corollary 2.3.2, ξ is indeed a compactly supported Lipschitz continuous function. The last inequality follows from (2.1.18). □

Now we can prove the following local energy inequalities.

Proposition 2.1.13. *Let the assumptions be as in Lemma 2.1.3, then the following local energy inequalities hold in dimension $n=4$, up to a suitable subsequence in Lemma 2.1.3,*

$$\begin{aligned}
&\limsup_{k\to\infty}\sup_t\int_{\mathbb{R}^4}|u_k(t)|^2\varphi(t)dx+\iint_D|\nabla u|^2\varphi dxdt+\iint_D\varphi d\lambda\\
\le&\iint_D|u|^2|\partial_t\varphi+\Delta\varphi|dxdt+3\sum_{i=1}^n\iint_D|\nabla\varphi_i|d\omega+2\sum_{i=1}^n\iint_D|u|^3|\nabla\varphi_i|dxdt\\
&+2\sum_{i=1}^n\iint_D|\nabla\varphi_i||p-\gamma_i|^{3/2}dxdt+\iint_D f\cdot u\varphi dxdt,
\end{aligned}\tag{2.1.19}$$

$$
\begin{aligned}
&\limsup_{k\to\infty}\sup_t\int_{\mathbb{R}^4}|u_k(t)|^2\varphi(t)dx+\iint_D|\nabla u|^2\varphi dxdt+\iint_D\varphi d\lambda\\
\le&\iint_D|u|^2|\partial_t\varphi+\Delta\varphi|dxdt+3\iint_D|\nabla\varphi|d\omega+\iint_D|u|^2(u\cdot\nabla)\varphi dxdt\\
&+2\iint_D|u-\beta|^3|\nabla\varphi|dxdt+2\iint_D p(u\cdot\nabla)\varphi dxdt+\iint_D f\cdot u\varphi dxdt,
\end{aligned}
\tag{2.1.20}
$$

for any $n\in\mathbb{N}$, *any functions* $\{\gamma_i\}_{1\le i\le n}\subset L^{3/2}([0,T],\mathbb{R})$ *and* $\beta\subset L^3([0,T],\mathbb{R}^4)$, *any nonnegative cut-off functions* $\varphi\in C_c^\infty(D)$ *with* $\varphi(\cdot,0)=0$ *and* $\{\varphi_i\}_{1\le i\le n}\subset C_c^\infty(D)$ *with* $\varphi=\sum_{i=1}^n\varphi_i$.

In dimension $n=3$, (u,p) *is a suitable weak solution in the sense of Definition 1.2.2 up to a suitable subsequence in Lemma 2.1.3.*

Proof. To prove local energy inequalities (2.1.19) and (2.1.20), we pass k to infinity in the local energy inequalities for approximation sequence u_k. For the cutoff function φ defined above, the local energy inequalities (2.1.5) reduces to

$$
\begin{aligned}
&\sup_t\int_{\mathbb{R}^4}|u_k(t)|^2\varphi(t)dx+\iint_D|\nabla u_k|^2\varphi dxdt\\
\le&\iint_D|u_k|^2|\partial_t\varphi+\Delta\varphi|dxdt+\iint_D|u_k|^2(\tilde u_k\cdot\nabla)\varphi dxdt\\
&+\iint_D 2p_k(u_k\cdot\nabla)\varphi dx+\iint_D f\cdot u_k\varphi dxdt.
\end{aligned}
\tag{2.1.21}
$$

Since $u_k\to u$ in $L^2(D)$, the convergence of the third and the last terms is straightforward. The convergence of the second term is given by Lemma 2.1.5 and (2.1.6) in Lemma 2.1.8. The difference between the two inequalities and the technical difficulties come from the rest terms, namely the cubic term of u and the term involving p.

For the cubic term of u in the local energy inequality (2.1.19), note that

$$
\iint_D|\tilde u_k|^3|\nabla\varphi|dxdt=\|h_1+h_2\|^3_{L^3(D)},
$$

where

$$h_1(x,t) = \int_{\mathbb{R}^4} u_k(x-y,t)\chi_k(y)\big(|\nabla\varphi(x,t)|^{1/3} - |\nabla\varphi(x-y,t)|^{1/3}\big)dy,$$
$$h_2(x,t) = \int_{\mathbb{R}^4} u_k(x-y,t)\chi_k(y)|\nabla\varphi(x-y,t)|^{1/3}dy.$$

For h_1, note that $d_k := \operatorname{diam}(\operatorname{supp}\chi_k) \to 0$ and that $x \to |x|^{1/3}$ is $\frac{1}{3}$−Hölder continuous, then Young's inequality for convolution yields

$$\begin{aligned}
&\|h_1\|_{L^3}\\
\leq&\Big\| \int_{\mathbb{R}^4} u_k(x-y,t)\chi_k(y)\frac{|\nabla\varphi(x,t)|^{1/3} - |\nabla\varphi(x-y,t)|^{1/3}}{|y|^{1/3}} d_k^{1/3} dy\Big\|_{L^3(D)}\\
\leq&Cd_k^{1/3}\|\varphi\|_{C^2}\|\tilde{u}_k\|_{L^3(D)}\\
\leq&Cd_k^{1/3}\|\varphi\|_{C^2}\|u_k\|_{L^3(D)}.
\end{aligned}$$

We can then deduce that h_1 part converges to zero in L^3 when k tends to infinity. For h_2, Young's inequality for convolution yields

$$\|h_2\|_{L^3(D)} = \big\|\big(u_k|\nabla\varphi|^{1/3}\big) * \chi_k\big\|_{L^3(D)} \leq \big\|u_k|\nabla\varphi|^{1/3}\big\|_{L^3(D)}.$$

Then these estimates for h_1 and h_2 yield

$$\begin{aligned}
&\limsup_{k\to\infty}\iint_D |u_k|^2(\tilde{u}_k\cdot\nabla)\varphi dxdt\\
\leq&\frac{2}{3}\limsup_{k\to\infty}\iint_D |u_k|^3|\nabla\varphi|dxdt + \frac{1}{3}\limsup_{k\to\infty}\iint_D |\tilde{u}_k|^3|\nabla\varphi|dxdt\\
\leq&\iint_D |\nabla\varphi||u|^3dxdt + \iint_D |\nabla\varphi|d\omega.
\end{aligned} \tag{2.1.22}$$

For the term involving pressure in the local energy inequality (2.1.19), we use Lemma 2.1.12 and the fact that u_k is weakly divergence-free to bound

$$
\begin{aligned}
&\limsup_{k\to\infty}\iint_D p_k(u_k\cdot\nabla)\varphi dxdt\\
=&\sum_{i=1}^n \limsup_{k\to\infty}\iint_D p_k u_k\cdot\nabla\varphi_i dxdt\\
=&\sum_{i=1}^n \limsup_{k\to\infty}\iint_D (p_k-\gamma_i)u_k\cdot\nabla\varphi_i dxdt\\
\le&\frac{1}{3}\sum_{i=1}^n\iint_D |u|^3|\nabla\varphi_i|dxdt+\sum_{i=1}^n\iint_D|\nabla\varphi_i|d\omega\\
&+\sum_{i=1}^n\frac{2}{3}\iint_D|\nabla\varphi_i||p-\gamma_i|^{3/2}dxdt.
\end{aligned}
$$

For the cubic term of u in the local energy inequality (2.1.20), we use the fact that $u_k, \tilde{u}_k$ and u are weakly divergence-free. Thus,

$$
\begin{aligned}
&\iint_D |u_k|^2(\tilde{u}_k\cdot\nabla)\varphi dxdt\\
=&\iint_D |u_k-\beta+\beta|^2(\tilde{u}_k\cdot\nabla)\varphi dxdt\\
=&\iint_D \big[|u_k-\beta|^2+2(u_k-\beta)\cdot\beta\big](\tilde{u}_k\cdot\nabla)\varphi dxdt\\
=&\iint_D |u_k-\beta|^2\big[(\tilde{u}_k-\beta)\cdot\nabla\big]\varphi dxdt\\
&+\iint_D |u_k-\beta|^2(\beta\cdot\nabla)\varphi dxdt\\
&+2\iint_D \big[(u_k-\beta)\cdot\beta\big](\tilde{u}_k\cdot\nabla)\varphi dxdt.
\end{aligned}
\tag{2.1.23}
$$

Next, we argue that the individual terms above can be bounded by the weak limit u and the concentration mass ω. For the term in the third line of (2.1.23), since $\tilde{u}_k-\beta=(u_k-\beta)*\chi_k$, we can apply the same trick by replacing u_k and $\tilde{u}_k$ with $u_k-\beta$ and $\tilde{u}_k-\beta$ and use

Young's inequality for convolution, therefore it is sufficient to look at the following term

$$\begin{aligned}
&\iint_D |u_k-\beta|^2\big[(u_k-\beta)\cdot\nabla\big]\varphi dxdt\\
\le&\iint_D |u_k-u|^3|\nabla\varphi|dxdt+\iint_D |u-\beta|^3|\nabla\varphi|dxdt\\
&+\iint_D 3|u_k-u|^2|u-\beta||\nabla\varphi|dxdt\\
&+\iint_D 3|u_k-u||u-\beta|^2|\nabla\varphi|dxdt\\
\le&\iint_D |u_k-u|^3|\nabla\varphi|dxdt+\iint_D |u-\beta|^3|\nabla\varphi|dxdt\\
\to&\iint_D |\nabla\varphi|d\omega+\iint_D |u-\beta|^3|\nabla\varphi|dxdt \quad \text{as } k\to\infty.
\end{aligned}$$

Now, we can pass $k\to\infty$ in the remaining two terms in the last line of (2.1.23). Hence we have

$$\begin{aligned}
&\limsup_{k\to\infty}\iint_D |u_k|^2(\tilde{u}_k\cdot\nabla)\varphi dxdt\\
\le&\iint_D |\nabla\varphi|d\omega+\iint_D |u-\beta|^3|\nabla\varphi|dxdt\\
&+\iint_D |u-\beta|^2(\beta\cdot\nabla)\varphi dxdt+2\iint_D \big[(u-\beta)\cdot\beta\big](u\cdot\nabla)\varphi dxdt\\
=&\iint_D |\nabla\varphi|d\omega+\iint_D |u-\beta|^3|\nabla\varphi|dxdt\\
&+\iint_D |u|^2(u\cdot\nabla)\varphi dxdt-\iint_D |u-\beta|^2\big[(u-\beta)\cdot\nabla\big]\varphi dxdt\\
\le&\iint_D |\nabla\varphi|d\omega+2\iint_D |u-\beta|^3|\nabla\varphi|dxdt+\iint_D |u|^2(u\cdot\nabla)\varphi dxdt.
\end{aligned}$$

Finally, for the term involving pressure in the local energy inequality

(2.1.20), Lemma 2.1.12 yields

$$\limsup_{k\to\infty} \iint_D p_k(u_k \cdot \nabla)\varphi dxdt \le \iint_D p(u \cdot \nabla)\varphi dxdt + \iint_D |\nabla\varphi| d\omega.$$

In dimension $n = 3$, it is easier to prove the local energy inequality (1.2.2), since $\{u_k\}_{k\in\mathbb{N}}$ converges in $L^3_{t,x}$ and $\{p_k\}_{k\in\mathbb{N}}$ converges in $L^{3/2}_{t,x}$. □

2.2 Partial regularity theory

Partial regularity theory contains deep results of natural scaling and local energy inequalities of the Navier-Stokes equations. In this section, we show that weak solution sets have the same scaling invariance as classical solutions, then we combine the ideas from Caffarelli, Kohn and Nirenberg [5] and Lin [27] to establish the partial regularity theory in dimension $n = 4$ with the presence of concentration measures.

As we mentioned in the introduction, Scheffer proved $\mathcal{H}^3(S) < \infty$ in dimension $n = 4$. An interesting point is that Scheffer overcame the loss of compactness in $L^3_{t,x}$ by proving uniform local $L^3_{t,x}$ estimate for the approximate solutions u_k[1], then one can pass the local estimate to the weak limit without splitting the concentration measures and the weak limit $u \in L^3$. In Scheffer's approach, local $L^3_{t,x}$ estimate gives the bound for $\mathcal{H}^3$ measure, while in this thesis, the $L^2_t H^1_x$ estimate gives more refined bound for $\mathcal{H}^2$ measure.

The case $n = 3$ follows from a simpler argument, so we present the main results in dimension $n = 3$ without detailed proofs. Interested readers may study [5] and [27] for detailed proofs.

2.2.1 Dimensionless estimates in dimension $n = 4$. The Navier-Stokes equations have a nice scaling property. If (u, p) solves (1.1.3)

[1] One can see Lemma 2.6 in Scheffer [43] for details.

with force f, then u_r, p_r defined by

$$u_r(x,t) = ru(rx, r^2t) \quad p_r(x,t) = r^2p(rx, r^2t)$$

solve (1.1.3) with force f_r defined by

$$f_r(x,t) = r^3 f(rx, r^2t).$$

The weak solution sets also have a similar scaling property.

Lemma 2.2.1. *Let $n = 4$. If (u, p, λ, ω) is a weak solution set of the Navier-Stokes equations (1.1.3) with external force f, then for any $r > 0$, the scaled quadruple $(u_r, p_r, \lambda_r, \omega_r)$ is also a weak solution set of (1.1.3) with external force f_r, where u_r, p_r and f_r are defined as above and λ_r, ω_r are defined as*

$$\iint_E d\lambda_r := r^{-2} \iint_{\{(rx,r^2t)|(x,t)\in E\}} d\lambda$$
$$\iint_E d\omega_r := r^{-3} \iint_{\{(rx,r^2t)|(x,t)\in E\}} d\omega$$

for any $E \subset \mathbb{R}^4 \times \mathbb{R}$.

For a weak solution set (u, p, λ, ω), we give short-hand notations for the following scale-invariant quantities.

$$\begin{aligned} A(x_0,t_0,r) &= \limsup_{k\to+\infty} \sup_{t_0-r^2<t<t_0} r^{-2} \int_{B_r(x_0)} |u_k(t)|^2 dx \\ \delta(x_0,t_0,r) &= r^{-2} \iint_{Q_r(x_0,t_0)} |\nabla u|^2 dxdt \qquad (2.2.1) \\ \delta_c(x_0,t_0,r) &= r^{-2} \iint_{Q_r(x_0,t_0)} d\lambda \end{aligned}$$

$$
\begin{aligned}
G(x_0,t_0,r) &= r^{-3}\iint_{Q_r(x_0,t_0)} |u|^3 dxdt\\
G_c(x_0,t_0,r) &= r^{-3}\iint_{Q_r(x_0,t_0)} d\omega\\
H(x_0,t_0,r) &= r^{-3}\iint_{Q_r(x_0,t_0)} |u-\tilde{u}_{r,x_0}|^3 dxdt\\
K(x_0,t_0,r) &= r^{-3}\iint_{Q_r(x_0,t_0)} |p|^{3/2} dxdt\\
L(x_0,t_0,r) &= r^{-3}\iint_{Q_r(x_0,t_0)} |p-\tilde{p}_{r,x_0}|^{3/2} dxdt\\
F_1(x_0,t_0,r) &= r^{3q-6}\iint_{Q_r(x_0,t_0)} |f|^q dxdt\\
F_2(x_0,t_0,r) &= \iint_{Q_r(x_0,t_0)} |f|^2 dxdt
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{u}_{r,x_0}(t) &= \frac{1}{\mathcal{L}(B_r)}\int_{B_r(x_0)} u(x,t)dx\\
\tilde{p}_{r,x_0}(t) &= \frac{1}{\mathcal{L}(B_r)}\int_{B_r(x_0)} p(x,t)dx
\end{aligned}
$$

and $Q_r(x,t)$ is the parabolic cylinder centered at (x,t) given by

$$Q_r(x,t) := B_r(x)\times(t-r^2,t).$$

When $(x_0,t_0)=(0,0)$, we abbreviate $A(0,0,r)$ to $A(r)$. This convention also applies to other quantities and parabolic cylinders. For technical reasons, we also need another quantity L' which is not scale-invariant.

$$L'(x_0,t_0,r) = r^{-5/2}\iint_{Q_r(x_0,t_0)} |p-\tilde{p}_{r,x_0}|^{3/2} dxdt = r^{1/2}L(x_0,t_0,r)$$

Note that already in the work [5] of Caffarelli, Kohn and Nirenberg, a quantity similar to L' that is not scale-invariant plays an important role.

A crucial component of proving partial regularity in space dimension 4 is interpolation inequalities. Next we introduce three interpolation inequalities based on the above dimensionless quantities.

Lemma 2.2.2. *Suppose that* (u, p, λ, ω) *is a weak solution set of the Navier-Stokes equations* (1.1.3) *in space dimension* 4 *in* $Q_r(x_0, t_0)$. *Then there exists an absolute constant* $C_1 > 0$, *which is independent of* $(x_0, t_0) \in \mathbb{R}^4 \times \mathbb{R}$ *and* $r > 0$, *such that*

$$\begin{aligned} G(x_0, t_0, r) &\le C_1 A^{3/2}(x_0, t_0, r) + C_1 \delta^{3/2}(x_0, t_0, r), \\ G_c(x_0, t_0, r) &\le C_1 A^{1/2}(x_0, t_0, r) \delta_c(x_0, t_0, r), \\ H(x_0, t_0, r) &\le C_1 A^{1/2}(x_0, t_0, r) \delta(x_0, t_0, r). \end{aligned}$$

Proof. Since all quantities here are scale-invariant, it suffices to prove these inequalities for $r = 1$. By Lebesgue interpolation inequality,

$$\begin{aligned} \|u\|_{L^3(B_1(x_0))} &\le \|u\|_{L^4(B_1(x_0))}^{2/3} \|u\|_{L^2(B_1(x_0))}^{1/3}, \\ \|u - \tilde{u}_{1,x_0}\|_{L^3(B_1(x_0))} &\le \|u - \tilde{u}_{1,x_0}\|_{L^4(B_1(x_0))}^{2/3} \|u - \tilde{u}_{1,x_0}\|_{L^2(B_1(x_0))}^{1/3} \\ &\le \|u - \tilde{u}_{1,x_0}\|_{L^4(B_1(x_0))}^{2/3} \|u\|_{L^2(B_1(x_0))}^{1/3}. \end{aligned}$$

By Sobolev embedding and Sobolev-Poincaré inequality,

$$\begin{aligned} \|u\|_{L^4(B_1(x_0))} &\lesssim \|u\|_{L^2(B_1(x_0))} + \|\nabla u\|_{L^2(B_1(x_0))} \\ \|u - \tilde{u}_{1,x_0}\|_{L^4(B_1(x_0))} &\lesssim \|\nabla u\|_{L^2(B_1(x_0))}. \end{aligned}$$

Then we integrate in time and use Young's inequality,

$$\begin{aligned} \iint_{Q_1(x_0,t_0)} |u|^3 dxdt &\lesssim A^{3/2}(x_0, t_0, 1) + A^{1/2}(x_0, t_0, 1) \delta(x_0, t_0, 1) \\ &\lesssim A^{3/2}(x_0, t_0, 1) + \delta^{3/2}(x_0, t_0, 1). \end{aligned}$$

In the first inequality, we use lower semi-continuity of the weak-$*$ convergence to bound $\|u\|_{L_t^\infty L_x^2}$ with $\limsup \|u_k\|_{L_t^\infty L_x^2}$. Similarly, we also have

$$\iint_{Q_1(x_0,t_0)} |u - \tilde{u}_{r,x_0}|^3 dxdt \lesssim A^{1/2}(x_0,t_0,1)\delta(x_0,t_0,1).$$

The second interpolation inequality follows directly from Lemma 2.1.8.

As we mentioned, these quantities are scale-invariant. Then we can obtain the inequalities for $r \neq 1$ via scaling. □

The second key ingredient is the local energy inequalities (2.1.19) and (2.1.20). To use these local energy inequalities, we also need different estimates for the pressure term. We prove a 4-dimensional analogue of Lemma 3.2 in [5].

Lemma 2.2.3. *Suppose that* (u, p, λ, ω) *is a weak solution set of the Navier-Stokes equations* (1.1.3) *in space dimension* 4 *in* $Q_\rho(x_0,t_0)$. *Then there exists an absolute constant* $C_2 > 0$, *which is independent of* $(x_0,t_0) \in \mathbb{R}^4 \times \mathbb{R}$ *and* $\rho > 0$, *such that*

$$\begin{aligned} L'(x_0,t_0,r) \leq & C_2 r^{-5/2} \iint_{Q_{2r}(x_0,t_0)} |u|^3 dxdt \\ & + C_2 r^5 \Big(\sup_{t_0-r^2<t<t_0} \int_{2r<|y-x_0|<\rho} \frac{|u|^2}{|y-x_0|^5} dy \Big)^{3/2} \\ & + C_2 \frac{r^3}{\rho^{11/2}} \iint_{Q_\rho(x_0,t_0)} \big(|u|^3 + |p|^{3/2}\big) dxdt, \end{aligned} \tag{2.2.2}$$

where $0 < r \leq \frac{\rho}{2}$.

Proof of Lemma 2.2.3. It suffices to prove the estimate when $(x_0,t_0) = (0,0)$. Choose a cutoff function $\psi \in C_c^\infty(\mathbb{R}^4)$ such that $0 \leq \psi \leq 1$ and

$$\psi \equiv 1 \text{ in } B_{3\rho/4}, \quad \psi \equiv 0 \text{ in } \mathbb{R}^4 \backslash B_\rho, \quad |\nabla \psi| \lesssim \rho^{-1}, \quad |\nabla^2 \psi| \lesssim \rho^{-2}.$$

Then we localize the pressure equation and integrate by parts to move the differentiation from u and p to ψ,

$$\begin{aligned}
p(x,t)\psi(x) =&(-\Delta)^{-1}(-\Delta)\big(p(x,t)\psi(x)\big)\\
=&\frac{1}{4\pi^2}\int_{\mathbb{R}^4}\frac{1}{|x-y|^2}\big(\psi\partial_i\partial_j(u_iu_j)-2\nabla\psi\cdot\nabla p-p\Delta\psi\big)dy\\
=&\frac{1}{4\pi^2}\int_{\mathbb{R}^4}u_iu_j\psi\partial_i\partial_j\Big(\frac{1}{|x-y|^2}\Big)dy\\
+&\frac{1}{4\pi^2}\int_{\mathbb{R}^4}u_iu_j\Big(\frac{\partial_i\partial_j\psi}{|x-y|^2}+\partial_j\psi\frac{4(x_i-y_i)}{|x-y|^4}\Big)dy\\
+&\frac{1}{4\pi^2}\int_{\mathbb{R}^4}p\Big(\frac{\Delta\psi}{|x-y|^2}+\frac{4(x-y)\cdot\nabla\psi}{|x-y|^4}\Big)dy\\
=&p_1(x,t)+p_2(x,t)+p_3(x,t)+p_4(x,t),
\end{aligned}\tag{2.2.3}$$

where by the fact that ψ is supported in B_ρ,

$$\begin{aligned}
p_1(x,t)&=\frac{1}{4\pi^2}\int_{B_{2r}}u_iu_j\psi\partial_i\partial_j\Big(\frac{1}{|x-y|^2}\Big)dy,\\
p_2(x,t)&=\frac{1}{4\pi^2}\int_{B_\rho\setminus B_{2r}}u_iu_j\psi\partial_i\partial_j\Big(\frac{1}{|x-y|^2}\Big)dy,\\
p_3(x,t)&=\frac{1}{4\pi^2}\int_{B_\rho}u_iu_j\Big(\frac{\partial_i\partial_j\psi}{|x-y|^2}+\partial_j\psi\frac{4(x_i-y_i)}{|x-y|^4}\Big)dy,\\
p_4(x,t)&=\frac{1}{4\pi^2}\int_{B_\rho}p\Big(\frac{\Delta\psi}{|x-y|^2}+\frac{4(x-y)\cdot\nabla\psi}{|x-y|^4}\Big)dy.
\end{aligned}\tag{2.2.4}$$

Now, we decompose $L'(x_0,t_0,r)$ into four terms involving p_1,p_2,p_3 and p_4 respectively and estimate them separately,

$$L'(x_0,t_0,r)\le\sum_{l=1}^{4}r^{-5/2}\iint_{Q_r}|p_l-\tilde{p}_{l,r}|^{3/2}dxdt.\tag{2.2.5}$$

We interpret p_1 as $p_1=T_{ij}\big(u_iu_j\psi\big)$, where singular integral operators $\{T_{ij}\}_{1\le i,j\le 4}$ are given by

$$T_{ij}\zeta = \Big(\partial_i\partial_j\Big(\frac{1}{|x|^2}\Big)\Big) * \zeta. \tag{2.2.6}$$

From Calderón-Zygmund theory we know $\{T_{ij}\}_{1\le i,j\le 4}$ are bounded linear operators from $L^q(\mathbb{R}^4)$ to $L^q(\mathbb{R}^4)$ for any $1 < q < \infty$, hence let

$$\zeta(y,t) = u_i(y,t)u_j(y,t)\psi(y)\mathbf{1}_{\{y\in B_{2r}\}}$$

and it yields

$$\int_{B_r} |p_1|^{3/2}dx \lesssim \int_{B_{2r}} |u|^3 dx.$$

By a simple computation and integrating in time, we have

$$\iint_{Q_r} |p_1 - \tilde{p}_{1,r}|^{3/2}dxdt \le C_2 \iint_{Q_{2r}} |u|^3 dxdt. \tag{2.2.7}$$

We estimate the remaining terms by bounding the L^∞-norm of the space derivatives of the pressure p.

For p_2, we can control its derivative as follows. When $(x,t)\in Q_r$,

$$|\nabla p_2(x,t)| \lesssim \int_{2r<|y|<\rho} \frac{\psi|u|^2}{|x-y|^5}dx \lesssim \int_{2r<|y|<\rho} \frac{|u|^2}{|y|^5}dx.$$

The second inequality follows from $2|x-y| > |y|$ when $x\in B_r, y\in B_{2r}^c$. Then we can estimate the second term in (2.2.5) by mean value theorem as follows,

$$\begin{aligned}\iint_{Q_r} |p_2 - \tilde{p}_{2,r}|^{3/2}dxdt &\le \frac{\pi^2}{2}r^4\int_{-r^2}^0 \|p_2 - \tilde{p}_{2,r}\|_{L^\infty(B_r)}^{3/2}dt \\ &\lesssim \frac{\pi^2}{2}r^{11/2}\int_{-r^2}^0 \|\nabla p_2\|_{L^\infty(B_r)}^{3/2}dt \\ &\lesssim r^{15/2}\Big(\sup_{-r^2<t<0}\int_{2r<|y|<\rho}\frac{|u|^2}{|y|^5}dx\Big)^{3/2}.\end{aligned} \tag{2.2.8}$$

Similarly, for p_3 and p_4, note that $\nabla\psi = 0$ and $\nabla^2\psi = 0$ in $B_{3\rho/4}$. Moreover, when $x \in B_r$ and $y \in B_\rho \backslash B_{3\rho/4}$, $|x-y| > \frac{\rho}{4}$. Hence for $(x,t) \in Q_r$,

$$|\nabla p_3(x,t)| \lesssim \int_{B_\rho \backslash B_{3\rho/4}} |u|^2 \Big(\frac{|\nabla^2\psi|}{|x-y|^3} + \frac{|\nabla\psi|}{|x-y|^4} \Big) dy \lesssim \rho^{-5} \int_{B_\rho \backslash B_{3\rho/4}} |u|^2 dy,$$
$$|\nabla p_4(x,t)| \lesssim \int_{B_\rho \backslash B_{3\rho/4}} |p| \Big(\frac{|\nabla^2\psi|}{|x-y|^3} + \frac{|\nabla\psi|}{|x-y|^4} \Big) dy \lesssim \rho^{-5} \int_{B_\rho \backslash B_{3\rho/4}} |p| dy. \tag{2.2.9}$$

Thus,

$$\begin{aligned}\sum_{l=3}^{4} \iint_{Q_r} |p_l - \tilde{p}_{l,r}|^{3/2} dx dt &\leq \sum_{l=3}^{4} \frac{\pi^2}{2} r^4 \int_{-r^2}^{0} \|p_l - \tilde{p}_{l,r}\|^{3/2}_{L^\infty(B_r)} dt \\ &\lesssim \sum_{l=3}^{4} \frac{\pi^2}{2} r^{11/2} \int_{-r^2}^{0} \|\nabla p_l\|^{3/2}_{L^\infty(B_r)} dt \\ &\lesssim \Big(\frac{r}{\rho}\Big)^{11/2} \int_{Q_\rho} |u|^3 + |p|^{3/2} dx dt.\end{aligned} \tag{2.2.10}$$

The second inequality follows from mean value theorem and the last one follows from (2.2.9) and Hölder's inequality.

Finally, combining the estimates (2.2.7), (2.2.8) and (2.2.10) and dividing them by $r^{5/2}$ yield (2.2.2). □

To obtain partial regularity theory in dimension $n = 4$, we also need another estimate for the pressure p.

Lemma 2.2.4. *Suppose that (u, p, λ, ω) is a weak solution set of the Navier-Stokes equations (1.1.3) in space dimension 4 in $Q_r(x_0, t_0)$. Then there exists an absolute constant $C_3 > 0$, which is independent of*

$(x_0,t_0)\in\mathbb{R}^4\times\mathbb{R}$ and $r>0$, such that

$$K(x_0,t_0,\theta r)\le C_1C_3\theta^{-3}A^{1/2}(x_0,t_0,r)\delta(x_0,t_0,r)+C_3\theta K(x_0,t_0,r)$$

for any $\theta\in(0,\frac{1}{2}]$. The constant $C_1>0$ is absolute and comes from Lemma 2.2.2.

Proof. Again it suffices to prove the estimate for $(x_0,t_0)=(0,0)$. Choose a cutoff function $\psi\in C_c^\infty(\mathbb{R}^4)$ such that $0\le\psi\le 1$ and

$$\psi\equiv 1 \text{ in } B_{3r/4},\quad \psi\equiv 0 \text{ in } \mathbb{R}^4\backslash B_r,\quad |\nabla\psi|\lesssim r^{-1},\quad |\nabla^2\psi|\lesssim r^{-2}.$$

The pressure equation can be written as

$$-\Delta p=\partial_i\partial_j\big[(u_i-\tilde u_{i,r})(u_j-\tilde u_{j,r})\big].$$

We can localize this equation like (2.2.3) and (2.2.4) to obtain

$$\begin{aligned}
&p(x,t)\psi(x)=p_1(x,t)+p_2(x,t)+p_3(x,t),\\
&\quad p_1(x,t)=\frac{1}{4\pi^2}\int_{B_r}(u_i-\tilde u_{i,r})(u_j-\tilde u_{j,r})\psi\partial_i\partial_j\Big(\frac{1}{|x-y|^2}\Big)dy,\\
&\quad p_2(x,t)=\frac{1}{4\pi^2}\int_{B_r}(u_i-\tilde u_{i,r})(u_j-\tilde u_{j,r})\Big(\frac{\partial_i\partial_j\psi}{|x-y|^2}+\partial_j\psi\frac{4(x_i-y_i)}{|x-y|^4}\Big)dy,\\
&\quad p_3(x,t)=\frac{1}{4\pi^2}\int_{B_r}p\Big(\frac{\Delta\psi}{|x-y|^2}+\frac{4(x-y)\cdot\nabla\psi}{|x-y|^4}\Big)dy.
\end{aligned}$$

For p_1, Calderón-Zygmund theory yields

$$\int_{B_{\theta r}}|p_1|^{3/2}dx\lesssim\int_{B_r}|(u_i-\tilde u_{i,r})(u_j-\tilde u_{j,r})|^{3/2}dx\lesssim\int_{B_r}|u-\tilde u_r|^3dx.$$

Integrating in time gives

$$\iint_{Q_{\theta r}}|p_1|^{3/2}dxdt\lesssim\iint_{Q_r}|u-\tilde u_r|^3dxdt.$$

For p_2, note that $\nabla\psi$ is supported in $B_r \backslash B_{3r/4}$. Then for $x \in B_{\theta r}$, $|x - y| > \frac{r}{4}$ and the bounds of ψ give

$$|p_2| \lesssim r^{-4} \int_{B_r} |u - \tilde{u}_r|^2 dx.$$

Then integrate in $Q_{\theta r}$ to obtain

$$\iint_{Q_{\theta r}} |p_2|^{3/2} dxdt \lesssim \int_{-(\theta r)^2}^{0} (\theta r)^4 \|p_2\|_{L^\infty(B_{\theta r})}^{3/2} dt \lesssim \theta^4 \iint_{Q_r} |u - \tilde{u}_r|^3 dxdt.$$

For p_3, likewise, we have

$$\iint_{Q_{\theta r}} |p_3|^{3/2} dxdt \lesssim \theta^4 \iint_{Q_r} |p|^{3/2} dxdt.$$

Combining the estimates for p_1, p_2 and p_3 and applying the interpolation inequality Lemma 2.2.2, we have

$$K(\theta r) \lesssim C_1 \theta^{-3} A^{1/2}(r) \delta(r) + C_3 \theta K(r),$$

as claimed. □

Two types of cutoff functions are introduced in following lemmas, respectively for two local partial regularity results which we will show later. Similar cutoff functions have been used by Scheffer [44] and Caffarelli, Kohn, and Nirenberg [5].

Lemma 2.2.5. *Let $r_n = 2^{-n}$ and $Q_n = Q_{r_n}$. In dimension $n = 4$, $\{\phi_n\}_{n \in \mathbb{N}}$ is a sequence of localized solutions of backward heat equations given by*

$$\phi_n(x,t) = \chi(x,t)\Psi_n(x,t) \quad (x,t) \in \mathbb{R}^4 \times (-\infty, 0),$$

where $\{\Psi_n\}_{n\in\mathbb{N}}$ *are the solutions of backward heat equations given by*

$$\Psi_n(x,t) = \frac{1}{(r_n^2 - t)^2} \exp\Big(- \frac{|x|^2}{4(r_n^2 - t)}\Big)$$

and χ *is a cut-off function such that*

$$\chi \equiv 1 \text{ in } Q_{1/4}, \quad \chi \equiv 0 \text{ in } \mathbb{R}^4 \times (-\infty, 0)\backslash Q_{1/3},$$

then the following statements hold for any integer $n \in \mathbb{N}$*:*

(i) $\partial_t \phi_n + \Delta\phi_n = 0$ *in* $Q_{1/4}$*;*

(ii) $|\partial_t \phi_n + \Delta\phi_n| \le C_4$ *in* $\mathbb{R}^4 \times (-\infty, 0)$*;*

(iii) $C_4^{-1} r_n^{-4} \le \phi_n \le C_4 r_n^{-4}$ *and* $|\nabla \phi_n| \le C_4 r_n^{-5}$ *in* Q_n*;*

(iv) $\phi_n \le C_4 r_k^{-4}$ *and* $|\nabla\phi_n| \le C_4 r_k^{-5}$ *in* $Q_{k-1}\backslash Q_k$ *for any* $2 \le k \le n$*.*

Note that the constant $C_4 > 0$ *is absolute.*

Proof. The first statement is obvious. For the second we compute

$$\begin{aligned}\partial_t \phi_n + \Delta\phi_n &= \Psi_n(\partial_t\chi + \Delta\chi) + \chi(\partial_t \Psi_n + \Delta\Psi_n) + 2\nabla\Psi_n \cdot \nabla\chi \\ &= \Psi_n(\partial_t\chi + \Delta\chi) + 2\nabla\Psi_n \cdot \nabla\chi.\end{aligned}$$

Because any derivative of χ vanishes in $Q_{1/4}$ and $\Psi_n, \nabla\Psi_n$ are uniformly bounded in $(x,t) \in \mathbb{R}^4 \times (-\infty, 0)\backslash Q_{1/4}$, we can deduce that $|\partial_t\phi_n + \Delta\phi_n|$ is bounded uniformly in $(x,t) \in \mathbb{R}^4 \times (-\infty, 0)$ and in $n \in \mathbb{N}$.

For the third, if $(x,t) \in Q_n$, then $r_n^2 \le r_n^2 - t \le 2r_n^2$ and $|x|^2 \le r_n^2$. We compute

$$\nabla\phi_n(x,t) = \Big(\frac{\nabla\chi(x,t)}{(r_n^2 - t)^2} - \frac{x\chi(x,t)}{2(r_n^2 - t)^3}\Big) \exp\Big(- \frac{|x|^2}{4(r_n^2 - t)}\Big).$$

The terms $\chi, \nabla\chi$ and $\exp\Big(-\frac{|x|^2}{4(r_n^2-t)}\Big)$ are bounded from above and from below uniformly in $(x,t) \in Q_n$ and in $n \in \mathbb{N}$. Then the third statement follows from

$$\frac{1}{(r_n^2-t)^2} \leq r_n^{-4} \qquad \frac{|x|}{(r_n^2-t)^3} \leq r_n^{-5}.$$

For the fourth, if $(x,t) \in Q_{k-1} \backslash Q_k$ and $t \leq -r_k^2$, we have $|x|^2 \leq r_{k-1}^2$ and $r_n^2 - t \geq r_n^2 + r_k^2$, then this statement follows from the argument for the third one. If $(x,t) \in Q_{k-1} \backslash Q_k$ and $t > -r_k^2$, then $r_k^2 \leq |x|^2 \leq r_{k-1}^2$ and $r_n^2 \leq r_n^2 - t \leq r_n^2 + r_k^2$, thus

$$\phi_n(x,t) \leq \frac{\chi}{(r_n^2-t)^2} \exp\Big(-\frac{r_k^2}{4(r_n^2-t)}\Big) \leq \chi r_k^{-4} \alpha^4 e^{-\alpha^2/4},$$

where $\alpha = r_k(r_n^2-t)^{-1/2}$ and the function $\alpha^4 e^{-\alpha^2/4}$ is uniformly bounded. The bound for $\nabla\phi_n$ follows similarly. $\square$

Lemma 2.2.6. *In dimension $n = 4$, fix $r > 0$. For any $0 < \theta \leq \frac{1}{2}$ we define*

$$\phi_\theta(x,t) = \frac{1}{[(\theta r)^2 - t]^2} \exp\Big(-\frac{|x|^2}{4[(\theta r)^2 - t]}\Big)\chi\Big(\frac{x}{r}, \frac{t}{r^2}\Big) \quad (x,t) \in \mathbb{R}^4 \times (-\infty, 0),$$

where $\chi \in C_c^\infty(B_1 \times (-1,1))$ is a cutoff function such that $\chi \equiv 1$ in $B_{1/2} \times (-\frac{1}{4}, \frac{1}{4})$. Then there exists an absolute constant $C_5 > 0$ such that

(i) $C_5^{-1}(\theta r)^{-4} \leq \phi_\theta \leq C_5(\theta r)^{-4}$ in $Q_{\theta r}$;

(ii) In Q_r, we have following bounds,

$$\begin{aligned} \phi_\theta &\leq C_5(\theta r)^{-4}, \\ |\nabla\phi_\theta| &\leq C_5(\theta r)^{-5}, \\ |\partial_t\phi_\theta + \Delta\phi_\theta| &\leq C_5 r^{-6}. \end{aligned}$$

Proof. This proof is analogue to the proof of Lemma 2.2.5. □

These estimates will be fundamental for the local partial regularity results of the Navier-Stokes equations in space dimension 4. They involve some constants C_1, C_2, C_3, C_4 and C_5. All of these constants are absolute.

2.2.2 Partial regularity results. The first partial regularity result states that u is locally bounded if u, p, f and concentration measure ω satisfy a local smallness condition. In dimension $n = 4$, we prove the following result.

Proposition 2.2.7. *There exist an absolute constant $\varepsilon > 0$ and, for any fixed $q > 3$, constants $\kappa = \kappa(\varepsilon, q)$ and $C = C(\varepsilon, q)$ depending on ε and q with the following property. If a weak solution set (u, p, λ, ω) of the Navier-Stokes equations* (1.1.3) *in $Q_1(0,0)$ in dimension $n = 4$ satisfies*

$$\begin{aligned} \iint_{Q_1} \left(|u|^3 + |p|^{3/2}\right) dxdt + \iint_{Q_1} d\omega \leq \varepsilon \\ \iint_{Q_1} |f|^q dxdt \leq \kappa, \end{aligned} \tag{2.2.11}$$

then $\|u\|_{L^\infty(Q_{1/2}(0,0))} < C$.

Its analogue in dimension $n = 3$ does not involve concentration measures and it was proved by Caffarelli, Kohn and Nirenberg [5].

Proposition 2.2.8 (cf. [5], Proposition 1)**.** *There exist an absolute constant $\varepsilon > 0$ and, for any fixed $q > \frac{5}{2}$, constants $\kappa = \kappa(\varepsilon, q)$ and $C = C(\varepsilon, q)$ depending on ε and q with the following property. If a suitable weak solution (u, p) of the Navier-Stokes equations* (1.1.3) *in*

$Q_1(0,0)$ *in dimension* $n=3$ *satisfies*

$$\begin{aligned} \iint_{Q_1} \left(|u|^3 + |p|^{3/2}\right) dxdt \le \varepsilon \\ \iint_{Q_1} |f|^q dxdt \le \kappa, \end{aligned} \tag{2.2.12}$$

then $\|u\|_{L^\infty(Q_{1/2}(0,0))} < C$.

Now we prove Proposition 2.2.7.

Proof of Proposition 2.2.7. Let $r_n = 2^{-n}$ and $Q_n = Q_{r_n}$, $n \ge 2$. The strategy is to iteratively prove the following estimates

$$G(x_0,t_0,r_n) + G_c(x_0,t_0,r_n) + L'(x_0,t_0,r_n) \le \varepsilon^{2/3} r_n^3 \tag{2.2.13}$$

$$A(x_0,t_0,r_n) + \delta(x_0,t_0,r_n) + \delta_c(x_0,t_0,r_n) \le C_B \varepsilon^{2/3} r_n^2 \tag{2.2.14}$$

for all $n \in \mathbb{N}$ and any $(x_0,t_0) \in Q_{1/2}(0,0)$. Without loss of generality, we set $(x_0,t_0) = (0,0)$ and omit these two entries for simplicity. We use $\sum_{k=1}^n A(r_k)$, $\sum_{k=1}^n \delta(r_k)$ and $\sum_{k=1}^n \delta_c(r_k)$ to control $G(r_{n+1})$, $G_c(r_{n+1})$ and $L'(r_{n+1})$ by means of the interpolation inequalities in Lemma 2.2.2 and the estimate for the pressure p in Lemma 2.2.3. Conversely, we bound $A(r_{n+1})$, $\delta(r_{n+1})$ and $\delta_c(r_{n+1})$ through $\sum_{k=1}^n G(r_k)$, $\sum_{k=1}^n G_c(r_k)$ and $\sum_{k=1}^n L'(r_k)$, by means of the local energy inequality (2.1.19).

In the rest of this proof, we use $(2.2.13)_k$ to denote the inequality (2.2.13) with index $k \in \mathbb{N}$. This notation also applies to (2.2.14).

Claim 1: *The inequality* $(2.2.13)_1$ *holds.*

Proof of Claim 1. Hölder's inequality gives

$$\begin{aligned} &G(r_1) + G_c(r_1) + L(r_1) \\ &\quad \le \iint_{Q_{1/2}(x_0,t_0)} \left(8|u|^3 + 16|p|^{3/2}\right) dxdt + \iint_{Q_{1/2}(x_0,t_0)} d\omega \end{aligned}$$

Then we impose the first condition on $\varepsilon > 0$,

$$\varepsilon \leq 2^{-21} \tag{2.2.15}$$

Now we can invoke initial smallness condition and it yields

$$G(r_1) + G_c(r_1) + L(r_1) \leq 16\varepsilon \leq \varepsilon^{2/3} r_1^3.$$

□

Claim 2: $\{(2.2.13)_k\}_{1\leq k\leq n}$ *implies* $(2.2.14)_{n+1}$.

Proof of Claim 2. Let ϕ_n be the localized solution of the backward heat equation

$$\phi_n(x,t) = \frac{\chi(x,t)}{(r_n^2 - t)^2} \exp\Big(- \frac{|x|^2}{4(r_n^2 - t)}\Big)$$

with χ as given in Lemma 2.2.5. Define smooth cutoff functions $\{\eta_k\}_{k\in\mathbb{N}}$ such that

$$\eta_k \equiv 1 \text{ in } Q_{7r_k/8}, \quad \eta_k \equiv 0 \text{ in } \mathbb{R}^4 \times (-\infty, 0)\backslash Q_k, \quad |\nabla \eta_k| \leq C' r_k^{-1}.$$

Then define $\varphi_k := \phi_n(\eta_k - \eta_{k+1})$ for $1 \leq k \leq n-1$ and $\varphi_n := \phi_n \eta_n$. It is easy to check the bound

$$|\nabla \varphi_k| = |\phi_n \nabla \eta_k + \eta_k \nabla \phi_n| \leq C_4 C' r_k^{-5} \quad \text{for any } k \leq n \tag{2.2.16}$$

and the fact $\phi_n = \sum_{k=1}^n \varphi_k$.

We use ϕ_n as the cutoff function in the local energy inequality (2.1.19) and choose the functions $\gamma_k = \tilde{p}_{r_k}$. This yields

$$\limsup_{k\to\infty} \sup_t \int_{B_{1/2}} |u_k|^2 \phi_n dx + \iint_{Q_{1/2}} \phi_n \big(|\nabla u|^2 dxdt + d\lambda\big)$$
$$\leq I_1 + 3I_2 + 2I_3 + I_4,$$

where

$$I_1 = \iint_{Q_{1/2}} |u|^2 |\partial_t \phi_n + \Delta \phi_n| dxdt,$$

$$I_2 = \sum_{k=1}^{n} \iint_{Q_{1/2}} |\nabla \varphi_k| \big(|u|^3 dxdt + d\omega \big),$$

$$I_3 = \sum_{k=1}^{n} \iint_{Q_{1/2}} |\nabla \varphi_k| |p - \tilde{p}_{r_k}|^{3/2} dxdt,$$

$$I_4 = \iint_{Q_{1/2}} |u||f||\phi_n| dxdt.$$

With the bounds in Lemma 2.2.5, we can deduce

$$C_4^{-1} r_{n+1}^{-2} \big(A(r_{n+1}) + \delta(r_{n+1}) + \delta_c(r_{n+1}) \big) \le I_1 + 3I_2 + 2I_3 + I_4 \quad (2.2.17)$$

For I_1, we use the bounds in Lemma 2.2.5, Hölder's inequality and the initial smallness condition (2.2.12),

$$I_1 \le C_4 \iint_{Q_{1/2}} |u|^2 dxdt \le C_4 \Big(\iint_{Q_1} |u|^3 dxdt \Big)^{2/3} \le C_4 \varepsilon^{2/3}.$$

For I_2, we need to decompose the integral over $Q_{1/2}$ into integrals over parabolic rings. Then for each subintegral we use the bounds in Lemma 2.2.5 and our induction hypothesis $\{(2.2.13)_k\}_{1 \le k \le n}$ to obtain

$$\begin{aligned} I_2 =& \sum_{k=2}^{n} \iint_{Q_{k-1} \setminus Q_k} |\nabla \varphi_k| \big(|u|^3 dxdt + d\omega \big) + \iint_{Q_n} |\nabla \varphi_k| \big(|u|^3 dxdt + d\omega \big) \\ \le& C_4 C' \Big(\sum_{k=2}^{n} r_k^{-5} \iint_{Q_{k-1}} \big(|u|^3 dxdt + d\omega \big) + r_n^{-5} \iint_{Q_n} \big(|u|^3 dxdt + d\omega \big) \Big) \\ \le& C_4 C' 2^5 \sum_{k=1}^{n} r_k \varepsilon^{2/3}. \end{aligned}$$

Similarly, by doing the decomposition and using the bounds in Lemma 2.2.5, Hölder's inequality, the initial smallness condition (2.2.12) and the induction hypothesis $\{(2.2.13)_k\}_{1\le k\le n}$, we have

$$\begin{aligned} I_4 \le & \sum_{k=2}^{n} \iint_{Q_{k-1}\setminus Q_k} |u||f||\varphi_k| dxdt + \iint_{Q_n} |u||f||\varphi_k| dxdt \\ \le & C_4 C' \sum_{k=1}^{n} r_k^{-4} \|u\|_{L^3(Q_k)} \|f\|_{L^q(Q_k)} \|1\|_{L^{\frac{3q}{2q-3}}(Q_k)} dxdt \\ \le & C_4 C' \sum_{k=1}^{n} r_k^{2-6/q} \varepsilon^{2/9} \kappa^{1/q}, \end{aligned}$$

and

$$\begin{aligned} I_3 \le & C_4 C' \sum_{k=2}^{n} r_k^{-5} \iint_{Q_{k-1}\setminus Q_k} |p - \tilde{p}_{r_k}|^{3/2} dxdt \\ & + C_4 C' r_n^{-5} \iint_{Q_n} |p - \tilde{p}_{r_n}|^{3/2} dxdt \\ \le & C_4 C' \sum_{k=1}^{n} r_k^{1/2} \varepsilon^{2/3}. \end{aligned}$$

Now we can combine the estimates for I_1, I_2, I_3, I_4 and the inequality (2.2.17) to deduce

$$A(r_{n+1}) + \delta(r_{n+1}) + \delta_c(r_{n+1}) \le C_B \varepsilon^{2/3} r_{n+1}^2.$$

with constants $\kappa = \varepsilon^{4q/9}$ and C_B to be

$$C_B = C_4^2 \Big(1 + 96 C' \sum_{k=1}^{n} r_k + 2C' \sum_{k=1}^{n} r_k^{1/2} + C' \sum_{k=1}^{n} r_k^{2-6/q}\Big).$$

Here, C' and C_4 are absolute, C_B only depends on $q > 3$. This concludes the proof of $(2.2.14)_{n+1}$. □

Claim 3: $\{(2.2.14)_k\}_{2\le k\le n}$ *implies* $(2.2.13)_n$.

Proof of Claim 3. For simplicity, let $(x_0,t_0)=(0,0)$. The interpolation inequality Lemma 2.2.2 yields for any $2\le k\le n$,

$$G(r_k)+G_c(r_k)\le C_1A^{3/2}(r_k)+C_1\delta^{3/2}(r_k)+C_1\delta_c^{3/2}(r_k)\le C_1C_B\varepsilon r_k^3. \tag{2.2.18}$$

Let $\rho=r_1, r=r_n$, Lemma 2.2.3 yields

$$\begin{aligned}
L'(r_n)\le & C_2r_n^{-5/2}\iint_{Q_{n-1}}|u|^3dxdt+C_2r_n^5\Big(\sup_{-r_n^2<t<0}\int_{2r_n<|y|<r_1}\frac{|u|^2}{|y|^5}dy\Big)^{3/2}\\
&+C_2\frac{r_n^3}{r_1^{11/2}}\iint_{Q_{r_1}}\big(|u|^3+|p|^{3/2}\big)dxdt\\
\le & 8C_2r_n^{1/2}G(r_{n-1})+C_2r_n^5\Big(\sup_{-r_n^2<t<0}\int_{2r_n<|y|<r_1}\frac{|u|^2}{|y|^5}dy\Big)^{3/2}+C_2\varepsilon r_n^3\\
\le & C_2\big(8C_1C_B^{3/2}r_n^{1/2}+1\big)\varepsilon r_n^3+C_2r_n^5\Big(\sup_{-r_n^2<t<0}\int_{2r_n<|y|<r_1}\frac{|u|^2}{|y|^5}dy\Big)^{3/2}.
\end{aligned} \tag{2.2.19}$$

In the first line of (2.2.19), for the first term, we use the interpolation inequality in Lemma 2.2.2 and our induction hypothesis $(2.2.14)_{n-1}$. For the third term, we use the initial smallness condition (2.2.12). For the second term, we decompose this integral into integrals over rings and estimate it using induction hypothesis $\{(2.2.14)_k\}_{2\le k\le n}$,

$$\begin{aligned}
\sup_{-r_n^2<t<0}\int_{2r_n<|y|<r_1}\frac{|u|^2}{|y|^5}dy &\le \sum_{k=2}^{n-1}\sup_{-r_{k-1}^2<t<0}\int_{r_k<|y|<r_{k-1}}\frac{|u|^2}{|y|^5}dy\\
&\le 4\sum_{k=2}^{n-1}r_k^{-3}A(r_{k-1})\\
&\le 16C_B\varepsilon^{2/3}r_n^{-1}.
\end{aligned} \tag{2.2.20}$$

In the second inequality we use $|y|^{-5} \leq r_k^{-5}$ when $r_k < |y| < r_{k-1}$. The third inequality follows from our induction hypothesis $\{(2.2.14)_k\}_{2\leq k\leq n}$.

Hence, from (2.2.18), (2.2.19) and (2.2.20), we can deduce that

$$G(r_n)+G_c(r_n)+L'(r_n) \leq \left[C_2(8C_1C_B^{3/2}r_n^{1/2}+1+64C_B^{3/2}r_n^{1/2})+C_1C_B\right]\varepsilon r_n^3.$$

Now we impose the second condition on $\varepsilon > 0$,

$$\left[C_2(8C_1C_B^{3/2}r_n^{1/2} + 1 + 64C_B^{3/2}r_n^{1/2}) + C_1C_B\right]\varepsilon^{1/3} < 1.$$

It yields

$$G(r_n) + G_c(r_n) + L'(r_n) \leq \varepsilon^{2/3}r_n^3.$$

Because C_1 and C_2 are absolute and C_B only depends on q, the choice of ε is uniform for any $(x_0, t_0) \in Q_{1/2}(0, 0)$. □

Now we can deduce that $(2.2.14)_k$ holds for any $k \geq 2$. This gives

$$\sup_{t_0-r_n^2<t<t_0} r_n^{-4}\int_{B_{r_n}(x_0)} |u|^2dx \leq r_n^{-2}A(x_0, t_0, r_n) \leq C_B\varepsilon^{2/3}$$

for any $(x_0, t_0) \in Q_{1/2}(0, 0)$ and $n \geq 2$. Hence

$$|u(x, t)| \leq C_B^{1/2}\varepsilon^{1/3},$$

given that $(x, t) \in Q_{1/2}(0, 0)$ is a Lebesgue point of u. □

Remark 2.2.9. Observe that the above proof uses the term L'. If instead of L', we were to carry out the estimate (2.2.19) with L, we would obtain $\varepsilon r_n^{5/2}$ when estimating the term in the second line of (2.2.19), which cannot be bounded by $\varepsilon^{2/3}r_n^3$ uniformly in $n \in \mathbb{N}$.

The second partial regularity result corresponds to a version of Proposition 2 in [5] in space dimension 4 with concentration measures. We use an idea from Lin's work [27] where he gave a simpler proof for the results in [5]. As a consequence, we are able improve Scheffer's result in [43], to show that the 2-dimensional parabolic Hausdorff measure of the singular set of u in space dimension 4 is finite.

Proposition 2.2.10. *There exists an absolute constant $\tau > 0$ with the following property. Suppose that (u, p, λ, ω) is a weak solution set of the Navier-Stokes equations* (1.1.3) *in some cylinder $Q_\rho(x_0, t_0)$ in dimension $n = 4$ with $f \in L^q_{loc}(Q_\rho(x_0, t_0))$ for some $q > 3$. If*

$$\limsup_{r\to 0} \frac{1}{r^2} \iint_{Q_r(x_0,t_0)} \left(|\nabla u|^2 dxdt + d\lambda\right) \le \tau,$$

then $\|u\|_{L^\infty(Q_{r_0}(x_0,t_0))} < Cr_0^{-1}$ for some $0 < r_0 < \rho$. Note that $C = C(\varepsilon, q)$ depends on ε and q in Proposition 2.2.7.

The analogue in dimension $n = 3$ is given below.

Proposition 2.2.11 (cf. [5], Proposition 2)**.** *There exists an absolute constant $\tau > 0$ with the following property. Suppose that (u, p) is a suitable weak solution of the Navier-Stokes equations* (1.1.3) *in some cylinder $Q_\rho(x_0, t_0)$ in dimension $n = 3$ with $f \in L^q_{loc}(Q_\rho(x_0, t_0))$ for some $q > \frac{5}{2}$. If*

$$\limsup_{r\to 0} \frac{1}{r^2} \iint_{Q_r(x_0,t_0)} |\nabla u|^2 dxdt \le \tau,$$

then $\|u\|_{L^\infty(Q_{r_0}(x_0,t_0))} < Cr_0^{-1}$ for some $0 < r_0 < \rho$. Note that $C = C(\varepsilon, q)$ depends on ε and q in Proposition 2.2.8.

Now we prove Proposition 2.2.10.

Proof of Proposition 2.2.10. Because $q > 3$, there holds $3q - 6 > 3$. Then there exists $r_1 > 0$, such that for any $0 < r \le r_1$,

$$F_1(r, x_0, t_0) = r^{3q-6} \iint_{Q_r(x_0,t_0)} |f|^q dxdt \le \kappa.$$

Claim 1: *For some* $0 < r_2 \le r_1$,

$$r_2^{-3} \iint_{Q_{r_2}(x_0,t_0)} \left(|u|^3 + |p|^{3/2}\right) dxdt + r_2^{-3} \iint_{Q_{r_2}} d\omega \le \varepsilon. \tag{2.2.21}$$

If this claim holds, Proposition 2.2.7 yields $\|u\|_{L^\infty(Q_{r_2/2}(x_0,t_0))} < C r_2^{-1}$ immediately.

Proof of Claim 1. We use the local energy inequality (2.1.20) to derive the smallness condition (2.2.21). Fix $r \in (0,\rho)$, $\theta \in (0,\frac{1}{2}]$ and $(x_0,t_0) = (0,0)$. We choose cutoff function ϕ_θ as stated in Lemma 2.2.6. Then from (2.1.20) we deduce

$$\frac{1}{C_5(\theta r)^2} \limsup_{k\to\infty} \sup_{-(\theta r)^2 < t < 0} \int_{B_{\theta r}} |u_k|^2 dx + \frac{1}{C_5(\theta r)^2} \iint_{Q_{\theta r}} \left(|\nabla u|^2 dxdt + d\lambda\right) \le I_1 + I_2 + 3I_2' + 2I_3 + I_4, \tag{2.2.22}$$

where

$$\begin{aligned}
I_1 &= (\theta r)^2 \iint_{Q_r} |u|^2 (\partial_t \phi_\theta + \Delta \phi_\theta) dxdt,\\
I_2 &= (\theta r)^2 \iint_{Q_r} |u|^2 u \cdot \nabla \phi_\theta dxdt,\\
I_2' &= (\theta r)^2 \iint_{Q_r} |\nabla \phi_\theta| \left(|u - \tilde{u}_r|^3 dxdt + d\omega\right),\\
I_3 &= (\theta r)^2 \iint_{Q_r} p u \cdot \nabla \phi_\theta dxdt,\\
I_4 &= (\theta r)^2 \iint_{Q_r} f \cdot u \phi_\theta dxdt.
\end{aligned}$$

For I_1, I_3 and I_4, we simply use Hölder's inequality and the bounds in Lemma 2.2.6. For I_2', we use the interpolation inequalities in Lemma 2.2.2.

Thus, we obtain

$$\begin{aligned} I_1 &\le C_5\theta^2 G^{2/3}(r),\\ I_3 &\le C_5\theta^{-3}K^{2/3}(r)G^{1/3}(r),\\ I_4 &\le C_5\theta^{-2}F_2^{1/2}(r)G^{1/3}(r),\\ I_2' &\le C_5\theta^{-3}[H(r)+G_c(r)]. \end{aligned} \tag{2.2.23}$$

For I_2, we reduce the estimates on u to estimates on $u-\tilde{u}_r$, then we use the interpolation inequality in Lemma 2.2.2. Note that $|u|^2 \in L_t^2 W_x^{1,1}(Q_r)$, because $u \in L_t^2 H_x^1(Q_r) \cap L_t^\infty L_x^2(Q_r)$. We have

$$\begin{aligned} I_2 =&(\theta r)^2 \iint_{Q_r} |u|^2(u-\tilde{u}_r)\cdot\nabla\phi_\theta dxdt + (\theta r)^2 \iint_{Q_r} |u|^2 \tilde{u}_r \cdot \nabla\phi_\theta dxdt\\ \le& C_5(\theta r)^{-3}\iint_{Q_r} |u-\tilde{u}_r|^3 dxdt - 2(\theta r)^2 \iint_{Q_r} u\cdot\big((\tilde{u}_r\cdot\nabla)u\big)\phi_\theta dxdt\\ &- 2(\theta r)^2 \iint_{Q_r} \big[\big((u-\tilde{u}_r)\cdot\nabla\big)u\big]\cdot\tilde{u}_r\phi_\theta dxdt, \end{aligned} \tag{2.2.24}$$

where we use the bounds in Lemma 2.2.6 and integration by parts. Notice that u and $u-\tilde{u}_r$ are divergence-free. Furthermore, we have

$$\begin{aligned} &\iint_{Q_r} \big[\big((u-\tilde{u}_r)\cdot\nabla\big)u\big]\cdot\tilde{u}_r\phi_\theta dxdt\\ \le& C_5(\theta r)^{-4}\int_{-r^2}^0 r^{-4}\int_{B_r}|u|dx\int_{B_r}|u-\tilde{u}_r||\nabla u|dxdt\\ \le& C_5\theta^{-4}r^{-5}\Big(\iint_{Q_r}|u|^3dxdt\Big)^{1/3}\Big(\iint_{Q_r}|\nabla u|^2dxdt\Big)^{1/2}\Big(\sup_{-r^2<t<0}\int_{B_r}|u|^2dx\Big)^{1/2}. \end{aligned} \tag{2.2.25}$$

The first inequality follows from the bounds in Lemma 2.2.6. The remaining inequalities follow from Hölder's inequality. Analogously,

$$\begin{aligned}&\iint_{Q_r} u\cdot\big((\tilde{u}_r\cdot\nabla)u\big)\phi_\theta dxdt\\ \le& C_5\theta^{-4}r^{-5}\Big(\iint_{Q_r}|u|^3dxdt\Big)^{1/3}\Big(\iint_{Q_r}|\nabla u|^2dxdt\Big)^{1/2}\Big(\sup_{-r^2<t<0}\int_{B_r}|u|^2dx\Big)^{1/2}.\end{aligned} \tag{2.2.26}$$

Now, from (2.2.24), (2.2.25) and (2.2.26), we deduce

$$I_2 \le 2C_5\theta^{-2}G^{1/3}(r)A^{1/2}(r)\delta^{1/2}(r) + C_5\theta^{-3}H(r). \tag{2.2.27}$$

Now, we are in a position to plug (2.2.23) and (2.2.27) into (2.2.22) and to invoke the interpolation inequality in Lemma 2.2.2,

$$\begin{aligned}&A(\theta r)+2[\delta(\theta r)+\delta_c(\theta r)]\\ \lesssim& C_5^2\Big(\theta^2G^{2/3}(r)+\theta^{-3}K^{2/3}(r)G^{1/3}(r)\\ &+2\theta^{-2}F_2^{1/2}(r)G^{1/3}(r)+\theta^{-3}[H(r)+G_c(r)]\\ &+3\theta^{-2}G^{1/3}(r)A^{1/2}(r)\delta^{1/2}(r)\Big)\\ \le& C_5^2\Big(4\theta^2G^{2/3}(r)+\theta^{-8}K^{4/3}(r)+\theta^{-6}F_2(r)\\ &+C_1\theta^{-3}A^{1/2}(r)[\delta(r)+\delta_c(r)]+\frac{9}{4}\theta^{-6}A(r)\delta(r)\Big)\\ \le& C_5^2\Big[4C_1\theta^2A(r)+5C_1\theta^2[\delta(r)+\delta_c(r)]+C_1\theta^{-8}A(r)[\delta(r)+\delta_c(r)]\\ &+\frac{9}{4}\theta^{-6}A(r)\delta(r)+\theta^{-8}K^{4/3}(r)+\theta^{-6}F_2(r)\Big].\end{aligned} \tag{2.2.28}$$

In the second inequality, we use Young's inequality to move $\theta G^{1/3}(r)$ to the first term. In the third inequality, we use the interpolation inequality in Lemma 2.2.2. And Young's inequality moves $\theta^{-3}A^{1/2}(r)\delta^{1/2}(r)$ to the term $4\theta^{-6}A(r)\delta(r)$.

On the other hand, with Lemma 2.2.4 we deduce

$$\begin{aligned}K^{4/3}(\theta r) &\lesssim C_1^{4/3}C_3^{4/3}\theta^{-4}A^{2/3}(r)\delta^{4/3}(r) + C_3^{4/3}\theta^{4/3}K^{4/3}(r)\\ &\lesssim C_1^2C_3^2\theta^{-12}A(r)\delta(r) + \theta^{12}\delta^2(r) + C_3^{4/3}\theta^{4/3}K^{4/3}(r).\end{aligned} \tag{2.2.29}$$

Taking the sum of (2.2.28) and $\theta^{-9} \times$ (2.2.29) gives

$$\begin{aligned}A(\theta r)+&\theta^{-9}K^{4/3}(\theta r) + 2[\delta(\theta r) + \delta_c(\theta r)]\\ &\lesssim 4C_1C_5^2\theta^2A(r) + 5C_1C_5^2\theta^2[\delta(r) + \delta_c(r)] + \theta^3\delta^2(r)\\ &\quad + \Big(\frac{9}{4}C_5^2 + C_1^2C_3^2\theta^{-15} + C_1C_5^2\theta^{-2}\Big)\theta^{-6}A(r)[\delta(r) + \delta_c(r)]\\ &\quad + (C_5^2\theta + C_3^{4/3}\theta^{4/3})\theta^{-9}K^{4/3}(r) + C_5^2\theta^{-6}F_2(r).\end{aligned} \tag{2.2.30}$$

Since C_1, C_3 and C_5 are absolute positive constants, we can fix $\theta \in (0, \frac{1}{2}]$ such that

$$\begin{aligned}4C_1C_5^2\theta^2 &\lesssim \frac{1}{4},\\ C_5^2\theta + C_3^{4/3}\theta^{4/3} &\lesssim \frac{1}{2},\\ \big(5C_1C_5^2 + 1\big)\theta^2 &\lesssim \frac{1}{8}.\end{aligned} \tag{2.2.31}$$

Then we choose $\tau \in (0,1)$ such that

$$\Big(\frac{9}{4}C_5^2 + C_1^2C_3^2\theta^{-10} + C_1C_5^2\theta^{-2}\Big)\theta^{-6} \cdot 2\tau \lesssim \frac{1}{4}. \tag{2.2.32}$$

Because $\limsup_{r\to 0} \delta(r) + \delta_c(r) \le \tau$ and $\lim_{r\to 0} F_2(r) = 0$, we can choose $r' > 0$ such that for any $0 < r \le r'$,

$$\delta(r) + \delta_c(r) \le 2\tau, \quad C_5^2\theta^{-6}F_2(r) \le \frac{\tau}{4}.$$

Let $E(r) = A(r) + \theta^{-9}K^{4/3}(r) + 2[\delta(r) + \delta_c(r)]$, then (2.2.30) yields for any $0 < r \leq r'$,

$$E(\theta r) \lesssim \frac{1}{2}E(r) + \frac{\tau}{2}. \tag{2.2.33}$$

Iterating this inequality yields for any $k \in \mathbb{N}$

$$A(\theta^k r') + \theta^{-9}K^{4/3}(\theta^k r') + 2[\delta(\theta^k r') + \delta_c(\theta^k r')] = E(\theta^k r') \lesssim \frac{1}{2^k}E(r') + \tau.$$

Then there exists some $r_2 > 0$ such that

$$A(r_2) + \delta(r_2) + \delta_c(r_2) + \theta^{-9}K^{4/3}(r_2) \lesssim 4\tau.$$

Again by the interpolation inequality in Lemma 2.2.2, we can bound $G(r_2) + G_c(r_2)$ with $A(r_2) + \delta(r_2) + \delta_c(r_2)$. Then we can impose another condition on τ to ensure (2.2.21). This additional condition on the choice of τ depends on ε, C_1 and θ, so it does not produce any circular reasoning. This concludes the proof. □

□

Now, we give definitions to singular set of weak solution set and parabolic Hausdorff measure of a space-time set.

Definition 2.2.12. Suppose that (u, p, λ, ω) is a weak solution set of the Navier-Stokes equation in $\mathbb{R}^4 \times [0, T]$. A point $(x, t) \in \mathbb{R}^4 \times (0, T]$ is called a regular point if there exists $r > 0$ such that $u \in L^\infty(Q_r(x, t))$. Otherwise, (x, t) is called a singular point. The singular set is the set of all singular points.

Definition 2.2.13. Given a set $D \subset \mathbb{R}^4 \times \mathbb{R}$, for a fixed positive real number s, s-dimensional parabolic Hausdorff measure is defined as

$$\mathcal{P}^s(D) = \lim_{\delta \to 0^+} \mathcal{P}^s_\delta(D),$$

where

$$\mathcal{P}^s_\delta(D) = \inf\Big\{\sum_{i=1}^{\infty} r_i^s \Big| D \subset \bigcup_{i\in\mathbb{N}^+} Q^*_{r_i}(x_0,t_0), 0<r_i<\delta, (x_0,t_0)\in\mathbb{R}^4\times\mathbb{R}\Big\}.$$

Here, $Q^*_r(x,t)$ is centered parabolic cylinder defined by

$$Q^*_r(x,t) := B_r(x)\times\Big(t-\frac{r^2}{2}, t+\frac{r^2}{2}\Big).$$

Theorem 1.2.5 follows from Lemma 2.1.3, Proposition 2.1.13, Proposition 2.2.10 and the following standard covering argument. Theorem 1.2.1 follows similarly from Lemma 2.1.3, Proposition 2.1.13 and Proposition 2.2.11.

Proof of Theorem 1.2.5. We assume S is bounded and $S\subset B_{\rho_0}\times[0,T]$ for some $\rho_0>0$. Let $D' := \overline{B_{\rho_0+1}(\mathbb{R}^4)}\times[0,T]$. Let V be a parabolic neighborhood (neighborhood given by parabolic cylinders) of S in D' and fix $\delta>0$. According to Proposition 2.2.10, for each $(x,t)\in S$, we choose $Q_r(x,t)\subset V$ with $r<\delta$ such that

$$r^{-2}\iint_{Q_r(x,t)}\big(|\nabla u|^2 dxdt + d\lambda\big) > \tau.$$

Because S is bounded, we can use Vitali Covering lemma to obtain a family of disjoint parabolic cylinders $\{Q_{r_i}(x_i,t_i)\}_{i\in\Lambda}$ such that

$$S\subset\bigcup_{i\in\Lambda} Q_{5r_i}(x_i,t_i).$$

Here Λ is a finite set. Then

$$\sum_{i\in\Lambda} r_i^2 \le \frac{1}{\tau}\sum_{i\in\Lambda}\iint_{Q_{r_i}(x_i,t_i)}\big(|\nabla u|^2dxdt+d\lambda\big) \le \frac{1}{\tau}\iint_V\big(|\nabla u|^2dxdt+d\lambda\big).$$

Since δ is arbitrary, we know the Lebesgue measure of S is zero and

$$\mathcal{P}^2(S) \leq \frac{5}{\tau} \iint_V \left(|\nabla u|^2 dxdt + d\lambda\right). \tag{2.2.34}$$

In case that S is unbounded, we look at $S \cap B_r \times [0, T]$ with $r \to \infty$, then $\mathcal{P}^2(S \cap B_r \times [0, T])$ is bounded uniformly in r, which concludes our proof. □

Remark 2.2.14. Dong, Gu [12] and Wang, Wu [53] proved that suitable weak solutions satisfy $\mathcal{P}^2(S) = 0$, but they were not able to show that such solutions exist. Here, we can only prove $\mathcal{P}^2(S) < \infty$, since the presence of the concentration measures leads to nontriviality of the Hausdorff measure of the singular set S in this covering argument.

2.3 A technical lemma

In this section, we prove that certain fractional power of any nonnegative smooth function is Lipschitz continuous, which is a direct consequence of the following lemma proved by Fefferman and Phong [15].

Lemma 2.3.1 (Fefferman and Phong [15]; Lemma 4, Guan [16]). *If $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^{3,1}$ nonnegative function, with $\|f\|_{C^4} \leq A$, then there is $N \in \mathbb{N}$ (only depends on n) and functions $g_1, g_2, \ldots, g_N \in C^{1,1}$, with $\|g_j\|_{C^2} \leq C$, such that*

$$f = \sum_{j=1}^{N} g_j^2,$$

where the constant C depends on n and A.

Corollary 2.3.2. *Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a $C^{3,1}$ nonnegative function, then $h := f^\alpha$ is Lipschitz continuous for any $\alpha \in [\frac{1}{2}, 1]$.*

Proof. This result follows from Lemma 2.3.1 and the following bound,

$$
\begin{aligned}
|h'| &= \frac{2\alpha \left| \sum_{j=1}^{N} g_j g_j' \right|}{\left(\sum_{i=1}^{N} g_i^2 \right)^{1-\alpha}} \\
&\leq \sum_{j=1}^{N} \frac{2\alpha \left| g_j^{2\alpha-1} g_j' \right|}{\left(\sum_{i=1}^{N} (g_i/g_j)^2 \right)^{1-\alpha}} \\
&\leq \sum_{j=1}^{N} 2\alpha \left| g_j^{2\alpha-1} g_j' \right|.
\end{aligned}
$$

□

Chapter 3

A toy model in dimension $n = 3$

This chapter studies how supercritical divergence-free drifts influence the regularity of the solutions to general parabolic and elliptic equations. Instead of the toy model (1.3.4), we consider the following closely related Cauchy problem with solenoidal drift $u : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}^n$

$$\begin{cases} \partial_t v - \Delta v + (u \cdot \nabla) v = 0, & \text{in } \mathbb{R}^n \times [0, \infty) \\ v(x, 0) = v_0(x) \end{cases}, \tag{3.0.1}$$

where $v : \mathbb{R}^n \times [0, \infty) \to \mathbb{R}$ is a scalar-valued function.

The Cauchy problem (3.0.1), the toy model (1.3.4) have the same scaling invariance as the Navier-Stokes equations. To properly classify the regularity requirement on u, note that: for any $\lambda > 0$, if v is a solution of (3.0.1) with drift u, then $v_\lambda(x, t) := \lambda v(\lambda x, \lambda^2 t)$ is a solution of (3.0.1) with drift $u_\lambda(x, t) := \lambda u(\lambda x, \lambda^2 t)$. This scaling invariance also leads to a natural classification of function spaces with respect to this equation: for a function space B, we say B is subcritical if $\|u_\lambda\|_B \to 0$ as $\lambda \to 0$. B is called critical, if $\|u_\lambda\|_B = \|u\|_B$ for any λ. B is called supercritical, if $\|u_\lambda\|_B \to \infty$ as $\lambda \to 0$.

Establishing Hölder continuity in its analougue of divergence form, i.e. $u = 0$, was the key to solve Hilbert 19th problem. This was down independently by E. De Giorgi [10] and J. Nash [37]. Their techniques were later simplified by Moser [32] and generalized to parabolic equations of non-divergence form by Krylov and Safonov [20], Aronson [2] and other mathematicians, and thus becomes a standard tool called De Giorgi-Nash-Moser iteration.

If we assume the drifts to be subcritical, for example, $u \in L^p(\mathbb{R}^n)$

with $p > n$, Aronson [2] in 1968 used De Giorgi-Nash-Moser iteration and showed that weak solutions of the parabolic equation (3.0.1) have many nice properties similar to parabolic equations in divergence form, including upper and lower Gaussian bounds for the fundamental solutions, Harnack's inequality and Hölder continuity of weak solutions. Fabes and Stroock [13] gave another proof of these properties for the equation of divergence form using De Giorgi-Nash-Moser iteration. We remark that one can follow this approach to prove that [P1], [P2] and [P3] also hold for equation (3.0.1) with critical drifts $u \in L_t^\infty L_x^n$. Before Aronson's work [2], these properties were proved for the parabolic equation of divergence form by Moser in [34], [35] and [36], for (3.0.1) with bounded drifts by Krylov and Safonov [20], or for (3.0.1) with Hölder continuous drifts by Aronson in [1], etc.

Theorem 3.0.1 (Aronson, [2]; Fabes and Stroock [13]). *Let $n \geq 2$. Given that $u \in L^p(\mathbb{R}^n)$ with $p \geq n$, the parabolic equation* (3.0.1) *have the following properties:*

(1) [P1] **Gaussian bounds** (cf. [2], page 613). *There exist $c_1, c_2, C > 0$ depending on u, such that the fundamental solution G of* (3.0.1) *satisfies*

$$C^{-1} g_1(x-y, t-s) \leq G(x,t;y,s) \leq C g_2(x-y, t-s) \tag{3.0.2}$$

for any $x, y \in \mathbb{R}^n$, $t, s \geq 0$ with $t > s$, where g_i is the fundamental solution of the heat equation $\partial_t v - c_i \Delta v = 0$ for $i = 1, 2$.

(2) [P2] **Harnack inequality** (cf. [2], Theorem H). *Let v be a nonnegative weak solution of the equation* (3.0.1) *in bounded space-time domain $Q \subset \mathbb{R}^n \times [0, \infty)$. Suppose $Q' \subset Q$ is convex and $d := \operatorname{dist}(Q', \partial Q) > 0$, then we have*

$$\sup_Q v \leq C \inf_Q v, \tag{3.0.3}$$

where $C > 0$ depends on d and $\|u\|_{L^p(\mathbb{R}^n)}$.

(3) [P3] **Hölder continuity** (cf. [2], Theorem C). *Let v be a bounded weak solution of the equation (3.0.1) in bounded space-time domain $\Omega \times [t_0, t_1] \subset \mathbb{R}^n \times [0, \infty)$ with $t_1 > t_0$, then $v \in C^\alpha(\Omega' \times [t_0 + t', t_1])$ for any Ω' with $\operatorname{dist}(\Omega, \Omega') > 0$, any $t' \in (0, t_1 - t_0)$ and some $\alpha > 0$. The C^α-norm and the Hölder exponent depend on Ω, Ω', $\frac{t'}{t_1 - t_0}$, $\|v\|_{L^2_{x,t}(\Omega \times [t_0, t_1])}$ and $\|u\|_{L^p(\mathbb{R}^n)}$.*[1]

In this chapter, we construct general supercritical drifts such that [P1], [P2] and [P3] fail, namely to prove the following theorem.

Theorem 3.0.2. *Let $n \geq 3$. For any $\lambda \in (0, n-1)$, there exists an time-independent divergence-free drift $u \in L^{n-\lambda}(B)$ with the following property, where $T \in (0, \infty)$ and B is the unit ball of $\mathbb{R}^n$. The parabolic equation (3.0.1) in $B \times [0, T]$ has a bounded weak solution which is not continuous at the origin at the finite time $T > 0$.*

Remark 3.0.3. This tells us that the regularity assumption on u in Theorem 3.0.1 is sharp. Our result Theorem 3.0.2 is sharp in dimension since it is false when $n = 2$. Indeed, Silvestre, Vicol and Zlatoš [48] proved that, for time-independent drift u, we have the following a-prior continuity estimate for v (cf. Theorem 1.4 in [48]), for any $r \in (0,1)$ and $t \in (0, T)$,

$$\sup_{x \in B_r(0)} |v(x,t) - v(0,t)| \leq \frac{C(1 + \|u\|_{L^1_{\text{loc}}})\|v_0\|_{C^2 \cap W^{4,1}}}{\sqrt{-\log r}} \Big(1 + \frac{1}{t}\Big).$$

Remark 3.0.4. Results analogues to Theorem 3.0.2 hold for many more general supercritical spaces. One can see this in Theorem 1.3.3 or by slightly modifying our construction of u.

[1] The dependence of the constant $C > 0$ is important, but this is not explicitly mentioned in [2]. For detailed estimates one can check the proofs of Theorem 2 and Theorem 3 in [3].

Remark 3.0.5. Since the loss of continuity is a local property, it is not necessary to specify initial and boundary conditions. As we shall see, the loss of continuity can be established for various smooth initial conditions and boundary data.

Following the approach in [13], one can prove [P1] $\rightarrow$ [P2] $\rightarrow$ [P3]. Therefore, a natural corollary of Theorem 3.0.2 is that the Gaussian bounds (3.0.2) and the Harnack inequality [P2] fail for the supercritical drifts that we construct. Consequently, Theorem 3.0.2 means that supercritical drifts can significantly change the fundamental structure of the semigroup generated by the parabolic equation (3.0.1), including the behavior of its fundamental solution.

The elliptic counterpart of (3.0.1) is given by

$$-\Delta v + (u \cdot \nabla) v = 0. \tag{3.0.4}$$

For any $\lambda > 0$, if v is a solution of (3.0.4) with drift u, $v_\lambda(x) := \lambda v(\lambda x)$ is also a solution of (3.0.4) with drift $u_\lambda(x) := \lambda u(\lambda x)$. De Giorgi's work [10] for second-order elliptic equations of divergence form was generalized to (3.0.4) by Stampacchia [50] in 1965 in the case of subcritical or critical drifts u.

Theorem 3.0.6 (Stampacchia, [50])**.** *Let $n \geq 2$. For any $\lambda \geq 0$, any bounded domain $\Omega \subset \mathbb{R}^n$ and $u \in L^{n+\lambda}(\Omega)$, any weak solution of the elliptic equation* (3.0.4) *in Ω is of $C^\alpha(\Omega')$ for any $\Omega' \subset \Omega$ and some $\alpha(\Omega, \Omega', u) > 0$. Moreover, Harnack's inequality holds for non-negative weak solutions.*

To be consistent with the parabolic case, we denote Harnack's inequality and Hölder continuity in the elliptic case by [E2] and [E3]. Independent of Stampacchia's work, similar problems have been studied by Ladyzhenskaya and Uraltseva [23], [24], Morrey [30], [31], Moser [32], [33] and Serrin [47]. Later, there have been many efforts trying to prove [E2] and [E3] for more general drifts, such as [9], [40] and

[38]. As far as we know, [E2] and [E3] have been only been proved in general in the case of subcritical or critical drifts. More recently, Seregin, Silvestre, Šverák and Zlatoš [46] proved, among other things, that $u \in \mathrm{BMO}^{-1}(\mathbb{R}^n)$ suffices to give [E2] and [E3]. They also conjectured on page 510 of [46] that the divergence-free condition on u is not sufficient to get a C^α-bound on the weak solutions of (3.0.4) under the assumptions $\|v\|_{L^\infty} < C$ and $\|u\|_{L^{n-\lambda}} < C$ with $\lambda > 0$.

The second contribution of this paper is to prove that the choice of $n + \lambda$ in Theorem 3.0.6 is sharp and thus to confirm the conjecture of Seregin, Silvestre, Šverák and Zlatoš [46]. We prove

Theorem 3.0.7. *Let $n \geq 3$. For any $\lambda \in (0, n-1)$, there exists $u \in L^{n-\lambda}(\mathbb{R}^n)$ satisfying $\operatorname{div} u = 0$, such that there is a bounded weak solution v of the elliptic equation (3.0.4) in the unit ball of $\mathbb{R}^n$ which is not continuous at the origin.*

Remark 3.0.8. This result Theorem 3.0.7 is also sharp in dimension since it is false when $n = 2$. Indeed, for the solution v of the elliptic equation (3.0.4) in the unit ball $B \subset \mathbb{R}^2$, Seregin, Silvestre, Šverák and Zlatoš [46] showed the following a prior continuity estimate for v (cf. Theorem 1.4 in [46]): for any $r \in (0, 1)$,

$$\sup_{x \in B_r(0)} |v(x) - v(0)| \leq \frac{C(1 + \|u\|_{L^1(B)})^{1/2}}{\sqrt{-\log r}} \|v\|_{L^\infty(B)}.$$

Hölder continuity can easily be derived from Harnack's inequality. Thus Theorem 3.0.7 immediately implies that the Harnack inequality for the elliptic equation (3.0.4) fails for the supercritical drifts constructed in Theorem 3.0.7.

Before our result, we are only aware of an example showing the loss of continuity for a supercritical drift $u \in L^1$ in dimension $n = 3$ constructed by Seregin, Silvestre, Šverák and Zlatoš in [46].

3.1 Construction of supercritical drifts

In this section we construct suitable supercritical drifts. Our construction is inspired by the work of Silvestre, Vicol and Zlatoš [48], where they proved the loss of continuity for the solutions of the parabolic equations with fractional Laplacian in dimension $n = 2$. We design supercritical drifts in a way such that the velocity field in a cone area transports the scalar v to the origin of the cone. Then the loss of continuity takes place at the origin, where the drift u is singular.

Let $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ be the Cartesian coordinate in $\mathbb{R}^n$ and we denote the unit vectors by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4, \ldots)$. Consider the scalar equation (3.0.1) in a mixed coordinate system of $\mathbb{R}^n$

$$(r, z, \theta, \varphi_1, \ldots, \varphi_{n-3}) \in [0, \infty) \times \mathbb{R} \times \Big[-\frac{\pi}{2}, \frac{3\pi}{2}\Big) \times [0, \pi]^{n-3}$$

with the transformation

$$\begin{cases} x_1 = z \\ x_2 = r\cos\theta \\ x_3 = r\sin\theta\cos\varphi_1 \\ \quad\ldots \\ x_{n-1} = r\sin\theta\sin\varphi_1\ldots\cos\varphi_{n-3} \\ x_n = r\sin\theta\sin\varphi_1\ldots\sin\varphi_{n-3} \end{cases}. \tag{3.1.1}$$

We denote the unit vectors in this mixed coordinate system by $(\mathbf{e}_r, \mathbf{e}_z, \mathbf{e}_\theta, \mathbf{e}_{\varphi_1}, \ldots)$. In this paper, we use $|\cdot|$ to denote standard Euclidean norm. Since the drift we construct only depends on r and z, for the simplicity of notation, we also use $|(r, z)| := \sqrt{r^2 + z^2}$.

For the parabolic case, namely throughout Section 3.2 and Section 3.3, we consider functions which only depend on r, z, θ and t. For those functions f which are independent of $\varphi_i, 1 \leq i \leq n-3$, we will only write the first four effctive arguments $f(r, z, \theta, t)$. The gradient of f is given by

$$\nabla f = \mathbf{e}_r \partial_r f + \frac{\mathbf{e}_\theta}{r} \partial_\theta f + \mathbf{e}_z \partial_z f$$

and the Laplacian is given by

$$\Delta f = \frac{1}{r^{n-2}} \partial_r (r^{n-2} \partial_r f) + \frac{n-3}{r^2 \tan \theta} \partial_\theta f + \frac{1}{r^2} \partial_{\theta\theta} f + \partial_{zz} f.$$

We also present a method to construct divergence-free vector fields with radial symmetry in hyperspherical part $\mathbb{R}^{n-1}$ from scalar-valued functions.

Lemma 3.1.1. *Let $n \geq 3$. For any locally integrable function $\Psi : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ which belongs to C^2 in any compact subset of $([0, \infty) \times \mathbb{R}) \backslash \{0\}$ and the velocity field $u := (u_r, u_z, 0, \dots, 0) : \mathbb{R}^n \to \mathbb{R}^n$ pointwisely defined by*

$$(u_r, u_z) := \Big(- \frac{\partial_z \Psi}{r^{n-2}}, \frac{\partial_r \Psi}{r^{n-2}} \Big), \tag{3.1.2}$$

assume the pointwise derivative ∇u belongs to $L^1(\mathbb{R}^n)$. Then $u \in W^{1,1}_{loc}(\mathbb{R}^n)$ and u is a weakly divergence-free drift. The scalar function Ψ is called a generalized stream function.

Proof. First, we compute the divergence operator for velocity field $u := (u_r, u_z, 0, \dots, 0)$ which only depends on r and z in the mixed coordinate system. Let $x := (y, z) \in \mathbb{R}^{n-1} \times \mathbb{R}$, the Cartesian coordinate of u is given by

$$\Big(\frac{y}{|y|} u_r, u_z \Big).$$

Then computing its divergence directly in Cartesian coordinate gives

$$\operatorname{div} u = \partial_r u_r + \frac{n-2}{r} u_r + \partial_z u_z.$$

Note that $\operatorname{div} u = 0$ formally corresponds to

$$\partial_r(r^{n-2}u_r) + \partial_z(r^{n-2}u_z) = 0. \tag{3.1.3}$$

Since the set $\{0\}$ has $W^{1,1}(\mathbb{R}^n)$-capacity zero and $\nabla u \in L^1_{\mathrm{loc}}([0,\infty)\times\mathbb{R})$, $u \in W^{1,1}_{\mathrm{loc}}(D)$. One can easily verify $\operatorname{div} u = 0$ by a formal computation. □

Now, we construct divergence-free drifts from their generalized stream functions Ψ.

Lemma 3.1.2. *Fix any $\lambda \in (0, n-2)$ and any $\alpha \in (0, \frac{\lambda}{n-\lambda})$, define the generalized stream function $\Psi|_D : D := [0,2]\times[-2,2] \to \mathbb{R}$ as follows. Let*

$$\Psi = \frac{1}{2(n-2-\alpha)}\Psi_s, \tag{3.1.4}$$

where Ψ_s is given by

$$\Psi_s(r,z) = \begin{cases} -r^{n-1}, & 0 \le \frac{3r}{4} \le -z \\ (r+z)^{n-2-\alpha} - (r-z)^{n-2-\alpha}, & -\frac{r}{2} \le z \le \frac{r}{2} \\ r^{n-1}, & 0 \le \frac{3r}{4} \le z \end{cases}$$

and

$$\begin{aligned} \Psi_s(r,z) =& -r^{n-1}\varrho\Big(\frac{-4z-2r}{r}\Big) \\ &+ \big[(r+z)^{n-2-\alpha} - (r-z)^{n-2-\alpha}\big]\varrho\Big(\frac{3r+4z}{r}\Big),\ \frac{r}{2} \le -z \le \frac{3r}{4}, \\ \Psi_s(r,z) =& r^{n-1}\varrho\Big(\frac{4z-2r}{r}\Big) \\ &+ \big[(r+z)^{n-2-\alpha} - (r-z)^{n-2-\alpha}\big]\varrho\Big(\frac{3r-4z}{r}\Big),\ \frac{r}{2} \le z \le \frac{3r}{4}. \end{aligned}$$

Here, $\varrho : \mathbb{R} \to \mathbb{R}$ is a non-negative smooth function with $\varrho(s) = 0$ for

$s \le 0$ and $\varrho(s) = 1$ for $s \ge 1$. In the region $\{r \ge 2\} \bigcup \{|z| \ge 2\}$, we smoothly extend $\Psi|_D$ such that (u_r, u_z) obtained from (3.1.2) is smooth in $\{r \ge 2\} \bigcup \{|z| \ge 2\}$ and of exponential decay at infinity. Then Ψ is smooth away from $x = 0$, and $u = (u_r, u_z, 0, \dots, 0)$ is in $L^{n-\lambda}(\mathbb{R}^n)$ with respect to the standard metric $r^{n-2} dr dz$ and weakly divergence-free. Furthermore, u satisfies the bound

$$|u| \le \frac{C(\lambda)}{|x|^{1+\alpha}}.$$

Proof. Note that Ψ is continuous and odd in z. From (3.1.2), we can compute the velocity in the following subregion

$$(u_r, u_z) = \begin{cases} \left(0, -\frac{n-1}{2(n-2-\alpha)}\right), & 0 \le \frac{3r}{4} \le -z, \\ \Big(-\frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}}, & \\ \quad \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}}\Big), & -\frac{r}{2} \le z \le \frac{r}{2}, \\ \left(0, \frac{n-1}{2(n-2-\alpha)}\right), & 0 \le \frac{3r}{4} \le z, \end{cases} \tag{3.1.5}$$

then direct computations show that $u|_D \in L^{n-\lambda}(D)$, $\nabla u|_D \in L^1(D)$ and u satisfies the pointwise bound. Because u is in C^∞ outside D and of exponential decay at infinity, $u \in W^{1,1}(\mathbb{R}^n)$ is weakly divergence-free follows from the fact $\nabla u \in L^1(\mathbb{R}^n)$. □

Next, we truncate u at origin to obtain bounded drifts.

Corollary 3.1.3. *Let Ψ, u, λ and α be as in Lemma 3.1.2. For any $\epsilon > 0$, there exists $\Psi_\epsilon : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ with*

$$\Psi_\epsilon = \Psi \text{ in } \big([0,\infty) \times \mathbb{R}\big) \backslash \{|(r,z)| < \epsilon, -r \le z \le r\}, \tag{3.1.6}$$

such that $u_\epsilon \in C^\infty(\mathbb{R}^n)$*,* u_ϵ *is bounded in* $L^{n-\lambda}(\mathbb{R}^n)$ *uniformly in* ϵ*, and*

$$|u_\epsilon| \le \frac{C(\lambda)}{|x|^{1+\alpha}}, \quad \textit{uniformly in } \epsilon,$$

where $u_\epsilon = (u_{\epsilon,r}, u_{\epsilon,z}, 0, \dots, 0)$ *is given by* Ψ_ϵ *via* (3.1.2)*. Moreover,* Ψ_ϵ *is odd in* z*.*

Proof. In the region $\{|(r,z)| < \epsilon, -r \le z \le r\}$, we smoothly extend Ψ_ϵ while preserving the symmetry. Because u in (3.1.5) has only one singularity at the origin $(r,z) = (0,0)$, one just needs to truncate this singularity while making Ψ_ϵ equal to $\pm r^{n-2}$ in r close to $\{r = 0\}$. □

3.2 The evolution in parabolic case

In this section, we prove the loss of continuity in the parabolic case. Instead of considering the drift-diffusion equation (3.0.1) in $\mathbb{R}^n \times [0,T]$ for some $T > 0$, we study its evolution in $D_n \times [0,T]$ with zero boundary condition, where $D_n \subset \mathbb{R}^n$ is the bounded domain defined by

$$D_n := \big\{(r, z, \theta, \varphi_1, \dots, \varphi_{n-3}) \in \mathbb{R}^n \,\big|\, (r,z) \in [0,2] \times [-2,2]\big\}.$$

This is sufficient because of the local nature of the properties [P2] and [P3]. Note that the drift u and all other functions do not depend on the variables φ_i for $1 \le i \le n-3$. Thus we omit these coordinates for the sake of simplicity from now on. The main result of this section is

Theorem 3.2.1. *Given any* $\lambda \in (0, n-2)$ *and any* $\alpha \in (0, \frac{\lambda}{n-\lambda})$*, there exist initial data* $v_0 \in C_c^\infty(D_n)$ *with* $\|v_0\|_{C^2(D_n)} \le C$*, independent of* $\varphi_i, 1 \le i \le n-3$*, and* $\kappa > 0$ *which satisfy the following property. For any* $\delta \in (0, \frac{1}{2})$ *and any* $\epsilon \in (0, \delta]$*, for the drift* $u_{\epsilon/2}$ *given by Corollary 3.1.3, the unique classical solution* v_ϵ *of* (3.0.1) *in the space-time domain* $D_n \times [0, \frac{1}{2+\alpha}]$ *with initial data* v_0 *and zero boundary condition* *satisfies*

$$v_\epsilon\Big(\delta, 0, 0, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \geq \kappa, \quad v_\epsilon\Big(\delta, 0, \pi, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \leq -\kappa,$$

and

$$v_\epsilon(r, z, \theta, t) \geq 0 \quad \textit{for } |\theta| \leq \frac{\pi}{2},$$
$$v_\epsilon(r, z, \theta, t) \leq 0 \quad \textit{for } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}.$$

Here, C is absolute. κ is independent of ϵ and δ, but depends on λ and v_0.

Theorem 3.0.2 follows from Theorem 3.2.1, since we can use the solutions v_ϵ for the regularized drifts $u_{\epsilon/2}$ to approximate the solution v we need in Theorem 3.0.2 with the following convergence argument.

Proof of Theorem 3.0.2. Let $T = \frac{1}{2+\alpha}$. $\{v_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $L_t^\infty L_x^2 \bigcap L_t^2 H_x^1$ by standard energy estimates using the fact that $u_{\epsilon/2}$ is divergence-free. Thus a subsequence $v_\epsilon \to v$ weakly in $L_t^2 H_x^1$ and weakly-$*$ in $L_t^\infty L_x^2$, as $\epsilon \to 0$, and we may choose further subsequences later. Moreover, by the maximum principle, we also know $\{v_\epsilon\}_{\epsilon>0}$ is uniformly bounded in $D_n \times [0, T]$.

Because of the smoothness of $u_{\epsilon/2}$ and the relation (3.1.6), $u_{\epsilon/2}$ is uniformly bounded in L^n in any compact subset of $(D_n \backslash \{0\}) \times [0, T]$. Moreover, from Theorem 3.2.1, $v_\epsilon \geq 0$ in the domain $(D_n \bigcap \{|\theta| < \frac{\pi}{2}\}) \times [0, T]$. By Theorem 3.0.1, moreover $\{v_\epsilon\}_{\epsilon>0}$ is uniformly Hölder continuous in any compact subset of $(D_n \bigcap \{|\theta| < \frac{\pi}{2}\}) \times [0, T]$. Indeed, we may use Harnack's inequality (3.0.3) with a constant C that is uniform in ϵ. This is because $\|u_{\epsilon/2}\|_{L^n(\Omega)} < C(\Omega)$ for any compact subset Ω of $D_n \backslash \{0\}$[2]. Then we can prove uniform Hölder continuity.

[2] Because u defined by (3.1.4) merely belongs to $L^{n-}(D_n)$, $\{u_{\epsilon/2}\}_{\epsilon>0}$ cannot be bounded uniformly in $L^n(D_n)$. Therefore, the constant $C(\Omega)$ cannot be made

Note that proving Hölder continuity from Harnack's inequality is a local argument, where we do not need any information on the boundary close to $r = 0$. In contrast, if we wanted to use Schauder estimate, we would need some control on the boundary close to $r = 0$.

Clearly, $u_{\epsilon/2}$ converges to u in $L^{n-\lambda}$ as $\epsilon \to 0$, where u is given by Lemma 3.1.2. With all the convergence information above, we can deduce that $v \in L_t^\infty L_x^2 \bigcap L_t^2 H_x^1$ solves (3.0.1) with drift u in the sense of distributions, thus v is a weak solution.

From Theorem 3.2.1, we know that for any $\delta \in (0, \frac{1}{2})$ and any $\epsilon \in (0, \delta]$,

$$v_\epsilon\Big(\delta, 0, 0, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \geq \kappa, \quad v_\epsilon\Big(\delta, 0, \pi, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \leq -\kappa.$$

Fix any sequence $\{\delta_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^+$ with $\lim_{k\to\infty} \delta_k = 0$. By the uniform Hölder continuity in any compact subset of $(D_n \bigcap \{|\theta| < \frac{\pi}{2}\}) \times [0, T]$ and a diagonal argument, we can extract a subsequence from $\{v_\epsilon\}_{\epsilon>0}$ converging to v such that

$$v\Big(\delta_k, 0, 0, \frac{1-\delta_k^{2+\alpha}}{2+\alpha}\Big) \geq \kappa, \quad v\Big(\delta_k, 0, \pi, \frac{1-\delta_k^{2+\alpha}}{2+\alpha}\Big) \leq -\kappa, \text{ for any } k,$$

holds in the limit when $\epsilon \to 0$. Here, the second inequality can be proved using exactly the same argument as the first inequality. Letting $k \to \infty$ shows that v is not continuous at the point $(r, z, \theta, t) = (0, 0, 0, \frac{1}{2+\alpha})$. Note that the loss of continuity is a local behavior, so we can state this result for general space domains, for example, the unit ball in the statement of Theorem 3.0.2.

□

To prepare for the proof of Theorem 3.2.1, we need two lemmas.

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uniform in different Ω. This is the reason why we must get rid of the point $x = 0$ and use a diagonal argument in this proof.

Lemma 3.2.2. *Given the drift $u \in L^{n-\lambda}(\mathbb{R}^n)$ from (3.1.2) and (3.1.4), let $\eta \in C^\infty(\mathbb{R})$ be an even function which is supported on $[-1,1]$. Assume furthermore $\eta(0) = 1$ and that η is strictly decreasing on $[0,1]$. Then there exists $r_0 \in (0, \frac{1}{2})$ with the following property. Define*

$$\begin{aligned} \phi(r,z) &:= \eta\Big(\frac{|(r-1,z)|}{r_0}\Big), \\ g(r,z,\theta,\varphi_1,\dots,\varphi_{n-3},t) &:= \phi\Big(\frac{r}{h(t)}, \frac{z}{h(t)}\Big)\eta\Big(\frac{8\theta}{\pi}\Big), \end{aligned} \tag{3.2.1}$$

where $h(t) := \big(1-(2+\alpha)t\big)^{\frac{1}{2+\alpha}}$, then g satisfies

$$\partial_t g + u \cdot \nabla g \le 0, \quad \text{in } D_n \times \Big[0, \frac{1}{2+\alpha}\Big]. \tag{3.2.2}$$

Lemma 3.2.3. *There exist a function $\eta \in C^\infty(\mathbb{R})$ satisfying all requirements in Lemma 3.2.2 and a constant $c_0 > 0$ depending on η, such that the function g defined in (3.2.1) satisfies*

$$-\Delta g \le c_0[h(t)]^{-2} g.$$

Now we prove the first main result.

Proof of Theorem 3.2.1. For the smooth drift $u_{\epsilon/2} \in C^\infty(\mathbb{R}^n)$ given in Corollary 3.1.3, standard theory yields existence and uniqueness of a classical solution v_ϵ of (3.0.1) in the domain $D_n \times [0, \frac{1}{2+\alpha}]$ with zero Dirichlet boundary condition and initial data

$$v_\epsilon(r,z,\theta,0) := g(r,z,\theta,0) - g(r,z,\pi-\theta,0).$$

Super-solutions and sub-solutions satisfy the standard maximum principle. Since this classical solution v_ϵ is unique, it must preserve all space symmetry of the initial data at any time t. At time $t = 0$, the initial data satisfies $v_\epsilon(r,z,\theta,0) = -v_\epsilon(r,z,\pi-\theta,0)$, then at all time t, we know

$$v_\epsilon(r, z, \theta, t) = -v_\epsilon(r, z, \pi - \theta, t).$$

and thus we have

$$v_\epsilon\Big(r, z, \frac{\pi}{2}, t\Big) = v_\epsilon\Big(r, z, -\frac{\pi}{2}, t\Big) = 0. \tag{3.2.3}$$

Define

$$\underline{v}(r, z, \theta, t) = \exp\Big(- c_0 \int_0^t [h(s)]^{-2} ds\Big)\big(g(r, z, \theta, t) - g(r, z, \pi - \theta, t)\big),$$

where h, g and c_0 are defined in Lemma 3.2.2 and Lemma 3.2.3. Because $u = u_{\epsilon/2}$ in $\{|(r, z)| \geq \frac{\epsilon}{2}\}$ and $\underline{v}(r, z, \theta, t) = 0$ for any $\{|(r, z)| \leq \frac{\epsilon}{2}\}$ and any $t \leq h^{-1}(\epsilon)$, we have that, in the region $\{|\theta| \leq \frac{\pi}{2}, t \leq h^{-1}(\epsilon)\}$,

$$\partial_t \underline{v} - \Delta \underline{v} + (u_{\epsilon/2} \cdot \nabla)\underline{v} \leq \exp\Big(- c_0 \int_0^t [h(s)]^{-2} ds\Big)(\partial_t g + u \cdot \nabla g) \leq 0.$$

Here, we use Lemma 3.2.2, Lemma 3.2.3 and the fact that $g(r, z, \pi - \theta, t) = 0$ in the region $\{|\theta| \leq \frac{\pi}{2}, t \leq h^{-1}(\epsilon)\}$. Therefore, $\underline{v}$ is a sub-solution of (3.0.1) in $D' := D_n \bigcap \{|\theta| \leq \frac{\pi}{2}\}$ until the time $t = h^{-1}(\epsilon)$. Clearly, by (3.2.3) and the definition of g, $\underline{v}|_{\partial D'} = 0$ and $v_\epsilon|_{\partial D'} \geq 0$, and then $v_\epsilon - \underline{v} \geq 0$ on $\partial D'$. Since $v_\epsilon - \underline{v}$ is a super-solution of (3.0.1) in D' until the time $t = h^{-1}(\epsilon)$, maximum principle yields that, for any $\delta \in (0, \frac{1}{2})$ and any $\epsilon \in (0, \delta]$, we have

$$\begin{aligned} v_\epsilon(\delta, 0, 0, h^{-1}(\delta)) \geq \underline{v}(\delta, 0, 0, h^{-1}(\delta)) &= \exp\Big(- c_0 \int_0^{h^{-1}(\delta)} [h(s)]^{-2} ds\Big) \\ &\geq \exp\Big(- c_0 \int_0^{\frac{1}{2+\alpha}} (1 - (2 + \alpha)s)^{\frac{-2}{2+\alpha}} ds\Big) := \kappa. \end{aligned} \tag{3.2.4}$$

Here, we use the fact $\frac{-2}{2+\alpha} > -1$. The bound of $v_\epsilon(\delta, 0, \pi, h^{-1}(\delta))$ can be proved by the same argument in the symmetric space-time domain. $\kappa > 0$ is independent of ϵ and δ. $\square$

Next, we come back to Lemma 3.2.2 and Lemma 3.2.3.

Proof of Lemma 3.2.2. Note that h is the solution of the solution of the following ODE

$$h' = -\frac{1}{h^{1+\alpha}}, \quad h(0) = 1. \tag{3.2.5}$$

Define a vector field $w = w_r \mathbf{e}_r + w_z \mathbf{e}_z$ with

$$(w_r, w_z) = \Big(-\frac{r}{[h(t)]^{2+\alpha}}, -\frac{z}{[h(t)]^{2+\alpha}} \Big).$$

The function g solves $\partial_t g + w \cdot \nabla g = 0$. This is immediate from

$$\partial_t g = \nabla\phi\Big(\frac{r}{h(t)}, \frac{z}{h(t)}\Big) \cdot \Big(-\frac{rh'(t)}{[h(t)]^2}, -\frac{zh'(t)}{[h(t)]^2} \Big) \eta\Big(\frac{8\theta}{\pi}\Big),$$
$$w \cdot \nabla g = \frac{1}{h(t)} \nabla\phi\Big(\frac{r}{h(t)}, \frac{z}{h(t)}\Big) \cdot \Big(-\frac{r}{[h(t)]^{2+\alpha}}, -\frac{z}{[h(t)]^{2+\alpha}} \Big) \eta\Big(\frac{8\theta}{\pi}\Big).$$

To check the inequality $\partial_t g + u \cdot \nabla g \leq 0$. Equivalently, we need to verify

$$(w - u) \cdot \nabla g \geq 0.$$

Note that $\eta \geq 0$. By the definitions of w and g, it suffices to show

$$\Big(\frac{(r, z)}{[h(t)]^{2+\alpha}} + (u_r, u_z) \Big) \cdot (r - h(t), z) \geq 0 \tag{3.2.6}$$

in the region $\{(r, z, t) \,|\, |(r - h(t), z)| \leq h(t) r_0\}$ where the support of g lies. Here, we use $\eta' \leq 0$.

Now we can impose our first condition $r_0 < \frac{1}{4}$ to ensure that the space support of $g(\cdot, \cdot, \theta, t)$ stays in the region $\{(r, z) \,|\, -\frac{r}{2} \leq z \leq \frac{r}{2}\}$ for any $t \in [0, \frac{1}{2+\alpha}]$. As we computed in (3.1.5), the velocity (u_r, u_z) in this region is given by

$$\Big(- \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}}, \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}} \Big).$$

By change of variable, it suffices to prove

$$\Big[\Big(- \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}}, \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}} \Big) + (r,z)\Big] \cdot (r-1,z) \geq 0 \tag{3.2.7}$$

in the region $\{(r,z) \,|\, |(r-1,z)| \leq r_0\}$. This is equivalent to proving that

$$f(r,z) := -\frac{r-z-1}{2(r+z)^{\alpha+3-n}} - \frac{r+z-1}{2(r-z)^{\alpha+3-n}} + r^{n-1}(r-1) + z^2 r^{n-2} \geq 0$$

in the region $\{(r,z) \,|\, |(r-1,z)| \leq r_0\}$. We can check by elementary computation that f achieves a local minimum at $(1,0)$ with

$$\begin{aligned} \partial_{rz} f(1,0) &= 0, \\ \partial_{rr} f(1,0) &= 4 + 2\alpha > 0, \\ \partial_{zz} f(1,0) &= 2(n-2-\alpha) > 0. \end{aligned}$$

Therefore, we can choose $r_0 > 0$ small enough to ensure (3.2.7).

□

Proof of Lemma 3.2.3. By scaling, it suffices to prove the following inequality

$$-\Delta \bar{g} \leq c_0 \bar{g} \tag{3.2.8}$$

for the function

$$\bar{g}(r,z,\theta) := \eta\Big(\frac{|(r-1,z)|}{r_0}\Big)\eta\Big(\frac{8\theta}{\pi}\Big).$$

Now fix

$$\eta(s) := \begin{cases} 0, & |s| \geq 1 \\ \exp\left(\frac{|s|}{|s|-1}\right), & \frac{1}{2} \leq |s| < 1 \\ \text{any smooth even monotone extension}, & \frac{1}{4} \leq |s| \leq \frac{1}{2} \\ 1 - s^2, & |s| \leq \frac{1}{4} \end{cases},$$

Let $\rho := \frac{|(r-1,z)|}{r_0}$, then we compute

$$\begin{aligned} \Delta\bar{g} =& \Big(\partial_{rr} + \frac{n-2}{r}\partial_r + \frac{1}{r^2}\partial_{\theta\theta} + \frac{n-3}{r^2\tan\theta}\partial_\theta + \partial_{zz}\Big)\bar{g} \\ =& \frac{1}{r_0^2}\eta''(\rho)\eta\Big(\frac{8\theta}{\pi}\Big) + \frac{n-1}{r_0^2\rho}\eta'(\rho)\eta\Big(\frac{8\theta}{\pi}\Big) - \frac{n-2}{r_0^2 r\rho}\eta'(\rho)\eta\Big(\frac{8\theta}{\pi}\Big) \\ &+ \frac{64}{\pi^2 r^2}\eta(\rho)\eta''\Big(\frac{8\theta}{\pi}\Big) + \frac{8(n-3)}{\pi r^2\tan\theta}\eta(\rho)\eta'\Big(\frac{8\theta}{\pi}\Big) \\ \geq& \frac{64}{\pi^2 r^2}\eta(\rho)\Big(\eta''\Big(\frac{8\theta}{\pi}\Big) + \frac{\pi(n-3)}{8\tan\theta}\eta'\Big(\frac{8\theta}{\pi}\Big)\Big) \\ &+ \frac{1}{r_0^2}\eta\Big(\frac{8\theta}{\pi}\Big)\Big(\eta''(\rho) + \frac{n-1}{\rho}\eta'(\rho)\Big). \end{aligned} \tag{3.2.9}$$

Here we eliminate the third term in the second line of (3.2.9), because $-\eta', \eta \geq 0$ on $[0,\infty)$.

By the definition of η, we have that for $\frac{1}{2} \leq s \leq 1$, the following two terms are bounded from below for $c_1 < 0$ small enough,

$$\Big(\eta''(s) + \frac{\pi(n-3)}{8\tan(\frac{\pi s}{8})}\eta'(s)\Big)\frac{1}{\eta(s)} = \frac{1}{(s-1)^4} + \frac{2}{(s-1)^3} - \frac{\pi(n-3)}{8\tan(\frac{\pi s}{8})(s-1)^2} \geq c_1,$$

$$\Big(\eta''(s) + \frac{n-1}{s}\eta'(s)\Big)\frac{1}{\eta(s)} = \frac{1}{(s-1)^4} + \frac{2}{(s-1)^3} - \frac{n-1}{(s-1)^2 s} \geq c_1.$$

This is indeed true since the first terms on the right hand side dominate when s is close to 1.

The same lower bound is also true when $|s| \leq \frac{1}{4}$, i.e.

$$\Big(\eta''(s) + \frac{\pi(n-3)}{8\tan(\frac{\pi s}{8})}\eta'(s)\Big)\frac{1}{\eta(s)} = \frac{-2}{1-s^2}\Big(1 + \frac{2\pi(n-3)s}{8\tan(\frac{\pi s}{8})}\Big) \geq c_1,$$
$$\Big(\eta''(s) + \frac{n-1}{s}\eta'(s)\Big)\frac{1}{\eta(s)} = \frac{-2n}{1-s^2} \geq c_1.$$

In the region $\frac{1}{4} \leq |s| \leq \frac{1}{2}$, the same lower bound is also true, because $\eta(s)$ is larger than a positive constant and $|\eta''(s)|, |\frac{1}{s}\eta'(s)|$ are bounded.

Combining all these estimates with (3.2.9), we have

$$\Delta\bar{g} \geq \frac{c_1}{r_0^2}\eta\Big(\frac{8\theta}{\pi}\Big)\eta(\rho) + \frac{64c_1}{\pi^2 r^2}\eta\Big(\frac{8\theta}{\pi}\Big)\eta(\rho) = \Big(\frac{c_1}{r_0^2} + \frac{64c_1}{\pi^2 r^2}\Big)\bar{g}.$$

The proof is complete. □

3.3 A parabolic toy model for the Navier-Stokes equations

In this section, we give the proof of Theorem 1.3.3. Its difference from Theorem 3.0.2 is that we need a drift $u \in L_t^q H_x^1(B \times [0,T])$ for $1 \leq q < \frac{2(2+\alpha)}{2\alpha+1}$. To achieve this goal, we will do some space-time truncation to get a time-dependent drift u.

For (3.0.1) in dimension $n = 3$, we have constructed the drift $\bar{u} := (\bar{u}_r, \bar{u}_z, 0) \in L^{3-\lambda}(\mathbb{R}^3)$ with

$$(\bar{u}_r, \bar{u}_z) = \begin{cases} (0, -\frac{1}{1-\alpha}), & 0 \leq \frac{3r}{4} \leq -z \\ \text{interpolation region}, & \frac{r}{2} \leq -z \leq \frac{3r}{4} \\ \big(-\frac{(r+z)^{-\alpha}}{2r} - \frac{(r-z)^{-\alpha}}{2r}, \frac{(r+z)^{-\alpha}}{2r} - \frac{(r-z)^{-\alpha}}{2r}\big), & -\frac{r}{2} \leq z \leq \frac{r}{2} \\ \text{interpolation region}, & \frac{r}{2} \leq z \leq \frac{3r}{4} \\ (0, \frac{1}{1-\alpha}), & 0 \leq \frac{3r}{4} \leq z \end{cases}$$

from its generalized function $\bar{\Psi} = \Psi$, defined in Lemma 3.1.2 with

$n = 3$. To compensate the term $2r^{-1}\partial_r v$ in the toy model of the axis-symmetric Navier-Stokes equations (1.3.4), define another generalized stream function $\Phi_0 : D \to \mathbb{R}$ by

$$\Phi_0(r,z) = \begin{cases} -r^2, & 0 \le \frac{3r}{4} \le -z \\ -r^2\varrho\left(\frac{-4z-2r}{r}\right) + 2z\varrho\left(\frac{3r+4z}{r}\right), & \frac{r}{2} \le -z \le \frac{3r}{4} \\ 2z, & -\frac{r}{2} \le z \le \frac{r}{2} \\ r^2\varrho\left(\frac{4z-2r}{r}\right) + 2z\varrho\left(\frac{3r-4z}{r}\right), & \frac{r}{2} \le z \le \frac{3r}{4} \\ r^2, & 0 \le \frac{3r}{4} \le z \end{cases}$$

and extend it outside D smoothly with exponential decay.

Now we do space-time truncation such that the drift belongs to the natural energy space of the Navier-Stokes equations. Define a cutoff function

$$\bar{\varrho}(r,t) := \varrho\Big(\frac{8r}{h(t)} - 1\Big),$$

where ϱ and h are defined in Lemma 3.1.2 and Lemma 3.2.2 respectively. Define

$$\Phi := (\Phi_0 + \bar{\Psi})\bar{\varrho} \tag{3.3.1}$$

and $\tilde{u} := (\tilde{u}_r, \tilde{u}_z, 0)$ via $(\tilde{u}_r, \tilde{u}_z) = \big(-\frac{\partial_z \Phi}{r^{n-2}}, \frac{\partial_r \Phi}{r^{n-2}}\big)$. Using the same argument in the proof of Lemma 3.1.2, one can check $\tilde{u} \in L_t^\infty L_x^{3-\lambda}(\mathbb{R}^3)$. And direct computations verify $\tilde{u} \in L_t^q H_x^1(\mathbb{R}^3)$ with $1 \le q < \frac{2(2+\alpha)}{2\alpha+1}$ for $\alpha > 0$ small enough.

As in Corollary 3.1.3, we do space truncation to get smooth drifts $\{\Phi_{0,\epsilon}\}_{\epsilon>0}$. Then

$$\Phi_\epsilon := (\Phi_{0,\epsilon} + \bar{\Psi}_\epsilon)\bar{\varrho} \tag{3.3.2}$$

gives a smooth time-dependent drift $\tilde{u}_\epsilon := (\tilde{u}_{r,\epsilon}, \tilde{u}_{z,\epsilon}, 0)$ via $(\tilde{u}_{r,\epsilon}, \tilde{u}_{z,\epsilon}) = \big(-\frac{\partial_z \Phi_\epsilon}{r^{n-2}}, \frac{\partial_r \Phi_\epsilon}{r^{n-2}}\big)$. Here, $\bar{\Psi}_\epsilon = \Psi_\epsilon$ is defined in Corollary 3.1.3 with $n = 3$.

The velocity field $\tilde{u}_\epsilon$ satisfies

$$\tilde{u}_\epsilon = \bar{u} + \Big(-\frac{2}{r}, 0 \Big) \quad \text{for } -\frac{r}{2} \leq z \leq \frac{r}{2}, r \geq \frac{h(t)}{4}. \tag{3.3.3}$$

In this setting, the following theorem is an analogue of Theorem 3.2.1.

Theorem 3.3.1. *Given any $\lambda \in (0,1)$ and any $\alpha \in (0, \frac{\lambda}{3-\lambda})$, there exist initial data $v_0 \in C_c^\infty(D_3)$ with $\|v_0\|_{C^2(D_3)} \leq C$, and $\kappa > 0$ which satisfy the following property. For any $\delta \in (0, \frac{1}{2})$ and any $\epsilon \in (0, \delta]$, for the drift $\tilde{u}_{\epsilon/2}$ given by (3.3.2), the unique classical solution v_ϵ of (1.3.4) in the space-time domain $D_3 \times [0, \frac{1}{2+\alpha}]$ with initial data v_0 and zero boundary condition satisfies*

$$v_\epsilon\Big(\delta, 0, 0, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \geq \kappa, \quad v_\epsilon\Big(\delta, 0, \pi, \frac{1-\delta^{2+\alpha}}{2+\alpha}\Big) \leq -\kappa,$$

and

$$\begin{aligned} v_\epsilon(r, z, \theta, t) \geq 0 \quad & \textit{for } |\theta| \leq \frac{\pi}{2}, \\ v_\epsilon(r, z, \theta, t) \leq 0 \quad & \textit{for } \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}. \end{aligned}$$

Here, C is absolute. κ is independent of ϵ and δ, but depends on λ and v_0.

The proofs of Theorem 1.3.3 and Theorem 3.3.1 are essentially the same argument in the previous section with the following two facts. The first fact is that, from (3.3.3), in the region $\big\{(r,z,\theta,t) \in D_3 \times [0, \frac{1}{2+\alpha}] \,|\, -\frac{r}{2} \leq z \leq \frac{r}{2}, r \geq \frac{h(t)}{4}\big\}$, (3.0.1) with the drift u given by Lemma 3.1.2 is identical to (1.3.4) with the drift $\tilde{u}$ given by (3.3.1). The second fact is that the effective dynamics we rely on to prove Theorem 3.2.1 and Theorem 3.0.2 exactly lies in this region.

3.4 The elliptic case

We consider the elliptic equation (3.0.4) in the unit ball $B_1(0, \mathbb{R}^n) \subset \mathbb{R}^n$. We use the stochastic formulation of (3.0.4) to prove the loss of continuity. Similar idea has been used by Seregin, Silvestre, Šverák and Zlatoš [46] to prove the loss of continuity for a supercritical drift $u \in L^1$ in dimension $n = 3$. For some technical convenience, we work with Cartesian coordinate system in this section. The main result of this section is as follows.

Theorem 3.4.1. *There exist absolute constants $C, \alpha_0 > 0$ and another constant $\kappa > 0$ with the following property. For any $\lambda \in (0, n-1)$, any $\alpha \in (0, \alpha_0)$, any $\delta \in (0,1)$ and any $\epsilon \in (0, \frac{\delta}{2})$, consider the elliptic equation* (3.0.4) *with the drift Cu_ϵ in the unit ball with the following smooth boundary condition $\gamma : \partial B_1(0, \mathbb{R}^n) \to \mathbb{R}$,*

$$\gamma(x) = \begin{cases} 1, \quad x_2 > \frac{1}{2} \\ \textit{smooth extension which is odd and monotone in } x_2,\ x_2 \in (-\frac{1}{2}, \frac{1}{2}) \\ -1, \quad x_2 < -\frac{1}{2} \end{cases}$$

then the unique classical solution v_ϵ of this elliptic equation satisfies

$$v_\epsilon(0, \delta, 0, \ldots, 0) \geq \kappa$$

and

$$v_\epsilon(x) \geq 0, \quad \textit{for } x \textit{ with } x_2 \geq 0.$$

Here, u_ϵ is given by Corollary 3.1.3 and $\kappa > 0$ is independent of ϵ and δ.

To prove Theorem 3.0.7, one just needs to follow the proof of Theorem 3.0.2 line by line, with the help of Theorem 3.4.1. To prove Theorem 3.4.1, we need the following technical lemma.

Lemma 3.4.2. *There exist absolute constants* $\nu_0, \alpha_0 \in (0,1)$ *with the following property. For any* $\mu > 0$, $l \in (0,1)$, $\nu \in (0, \nu_0]$, $C > 0$ *and* $a > 0$, *define*

$$\Sigma_{\nu,\mu} := \Big\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,\Big|\, x_2 \in \Big[\frac{\mu}{2}, 2\mu\Big], \max\big\{|x_1|, |x_3|, \ldots, |x_n|\big\} \leq \nu x_2\Big\}$$

and

$$\Gamma_{\nu,\mu} := \big\{(x_1, x_2, \ldots, x_n) \in \Sigma_{\nu,\mu} \,\big|\, x_2 = 2\mu\big\},$$
$$S_{\nu,\mu,l} := \big\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \,\big|\, x_2 = \mu, \max\big\{|x_1|, |x_3|, \ldots, |x_n|\big\} \leq l\nu\mu\big\}.$$

Suppose we have two continuous functions $\rho : [0,a] \to \Sigma_{\nu,\mu}$, $b : [0,a] \to \mathbb{R}^n$ *and for any* $t \in [0,a]$,

$$\rho(t) = \rho(0) - C\int_0^t u(\rho(s))ds + b(t) - b(0), \qquad (3.4.1)$$
$$\sup_t |b(t) - b(0)| \leq \frac{(1-l)\nu\mu}{8}. \qquad (3.4.2)$$

Moreover, suppose ρ *satisfies*

$$\rho(0) \in S_{\nu,\mu,l},$$
$$\rho(a) \in \partial\Sigma_{\nu,\mu},$$

then $\rho(a) \in \Gamma_{\nu,\mu}$. *Here,* u *is given by Lemma 3.1.2.*

Now we can prove Theorem 3.4.1 from the viewpoint of Brownian motion with drifts.

Proof of Theorem 3.4.1. Consider the elliptic equations with regularized drifts u_ϵ

$$-\Delta v_\epsilon + C(u_\epsilon \cdot \nabla)v_\epsilon = 0 \qquad (3.4.3)$$

with the boundary condition $v_\epsilon|_{\partial B_1(0,\mathbb{R}^n)} = \gamma$. Let $(\Theta, \mathcal{F}, \mathbb{P})$ be a probability space and $\{B_t\}_{t\geq 0}$ be a n-dimensional Brownian motion. For any $x \in B_1(0, \mathbb{R}^n)$, let $\{X_t^{x,\epsilon}\}_{t \geq 0}$ be the stochastic process given by the stochastic differential equation

$$dX_t^{x,\epsilon} = -Cu_\epsilon(X_t^{x,\epsilon})dt + dB_t, \tag{3.4.4}$$

$$X_0^{x,\epsilon} = x. \tag{3.4.5}$$

The solution of the stochastic differential equation is given by

$$X_t^{x,\epsilon} = X_0^{x,\epsilon} - \int_0^t Cu_\epsilon(X_s^{x,\epsilon})ds + B_t.$$

Define two stopping time τ and ϑ by

$$\tau := \inf\{t \geq 0 \,|\, X_t^{x,\epsilon} \in \partial B_1(0, \mathbb{R}^n)\},$$
$$\vartheta := \inf\{t \geq 0 \,|\, X_t^{x,\epsilon} \in \partial B_1(0, \mathbb{R}^n), X_t^{x,\epsilon} \cdot \mathbf{e}_2 \leq 0\}.$$

Note that $\mathbb{P}(\tau < \infty) = 1$. By Theorem 9.2.13 of [39], we know the solution v_ϵ is given by

$$v_\epsilon(x) = \mathbb{E}[\gamma(X_\tau^{x,\epsilon})] = \int_A \gamma(X_\tau^{x,\epsilon})d\mathbb{P} + \int_{A^c} \gamma(X_\tau^{x,\epsilon})d\mathbb{P}, \tag{3.4.6}$$

where

$$A := \{\omega \in \Theta \,|\, X_t^{x,\epsilon}(\omega) \cdot \mathbf{e}_2 > 0 \text{ for any } t \in [0, \tau(\omega)]\}.$$

If $x_2 > 0$, let $\tilde{x} := X_\vartheta^{x,\epsilon}$. For any $\omega \in A^c$, $\vartheta(\omega) < \infty$ and $\tilde{x}_2 = 0$. We can deduce

$$\int_{A^c} \gamma(X_\tau^{x,\epsilon})d\mathbb{P} = \int_{A^c} \gamma(X_{\tau-\vartheta}^{\tilde{x},\epsilon})d\mathbb{P} = 0, \tag{3.4.7}$$

because the symmetry of u_ϵ with respect to $x_2 = 0$ and the oddness of γ in x_2. Here we also use that Brownian motion has independent increments.

Fix a small number $\nu \in (0, \nu_0)$, where ν_0 is given in Lemma 3.4.2. For any $y = (y_1, y_2, \dots, y_n) \in B_1(0, \mathbb{R}^n)$ with $y_2 > 0$, we introduce more convenient notations for the sets defined in Lemma 3.4.2

$$\begin{aligned}\Sigma_y &:= \Sigma_{\nu y_2^{\alpha/4}, y_2},\\ \Gamma_y &:= \Gamma_{\nu y_2^{\alpha/4}, y_2},\\ S_y &:= S_{\nu y_2^{\alpha/4}, y_2, 2^{-\alpha/4}}.\end{aligned}$$

Note that $\Gamma_y = S_{2y}$.

Claim 1: There exist absolute constants C, $C_1 > 0$ with the following property. For any $y \in B_1(0, \mathbb{R}^n)$ with $y_2 \in [2\epsilon, 1]$ and $y \in S_y$, the solution $X_t^{y,\epsilon}$ of (3.4.4) with drift Cu_ϵ satisfies

$$\mathbb{P}(X_\sigma^{y,\epsilon} \in \Gamma_y \cup \partial B_1(0, \mathbb{R}^n)) \geq 1 - \frac{4n}{\sqrt{2\pi}} \exp(-C_1 y_2^{-\alpha/2}) \geq \frac{1}{2}, \quad (3.4.8)$$

where the stopping time σ is defined by

$$\sigma := \inf\{t \geq 0 \,|\, X_t^{y,\epsilon} \in \partial(B_1(0, \mathbb{R}^n) \cap \Sigma_y)\}.$$

Since τ and σ are almost surely finite, we only need to look at those occurences for which τ and σ are finite. With the help of Claim 1, we can estimate the value $v_\epsilon(p)$ for a fixed point $p \in B_1(0, \mathbb{R}^n)$ with $p_2 \geq 2\epsilon$ and $p_i = 0$ for $i \neq 2$ from (3.4.6). It suffices to estimate $\mathbb{E}[\gamma(X_\tau^{p,\epsilon})]$. Since the Brownian motion trajectory is almost surely continuous and u_ϵ is bounded, the function $t \to X_t^{p,\epsilon}$ is almost surely continuous. Define $F(\omega) := \{X_s^{p,\epsilon}(\omega), s \in [0, \tau(\omega)]\}$. Consider the sets $\Sigma_{2^k p}$ for $1 \leq k \leq k_0$, where $k_0 := \lfloor -\log_2(p_2) \rfloor$. Let

$$D := \left\{\omega \in \Theta \,\middle|\, \forall\, k \text{ with } S_{2^k p} \cap F(\omega) \neq \emptyset, t \to X_t^{p,\epsilon} \text{ exits } \Sigma_{2^k p} \text{ from } \Gamma_{2^k p}\right\}.$$

By Claim 1, we have

$$\begin{aligned}
\mathbb{P}(D) &\geq \prod_{k=1}^{k_0}\Big(1-\frac{4n}{\sqrt{2\pi}}\exp\big(-C_1(p_2 2^k)^{-\alpha/2}\big)\Big)\\
&\geq \prod_{k=1-k_0}^{0}\Big(1-\frac{4n}{\sqrt{2\pi}}\exp\big(-C_1 2^{-k\alpha/2}\big)\Big)\\
&\geq \prod_{k=0}^{\infty}\Big(1-\frac{4n}{\sqrt{2\pi}}\exp\big(-C_1 2^{k\alpha/2}\big)\Big)\\
&\geq \kappa > 0.
\end{aligned}$$

Here, $\kappa > 0$ is independent of $\epsilon > 0$ and $p_2 \in [2\epsilon, 1]$.

Since $\Gamma_{2^k p} = S_{2^{k+1}p}$ and $p \in S_p$, for any $\omega \in D$, $X_t^{p,\epsilon}(\omega)$ is a trajectory which exits $B_1(0,\mathbb{R}^n)$ from the part $\{x \in \partial B_1(0,\mathbb{R}^n) \,|\, x_2 > \frac{1}{2}\}$, i.e. for all $\omega \in D$, we have $X^{p,\epsilon}_{\tau(\omega)}(\omega)\cdot \mathbf{e}_2 > \frac{1}{2}$. From (3.4.6) and (3.4.7),

$$v_\epsilon(p) = \mathbb{E}[\gamma(X_\tau^{p,\epsilon})] = \int_A \gamma(X_\tau^{p,\epsilon})d\mathbb{P} \geq \int_D \gamma(X_\tau^{p,\epsilon})d\mathbb{P} \geq \kappa.$$

Here, we have the following picture: For those occurences $\omega \in D$, $\gamma(X^{p,\epsilon}_{\tau(\omega)}(\omega)) = 1$. For $\omega \in A$, $\gamma(X^{p,\epsilon}_{\tau(\omega)}(\omega)) \geq 0$. Both follow from the definition of γ. Finally, we conclude that $v_\epsilon(p) \geq \kappa$ for any $p \in B_1(0,\mathbb{R}^n)$ with $p_2 \in [2\epsilon, 1]$ and $p_i = 0, i \neq 2$.

The fact that $v_\epsilon(x) \geq 0$ for $x \in B_1(0,\mathbb{R}^n)$ with $x_2 \geq 0$ follows from maximum principle and symmetry of γ.

Proof of Claim 1. For 1-dimensional Brownian motion B_t^1, by reflection principle, we have for any $a > 0$ and any $t > 0$

$$\mathbb{P}\Big(\sup_{0\leq s\leq t} B_s^1 \geq a\Big) = 2\mathbb{P}\Big(B_t^1 \geq a\Big),$$

then

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s\le t}|B_s^1|\ge a\Big)\le&\mathbb{P}\Big(\sup_{0\le s\le t}B_s^1\ge a\Big)+\mathbb{P}\Big(\inf_{0\le s\le t}B_s^1\le -a\Big)\\ =&4\mathbb{P}\Big(B_t^1\ge a\Big)=\frac{4}{\sqrt{2\pi t}}\int_a^\infty \exp\Big(-\frac{s^2}{2t}\Big)ds.\end{aligned}$$

For n-dimensional Brownian motion B_t and $(nt)^{-\frac{1}{2}}a \ge 1$,

$$\begin{aligned}\mathbb{P}\Big(\sup_{0\le s\le t}|B_s|\ge a\Big)\le&\sum_{i=1}^n\mathbb{P}\Big(\sup_{0\le s\le t}|B_s^i|\ge \frac{a}{\sqrt{n}}\Big)\\ \le&\frac{4n}{\sqrt{2\pi t}}\int_{n^{-1/2}a}^\infty \exp\Big(-\frac{s^2}{2t}\Big)ds\\ =&\frac{4n}{\sqrt{2\pi}}\int_{(nt)^{-1/2}a}^\infty \exp\Big(-\frac{\zeta^2}{2}\Big)d\zeta\\ \le&\frac{4n}{\sqrt{2\pi}}\exp\Big(-\frac{a^2}{2nt}\Big).\end{aligned}$$

Hence for any $t > 0$ and any $a > 0$,

$$\mathbb{P}\Big(\sup_{0\le s\le t}|B_s|\le a\Big)\ge 1-\frac{4n}{\sqrt{2\pi}}\exp\Big(-\frac{a^2}{2nt}\Big).$$

Consider the case $\Sigma_y \subset B_1(0,\mathbb{R}^n)$. For those trajectories $B_t(\omega)$ with

$$\sup_{0\le s\le y_2^{2+\alpha}}|B_s(\omega)|\le \frac{(1-2^{-\alpha/4})\nu y_2^{\alpha/4+1}}{8}, \tag{3.4.9}$$

we can choose $C > 0$ big enough such that $t \to X_t^{y,\epsilon}(\omega)$ must exit Σ_y at some time $\sigma(\omega) \in (0, C_0^{-1}y_2^{2+\alpha})$ with $C_0 > 0$ to be determined, because $u_\epsilon \cdot \mathbf{e}_2 \sim y_2^{-(1+\alpha)}$ in Σ_y, from the definition of the velocity field Cu_ϵ. Since $y \in S_y$ and $u = u_\epsilon$ in Σ_y, we can apply Lemma 3.4.2 and we can deduce $X^{y,\epsilon}_{\sigma(\omega)}(\omega) \in \Gamma_y$. This leads to

$$\mathbb{P}\big(X^{y,\epsilon}_{\sigma} \in \Gamma_y\big) \geq 1 - \frac{4n}{\sqrt{2\pi}} \exp\big(-C_1 y_2^{-\alpha/2}\big), \quad C_1 := \frac{C_0(1-2^{-\alpha/4})^2\nu^2}{32n}.$$

We can choose $C_0 > 0$ such that $\mathbb{P}\big(X^{y,\epsilon}_{\sigma} \in \Gamma_y\big) \geq \frac{1}{2}$ for any $y_2 \in [2\epsilon, 1]$.

For the case where Σ_y is not a subset of $B_1(0,\mathbb{R}^n)$, the trajectory $t \to X^{y,\epsilon}_t(\omega)$ with (3.4.9) must exit $\Sigma_y \cap B_1(0,\mathbb{R}^n)$ no later than the time when it exits Σ_y. The probability of exiting $\Sigma_y \cap B_1(0,\mathbb{R}^n)$ through $\partial(\Sigma_y \cap B_1(0,\mathbb{R}^n)) \backslash (\Gamma_y \cup \partial B_1(0,\mathbb{R}^n))$ is smaller than $\frac{4n}{\sqrt{2\pi}} \exp\big(-C_1 y_2^{-\alpha/2}\big)$. This concludes the proof of Claim 1. □

□

Finally, we prove Lemma 3.4.2.

Proof of Lemma 3.4.2. For any $i \in \{1,3,4,\ldots,n\}$, define

$$\begin{aligned}
\Gamma^{i+}_{\nu,\mu} &:= \big\{(x_1,x_2,\ldots,x_n) \in \Sigma_{\nu,\mu} \,\big|\, x_i = \nu x_2\big\},\\
\Gamma^{i-}_{\nu,\mu} &:= \big\{(x_1,x_2,\ldots,x_n) \in \Sigma_{\nu,\mu} \,\big|\, x_i = -\nu x_2\big\},\\
\Gamma'_{\nu,\mu} &:= \Big\{(x_1,x_2,\ldots,x_n) \in \Sigma_{\nu,\mu} \,\Big|\, x_2 = \frac{\mu}{2}\Big\},
\end{aligned}$$

then

$$\partial\Sigma_{\nu,\mu} = \Gamma_{\nu,\mu} \bigcup \Gamma'_{\nu,\mu} \bigcup \Big(\bigcup_i \Gamma^{i+}_{\nu,\mu}\Big) \bigcup \Big(\bigcup_i \Gamma^{i-}_{\nu,\mu}\Big).$$

We prove $\rho(a) \in \Gamma_{\nu,\mu}$ by contradiction. Suppose $\rho(a) \in \Gamma^{i+}_{\nu,\mu}$ for some $i \in \{1,3,4,\ldots,n\}$ and we make the following claim.

Claim 1: For any $x \in \Sigma_{\nu,\mu}$ with $x_i \geq 0$, we have

$$u(x) \cdot n_{i+} \geq 0, \tag{3.4.10}$$

where n_{i+} is the outer normal vector of $\Sigma_{\nu,\mu}$ on the boundary component $\Gamma^{i+}_{\nu,\mu}$.

Because $\rho(a) \in \Gamma^{i+}_{\nu,\mu}$ and $\rho(0) \in S_{\nu,\mu,l}$, there exists $t_0 \in (0,a)$, such that

$$\begin{aligned} n_{i+} \cdot (\rho(a) - \rho(t_0)) \geq \frac{(1-l)\nu\mu}{3}, \\ \rho(t) \cdot \mathbf{e}_i \geq 0 \quad \text{for any } t \in [t_0, a]. \end{aligned} \tag{3.4.11}$$

We multiply (3.4.1) with n_{i+}

$$n_{i+} \cdot (\rho(a) - \rho(t_0)) = -C \int_{t_0}^{a} n_{i+} \cdot u(\rho(s)) ds + n_{i+} \cdot (b(a) - b(t_0)),$$

then by (3.4.11), we can deduce

$$\frac{(1-l)\nu\mu}{3} + C \int_{t_0}^{a} n_{i+} \cdot u(\rho(s)) ds \leq \frac{(1-l)\nu\mu}{4},$$

which contradicts (3.4.10) and $\rho(t) \cdot \mathbf{e}_i \geq 0$ for any $t \in [t_0, a]$.

The possibility $\rho(a) \in \Gamma^{i-}_{\nu,\mu}$ or $\rho(a) \in \Gamma'_{\nu,\mu}$ can be excluded similarly.

Now it suffices to prove Claim 1.

Proof of Claim 1. The outer normal vector on the boundary component $\Gamma^{i+}_{\nu,\mu}$ is given by

$$n_{i+} = \frac{1}{\sqrt{1+\nu^2}} \mathbf{e}_i + \frac{-\nu}{\sqrt{1+\nu^2}} \mathbf{e}_2$$

and the velocity components $u(x) \cdot \mathbf{e}_1$ and $u(x) \cdot \mathbf{e}_j, j \geq 2$ are given by

$$\begin{aligned} u(x) \cdot \mathbf{e}_1 &= \frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}}, \\ u(x) \cdot \mathbf{e}_j &= \Big(-\frac{(r+z)^{n-3-\alpha}}{2r^{n-2}} - \frac{(r-z)^{n-3-\alpha}}{2r^{n-2}} \Big) \cdot \frac{x_j}{r}, \quad 2 \leq j \leq n. \end{aligned}$$

If $i = 1$, for any $x \in \Sigma_{\nu,\mu}$ with $x_1 > 0$, we know $\cos\theta \in \big[(1+\nu^2(n-$

$2))^{-\frac{1}{2}}, 1\big]$. Let $\zeta = \frac{z}{r} \in [0, \nu]$, then (3.4.10) is equivalent to

$$(1 + \nu\cos\theta)(1+\zeta)^{n-3-\alpha} - (1 - \nu\cos\theta)(1-\zeta)^{n-3-\alpha} \geq 0, \text{ for } \zeta \in [0, \nu]. \tag{3.4.12}$$

By elementary analysis, there exist $\alpha_0, \nu_0 > 0$ such that (3.4.12) and hence (3.4.10) hold for any $\nu \in (0, \nu_0]$ and $\alpha \in (0, \alpha_0]$. Indeed, we just need to look at the following function β in ζ, ν with parameter $\xi \in \big[(1 + \nu^2(n-2))^{-\frac{1}{2}}, 1\big]$,

$$\beta(\zeta, \nu) := (1 + \xi\nu)(1 + \zeta)^{n-3-\alpha} - (1 - \xi\nu)(1 - \zeta)^{n-3-\alpha}.$$

For $n \geq 4$, we have $\partial_\zeta \beta \geq 0$ and $\beta(0, \nu) \geq 0$, so (3.4.12) holds. For $n = 3$, $\partial_\zeta \beta \leq 0$, it suffices to prove $\beta(\nu, \nu) \geq 0$. Now we look at the function $(1 + \xi\nu)(1 + \nu)^{-\alpha} - (1 - \xi\nu)(1 - \nu)^{-\alpha}$, its derivative in ν is positive for small $\nu > 0$, so $\beta(\nu, \nu) \geq 0$ and thus (3.4.12) holds.

If $i \geq 3$, this is straightforward since (3.4.10) is equivalent to $x_i \leq \nu x_2$.

□

□

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