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Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow

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Abstract. We consider the Landau-Lifshitz flow on a bounded planar domain. An ε-regularity type a-priori estimate provides the analytic tool for the subsequent geometric description of the flow at isolated singularities. At forward isolated singularities where the energy is not left continuous the flow concentrates energy and develops bubbles. As in J.Qing's bubbling-energy-equality for the harmonic map flow, the energy loss at such a singularity can be recovered as a finite sum of energies of tangent bubbles. We then clarify a known uniqueness result for the Landau-Lifshitz flow and show how non-uniqueness of extensions of the flow after point singularities is related to backward bubbling. Finally the ε-regularity estimate also yields a partial compactness result for sequences of smooth solutions to the Landau-Lifshitz flow with uniformly bounded energy, defined on a planar domain.

Mathematics Subject Classification (2000): 58E20, 35B65, 35B60, 35K55

1. Introduction

The Landau-Lifshitz flow $u : \overline{\Omega} \times \mathbb{R}_+ \to S^2$ is defined by

$$\partial_t u = -\alpha\, u \times (u \times \triangle u) + \beta\, u \times \triangle u \;\text{ in } \Omega \times \mathbb{R}_+\,, \tag{1}$$

$$u = u_0 \text{ on } (\Omega \times \{0\}) \cup (\partial\Omega \times \mathbb{R}_+)\,, \tag{2}$$

where $\alpha > 0, \beta \in \mathbb{R}$ and where "$\times$" denotes the usual vector product in $\mathbb{R}^3$. Here $\Omega \subset \mathbb{R}^2$ denotes a smooth bounded domain and $S^2 \subset \mathbb{R}^3$ is the standard sphere.

In physics these equations describe an isotropic Heisenberg spin chain phenomenon in non-equilibrium magnetism (see [22]). The map u describes the spin density, $\alpha > 0$ is the Gilbert damping constant and $\beta \in \mathbb{R}$ is an exchange constant.

By using that $|u| \equiv 1$, from the identity $a \times (b \times c) = (a \cdot c)\, b - (a \cdot b)\, c$ for $a, b, c \in \mathbb{R}^3$ it is easy to see that for sufficiently regular solutions equation (1) is equivalent to

$$\gamma_1\, \partial_t u - \gamma_2\, u \times \partial_t u = \triangle u + |\nabla u|^2 u \quad \text{in } \Omega \times \mathbb{R}_+\,, \tag{3}$$

where $\gamma_1 := \frac{\alpha}{\alpha^2+\beta^2} > 0$ and $\gamma_2 := \frac{\beta}{\alpha^2+\beta^2} \in \mathbb{R}$. (See [17].) By scaling the time variable, we may assume $\gamma_1 = 1$ and $\gamma := \gamma_2 \in \mathbb{R}$.

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For $\beta = 0$, or equivalently $\gamma = 0$, equation (3) is a harmonic map flow into S^2 with time scaled by $\alpha > 0$. (Compare [17].)
All our results also hold for the harmonic map flow with a general Riemannian manifold N as target. This flow is given by

$$\partial_t u - \triangle u = A(u)\big(\nabla u, \nabla u\big) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{4}$$
$$u = u_0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial\Omega \times \mathbb{R}_+)\,, \tag{5}$$

where $A(u)\big(\nabla u, \nabla u\big) = \sum_{i=1}^{2} A(u)\big(\partial_i u, \partial_i u\big)$ and A denotes the second fundamental form of $N \hookrightarrow \mathbb{R}^n$. If $N = S^2 \hookrightarrow \mathbb{R}^3$ is the standard sphere, then $A(u)\big(\nabla u, \nabla u\big) = |\nabla u|^2\, u$ and we recover (3) for $\gamma_1 = 1$ and $\gamma_2 = 0$.

Standard methods imply the existence of a smooth "short-time" solution

$$u \in C^\infty(\Omega \times]0, T[; N)$$

to (2)-(3) for initial and boundary data $u_0 \in H^{1,2}(\Omega, N)$ and for sufficiently small $T = T(u_0, N) > 0$ (See [18,31,32,17,19]).

At the maximal existence time there are at most finitely many point singularities (compare Section 3, but also [31] and [18] for the harmonic map flow and [17] or [19] for the Landau-Lifshitz flow). By iterating the above short time existence result, a short time smooth solution can be extended to a global weak solution which is smooth except for finitely many point singularities and has decreasing energy

$$t \mapsto E\big(u(t)\big) := \frac{1}{2}\int_\Omega \big|\nabla u\big|^2(x,t)\, dx\,. \tag{6}$$

We will refer to this extension as the Struwe-solution. It was first constructed by M.Struwe in [31] for the harmonic map flow on Riemann surfaces. The construction was generalized to two dimensional domain manifolds with boundary by K.C.Chang in [2] and to the Landau-Lifshitz flow by B.Guo and M.C.Hong in [17]. See also [19] for the case with boundary.

Like the harmonic map flow (see [31] and [28]), at isolated singularities, where the energy is not left continuous, the Landau-Lifshitz flow splits off "bubbles", i.e. non-constant harmonic maps $\varphi : S^2 \to S^2$, which account for the loss of energy at the singular time.

The Struwe-solution is unique in the class of solutions that are smooth except for isolated point singularities and with decreasing energy. A.Freire showed in [15] and [16] that it is still unique in the class

$$H^{1,2}_{loc}\big(\overline{\Omega} \times \mathbb{R}_+; N\big) \cap L^\infty\big(\mathbb{R}_+; H^{1,2}(\Omega; N)\big)$$

with initial and boundary data

$$u_0 \in H^{1,2}(\Omega; N) \cap H^{3/2,2}(\partial\Omega; N)\,,$$

if the energy (6) is decreasing.

This uniqueness result was extended by Y. Chen and by B. Guo and S. Ding in [10,9,14,5] to the Landau-Lifshitz flow, but the essential assumption that the energy should be decreasing is erronously omitted. Note that neither $C^\infty(M\times[0,T],\mathbb{R}^3)$ nor $L^\infty([0,T];C^1(M,\mathbb{R}^3))$ are dense in $L^\infty([0,T],H^{1,2}(M,S^2))$, with the usual norm

$$\|u\|_{L^\infty(H^{1,2})} := \sup_{t\in[0,T]} \|u(.,t)\|_{H^{1,2}(\Omega)}\,.$$

For $C^\infty(M\times[0,T],\mathbb{R}^3)$ this is obvious and for $L^\infty([0,T];C^1(M,\mathbb{R}^3))$ we will give a counterexample in the appendix. These density statements are however used in [6,14,9] (see [6] (3.31)–(3.33) p.118; [14] p.150; [9], (3.6)–(3.7) p.429). If the energy is assumed to be decreasing, the proof can however be restored by the same argument as in [16] p.331.

If the energy is not assumed to be decreasing, then weak $H^{1,2}_{loc}\cap L^\infty(H^{1,2})$-solutions can no longer be expected to be unique. In Section 5 we show how non-uniqueness of extensions of solutions after point sigularities is related to "backward bubbling". Recently explicit examples of non-uniqueness and backward bubbling have been constructed for the case of the harmonic map flow in two space dimensions by M.Bertsch, R.Dal Passo and R.Van der Hout in [1] and independently by P.Topping in [36].

2. The energy estimates

While the harmonic map flow (4) can be interpreted as the L^2-gradient flow of the energy functional $u\mapsto E(u)$ on $H^{1,2}(\Omega;N)$ with fixed boundary data u_0, the Landau-Lifshitz flow does not appear to be a gradient flow. The energy is however still decreasing along a regular Landau-Lifshitz flow. Let

$$E\big(u(t),B^\Omega_R(x)\big) := \frac{1}{2}\int_{B_R(x)\cap\Omega} |\nabla u|^2(y,t)\,dy$$

be the local energy. Then we have the following straightforward but fundamental estimates.

Lemma 1. *Let $u\in C^2(\Omega\times[0,T[;S^2)$ be a solution of (2)-(3). Then we have the energy equality*

$$\text{(7)}\qquad \frac{d}{dt}E\big(u(t)\big) = -\gamma_1\int_\Omega |\partial_t u|^2\,dx \quad \textit{for}\ \ 0\le t<T\,,$$

and the local energy estimate

(8)

$$E\big(u(t_2),B^\Omega_R(x_0)\big) \le E\big(u(t_1),B^\Omega_{2R}(x_0)\big) + \frac{C}{\gamma_1 R^2}\int_{t_1}^{t_2} E\big(u(t),B^\Omega_{2R}(x_0)\big)\,dt,$$

for $0\le t_1\le t_2<T$ and with $B^\Omega_R(x) := B_R(x)\cap\Omega$.

Proof. (7) is obtained by multiplying (3) with $\partial_t u$ and integrating by parts. (8) follows from (3) by multiplying with $\partial_t u\,\varphi^2$, where φ is a standard cut-off function, and then integrating by parts and absorbing the $\partial_t u$-term, similar to [31,32].
Note that $\partial_t u\equiv 0$ on $\partial\Omega\times\mathbb{R}_+$; thus we may still integrate by parts and the usual proofs are still valid in the case $\partial\Omega\neq\emptyset$. (Compare [31,17] for the case $\partial\Omega=\emptyset$.) □

3. An a priori estimate, higher regularity and extensions

We start with higher regularity. Set

$$P_R(z) := B_R(x) \times]t - R^2, t[\text{ and } P_R^{\Omega}(z) := \big(B_R(x) \cap \Omega\big) \times]t - R^2, t[\,.$$

The following Proposition says that any $L^\infty(H^{1,2})$-solution of (2)-(3) with bounded gradient is actually smooth.

Proposition 2. *Consider* $u_0 \in H^{1,2}(\Omega; N) \cap C^{k+\lambda}(\overline{\Omega}; N)$ *for* $k \geq 2$ *and* $\lambda \in]0, 1[$. *Let* $u \in L^\infty([0, T], H^{1,2}(\Omega; N))$ *be a solution of (2)-(3) (or (4)-(5)). Assume*

$$\sup_{P_R^{\Omega}(z_0)} |\nabla u| \leq C_0$$

for $z_0 \in \overline{\Omega} \times]0, T]$ *and some fixed* $R \in]0, \sqrt{T}[$. *Then for any* $0 < \delta < 1$, *we have*

$$u \in C^{k+\lambda,(k+\lambda)/2}\left(\overline{P_{\delta R}^{\Omega}}(z_0)\right) \cap C^\infty\left(P_R^{\Omega}(z_0)\right),$$

with bounds depending only on δ, R, C_0, *the parameters* γ_1, γ_2 *of equation (3), the curvature of* $\partial\Omega$ *and the geometry of the target (i.e. the metric of the target and its covariant derivatives).*

Proof. The proof is a standard bootstrap argument applied to the system (3) with initial and boundary data (2). This system is strongly parabolic. Indeed, it may be written as

$$\partial_t u - M(u) \triangle u = |\nabla u|^2 M(u)\, u\,, \tag{9}$$

where the matrix valued function

$$u \mapsto M(u) = \left(\gamma_1 I - \gamma_2 \begin{pmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{pmatrix}\right)^{-1}$$

is continuous and bounded and $M(u)\triangle u$ is strictly elliptic. More precisely

$$\alpha\, |\xi|^2 < \xi^T M(u)\, \xi < \frac{1}{\gamma_1} |\xi|^2 \quad \forall \xi \in \mathbb{R}^2\,.$$

(See [13] p.12, and [19]. Compare also [21] section VII.8 for definitions and results on parabolic systems.) Theorem 10.4 in Section VII.10 of [21] (see also Theorem 9.1 in Section IV.9 of [21]) then provides Sobolev estimates with constants that still depend on the modulus of continuity of the coefficients and thus of u. The assumption $\sup_{P_R^{\Omega}(z_0)} |\nabla u| \leq C_0$ however does not include time derivatives. To obtain bounds on the modulus of continuity with respect to time, observe that equation (9) may be written in divergence form with bounded coefficients, i.e. the map $v := u$ solves

$$\partial_t v - \operatorname{div}\big(M(u)\, \nabla v\big) + \big(DM(u) \partial_k u\big)\, \partial_k v \;=\; |\nabla u|^2\, u\,.$$

Then, still under the assumption

$$\sup_{P_R^{\Omega}(z_0)} |\nabla u| \le C_0$$

Theorem 3.1 in Section VII.3 or Theorem 1.1 in Section V.1 of [21] yield Hölder-estimates for $v = u$, that provide adequate bounds for the modulus of continuity of u in the previous Sobolev-estimates. Upon differentiating (3), we obtain an equation of the same type. The above Sobolev estimates may then be iterated to obtain smoothness. □

By using scaling properties of the equation, we may now derive an a-priori sup-estimate for ∇u from the previous higher estimates.
$C^{2,\lambda}(\partial\Omega; N)$ will denote the space of maps $v : \partial\Omega \to N$ that admit an extension to $C^{2,\lambda}(\Omega; N)$.

Theorem 3. *Consider* $u_0 \in H^{1,2}(\Omega; N) \cap C^{2,\lambda}(\partial\Omega; N)$ *for* $0 < \lambda < 1$. *Let* $u \in W_2^{2,1}(\Omega \times [0, T]; N)$ *be a solution of (2)-(3). Then there are constants*

$$\varepsilon_0 = \varepsilon_0(\Omega, \|u_0\|_{C^2(\partial\Omega)}) > 0 \text{ and } C_0 = C_0(\Omega, \|u_0\|_{C^2(\partial\Omega)}) > 0$$

such that, if for $z_0 = (x_0, t_0) \in \overline{\Omega}\times]0, T]$ *and* $0 < R_0 < \min\{1, \sqrt{t_0}\}$ *there holds*

$$\nabla u \text{ is continuous on } \overline{P_{R_0}^{\Omega}(z_0)} \text{ and } \sup_{t_0 - R_0^2 < t < t_0} E\big(u(t), B_{R_0}^{\Omega}(x_0)\big) < \varepsilon_0\,,$$

then

$$\sup_{P_{\delta R_0}^{\Omega}(z_0)} |\nabla u| \le \frac{C_0}{(1-\delta)\, R_0}\,,$$

for any $\delta \in]0, 1[$. *If* $\gamma_2 = 0$ *and the target is a manifold* N, *the constants* C_0 *and* ε_0 *also depend on the geometry of* N.

Proof. Without loss of generality $(x_0, t_0) = 0$. We set $P_R := P_R^{\Omega}(0)$ and

$$e(u) := \frac{1}{2}|\nabla u|^2\,.$$

We would like to consider $z_1 = \arg\sup_{P_{R_0}} e(u)$. Difficulties however arise if $z_1 \in \partial P_R$. This is elegantly avoided by considering (10) below, as in Schoen [30], proof of Theorem 2.2. (Schoen's method was extended to the parabolic context in [33, 34].)
Since ∇u is continuous, there is $\sigma_0 \in [0, R_0[$ such that

$$(R_0 - \sigma_0)^2 \sup_{P_{\sigma_0}} e(u) = \max_{0 \le \sigma \le R_0} \Big((R_0 - \sigma)^2 \sup_{P_\sigma} e(u) \Big). \tag{10}$$

Moreover there is $z_* = (x_*, t_*) \in \overline{P_{\sigma_0}}$ such that

$$e_0 := e(u(z_*)) = \sup_{P_{\sigma_0}} e(u)\,.$$

Set $\rho_0 := \frac{1}{2}(R_0 - \sigma_0)$. Since $P_{\rho_0}(z_*) \subset P_{\sigma_0+\rho_0} \subset P_{R_0}$, we have

$$\sup_{P_{\rho_0}(z_*)} e(u) \le \frac{1}{(R_0 - (\sigma_0 + \rho_0))^2} (R_0 - (\sigma_0 + \rho_0))^2 \sup_{P_{\rho_0+\sigma_0}} e(u)$$

$$\le \frac{4}{(R_0 - \sigma_0)^2} (R_0 - \sigma_0)^2 e_0 \le 4e_0 \,.$$

Let $r_0 := \sqrt{e_0}\rho_0$ and consider the rescaled map

$$v(y,s) := u(x_* + e_0^{-1/2}\, y, t_* + {e_0}^{-1} s) \quad \text{ for } (y,s) \in P_{r_0}^{\Omega^*} \,,$$

where $\Omega^* := e_0^{1/2}(\Omega - x_*)$ and $P_{r_0}^{\Omega^*} := P_{r_0} \cap (\Omega^* \times \mathbb{R})$. By scaling invariance v satisfies (3) on $P_{r_0}^* := P_{r_0}^{\Omega^*}$ with boundary data

$$v(y,s) = v_0(y) := u_0(x_* + e_0^{-1/2}\, y) \quad \text{on } P_{r_0} \cap \partial\Omega^* \,.$$

Moreover, by construction,

$$e(v)(0,0) = 1 \,, \quad \sup_{P_{r_0}^*} e(v) \le 4 \tag{11}$$

and also

$$e(v_0) = \frac{1}{2} |\nabla v_0|^2 \le 4 \,.$$

We may choose coordinates on the target such that $v_0(0) = 0$ and then $\sup_{P_{r_0}^*} |v_0| \le 4\, r_0$.

Now we claim $r_0 \le 2$. This will prove the theorem, since by definition of r_0, we then have $(R_0 - \sigma_0)^2\, e_0 \le 16$.

Assume $r_0 > 2$. Then $e_0 = \frac{r_0^2}{\rho_0^2} \ge 4\frac{2}{R_0^2} > 8$, since $0 < R_0 < 1$.

Since $r_0 > 2$, by (11) and Proposition 2 with $\delta = \frac{1}{2}$, all higher derivatives of v are bounded on P_1^*. In particular for a constant C depending only on $\alpha > 0$, the curvature of $\partial\Omega^*$ and possibly the geometry of N, there holds,

$$\sqrt{|\partial_t e(v)|},\ |\nabla e(v)| \le C < \infty \text{ on } P_1^* \,.$$

Therefore

$$\inf_{P_{r_1}^*} e(v) \ge \frac{1}{2} \quad \text{ for } r_1 := \min\{\frac{1}{4C}, 1\} \,. \tag{12}$$

Since $e_0 > 8$ by assumption, the curvature of $\partial\Omega^*$ is bounded in terms of the maximum of the curvature of $\partial\Omega$.

Moreover since Ω is compact, there is $c_0 > 0$, such that

$$|B_r(x) \cap \Omega| \ge c_0\, r^2 \quad \text{for all } x \in \partial\Omega \,,\ 0 < r < 1$$

and also

$$|B_r(y) \cap \Omega^*| \ge c_0\, r^2 \quad \text{for all } y \in \partial\Omega^* \,,\ 0 < r < 1 \,.$$

Set $C_* = \frac{2}{c_0 r_1^2}$ and $\varepsilon_0 := \min\{\frac{1}{2}, \frac{1}{2C_*}\}$. Since $r_0 = \sqrt{e_0}\rho_0 > 2 > r_1$, we have $\frac{r_1}{\sqrt{e_0}} + \sigma_0 \le \rho_0 + \sigma_0 \le R_0$ and $(\frac{r_1}{\sqrt{e_0}})^2 + \sigma_0^2 \le (\rho_0+\sigma_0)^2 \le R_0^2$. The above lower bound (12) then implies

$$\begin{aligned}1 = e(v)(0,0) &\le \frac{2}{c_0 r_1^2} \sup_{-r_1^2<s<0} \int_{B_{r_1}\cap\Omega^*} e(v)(y,s)\,dy \\ &\le C_* \sup_{t_*-r_1^2 e_0^{-1}<t<t_*} \int_{B_{e_0^{-1/2} r_1}(x_*)\cap\Omega} e(u)(x,t)\,dx \\ &\le C_* \sup_{-(\frac{r_1^2}{e_0}+\sigma_0^2)<t<0} \int_{B_{\frac{r_1}{\sqrt{e_0}}+\sigma_0}} e(u)(x,t)\,dx\,.\end{aligned}$$

The last estimate yields the desired contradiction, since the right hand side is smaller than $C_*\,\varepsilon_0 \le \frac{1}{2}$. □

The regularity assumption for $u_{0|\partial\Omega}$ can (at least) be reduced to $u_0 \in C^2(\partial\Omega; N)$ (compare [19], Theorem 2.7).
By combining the above estimates, we obtain a simple criterion that tells us when a solution in $C^\infty(\overline{\Omega}\times]0,T[;N)$ admits an extension to $C^\infty(\overline{\Omega}\times]0,T];N)$.

Corollary 4. *Consider* $u_0 \in H^{1,2}(\Omega;N) \cap C^{k+\lambda}(\overline{\Omega};N)$ *and let*

$$u \in W_2^{2,1}(\Omega\times]0,T[,N) \cap C^{k+\lambda,(k+\lambda)/2}(\overline{\Omega}\times]0,T[,N)$$

be a solution of (2)-(3) for $k \ge 2$ *and* $\lambda \in]0,1[$. *Let* $\varepsilon_0 > 0$ *be the constant from Theorem 3 and suppose for* $x_0 \in \overline{\Omega}$ *and* $0 < R_0 < \min\{1, \sqrt{T}\}$ *there holds*

$$\sup_{T-R_0^2<t<T} E\big(u(t), B_{R_0}^{\Omega}(x_0)\big) < \varepsilon_0\,.$$

Then u *admits an extension*

$$\begin{aligned}&u \in C^{k+\lambda,(k+\lambda)/2}\big((B_{\delta R_0}(x_0)\cap\overline{\Omega})\times]0,T];\mathbb{R}^3\big)\\ &\cap C^\infty\big((B_{(1/2)R_0}(x_0)\cap\Omega)\times]0,T];\mathbb{R}^3\big)\end{aligned}$$

for any $\delta \in]0,1/2[$.

Proof. By Theorem 3 we have

$$\sup_{P^{\Omega}_{(1/2)\delta R_0}(\tilde{z}_0)} |\nabla u| \le \frac{2C_0}{(1-\delta)\ R_0}\,,$$

for any $\delta \in]0,1[$ and any $\tilde{z}_0 = (x_0,\tilde{t})$ with $\tilde{t} \in]T-1/2R_0^2, T[$, with constant C_0 independent of $\tilde{t}$. By Proposition 2 the corresponding Höldernorms of u are bounded on $(B_{\delta R_0}(x_0)\cap\Omega)\times]T-1/2R_0^2,T[$ for any $\delta \in]0,1/2[$, which proves the claim. □

4. Bubbling at parabolically isolated point singularities

Corollary 4 combined with the local energy estimate in Lemma 1 implies that at the maximal smooth existence time $T_* > 0$ of the Landau-Lifshitz flow (2)-(3) with initial data $u_0 \in H^{1,2}(\Omega, S^2) \cap C^{2,\lambda}(\partial\Omega; S^2)$, there are at most finitely many singular points $(x_k, T_*) \in \overline{\Omega}$, $k = 1, \ldots, K$ over which the flow does not admit a smooth extension. Each singular point (x_k, T_*) is characterized by energy concentration in the sense that for any $R > 0$

$$\limsup_{t \nearrow T_*} E\big(u(t), B_R^\Omega(x_k)\big) \geq \varepsilon_0 \,.$$

(See [31,32].) Note that the above singularities are isolated in the following sense. We call $z_0 = (x_0, t_0) \in \overline{\Omega} \times \mathbb{R}_+$ *parabolically isolated* for u, if there is $R > 0$, such that

$$\overline{P_R^\Omega(z_0)} \setminus \{z_0\} \subset Reg(u)\,,$$

where $Reg(u)$ denotes the set of regular points of u, that is the points $z \in \overline{\Omega} \times \mathbb{R}_+$ for which u is smooth on a neighborhood of z. The complement $Sing(u) := \big(\overline{\Omega} \times \mathbb{R}_+\big) \setminus Reg(u)$ is the set of singular points.

The following is the analogue for the Landau-Lifshitz flow of J.Qing's result in [28] for the harmonic map flow.

Theorem 5. *Assume* $u_0 \in H^{1,2}(\Omega; S^2) \cap C^{2,\lambda}(\partial\Omega; S^2)$ *for* $0 < \lambda < 1$. *Let* $u \in H^{1,2}_{loc}(\Omega \times \mathbb{R}_+; S^2)$ *be a distributional solution of (2)-(3) with* $E\big(u(t)\big) \leq E_0$ *for a.e.* $t \in [0, T]$.
Consider a parabolically isolated point $z_0 = (x_0, t_0) \in \mathbb{S}(u)$, *such that u does not admit any smooth extension to* $\overline{P_R^\Omega(z_0)}$ *for any* $R > 0$.

Then there are $R_0 > 0$ *and finitely many non-constant harmonic maps* $\varphi_l : S^2 \to S^2$ $(l = 1 \ldots K)$, *such that for any* $0 < R \leq R_0$

$$\lim_{t \nearrow t_0} E\big(u(t), B_R^\Omega(x_0)\big) \;=\; E\big(u(t_0), B_R^\Omega(x_0)\big) \;+\; \sum_{l=1}^{K} E(\varphi_l)\,,$$

and $\frac{1}{4\pi}\sum_{l=1}^{K} E(\varphi_l) \in \mathbb{N}$. *Letting* $\omega_l : \mathbb{R}^2 \cup \{\infty\} \to S^2$ *be the pullback to* $\mathbb{R}^2$ *of* $\varphi_l : S^2 \to S^2$ *by stereographic projection and* $\omega_l(\infty)$ *the image under* φ_l *of the north-pole, there are sequences* $t_j \nearrow t_0$, $x_j^l \to x_0$, $0 \leq \lambda_j^l \to 0$ *as* $j \to \infty$ *for* $l = 1, \ldots, K$, *such that*

$$u(x, t_j) - \sum_{l=1}^{K} \Big(\omega_l\big(\frac{x - x_j^l}{\lambda_j^l}\big) - \omega_l(\infty) \Big) \overset{(j\to\infty)}{\to} u(x, t_0)$$

strongly in $H^{1,2}(B_{R_0}^\Omega(x_0), \mathbb{R}^3)$.
If $x_0 \in \partial\Omega$, *then* $\lim_j \frac{\mathrm{dist}(x_j^l, \partial\Omega)}{\lambda_j^l} = \infty$.
Finally for $k \neq l$, $\max\big\{\frac{\lambda_j^k}{\lambda_j^l}, \frac{\lambda_j^l}{\lambda_j^k}, \frac{|x_j^l - x_j^k|}{\lambda_j^k + \lambda_j^l}\big\} \to \infty$ *as* $j \to \infty$.

The proof is almost literally the same as in [28] and we therefore omit it. (For more details see [19].)
The maps φ_l often are referred to as "bubbles", the phenomenon described by Theorem 5 as "bubbling".
The assumption that u should not admit a smooth extension to $\overline{P_R^\Omega(z_0)}$ for any $R > 0$ may be replaced by the condition that the function

$$t \mapsto E\big(u(t); B_R(x_0)\big)$$

is not left-continuous at t_0 for any $R > 0$. Without these assumptions, the case $K = 0$ is not excluded. Indeed, as we will indicate in Theorem 6 and as the examples of M.Bertsch et al. [1] and also of P.Topping [36] demonstrate, it is possible that there are parabolically isolated singularities without "forward" bubbling.

5. Non-Uniqueness and backward bubbling

In the following the map u should be thought of as a smooth continuation after a (first) singularity t_0 of a solution of the Landau-Lifshitz or of the harmonic map flow with point singularities at t_0. By relaxing the initial condition

$$\lim_{t \searrow t_0} u(\,.\,,t) = u(\,.\,,t_0) \text{ in } H^{1,2}(\Omega, N)\,,$$

continuations different from the Struwe-solution can be found, whose energy is not right continuous at time t_0. The following theorem is the analogue of Theorem 5 in this setting. We may shift time so that $t_0 = 0$. Set

$$a_0 := \inf\{E(v) \mid v : S^2 \to N \text{ is harmonic and non-constant}\}\,.$$

If N is compact, we have $a_0 > 0$ (see [29], Theorem 3.3).

Theorem 6. *Let* $u \in C^\infty\big(\Omega\times]0,T];N\big)\cap C^\infty\big((\overline{\Omega}\setminus\{\overline{x}_1,\dots,\overline{x}_K\})\times[0,T];N\big)\cap H^{1,2}\big(\Omega\times]0,T[;N\big)$ *be a solution of (2)-(3) for* $u_0 \in C^\infty\big(\overline{\Omega}\setminus\{\overline{x}_1,\dots,\overline{x}_K\};N\big)\cap H^{1,2}(\Omega;N)$*. Assume*

$$E(u(t)) \,\le\, C_0 \quad \forall t \in [0,T]\,.$$

(i) If $\limsup_{t\searrow 0} E\big(u(t)\big) > E(u_0)$*, then* $\limsup_{t\searrow 0} E\big(u(t)\big) \ge E(u_0) + a_0$*. More precisely for any* $\overline{x} \in \{\overline{x}_1,\dots\overline{x}_K\}$*, there either exists* $R_0 > 0$*, such that*

$$\limsup_{t\searrow 0}\; E\big(u(t), B^\Omega_{R_0}(x)\big) = E(u_0, B^\Omega_{R_0}(x))$$

or

$$\limsup_{t\searrow 0}\; E\big(u(t), B^\Omega_R(\overline{x})\big) \ge E(u_0, B^\Omega_R(\overline{x})) + a_0 \textit{ for all } R > 0\,.$$

In the second case the $\{u(\,.\,,t)\}_{t>0}$ *has a 2-bubble in* C^∞ *at* $\overline{x}$ *as* $t \searrow 0$ *in the sense that for suitable* $a > 0$ *and sequences* $R_j \searrow 0$, $t_j \searrow 0$, $x_j \to \overline{x}$ *as* $j \to \infty$*, there holds*

$$v_j(y,s) := u(x_j + R_j y, t_j + R_j^2 s) \overset{(j\to\infty)}{\to} v \in C^\infty\big(\mathbb{R}^2 \times [0,a];N\big)\,,$$

where v is independent of time. The limit $v : \mathbb{R}^2 \to N$ is a non-constant smooth harmonic map with finite energy and thus extends to a smooth harmonic map $\varphi : S^2 \to N$.
If $\overline{x} \in \partial\Omega$, then $\frac{\mathrm{dist}(x_j,\partial\Omega)}{R_j} \to +\infty$ as $j \to \infty$.

(ii) If $\limsup_{t\searrow 0} E\big(u(t)\big) \le E(u_0)$, then $\lim_{t\searrow 0} E\big(u(t)\big) = E(u_0)$ and u is the Struwe-solution on $\Omega \times [0,T]$.

In the case of the harmonic map flow, there are examples of solutions with non right continuous energy and backward bubbling as in (i). Indeed, M. Bertsch, R. Dal Passo and R. van der Hout have constructed a rotationally symmetric solution u to the 2-dimensional harmonic map flow that concentrates energy and bubbles at a time $t_* > 0$ as $t \searrow t_*$ (see [1]). More precisely for rotationally symmetric initial and boundary data $u_0 : B \to S^2 \subset \mathbb{R}^3$ defined on the unit ball $B = B_1 \subset \mathbb{R}^2$ similar to those considered in [4] by K.C. Chang, W.Y. Ding and R. Ye, they found particular times $t_1 < t_2$, such that for any $\tau > t_2$ there is a solution to the harmonic map flow which is smooth on $(B \times [0,\infty)) \setminus (\{0\} \times \{t_1, \tau\})$, concentrates energy at $(0, t_1)$ as $t \nearrow t_1$, has right continuous energy at t_1 and left continuous energy at τ, but concentrates energy and bubbles backwards as $t \searrow \tau$. In particular they thereby constructed infinitely many solutions to the given initial and boundary value problem and each of them corresponds to the Struwe-solution on $B \times [0, \tau]$.
Independently P. Topping also constructed a solution of the two dimensional harmonic map flow with backwards bubbling. He further sketched the construction of a solution with neither left nor right continuous energy at an isolated singularity (x_*, t_*), where energy concentrates at x_* both for $t \nearrow t_*$ and $t \searrow t_*$ (see [36] Paragraph 5). Note that the energy bound does not "forbid" such singularities to accumulate.
It is not known whether "Bertsch-DalPasso-VanDerHout-type" point singularities with left but not right continuous energy or "Topping-type" singularities with neither left nor right continuous energy, as described above, also arise for the Landau-Lifshitz flow.
We now return to the proof of Theorem 6:

Proof. (ii) follows from A.Freire's uniqueness result (see [15,16]) and its extensions to the Landau-Lifshitz flow (see [10,9,14,5]).
(i) Fix $\overline{x} \in \{\overline{x}_1, \dots, \overline{x}_K\}$. Assume we do not have

$$\lim_{t\searrow t_0} \nabla u(\,.\,,t) = \nabla u_0(\,.\,) \quad \text{in } L^2(B_R(\overline{x}); \mathbb{R}^{2n})$$

for any $R > 0$. By the Vitali Convergence Theorem there is $\varepsilon_1 = \varepsilon_1(\overline{x}) > 0$ such that for all $R > 0$ there holds

$$\lim_{\delta\searrow 0} \sup_{0<t<\delta} E\big(u(t), B_R^\Omega(\overline{x})\big) \ge \varepsilon_1 > 0\,. \tag{13}$$

Since u is smooth on $\overline{\Omega} \times [\delta, T]$ for any $0 < \delta < T$, we have

$$\lim_{R\to 0} \sup_{x\in\Omega, t\in[\delta,T]} E\big(u(t), B_R^\Omega(x)\big) = 0\,. \tag{14}$$

Choose $R_0 > 0$ such that $B_{4R_0}(\overline{x}) \cap \{\overline{x}_1, \dots, \overline{x}_K\} = \{\overline{x}\}$. We will write B_R for $B_R(\overline{x})$.
By (13) and (14) there are sequences $0 < t_n \to 0$ and $0 < R_n \to 0$ as $n \to \infty$, such that

$$\sup_{x \in B_{2R_0}} E\big(u(t_n), B^{\Omega}_{2R_n}(x)\big) = \frac{1}{4} \min\{\varepsilon_1, \varepsilon_0\}, \tag{15}$$

where ε_0 is the constant from Theorem 3. We may assume $R_0 > R_n$ for all n and also $\varepsilon_1 \le \varepsilon_0$. By the local energy inequality for u, for any x we have

$$E\big(u(t), B^{\Omega}_{R_n}(x)\big) \le E\big(u(t_n), B^{\Omega}_{2R_n}(x)\big) + c\frac{t - t_n}{R_n^2} C_0 \quad \text{for } t > t_n\,. \tag{16}$$

For $t \in [t_n, t_n + \delta_n]$ and $\delta_n := \frac{\varepsilon_1 R_n^2}{4cC_0}$, this leads to

$$\sup_{x \in B^{\Omega}_{2R_0},\ t_n \le t \le t_n + \delta_n} E\big(u(t), B^{\Omega}_{R_n}(x)\big) \le \frac{\varepsilon_1}{2}\,. \tag{17}$$

By (15) there is a sequence $(x_n)_n \subset \overline{B_{2R_0}}$, such that

$$\frac{\varepsilon_1}{4} = E\big(u(t_n), B^{\Omega}_{2R_n}(x_n)\big)\,. \tag{18}$$

By compactness we may assume it converges. We now claim that the limit is $\overline{x}$. Indeed if $x_n \to x_* \neq \overline{x}$, then $B_{2R_n}(x_n) \subset \overline{B_{4R_0}} \setminus B_r(\overline{x})$ for a $r \in]0, R_0[$ and sufficiently large n. Therefore

$$E\big(u(t_n), B^{\Omega}_{2R_n}(x_n)\big) \le 2R_n^2 \sup_{(B_{4R_0} \setminus B_r(\overline{x})) \times [0,T]} |\nabla u|^2 \to 0$$

as $n \to \infty$, since u is assumed to be smooth away from its point singularities at $t = 0$. This is in contradiction with (18). Thus $x_n \to \overline{x}$ and may assume $B_{R_n}(x_n) \subset B^{\Omega}_{R_0} = B^{\Omega}_{R_0}(\overline{x})$ for all n.
Consider now the rescaled maps

$$v_n(y, s) := u(x_n + R_n y, t_n + R_n^2 s)$$

for

$$(y, s) \in \big(B_{R_0/R_n}(0) \cap \Omega_n\big) \times [0, (T - t_n)/R_n^2]\,,$$

where $\Omega_n := (1/R_n)(\Omega - x_n)$.
By construction v_n is smooth on $(B_{R_0/R_n}(0) \cap \Omega_n) \times [0, \frac{T - t_n}{R_n^2}]$. Also

$$\int_0^{\tau}, \int_{B_{R_0/R_n}(0) \cap \Omega_n} |\partial_t v_n|^2 dy ds = \int_{t_n}^{t_n + \tau R_n^2} \int_{B_{R_0}(x_n) \cap \Omega} |\partial_t u|^2 dx dt \overset{(n \to \infty)}{\to} 0$$

for any $\tau > 0$. Here we used that $\partial_t u \in L^2\big(\Omega \times [0, T]; N\big)$, which follows from the energy estimate for u. Set $a = \frac{\varepsilon_1}{4C_0 c}$. Then we conclude

(19) $$\frac{\varepsilon_1}{4} = E\big(v_n(0), B_2^{\Omega_n}(0)\big)$$

and

(20) $$\sup_{(y,s)\in B^{\Omega_n}_{R_0/R_n}(0)\times[0,a]} E\big(v_n(s), B_1^{\Omega_n}(y)\big)$$

$$\leq \sup_{(x,t)\in B^{\Omega}_{R_0}(x_n)\times[t_n,t_n+\delta_n]} E\big(u(t_n+R_n^2 s), B^{\Omega}_{R_n}(x_n+R_n y)\big) \leq \frac{\varepsilon_1}{2}\,,$$

uniformly in n.
Finally the maps v_n are smooth solutions of either the Landau-Lifshitz or of the harmonic map flow on $(B_{R_0/R_n}(0)\cap\Omega_n)\times[0,a]$ for sufficiently large n.
Now we show that the sequence $(v_n)_n$ admits a subsequence that converges smoothly to a bubble.

First assume $B_{2R_0}(\overline{x})\cap\partial\Omega=\emptyset$. Then also $B_1\cap\partial\Omega_n=\emptyset$ and Theorem 3 combined with Proposition 2 provide uniform C^k-estimates for $(v_n)_n$ on $\overline{B_{1/2}(y)}\times[\delta,a]$ for any $\delta>0$ that only depend on $\delta>0$ and a and $R=1$.
Thus there is a subsequence again denoted by $(v_n)_n$ and some

$$v\in C^\infty\big(\mathbb{R}^2\times]0,a];N\big)\,,$$

such that $v_n\to v$ in $C^\infty_{loc}(\mathbb{R}^2\times]0,a];\mathbb{R}^n)$ and $\partial_s v_n\to 0$ in $L^2_{loc}(\mathbb{R}^2\times[0,a];\mathbb{R}^n)$. (Here "loc" means that for any $R>0$ and sufficiently large $n_0=n_0(R)$, so that $R_0/R_{n_0}>R$, the sequence $(v_n)_{n\geq n_0}$ convergences on any $\overline{B_R(0)}\times[\delta,a]$ for $\delta>0$ in the respective norm (actually on $\overline{B_R(0)}\times[0,a]$ for L^2_{loc}).)
By construction v is a time independent solution of the Landau-Lifshitz or harmonic map flow with finite energy (bounded by C_0).
Assume next that v is a trivial solution, i.e. $v\equiv const.$ on $\mathbb{R}^2\times]0,a]$. By (19) the limit $v(\,.\,,0)$ is non-constant, but convergence is not necessarily uniform in a neighborhood of $s=0$. However by the local energy estimate for v_n (which is smooth on $\overline{B}_3(0)\times[0,a]$ for n such that $\frac{R_0}{R_n}>3$), we have for any $\varphi\in C^\infty(\mathbb{R}^2)$ with $spt\varphi\subset B_3(0)$, $\varphi\equiv 1$ on $B_2(0)$, $0\leq\varphi\leq 1$

$$\frac{1}{2}\int_0^s\int_{\mathbb{R}^2}\partial_t|\nabla v_n|^2\,\varphi^2\,dydt$$

$$= -\int_0^s\int_{\mathbb{R}^2}\nabla v_n\,\partial_t v_n\,\nabla\varphi\varphi\,dydt\;-\;\gamma_1\int_0^s\int_{\mathbb{R}^2}|\partial_t v_n|^2\,\varphi^2\,dydt \overset{(n\to\infty)}{\to} 0\,,$$

since $\partial_t v_n\to 0$ and $\nabla v_n\to\nabla v=0$ in $L^2\big(B_3(0)\times[0,a];N\big)$. Thus

$$\int_{\mathbb{R}^2}\big(|\nabla v_n|^2(y,s)-|\nabla v_n|^2(y,0)\big)\,\varphi^2\,dy \overset{(n\to\infty)}{\to} 0\;\;\forall s\in[0,a]\,,$$

which is in contradiction with $\int_{\mathbb{R}^2}|\nabla v_n|^2(y,0)\,\varphi^2\,dy\geq\frac{\varepsilon_0}{4}$ for all n and

$$\int_{\mathbb{R}^2}|\nabla v_n|^2(y,s)\,\varphi^2\,dy \overset{(n\to\infty)}{\to} 0 \qquad \text{for all } s\in]0,a]\,.$$

Note that in the case of the Landau-Lifshitz flow $\partial_s v = 0$ implies $-\triangle v = |\nabla v|^2 v$ and in particular $v \times \triangle v = 0$.
Further for any $\tilde{\Omega} \subset \Omega$

$$E\big(u(t), \tilde{\Omega}\big) = \int_{\tilde{\Omega} \setminus \cup_{j=1}^{K} B_R(x_j)} \frac{1}{2} \, |\nabla u|^2(x,t) dx + \int_{\bigcup_{j=1}^{K} B_R(x_j)} \frac{1}{2} |\nabla u|^2(x,t) dx \,,$$

for all $R > 0$. The first term converges for $t \searrow 0$ and

$$\limsup_{t \searrow 0} \int_{\bigcup_{j=1}^{K} B_R(x_j)} \frac{1}{2} |\nabla u|^2(x,t) dx \geq \lim_{n \to \infty} \int_{B_{R/2R_n}(x_j)} \frac{1}{2} |\nabla v_n|^2(y) dy \geq a_0$$

Thus

$$\limsup_{t \searrow 0} E\big(u(t), \tilde{\Omega}\big) \;\geq\; \int_{\tilde{\Omega} \setminus \cup_{j=1}^{K} B_R(x_j)} \frac{1}{2} \, |\nabla u|^2(x, t_0) \, dx \;+\; a_0 \quad \text{for any } R > 0 \,.$$

After passing to the limit $R \to 0$, we obtain the claim, if we choose $\tilde{\Omega} = \Omega$ or $\tilde{\Omega} = B_{R_1}(\overline{x})$ for any $R_1 > 0$.

If $\overline{x} \in \partial \Omega$, the above construction also works. Note that we may still integrate by parts to obtain the energy inequality, since $\partial_t u \equiv 0$ on $\partial \Omega \times [0, T]$. Moreover the rescaled solutions v_n satisfy

$$v_n(y,s) = v_{0,n}(y,s) := u_0(x_n + R_n y) \quad \text{ on } \Omega_n \cap B_{R_0/R_n}(\overline{x} - x_n) \,.$$

The uniform C^k-estimates on $\big(\overline{B_{1/2}(y)} \cap \Omega_n\big) \times [\delta, a]$ now also depend on the curvature of $\partial \Omega_n$ and on $\|v_{0,n}\|_{C^k}$. The boundary $\partial \Omega_n$ is however "flattened" by the scaling and its curvature tends to 0 as $n \to \infty$. On the other hand $\|v_{0,n} - u_0(\overline{x})\|_{C^k(B_1)} \to 0$ as $n \to \infty$ on any ball B_1 contained in the domain.

After passing to subsequences, we may assume

$$\frac{\operatorname{dist}(x_n, \partial \Omega)}{R_n} \;=\; \operatorname{dist}(0, \partial \Omega_n) \to \delta_* \in [0, \infty] \;\; \text{as } n \to \infty \,.$$

If $\delta_* = \infty$, the limit v of the locally rescaled sequence $(v_n)_n$ is defined on all of $\mathbb{R}^2$ and the proof may be completed as in the interior case.
If $\delta_* < \infty$, then after a rotation of the coordinates, we may assume the x_1-axis is parallel to the tangent space of $\partial \Omega$ at $\overline{x}$ and the x_2-axis points to the interior of Ω. Then the limit v of the locally rescaled sequence v_n is defined on the upper half plane $\mathbb{R}^2_{\delta_*} := \{y \in \mathbb{R}^2 \mid y_2 \geq -\delta_*\}$ and has constant boundary value $u_0(\overline{x})$. Since $\mathbb{R}^2_{\delta_*}$ is conformal to a closed disk $B_1(0) \subset \mathbb{C}$, the pullback of v to $B_1(0)$ is harmonic and by J.Sacks and K.Uhlenbeck's Removable Singularity Theorem (see [29]), it extends to a non-constant harmonic map with constant boundary values. This is in contradiction with a result by L. Lemaire ([23], Theorem 3.2), which says that any harmonic map with constant boundary values from a contractible surface with boundary, is itself constant.
This rules out the case $\delta_* < \infty$. Thus $\frac{\operatorname{dist}(x_n, \partial\Omega)}{R_n} \to \infty$ as $n \to \infty$. □

6. Partial compactness of sequences of smooth flows

The estimates from Section 3 can also be applied in the following situation. Let $(u_k)_k$ be a sequence of smooth solutions of the harmonic map flow or Landau-Lifshitz flow on a smooth bounded domain $\Omega \subset \mathbb{R}^2$

$$\gamma_1 \partial_t u_k - \gamma_2 u_k \times \partial_t u_k - \triangle u_k = |\nabla u_k|^2 u_k \quad \text{in } \Omega \times \mathbb{R}_+ \,, \tag{21}$$

$$u = u_{0,k} \text{ on } (\Omega \times \{0\}) \cup (\partial\Omega \times \mathbb{R}_+) \,, \tag{22}$$

with uniformly bounded energy

$$\sup_k E(u_{0,k}) \le E_0 \,, \tag{23}$$

and convergence at the boundary

$$u_{0,k} \overset{(k \to \infty)}{\to} u_0 \quad \text{in } C^{2,\lambda}(\partial\Omega; S^2) \,, \tag{24}$$

for some $\lambda \in]0,1[$. Assumption (23) together with the energy estimate (Lemma 1) imply

$$\sup_k \sup_{t \in \mathbb{R}_+} E(u_k(t)) \le E_0 \,.$$

By Proposition 2 and Theorem 3, we obtain uniform estimates on the following "regular set":
$Reg\big((u_k)_k\big)$ is the set of points $z_0 = (x_0, t_0) \in \overline{\Omega} \times]0, \infty[$ for which there is $R_0 > 0$ with

$$\limsup_{k \to \infty} \sup_{t_0 - R_0^2 < t < t_0} E\big(u_k(t), B^{\Omega}_{R_0}(x_0)\big) < \varepsilon_0 \,, \tag{25}$$

where $\varepsilon_0 > 0$ is the constant from Proposition 3.
The complement, denoted as

$$Sing\big((u_k)_k\big) = \Big(\overline{\Omega} \times \mathbb{R}_+\Big) \setminus Reg\big((u_k)_k\big) \,,$$

is the "energy concentration" set. Note that $Reg\big((u_k)_k\big)$ is open, which easily follows from the local energy estimate (Lemma 1).

Corollary 7. *The sequence $(u_k)_k$ of smooth solutions of (21)-(22) with (23) and (24) is uniformly bounded in*

$$C^\infty(Reg\big((u_k)_k\big) \cap (\Omega \times]0, \infty[), \mathbb{R}^3) \text{ and } C^{2,\lambda}(Reg\big((u_k)_k\big), \mathbb{R}^3)$$

and a subsequence converges smoothly to a solution of the Landau-Lifshitz flow on $Reg\big((u_k)_k\big) \cap (\Omega \times]0, \infty[)$.

The local energy estimate (26) implies

$$\sharp\Big(Sing\big((u_k)_k\big) \cap \big(\overline{\Omega} \times \{t\}\big)\Big) \le K_0 \quad \text{for all } t \ge 0\,,$$

where K_0 does not depend on $t \ge 0$. (See [19]. Compare also [34] and [11].) Remember that by Lemma 1, we have for any $0 \le s < t$

$$E\big(u_k(t), B_R^{\Omega}(x_0)\big) \le E\big(u_k(s), B_{2R}^{\Omega}(x_0)\big) + \frac{C(t-s)E_0}{\gamma_1 R^2}\,. \tag{26}$$

Set $\delta_0 := \frac{\gamma_1 \varepsilon_0}{2CE_0}$, where ε_0 is the constant from Theorem 3. After increasing C if necessary, we may assume $0 < \delta_0 < 1$.

Lemma 8. *Let $(u_k)_k$ be a sequence of smooth solution as in (21)-(22) with (23) and (24). Then the following assertions are equivalent:*

(i) $z_0 = (x_0, t_0) \in Reg\big((u_k)_k\big)\,.$

(ii) $\exists \delta, R > 0 :\ \limsup_{k\to\infty}\ \sup_{t_0-\delta<t<t_0} E\big(u_k(t), B_R^{\Omega}(x_0)\big) < \varepsilon_0\,.$

(iii) $\exists \delta > 0 :\ \lim_{R\searrow 0} \limsup_{k\to\infty}\ \sup_{t_0-\delta<t<t_0} E\big(u_k(t), B_R^{\Omega}(x_0)\big) = 0\,.$

(iv) $\exists R > 0 :\ \limsup_{k\to\infty} \frac{1}{R^2} \int_{t_0-R^2}^{t_0} \int_{B_R^{\Omega}(x_0)} |\nabla u_k|^2\, dx\, dt < \frac{1}{4}\, \delta_0\, \varepsilon_0\,.$

(v) $\exists \delta, R > 0 :\ \limsup_{k\to\infty}\ \sup_{t_0-\delta<t<t_0+\delta} E\big(u_k(t), B_R^{\Omega}(x_0)\big) < \varepsilon_0\,.$

Following $(i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (ii)$, the proof uses (26) and Theorem 3 and is straightforward (see [19]).
From characterization (iv) of the regular set in Lemma 8, it is easy to see that $Sing\big((u_k)_k\big)$ has locally finite 2-dimensional parabolic Hausdorff measure (see [19]). It then follows that the accumulations points of sequences as in Corollary 7 are weak solution of the Landau-Lifshitz flow on all $\Omega \times \mathbb{R}_+$ in

$$H^{1,2}(\Omega \times \mathbb{R}_+; S^2) \cap L^\infty\big(\mathbb{R}_+; H^{1,2}(\Omega; S^2)\big)\,.$$

A description of the bubbling behaviour of smooth sequences of the Landau-Lifshitz flow on planar domains with uniformly bounded energy, as in [19], will be presented elsewhere.
In [12,13] and [14] partial compactness for a similar problem is studied. Their arguments are however not conclusive.
Interesting results on the dynamics of the energy concentration set in the case of the harmonic map flow also in higher dimensions are found in [25–27] and [24].

Appendix

Let $\Omega \subset \mathbb{R}^2$ be open. We will give an example showing that the space

$$L^\infty(C^1) := L^\infty\big([0,T]; C^1(\Omega; \mathbb{R}^3)\big)$$

is neither dense in

$$L^\infty\big([0,T]; H^{1,2}(\Omega; S^2)\big)$$

with norm $\|u\|_{L^\infty(H^{1,2})} := \sup_{[0,T]} \|u\|_{H^{1,2}(\Omega)}$, nor in

$$L^\infty\big([0,T]; H^{1,2}(\Omega;\mathbb{R}^3)\big).$$

For $\varphi \in L^\infty\big([0,T], C^1(\Omega;\mathbb{R}^3)\big)$, we require $\sup_{[0,T]} \|\varphi(.,t)\|_{C^1(\Omega)} < \infty$.
Set $\Omega = B_1 = B_1(0) \subset \mathbb{R}^2$ and $T = 2$. Let $N = (0,0,1)$ be the north pole of the standard sphere $S^2 \subset \mathbb{R}^3$ centered at the origin and $S = (0,0,-1)$ the south pole. Set

$$u_0(x) := \begin{cases} N, & \text{if } x \in B_1 \setminus B_{1/2}\,, \\ S, & \text{if } x = 0\,, \end{cases}$$

and extend u_0 smoothly to B_1, such that $u_0(B_1)$ covers all of S^2. Then $E(u_0) \geq \text{Area}(S^2) = 4\pi$ (see [35], Proposition 1.1). We can also extend u_0 outside B_1 by $u_0 \equiv N$. Further set

$$u(x,t) := \begin{cases} u_0(\frac{x}{1-t}) & \text{if } t \in [0,1[, \\ N & \text{if } t \in [1,2]. \end{cases}$$

Then for any $\varphi \in L^\infty([0,2]; C^1(B_1;\mathbb{R}^3))$, letting $\sup_{B_1\times[0,2]} |\nabla\varphi| = C < \infty$, we have

$$\begin{aligned} &\sup_{t\in[0,2]} \|\nabla\varphi(.,t) - \nabla u(.,t)\|^2_{L^2(B_1)} \\ &\qquad \geq \sup_{t\in[0,1[} \int_{B_1} \big(|\nabla u|^2(x,t) - 2C|\nabla u|(x,t)\big)\,dx \\ &\qquad \geq 2E_0 - \inf_{t\in[0,1[} 2C(1-t)\int_{B_1} |\nabla u_0|\,dx = 2E_0\,, \end{aligned}$$

showing

$$\inf_{\varphi\in L^\infty(C^1)} \|u-\varphi\|_{L^\infty(H^{1,2})} \geq \sqrt{8\pi}$$

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