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ON RANDOM WALKS ON HOMOGENEOUS SPACES

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Abstract

Ten years back, Benoist–Quint established breakthrough results concerning actions of non-amenable groups on finite volume homogeneous spaces of real Lie groups. Taking their work as starting point, the purpose of this thesis is to further explore and advance the theory of random walks on homogeneous spaces in several directions.

We start our investigations with basic aspects of convergence of random walks on homogeneous spaces, obtaining aperiodicity and uniform Cesàro convergence results. Under a spectral gap assumption, we prove that almost every starting point is exponentially generic for the random walk, where, notably, the exponential speed of convergence is uniform over all compactly supported smooth test functions.

Secondly, in joint work with Çağrı Sert, we consider random walks driven by stochastic processes which are not necessarily i.i.d.; specifically, Markov random walks. Under an expansion assumption, using bootstrapping techniques and joint equidistribution results, we establish equidistribution statements for this more general class of random walks. This has consequences for Diophantine approximation problems on a certain class of fractals, namely on graph-directed self-similar sets.

In the third part, we investigate i.i.d. random walks on homogeneous spaces given by probability measures that are spread out, i.e. admit a convolution power non-singular to Haar measure. Drawing on the existing theory of general state space Markov chains, we obtain a complete picture of the convergence properties of this special type of random walks. In particular, we establish non-averaged convergence in law in the finite volume case, answering an open question of Benoist–Quint for this type of measures. In the infinite volume case, we prove recurrence for spread out random walks on homogeneous spaces of at most quadratic growth, which settles one direction in the long-standing "quadratic growth conjecture".

Finally, in joint work with Çağrı Sert and Ronggang Shi, we tackle the problem of unifying the results of Benoist–Quint and other more recent results in the same area, like the work of Simmons–Weiss, which relies on a set of assumptions orthogonal to the setting of Benoist–Quint. To this end, we introduce a new class of probability measures defined by a uniform expansion property in finite-dimensional representations. Using the recent work of Eskin–Lindenstrauss, we are able to prove measure rigidity, strong recurrence properties, orbit closure descriptions, as well as equidistribution results for this class of expanding random walks. As consequence, we also obtain new results on Birkhoff genericity for certain diagonalizable flows, which in turn has implications for weighted Diophantine approximation problems on self-affine sets.

Résumé

Il y a dix ans, Benoist et Quint ont établi des résultats décisifs concernant les actions de groupes non moyennables sur des espaces homogènes de volumes finis de groupes de Lie réels. En prenant leur travail comme point de départ, le but de cette thèse est d'explorer et de faire progresser la théorie des marches aléatoires sur des espaces homogènes dans plusieurs directions.

Nous commençons nos recherches par les aspects de base de la convergence de marches aléatoires sur espaces homogènes, obtenant des résultats d'apériodicité et de convergence de Cesàro uniforme. Sous une hypothèse de trou spectral, nous démontrons que presque tout point de départ est exponentiellement générique pour la marche aléatoire, où, notamment, la vitesse exponentielle de convergence est uniforme sur toutes les fonctions tests lisses à support compact.

Deuxièmement, en collaboration avec Çağrı Sert, nous considérons des marches aléatoires issues de processus stochastiques qui ne sont pas nécessairement iid ; plus précisément, des marches aléatoires markoviennes. Sous une hypothèse d'expansion, en utilisant des techniques d'amorçage et des résultats d'équidistribution jointe, nous établissons un théorème d'équidistribution pour cette classe plus générale de marches aléatoires. Cela a des conséquences pour les problèmes d'approximation diophantienne sur une certaine classe de fractales, à savoir sur les ensembles autosimilaires dirigés par un graphe au sens de Mauldin et Williams.

Dans la troisième partie, nous étudions des marches aléatoires iid sur des espaces homogènes données par des mesures de probabilité qui sont étalées, c'est-à-dire admettent une puissance de convolution non singulière par rapport à la mesure de Haar. En nous inspirant de la théorie des chaînes de Markov dans l'espace d'états général, nous obtenons une image complète des propriétés de convergence de ce type particulier de marches aléatoires. En particulier, nous établissons la convergence en loi non moyennée dans le cas de volume fini, répondant à une question ouverte de Benoist et Quint pour ce type de mesures. Dans le cas du volume infini, nous démontrons la récurrence pour des marches aléatoires étalées sur des espaces homogènes de croissance au plus quadratique, ce qui établit une direction de la « conjecture de croissance quadratique ».

Enfin, en collaboration avec Çağrı Sert et Ronggang Shi, nous abordons le problème de l'unification des résultats de Benoist et Quint et d'autres résultats plus récents dans le même domaine, comme le travail de Simmons et Weiss, qui s'appuie sur un ensemble d'hypothèses orthogonales au cadre de Benoist et Quint. À cette fin, nous introduisons une nouvelle classe de mesures de probabilité définies par une propriété d'expansion uniforme dans les représentations de dimension finie. En utilisant les travaux récents d'Eskin et Lindenstrauss, nous sommes en mesure de prouver la rigidité des mesures, des propriétés de récurrence forte, une description des adhérences d'orbite, ainsi que des résultats d'équidistribution pour cette classe de marches aléatoires expansives. En conséquence, nous obtenons également de nouveaux résultats sur la généricité de Birkhoff de certains flots diagonalisables, ce qui à son tour a des implications pour les problèmes d'approximation diophantienne pondérée sur les ensembles auto-affines.

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Contents

CHAPTER 0

Introduction

0.1. A Bit of History

Originally motivated by applications to number theory, the rigidity properties of subgroup actions on a homogeneous space $X = G/\Lambda$, where G is a real Lie group and $\Lambda < G$ a discrete subgroup, have been an active field of research over the last fifty years. Among the first striking results was Margulis' resolution of the Oppenheim conjecture [**84, 86**] via a reformulation into an orbit closure problem for the action of $\mathrm{SO}(2,1)$ on $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ noticed by Raghunathan. Raghunathan had conjectured, more generally, that orbit closures for unipotent subgroups are closed orbits of larger subgroups. After more partial results by Dani, Margulis, and Shah, Raghunathan's conjecture was settled in full generality in celebrated work of Ratner [**109, 110, 111, 112**].

In absence of unipotent elements, the dynamics of subgroup actions are harder to understand—already the case of actions on a torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ by non-amenable subgroups of $\mathrm{SL}_d(\mathbb{Z})$ poses serious challenges. The very first difficulty arising in this setup is the potential lack of invariant measures. What has proved to be a fruitful approach for overcoming this issue is taking a probabilistic viewpoint of random walks and stationary measures, techniques mainly pioneered by Furstenberg starting in the sixties [**49, 51, 52, 53**]. Using this random walks approach, Guivarc'h–Starkov [**57**] made first contributions to understanding the action of subgroups $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ on $\mathbb{T}^d$, and Bourgain–Furman–Lindenstrauss–Mozes [**20**] proved a quantitative result which answered many remaining questions.

For subgroup actions on a general homogeneous space $X = G/\Lambda$, a major breakthrough came around 2010 with a series of papers by Benoist–Quint [**5, 7, 8, 9**]. Applying several novel techniques, they were able to give a complete classification of stationary measures, descriptions of orbit closures, and prove equidistribution statements for random walks under the assumption of semisimplicity of the Zariski closure of the acting group Γ. One crucial new ingredient in the proof of their measure classification result was the so-called "exponential drift" argument (as compared to the "polynomial drift" argument of Ratner), which was further developed in the seminal work of Eskin–Mirzakhani [**42**] on stationary measures for the $\mathrm{SL}_2(\mathbb{R})$-action on moduli space. Bringing back to homogeneous dynamics ideas from the setting of random walks on moduli space, Eskin–Lindenstrauss [**39**] have recently obtained a theorem which generalizes the measure classification results of Benoist–Quint.

Building upon much of the work outlined above, the goal of this thesis is to further advance the theory of subgroup actions and random walks on homogeneous spaces in various directions. In the following section §0.2, we start by introducing the basic central concepts involved. Afterwards, in §0.3, we state two major theorems of Benoist–Quint together with ensuing questions that will serve as motivation throughout the whole thesis. The structure of this dissertation and its main contributions will be summarized in §0.4.

0.2. Background

Let G be a locally compact σ-compact metrizable group acting continuously on a locally compact σ-compact metrizable space X. A choice of Borel probability measure μ on G defines a *random walk* on X: A step corresponds to sampling a random group element $g \in G$ according to μ and then moving from the current location $X \ni x$ to gx. When the starting point of the random walk is the point x, the location Φ_n after n steps can be represented as

$$\Phi_n = g_n \cdots g_1 x,$$

where $(g_k)_{k\in\mathbb{N}}$ is a sequence of random group elements that are independent and identically distributed (i.i.d.) with common law μ. Conditioned on the starting point being x, the law of Φ_n is thus given by

$$\mathcal{L}_x(\Phi_n) = \mu^{*n} * \delta_x,$$

where δ_x denotes the Dirac mass at x, μ^{*n} is the n-fold convolution power of μ, and the convolution of probability measures μ on G and ν on X via the given action is defined by $\mu * \nu = \int_G g_*\nu \,\mathrm{d}\mu(g)$. In other words, this means that the convolution is defined by the property that

$$\int_X f \,\mathrm{d}(\mu * \nu) = \int_G \int_X f(gx) \,\mathrm{d}\nu(x) \,\mathrm{d}\mu(g)$$

for non-negative measurable functions f on X.

More generally, not starting the random walk at a fixed point $x \in X$ but according to some distribution ν on X, the law of Φ_n is given by the analogous formula

$$\mathcal{L}_\nu(\Phi_n) = \mu^{*n} * \nu.$$

Of special interest are probability measures ν on X which stay "invariant" under steps of the random walk. The natural formalization of such invariance is the requirement that $\mathcal{L}_\nu(\Phi_n) = \mathcal{L}_\nu(\Phi_0)$ for all $n \in \mathbb{N}$, which, in light of the above, is equivalent to the simple equation

$$\mu * \nu = \nu.$$

Probability measures ν on X satisfying the relation above are said to be *μ-stationary*. The set of μ-stationary probability measures is a convex subset of the space $\mathcal{P}(X)$ of all probability measures on X. Probability measures extremal in the set of μ-stationary measures are called *μ-ergodic*. In other words, a μ-stationary probability measure ν is μ-ergodic if and only if it cannot be written as proper convex combination $\nu = s\nu_1 + (1-s)\nu_2$, where ν_1, ν_2 are distinct μ-stationary probability measures on X and s is a real number with $0 < s < 1$.

Besides stationary measures, a further aspect that is crucial for the understanding of random walks is their long-term behavior, reflected for example in the convergence of the n-step distributions $\mu^{*n} * \delta_x$ or the asymptotics of individual trajectories $(g_n \cdots g_1 x)_{n\in\mathbb{N}}$ almost surely (a.s.) with respect to the product measure $\mu^{\otimes\mathbb{N}}$. Important in this regard is the notion of convergence used. Mainly, we are going to focus on convergence in the *weak* topology* on $\mathcal{P}(X)$, in which a basis of neighborhoods of a probability measure ν_0 on X is given by sets of the form

$$N_{f_1,\dots,f_k;\varepsilon}(\nu_0) = \left\{\nu \in \mathcal{P}(X) \,\middle|\, \left|\int_X f_i \,\mathrm{d}\nu - \int_X f_i \,\mathrm{d}\nu_0\right| < \varepsilon \text{ for } 1 \le i \le k\right\}$$

for compactly supported continuous functions $f_1, \dots, f_k$ on X and $\varepsilon > 0$. Then convergence of a sequence of probability measures $(\nu_j)_j$ on X towards a probability measure ν on X is equivalent to the requirement that

$$\int_X f \,\mathrm{d}\nu_j \longrightarrow \int_X f \,\mathrm{d}\nu \tag{0.2.1}$$

as $j \to \infty$ for every compactly supported continuous function f on X. It is known from probability theory that weak* convergence $\nu_j \to \nu$ for probability measures ν_j and ν on X implies that (0.2.1) holds for all bounded continuous functions f on X. The space $\mathcal{P}(X)$ of probability measures on X is compact in the weak* topology if and only if X is compact. To deal with non-compact spaces X, one considers the one-point compactification $\overline{X} = X \cup \{\infty\}$ of X. Then $\mathcal{P}(X)$ admits a natural embedding into the compact space $\mathcal{P}(\overline{X})$ by virtue of identification with $\{\nu \in \mathcal{P}(\overline{X}) \mid \nu(X) = 1\}$. Consequently, it is always possible to extract convergent subsequences from a sequence $(\nu_j)_j$ of probability measures on X, the only danger being that the limit measure ν might no longer be a probability measure on X, but assign positive mass to the point at infinity—a phenomenon known as "escape of mass".

Let us now specialize to the setting of our main interest. Namely, we let G be a real Lie group, $\Lambda < G$ a discrete subgroup, and X the homogeneous space G/Λ. Then there is a natural action of G on X given by left translation. When $X = G/\Lambda$ admits a non-trivial finite G-invariant measure, we say that Λ is a *lattice* in G. In this case, there is a unique G-invariant probability measure m_X on X, which we refer to as the *Haar measure* on X. A good example to have in mind is $G = \mathrm{SL}_d(\mathbb{R})$ with its lattice $\Lambda = \mathrm{SL}_d(\mathbb{Z})$. When it exists, the Haar measure m_X on X is a desirable candidate when studying the limiting behavior of random walks (or other dynamical systems) on X, in that it represents the most uniform distribution possible. Therefore, limiting behavior governed by m_X is generally referred to as "equidistribution". Often, however, this is prevented by algebraic obstructions, for example when the random walk starts inside a lower-dimensional invariant subspace of X. To make the latter precise, we say that a probability measure ν on X is *homogeneous* if there exist a point $x \in X$ and a closed subgroup H of G preserving ν such that $\nu(Hx) = 1$. The orbit Hx is then automatically closed and is called a *homogeneous subspace* of X. If $x = g\Lambda$, then Hx is a homogeneous subspace of X if and only if $H \cap g\Lambda g^{-1}$ is a lattice in H. In this case, the map

$$\begin{aligned} H/(H \cap g\Lambda g^{-1}) &\to Hx \subset X, \\ [h] &\mapsto hx \end{aligned}$$

is an equivariant homeomorphic embedding, and the homogeneous measure on the closed orbit Hx corresponds to the Haar measure on $H/(H \cap g\Lambda g^{-1})$. We thus obtain a one-to-one correspondence between homogeneous measures on X

and homogeneous subspaces of X. For a closed subgroup Γ of G, a homogeneous subspace Y of X is said to be Γ-*ergodic* if Γ preserves the corresponding homogeneous probability measure ν_Y and the action of Γ on (Y, ν_Y) is ergodic.

0.3. Theorems of Benoist and Quint

Using the notions introduced in §0.2, we are now ready to formulate two fundamental theorems of Benoist–Quint which serve as motivation throughout the thesis. The first result concerns the classification of ergodic stationary measures. We will denote by Γ_μ and Γ_μ^+ the closed subgroup and subsemigroup of G, respectively, generated by the support of μ.

THEOREM A (Benoist–Quint [**8**]). *Let G be a real Lie group, $\Lambda < G$ a lattice, $X = G/\Lambda$, and μ a compactly supported probability measure on G. Suppose that the Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ is Zariski connected, semisimple, and has no compact factors. Then every μ-ergodic μ-stationary probability measure on X is Γ_μ-invariant and homogeneous.*

In other words, all ergodic stationary measures are nice, algebraically constructed objects, and no pathologies can arise. Informally, statements of this type are thus referred to as "measure rigidity".

Secondly, random walks on X equidistribute with respect to a homogeneous probability measure on X naturally determined by the starting point of the random walk.

THEOREM B (Benoist–Quint [**9**]). *Retain the notation and assumptions from Theorem A. Then for every $x \in X = G/\Lambda$ there is a Γ_μ-ergodic homogeneous subspace $Y_x \subset X$ with corresponding homogeneous probability measure ν_x such that the following hold:*

(i) *The orbit closures $\overline{\Gamma_\mu x}$ and $\overline{\Gamma_\mu^+ x}$ are both equal to Y_x.*

(ii) *One has the convergence*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x \longrightarrow \nu_x$$

as $n \to \infty$ in the weak topology.*

(iii) *For $\mu^{\otimes\mathbb{N}}$-almost every (a.e.) sequence $(g_j)_j$ one has*

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \longrightarrow \nu_x$$

as $n \to \infty$ in the weak topology.*

We observe that the convergence in point (ii) of the above theorem is convergence on average of the n-step distributions of the random walk starting at x, also referred to as "Cesàro convergence in law". Point (iii) states a.s. pathwise equidistribution of the random walk with respect to ν_x. Part (i), on the other hand, is an entirely deterministic claim, describing all orbit closures for the action of Γ_μ on X. As warm-up exercise, let us convince ourselves that the latter is an immediate consequence of the other two dynamical statements on equidistribution of random walks.

LEMMA 0.3.1. *Let G be a locally compact σ-compact metrizable group acting continuously on a locally compact σ-compact metrizable space X. Let μ be a probability measure on G and fix $x \in X$. If ν_x is a probability measure on X such that its support $Y_x = \operatorname{supp}(\nu_x)$ is Γ_μ-invariant and contains x, then for the statements in Theorem* B, (iii) *implies* (ii) *implies* (i).

PROOF. We first assume that (iii) holds. Then by definition of weak* convergence, for every sequence $(g_j)_j$ of group elements drawn from some fixed full measure subset of $G^{\mathbb{N}}$ with respect to $\mu^{\otimes\mathbb{N}}$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(g_k \cdots g_1 x) \longrightarrow \int_X f \, \mathrm{d}\nu_x$$

as $n \to \infty$ for every compactly supported continuous function f on X. Since the left-hand side in the convergence above is uniformly bounded by the supremum norm $\|f\|_\infty$ of f, an application of Lebesgue's dominated convergence theorem yields

$$\frac{1}{n}\sum_{k=0}^{n-1}\int_X f \, \mathrm{d}(\mu^{*k} * \delta_x) = \frac{1}{n}\sum_{k=0}^{n-1}\int_{G^{\mathbb{N}}} f(g_k \cdots g_1 x) \, \mathrm{d}\mu^{\otimes\mathbb{N}}((g_j)_j) \longrightarrow \int_X f \, \mathrm{d}\nu_x$$

as $n \to \infty$, which is (ii).

To show that (ii) implies (i), note first that $\overline{\Gamma_\mu^+ x} \subset \overline{\Gamma_\mu x} \subset Y_x$ holds because Y_x is closed, Γ_μ-invariant and $x \in Y_x$. For the remaining reverse inclusion, one only has to note that

$$\operatorname{supp}(\mu^{*k} * \delta_x) = \overline{\operatorname{supp}(\mu)^k x}.$$

This relation implies that the support of $\frac{1}{n}\sum_{k=0}^{n-1}\mu^{*k} * \delta_x$ is contained in $\overline{\Gamma_\mu^+ x}$ for every $n \in \mathbb{N}$. Assuming the weak* convergence in (ii) it thus follows at once that $Y_x \subset \overline{\Gamma_\mu^+ x}$. □

Taking the two theorems above as starting point, the aim of this thesis is to further investigate random walks on homogeneous spaces. Guiding questions, already listed by Benoist–Quint at the end of their survey article [**6**], are the following.

QUESTION 1. Is the description of orbit closures in Theorem B still true when the Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ is only supposed to be spanned by its one-parameter unipotent subgroups?

QUESTION 2. Are the measure classification in Theorem A and the convergence statements in Theorem B still valid when μ is not supposed to have compact support?

QUESTION 3. Can the Cesàro average be removed from statement (ii) in Theorem B, i.e. does one also have $\lim_{n\to\infty}\mu^{*n} * \delta_x = \nu_x$?

Although not definitively resolving any of them, we will make significant contributions to the problems addressed by these questions and further ones, hopefully leading to a better understanding of subgroup actions and random walks on homogeneous spaces.

0.4. Outline of the Thesis

Besides this introduction, this dissertation consists of four largely independent chapters, each of which corresponds to a research article, published or in preprint stage.

Chapter 1, which has appeared as preprint on the arXiv [**103**], starts by considering some aspects of Question 3 at the end of §0.3. First, we look into the obvious obstruction to the upgrade from Cesàro to non-averaged convergence: periodicity. We give examples where it occurs and conditions under which it does not. Secondly, we prove that non-averaged convergence holds generically: We establish convergence of the n-step distributions towards Haar measure with exponential speed from almost every starting point. Using a certain type of height function, the exponential speed is seen to hold uniformly across all compactly supported smooth test functions. Finally, we establish a strong uniformity property for Cesàro convergence in law towards Haar measure for uniquely ergodic random walks.

In Chapter 2, joint work with Çağrı Sert that was published in revised form by the Transactions of the American Mathematical Society [**105**], we study random walks whose steps are not necessarily i.i.d., but have Markovian dependence. Employing techniques based on renewal and joint equidistribution arguments, we obtain pathwise equidistribution statements for this more general class of random walks. For the involved arguments to work, we first have to establish a positive answer to Question 2 in a basic special case, for which we rely on the recent measure classification results of Eskin–Lindenstrauss [**39**]. Finally, following a strategy of Simmons–Weiss [**129**], we apply these results to Diophantine approximation problems on fractals and show that almost every point with respect to Hausdorff measure on a graph-directed self-similar set is of generic type, so in particular, well approximable.

In Chapter 3, published in revised form in Ergodic Theory and Dynamical Systems [**104**], we tackle Question 3 for a special class of probability measures μ. Namely, this chapter considers the case that μ is spread out, meaning that there exists a convolution power μ^{*n} that is non-singular with respect to Haar measure on G. Systematically exploiting the theory of Markov chains on general state spaces, we conduct a detailed analysis of random walks on homogeneous spaces with spread out increment distribution. For finite volume spaces, we arrive at a complete picture of the asymptotics of the n-step distributions: They equidistribute towards Haar measure, often exponentially fast and locally uniformly in the starting position. In addition, many classical limit theorems are shown to hold. In the infinite volume case, we prove recurrence and a ratio limit theorem for symmetric spread out random walks on homogeneous spaces of at most quadratic growth, which settles one direction in the long-standing "quadratic growth conjecture".

In Chapter 4, joint work with Çağrı Sert and Ronggang Shi that is available as preprint on the arXiv [**106**], we investigate the problem underlying Question 1, namely whether the assumption of semisimplicity of the Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ can be relaxed. For a semisimple subgroup H of G without compact factors and with finite center, we define the notion of H-expanding probability measures μ on H and, applying the recent work of Eskin–Lindenstrauss [**39**], prove that μ-stationary probability measures on $X = G/\Lambda$ are homogeneous. Transferring a construction by Benoist–Quint [**7**] and drawing on ideas of Eskin–Mirzakhani–Mohammadi [**43**], we construct Lyapunov/Margulis functions to show that H-expanding random walks on X satisfy a recurrence condition and that homogeneous subspaces are repelling. Combined with a countability result, this allows us to prove equidistribution of trajectories in X for H-expanding random walks and to obtain orbit closure descriptions. In all of this, we work with appropriate moment conditions on μ instead of an assumption of compact support, thus giving a partial affirmative answer also for Question 2 for the class of H-expanding measures. Finally, elaborating on an idea of Simmons–Weiss [**129**], we deduce Birkhoff genericity of a class of measures with respect to some diagonal flows and extend their applications to Diophantine approximation on similarity fractals to a non-conformal and weighted setting.

CHAPTER 1

Aspects of Convergence of Random Walks on Finite Volume Homogeneous Spaces

Let G be a real Lie group, Λ a lattice in G, and X the homogeneous space G/Λ. Recall that a probability measure μ on G defines a random walk on X, whose n-step distribution when starting at $x_0 \in X$ is given by the convolution

$$\mu^{*n} * \delta_{x_0},$$

which is the push-forward of the product measure $\mu^{\otimes n} \otimes \delta_{x_0}$ under the multiplication map $G^n \times X \ni (g_n, \dots, g_1, x) \mapsto g_n \cdots g_1 x \in X$. Benoist–Quint's Theorem B in the Introduction states that under certain conditions, the Cesàro averages of these n-step distributions converge in the weak* topology towards the homogeneous probability measure ν_{x_0} supported on the Γ_μ-orbit closure of the starting point x_0, where Γ_μ denotes the closed subgroup of G generated by the support of μ. A major open question, stated in the Introduction as Question 3, is the following.

Question. In the setting of Benoist–Quint's theorems, is it also true that

$$\mu^{*n} * \delta_{x_0} \longrightarrow \nu_{x_0} \tag{1.0.1}$$

as $n \to \infty$?

Answers are available only in special cases: Breuillard [**22**] established (1.0.1) for certain measures μ supported on unipotent subgroups, Buenger [**23**] proved it for some sparse solvable measures, and in Chapter 3 we are going to deal with the case of spread out measures.

The purpose of this chapter is to discuss three (largely independent) aspects of random walk convergence related to Benoist–Quint's Theorem B and the question above, mainly having in mind the case that G is a semisimple real Lie group. We are going to use the following terminology.

Definition 1.0.1. Let ν be a probability measure on X and $x_0 \in X$. We say that the random walk on X given by the probability measure μ on G *converges to ν on average* (resp. *converges to ν*) from the starting point x_0 if $\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_{x_0} \to \nu$ (resp. $\mu^{*n} * \delta_{x_0} \to \nu$) as $n \to \infty$ in the weak* topology.

Convergence on average is also commonly referred to as "Cesàro convergence". We use the two terms interchangeably.

The chapter is organized as follows.

In §1.1, we look into the obvious obstruction to the upgrade from Cesàro convergence to non-averaged convergence: periodicity. We show in Example 1.1.1 how (1.0.1) can fail when x_0 has finite orbit under the closed subsemigroup Γ_μ^+ of G generated by $\mathrm{supp}(\mu)$. Using a product construction, we can also produce a counterexample in which the orbit closure $\overline{\Gamma_\mu^+ x_0}$ has positive dimension (Example 1.1.2). In both cases, the periodic behavior occurs at the level of the connected components of the orbit closure. As it turns out, this is no coincidence: If, in the setting of Theorem B, the orbit closure $\overline{\Gamma_\mu^+ x_0}$ is connected, there can be no periodicity (Theorem 1.1.5) and we can show that the Cesàro convergence towards ν_{x_0} also holds along arithmetic progressions (Corollary 1.1.7).

In §1.2, we establish effective convergence of random walks to the Haar measure m_X on X for typical starting points x_0: When $\operatorname{supp}(\mu)$ generates a Zariski dense subgroup of a semisimple real Lie group G without compact factors and with finite center, for any fixed L^2-function f on X the convergence

$$\int_X f \,\mathrm{d}(\mu^{*n} * \delta_{x_0}) \overset{n\to\infty}{\longrightarrow} \int_X f \,\mathrm{d}m_X$$

not only holds but is in fact exponentially fast for m_X-almost every $x_0 \in X$ (Theorem 1.2.2, Proposition 1.2.4). The proof relies on an L^2-spectral gap of the convolution operator

$$\pi(\mu)\colon f \mapsto \Big(x \mapsto \int_G f(gx)\,\mathrm{d}\mu(g)\Big)$$

acting on measurable functions on X. Taking into account regularity of the function f, the above can be further strengthened to the statement that almost every $x \in X$ is "exponentially generic" (Definition 1.2.12): Up to a constant factor depending on derivatives of f, the exponential speed of convergence holds uniformly over all compactly supported smooth functions (Theorem 1.2.13). Key to this upgrade are the definition of suitable Sobolev norms and a functional analytic argument involving relative traces, first exploited in a dynamical context by Einsiedler–Margulis–Venkatesh [**36**].

Finally, in §1.3 we prove that convergence on average to m_X happens locally uniformly in x_0 in a strong way when the random walk is uniquely ergodic and admits a Lyapunov function (Theorem 1.3.13). For example, this is the case when G is a connected semisimple real algebraic group and $\operatorname{supp}(\mu)$ generates a non-discrete Zariski dense subgroup, and also in the setup of Simmons–Weiss [**129**], which has connections to Diophantine approximation problems on fractals. To this end, we introduce the new concept of "$(K_n)_n$-uniform recurrence" (Definition 1.3.10), which refines recurrence properties of random walks previously studied in [**7, 40**].

Standing Assumptions & Notation. As many of our arguments work in greater generality, in the remainder of the chapter we will relax the assumptions stated at the beginning. The following minimal setup shall be in place whenever nothing else is specified: G is a locally compact σ-compact metrizable group acting continuously and ergodically on a locally compact σ-compact metrizable space X endowed with a G-invariant probability measure m_X; μ is a Borel probability measure on G; and Γ_μ and Γ_μ^+ denote the closed subgroup and subsemigroup of G generated by $\operatorname{supp}(\mu)$, respectively.

1.1. Periodicity

In this section, we start with two simple counterexamples to (1.0.1), which illustrate ways in which a random walk may exhibit periodic behavior (§1.1.1). Analyzing these examples for their common feature, we are led to a simple condition ensuring aperiodicity, stated and proved in §1.1.2.

1.1.1. Examples. The first example with periodicity is on finite periodic orbits. In the following, for $d \geq 2$ we denote by $\mathbf{1}_d$ the $d \times d$-identity matrix.

Example 1.1.1. Consider the *principal congruence lattice*

$$\Lambda = \Lambda(2) = \{g \in \mathrm{SL}_2(\mathbb{Z}) \mid g \equiv \mathbf{1}_2 \bmod 2\}$$

in $G = \mathrm{SL}_2(\mathbb{R})$. Being the kernel of the reduction homomorphism from $\mathrm{SL}_2(\mathbb{Z})$ to $\mathrm{SL}_2(\mathbb{Z}/2\mathbb{Z})$, we recognize $\Lambda(2)$ as a finite-index normal subgroup of $\mathrm{SL}_2(\mathbb{Z})$. In particular, $\Lambda(2)$ is a lattice in G. Let $\mu = \frac{1}{2}(\delta_{h_1} + \delta_{h_2})$ with

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ h_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Then the closed subgroup generated by $\mathrm{supp}(\mu) = \{h_1, h_2\}$ is $\Gamma_\mu = \mathrm{SL}_2(\mathbb{Z})$, which is Zariski dense in G. The Γ_μ-orbit of $x_0 = \mathbf{1}_2\Lambda \in G/\Lambda$ is

$$\begin{aligned} \mathcal{O} = \Big\{ & x_0, h_1x_0, h_2x_0, h_2h_1x_0 = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)x_0, h_1h_2x_0 = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)x_0, \\ & h_1h_2h_1x_0 = h_2h_1h_2x_0 = \left(\begin{smallmatrix} 2 & -1 \\ 1 & 0 \end{smallmatrix}\right)x_0 \Big\}, \end{aligned}$$

with transitions as shown in the following diagram:

$$\begin{tikzcd}
 & h_2h_1x_0 \arrow[r, bend left, "h_2"] \arrow[dl, bend left, "h_1"] & h_1x_0 \arrow[l, bend left, "h_2"] \arrow[dr, bend left, "h_1"] & \\
h_1h_2h_1x_0 \arrow[ur, bend left, "h_1"] \arrow[dr, bend left, "h_2"] & & & x_0 \arrow[ul, bend left, "h_1"] \arrow[dl, bend left, "h_2"] \\
 & h_1h_2x_0 \arrow[ul, bend left, "h_2"] \arrow[r, bend left, "h_1"] & h_2x_0 \arrow[l, bend left, "h_1"] \arrow[ur, bend left, "h_2"] &
\end{tikzcd}$$

Consequently, we see that the random walk with starting point x_0 alternates between the two sets

$$\mathcal{O}_1 = \{x_0, h_1h_2x_0, h_2h_1x_0\} \text{ and } \mathcal{O}_2 = \{h_1x_0, h_2x_0, h_1h_2h_1x_0\}.$$

The 2-step random walks on these sets constitute irreducible, aperiodic, finite state Markov chains, so that

$$\mu^{*2n} * \delta_{x_0} \longrightarrow \frac{1}{3}\sum_{p \in \mathcal{O}_1} \delta_p,$$

$$\mu^{*(2n+1)} * \delta_{x_0} \longrightarrow \frac{1}{3}\sum_{p \in \mathcal{O}_2} \delta_p,$$

as $n \to \infty$ in the weak* topology. $\diamond$

In the example above, the support of μ generates a Zariski dense subgroup of G and the lattice Λ in G is irreducible.[1] By the work of Benoist–Quint ([**9**, Corollary 1.8]), these properties force any orbit closure $\overline{\Gamma_\mu^+ x_0}$ to be either finite or all of X. As soon as intermediate orbit closures are possible, however, one can also construct examples with periodic behavior on non-discrete orbit closures.

Example 1.1.2. Let G, Λ, $X = G/\Lambda$, h_1, h_2, x_0 be as in Example 1.1.1, set $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and choose a diagonal matrix $a \in \mathrm{SL}_2(\mathbb{R})$ such that the diagonal entries of a^2 are irrational. We are going to consider the random walk on the product space

$$X \times X = (G \times G)/(\Lambda \times \Lambda)$$

given by the probability measure $\mu = \frac{1}{4}\sum_{i=1}^4 \delta_{g_i}$ on $G \times G$ with

$$\begin{aligned} g_1 &= (h_1, ah_1a^{-1}),\ g_2 = (h_1, \mathbf{1}_2),\\ g_3 &= (h_2, ah_2a^{-1}),\ g_4 = (h_2, \mathbf{1}_2). \end{aligned}$$

The (closed) subgroup generated by the support of this measure μ is given by $\Gamma_\mu = \Gamma \times a\Gamma a^{-1} = \mathrm{SL}_2(\mathbb{Z}) \times a\,\mathrm{SL}_2(\mathbb{Z})a^{-1}$. Indeed, the correct entry in the second copy of G can be arranged using a finite product of $g_1^{\pm 1}, g_3^{\pm 1}$, and then the entry in the first copy can be corrected using $g_2^{\pm 1}, g_4^{\pm 1}$. By Theorem B we thus know that for the starting point $(x_0, x_0) \in X \times X$ we have the weak* convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_{(x_0,x_0)} \longrightarrow \nu_{(x_0,x_0)}$$

as $n \to \infty$, where $\nu_{(x_0,x_0)}$ is the homogeneous probability measure on the closure of the $\Gamma \times a\Gamma a^{-1}$-orbit of (x_0, x_0).

Let us identify this orbit closure. In the first copy of X, we recognize the finite orbit $\mathcal{O}$ from Example 1.1.1. In the second copy, we see the action of irrational conjugates of h_1, h_2. As the acting group has product structure, the orbit closure in question is the product of these two orbit closures in the components:

$$\overline{(\Gamma \times a\Gamma a^{-1})(x_0, x_0)} = \mathcal{O} \times \overline{a\Gamma a^{-1}x_0}.$$

Since the orbit $a\Gamma a^{-1}x_0$ is infinite by our choice of the matrix a, it follows from [**9**, Corollary 1.8] that $\overline{a\Gamma a^{-1}x_0} = X$, so that

$$\overline{(\Gamma \times a\Gamma a^{-1})(x_0, x_0)} = \mathcal{O} \times X \text{ and } \nu_{(x_0,x_0)} = m_{\mathcal{O}} \otimes m_X$$

for the normalized counting measure $m_{\mathcal{O}}$ on $\mathcal{O}$ and the Haar measure m_X on X. However, in analogy to Example 1.1.1, the random walk is found to alternate between the sets

$$\mathcal{O}_1 \times X \text{ and } \mathcal{O}_2 \times X,$$

[1] Loosely speaking, irreducibility of Λ means that it does not arise from a product construction. The precise definition is given before Corollary 4.0.2 in Chapter 4; see also [**107**, Definition 5.20].

in the sense that $\mathrm{supp}(\mu^{*2n} * \delta_{(x_0,x_0)}) \subset \mathcal{O}_1 \times X$ and $\mathrm{supp}(\mu^{*(2n+1)} * \delta_{(x_0,x_0)}) \subset \mathcal{O}_2 \times X$ for all $n \in \mathbb{N}$. Hence, we conclude that the random walk starting from (x_0, x_0) does not converge to $\nu_{(x_0,x_0)}$. ◇

Remark 1.1.3. The same behavior as in the previous example can be arranged inside a homogeneous space $X' = G'/\Lambda'$ that is the quotient of a semisimple real Lie group G' by an irreducible lattice Λ'. Indeed, this is only a matter of choosing suitable embeddings $G \times G \hookrightarrow G'$ and $X \times X \hookrightarrow X'$, where G and X are as in Example 1.1.2. Concretely, one can for example consider the 4×4-congruence lattice

$$\Lambda' = \Lambda(2) = \{g \in \mathrm{SL}_4(\mathbb{Z}) \mid g \equiv \mathbf{1}_4 \bmod 2\}$$

in $G' = \mathrm{SL}_4(\mathbb{R})$ and the diagonal embeddings

$$\begin{aligned} G \times G &\hookrightarrow G', & X \times X &\hookrightarrow X', \\ (g, h) &\mapsto \begin{pmatrix} g & \\ & h \end{pmatrix}, & (g\Lambda, h\Lambda) &\mapsto \begin{pmatrix} g & \\ & h \end{pmatrix}\Lambda'. \end{aligned}$$

We therefore see that Example 1.1.2, i.e. periodic behavior on a non-discrete orbit closure, can be realized inside $X' = G'/\Lambda'$. Of course, after applying this embedding, the subgroup generated by the support of μ will no longer be Zariski dense in G'. ◇

1.1.2. An Aperiodicity Criterion. Inspecting the examples above, one may notice that their common salient feature is that the orbit closure $\overline{\Gamma_\mu^+ x_0}$ is disconnected. This naturally raises the question whether periodic behavior can also occur when this orbit closure is connected. In what follows, we answer this question in the negative. We shall use the following formalization of periodicity.

Definition 1.1.4. Assume that the random walk on X given by μ converges on average to a probability measure ν on X from the starting point x_0 in X. We say that this convergence is *periodic* if there exists an integer $d \geq 2$ and pairwise disjoint measurable subsets $D_0, \dots, D_{d-1} \subset X$ with $\nu(\partial D_i) = 0$ for $0 \leq i < d$ and such that $(\mu^{*n} * \delta_{x_0})(D_{n \bmod d}) = 1$ for every $n \in \mathbb{N}$. Otherwise, we call the convergence *aperiodic*.

The requirement on the boundaries of the sets D_i is needed to ensure that the cyclic behavior is witnessed by the limit measure ν. Without a condition of this sort, one could try to artificially define D_i as the set of all points in X that can be reached from x_0 precisely in $n \equiv i \bmod d$ steps. Indeed, this construction is possible for example when μ is finitely supported with the property that its support freely generates a discrete subsemigroup Γ_μ^+ of G and the starting point $x_0 \in X$ has a free Γ_μ^+-orbit. The latter is the case e.g. for $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, $\mu = \frac{1}{2}(\delta_{h_1} + \delta_{h_2})$ with $h_1 = \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right)$ and $h_2 = \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right)$, and $x_0 = a\,\mathrm{SL}_2(\mathbb{Z})$ for a diagonal matrix $a \in \mathrm{SL}_2(\mathbb{R})$ such that the diagonal entries of a^2 are irrational.

We are now ready to state the announced aperiodicity theorem.

Theorem 1.1.5. *Retain the notation and assumptions from Theorem* B *and let $x_0 \in X$ be such that the orbit closure $\overline{\Gamma_\mu^+ x_0}$ is connected. Then the Cesàro convergence to ν_{x_0} of the random walk on X given by μ starting from x_0 is aperiodic.*

For the proof we need the following simple lemma.

LEMMA 1.1.6. *Let H be a Zariski connected real algebraic group and S a subset of H generating a Zariski dense subsemigroup. Then for every positive integer $d \in \mathbb{N}$, also the d-fold product set $S^d = \{g_d \cdots g_1 \mid g_1, \ldots, g_d \in S\}$ generates a Zariski dense subsemigroup of H. In particular, if* $\operatorname{supp}(\mu)$ *generates a Zariski dense subsemigroup for some probability measure μ on H, the same is true for* $\operatorname{supp}(\mu^{*d})$.

PROOF. Let $O \subset H$ be a non-empty Zariski open subset and consider the map $\phi \colon H \to H$, $g \mapsto g^d$. Since ϕ is Zariski continuous, $\phi^{-1}(O)$ is Zariski open. Moreover, this preimage is non-empty because O is dense in the Lie group topology on H and ϕ is a diffeomorphism near the identity. By the assumption that S generates a Zariski dense subsemigroup, we can thus find an element $g \in \phi^{-1}(O)$ that is the product of finitely many elements of S. It follows that $\phi(g) = g^d$ lies in the intersection of O with the subsemigroup of H generated by S^d.

The second claim involving μ immediately follows from the above together with the inclusion $\operatorname{supp}(\mu^{*d}) \supset \operatorname{supp}(\mu)^d$. □

PROOF OF THEOREM 1.1.5. Suppose $d \in \mathbb{N}$ is an integer such that there are pairwise disjoint $D_0, \ldots, D_{d-1} \subset X$ with $\nu_{x_0}(\partial D_i) = 0$ for all $0 \leq i < d$ and such that $(\mu^{*n} * \delta_{x_0})(D_{n \bmod d}) = 1$ for all $n \in \mathbb{N}$ as in the definition of periodicity. We have to show that $d = 1$.

First note that from Theorem B and the properties of the sets D_i it follows that

$$\nu_{x_0}(D_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\mu^{*k} * \delta_{x_0})(D_0) = \frac{1}{d}, \tag{1.1.1}$$

where the application of weak* convergence to the set D_0 is justified since it has negligible boundary with respect to the limit measure ν_{x_0}. In view of Lemma 1.1.6, Theorem B also applies to the d-step random walk given by μ^{*d}. Assuming for the moment that the limit measure for this d-step random walk starting from x_0 coincides with ν_{x_0}, we deduce that

$$\nu_{x_0}(D_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} (\mu^{*dk} * \delta_{x_0})(D_0) = 1. \tag{1.1.2}$$

Together, (1.1.1) and (1.1.2) imply $d = 1$, the desired conclusion.

It thus remains to show that the d-step random walk starting from x_0 does indeed have the same limit measure as the 1-step random walk. Denoting by $\Gamma^+ = \Gamma^+_\mu$ and Γ^+_d the closed subsemigroups of G generated by $\operatorname{supp}(\mu)$ and $\operatorname{supp}(\mu^{*d})$, respectively, this statement is equivalent to the equality $\overline{\Gamma^+ x_0} = \overline{\Gamma^+_d x_0}$ of orbit closures. To prove this, let $g \in \operatorname{supp}(\mu)$ be arbitrary. We claim that

$$\overline{\Gamma^+ x_0} = \bigcup_{k=0}^{d-1} g^{-k} \overline{\Gamma^+_d x_0}.$$

Indeed, since $\overline{\Gamma^+ x_0}$ is homogeneous, it is invariant under the group generated by Γ^+. As $\overline{\Gamma^+ x_0}$ clearly contains $\overline{\Gamma^+_d x_0}$, the inclusion "$\supset$" follows. For the reverse inclusion let $g_n, \dots, g_1 \in \operatorname{supp}(\mu)$ for some $n \in \mathbb{N}$. Choose $0 \le k < d$ such that $n + k \equiv 0 \bmod d$. Then $g^k g_n \cdots g_1 x_0 \in \overline{\Gamma_d^+ x_0}$ and hence $g_n \cdots g_1 x_0 \in g^{-k}\overline{\Gamma_d^+ x_0}$, giving the claim.

We already noted that Theorem B applies to μ^{*d}. In particular, the orbit closure $\overline{\Gamma_d^+ x_0}$ and its translates by g^{-k}, $0 \le k < d$, are submanifolds of $\overline{\Gamma^+ x_0}$. Necessarily, all these translates have the same dimension, and since together they make up $\overline{\Gamma^+ x_0}$ by the claim above, their shared dimension coincides with that of $\overline{\Gamma^+ x_0}$. This implies that $\overline{\Gamma_d^+ x_0}$ is open in $\overline{\Gamma^+ x_0}$. However, it is also closed, so that the assumed connectedness of $\overline{\Gamma^+ x_0}$ forces $\overline{\Gamma^+ x_0} = \overline{\Gamma_d^+ x_0}$. This completes the proof. $\square$

We end this section by recording a corollary of the proof above.

Corollary 1.1.7. *Retain the notation and assumptions from Theorem* B *and denote by* $\Gamma^+ = \Gamma_\mu^+$ *the closed subsemigroup of* G *generated by* $\operatorname{supp}(\mu)$. *Suppose that* $\overline{\Gamma^+ x_0}$ *is connected. Let* $d \in \mathbb{N}$ *and denote by* Γ_d^+ *the closed subsemigroup generated by* $\operatorname{supp}(\mu^{*d})$. *Then* $\overline{\Gamma^+ x_0} = \overline{\Gamma_d^+ x_0}$, *and for the homogeneous probability measure* ν_{x_0} *on this orbit closure we have for arbitrary* $r \in \mathbb{N}_0$ *that*

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*(dk+r)} * \delta_{x_0} \longrightarrow \nu_{x_0} \tag{1.1.3}$$

as $n \to \infty$ *in the weak* topology.*

Proof. The statement about orbit closures was established as part of the proof of Theorem 1.1.5. From Theorem B we thus get the weak* convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*dk} * \delta_{x_0} \overset{n\to\infty}{\longrightarrow} \nu_{x_0}, \tag{1.1.4}$$

which is (1.1.3) for $r = 0$. Given $f \in C_c(X)$, the general case follows by applying (1.1.4) to the compactly supported continuous function f_r defined by

$$f_r(x) = \int_G f(gx)\,\mathrm{d}\mu^{*r}(g) = \int_{G^r} f(g_r \cdots g_1 x)\,\mathrm{d}\mu^{\otimes r}(g_1, \dots, g_r)$$

for $x \in X$. $\square$

This corollary sharpens the convergence statement in Theorem B in the case of a connected orbit closure: The Cesàro convergence to ν_{x_0} holds along arbitrary arithmetic progressions. Although this does not provide an answer to Question 3, it at least allows the following conclusion to be drawn: If $(n_i)_i$ is a sequence of indices such that $\mu^{*n_i} * \delta_{x_0}$ converges to a weak* limit different from ν_{x_0} as $i \to \infty$, then $(n_i)_i$ cannot contain a density 1 subset of an infinite arithmetic progression.

1.2. Spectral Gap

In this section, we will explain how a spectral gap of the convolution operator $\pi(\mu)$ associated to a random walk entails the convergence of $\mu^{*n} * \delta_x$ towards m_X for m_X-a.e. $x \in X$. In its simplest form, the involved argument works in great generality and also produces an exponential rate of convergence from almost every starting point when the test function f is fixed. This is carried out in §1.2.1. The following subsections §§1.2.2–1.2.4 are dedicated to a substantial refinement of this spectral gap argument for random walks on homogeneous spaces of real Lie groups, making the exponentially fast convergence uniform over smooth test functions.

1.2.1. Generic Points. Recall that $\pi(\mu)\colon L^\infty(X,m_X)\to L^\infty(X,m_X)$ is defined by

$$\pi(\mu)f(x) \coloneqq \int_X f\,\mathrm{d}(\mu*\delta_x) = \int_G f(gx)\,\mathrm{d}\mu(g)$$

for $f\in L^\infty(X,m_X)$ and $x\in X$. This convolution operator extends to a continuous contraction on each L^p-space (see [**10**, Corollary 2.2]). We shall study its behavior on $L^2(X,m_X)$. By ergodicity, the G-fixed functions are the constant functions, so we restrict our attention to their orthogonal complement of L^2-functions with mean 0, denoted by $L^2_0(X,m_X)$.

Definition 1.2.1. We say that μ has a *spectral gap* on X if the associated convolution operator $\pi(\mu)$ restricted to $L^2_0(X,m_X)$ has spectral radius strictly less than 1.

We note that by the spectral radius formula, μ having a spectral gap on X can be reformulated as the requirement that

$$\lim_{n\to\infty}\sqrt[n]{\|\pi(\mu)|^n_{L^2_0}\|_{\mathrm{op}}}<1.$$

Given the existence of a spectral gap, we obtain an almost everywhere convergence result in a quite general setup.

Theorem 1.2.2. *Suppose that μ has a spectral gap on X. Then m_X-a.e. starting point $x\in X$ is* generic *for the random walk on X given by μ, meaning that*

$$\mu^{*n}*\delta_x\longrightarrow m_X$$

as $n\to\infty$ in the weak topology. This convergence is exponentially fast in the sense that for every fixed $f\in L^2(X,m_X)$ we have*

$$\limsup_{n\to\infty}\left|\int_X f\,\mathrm{d}(\mu^{*n}*\delta_x)-\int f\,\mathrm{d}m_X\right|^{1/n}\le\rho(\pi(\mu)|_{L^2_0})^{1/2}$$

for m_X-a.e. $x\in X$, where $\rho(\pi(\mu)|_{L^2_0})$ denotes the spectral radius of $\pi(\mu)$ restricted to $L^2_0(X,m_X)$.

Proof. By separability of $C_c(X)$, for the statement about weak* convergence it suffices to prove m_X-a.s. convergence for one fixed function $f\in C_c(X)$. Consequently, it is enough to prove the second assertion of the theorem. To this end, fix a function $f\in L^2(X,m_X)$ and a number $\rho(\pi(\mu)|_{L^2_0})<\alpha<1$, and consider the L^2_0-function $f_0=f-\int f\,\mathrm{d}m_X$. Then in view of the spectral radius formula we have

$$\left\|\pi(\mu)^n f-\int f\,\mathrm{d}m_X\right\|_{L^2}=\|\pi(\mu)^n f_0\|_{L^2}\le\|\pi(\mu)|^n_{L^2_0}\|_{\mathrm{op}}\|f_0\|_{L^2}\le\alpha^n\|f_0\|_{L^2}$$

for sufficiently large $n\in\mathbb{N}$. A standard Borel–Cantelli argument now implies the statement. As we shall need similar estimates later on, we quickly go through the details: By Chebyshev's inequality, the above implies that for large n we have

$$m_X\Big(\Big\{x \in X \;\Big|\; \Big|\pi(\mu)^n f(x) - \int f \,\mathrm{d}m_X\Big| \geq \alpha^{n/2}\|f_0\|_{L^2}\Big\}\Big) \leq \frac{\|\pi(\mu)^n f - \int f \,\mathrm{d}m_X\|_{L^2}^2}{\alpha^n \|f_0\|_{L^2}^2} \leq \alpha^n.$$

By Borel–Cantelli it follows that for m_X-a.e. $x \in X$, the inequality

$$\Big|\pi(\mu)^n f(x) - \int f \,\mathrm{d}m_X\Big| \geq \alpha^{n/2}\|f_0\|_{L^2}$$

holds only for finitely many $n \in \mathbb{N}$. Since $\pi(\mu)^n f(x) = \int f \,\mathrm{d}(\mu^{*n} * \delta_x)$, letting α approach $\rho(\pi(\mu)|_{L_0^2})$ gives the result. □

Remark 1.2.3. In the second conclusion of Theorem 1.2.2, how long it takes for the exponential rate of convergence to kick in depends on the point x. However, the measure of sets on which one has to wait for a long time can be controlled as follows: Given $\rho(\pi(\mu)|_{L_0^2}) < \alpha < 1$, choose $N \in \mathbb{N}$ such that $\|\pi(\mu)|_{L_0^2}^n\|_{\mathrm{op}} \leq \alpha^n$ for all $n \geq N$. Then if we denote

$$B_{\alpha,n,f} = \Big\{x \in X \;\Big|\; \Big|\pi(\mu)^{n'} f(x) - \int f \,\mathrm{d}m_X\Big| \geq \alpha^{n'/2}\|f_0\|_{L^2} \text{ for some } n' \geq n\Big\},$$

the proof above gives the bound

$$m_X(B_{\alpha,n,f}) \leq \frac{\alpha^n}{1-\alpha}$$

for every $n \geq N$. In other words, the measure of the set on which the exponential convergence does not start during the first n steps decays exponentially in n. ◇

We now demonstrate that the previous result covers the case of our main interest.

Proposition 1.2.4. *Let G be a connected semisimple real Lie group without compact factors and with finite center, $\Lambda < G$ a lattice, and X the homogeneous space G/Λ endowed with the Haar measure m_X. Suppose that* $\mathrm{Ad}(\Gamma_\mu^+)$ *is Zariski dense in* $\mathrm{Ad}(G)$*. Then μ has a spectral gap on X.*

Proof. Consider the regular representation of G on $L_0^2(X, m_X)$. By [**3**, Lemma 3] it does not weakly contain the trivial representation. From this, in view of [**125**, Theorem C], the result follows if we can argue that the projection of μ to any simple factor of G is not supported on a closed amenable subgroup. However, since amenability passes to the Zariski closure (see e.g. [**136**, Theorem 4.1.15]) the latter would imply that one of the simple factors of $\mathrm{Ad}(G)$ is amenable, hence compact by a classical result of Furstenberg (see e.g. [**136**, Proposition 4.1.8]). □

1.2.2. Good Height Functions. Inspecting the proof of Theorem 1.2.2, one observes that every step is effective, with explicit bounds and good control over the measure of exceptional sets, except for the very first one: separability of the space $C_c(X)$ of compactly supported continuous functions. In the remainder of this section, we aim to also make effective this step, the goal being exponentially fast convergence $\mu^{*n} * \delta_x \to m_X$ from almost every starting point, uniformly over functions f on X. As merely continuous functions can behave arbitrarily badly (with respect to the convergence problem at hand), there is no hope of achieving this feat for all $f \in C_c(X)$. We shall therefore restrict our attention to smooth functions of compact support, and take into account their regularity by considering not just their L^2-, but also certain Sobolev norms. Built into the definition of these norms will be what we call a "good height function", the concept of which is introduced in this subsection.

Our setup is as follows: Let G be a real Lie group with Lie algebra $\mathfrak{g}$. We endow $\mathfrak{g}$ with a scalar product, which we use to define a right-invariant metric d^G on G. Given a lattice $\Lambda < G$, this metric descends to a metric d^X on $X = G/\Lambda$ such that the projection $G \to X$ is locally an isometry. Moreover, we fix an orthonormal basis of $\mathfrak{g}$, using which we will identify $\mathfrak{g}$ with $\mathbb{R}^{\dim \mathfrak{g}}$. Here is the crucial definition.

Definition 1.2.5. We call a measurable function $\mathrm{ht}\colon X \to (0, \infty)$ a *good height function* if there exists $0 < R \leq 1$ and a function $r\colon X \to (0, R]$ with the following properties:

(i) The restriction of the exponential map $\exp\colon (-R, R)^{\dim \mathfrak{g}} \to G$ is a diffeomorphism onto its image and $\exp((-r/2, r/2)^{\dim \mathfrak{g}}) \subset B_r^G(e)$ for all $r \leq R$, where $B_r^G(e)$ denotes the open ball of radius r around the identity $e \in G$ with respect to the metric d^G on G.
(ii) For all $x \in X$, the projection $G \supset B_{r(x)}^G(e) \to X$, $g \mapsto gx$ is injective.
(iii) There exist constants $c, \kappa > 0$ such that $r(x) \geq c\,\mathrm{ht}(x)^{-\kappa}$ for all $x \in X$.
(iv) There exists a constant $\sigma > 1$ such that $\mathrm{ht}(x) \leq \sigma\,\mathrm{ht}(gx)$ for all $x \in X$ and all $g \in B_{r(x)}^G(e)$.

The definition suggests to think of a good height function as reciprocal of the injectivity radius. And indeed, this viewpoint allows their construction on any homogeneous space $X = G/\Lambda$.

Proposition 1.2.6. *Let G be a real Lie group and Λ a lattice in G. Then $X = G/\Lambda$ admits a good height function.*

Proof. Choose $R > 0$ such that condition (i) of the definition is satisfied and set $r(x) = \min(R, r_{\mathrm{inj}}(x))$, where $r_{\mathrm{inj}}(x)$ is the injectivity radius at $x \in X$, i.e. the maximal radius such that (ii) holds at x. Define

$$\mathrm{ht}(x) = r(x)^{-1}.$$

Then the only thing that needs to be verified is the validity of (iv). We claim that it holds with $\sigma = 2$. This will follow if we can show that

$$r_{\mathrm{inj}}(gx) \leq 2 r_{\mathrm{inj}}(x) \tag{1.2.1}$$

whenever $g \in B_{r(x)}^G(e)$. To this end, let $r > r_{\mathrm{inj}}(x)$. Then by definition, there are distinct $g_1, g_2 \in B_r^G(e)$ such that $g_1 x = g_2 x$. As $g \in B_{r(x)}^G(e)$, right-invariance of the metric implies

$$\mathrm{d}^G(g_i g^{-1}, e) = \mathrm{d}^G(g_i, g) \leq \mathrm{d}^G(g_i, e) + \mathrm{d}^G(g, e) < r + r(x) < 2r$$

for $i = 1, 2$, and we also have $(g_1 g^{-1})gx = (g_2 g^{-1})gx$. This shows that $r_{\mathrm{inj}}(gx) \leq 2r$, and as $r > r_{\mathrm{inj}}(x)$ was arbitrary, we see that (1.2.1) holds. $\square$

Often, however, one might want to work with different, naturally occurring height functions. The flexibility in our definition of a good height function accommodates this possibility.

In the examples below, we denote by $\lambda_1(\Lambda)$ the length of a shortest non-zero vector in a lattice $\Lambda < \mathbb{R}^d$.

EXAMPLE 1.2.7. Let $G = \mathrm{SL}_d(\mathbb{R})$ and $\Lambda = \mathrm{SL}_d(\mathbb{Z})$. Then $X = G/\Lambda$ can be identified with the space of lattices in $\mathbb{R}^d$ with covolume 1 via

$$X \ni g\,\mathrm{SL}_d(\mathbb{Z}) \longleftrightarrow g\mathbb{Z}^d \subset \mathbb{R}^d.$$

Then the function $\mathrm{ht} = \lambda_1^{-1}$, defined on X via the above identification, is a good height function. Indeed, one can first choose $R > 0$ such that (i) is satisfied, and then set $r(x) = \min(R, r_{\mathrm{inj}}(x))$ as in the proof of Proposition 1.2.6. Then (ii) is automatically satisfied, and (iv) is valid for a suitable choice of σ due to the inequality $\lambda_1(gx) \leq \|g\|\lambda_1(x)$ for $g \in G$ and $x \in X$, where $\|\cdot\|$ denotes any matrix norm. To see that also (iii) holds, let $x = g\Lambda$ and suppose that $hx = x$ for some $h \in G$ with $h \neq e$. Then for all $\gamma \in \mathrm{SL}_d(\mathbb{Z})$, the matrix $(g\gamma)^{-1}h(g\gamma)$ fixes the lattice $\mathbb{Z}^d$ but is not the identity, so that

$$\|g\gamma\|^{\kappa_1}\|h-e\| \geq \|(g\gamma)^{-1}(h-e)(g\gamma)\| = \|(g\gamma)^{-1}h(g\gamma)-e\| \geq c_1$$

for some constants $c_1, \kappa_1 > 0$. For a basis change $\gamma \in \mathrm{SL}_d(\mathbb{Z})$ such that $g\gamma$ consists of a reduced basis of the lattice x we have $\|g\gamma\| \leq c_2\lambda_1(x)^{-\kappa_2}$ for some $c_2, \kappa_2 > 0$ (cf. e.g. [**128**, Chapter III]). With this choice, the above inequality implies

$$\|h-e\| \geq c\lambda_1(x)^{\kappa}$$

for $c = c_1/c_2$ and $\kappa = \kappa_1\kappa_2$. Since near the identity, the metric d^G on G is Lipschitz-equivalent to the distance induced by $\|\cdot\|$, this establishes (iii). ◇

A similar construction is possible in a more general context.

EXAMPLE 1.2.8 ([**36**]). Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a semisimple $\mathbb{Q}$-group $\mathbf{G}$ and Λ an arithmetic lattice in G. Choose a rational $\mathrm{Ad}(\Lambda)$-stable lattice $\mathfrak{g}_{\mathbb{Z}} \subset \mathfrak{g}$. Then, using similar reasoning as in the previous example, the function ht on $X = G/\Lambda$ defined by

$$\mathrm{ht}(x) = \lambda_1(\mathrm{Ad}(g)\mathfrak{g}_{\mathbb{Z}})^{-1}$$

for $x = g\Lambda \in X$ is seen to be a good height function (cf. [**36**, §3.6]). ◇

1.2.3. Sobolev Norms. Given a good height function ht on X, the associated *Sobolev norm* of degree $\ell \geq 0$ of a compactly supported smooth function $f \in C_c^\infty(X)$ is defined by

$$\mathcal{S}_\ell(f)^2 = \sum_{\deg \mathcal{D} \leq \ell} \|\mathrm{ht}(\cdot)^\ell \mathcal{D}f\|_{L^2}^2,$$

where the sum runs over differential operators $\mathcal{D}$ given by monomials of degree at most ℓ in elements of the fixed orthonormal basis of $\mathfrak{g}$ in the universal enveloping algebra.

In other words, the differential operators $\mathcal{D}$ appearing above are $\partial_{v_1}\cdots\partial_{v_k}$ for any k-tuple $(v_1, \ldots, v_k)$ of elements of the fixed basis of $\mathfrak{g}$, $0 \leq k \leq \ell$, where

∂_v for $v \in \mathfrak{g}$ is defined by

$$\partial_v f(x) = \lim_{t\to 0} \frac{f(\exp(tv)x) - f(x)}{t}$$

for $f \in C_c^\infty(X)$ and $x \in X$.

Here are two immediate observations.

LEMMA 1.2.9. *Let* ht *be a good height function on X and $\mathcal{S}_\ell$ the associated Sobolev norms.*

(i) *The norms $\mathcal{S}_\ell$ are induced by inner products $\langle\cdot,\cdot\rangle_\ell$ on $C_c^\infty(X)$.*
(ii) *Given $0 \le \ell_0 \le \ell_1$, there exists a constant $\tilde{c} > 0$ such that $\mathcal{S}_{\ell_0} \le \tilde{c}\mathcal{S}_{\ell_1}$.*

PROOF. Part (i) is clear. Part (ii) is also immediate from the definition of the Sobolev norms, once we know that a good height function must be bounded away from 0. The latter, however, follows directly from property (iii) in the definition of a good height function, as the function r appearing there is assumed to be bounded. □

The proof of our convergence result in §1.2.4 will depend on the following proposition.

PROPOSITION 1.2.10 ([**36**]). *For the Sobolev norms associated to a good height function on X, there exists a non-negative integer $\ell_0 \ge 0$ and a constant $C > 0$ with the following properties:*

(i) (*Sobolev embedding estimate,* [**36**, (3.9)]) *For every $f \in C_c^\infty(X)$ it holds that $\|f\|_\infty \le C\mathcal{S}_{\ell_0}(f)$.*
(ii) (*Finite relative traces,* [**36**, (3.10)]) *For all integers $\ell \ge 0$ the relative trace* $\mathrm{Tr}(\mathcal{S}_\ell^2|\mathcal{S}_{\ell+\ell_0}^2)$ *is finite, meaning that for any orthogonal basis $(e^{(k)})_k$ in the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_{\ell+\ell_0}$*

$$\mathrm{Tr}(\mathcal{S}_\ell^2|\mathcal{S}_{\ell+\ell_0}^2) := \sum_k \frac{\mathcal{S}_\ell(e^{(k)})^2}{\mathcal{S}_{\ell+\ell_0}(e^{(k)})^2} < \infty.$$

We refer to Bernstein–Reznikov [**13**] for a systematic treatment of relative traces. In particular, it is proved in this reference that the above expression is independent of the choice of orthogonal basis.

The proofs in [**36**] of the statements in the above proposition are given for the height function from Example 1.2.8. However, the only properties used are those in our definition of a good height function. In fact, the arguments only depend on validity of the second statement in [**36**, Lemma 5.1], which holds in our context, as we demonstrate below.

LEMMA 1.2.11. *Let* ht *be a good height function on X. Then there exists a non-negative integer $\ell_0 \ge 0$ and a constant $C > 0$ such that for every non-negative integer $\ell \ge 0$ and every differential operator $\mathcal{D}$ given by a monomial of degree at most ℓ in elements of the fixed basis of $\mathfrak{g}$ we have*

$$|\mathrm{ht}(x)^\ell \mathcal{D}f(x)| \le C\mathcal{S}_{\ell+\ell_0}(f)$$

for every $f \in C_c^\infty(X)$ and $x \in X$.

PROOF. We inspect the function $F = \mathcal{D}f$ in a chart around x given by the exponential map: We set $\varepsilon = r(x)/2$, where $r\colon X \to (0, R]$ is the function from the definition of a good height function, $d = \dim \mathfrak{g}$, and consider

$$\tilde{F}\colon (-\varepsilon, \varepsilon)^d \to \mathbb{R},\ v \mapsto F(\exp(v)x).$$

Then by the first statement of [**36**, Lemma 5.1], which is simply a Sobolev embedding estimate on $\mathbb{R}^d$, we know

$$|F(x)| = |\tilde{F}(0)| \leq C_1 2^d r(x)^{-d} \mathcal{S}_{d,\varepsilon}(\tilde{F}), \tag{1.2.2}$$

where $C_1 > 0$ is a constant depending only on the dimension d of $\mathfrak{g}$ and $\mathcal{S}_{d,\varepsilon}$ is the standard degree d Sobolev norm on the open subset $(-\varepsilon, \varepsilon)^d$ of $\mathbb{R}^d$, i.e.

$$\mathcal{S}_{d,\varepsilon}(\tilde{F})^2 = \sum_{|\boldsymbol{\alpha}| \leq d} \|\partial_{\boldsymbol{\alpha}} \tilde{F}\|^2_{L^2((-\varepsilon,\varepsilon)^d)},$$

where the sum is over all multi-indices $\boldsymbol{\alpha}$ of degree at most d and $\partial_{\boldsymbol{\alpha}} \tilde{F}$ is the corresponding standard partial derivative of $\tilde{F}$. Using property (iii) in the definition of a good height function, (1.2.2) implies that

$$|\mathrm{ht}(x)^\ell F(x)| \leq C_2 \, \mathrm{ht}(x)^{\ell+\ell_0} \mathcal{S}_{d,\varepsilon}(\tilde{F}), \tag{1.2.3}$$

where $C_2 > 0$ is another constant and we used that ht is bounded away from 0 to replace κd appearing in the exponent by $\ell_0 = \max(\lceil \kappa d \rceil, d)$. Using properties (i) and (ii) in the definition of a good height function, we find $C_3 > 0$ such that

$$\mathcal{S}_{d,\varepsilon}(\tilde{F}) \leq C_3 \sqrt{\sum_{\deg \mathcal{D}' \leq d} \|\mathcal{D}' F|_{B^X_{r(x)}(x)}\|^2_{L^2}}. \tag{1.2.4}$$

To see this, one needs to note two things: firstly, that by the chain rule the partial derivatives of $\tilde{F}$ at a point $v \in (-\varepsilon, \varepsilon)^d$ in the chart can be expressed as linear combinations of derivatives $\mathcal{D}'F$ appearing on the right-hand side in (1.2.4) evaluated at the corresponding point $x' = \exp(v)x$, with fixed coefficient functions depending only on finitely many derivatives of the exponential map on $(-\varepsilon, \varepsilon)^d$; and secondly, that the Haar measure m_X is a smooth measure, meaning that it has a smooth and nowhere vanishing density w.r.t. Lebesgue measure in the chart.

Combining (1.2.3), (1.2.4), condition (iv) in the definition of a good height function, and plugging back in the definition of F, we finally arrive at

$$|\mathrm{ht}(x)^\ell \mathcal{D}f(x)| \leq C_4 \sqrt{\sum_{\deg \mathcal{D}' \leq d} \|\mathrm{ht}(\cdot)^{\ell+\ell_0} \mathcal{D}' \mathcal{D} f|_{B^X_{r(x)}(x)}\|^2_{L^2}} \leq C_4 \mathcal{S}_{\ell+\ell_0}(f),$$

for yet another constant $C_4 > 0$, which is the one appearing in the lemma. □

1.2.4. Exponentially Generic Points. Now we are ready to define the notion of effective genericity we wish to establish, and to prove the main convergence result of this section.

Until the end of this section, we fix a good height function ht on X. Moreover, given a bounded measurable function f on X and $n \in \mathbb{N}$ we will use the notation

$$D_n(f)(x) = \pi(\mu)^n f(x) - \int f \, \mathrm{d}m_X$$

for $x \in X$. We refer to $D_n(f)$ as the *time n discrepancy* for the function f.

DEFINITION 1.2.12. We say that a point $x \in X$ is (ℓ, β)-*exponentially generic* if $\ell \geq 0$ is a non-negative integer and β a real number in $(0,1)$ satisfying

$$\limsup_{n\to\infty} \sup_{f\in C_c^\infty(X)\setminus\{0\}} \left(\frac{|D_n(f)(x)|}{\mathcal{S}_\ell(f)}\right)^{1/n} \leq \beta,$$

where $\mathcal{S}_\ell$ is the degree ℓ Sobolev norm associated to ht.

With this terminology, we have the following result, which quantifies the dependence on the function f in the effective part of Theorem 1.2.2.

THEOREM 1.2.13. *Let G be a real Lie group, $\Lambda < G$ a lattice and $X = G/\Lambda$ endowed with the Haar measure m_X. Suppose that μ has a spectral gap on X. Then there exists a non-negative integer $\ell_1 \geq 0$ such that m_X-almost every point $x \in X$ is $(\ell_1, \rho(\pi(\mu)|_{L^2_0})^{1/2})$-exponentially generic.*

Our argument uses ideas from the proof of [**36**, Proposition 9.2]. Recall that $\langle\cdot,\cdot\rangle_\ell$ denotes the inner product associated to the Sobolev norm $\mathcal{S}_\ell$.

PROOF. Set $\ell_1 = 2\ell_0$ with ℓ_0 from Proposition 1.2.10. We denote by $\mathcal{H}$ the completion of $C_c^\infty(X)$ with respect to $\mathcal{S}_{\ell_1}$.

The first step of the proof is to argue that $\mathcal{H}$ admits an orthonormal basis $(e^{(k)})_k$ with respect to $\mathcal{S}_{\ell_1}$ that is also orthogonal with respect to $\mathcal{S}_{\ell_0}$. To this end, let us endow $\mathcal{H}$ with the scalar product $\langle\cdot,\cdot\rangle_{\ell_1}$ associated to $\mathcal{S}_{\ell_1}$. This makes $\mathcal{H}$ into a Hilbert space. As a consequence of Lemma 1.2.9(ii), $\langle\cdot,\cdot\rangle_{\ell_0}$ defines a bounded positive definite Hermitian form on $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\ell_1})$. Using Riesz representation it follows that there is a bounded positive self-adjoint operator T on $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\ell_1})$ such that

$$\langle v, w\rangle_{\ell_0} = \langle Tv, w\rangle_{\ell_1}$$

for all $v, w \in \mathcal{H}$. Finiteness of the relative trace $\mathrm{Tr}(\mathcal{S}^2_{\ell_0}|\mathcal{S}^2_{\ell_1})$ from Proposition 1.2.10(ii) then translates into the statement that T is a trace-class operator on $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\ell_1})$ (cf. [**38**, Proposition 6.44]); in particular, the operator T is compact (cf. [**38**, Proposition 6.42]). By the spectral theorem, T is thus diagonalizable. Hence, an orthonormal basis $(e^{(k)})_k$ of $(\mathcal{H}, \langle\cdot,\cdot\rangle_{\ell_1})$ consisting of eigenvectors of T is a basis with the desired properties.

Next, fix a number $\rho(\pi(\mu)|_{L^2_0}) < \alpha < 1$. As in the proof of Theorem 1.2.2, using Chebyshev's inequality we find that for every $k \geq 0$ and large enough n we have

$$m_X\Big(\Big\{x \in X \;\Big|\; |D_n(e^{(k)})(x)| \geq \alpha^{n/2}\mathcal{S}_{\ell_0}(e^{(k)})\Big\}\Big) \leq \frac{\|e_0^{(k)}\|^2_{L^2}}{\mathcal{S}_{\ell_0}(e^{(k)})^2}\alpha^n \leq \frac{\|e^{(k)}\|^2_{L^2}}{\mathcal{S}_{\ell_0}(e^{(k)})^2}\alpha^n,$$

where $e_0^{(k)} = e^{(k)} - \int e^{(k)}\,\mathrm{d}m_X$. Since the relative trace $\mathrm{Tr}(\mathcal{S}_0^2|\mathcal{S}^2_{\ell_0})$ is finite by Proposition 1.2.10, the terms on the right-hand side above are summable over $k, n \geq 0$. Borel–Cantelli thus implies that

$$A = \limsup_{k,n\geq 0}\Big\{x \in X \;\Big|\; |D_n(e^{(k)})(x)| \geq \alpha^{n/2}\mathcal{S}_{\ell_0}(e^{(k)})\Big\}$$

is a null set. We claim that any $x \notin A$ is $(\ell_1, \alpha^{1/2})$-exponentially generic. Fix such a point x and let $f \in C_c^\infty(X) \setminus \{0\}$. Write $f = \sum_k f_k e^{(k)}$ for the expansion of f in terms of the orthonormal basis $(e^{(k)})_k$. Then, using the triangle inequality, we can estimate the time n discrepancy for f as follows:

$$|D_n(f)(x)| \leq \sum_k |f_k||D_n(e^{(k)})(x)|. \tag{1.2.5}$$

The exchange of integral and summation involved in the above estimate is justified by part (i) of Proposition 1.2.10: It ensures that the functions $e^{(k)}$ are defined pointwise and the series expansion of f converges uniformly. Since $x \notin A$ we know that for large n the inequality $|D_n(e^{(k)})(x)| < \alpha^{n/2}\mathcal{S}_{\ell_0}(e^{(k)})$ holds for all k. For such n, an application of the Cauchy–Schwarz inequality implies that (1.2.5) is strictly less than

$$\alpha^{n/2}\Big(\sum_k |f_k|^2\Big)^{1/2}\Big(\sum_k \mathcal{S}_{\ell_0}(e^{(k)})^2\Big)^{1/2} = \alpha^{n/2}\mathcal{S}_{\ell_1}(f)\operatorname{Tr}(\mathcal{S}_{\ell_0}^2|\mathcal{S}_{\ell_1}^2)^{1/2}. \tag{1.2.6}$$

Again by Proposition 1.2.10, the relative trace $\operatorname{Tr}(\mathcal{S}_{\ell_0}^2|\mathcal{S}_{\ell_1}^2)$ is finite. Hence, in view of our definition of exponential genericity, combining (1.2.5) and (1.2.6) establishes the claim. Letting $\alpha \searrow \rho(\pi(\mu)|_{L^2_0})$ gives the theorem. $\square$

Remark 1.2.14. In analogy to Remark 1.2.3, we can control the measure of the set of points where exponentially generic behavior is not observed for a given number of steps: If we define

$$B_{\alpha,n} = \Big\{x \in X \;\Big|\; |D_{n'}(f)(x)| \geq \alpha^{n'/2}\mathcal{S}_{\ell_1}(f)\operatorname{Tr}(\mathcal{S}_{\ell_0}^2|\mathcal{S}_{\ell_1}^2)^{1/2} \text{ for some } n' \geq n, f \in C_c^\infty(X)\Big\}$$

for $\rho(\pi(\mu)|_{L^2_0}) < \alpha < 1$ and $n \in \mathbb{N}$, and $N \in \mathbb{N}$ is chosen in such a way that $\|\pi(\mu)|^n_{L^2_0}\|_{\mathrm{op}} \leq \alpha^n$ for all $n \geq N$, then for every $n \geq N$ it holds that

$$m_X(B_{\alpha,n}) \leq \operatorname{Tr}(\mathcal{S}_0^2|\mathcal{S}_{\ell_0}^2)\frac{\alpha^n}{1-\alpha}.$$

Indeed, we have $B_{\alpha,n} \subset \bigcup_{n'\geq n, k\geq 0}\{x \in X \mid |D_{n'}(e^{(k)})(x)| \geq \alpha^{n'/2}\mathcal{S}_{\ell_0}(e^{(k)})\}$, as the proof of Theorem 1.2.13 demonstrates. Thus, again, the measure of the set of "bad points", on which exponential genericity takes more than n steps to manifest, is itself exponentially small in n. $\diamond$

1.3. Uniform Cesàro Convergence

In this last section of the chapter, we explore the situation where the only possible limit in Theorem B is the Haar measure m_X. In this setting, by analogy with the case of unique ergodicity in classical ergodic theory, it is reasonable to expect the Cesàro convergence in part (ii) of Theorem B to hold (locally) uniformly in the starting point x_0. We shall prove in §1.3.1 below that this indeed holds true. In §1.3.2, we conclude the chapter by showing that in many naturally occurring situations something even stronger than locally uniform can be achieved.

Before continuing with the pertinent definitions, let us recall that even though the setup of Theorem B is our motivation and useful to have in mind, formally we are working with the following more general setup: (X, m_X) is merely required to be a space with a continuous G-action for which m_X is invariant and ergodic.

Definition 1.3.1. The random walk on X induced by μ is called *uniquely ergodic* if m_X is the unique μ-stationary probability measure on X.

In particular, for a random walk to be uniquely ergodic, there must be no finite Γ_μ-orbits in X. In the case that $X = G/\Lambda$ for a lattice Λ in G, this happens if and only if Γ_μ is not virtually contained in a conjugate of Λ.[2] In fact, in many cases of interest, finite orbits are the only obstruction to unique ergodicity: For example, this is true when G is a connected semisimple Lie group without compact factors, Λ is an irreducible lattice, $X = G/\Lambda$, and $\mathrm{Ad}(\Gamma_\mu^+)$ is Zariski dense in $\mathrm{Ad}(G)$ (see [**9**, Corollary 1.8]); and also in the setting of [**129**], a special case of which is reproduced below as Example 1.3.8.

1.3.1. Locally Uniform Convergence. The notion of unique ergodicity introduced above coincides with the classical property of unique ergodicity of the Markov operator $\pi(\mu)$. When the space X is compact, this is enough to guarantee that the Cesàro convergence $\frac{1}{n}\sum_{k=0}^{n-1}\mu^{*k} * \delta_x \to m_X$ as $n \to \infty$ is uniform in x (see e.g. [**78**, §5.1]). Without compactness, we also need to assume a form of recurrence.

Definition 1.3.2. We say that the random walk on X given by μ is *locally uniformly recurrent* if for every compact subset $K \subset X$ and $\varepsilon > 0$ there exists a positive integer $n_0 \in \mathbb{N}$ and a compact subset $M \subset X$ with

$$\mu^{*n} * \delta_x(M) \geq 1 - \varepsilon$$

for all $n \geq n_0$ and $x \in K$. It is called *locally uniformly recurrent on average* if the above holds with the Cesàro averages $\frac{1}{n}\sum_{k=0}^{n-1}\mu^{*k} * \delta_x$ in place of $\mu^{*n} * \delta_x$.

It is a simple exercise to check that locally uniform recurrence implies locally uniform recurrence on average. In concrete examples, recurrence properties such as these are typically established by constructing a Lyapunov function; see §1.3.2 below.

The following well-known fact explains why these properties are referred to as "non-escape of mass".

Lemma 1.3.3. *Let the sequence $\{x_n\}_n$ of points in X be relatively compact and suppose that the random walk on X is locally uniformly recurrent (resp. on average). Then every weak* limit of the sequence $(\mu^{*n} * \delta_{x_n})_n$ (resp. $(\frac{1}{n}\sum_{k=0}^{n-1}\mu^{*k} * \delta_{x_n})_n$) is a probability measure.* □

The proof is immediate and left to the reader.

We are now ready to state and prove our first result on locally uniform Cesàro convergence.

Theorem 1.3.4. *Suppose that the random walk on X induced by the probability measure μ is uniquely ergodic and locally uniformly recurrent on average. Then for every $f \in C_c(X)$, every compact $K \subset X$, and every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and $x \in K$ we have*

$$\left|\frac{1}{n}\sum_{k=0}^{n-1}\int_X f\,\mathrm{d}(\mu^{*k} * \delta_x) - \int_X f\,\mathrm{d}m_X\right| < \varepsilon.$$

[2]A subgroup H of G is said to be *virtually contained* in a subgroup L of G if $H \cap L$ has finite index in H.

Equivalently, considering the space of probability measures on X as endowed with the weak topology, the sequence of functions*

$$X \ni x \mapsto \frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x$$

converges to m_X uniformly on compact subsets of X as $n \to \infty$.

Proof. The equivalence of the two formulations is due to the definition of neighborhoods in the weak* topology by finitely many compactly supported continuous test functions.

To prove the statement for individual functions, we proceed by contradiction. If the conclusion is false, then for some $f \in C_c(X)$, $K \subset X$ compact and $\varepsilon > 0$ there exist indices $n(j) \to \infty$ and $x_j \in K$ with

$$\left|\frac{1}{n(j)}\sum_{k=0}^{n(j)-1} \int_X f \,\mathrm{d}(\mu^{*k} * \delta_{x_j}) - \int_X f \,\mathrm{d}m_X\right| \geq \varepsilon \tag{1.3.1}$$

for all $j \in \mathbb{N}$. Let ν be a weak* limit point of the sequence

$$\left(\frac{1}{n(j)}\sum_{k=0}^{n(j)-1} \mu^{*k} * \delta_{x_j}\right)_j.$$

Then ν is μ-stationary, and a probability measure because of our recurrence assumption and the fact that all x_j lie in the fixed compact set K (Lemma 1.3.3). But by unique ergodicity this forces $\nu = m_X$, contradicting (1.3.1). $\square$

1.3.2. Lyapunov Functions & Stronger Uniformity. Loosely speaking, (Foster–)Lyapunov functions are functions enjoying certain contraction properties with respect to the random walk, to the effect that (on average) its dynamics are directed towards the “center” of the space, where the function takes values below some threshold. They were introduced into the study of random walks on homogeneous spaces by Eskin–Margulis [**40**], whose ideas were further developed by Benoist–Quint [**7**]. Although they can be defined in greater generality (which we shall do in later chapters), here we work with the following definition.

Definition 1.3.5. A proper continuous function $V\colon X \to [0,\infty)$ is called a *Lyapunov function* for the random walk on X induced by μ if there exist constants $\alpha < 1$, $\beta \geq 0$ such that $\pi(\mu)V \leq \alpha V + \beta$, where $\pi(\mu)$ is the convolution operator associated to μ introduced in §1.2.

Remark 1.3.6. Let us collect some immediate observations about Lyapunov functions.

(i) If V is a Lyapunov function, then so are cV and $V+c$ for any constant $c > 0$. In particular, one may impose an arbitrary lower bound on V, so that it is no restriction to assume that a Lyapunov function takes values ≥ 1, say.
(ii) Given a Lyapunov function $V'\colon X \to [0,\infty)$ for the n_0-step random walk (induced by the convolution power μ^{*n_0}), one can construct a

Lyapunov function V for the random walk given by μ itself by setting

$$V = \sum_{k=0}^{n_0-1} \alpha^{(n_0-1-k)/n_0} \pi(\mu)^k V'.$$

(iii) By enlarging α and using properness, the contraction inequality in the definition of a Lyapunov function V may be replaced by

$$\pi(\mu)V \leq \alpha V + \beta \mathbb{1}_K$$

for some compact $K \subset X$, where $\mathbb{1}_K$ denotes the indicator function of K (cf. [**90**, Lemma 15.2.8]). $\diamond$

Two examples in which a Lyapunov function exists are the following.

Example 1.3.7 ([**40**]). Identify $X = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$ with the space of unimodular lattices in $\mathbb{R}^2$ as in Example 1.2.7 and recall that we denote by $\lambda_1(x)$ the length of a shortest non-zero vector in $x \in X$. Then for every compactly supported probability measure μ on G whose support generates a Zariski dense subgroup there exist $\varepsilon, \delta > 0$ such that $V' = 1 + \varepsilon\lambda_1^{-\delta}$ is a Lyapunov function for the n_0-step random walk on X induced by μ^{*n_0} for some $n_0 \in \mathbb{N}$. This construction can be generalized to higher dimensions by taking into account the higher successive minima $\lambda_2, \ldots, \lambda_d$ of lattices in $\mathbb{R}^d$. A more advanced construction also ensures existence of Lyapunov functions for Zariski dense probability measures with finite exponential moments when $G = \mathbf{G}(\mathbb{R})$ is the group of real points of a Zariski connected semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$ such that G has no compact factors. $\diamond$

Example 1.3.8 ([**129**]). Let $G = \mathrm{SL}_{d+1}(\mathbb{R})$, $\Lambda = \mathrm{SL}_{d+1}(\mathbb{Z})$ and $X = G/\Lambda$. For $0 \leq i \leq m$ let $c_i > 1$ be positive real numbers, $y_i \in \mathbb{R}^d$ vectors such that $y_0 = 0$ and $y_1, \ldots, y_m$ span $\mathbb{R}^d$, $O_i \in \mathrm{SO}_d(\mathbb{R})$ and set

$$g_i = \begin{pmatrix} c_i O_i & y_i \\ 0 & c_i^{-d} \end{pmatrix} \in G.$$

Then for any choice of $p_0, \ldots, p_m > 0$ with $\sum_{i=0}^m p_i = 1$, the probability measure $\mu = \sum_{i=0}^m p_i \delta_{g_i}$ defines a uniquely ergodic random walk on X admitting a Lyapunov function. $\diamond$

It is well known that existence of a Lyapunov function as above guarantees locally uniform recurrence.

Lemma 1.3.9 ([**40**, Lemma 3.1]). *Suppose the random walk on X given by μ admits a Lyapunov function V. Then this random walk is locally uniformly recurrent.*

The intuitive reason for this behavior is simple: The defining contraction inequality means that after a step of the random walk, the value of the Lyapunov function V on average gets smaller by a constant factor, at least when starting outside some compact set K (cf. Remark 1.3.6(iii) above). It is an exercise to show that K can be taken to be an appropriate sublevel set for V, and one thinks about it as the "center" of the space. By the contraction property, the number of steps required to reach it is uniform over starting points x in compact subsets of X. This suggests that we might even let the starting points diverge, as long as this divergence is outcompeted by the geometric rate of contraction of V. We are led to the following notion of recurrence.

Definition 1.3.10. Let $(K_n)_n$ be a sequence of subsets of X. We say that the random walk on X given by μ is $(K_n)_n$*-uniformly recurrent* if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ and a compact subset $M \subset X$ with

$$\mu^{*n} * \delta_x(M) \geq 1 - \varepsilon$$

for all $n \geq n_0$ and $x \in K_n$. It is called $(K_n)_n$*-uniformly recurrent on average* if the above holds with the Cesàro averages $\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x$ in place of $\mu^{*n} * \delta_x$.

Remark 1.3.11. We point out that contrary to the locally uniform situation, for the two versions of this property (with/without average) it is generally not clear whether one implies the other. $\diamond$

We are now going to use Lyapunov functions to establish such recurrence properties for certain slowly growing exhaustions of X by compact sets K_n. We will need the notion of *Lyapunov exponent* of a function $\varphi\colon \mathbb{N} \to [1, \infty)$, which is defined as the exponential growth rate

$$\lambda(\varphi) = \limsup_{n\to\infty} \tfrac{1}{n} \log \varphi(n).$$

If $\lambda(\varphi) = 0$, we say that φ has *subexponential growth.*

Proposition 1.3.12. *Let $\varphi\colon \mathbb{N} \to [1, \infty)$ be a function. Suppose that the random walk on X induced by μ admits a Lyapunov function V with contraction factor $\alpha < 1$ and set $K_n = V^{-1}([0, \varphi(n)])$.*

- (i) *If φ has Lyapunov exponent $\lambda(\varphi) < \log(\alpha^{-1})$, then the random walk on X given by μ is $(K_n)_n$-uniformly recurrent. The number n_0 in the definition can be chosen independently of ε.*
- (ii) *If φ has subexponential growth, then the random walk on X given by μ is $(K_n)_n$-uniformly recurrent on average.*

The proof is a refinement of the methods in [**7, 40**].

Proof. Let α, β be the constants associated to V as in the definition of a Lyapunov function and set $B = \beta/(1-\alpha)$. We are going to use the same set M for both parts of the proposition, namely $M = V^{-1}([0, 2B/\varepsilon])$, which is compact since V is proper. Then for $n \in \mathbb{N}$ and $x \in K_n$ we find, by repeatedly using the contraction property of V,

$$\mu^{*n} * \delta_x(M^c) \leq \frac{\varepsilon}{2B}\pi(\mu)^n V(x) \leq \frac{\varepsilon}{2B}(\alpha^n V(x) + B) \leq \frac{\varepsilon}{2B}\alpha^n \varphi(n) + \frac{\varepsilon}{2}.$$

When the exponential growth rate of φ is less than $\log(\alpha^{-1})$, for some $n_0 \in \mathbb{N}$ we have $\alpha^n\varphi(n) \leq B$ for all $n \geq n_0$. This proves (i).

In order to prove (ii) we use a similar estimate, but have to ensure that the values $\mu^{*k} * \delta_x(M^c)$ are small for a sufficiently large proportion of $0 \leq k < n$. For $x \in K_n$ we find, as above,

$$\mu^{*k} * \delta_x(M^c) \leq \frac{\varepsilon}{2B}\alpha^k \varphi(n) + \frac{\varepsilon}{2}. \tag{1.3.2}$$

Using straightforward manipulations, we further see

$$\alpha^k \varphi(n) \leq B/2 \iff \frac{k}{n} \geq \log(\alpha^{-1})^{-1}\Big(\frac{1}{n}\log\varphi(n) - \frac{1}{n}\log(B/2)\Big),$$

the right-hand side of which tends to 0 as $n \to \infty$ by subexponential growth of φ. Hence, with $k(n) = \lfloor \varepsilon n/4 \rfloor$, we may choose n_0 large enough to ensure the above inequality holds for all $k \geq k(n)$ for $n \geq n_0$. For such n we conclude, using (1.3.2),

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x(M^c) &= \frac{1}{n}\sum_{k=0}^{k(n)-1} \mu^{*k} * \delta_x(M^c) + \frac{1}{n}\sum_{k=k(n)}^{n-1} \mu^{*k} * \delta_x(M^c) \\ &\leq \frac{k(n)}{n} + \frac{3\varepsilon}{4} \leq \varepsilon,\end{aligned}$$

which ends the proof of (ii). □

Theorem 1.3.4 can now be strengthened in the following way.

THEOREM 1.3.13. *In addition to the assumptions of Theorem* 1.3.4*, suppose that the random walk on X induced by μ admits a Lyapunov function V. Let $\varphi\colon \mathbb{N} \to [1, \infty)$ have subexponential growth. Then for every $f \in C_c(X)$ we have*

$$\lim_{n\to\infty} \sup_{V(x)\leq\varphi(n)} \left| \frac{1}{n}\sum_{k=0}^{n-1} \int_X f \,\mathrm{d}(\mu^{*k} * \delta_x) - \int_X f \,\mathrm{d}m_X \right| = 0.$$

PROOF. Using $(K_n)_n$-uniform recurrence on average for the compact sets $K_n = V^{-1}([0, \varphi(n)])$ from Proposition 1.3.12(ii), the proof of Theorem 1.3.4 goes through with the obvious modifications. □

CHAPTER 2

Markov Random Walks on Homogeneous Spaces and Diophantine Approximation on Fractals

Joint with Çağrı Sert

† For the introduction, let G be a connected simple real Lie group and Λ a lattice in G. In the previous chapters, we have been considering random walks on the homogeneous space $X = G/\Lambda$ given by a probability measure μ on G. Given a starting point $x_0 \in X$, trajectories of these i.i.d. random walks can be written as

$$(Y_n \cdots Y_1 x_0)_n, \tag{2.0.1}$$

where $(Y_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables in G with common law μ. In this chapter, the goal is to relax the i.i.d. assumption and consider more general increment processes $(Y_k)_{k\in\mathbb{N}}$, trying to identify a set of conditions ensuring that the random walk trajectory (2.0.1) almost surely equidistributes towards the Haar measure m_X on X for every $x_0 \in X$, meaning convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{Y_k\cdots Y_1 x_0} \longrightarrow m_X$$

in the weak* topology as $n \to \infty$.

2.0.1. I.I.D. Random Walks. Nevertheless, we first consider the classical case where the increments Y_k are i.i.d. Two different types of assumptions on the common distribution μ have previously been used to establish equidistribution statements in this context. The first one concerns the algebraic structure of the support of μ and was studied by Benoist–Quint. Let us restate their equidistribution result (Theorem B in the Introduction) for the special case of simple Lie groups in the language of this chapter.

Theorem 2.0.1 (Benoist–Quint [**9**]). *Let μ be a compactly supported probability measure on a connected simple Lie group G and suppose that the closed subgroup Γ_μ generated by $\mathrm{supp}(\mu)$ has the property that $\mathrm{Ad}(\Gamma_\mu)$ is Zariski dense in $\mathrm{Ad}(G)$. Let $(Y_k)_k$ be a sequence of i.i.d. random variables with common distribution μ. Then for every $x_0 \in X$ with infinite Γ_μ-orbit, the random walk trajectory $(Y_n \cdots Y_1 x_0)_n$ almost surely equidistributes towards m_X.*

The second set of assumptions involves the dynamics of the linearized random walk on the Lie algebra $\mathfrak{g}$ of G. In [**129**], Simmons–Weiss impose the following requirements on the adjoint action of $\Gamma_\mu = \overline{\langle \mathrm{supp}(\mu)\rangle}$ on $\mathfrak{g}$, phrased in terms of Oseledets subspaces (see Theorem 2.1.1 for their definition):

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(I) For every $1 \le k \le \dim(G) - 1$ there exists a proper non-trivial Γ_μ-invariant subspace $W_k \subset \mathfrak{g}^{\wedge k}$ such that, almost surely, W_k trivially intersects the Oseledets subspace $V^{\leqslant 0}$ of subexponential expansion, and $W \coloneqq W_1$ is complementary to the Oseledets subspace $V^{<\max}$ of non-maximal expansion.

(II) The adjoint action of Γ_μ on W is by similarities and satisfies

$$\int_G \log\|\mathrm{Ad}(g)|_W\| \,\mathrm{d}\mu(g) > 0.$$

(III) For $1 \le k \le \dim(G) - 1$, any non-trivial subspace $L \subset \mathfrak{g}^{\wedge k}$ with finite orbit under Γ_μ intersects W_k non-trivially.

A model example to have in mind is the action of the Borel subgroup (endowed with a suitable measure) on the Lie algebra of the upper unipotent subgroup in $\mathrm{SL}_2(\mathbb{R})$.

Modifying the arguments in [**5**], Simmons–Weiss prove the following theorem. For the statement, recall that a subgroup H of G is said to be *virtually contained* in a subgroup L of G if $H \cap L$ has finite index in H.

THEOREM 2.0.2 (Simmons–Weiss [**129**]). *Let μ be a compactly supported probability measure on a connected simple Lie group G such that the closed subgroup Γ_μ generated by* $\mathrm{supp}(\mu)$ *is not virtually contained in any conjugate of Λ and suppose that conditions* (I)–(III) *are satisfied. Let $(Y_k)_k$ be a sequence of i.i.d. random variables with distribution μ. Then for every $x_0 \in X$, the random walk trajectory $(Y_n \cdots Y_1 x_0)_n$ almost surely equidistributes towards m_X.*

Note that the virtual containment condition in the above theorem is equivalent to saying that there do not exist finite Γ_μ-orbits in X.

Simmons–Weiss' conditions (I) & (III) and Benoist–Quint's assumption of Zariski density of $\mathrm{Ad}(\Gamma_\mu)$ in the simple group $\mathrm{Ad}(G)$ are mutually exclusive. However, what the two settings have in common is that both imply what we shall call "uniform expansion on Grassmannians" (see §2.1.3): For every $1 \le k \le \dim(G) - 1$ and every non-zero pure wedge product $v = v_1 \wedge \cdots \wedge v_k$ in $\mathfrak{g}^{\wedge k}$, almost surely,

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|\mathrm{Ad}^{\wedge k}(Y_n \cdots Y_1)v\| > 0. \tag{2.0.2}$$

Elaborating on the recent measure classification results of Eskin–Lindenstrauss in [**39**], we show that this expansion property is sufficient to guarantee almost sure equidistribution. Moreover, their work allows replacing the compact support assumption on μ by *finite exponential moments in* $\mathfrak{g}$, meaning that $\mathrm{N}_a(g) = \max(\|\mathrm{Ad}(g)\|, \|\mathrm{Ad}(g)^{-1}\|)$ satisfies

$$\int_G \mathrm{N}_a(g)^\delta \,\mathrm{d}\mu(g) < \infty$$

for some $\delta > 0$. We prove the following.

THEOREM 2.0.3. *Let μ be a probability measure on a connected simple Lie group G with finite exponential moments in $\mathfrak{g}$ such that the closed subgroup Γ_μ generated by* $\mathrm{supp}(\mu)$ *is not virtually contained in any conjugate of Λ. Suppose that the i.i.d. process $(Y_k)_k$ with common law μ is uniformly expanding on Grassmannians. Then for every starting point $x_0 \in X$, the random walk trajectory $(Y_n \cdots Y_1 x_0)_n$ almost surely equidistributes towards m_X.*

We will establish this result in the slightly more general form of Theorem 2.1.12. The proof breaks down into the two usual steps:

- Classification of stationary measures (Theorem 2.1.9): This step essentially follows from the work of Eskin–Lindenstrauss [**39**], but an additional argument is required to upgrade their classification to the statement we need, namely that the only non-atomic μ-stationary probability measure on X is the Haar measure m_X.
- Ruling out escape of mass (Proposition 2.1.11): Here the key ingredient is Eskin–Margulis' work on non-divergence [**40**], which we exploit along the same lines as in the proof of [**129**, Theorem 2.1].

As one of the consequences of Theorem 2.0.3, we will show that assumptions (I)–(III) above can be relaxed to the following two conditions:

(I') For every $1 \le k \le \dim(G) - 1$ there exists a proper non-trivial Γ_μ-invariant subspace $W_k \subset \mathfrak{g}^{\wedge k}$ such that, almost surely, W_k trivially intersects the Oseledets subspace $V^{\leqslant 0}$ of subexponential expansion.

(III') For $1 \le k \le \dim(G) - 1$, any non-trivial Γ_μ-invariant subspace L of $\mathfrak{g}^{\wedge k}$ intersects W_k non-trivially.

A simple example in which (I') and (III') hold whereas (I)–(III) fail is given by $G = \mathrm{SL}_3(\mathbb{R})$, $\Lambda = \mathrm{SL}_3(\mathbb{Z})$ and $\mu = \frac{1}{3}(\delta_{g_1} + \delta_{g_2} + \delta_{g_3})$ for the matrices

$$g_1 = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1/6 \end{pmatrix},\ g_2 = \begin{pmatrix} 3 & & 1 \\ & 2 & \\ & & 1/6 \end{pmatrix} \text{ and } g_3 = \begin{pmatrix} 3 & & \\ & 2 & 1 \\ & & 1/6 \end{pmatrix}.$$

We postpone the justification to §2.1.3.

2.0.2. Markov Random Walks. The properties of random products of elements of G are much less understood when the increments Y_k do not form an i.i.d. process. The problem of equidistribution on homogeneous spaces, for instance, has not been studied beyond the case of i.i.d. random walks. In this chapter, we investigate this problem for Markovian increment processes and, as our main result, obtain equidistribution results analogous to the i.i.d. case.

THEOREM 2.0.4. *Let E be a finite subset of a connected simple Lie group G and let $(Y_k)_k$ be an irreducible Markov chain on E that is uniformly expanding on Grassmannians in the sense of* (2.0.2). *Suppose that for every $x \in X$ the random orbit $\{Y_n \cdots Y_1 x \mid n \in \mathbb{N}\}$ is almost surely infinite. Then for every starting point $x_0 \in X$, the random walk trajectory $(Y_n \cdots Y_1 x_0)_n$ almost surely equidistributes towards m_X.*

Note that when μ is finitely supported, Γ_μ denotes the closed subgroup of G generated by $\mathrm{supp}(\mu)$, and the Y_k are i.i.d. with distribution μ, the random orbit $\{Y_n \cdots Y_1 x \,|\, n \in \mathbb{N}\}$ is almost surely infinite if and only if the orbit $\Gamma_\mu x$ is infinite. Hence, the condition on almost surely infinite orbits in Theorem 2.0.4 is a natural analogue of the virtual containment condition in Theorem 2.0.3.

The proof of Theorem 2.0.4 relies on Theorem 2.0.3 and a renewal argument. Indeed, our strategy of proof will be to apply Theorem 2.0.3 to the blocks $Z_n = Y_{\tau_g^{n+1}-1} \cdots Y_{\tau_g^n}$ between consecutive hitting times τ_g^n and τ_g^{n+1} of a fixed state $g \in G$, which are i.i.d. by the Markov property of $(Y_k)_k$, and then deal with the excursions between such hitting times. By the strong recurrence properties of finite-state Markov chains, these excursions are rather short most of the time, so that their contribution can be precisely controlled thanks to a joint equidistribution phenomenon (see §2.2.3). Note that for this approach to work, it is crucial that Theorem 2.0.3 does not require μ to have compact support as in Theorems 2.0.1 and 2.0.2.

A concrete corollary of the previous result is the following Markovian version of [**129**, Theorem 1.1].

COROLLARY 2.0.5. *Let $G = \mathrm{SL}_{d+1}(\mathbb{R})$, $\Lambda = \mathrm{SL}_{d+1}(\mathbb{Z})$, and $X = G/\Lambda$. For $0 \leq i \leq r$ let $c_i > 1$ be real numbers, $y_i \in \mathbb{R}^d$ vectors such that $y_0 = 0$ and $y_1, \dots, y_r$ span $\mathbb{R}^d$, $O_i \in \mathrm{SO}_d(\mathbb{R})$, and set*

$$g_i = \begin{pmatrix} c_i O_i & y_i \\ 0 & c_i^{-d} \end{pmatrix} \in G.$$

Then for any irreducible Markov chain $(Y_k)_k$ on $E = \{g_0, \dots, g_r\} \subset G$ with one universally accessible state (i.e. a state that can be reached in a single step from everywhere with positive probability) and any starting point $x_0 \in X$, the random walk trajectory $(Y_n \cdots Y_1 x_0)_n$ almost surely equidistributes towards the Haar measure m_X.

We remark that, in this corollary, the assumption of having a universally accessible state plays the role of an aperiodicity condition, which allows deducing the dynamical property of uniform expansion on Grassmannians from the algebraic structure of the set E. Without such a condition, excursions from a fixed state might fail to witness this structure in full, and degenerate behavior may occur.

2.0.2.1. *Beyond Markov.* An advantage of an expansion condition such as (2.0.2) over one involving the measure μ is that it puts the i.i.d. case on equal footing with arbitrary increment processes. Consequently, the formulation of Theorem 2.0.4 suggests the natural question of equidistribution for more general, say ergodic and stationary, increment processes $(Y_k)_k$ on G. For example, one might expect Theorem 2.0.4 to hold true when, instead of being a Markov process, the distribution of $(Y_k)_k$ is a Gibbs measure of some Hölder continuous potential on $E^{\mathbb{N}}$. While our approach in this chapter can handle locally constant potentials (corresponding to generalized Markov measures), the general question remains open.

2.0.3. Applications to Diophantine Approximation on Fractals. By a classical theorem of Dirichlet, for any $\mathbf{v} \in \mathbb{R}^m$, there exist infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^m \times (\mathbb{Z} \setminus \{0\})$ such that $\|q\mathbf{v} - \mathbf{p}\|_\infty \leq |q|^{-1/m}$. If for some constant $c \in (0,1)$ there are only finitely many solutions $(\mathbf{p}, q) \in \mathbb{Z}^m \times (\mathbb{Z} \setminus \{0\})$ to the stronger inequality $\|q\mathbf{v} - \mathbf{p}\|_\infty \leq c|q|^{-1/m}$, then $\mathbf{v}$ is said to be *badly approximable*, and *well approximable* otherwise. The set of badly approximable points in $\mathbb{R}^m$ is of zero Lebesgue measure (but of full Hausdorff dimension).

In the study of Diophantine approximation on fractals, one is in particular interested in Diophantine properties of typical points of a fractal in $\mathbb{R}^m$ with respect to natural measures on that fractal; most prominently, Hausdorff measure. In the absence of algebraic obstructions, it is generally expected that these properties are the same as for Lebesgue-typical points of the ambient space $\mathbb{R}^m$. However, for badly approximable points this analogy remained poorly understood after the initial results of Einsiedler–Fishman–Shapira [**37**] that concerned a somewhat restricted class of fractals.

The recent breakthrough of Simmons–Weiss [**129**] contributed considerably to this problem, showing in particular that for an irreducible iterated function system (IFS) $\Phi = \{\phi_1, \dots, \phi_k\}$ of contracting similarities of $\mathbb{R}^m$ and any Bernoulli measure β on $\Phi^{\mathbb{N}}$ of full support, almost every point of the associated self-similar fractal is of generic type, where "almost every" is understood with respect to the pushforward of β by the *natural projection*

$$\Pi\colon \Phi^{\mathbb{N}} \to \mathbb{R}^m,\ (\phi_{i_j})_j \mapsto \lim_{n\to\infty} \phi_{i_0} \cdots \phi_{i_{n-1}}(x),$$

where $x \in \mathbb{R}^m$ is arbitrary. Thanks to a classical result of Hutchinson [**65**], this implies the same conclusion with respect to Hausdorff measure whenever the IFS satisfies the open set condition. Here, a point being of "generic type" intuitively means that, from a Diophantine approximation perspective, it behaves like a Lebesgue-typical point in $\mathbb{R}^m$. In particular, such points are well approximable. When $m = 1$, this property also implies that the blocks of the continued fraction expansion are distributed according to Gauss measure. For the precise definition see §2.3.2.

In our main applications below, following the strategy in [**129**] and making use of our Markovian equidistribution results, we extend the aforementioned results of [**129**] in two directions.

The first one concerns measures that are not necessarily Bernoulli. We will say that an IFS $\Phi = \{\phi_1, \dots, \phi_k\}$ of contracting similarities of $\mathbb{R}^m$ is *irreducible* if there does not exist a proper affine subspace W of $\mathbb{R}^m$ that is preserved by all ϕ_i. The *attractor* of Φ is the unique non-empty compact subset $\mathcal{K} \subset \mathbb{R}^m$ with $\mathcal{K} = \bigcup_{i=1}^k \phi_i(\mathcal{K})$. Equivalently, the attractor $\mathcal{K}$ can be written as the image of $\Phi^{\mathbb{N}}$ under the natural projection Π defined above.

THEOREM 2.0.6. *Let Φ be an irreducible IFS of finitely many contracting similarities of $\mathbb{R}^m$, $\mathcal{K}$ the associated attractor, and $\Pi\colon \Phi^{\mathbb{N}} \to \mathbb{R}^m$ the natural projection. Then for any Markov measure $\mathbb{P}$ on $\Phi^{\mathbb{N}}$ of full support, $\Pi_*\mathbb{P}$-a.e. point on $\mathcal{K}$ is of generic type, so in particular, well approximable.*

Under a strong separation condition, the statement about well approximable points in the above theorem also follows from Simmons–Weiss' [**129**, Theorem 8.4] on doubling measures. However, in general the measures in our theorem are not doubling on the attractor $\mathcal{K}$, even under the open set condition; see [**135**].

Secondly, we consider more general, no longer strictly self-similar fractals $\mathcal{K}$. Given an IFS Φ of contracting similarities, these fractals are obtained as images under the natural projection Π of *sofic subshifts* of the shift space $\Phi^{\mathbb{N}}$, which are by definition continuous factors of subshifts of finite type [**134**]. Accordingly, we call the associated fractals *sofic similarity fractals*.

In the literature, the iterated function systems appearing in the construction of such fractals are known as "graph-directed IFS", since a sofic shift can always be realized as image of the edge shift of a directed graph under a one-block factor map (see e.g. [**82**]). Each edge in the graph has as label one of the similarities in Φ and the possible paths in the graph determine the sequences appearing in the sofic shift. Since its introduction by Mauldin–Williams [**88**], this viewpoint has proved to be a fruitful approach and has been studied by many authors, among others Edgar–Mauldin [**34**], Olsen [**96**], Wang [**131**], and Mauldin–Urbanski in their monograph [**87**]. For an accessible introduction we refer to Edgar's book [**35**].

The advantage of this setup over the point of view of an abstract sofic shift is that classical properties of an IFS like the open set condition or irreducibility can be expressed in a more lucid and conceptual way. With these notions, which will be defined in §2.3.1, we have the following result.

THEOREM 2.0.7. *Let $\mathcal{K} \subset \mathbb{R}^m$ be a sofic similarity fractal constructed by a finite graph-directed IFS of contracting similarities that is irreducible and satisfies the open set condition. Let $s \geq 0$ denote the Hausdorff dimension of $\mathcal{K}$. Then almost every point on $\mathcal{K}$ with respect to s-dimensional Hausdorff measure is of generic type, so in particular, well approximable.*

Terminology, Notation, Conventions. In the whole chapter, G is a real Lie group with Lie algebra $\mathfrak{g}$ and X is a locally compact σ-compact metrizable space on which G acts continuously. Frequently, X will be the homogeneous space G/Λ for a discrete subgroup Λ of G. In case Λ is a lattice, we write m_X for the Haar measure on X. Throughout, we fix a scalar product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$, which induces scalar products on the exterior powers of $\mathfrak{g}$ given by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k\rangle = \det(\langle v_i, w_j\rangle)_{1\leq i,j\leq k}$$

for pure wedge products and extended bilinearly to all of $\mathfrak{g}^{\wedge k}$. The induced norms are all denoted by $\|\cdot\|$. This should cause no confusion.

In the sequel, we shall not take the point of view of stochastic processes as in the introduction, but rather work with the canonical coordinate process on the product space $B = G^{\mathbb{N}}$, governed by some probability measure thereon. In the i.i.d. case, that measure is the infinite product measure $\beta = \mu^{\otimes\mathbb{N}}$ for a Borel probability measure μ on G. In the Markovian case it will in fact be advantageous to not work directly in G, but with an abstract set E that is mapped to G via some *coding map* $E \ni e \mapsto g_e \in G$. The measures governing our processes will then be Markov measures on $\Omega = E^{\mathbb{N}}$. The shift map on Ω will be denoted by T. We shall also need to deal with the semigroup E^* of finite words over E. The length of a word w is denoted by $\ell(w)$. The coding map $e \mapsto g_e$ naturally extends to a homomorphism $E^* \to G$ given by $g_w = g_{e_{n-1}} \cdots g_{e_0} \in G$ for a word $w = e_{n-1} \dots e_0 \in E^*$. For $\omega = (\omega_j)_j \in \Omega$ and $n \in \mathbb{N}$ we shall write $\omega|_n$ for the finite word $\omega_{n-1} \dots \omega_0 \in E^*$.

An important special case of the above is the choice $E = G$ with the identity map as coding map. In this case, we have $g_{b|_n} = b_{n-1} \cdots b_0$ for $b = (b_j)_j \in B$ and $n \in \mathbb{N}$, and T is the shift map on B.

For $g \in \mathrm{GL}_d(\mathbb{R})$, we set $\mathrm{N}(g) = \max(\|g\|, \|g^{-1}\|)$ for some choice of operator norm on $\mathbb{R}^{d\times d}$. A probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ is said to have a *finite first moment* if

$$\int \log \mathrm{N}(g)\,\mathrm{d}\mu(g) < \infty,$$

and to have *finite exponential moments* if

$$\int \mathrm{N}(g)^{\delta}\,\mathrm{d}\mu(g) < \infty$$

for sufficiently small $\delta > 0$.

Given a finite-dimensional real vector space V, we write $\mathbb{P}(V)$ for the projective space associated to V. Under a *representation* of G on V we understand a continuous homomorphism ρ from G into the group $\mathrm{GL}(V)$ of invertible linear transformations of V. A probability measure μ on G is said to have a *finite first moment in* (V, ρ) or *finite exponential moments in* (V, ρ) if the pushforward $\rho_*\mu$ of μ by ρ has the corresponding property. When $(V, \rho) = (\mathfrak{g}, \mathrm{Ad})$, we shall omit the representation from the terminology and simply speak of finite first or exponential moments in $\mathfrak{g}$. In this case, we sometimes also use the abbreviation $\mathrm{N}_a(g) \coloneqq \mathrm{N}(\mathrm{Ad}(g))$ for $g \in G$.

We say that a sequence $(y_n)_n$ in a Polish space Y *equidistributes* towards a probability measure η on Y if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(y_k) = \int_Y f \,\mathrm{d}\eta$$

for every bounded continuous function f on Y. Equidistribution of sequences in the (locally compact) space X can be expressed in terms of weak* convergence as follows: By definition, a sequence $(\nu_n)_n$ of probability measures on X converges to a finite measure ν on X in the weak* topology if

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}\nu_n = \int_X f \,\mathrm{d}\nu \tag{2.0.3}$$

for every compactly supported continuous function f on X. The limit measure ν always satisfies $\nu(X) \leq 1$. When ν is a probability measure, weak* convergence of $(\nu_n)_n$ to ν implies that (2.0.3) holds for all bounded continuous functions. Consequently, a sequence $(x_n)_n$ in X equidistributes towards a probability measure ν on X if and only if the *empirical measures* $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{x_k}$ converge to ν in the weak* topology as $n \to \infty$.

2.1. I.I.D. Random Walks

In this section, we investigate i.i.d. random products satisfying certain expansion conditions. After recalling some classical facts about random matrix products in §2.1.1, these conditions are defined and studied in §2.1.2 and §2.1.3. Afterwards, we state and prove measure classification and equidistribution results in §2.1.4. The main result is Theorem 2.1.12, which implies Theorem 2.0.3. The employed arguments rely on Eskin–Lindenstrauss' results in [**39**].

Throughout this section, μ is a probability measure on G and Γ_μ denotes the closed subgroup of G generated by $\mathrm{supp}(\mu)$.

2.1.1. Preliminaries on Random Matrix Products. We start by recalling two fundamental results about exponential growth rates for random matrix products. Let $G = \mathrm{GL}_d(\mathbb{R})$ and assume that μ has a finite first moment.

The first result is Oseledets' multiplicative ergodic theorem. It makes a statement about the *Lyapunov exponents* $\lambda_1(\mu) \geq \cdots \geq \lambda_d(\mu)$ of μ, which are the real numbers defined by

$$\lambda_1(\mu) + \cdots + \lambda_i(\mu) \coloneqq \lim_{n\to\infty} \tfrac{1}{n}\mathbb{E}\Big[\log\|(g_{b|_n})^{\wedge i}\|\Big] \overset{\beta\text{-a.s.}}{=} \lim_{n\to\infty} \tfrac{1}{n}\log\|(g_{b|_n})^{\wedge i}\|$$

for $1 \leq i \leq d$, where the second equality follows from Kingman's subadditive ergodic theorem and ergodicity of the underlying Bernoulli shift.

THEOREM 2.1.1 (Oseledets [**99**]). *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment and let $1 \leq i_1 < \cdots < i_s \leq d$ be indices chosen such that $\lambda_{i_1}(\mu) > \cdots > \lambda_{i_s}(\mu)$ are the distinct Lyapunov exponents of μ. Then there exists a shift-invariant measurable subset $B' \subset B$ of full measure with respect to β such that for every $b \in B'$*

(i) *$((g_{b|_n})^*(g_{b|_n}))^{1/2n}$ converges to an invertible symmetric matrix L_b,*
(ii) *the eigenvalues of L_b are $\mathrm{e}^{\lambda_{i_1}(\mu)}, \dots, \mathrm{e}^{\lambda_{i_s}(\mu)}$, and*
(iii) *if $U_b^1, \dots, U_b^s$ denote the corresponding eigenspaces of L_b, the* Oseledets subspaces *$V_b^j := U_b^j \oplus \cdots \oplus U_b^s$ for $1 \leq j \leq s$ have the property that*

$$\lim_{n\to\infty} \tfrac{1}{n} \log\|g_{b|_n} v\| = \lambda_{i_j}(\mu)$$

whenever $v \in V_b^j \setminus V_b^{j+1}$, and satisfy the equivariance $V_b^j = b_1^{-1} V_{Tb}^j$.

The space $V_b^{<\max} := V_b^2$ is called the Oseledets subspace of non-maximal expansion. *The largest Oseledets subspace with non-positive exponent is denoted by $V_b^{\leqslant 0}$ and is called the* Oseledets subspace of subexponential expansion (*set $V_b^{\leqslant 0} = \{0\}$ if there are only positive exponents*).

We refer to Ruelle [**116**, §1] for an exposition.

In contrast to the random nature of Oseledets subspaces, the second result we wish to review describes exponential growth rates along a deterministic filtration.

THEOREM 2.1.2 (Furstenberg–Kifer [**50**], Hennion [**62**]). *Let μ be as above. Then there exists a partial flag $\mathbb{R}^d = F_1 \supset F_2 \supset \cdots \supset F_k \supset F_{k+1} = \{0\}$ of Γ_μ-invariant subspaces and a collection of real numbers $\lambda_1(\mu) = \beta_1(\mu) > \cdots > \beta_k(\mu)$ such that for every $v \in F_i \setminus F_{i+1}$ we have β-a.s.*

$$\lim_{n\to\infty} \tfrac{1}{n} \log\|g_{b|_n} v\| = \beta_i(\mu).$$

Moreover, the $\beta_i(\mu)$ are the values of

$$\alpha(\nu) := \int_{\mathbb{P}(\mathbb{R}^d)} \int_G \log \frac{\|gv\|}{\|v\|} \,\mathrm{d}\mu(g)\,\mathrm{d}\nu(\mathbb{R}v)$$

that occur when ν ranges over μ-ergodic μ-stationary probability measures on $\mathbb{P}(\mathbb{R}^d)$. If $\alpha(\nu) = \beta_i(\mu)$ for such a measure ν, then $v \in F_i \setminus F_{i+1}$ for ν-a.e. $\mathbb{R}v \in \mathbb{P}(\mathbb{R}^d)$.

When applying the above theorem, we will frequently use the notation $F^{\leqslant 0}$ for the maximal subspace F_i with exponent $\beta_i(\mu) \leq 0$.

2.1.2. Expansion on Projective Space. When all exponents β_i in Theorem 2.1.2 are positive, all non-zero vectors are expanded by the random matrix product at a uniform exponential rate.

DEFINITION 2.1.3 (Uniform expansion). Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ and P a closed Γ_μ-invariant subset of $\mathbb{P}(\mathbb{R}^d)$. Then μ is said to be *uniformly expanding on P* if for every $\mathbb{R}v \in P$, for β-a.e. $b \in B$ we have

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|g_{b|_n} v\| > 0.$$

In the literature, the idea of uniform expansion has been formalized in different ways, with some relationships established between them (see e.g. [**39**, Lemma 1.5], [**129**, §3]). In the following proposition, we prove the equivalence of the definition we are working with to some of its common variants.

PROPOSITION 2.1.4. *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment and let P be a closed Γ_μ-invariant subset of $\mathbb{P}(\mathbb{R}^d)$. The following properties are equivalent to uniform expansion of μ on P:*

(i) *There exists $N \in \mathbb{N}$ and a constant $C_1 > 0$ such that for every $\mathbb{R}v \in P$ and every $n \geq N$ we have*

$$\frac{1}{n}\int_G \log \frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu^{*n}(g) \geq C_1 > 0.$$

(ii) *There exists $N \in \mathbb{N}$ and a constant $C_2 > 0$ such that for every $\mathbb{R}v \in P$ we have*

$$\int_G \log \frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu^{*N}(g) \geq C_2 > 0.$$

(iii) *For every $\mathbb{R}v \in P$, for β-a.e. $b \in B$ we have*

$$\lim_{n\to\infty} \tfrac{1}{n} \log\|g_{b|_n} v\| > 0.$$

PROOF. We apply Theorem 2.1.2. One of its consequences is that the limit in (iii) exists β-a.s. for every $\mathbb{R}v \in P$. In particular, we see that (iii) is equivalent to uniform expansion of μ on P. Of the remaining implications, only (ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i) require a proof.

(ii) $\Longrightarrow$ (iii): Since the limit in (iii) exists, we may pass to a subsequence of indices and assume $N = 1$. The set $P \cap \mathbb{P}(F^{\leqslant 0})$ is a closed Γ_μ-invariant subset of $\mathbb{P}(\mathbb{R}^d)$. Assume it is non-empty. Then it supports a μ-ergodic μ-stationary probability measure ν and Theorem 2.1.2 implies that $\alpha(\nu)$ occurs as exponential growth rate on $F^{\leqslant 0}$. However, due to (ii) we have

$$\alpha(\nu) = \int_P \int_G \log \frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(\mathbb{R}v) \geq C_2 > 0,$$

a contradiction. Hence, $P \cap \mathbb{P}(F^{\leqslant 0})$ must be empty, which is equivalent to (iii).

(iii) $\Longrightarrow$ (i): We argue by contradiction. If (i) does not hold, then there exists a sequence $(\mathbb{R}v_j)_j$ in P and a sequence of integers $(n_j)_j$ with $n_j \to \infty$ such that

$$\limsup_{j\to\infty} \frac{1}{n_j}\int_G \log \frac{\|gv_j\|}{\|v_j\|}\,\mathrm{d}\mu^{*n_j}(g) \leq 0. \tag{2.1.1}$$

Passing to a subsequence if necessary, we may assume that

$$\frac{1}{n_j}\sum_{k=0}^{n_j-1} \mu^{*k} * \delta_{\mathbb{R}v_j} \longrightarrow \tilde{\nu}$$

as $j \to \infty$ in the weak* topology for some limit probability measure $\tilde{\nu}$ on $\mathbb{P}(\mathbb{R}^d)$ with support in P. Note that $\tilde{\nu}$ necessarily is μ-stationary. Using the additive cocycle property of $(g, \mathbb{R}v) \mapsto \log(\|gv\|/\|v\|)$ together with (2.1.1), it follows that

$$\begin{aligned}\int_{\mathbb{P}(\mathbb{R}^d)}\int_G \log\frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu(g)\,\mathrm{d}\tilde{\nu}(\mathbb{R}v) &= \lim_{j\to\infty}\frac{1}{n_j}\sum_{k=0}^{n_j-1}\int_G\int_G \log\frac{\|gg'v_j\|}{\|g'v_j\|}\,\mathrm{d}\mu(g)\,\mathrm{d}\mu^{*k}(g') \\ &= \lim_{j\to\infty}\frac{1}{n_j}\int_G \log\frac{\|gv_j\|}{\|v_j\|}\,\mathrm{d}\mu^{*n_j}(g) \le 0,\end{aligned}$$

the application of weak* convergence being justified since the function

$$\mathbb{R}v \mapsto \int_G \log\frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu(g)$$

on $\mathbb{P}(\mathbb{R}^d)$ is continuous by dominated convergence in view of the finite first moment assumption on μ. Consequently, there exists a μ-ergodic component ν of $\tilde{\nu}$ with support in P satisfying

$$\alpha(\nu) = \int_{\mathbb{P}(\mathbb{R}^d)}\int_G \log\frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(\mathbb{R}v) \le 0.$$

The last statement in Theorem 2.1.2 therefore implies $\nu(\mathbb{P}(F^{\leqslant 0})) = 1$. However, this is a contradiction to (iii), since as remarked before, this condition means that $P \cap \mathbb{P}(F^{\leqslant 0}) = \emptyset$. ☐

2.1.3. Expansion on Grassmannians. Here, we introduce our main expansion assumption and show that it is satisfied in the settings of Theorems 2.0.1 and 2.0.2.

Let Λ be a lattice in the real Lie group G and $X = G/\Lambda$. In [**39**], Eskin–Lindenstrauss introduce the uniform expansion assumption for the adjoint representation to obtain a description of the μ-ergodic μ-stationary probability measures on X (see [**39**, Theorem 1.7]). However, as they point out, this condition is not sufficient to ensure that all such measures on X are homogeneous. Below, we single out a stronger expansion assumption which guarantees that the only μ-ergodic μ-stationary probability measures on X are finite periodic orbit measures and the Haar measure m_X.

Let V be a real vector space of dimension d. For each $1 \le k \le d$, denote by $\mathrm{Gr}_k(V)$ the k-Grassmann variety of V. Let $\mathrm{Gr}_k(V) \hookrightarrow \mathbb{P}(V^{\wedge k})$ be the Plücker embedding. Its image is a closed subset of $\mathbb{P}(V^{\wedge k})$ given by $\mathbb{P}(\bigwedge_{\mathrm{p}}^k V)$, where we denote by $\bigwedge_{\mathrm{p}}^k V$ the set of non-zero pure wedge products in $V^{\wedge k}$. For a probability measure μ on $\mathrm{GL}_d(\mathbb{R})$, we denote by $\bigwedge_*^k \mu$ the pushforward of μ under the k^{th} exterior power representation. Note that all the $\bigwedge_*^k \mu$ have finite first moments if μ does, by virtue of the inequality $\mathrm{N}(g^{\wedge k}) \le \mathrm{N}(g)^k$.

Definition 2.1.5 (Expansion on Grassmannians). We say that a probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ is *uniformly expanding on Grassmannians* if $\bigwedge_*^k \mu$ is uniformly expanding on $\mathbb{P}(\bigwedge_{\mathrm{p}}^k \mathbb{R}^d) \subset \mathbb{P}(\bigwedge^k \mathbb{R}^d)$ for every $1 \le k \le d-1$.

We will usually impose this expansion condition on $\mathrm{Ad}_* \mu$. This accounts for the cases previously studied by Benoist–Quint in [**5**] (Proposition 2.1.6) and Simmons–Weiss [**129**] (Proposition 2.1.7).

PROPOSITION 2.1.6. *Let G be a real Lie group with non-compact simple identity component such that the Zariski closure G' of* $\mathrm{Ad}(G)$ *is Zariski connected. Suppose that μ has a finite first moment in $\mathfrak{g}$ and that* $\mathrm{Ad}(\Gamma_\mu)$ *is Zariski dense in G'. Then* $\mathrm{Ad}_* \mu$ *is uniformly expanding on Grassmannians.*

We remark that in the statement above, one cannot relax the requirement of simplicity to semisimplicity. Indeed, expansion fails for any vector corresponding under the Plücker embedding to a non-trivial proper Lie ideal in $\mathfrak{g}$.

PROOF. We are given that $\mathrm{Ad}(\Gamma_\mu)$ is Zariski dense in the non-compact simple real algebraic subgroup G' of $\mathrm{Aut}(\mathfrak{g})$. From Furstenberg's theorem on positivity of the top Lyapunov exponent (see [**51**, Theorem 8.6]) it follows that $\mathrm{Ad}_* \mu$ is uniformly expanding in every finite-dimensional algebraic representation (V, ρ) of G' without fixed vectors. Indeed, using complete reducibility one may assume that $\mathrm{Ad}(\Gamma_\mu)$ acts irreducibly, which already implies strong irreducibility in view of Zariski connectedness of G'. Applying Theorem 2.1.2 to the k^{th} exterior power of the standard representation for some $1 \le k \le \dim(G) - 1$, we find that $F^{\leqslant 0}$ consists of G'-fixed vectors only. Since a fixed element of $\bigwedge_{\mathrm{p}}^k \mathfrak{g}$ would give rise to a non-trivial proper Lie ideal of $\mathfrak{g}$, we conclude $\bigwedge_{\mathrm{p}}^k \mathfrak{g} \cap F^{\leqslant 0} = \emptyset$, which is uniform expansion on $\mathbb{P}(\bigwedge_{\mathrm{p}}^k \mathfrak{g})$. $\square$

PROPOSITION 2.1.7. *Suppose that μ has a finite first moment in $\mathfrak{g}$ and satisfies conditions* (I') *and* (III') *from* §2.0.1. *Then* $\mathrm{Ad}_* \mu$ *is uniformly expanding on Grassmannians.*

PROOF. Let $1 \le k \le \dim(G) - 1$ and apply Theorem 2.1.2 to the k^{th} exterior power of the adjoint representation. The obtained spaces F_i are Γ_μ-invariant. Since conditions (I') and (III') together force every invariant subspace to contain vectors exhibiting almost sure exponential growth, all the numbers $\beta_i(\mu)$ are positive, which is uniform expansion on $\mathbb{P}(\mathfrak{g}^{\wedge k})$. $\square$

Let us now explain the example at the end of §2.0.1 in greater detail.

EXAMPLE 2.1.8. Let $G = \mathrm{SL}_3(\mathbb{R})$, $\Lambda = \mathrm{SL}_3(\mathbb{Z})$ and $\mu = \frac{1}{3}(\delta_{g_1} + \delta_{g_2} + \delta_{g_3})$ for the matrices

$$g_1 = \begin{pmatrix} 3 & & \\ & 2 & \\ & & 1/6 \end{pmatrix},\ g_2 = \begin{pmatrix} 3 & & 1 \\ & 2 & \\ & & 1/6 \end{pmatrix} \text{ and } g_3 = \begin{pmatrix} 3 & & \\ & 2 & 1 \\ & & 1/6 \end{pmatrix}.$$

A calculation shows that the subspaces

$$V^{++} = \left\{ \left(\begin{smallmatrix} 0 & & t \\ & 0 & \\ & & 0 \end{smallmatrix} \right) \,\middle|\, t \in \mathbb{R} \right\},\ V^{+} = \left\{ \left(\begin{smallmatrix} 0 & & \\ & 0 & t \\ & & 0 \end{smallmatrix} \right) \,\middle|\, t \in \mathbb{R} \right\},$$

of $\mathfrak{g}$ are Γ_μ-invariant with Lyapunov exponent $\log(18)$ on V^{++} and $\log(12)$ on V^{+}. Thus there cannot exist a subspace $W \subset \mathfrak{g}$ satisfying (I) and (III). However, the space $W' = V^{++} \oplus V^{+}$ satisfies (I') and (III'). More generally, if U denotes the unipotent subgroup of $\mathrm{SL}_3(\mathbb{R})$ with Lie algebra W', the space W_k for $k \ge 1$ can be defined as the subspace of U-fixed vectors in $\mathfrak{g}^{\wedge k}$. As Γ_μ normalizes U, these spaces are Γ_μ-invariant. Property (I') holds in view of [**127**, Example 1.1] (see also §4.2.2.1 and Example 4.2.9 in Chapter 4). Finally, property (III') is satisfied thanks to the Lie–Kolchin theorem, since any Γ_μ-invariant subspace is seen to be U-invariant by passing to the Zariski closure. $\diamond$

2.1.4. Measure Classification and Equidistribution Under Expansion. We are now ready to establish equidistribution under the assumption of uniform expansion on Grassmannians in the adjoint representation.

As outlined in §2.0.1, the first step is the classification of stationary measures. The result is essentially a corollary of Eskin–Lindenstrauss' classification in [**39**]. As already indicated at the beginning of §2.1.3, the aspect that is new is that our stronger expansion condition allows to rule out exceptional stationary measures that can a priori occur in [**39**, Theorem 1.7].

THEOREM 2.1.9. *Let G be a real Lie group, Λ a discrete subgroup of G, and ν a μ-ergodic μ-stationary probability measure on $X = G/\Lambda$. Suppose that μ has a finite first moment in $\mathfrak{g}$, that $\mathrm{Ad}_*\,\mu$ is uniformly expanding on Grassmannians, and that Γ_μ acts transitively on the connected components of X. Then either*

(i) *ν is Γ_μ-invariant and supported on a finite Γ_μ-orbit, or*
(ii) *Λ is a lattice and ν is the Haar measure m_X on X.*

The proof combines ideas from the proofs of [**39**, Theorem 1.3] and [**129**, Proposition 3.2].

PROOF. In view of Proposition 2.1.4, we may apply [**39**, Theorem 1.7] with trivial Z.[3] The conclusion is that if we are not in case (i), ν must be of the form

$$\nu = \int_{G/H} g_*\nu_0 \,\mathrm{d}\lambda(g),$$

where H is a closed non-discrete subgroup of G, ν_0 is an H-homogeneous probability measure on X, and λ is a μ-stationary probability measure on G/H. Observe that H is unimodular, since ν_0 being H-homogeneous implies that H intersects a conjugate of Λ in a lattice.

If $\dim(H) = \dim(G)$, then λ is Γ_μ-invariant (being stationary on a countable set, see [**5**, Lemma 8.3]), and since Γ_μ acts transitively on the connected components of X by assumption, it follows that $\nu = m_X$.

Otherwise, we have $k := \dim(H) < \dim(G)$. Let $v_1, \dots, v_k$ be a basis of $\mathrm{Lie}(H)$ and consider the vector $v = v_1 \wedge \cdots \wedge v_k \in \bigwedge^k_{\mathrm{p}} \mathfrak{g}$ and the stabilizer $L = \mathrm{Stab}_G(v)$ of v in G. Since H is unimodular, it acts on v by ± 1. For simplicity, let us assume that H is connected, so that we have $H \leqslant L$. Thus λ projects to a μ-stationary probability measure $\hat\lambda$ on $G/L \cong G\rho \subset \mathfrak{g}^{\wedge k} \setminus \{0\}$. The measure $\beta \otimes \hat\lambda$ is then a probability measure on $B \times \mathfrak{g}^{\wedge k}$ preserved by the skew-product transformation

$$\hat{T}(b, w) = (Tb, \mathrm{Ad}^{\wedge k}(b_0)w),$$

where $b = (b_j)_j$ and T is the shift on B (see [**10**, Proposition 2.14]). However, since $\hat\lambda(\{0\}) = 0$, our expansion assumption implies that almost every trajectory under this transformation is divergent, contradicting Poincaré recurrence.

The general case where H might be disconnected can be dealt with by considering the vector $\omega = v \otimes v$ in the symmetric square representation, as is done in the proofs of Theorems 4.0.1 and 4.3.7 in Chapter 4. □

[3] We remark that in §4.3.1 of Chapter 4, [**39**, Theorem 1.7] is reproduced as Theorem 4.3.2.

REMARK 2.1.10. To apply [**39**, Theorem 1.7] in the proof above, we need uniform expansion on $\mathfrak{g}$. In the exterior powers of $\mathfrak{g}$ the proof only uses almost sure divergence, i.e. that for every $v \in \bigwedge_{\mathrm{p}}^k \mathfrak{g}$ with $2 \leq k \leq \dim(G) - 1$ we have

$$\lim_{n\to\infty} \|\mathrm{Ad}^{\wedge k}(g_{b|_n})v\| = \infty$$

for β-a.e. $b \in B$. However, this property in fact already implies uniform expansion.

To see this, note that if uniform expansion does not hold, then the compact set $\mathbb{P}(\bigwedge_{\mathrm{p}}^k \mathfrak{g}) \cap \mathbb{P}(F^{\leqslant 0})$ is non-empty and Γ_μ-invariant and therefore supports a μ-ergodic μ-stationary probability measure ν. Using Atkinson/Kesten's lemma (see e.g. [**18**, Lemma II.2.2]) the above almost sure divergence implies $\alpha(\nu) > 0$, which gives a contradiction in view of Theorem 2.1.2. $\diamond$

The second ingredient is non-escape of mass.

PROPOSITION 2.1.11. *Let G be a real Lie group with simple identity component such that the Zariski closure of* $\mathrm{Ad}(G)$ *is Zariski connected and Λ a lattice in G. Suppose that μ has finite exponential moments in $\mathfrak{g}$ and that* $\mathrm{Ad}_* \mu$ *is uniformly expanding on Grassmannians. Then, almost surely, there is no escape of mass for the random walk on $X = G/\Lambda$, in the sense that for every $x_0 \in X$ and $\varepsilon > 0$ there exists a compact set $K \subset X$ such that, β-a.s.,*

$$\limsup_{n\to\infty} \tfrac{1}{n}|\{0 \leq k < n \mid g_{b|_k}x_0 \notin K\}| \leq \varepsilon.$$

As is by now standard (see e.g. [**7, 9, 40**]), recurrence results of this type are most conveniently established by constructing what is known as "Lyapunov function" for the random walk (also referred to as "Margulis function" in this context), that is, a proper continuous function $f\colon X \to [0,\infty)$ which is contracted by μ in the sense that there are constants $c < 1$ and $d \geq 0$ such that

$$\int_G f(gx)\,\mathrm{d}\mu(g) \leq cf(x) + d$$

for all $x \in X$. The proof of the proposition above will thus boil down to showing the existence of such a Lyapunov function. Specifically, we are going to show that our expansion assumption allows using the construction of Eskin–Margulis in [**40**], a strategy that already appeared in the proof of [**129**, Theorem 2.1].

PROOF OF PROPOSITION 2.1.11. By [**9**, Lemma 3.10], it is enough to exhibit a Lyapunov function for the random walk. In order to use results from [**40**], we need to perform some initial reductions.

Setting $R = \ker(\mathrm{Ad})$, we know that $R \cap \Lambda$ has finite index in R and the image $\mathrm{Ad}(\Lambda)$ is a lattice in the Zariski closure G' of $\mathrm{Ad}(G)$ (see [**7**, Lemma 6.1]). Accordingly, the induced map from $X = G/\Lambda$ to $G'/\mathrm{Ad}(\Lambda)$ is proper. We may thus assume to begin with that G is a Zariski connected simple real algebraic group. If Λ is cocompact, there is nothing to prove. So we may moreover assume that Λ is nonuniform, placing us in the setting of [**40**].

We want to use the construction of a Lyapunov function given in [**40**, §3]. For this, what remains to argue is that "condition (A)", formulated at the end of [**40**, §2], is satisfied. The requirement is that for certain representations (V_i, ρ_i) of G and vectors $w_i \in V_i$, the following contraction property holds: For sufficiently small $\delta > 0$ there ought to exist $c < 1$ and $n \in \mathbb{N}$ such that

$$\int_G \|\rho_i(g)v\|^{-\delta} \, \mathrm{d}\mu^{*n}(g) \leq c\|v\|^{-\delta} \tag{2.1.2}$$

for all $v \in Gw_i$. The representations $\rho_i \colon G \to \mathrm{GL}(V_i)$ and vectors w_i occurring in the above condition are characterized by the property that the stabilizer of $\mathbb{R}w_i$ in G is some predetermined maximal parabolic subgroup P_i of G. In our case, we can thus take $V_i = \mathfrak{g}^{\wedge \dim(P_i)}$, $\rho_i \colon G \to \mathrm{SL}(V_i)$ the respective exterior power of Ad, and w_i to be a volume form of the Lie algebra $\mathfrak{p}_i$ of P_i (see [**76**, Proposition 7.83(b)]). However, using uniform expansion as input, the proof of [**40**, Lemma 4.2] precisely shows that (2.1.2) holds for all non-zero pure wedge products v. This finishes the proof. □

Combining the previous statements, we arrive at the main equidistribution result of this section.

Theorem 2.1.12. *Let G be a real Lie group with simple identity component such that the Zariski closure of $\mathrm{Ad}(G)$ is Zariski connected and Λ a lattice in G. Suppose that Γ_μ is not virtually contained in any conjugate of Λ, Γ_μ acts transitively on the connected components of $X = G/\Lambda$, μ has finite exponential moments in $\mathfrak{g}$, and $\mathrm{Ad}_* \mu$ is uniformly expanding on Grassmannians. Then for every $x_0 \in X$, the random walk trajectory $(g_{b|_n} x_0)_n$ equidistributes towards m_X for β-a.e. $b \in B$.*

Proof. The remaining argument is standard:

- The Breiman law of large numbers (see [**9**, Corollary 3.3]) applied to the one-point compactification of X shows that, almost surely, any weak* limit of the sequence $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_{b|_k} x_0}$ of empirical measures is μ-stationary.
- Non-escape of mass (Proposition 2.1.11) implies that all such weak* limits are probability measures on X.
- Since there are no finite orbits, using the classification of stationary measures (Theorem 2.1.9) we conclude that all (μ-ergodic components of) these limits coincide with the Haar measure m_X. Hence the result. □

Proof of Theorem 2.0.3. Using connectedness of G, we see that the conditions of Theorem 2.1.12 are satisfied. □

2.2. Markov Random Walks

We now turn our attention to Markov random walks. We first adopt a bootstrapping approach (§2.2.2, §2.2.3), upgrading statements about the random walk with i.i.d. increments $Z_n = Y_{\tau_g^{n+1}-1} \cdots Y_{\tau_g^n}$ to statements about the whole random walk. As preparation, we study the distribution of these excursions, which we call "renewal measures", in §2.2.1. In §2.2.4 we discuss expansion in the Markovian setting, and in §2.2.5 we prove our main result (Theorem 2.2.17) about expanding Markov chains, which implies Theorem 2.0.4. The final subsection §2.2.6 is dedicated to a concrete example that contains Corollary 2.0.5 and will be important in §2.3 about Diophantine approximation on fractals.

Let E be a countable set. A Markov chain on E is defined by a transition kernel P on E. This means that for every $e \in E$, $P(e, \cdot)$ is a probability distribution on E specifying the transition probabilities when the current state is e. We are going to write $p_{e',e} := P(e, e')$ for the probability of going from state e to e', and $p_w = p_{e_{n-1},e_{n-2}} \cdots p_{e_1,e_0}$ for a word $w = e_{n-1} \dots e_0 \in E^*$.[4] Let $\mathbb{P}_e$ be the associated Markov measure on $\Omega = E^{\mathbb{N}}$ starting at $e \in E$ (at time $n = 0$), characterized by the property that

$$\mathbb{P}_e\Big[\{e_0\} \times \cdots \times \{e_n\} \times E^{\mathbb{N}}\Big] = p_{e_n,e_{n-1}} \cdots p_{e_1,e_0} \delta_{e_0=e}$$

for $e_0, \dots, e_n \in E$. More generally, for an arbitrary probability distribution λ on E we write

$$\mathbb{P}_\lambda = \sum_{e \in E} \lambda(\{e\}) \mathbb{P}_e, \tag{2.2.1}$$

which is the unique Markov measure on Ω for the given Markov chain on E with starting distribution λ. Expectation with respect to the probability measures $\mathbb{P}_e$ and $\mathbb{P}_\lambda$ will be denoted by $\mathbb{E}_e$ and $\mathbb{E}_\lambda$, respectively.

The consecutive hitting times of a state e will be denoted by τ_e^n, defined by $\tau_e^0 = 0$ and

$$\tau_e^n(\omega) = \inf\{n > \tau_e^{n-1}(\omega) \mid \omega_n = e\}$$

for $\omega = (\omega_j)_j \in \Omega$ and $n \in \mathbb{N}$. We abbreviate the first hitting time τ_e^1 as τ_e.

We will only be interested in irreducible chains, i.e. ones where every state can be reached from every other in finite time with positive probability (formally, chains with $\mathbb{P}_e[\tau_{e'} < \infty] > 0$ for any two states $e, e' \in E$). Let us recall the classical notions of recurrence for Markov chains.

Definition 2.2.1. An irreducible Markov chain on E is called

- *recurrent* if $\mathbb{P}_e[\tau_e < \infty] = 1$ for every $e \in E$,
- *positive recurrent* if $\mathbb{E}_e[\tau_e] < \infty$ for every $e \in E$, and
- *exponentially recurrent* if for every $e \in E$ there exists $\delta > 0$ such that $\mathbb{E}_e[\exp(\delta\tau_e)] < \infty$.

It is well known that these forms of recurrence hold for all states as soon as one state has the respective property. Irreducible positive recurrent chains admit a unique stationary probability distribution π on E, given by

$$\pi(\{e'\}) = \frac{1}{\mathbb{E}_e[\tau_e]} \mathbb{E}_e\left[\sum_{k=0}^{\tau_e - 1} \mathbb{1}_{\omega_k = e'}\right] \tag{2.2.2}$$

for $e, e' \in E$. The Markov measure $\mathbb{P}_\pi$ is then invariant and ergodic under the shift map T on Ω. See e.g. Chung [**26**] for proofs of these classical facts.

For the sequel, we fix a coding map $E \ni e \mapsto g_e \in G$. Such a map allows us to define a stochastic process on G by

$$(Y_k)_k \colon \Omega \ni \omega \mapsto (g_{\omega_{k-1}})_k.$$

This process is generally not a Markov chain on G in the usual sense. Indeed, any generalized Markov measure on $\{g_e | e \in E\}^{\mathbb{N}}$ can be obtained in this manner as the distribution of $(Y_k)_k$. Similarly, the induced random walk $(Y_n \cdots Y_1 x_0)_n$

[4]Since we are studying random walks on X coming from a left action of G, we are using a right-to-left ordering throughout this section.

on X does not constitute a Markov chain. However, this flaw can be removed by embedding this random walk into a Markov chain on a larger space.

Definition 2.2.2. Given a Markov chain on E with transition kernel P, the *action chain* is the Markov chain on $E \times X$ defined by the transition kernel Q given by

$$Q(e,x) = P(e,\cdot) \otimes \delta_{g_e x}$$

for $(e,x) \in E \times X$.

The interpretation is that the E-coordinate contains the element to be applied next. A step into the future consists of the application of that group element to the X-coordinate and choosing the next element according to the transition kernel P in the E-coordinate.

It is evident by construction that in the X-coordinate of the action chain we obtain our random walks of interest of the form $(Y_n \cdots Y_1 x_0)_n$. A precise formulation of this statement is the following.

Lemma 2.2.3. *Let λ be any distribution on E and $x_0 \in X$. Write $\mathbb{P}_{\lambda\otimes\delta_{x_0}}$ for the Markov measure on $(E \times X)^{\mathbb{N}}$ for the action chain starting from $\lambda \otimes \delta_{x_0}$, and $\mathrm{pr}_E\colon (E \times X)^{\mathbb{N}} \to E^{\mathbb{N}}$, $\mathrm{pr}_X\colon (E\times X)^{\mathbb{N}} \to X^{\mathbb{N}}$ for the projections onto all the E- and X-coordinates, respectively. Then the pushforward of $\mathbb{P}_{\lambda\otimes\delta_{x_0}}$ by pr_E is $\mathbb{P}_\lambda$ and the pushforward by pr_X is the distribution of $(Y_n \cdots Y_1 x_0)_n$, where $(Y_k)_k$ is the Markov chain on E starting with Y_1 distributed according to λ.* $\square$

2.2.1. Renewal Measures. We say that a word $w = e_{n-1} \dots e_0 \in E^*$ is $(e'{\leftarrow}e)$-*admissible* if $e_0 = e$, $p_{e_k,e_{k-1}} > 0$ for $1 \le k \le n-1$ and $p_{e',e_{n-1}} > 0$, and simply that it is *admissible* if it is $(e'{\leftarrow}e)$-admissible for some $e, e' \in E$. Further, we call w an *e–renewal word* if it is $(e{\leftarrow}e)$-admissible and $e_k \neq e$ for $1 \le k \le n-1$, and denote the set of e–renewal words by E_e^{r}. A sequence $\omega \in \Omega = E^{\mathbb{N}}$ is said to be admissible if all words $\omega|_n$ for $n \in \mathbb{N}$ are. The set of all admissible sequences is going to be denoted E^∞, and E_e^∞ is the subset of such sequences starting with e.

Definition 2.2.4. Given a recurrent irreducible Markov chain on E and a state $e \in E$, we define the measure $\tilde{\mu}_e$ on the set E_e^{r} of e–renewal words by

$$\begin{aligned}\tilde{\mu}_e(\{w\}) &\coloneqq p_{e,e_{n-1}} p_w = p_{e,e_{n-1}} p_{e_{n-1},e_{n-2}} \cdots p_{e_1,e} \\ &= \mathbb{P}_e\Big[\{\omega \in \Omega \mid \omega_1 = e_1, \dots, \omega_{n-1} = e_{n-1}, \tau_e(\omega) = n\}\Big]\end{aligned}$$

for $w = e_{n-1} \dots e_1 e \in E_e^{\mathrm{r}}$. Then the *renewal measure* μ_e starting at $e \in E$ is defined to be the pushforward of $\tilde{\mu}_e$ to G via the coding map $E^* \to G$, $w \mapsto g_w$.

Note that recurrence implies that $\mathbb{P}_e$-a.s. we have $\tau_e < \infty$, so that under this assumption $\tilde{\mu}_e$ and μ_e are probability measures.

The following simple lemma formalizes the fact that consecutive excursions of a Markov chain are i.i.d.

Lemma 2.2.5. *For a recurrent irreducible Markov chain on E and any state $e \in E$, the map*

$$\begin{aligned}(E_e^\infty, \mathbb{P}_e) &\to \Big((E_e^{\mathrm{r}})^{\mathbb{N}}, \tilde{\mu}_e^{\otimes\mathbb{N}}\Big), \\ \omega &\mapsto \Big(\omega_{\tau_e^{n+1}-1} \dots \omega_{\tau_e^n}\Big)_n\end{aligned} \tag{2.2.3}$$

is an isomorphism (*mod* 0) *of probability spaces.*

PROOF. On the $\mathbb{P}_e$–full measure subset

$$\{\omega \in E_e^\infty \mid \tau_e^n(\omega) < \infty \text{ for all } n \in \mathbb{N}\}$$

of Ω the given map is a well-defined bijection. To see that it is measure-preserving it suffices to consider cylinder sets $\{w_0\} \times \cdots \times \{w_N\} \times (E_e^{\mathrm{r}})^{\mathbb{N}}$ for e–renewal words $w_0, \ldots, w_N$. But for such sets the statement follows directly by construction of $\tilde{\mu}_e$. □

Before moving on, let us shed some light on the relationship between the various renewal measures μ_e on G, knowledge of which will be of interest later on. For this, we denote by Γ_e^+ (resp. Γ_e) the closed subsemigroup (resp. subgroup) of G generated by the support of μ_e, and by Γ_E the image of the set of admissible words under the coding map $E \to G$. Note that Γ_E is in general not closed under multiplication. In case G is real algebraic, we write H_e and H_E for the Zariski closures of Γ_e^+ and Γ_E, respectively.

LEMMA 2.2.6. *Assume the Markov chain on E is irreducible and recurrent.*

(i) *Let $c \in E^*$ be $(e'{\leftarrow}e)$-admissible and $c' \in E^*$ be $(e{\leftarrow}e')$-admissible. Then the semigroups Γ_e^+ and $\Gamma_{e'}^+$ satisfy*

$$g_{c'}\Gamma_{e'}^+ g_c \subset \Gamma_e^+.$$

If G is real algebraic, we additionally have the following.

(ii) *The groups H_e and $H_{e'}$ are conjugate inside H_E. More precisely, with c, c' as in* (i) *we have*

$$g_{c'} H_{e'} g_{c'}^{-1} = g_c^{-1} H_{e'} g_c = H_e.$$

(iii) *If there exists $\tilde{e} \in E$ with both $e\tilde{e}$ and $e'\tilde{e}$ admissible, then $H_e = H_{e'}$.*

(iv) *If there exists $\tilde{e} \in E$ with $e\tilde{e}$ admissible for all $e \in E$, then all H_e coincide and H_E is contained in their normalizer.*

(v) *If there exists $\tilde{e} \in E$ with $\tilde{e}e$ admissible for all $e \in E$, then $H_e = H_E$ for all $e \in E$. In particular, H_E is a group.*

PROOF. For (i), simply note that for every $(e'{\leftarrow}e')$-admissible word $w \in E^*$ the word $c'wc$ is $(e{\leftarrow}e)$-admissible.

For (ii), taking the Zariski closure of both sides of the inclusion in (i), we get $g_{c'} H_{e'} g_c \subset H_e$. Since the word cc' is $(e'{\leftarrow}e')$-admissible, $g_{c'}\Gamma_{e'}^+ g_c$ is a semigroup and hence its Zariski closure $g_{c'} H_{e'} g_c$ is a subgroup of H_e. This implies

$$g_{c'} H_{e'} g_c = g_{c'} H_{e'} g_{c'}^{-1} = g_c^{-1} H_{e'} g_c \subset H_e.$$

By the symmetric argument, we also have $g_c H_e g_c^{-1} \subset H_{e'}$, which in combination with the above gives (ii).

For (iii) note that existence of such an element $\tilde{e}$ implies that c' can be chosen to be both $(e'{\leftarrow}e')$- and $(e{\leftarrow}e')$-admissible. Then $g_{c'} \in H_{e'}$ and we conclude using (ii).

In (iv), all the H_e coincide due to (iii). For every admissible word $w \in E^*$, the word $w\tilde{e}$ is $(e{\leftarrow}\tilde{e})$-admissible for some $e \in E$. Thus, using $g_{\tilde{e}} \in H_{\tilde{e}}$ and part (ii) we find

$$g_w H_{\tilde{e}} g_w^{-1} = g_{w\tilde{e}} H_{\tilde{e}} g_{w\tilde{e}}^{-1} = H_e = H_{\tilde{e}}.$$

This shows that Γ_E is contained in the normalizer of $H_{\tilde{e}}$. Hence, so is H_E.

In the setting of (v), $H_e = H_{\tilde{e}}$ for all $e \in E$ again follows from (iii). Clearly, we also have $H_{\tilde{e}} \subset H_E$. For the reverse inclusion let $w \in E^*$ be any admissible word, say $(e'{\leftarrow}e)$-admissible, and choose a $(e{\leftarrow}\tilde{e})$-admissible word $c \in E^*$. Then both c and wc are $(\tilde{e}{\leftarrow}\tilde{e})$-admissible. This implies $g_w \in H_{\tilde{e}}$, and hence $H_E \subset H_{\tilde{e}}$. $\square$

2.2.2. Stationary Measures. Next, we describe the structure of ergodic stationary measures for the action chain in terms of ergodic stationary measures for the renewal measures μ_e.

LEMMA 2.2.7. *Suppose the Markov chain on E is irreducible and positive recurrent and let π be its stationary distribution. If ν is a stationary probability measure for the action chain on $E \times X$, then*

$$\nu = \sum_{e \in E} \pi(\{e\}) \delta_e \otimes \nu_e, \tag{2.2.4}$$

where for each $e \in E$, ν_e is a μ_e-stationary probability measure on X, satisfying

$$\pi(\{e\})\nu_e = \sum_{e' \in E} \pi(\{e'\}) p_{e,e'} (g_{e'})_* \nu_{e'}. \tag{2.2.5}$$

If ν is ergodic, then the ν_e are μ_e-ergodic.

Furthermore, if $c \in E^*$ is $(e{\leftarrow}e')$-admissible, we have $(g_c)_*\nu_{e'} \ll \nu_e$, and if ν_e is Γ_e^+-invariant, then ν_e and $(g_c)_*\nu_{e'}$ belong to the same measure class.

PROOF. For any measurable subset $Y \subset X$ and $e \in E$ we have by stationarity of ν

$$\begin{aligned}
\pi(\{e\})\nu_e(Y) &= \nu(\{e\} \times Y) = \nu Q(\{e\} \times Y) \\
&= \int_{E \times X} Q\Big((e', x), \{e\} \times Y\Big) \, \mathrm{d}\nu(e', x) \\
&= \sum_{e' \in E} \pi(\{e'\}) p_{e,e'} \nu_{e'}(g_{e'}^{-1} Y),
\end{aligned}$$

which is precisely (2.2.5). Specializing to $Y = X$ shows that the projection of ν to E is a stationary probability measure for the abstract Markov chain on E. By uniqueness, it follows that this projection is π, or in other words that the ν_e are probability measures.

The fact that the ν_e are μ_e-stationary (and μ_e-ergodic if ν is ergodic) follows from [**9**, Lemma 3.4] applied to the Q-recurrent subsets $\{e\} \times X$ of $E \times X$.

The first statement about absolute continuity follows by noting that as a consequence of (2.2.5) and by induction, for every $e \in E$ and $n \in \mathbb{N}$ we have

$$\pi(\{e\})\nu_e = \sum_{\substack{w = e_{n-1} \dots e_0 \in E^* \\ (e \leftarrow e')\text{-admissible}}} \pi(\{e'\}) p_{e,e_{n-1}} p_w (g_w)_* \nu_{e'},$$

with all occurring factors positive. For the last claim let $c' \in E^*$ be $(e'{\leftarrow}e)$-admissible. Then we have $g_{cc'} \in \Gamma_e^+$, so that the above and Γ_e^+-invariance of ν_e imply

$$\nu_e = (g_{cc'})_* \nu_e \ll (g_c)_* \nu_{e'} \ll \nu_e. \qquad \square$$

2.2.3. Equidistribution. This subsection contains the joint equidistribution results alluded to in §2.0.2, which represent a key ingredient of our approach.

PROPOSITION 2.2.8. *Suppose the Markov chain on E is irreducible and positive recurrent and denote by π its stationary distribution. Let $x_0 \in X$, $e \in E$ and m be a probability measure on X invariant under g_w for every admissible word $w \in E^*$ starting with e. If the trajectory $(g_{b|_n} x_0)_n$ equidistributes towards m for $\mu_e^{\otimes \mathbb{N}}$-a.e. $b \in B$, then $(g_{\omega|_n} x_0, T^n \omega)_n$ equidistributes towards $m \otimes \mathbb{P}_\pi$ for $\mathbb{P}_e$-a.e. $\omega \in \Omega$.*

In the proof of Proposition 2.2.8 we will need part (i) of the following technical lemma. Part (ii) will be used in §2.3 about Diophantine approximation on fractals.

LEMMA 2.2.9.

(i) ([**129**, Proposition 5.1]) *Let $\mathbb{P} = \mu^{\otimes \mathbb{N}}$ be the Bernoulli measure on Ω associated to a probability measure μ on E. Assume that $(g_{\omega|_n} x_0)_n$ equidistributes towards a probability measure m on X for $\mathbb{P}$-a.e. $\omega \in \Omega$. Then $(g_{\omega|_n} x_0, T^n \omega)_n$ equidistributes towards $m \otimes \mathbb{P}$ for $\mathbb{P}$-a.e. $\omega \in \Omega$.*

(ii) *Denote by π the stationary distribution of a positive recurrent Markov chain on E and let λ be any starting distribution on E. Assume that $(\omega_n, g_{\omega|_n} x_0)_n$ equidistributes towards a probability measure on $E \times X$ of the form*
$$\sum_{e \in E} \pi(\{e\}) \delta_e \otimes m_e$$
for $\mathbb{P}_\lambda$-a.e. $\omega \in \Omega$. Then $(\omega_n, g_{\omega|_n} x_0, T^n \omega)_n$ equidistributes towards
$$\sum_{e \in E} \pi(\{e\}) \delta_e \otimes m_e \otimes \mathbb{P}_e$$
for $\mathbb{P}_\lambda$-a.e. $\omega \in \Omega$.

The first part of this lemma is essentially contained in the article [**129**] of Simmons–Weiss, whose proof relies on ideas going back to Kolmogorov and Doob (cf. [**10**, §A.3]). Our proof of the second part generalizes the argument to the Markovian case.

The method of proof is to show the desired almost sure convergence for a fixed test function and then use separability of an appropriate space of test functions to exchange the order of quantifiers. When the underlying space is locally compact, this test function space can be taken to be the space of compactly supported continuous functions. This is however not the case in our setup, so that we need to find a substitute. To this end, let us introduce the following concept: Given a locally compact second countable metrizable space $\tilde{X}$, we shall say that a continuous function f on $\tilde{X} \times \Omega$ *compactly depends on finitely many coordinates* if there exists $N \in \mathbb{N}$ and a compactly supported continuous function $\tilde{f}$ on $\tilde{X} \times E^{N+1}$ such that $f(x, \omega) = \tilde{f}(x, \omega_0, \dots, \omega_N)$ for all $(x, \omega) \in \tilde{X} \times \Omega$. The collection of all such functions is separable; we denote it by $C_{cf}(\tilde{X} \times \Omega)$.

PROOF OF LEMMA 2.2.9. It is deduced from [**44**, Propositions 3.4.4, 3.4.6] that it suffices to check convergence for test functions $f \in C_{cf}(\tilde{X} \times \Omega)$, where

$\tilde{X} = X$ in (i) and $\tilde{X} = E \times X$ in (ii). In view of separability of this function space, the proof of [**129**, Proposition 5.1] still yields part (i).

Similarly, to obtain (ii) it is enough to establish the desired $\mathbb{P}_\lambda$-a.s. convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} \varphi(\omega_k, g_{\omega|_k} x_0, T^k\omega) \overset{n\to\infty}{\longrightarrow} \sum_{e\in E} \pi(\{e\}) \int_{X\times\Omega} \varphi(e,x,\omega)\,\mathrm{d}(m_e \otimes \mathbb{P}_e)$$

for a single bounded continuous test function $\varphi\colon E\times X\times\Omega \to \mathbb{R}$ depending on finitely many coordinates, say on the first $N+1$ coordinates $\omega_0,\dots,\omega_N$ in Ω.

Introduce the functions

$$\begin{aligned} \varphi_X(e,x) &= \int_\Omega \varphi(e,x,\omega)\,\mathrm{d}\mathbb{P}_e(\omega), \text{ and} \\ h(e,x,\omega) &= \varphi(e,x,\omega) - \varphi_X(e,x). \end{aligned}$$

Applying $\mathbb{P}_\lambda$-a.s. equidistribution of $(\omega_n, g_{\omega|_n} x_0)_n$ to the function φ_X and setting $z_k = z_k(\omega) = (\omega_k, g_{\omega|_k} x_0, T^k\omega)$, we see that it remains to show that $\mathbb{P}_\lambda$-a.s. we have

$$\frac{1}{n}\sum_{k=0}^{n-1} h(z_k) \overset{n\to\infty}{\longrightarrow} 0. \tag{2.2.6}$$

Denote by $\mathcal{B}_n$ the σ-algebra of Borel subsets of $E\times X\times\Omega$ depending only on the first $n+1$ coordinates $\omega_0,\dots,\omega_n$ in Ω. Then by definition of z_k and assumption on φ we have for $k \le n-N$

$$\mathbb{E}_\lambda[h(z_k)|\mathcal{B}_n] = h(z_k). \tag{2.2.7}$$

Now suppose $k \ge n$. Then, using the Markov property and the definition of φ_X,

$$\begin{aligned} &\int_\Omega \varphi_X(\omega'_{k-n}, g_{\omega'|_{k-n}} g_{\omega|_n} x_0)\,\mathrm{d}\mathbb{P}_{\omega_n}(\omega') \\ &\quad= \int_\Omega\int_\Omega \varphi(\omega'_{k-n}, g_{\omega'|_{k-n}} g_{\omega|_n} x_0, \omega'')\,\mathrm{d}\mathbb{P}_{\omega'_{k-n}}(\omega'')\,\mathrm{d}\mathbb{P}_{\omega_n}(\omega') \\ &\quad= \int_\Omega \varphi(\omega'_{k-n}, g_{\omega'|_{k-n}} g_{\omega|_n} x_0, T^{k-n}\omega')\,\mathrm{d}\mathbb{P}_{\omega_n}(\omega'). \end{aligned} \tag{2.2.8}$$

Using the Markov property again, one can express the conditional expectation $\mathbb{E}_\lambda[h(z_k)|\mathcal{B}_n]$ as

$$\begin{aligned} \mathbb{E}_\lambda[h(z_k)|\mathcal{B}_n] &= \mathbb{E}_\lambda\Big[h(\omega_k, g_{\omega|_k} x_0, T^k\omega)\Big|\mathcal{B}_n\Big] \\ &= \int_\Omega h(\omega'_{k-n}, g_{\omega'|_{k-n}} g_{\omega|_n} x_0, T^{k-n}\omega')\,\mathrm{d}\mathbb{P}_{\omega_n}(\omega'). \end{aligned} \tag{2.2.9}$$

Combining (2.2.8) and (2.2.9), we deduce that for $k \ge n$ we have

$$\mathbb{E}_\lambda[h(z_k)|\mathcal{B}_n] = 0. \tag{2.2.10}$$

It follows from (2.2.7) and (2.2.10) that the random variables

$$M_n = \sum_{k=0}^{\infty} \mathbb{E}_\lambda[h(z_k)|\mathcal{B}_n]$$

form a martingale under $\mathbb{P}_\lambda$ differing by a bounded amount (at most $2N\|h\|_\infty$) from $\sum_{k=0}^{n-1} h(z_k)$. In particular, $(M_n)_n$ has bounded increments, so that [**10**, Corollary A.8] yields that $\mathbb{P}_\lambda$-a.s. $\frac{1}{n}M_n \to 0$ as $n \to \infty$, proving (2.2.6) and hence the lemma. $\square$

Proof of Proposition 2.2.8. For the sake of readability, we shall first ignore the second component $T^n\omega$ and only prove equidistribution of $(g_{\omega|_n}x_0)_n$. Afterwards, we explain the modifications needed to obtain the full statement.

Let f be a bounded continuous function on X. For $\ell \in \mathbb{N}$ we consider the function

$$F_\ell \colon X \times (E_e^{\mathrm{r}})^{\mathbb{N}} \to \mathbb{R},\ (x, (w_j)_j) \mapsto \begin{cases} f(g_{w_0|_\ell}x), & \ell(w_0) > \ell, \\ 0, & \ell(w_0) \le \ell, \end{cases}$$

where $\ell(w_0)$ denotes the length of the word w_0. Applying Lemma 2.2.9(i) to F_ℓ with $\mathbb{P} = \tilde{\mu}_e^{\otimes\mathbb{N}}$ and using the invariance assumption on m, we get

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} f(g_{w_k|_\ell}g_{w_{k-1}}\cdots g_{w_0}x_0)\mathbb{1}_{\ell(w_k)>\ell} &\longrightarrow \int F_\ell \,\mathrm{d}(m \otimes \tilde{\mu}_e^{\otimes\mathbb{N}}) \qquad (2.2.11)\\ &= \mathbb{P}_e[\tau_e > \ell]\int f\,\mathrm{d}m \end{aligned}$$

as $n \to \infty$ for $\tilde{\mu}_e^{\otimes\mathbb{N}}$-a.e. $(w_j)_j \in (E_e^{\mathrm{r}})^{\mathbb{N}}$.

Now let $\omega \in \Omega$ correspond to $(w_j)_j \in (E_e^{\mathrm{r}})^{\mathbb{N}}$ via (2.2.3) and denote by $T(n)$ the number of occurrences of e in ω before time n. In other words, $T(n)$ is the number of the w_j contributing to $\omega|_n$, so that the latter is some intermediate word between $w_{T(n)-2}\dots w_0$ and $w_{T(n)-1}\dots w_0$. Using this observation, for every $L \in \mathbb{N}$ we can write

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} f(g_{\omega|_k}x_0) = \frac{T(n)}{n}\sum_{\ell=0}^{L-1}\frac{1}{T(n)}\sum_{k=0}^{T(n)-1} f(g_{w_k|_\ell}g_{w_{k-1}}\cdots g_{w_0}x_0)\mathbb{1}_{\ell(w_k)>\ell} \quad (2.2.12)\\ + \frac{1}{n}\sum_{k=0}^{T(n)-1}\sum_{\ell=L}^{\ell(w_k)-1} f(g_{w_k|_\ell}g_{w_{k-1}}\cdots g_{w_0}x_0) \quad (2.2.13)\\ - \frac{1}{n}\sum_{k=n}^{\tau_e^{T(n)}(\omega)-1} f(g_{\omega|_k}x_0). \quad (2.2.14) \end{aligned}$$

Using Lemma 2.2.5 and the Birkhoff ergodic theorem, we have $\mathbb{P}_e$-a.s. $\tau_e^n/n \to \mathbb{E}_e[\tau_e]$. This in turn implies that $\mathbb{P}_e$-a.s. also $T(n)/n \to 1/\mathbb{E}_e[\tau_e]$. Together with (2.2.11) it follows that the right-hand side of (2.2.12) converges $\mathbb{P}_e$-a.s. to

$$\frac{1}{\mathbb{E}_e[\tau_e]}\sum_{\ell=0}^{L-1}\mathbb{P}_e[\tau_e > \ell]\int f\,\mathrm{d}m.$$

Using the ergodic theorem again, we also know that (2.2.13) is bounded by

$$\frac{\|f\|_\infty}{n}\sum_{k=0}^{T(n)-1}(\ell(w_k) - L)^+ \overset{n\to\infty}{\longrightarrow} \frac{\|f\|_\infty}{\mathbb{E}_e[\tau_e]}\mathbb{E}_e[(\tau_e - L)^+],$$

and (2.2.14) by

$$\frac{\|f\|_\infty \ell(w_{T(n)-1})}{n} \overset{n\to\infty}{\longrightarrow} 0,$$

where in both cases convergence holds $\mathbb{P}_e$-a.s.

Since $\sum_{\ell=0}^{\infty} \mathbb{P}_e[\tau_e > \ell] = \mathbb{E}_e[\tau_e]$ and, by positive recurrence, $\mathbb{E}_e[(\tau_e - L)^+] \to 0$ as $L \to \infty$, the above combine to imply the desired $\mathbb{P}_e$-a.s. convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} f(g_{\omega|_k} x_0) \longrightarrow \int f \,\mathrm{d}m$$

as $n \to \infty$.

We now upgrade the argument above to also obtain joint equidistribution. With the same initial reduction as in the proof of Lemma 2.2.9, it suffices to prove $\mathbb{P}_e$-a.s. convergence

$$\frac{1}{n}\sum_{k=0}^{n-1} f(g_{\omega|_k} x_0, T^k\omega) \overset{n\to\infty}{\longrightarrow} \int f \,\mathrm{d}(m \otimes \mathbb{P}_\pi)$$

for one fixed bounded continuous function f on $X \times \Omega$ depending on only finitely many coordinates. The argument is similar as above; only the functions F_ℓ need to be chosen in a slightly more intricate way: We set

$$F_\ell \colon X \times (E_e^{\mathrm{r}})^{\mathbb{N}} \to \mathbb{R},\; (x, (w_j)_j) \mapsto \begin{cases} f(g_{w_0|_\ell} x_0, T_\ell(w_j)_j), & \ell(w_0) > \ell, \\ 0, & \ell(w_0) \le \ell, \end{cases}$$

where $T_\ell(w_j)_j$ is obtained by first identifying $(w_j)_j$ with $\omega \in \Omega$ via (2.2.3) and then applying the ℓ-fold shift T^ℓ. These functions F_ℓ again satisfy the assumptions of part (i) of Lemma 2.2.9. We find

$$\frac{1}{n}\sum_{k=0}^{n-1} f(g_{w_k|_\ell} g_{w_{k-1}} \cdots g_{w_0} x_0, T_\ell(w_{j+k})_j) \mathbb{1}_{\ell(w_k)>\ell} \longrightarrow \int F_\ell \,\mathrm{d}(m \otimes \tilde{\mu}_e^{\otimes\mathbb{N}})$$

as $n \to \infty$ for $\tilde{\mu}_e^{\otimes\mathbb{N}}$-a.e. $(w_j)_j \in (E_e^{\mathrm{r}})^{\mathbb{N}}$, the limit equaling, again by the assumed invariance of m and Lemma 2.2.5,

$$\begin{aligned} \int F_\ell \,\mathrm{d}(m \otimes \tilde{\mu}_e^{\otimes\mathbb{N}}) &= \int_{\{\ell(w_0)>\ell\}} \int_X f(x, T_\ell(w_j)_j) \,\mathrm{d}m(x) \,\mathrm{d}\tilde{\mu}_e^{\otimes\mathbb{N}}((w_j)_j) \\ &= \int_X \int_{\{\tau_e>\ell\}} f(x, T^\ell\omega) \,\mathrm{d}\mathbb{P}_e(\omega) \,\mathrm{d}m(x). \end{aligned}$$

Noting that by the Markov property and the description (2.2.2) of π we have

$$\sum_{\ell=0}^{\infty} \int_{\{\tau_e>\ell\}} f(x, T^\ell\omega) \,\mathrm{d}\mathbb{P}_e(\omega) = \mathbb{E}_e\left[\sum_{k=0}^{\tau_e-1} \mathbb{E}_{\omega_k}[f(x,\cdot)]\right] = \mathbb{E}_e[\tau_e]\mathbb{E}_\pi[f(x,\cdot)]$$

for every $x \in X$, the remainder of the argument is the same as above. Indeed, together with dominated convergence this implies that the limit

$$\frac{1}{\mathbb{E}_e[\tau_e]} \int_X \sum_{\ell=0}^{L-1} \int_{\{\tau_e>\ell\}} f(x, T^\ell\omega) \,\mathrm{d}\mathbb{P}_e(\omega) \,\mathrm{d}m(x)$$

of (2.2.12) now converges to $\int f \,\mathrm{d}(m \otimes \mathbb{P}_\pi)$ as $L \to \infty$, and (2.2.13) and (2.2.14) still tend to 0. □

2.2.4. Moment and Expansion Conditions. We now express the notions of finite moments and expansion in the Markovian setting, in a way that will be convenient when combining the results of §2.1 and §2.2.3.

Let ρ be a representation of G on a finite-dimensional real vector space V. Recall that $\mathrm{N}(\rho(g)) = \max(\|\rho(g)\|, \|\rho(g)^{-1}\|)$, where $\|\cdot\|$ is the operator norm associated to a fixed norm on V.

DEFINITION 2.2.10. A Markov chain on E is said to have *finite first moments* in (V,ρ) if for every $e \in E$

$$\mathbb{E}_e[\log \mathrm{N}(\rho(g_{\omega|_{\tau_e}}))] < \infty,$$

and to have *finite exponential moments* in (V,ρ) if for every $e \in E$ there exists $\delta > 0$ such that

$$\mathbb{E}_e[\mathrm{N}(\rho(g_{\omega|_{\tau_e}}))^{\delta}] < \infty.$$

As usual, we suppress the representation from the notation when $(V,\rho) = (\mathfrak{g}, \mathrm{Ad})$. Note that the definition does not depend on the choice of norm on V.

In terms of renewal measures these conditions read as follows.

LEMMA 2.2.11. *A recurrent irreducible Markov chain on E has finite first* (*resp. exponential*) *moments in V if and only if all renewal measures μ_e have the corresponding property.* □

Let us mention a few simple examples in which the above moment conditions are satisfied.

EXAMPLE 2.2.12.

(i) If the state space E is finite, then any irreducible Markov chain on E has finite exponential moments in (V,ρ).

(ii) More generally, if the Markov chain on E is irreducible and positive (resp. exponentially) recurrent and the coding map $E \to G$ takes values in a bounded subset of G, then the Markov chain has finite first (resp. exponential) moments in (V,ρ). This conclusion stays valid when the coding map has sufficiently slow growth.

(iii) Suppose the Markov chain on E is positive recurrent and let π be its stationary distribution. Denote by $c\colon E \to G$ the coding map. If $c_*\pi$ has a finite first moment in (V,ρ), i.e. if

$$\sum_{e' \in E} \log \mathrm{N}(\rho(g_{e'}))\pi(\{e'\}) < \infty,$$

then the Markov chain has finite first moments in (V,ρ).

We omit the straightforward verifications. ◇

Next, we generalize the notion of uniform expansion from §2.1.

DEFINITION 2.2.13. Let $(Y_k)_k$ be a stochastic process with values in $\mathrm{GL}_d(\mathbb{R})$ and $P \subset \mathbb{P}(\mathbb{R}^d)$ a closed subset invariant under the support of the distribution of Y_k for all $k \in \mathbb{N}$. Then we call $(Y_k)_k$ *uniformly expanding on P* if for all $\mathbb{R}v \in P$, almost surely,

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|Y_n \cdots Y_1 v\| > 0.$$

We call $(Y_k)_k$ *uniformly expanding on Grassmannians* if $(Y_n^{\wedge k})_n$ is uniformly expanding on $\mathbb{P}(\bigwedge_{\mathrm{p}}^k \mathbb{R}^d)$ for all $1 \le k \le d-1$.

To efficiently deal with our setting involving an abstract Markov chain on E, different starting distributions, and a coding map, it will be convenient to introduce the following more concise terminology.

DEFINITION 2.2.14. Let λ be a starting distribution on E. Then we say that a Markov chain on E is *λ-expanding* (*under the coding map $e \mapsto g_e$*) if the stochastic process

$$(Y_k)_k \colon (\Omega, \mathbb{P}_\lambda) \ni \omega \mapsto (\mathrm{Ad}(g_{\omega_{k-1}}))_k$$

on $\mathrm{GL}(\mathfrak{g})$ is uniformly expanding on Grassmannians. When $\lambda = \delta_e$ for some $e \in E$ we also say that it is *e-expanding*.

For brevity, we will usually omit the coding map from the notation when using these notions of expansion.

Under a moment assumption as in Example 2.2.12(iii), e-expansion can be phrased in terms of the renewal measure μ_e.

LEMMA 2.2.15. *Suppose that the Markov chain on E is irreducible and recurrent, let $e \in E$ and denote by $c \colon E \to G$ the coding map.*

(i) *If the Markov chain is e-expanding, then $\mathrm{Ad}_* \mu_e$ is uniformly expanding on Grassmannians.*
(ii) *Suppose the Markov chain is additionally positive recurrent and denote by π its stationary distribution. If $c_*\pi$ has a finite first moment in $\mathfrak{g}$, then the Markov chain is e-expanding if and only if $\mathrm{Ad}_* \mu_e$ is uniformly expanding on Grassmannians.*

PROOF. Let $1 \le k \le \dim(G) - 1$. We $\mathbb{P}_e$-a.s. have $\tau_e^n/n \to \mathbb{E}_e[\tau_e] \in [1, \infty]$ as $n \to \infty$. By definition, e-expansion means that, $\mathbb{P}_e$-a.s.,

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|\mathrm{Ad}^{\wedge k}(g_{\omega|_n})v\| > 0. \tag{2.2.15}$$

From this it follows that $\mathbb{P}_e$-a.s. also

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|\mathrm{Ad}^{\wedge k}(g_{\omega|_{\tau_e^n}})v\| = \liminf_{n\to\infty} \frac{\tau_e^n}{n} \frac{1}{\tau_e^n} \log\|\mathrm{Ad}^{\wedge k}(g_{\omega|_{\tau_e^n}})v\| > 0. \tag{2.2.16}$$

This gives part (i). In the setting of (ii), we have $\mathbb{E}_e[\tau_e] \in [1, \infty)$, and the moment assumption allows applying Oseledets' theorem with the shift map on $(\Omega, \mathbb{P}_\pi)$ (we remark that Oseledets' theorem holds not only for i.i.d. processes, but more generally for stationary ones; see e.g. [**116**, Theorem 1.6]). We find that all the limit inferiors above are actually limits $\mathbb{P}_\pi$-, thus in particular $\mathbb{P}_e$-a.s., so that in this case (2.2.16) also implies (2.2.15). ☐

2.2.5. Expanding Markov Chains. We now combine the bootstrapping results from §2.2.2 and §2.2.3 with those of §2.1 to prove our main Markovian measure classification and equidistribution results. These will imply Theorem 2.0.4. Recall that for $e \in E$, Γ_e denotes the closed subgroup of G generated by the support of the renewal measure μ_e.

THEOREM 2.2.16. *Let G be a real Lie group, Λ a discrete subgroup of G, and X the homogeneous space G/Λ. Suppose that the Markov chain on E is irreducible and positive recurrent; denote by π its stationary distribution. Suppose furthermore that the Markov chain is π-expanding and has finite first moments in $\mathfrak{g}$. Let ν be an ergodic stationary probability measure for the action chain on $E \times X$ as in* (2.2.4)*. Then either*

(i) *for every $e \in E$ the measure ν_e is Γ_e-invariant and supported on a finite Γ_e-orbit, or*

(ii) *Λ is a lattice and all ν_e are the Haar measure m_X on X.*

Moreover, for every $(e' \leftarrow e)$-admissible word $c \in E^*$, we have $(g_c)_*\nu_e = \nu_{e'}$.

Proof. Note that by irreducibility of the Markov chain on E, π-expansion implies e-expansion for every $e \in E$. Thus, it follows by Lemma 2.2.7 and Theorem 2.1.9 that each ν_e is either supported on a finite Γ_e-orbit or is the Haar measure m_X on X. Irreducibility of the Markov chain together with the last statement of Lemma 2.2.7 imply that the same option applies to all $e \in E$.

The last claim is clear in case (ii). In case (i), Lemma 2.2.7 implies that $(g_c)_*\nu_e$ and $\nu_{e'}$ are of the same measure class. Being uniform measures on finite orbits, this forces $(g_c)_*\nu_e = \nu_{e'}$, as claimed. ☐

Theorem 2.2.17. *Let G be a real Lie group with simple identity component such that the Zariski closure of $\mathrm{Ad}(G)$ is Zariski connected, Λ a lattice in G, and $X = G/\Lambda$. Suppose that the Markov chain on E is irreducible and positive recurrent and has finite exponential moments in $\mathfrak{g}$. Denote by π its stationary distribution and let $e \in E$. Assume that Γ_e is not virtually contained in any conjugate of Λ, that Γ_e acts transitively on the connected components of X, and that the Markov chain is e-expanding. Then, for every $x_0 \in X$, $(g_{\omega|_n}x_0, T^n\omega)_n$ equidistributes towards $m_X \otimes \mathbb{P}_\pi$ for $\mathbb{P}_e$-a.e. $\omega \in \Omega$.*

Proof. Combine Lemma 2.2.15, Theorem 2.1.12, and Proposition 2.2.8. ☐

Proof of Theorem 2.0.4. We use the inclusion $E \hookrightarrow G$ as coding map and let λ denote the distribution of Y_1. By hypothesis, $(Y_k)_k$ is λ-expanding, hence e-expanding for every $e \in E$ satisfying $\lambda(\{e\}) > 0$. Moreover, since E is finite, the Markov chain on E is positive recurrent and has finite exponential moments in $\mathfrak{g}$. Now, in view of Lemma 2.2.18 below, the result follows by applying Theorem 2.2.17 to each such $e \in E$. ☐

Lemma 2.2.18. *Suppose that the state space E is finite and that the Markov chain on E is irreducible. If $x \in X$ and $e \in E$ are such that the random orbit $\{g_{\omega|_n}x \mid n \in \mathbb{N}\} \subset X$ is $\mathbb{P}_e$-a.s. infinite, then the orbit $\Gamma_e^+ x$ is infinite.*

Proof. Denote by E_e^{adm} the set of all admissible words starting with e and consider the set

$$\mathcal{O} = \{g_w x \mid w \in E_e^{\mathrm{adm}}\}.$$

By assumption it is infinite.

Since the state space is finite, we can choose $k \in \mathbb{N}$ such that any state can be reached from everywhere in at most k steps with positive probability. Then for every $w \in E_e^{\mathrm{adm}}$ there is an admissible word $c \in E^*$ of length at most $k-1$ such that cw is $(e \leftarrow e)$-admissible. It follows that $g_{cw}x \in \Gamma_e^+ x$ and hence

$$\mathcal{O} \subset \bigcup_{\substack{c \in E^* \text{ admissible} \\ \ell(c) \leq k-1}} g_c^{-1} \Gamma_e^+ x,$$

which forces $\Gamma_e^+ x$ to be infinite as well. ☐

2.2.6. An Example. To conclude this section, we are going to explain an example due to Simmons–Weiss [**129**] that is used to relate Diophantine properties of fractals to random walks. We prove Proposition 2.2.19, which can be considered a Markovian extension of [**129**, Theorem 6.4], and deduce Corollary 2.0.5.

Let $G = \mathrm{PGL}_d(\mathbb{R})$ and $\Lambda = \mathrm{PGL}_d(\mathbb{Z})$. Given positive integers m and n with $m+n=d$, let $\mathbb{R}^{m\times n}$ be the space of $m\times n$-matrices with real entries and define

$$a(t) = \begin{pmatrix} \mathrm{e}^{t/m}\mathbf{1}_m & \\ & \mathrm{e}^{-t/n}\mathbf{1}_n \end{pmatrix},\ u_M = \begin{pmatrix} \mathbf{1}_m & -M \\ & \mathbf{1}_n \end{pmatrix},\ \text{and } O_1 \oplus O_2 = \begin{pmatrix} O_1 & \\ & O_2 \end{pmatrix}$$

for $t \in \mathbb{R}$, $M \in \mathbb{R}^{m\times n}$ and $O_1 \in \mathrm{O}_m(\mathbb{R})$, $O_2 \in \mathrm{O}_n(\mathbb{R})$. We will denote the corresponding subgroups of G by $A = \{a(t) \mid t \in \mathbb{R}\}$, $U = \{u_M \mid M \in \mathbb{R}^{m\times n}\}$, $K = \{O_1 \oplus O_2 \mid O_1 \in \mathrm{O}_m(\mathbb{R}), O_2 \in \mathrm{O}_n(\mathbb{R})\}$, and set $P = AKU$. Note that A and K commute and normalize U; in particular, P is a group. An element $g \in P$ can be uniquely written as a product of the form $a(t)ku_M$ and we denote the corresponding values of t, k by $t(g)$ and $k(g)$, respectively. Finally, let $V^+ = \mathrm{Lie}(U)$ be the Lie algebra of U.

Proposition 2.2.19. *Suppose that E is finite and let π be the stationary distribution of an irreducible Markov chain on E. Suppose that the coding map $E \to G$, $e \mapsto g_e$ takes values in P, that*

$$\sum_{e'\in E} t(g_{e'})\pi(\{e'\}) > 0, \tag{2.2.17}$$

and that for some $e_0 \in E$ the Lie algebra of H_{e_0} contains V^+. Then the assumptions of Theorem 2.2.17 are satisfied for every $e \in E$.

Proof. Positive recurrence and finite exponential moments in $\mathfrak{g}$ follow from finiteness of the state space. Below, we are going to show that all renewal measures μ_e are in "(m,n)-upper block form" in the sense of [**129**, Definition 6.3]. Then [**129**, Theorem 6.4] (the proof of which does not use the assumption of compact support) implies that for every $e \in E$, Γ_e is not virtually contained in any conjugate of Λ and that Proposition 2.1.7 can be applied to μ_e, yielding e-expansion of the Markov chain.

To show that μ_e is in (m,n)-upper block form for every $e \in E$, we have to argue that $\int_G t(g)\,\mathrm{d}\mu_e(g) > 0$ and that the Lie algebra of H_e contains V^+.

Regarding positivity of the integral, we calculate, using that $t\colon P \to (\mathbb{R},+)$ is a homomorphism and (2.2.2),

$$\begin{aligned}\int_G t(g)\,\mathrm{d}\mu_e(g) = \mathbb{E}_e[t(g_{\omega|_{\tau_e}})] &= \sum_{e'\in E} t(g_{e'})\mathbb{E}_e\Bigg[\sum_{k=0}^{\tau_e-1} \mathbb{1}_{\omega_k=e'}\Bigg] \\ &= \mathbb{E}_e[\tau_e]\sum_{e'\in E} t(g_{e'})\pi(\{e'\}) > 0.\end{aligned}$$

Finally, in view of the assumption on H_{e_0}, the inclusion $V^+ \subset \mathrm{Lie}(H_e)$ follows from part (ii) of Lemma 2.2.6 and the fact that U is normalized by P. □

Proof of Corollary 2.0.5. By Proposition 2.2.19 and Lemma 2.2.6(v) we need only verify that the Lie algebra of H_E contains V^+. (Recall that part of the conclusion of Lemma 2.2.6 is that H_E is in fact a group; here it is the real algebraic subgroup of G generated by $g_0, \ldots, g_r$.) The argument for this is the same as in the proof of [**129**, Theorem 1.1]. Let us briefly reproduce it: For $0 \leq i \leq r$, we have $g_i = u_i' a_i k_i$ with

$$u_i' = \begin{pmatrix} \mathbf{1}_d & c_i^d y_i \\ 0 & 1 \end{pmatrix},\ a_i = \begin{pmatrix} c_i \mathbf{1}_d & 0 \\ 0 & c_i^{-d} \end{pmatrix},\ \text{and } k_i = \begin{pmatrix} O_i & 0 \\ 0 & 1 \end{pmatrix}.$$

Then, for $n \in \mathbb{N}$, we can write

$$H_E \ni g_0^{-n} g_i g_0^n = (k_0^{-n} a_0^{-n} u_i' a_0^n k_0^n) a_i (k_0^{-n} k_i k_0^n).$$

Noting that for $n \to \infty$ we have $a_0^{-n} u_i' a_0^n \to \mathbf{1}_{d+1}$ and passing to a subsequence along which $k_0^{n_j} \to \mathbf{1}_{d+1}$ as $j \to \infty$, it follows that $a_i k_i \in H_E$, so that also $u_i' \in H_E$. Thus, we see that $M_j := k_0^{-n_j} a_0^{-n_j} u_i' a_0^{n_j} k_0^{n_j} \in H_E \cap U$ for all j. Since $M_j \to \mathbf{1}_{d+1}$ as $j \to \infty$, this implies that

$$\mathrm{Lie}(H_E) \ni \log(M_j) = M_j - \mathbf{1}_{d+1}$$

for j large enough. As a computation shows, the right-hand side above converges in direction towards $\left(\begin{smallmatrix} \mathbf{0}_d & y_i \\ 0 & 0 \end{smallmatrix}\right)$. Since the y_i span $\mathbb{R}^d$ by assumption, we conclude that indeed $V^+ \subset \mathrm{Lie}(H_E)$. □

2.3. Diophantine Approximation on Fractals

As observed by Simmons–Weiss, equidistribution results as in §2.1 can be used to obtain statements about Diophantine approximation on fractals obtained as limit sets of similarity IFS. In this final section, using the analogous results for Markov random walks from §2.2, we deal with limit sets of graph-directed similarity IFS.

The first three subsections are of preparatory nature. We recall basic terminology and results on graph-directed IFS (§2.3.1), and make the connection between similarities, the homogeneous dynamics setting and Diophantine approximation (§2.3.2, §2.3.3). Our main Diophantine approximation results, which imply Theorems 2.0.6 and 2.0.7, will be stated and proved in §2.3.4.

2.3.1. Graph-Directed IFS. A *directed multigraph* is a tuple (V, E, i, t) consisting of non-empty sets V, E of *vertices* and *edges*, respectively, and functions $i, t\colon E \to V$ associating to an edge $e \in E$ the *initial vertex* $i(e) \in V$ and the *terminal vertex* $t(e) \in V$. The multigraph is *finite* if both sets V and E are. A non-empty word $w = e_0 \ldots e_{n-1} \in E^*$ or sequence $\omega = (e_j)_j \in E^{\mathbb{N}}$ is called a (*finite resp. infinite*) *path* if $t(e_{j-1}) = i(e_j)$ for all j. Denote the set of infinite paths by E^∞. We extend the initial vertex function i to paths by $i((e_j)_j) = i(e_0 \ldots e_{n-1}) = i(e_0)$, and the terminal vertex function t to finite paths by $t(e_0 \ldots e_{n-1}) = t(e_{n-1})$. We call the multigraph *connected* if for every pair of vertices $u, v \in V$ there exists a finite path from u to v (i.e. a path w with $i(w) = u$ and $t(w) = v$). Finally, we call a Markov chain on E (or an associated Markov measure on $E^{\mathbb{N}}$) *adapted* if the transition probabilities $(p_{e',e})_{e,e' \in E}$ satisfy $p_{e',e} > 0 \iff t(e) = i(e')$ for $e, e' \in E$. Observe that if the multigraph is connected, any adapted shift-invariant Markov measure on E^∞ is ergodic.

Remark 2.3.1. When E is finite, the space $E^\infty \subset E^{\mathbb{N}}$ of infinite paths is the subshift of finite type defined by the edge-incidence relation given by the multigraph. The notation is intentionally the same as for admissible sequences in §2.2, since these notions coincide for adapted Markov chains on E, to which we will from now on restrict our attention. $\diamond$

Recall that a *similarity* of $\mathbb{R}^D$ is a map $\phi\colon \mathbb{R}^D \to \mathbb{R}^D$ of the form $\phi(x) = rO(x) + b$ for some $r > 0$, $O \in \mathrm{O}_D(\mathbb{R})$ and $b \in \mathbb{R}^D$. The number $r = \|\phi'\|$ is the *similarity ratio* of ϕ. If $r < 1$, ϕ is said to be *contracting.*

Definition 2.3.2. Let (V, E, i, t) be a finite connected directed multigraph and suppose that for every $e \in E$ we are given a similarity $\phi_e\colon \mathbb{R}^D \to \mathbb{R}^D$. Then the tuple $(V, E, i, t, (\phi_e)_e)$ is called a *graph-directed similarity IFS.*

Ordinary similarity IFS represent the special case of graph-directed similarity IFS with a single vertex. We also emphasize that finiteness and connectedness of the directed multigraph are part of our definition of graph-directed similarity IFS.

It is customary to think of one copy of $\mathbb{R}^D$ being attached to each vertex, and the map ϕ_e going from the copy at $t(e)$ to the one at $i(e)$. This viewpoint is consistent with the formula $\phi_w = \phi_{e_0} \cdots \phi_{e_{n-1}}$ for words $w = e_0 \dots e_{n-1}$.[5]

We need to introduce some more terminology. A graph-directed similarity IFS is said to be

- *contracting* if $\sup_{e\in E}\|\phi'_e\| < 1$,
- to satisfy the *open set condition* if there exists a collection $(U_v)_{v\in V}$ of non-empty open subsets of $\mathbb{R}^D$ with $\phi_e(U_{t(e)}) \subset U_{i(e)}$ for every $e \in E$ and $\phi_e(U_{t(e)}) \cap \phi_{e'}(U_{t(e')}) = \emptyset$ for any distinct edges $e, e' \in E$ with $i(e) = i(e')$, and
- to be *irreducible* if there does not exist a collection $(W_v)_{v\in V}$ of proper affine subspaces of $\mathbb{R}^D$ with $\phi_e(W_{t(e)}) = W_{i(e)}$ for every $e \in E$.

Given a contracting graph-directed similarity IFS, one proves in complete analogy to the classical case that there is a unique collection $(\mathcal{K}_v)_{v\in V}$ of non-empty compact subsets of $\mathbb{R}^D$ such that

$$\mathcal{K}_v = \bigcup_{i(e)=v} \phi_e(\mathcal{K}_{t(e)})$$

for every $v \in V$ (see [**88**]). The union $\mathcal{K} = \bigcup_{v\in V} \mathcal{K}_v$ is called the *attractor* of the graph-directed IFS. It can alternatively be obtained as the image of E^∞ under the *natural projection*

$$\begin{aligned}\Pi\colon E^\infty &\to \mathbb{R}^D,\\ \omega = (\omega_j)_j &\mapsto \lim_{n\to\infty} \phi_{\omega_0} \cdots \phi_{\omega_{n-1}}(x),\end{aligned}$$

which is continuous and independent of the choice of $x \in \mathbb{R}^D$. Observe that the attractors $\mathcal{K}$ arising in this way are precisely what we called sofic similarity fractals in §2.0.3. Indeed, setting $\Phi = \{\phi_e \,|\, e \in E\}$, the image of E^∞ under the map $E^\infty \to \Phi^\mathbb{N}$, $\omega \mapsto (\phi_{\omega_j})_j$ is a sofic subshift of $\Phi^\mathbb{N}$.

Generalizing a classical result of Hutchinson [**65**], Wang [**131**] identified the Hausdorff measure on attractors of graph-directed similarity IFS satisfying the open set condition.

[5]Incidentally, this also explains the switch to a left-to-right indexing convention.

THEOREM 2.3.3 (Wang [**131**]). *Let $(V, E, i, t, (\phi_e)_e)$ be a contracting graph-directed similarity IFS satisfying the open set condition. Let $\mathcal{K}$ be the associated attractor, $s \geq 0$ its Hausdorff dimension, Π the natural projection, and denote s-dimensional Hausdorff measure by $\mathcal{H}^s$. Then $\mathcal{H}^s|_{\mathcal{K}}$ is proportional to $\Pi_*\mathbb{P}$ for some adapted shift-invariant Markov probability measure $\mathbb{P}$ on E^∞.*

2.3.2. Diophantine Approximation and Dani Correspondence. A matrix $M \in \mathbb{R}^{m\times n}$ is said to be

- *badly approximable* if there exists $c > 0$ such that for all $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ we have $\|M\mathbf{q} - \mathbf{p}\| \geq c\|\mathbf{q}\|^{-n/m}$,
- *well approximable* if it is not badly approximable, and
- *Dirichlet improvable* if there exists $0 < \varepsilon < 1$ such that for all sufficiently large Q there exist $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ with $\|\mathbf{q}\|_\infty \leq \varepsilon Q$ and $\mathbf{p} \in \mathbb{Z}^m$ with $\|M\mathbf{q} - \mathbf{p}\|_\infty \leq \varepsilon Q^{-n/m}$.

In the above, $\|\cdot\|_\infty$ denotes the supremum norm on $\mathbb{R}^{m\times n}$ and $\|\cdot\|$ an arbitrary norm. A general survey of Diophantine approximation can be found in [**11**]. For a more specific overview pertaining to the topic at hand we refer to [**129**, §7].

The Dani correspondence principle asserts that the Diophantine properties of a matrix $M \in \mathbb{R}^{m\times n}$ are encoded in the behavior of the forward orbit $(a(t)u_M x_0)_{t\geq 0}$ inside the homogeneous space $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, where $d = m + n$, x_0 denotes the base point in X, and we are using the notation from §2.2.6. To see this, it is useful to think of X as the space X_d of unimodular lattices in $\mathbb{R}^d$, via the identification

$$X \ni g\,\mathrm{SL}_d(\mathbb{Z}) \longleftrightarrow g\mathbb{Z}^d \in X_d.$$

The Mahler compactness criterion then says that a subset $A \subset X$ is relatively compact if and only if it is contained in one of the sets

$$K_\varepsilon = \{x \in X \mid \forall v \in x \setminus \{0\} \colon \|v\|_\infty \geq \varepsilon\}$$

for $0 < \varepsilon < 1$. Note that these sets themselves are compact, exhaust X, and satisfy $K_{\varepsilon_1}^\circ \supset K_{\varepsilon_2}$ for $0 < \varepsilon_1 < \varepsilon_2$.

THEOREM 2.3.4 (Dani correspondence). *The matrix $M \in \mathbb{R}^{m\times n}$ is*

(i) *badly approximable if and only if the forward orbit $\{a(t)u_M x_0 \mid t \geq 0\}$ is relatively compact, i.e. contained in K_ε for some $0 < \varepsilon < 1$,*
(ii) *Dirichlet improvable if and only if for some $0 < \varepsilon < 1$ the trajectory of $u_M x_0$ under $(a(t))_{t\geq 0}$ eventually leaves K_ε, i.e. if there exists $T \geq 0$ such that $\{a(t)u_M x_0 \mid t \geq T\}$ does not intersect K_ε.*

For the proofs, see Dani [**30**, Theorem 2.20] and Kleinbock–Weiss [**74**, Proposition 2.1].

COROLLARY 2.3.5. *If $\{a(t)u_M x_0 \mid t \geq 0\}$ is dense in X, then M is well approximable and not Dirichlet improvable.*

In fact, the random walk approach yields the following stronger property.

DEFINITION 2.3.6. A matrix $M \in \mathbb{R}^{m\times n}$ is said to be *of generic type* if the forward orbit $(a(t)u_M x_0)_{t\geq 0}$ is equidistributed in $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ with respect to the Haar measure m_X, where $d = m + n$ and x_0 denotes the base point in X.

2.3.3. Algebraic Similarities as Group Elements. Following [**129**, §10], we next interpret a class of similarities of $\mathbb{R}^{m\times n}$ as elements of $\mathrm{PGL}_d(\mathbb{R})$.

Recall the subgroups A, K, U and $P = AKU$ of $\mathrm{PGL}_d(\mathbb{R})$ defined in §2.2.6. The group P acts by left multiplication on the space P/AK, which is topologically identified with $U \cong \mathbb{R}^{m\times n}$ via

$$\mathbb{R}^{m\times n} \ni B \longleftrightarrow u_{-B}AK \in P/AK.$$

The obtained action of P on $\mathbb{R}^{m\times n}$ is faithful and is described as follows: For $B \in \mathbb{R}^{m\times n}$ we have

$$\begin{aligned} a(t) \cdot B &= \mathrm{e}^{t(1/m+1/n)}B, \\ k \cdot B &= O_1 B O_2^{-1}, \\ u_M \cdot B &= B - M, \end{aligned}$$

for $a(t) \in A$, $k = O_1 \oplus O_2 \in K$ and $u_M \in U$. Thus, P can be identified with the group of *algebraic similarities* of $\mathbb{R}^{m\times n}$, i.e. similarities of the form $B \mapsto rO_1BO_2 + M$ for some $r > 0$, $O_1 \in \mathrm{O}_m(\mathbb{R})$, $O_2 \in \mathrm{O}_n(\mathbb{R})$ and $M \in \mathbb{R}^{m\times n}$. Note that when $m = 1$ or $n = 1$, all similarities of $\mathbb{R}^{m\times n}$ are algebraic.

2.3.4. The Approximation Result. We are now ready to formulate and prove the graph-directed version of [**129**, Theorem 8.1].

Theorem 2.3.7. *Let $(V, E, i, t, (\phi_e)_e)$ be a contracting irreducible graph-directed IFS of algebraic similarities of $\mathbb{R}^{m\times n}$ satisfying the open set condition. Let $\mathcal{K}$ denote its attractor and $s \geq 0$ its Hausdorff dimension. Then almost every point on $\mathcal{K}$ with respect to s-dimensional Hausdorff measure is of generic type, so in particular, well approximable and not Dirichlet improvable.*

Proof of Theorem 2.0.7. As already remarked, in the case $n = 1$ all similarities are algebraic. Now the result follows by an application of Theorem 2.3.7. ☐

By virtue of Wang's Theorem 2.3.3, Theorem 2.3.7 above is a consequence of the following result.

Theorem 2.3.8. *Let $(V, E, i, t, (\phi_e)_e)$ be an irreducible graph-directed similarity IFS on $\mathbb{R}^{m'\times n'}$ consisting of algebraic similarities, and $\mathbb{P}$ an adapted shift-invariant Markov measure on E^∞ for which the IFS is* contracting on average*, in the sense that*

$$\sum_{e\in E} \log\|\phi_e'\|\pi(\{e\}) < 0,$$

where π denotes the projection of $\mathbb{P}$ to the first coordinate. Then the natural projection $\Pi\colon E^\infty \to \mathbb{R}^{m'\times n'}$ is well-defined $\mathbb{P}$-almost everywhere and almost every point with respect to $\Pi_*\mathbb{P}$ is of generic type.

Proof. The natural projection is well-defined at $\omega \in E^\infty$ whenever the contraction ratios $\|\phi'_{\omega_0\dots\omega_{n-1}}\|$ decay exponentially. Recalling that adapted shift-invariant Markov measures are ergodic, it follows from the Birkhoff ergodic theorem and the contraction-on-average assumption that this is the case $\mathbb{P}$-a.s. By definition, what we need to show is that the forward orbit $(a(t)u_{\Pi(\omega)}x_0)_{t\geq 0}$ is equidistributed with respect to the Haar measure m_X on the homogeneous space $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z}) = \mathrm{PGL}_d(\mathbb{R})/\mathrm{PGL}_d(\mathbb{Z})$ for $\mathbb{P}$-a.e. $\omega \in E^\infty$, where $d = m' + n'$ and x_0 denotes the base point in X.

To see this, we follow Simmons–Weiss' strategy in the proof of [**129**, Theorem 8.11] and connect the above orbit with certain random walk trajectories. First note that $\mathbb{P}$ defines an irreducible finite-state Markov chain on the set E of edges. Using the construction in §2.3.3, we can view the algebraic similarities ϕ_e as elements of $P \leqslant G = \mathrm{PGL}_d(\mathbb{R})$. Defining the coding map

$$E \ni e \mapsto g_e \coloneqq \phi_e^{-1} \in P,$$

we are then in the setting of §2.2. We claim that (after a conjugation) the assumptions of Proposition 2.2.19 are satisfied. Indeed, validity of (2.2.17) follows from the contraction-on-average assumption on the ϕ_e (notice the inverse in the definition of the g_e), and the assumption on the Lie algebra of H_{e_0} for some $e_0 \in E$ is satisfied after conjugating the coding map by an element of P so that H_{e_0} contains an element $h_0 \in AK$ with $t(h_0) > 0$, as the corresponding argument in [**129**, §10.1] shows. One just needs to observe that the irreducibility assumption on the graph-directed IFS forces the IFS consisting of the atoms of the renewal measure μ_{e_0} to be irreducible. (An invariant affine subspace W for the support of μ_{e_0} gives rise to an invariant collection of subspaces $(W_v)_v$ in the graph-directed sense by choosing for each vertex v a path w_v from $i(e_0)$ to v starting with e_0 and setting $W_v = \phi_{w_v}^{-1}(W)$.) We conclude that Theorem 2.2.17 can be applied for every $e \in E$. Writing $\mathbb{P}$ as convex combination of the measures $\mathbb{P}_e$ as in (2.2.1), we thus obtain $\mathbb{P}$-a.s. equidistribution of $(g_{\omega|_n} x_0)_n$ towards m_X.

We shall use this to argue that the sequence

$$\Big(k(g_{\omega|_n})^{-1} u_{\Pi(T^n\omega)} g_{\omega|_n} x_0, \omega_n\Big)_n \tag{2.3.1}$$

equidistributes towards $m_X \otimes \pi$ for $\mathbb{P}$-a.e. $\omega \in E^\infty$, where $k(\cdot)$ denotes the K-component of an element of $P = AKU$. To this end, we consider the Markov random walk on $X \times K$ given by the coding map $E \ni e \mapsto (g_e, k(g_e)) \in G \times K$ and the associated action chain trajectories

$$(y_n)_n = (\omega_n, g_{\omega|_n} x_0, k(g_{\omega|_n}))_n$$

in $E \times X \times K$. Since $\mathbb{P}$-a.s. the random walk trajectory $(g_{\omega|_n} x_0)_n$ equidistributes towards m_X, no escape of mass can occur for the sequence $\frac{1}{n}\sum_{k=0}^{n-1} \delta_{y_k}$ of empirical measures. The Breiman law of large numbers (see [**9**, Corollary 3.3]) thus implies that $\mathbb{P}$-a.s. every weak* limit ν of this sequence of empirical measures is a probability measure on $E \times X \times K$ that is stationary for the action chain. By Lemma 2.2.7, ν decomposes as

$$\nu = \sum_{e \in E} \pi(\{e\}) \delta_e \otimes \nu_e$$

for μ_e-stationary probability measures ν_e on $X \times K$. Using equidistribution of $(g_{\omega|_n} x_0)_n$ once more, we see that the ν_e project to m_X in the first coordinate. Moreover, for every $e \in E$ the closed subgroup Γ_e generated by the support of the renewal measure μ_e is non-compact and therefore acts mixingly on X by the Howe–Moore theorem. Thus, its action on $X \times K_e$ is ergodic, where K_e denotes the compact group $\overline{k(\Gamma_e)}$. These observations put us in a position to apply [**129**, Proposition 5.3]. The conclusion is that $\nu_e = m_X \otimes m_{K_e}$, where

m_{K_e} is the Haar measure on K_e. Hence the limit ν is unique, so that $(y_n)_n$ equidistributes $\mathbb{P}$-a.s. towards

$$\sum_{e \in E} \pi(\{e\})\delta_e \otimes m_X \otimes m_{K_e}.$$

Now part (ii) of Lemma 2.2.9 implies that

$$(y_n, T^n\omega)_n$$

equidistributes towards the probability measure

$$\sum_{e \in E} \pi(\{e\})\delta_e \otimes m_X \otimes m_{K_e} \otimes \mathbb{P}_e$$

on $E \times X \times K \times E^\infty$ for $\mathbb{P}$-a.e. $\omega \in E^\infty$.

The natural projection Π is not necessarily continuous in the contracting-on-average case. However, a standard argument involving Lusin's theorem still shows that the equidistribution of $(y_n, T^n\omega)_n$ established above entails $\mathbb{P}$-a.s. equidistribution of

$$(y_n, \Pi(T^n\omega))_n$$

towards

$$\sum_{e \in E} \pi(\{e\})\delta_e \otimes m_X \otimes m_{K_e} \otimes \Pi_*\mathbb{P}_e$$

(cf. the proof of [**129**, Proposition 5.2]). Applying the continuous map

$$F\colon E \times X \times K \times \mathbb{R}^{m' \times n'} \to X \times E,$$
$$(e, x, k, M) \mapsto (k^{-1}u_M x, e),$$

we finally obtain equidistribution of (2.3.1) towards

$$F_*\Big(\sum_{e \in E} \pi(\{e\})\delta_e \otimes m_X \otimes m_{K_e} \otimes \Pi_*\mathbb{P}_e\Big) = m_X \otimes \pi.$$

Having established the necessary equidistribution for random walk trajectories, the final ingredient needed to finish the proof is the connection to the geodesic flow trajectory of $u_{\Pi(\omega)}x_0$. It comes from the relationship

$$k(g_{\omega|_n})^{-1} u_{\Pi(T^n\omega)} g_{\omega|_n} x_0 = a(t_n) u_{\Pi(\omega)} x_0, \tag{2.3.2}$$

where $t_n = t(g_{\omega|_n})$. To verify this formula, one first notes that the AK-components of both sides agree. To see that the U-components do as well, one applies the inverses of $g_{\omega|_n}$ and $u_{\Pi(T^n\omega)}^{-1} k(g_{\omega|_n}) a(t_n) u_{\Pi(\omega)}$ interpreted as algebraic similarities to the matrix $\Pi(T^n\omega)$ and observes that the result is $\Pi(\omega)$ in both cases.

Given a bounded continuous function f on X, it now remains to apply equidistribution of (2.3.1) towards $m_X \otimes \pi$ to the function f' on $X \times E$ defined by $f'(x, e) = \int_0^{t(g_e)} f(a(t)x)\,\mathrm{d}t$. As in the proof of [**129**, Theorem 8.11], in view of (2.3.2) this yields

$$\lim_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} f(a(t) u_{\Pi(\omega)} x_0)\,\mathrm{d}t = \frac{\int_{X \times E} f'\,\mathrm{d}(m_X \otimes \pi)}{\int_E t(g_e)\,\mathrm{d}\pi(e)} = \int_X f\,\mathrm{d}m_X.$$

Using that the sequence $(t_n)_n$ has bounded gaps, this proves equidistribution of $(a(t)u_{\Pi(\omega)}x_0)_{t \geq 0}$ with respect to m_X. $\square$

Proof of Theorem 2.0.6. Consider a directed multigraph with a single vertex v_0 and edge set $E = \Phi$ (with $t(\phi) = i(\phi) = v_0$ for all $\phi \in \Phi$). Since the Markov measure $\mathbb{P}$ has full support, it defines an irreducible Markov chain on Φ. Let π be its stationary distribution and $\mathbb{P}_\pi$ the associated Markov measure. Then Theorem 2.3.8 can be applied to $\mathbb{P}_\pi$ and yields the desired conclusion for $\Pi_*\mathbb{P}_\pi$-a.e. point on $\mathcal{K}$. Noting that $\pi(\{\phi\}) > 0$ for all $\phi \in \Phi$ by irreducibility and using (2.2.1) once for π and once for the projection of $\mathbb{P}$ to the first coordinate, we deduce that the conclusion holds $\Pi_*\mathbb{P}_\phi$-a.s. for every $\phi \in \Phi$, and thus also $\Pi_*\mathbb{P}$-a.s. $\square$

CHAPTER 3

Spread Out Random Walks on Homogeneous Spaces

† Let G be a locally compact σ-compact metrizable group, $\Lambda < G$ a discrete subgroup, and X the homogeneous space G/Λ. We recall that any Borel probability measure μ on G defines a random walk on X, whose location Φ_n after n steps when starting at $x_0 \in X$ is given by

$$\Phi_n = Y_n \cdots Y_1 x_0, \tag{3.0.1}$$

where $(Y_k)_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables in G with common law μ. As usual, we are interested in a quantitative description of the asymptotics of the random walk in terms of some natural, "stable" limiting distribution on X.

In this chapter, we will not a priori restrict to the case that the space X has finite volume. In this more general setting, by a *Haar measure* m_X on X we mean a non-trivial G-invariant Radon measure on X (if one exists). In case X admits a finite Haar measure, we are in the setup of the previous chapters: Λ is a lattice in G, we assume that m_X is normalized to be a probability measure, and say that X has finite volume. Otherwise, we say that X has infinite volume. According to this distinction, the discussion in this chapter splits into two cases.

3.0.1. Finite Volume Spaces. In the finite volume case, the following version of Question 3 from the Introduction will serve as our motivation.

Question.

(Q1) Do the laws $\mathcal{L}_{x_0}(\Phi_n) = \mu^{*n} * \delta_{x_0}$ converge as $n \to \infty$?
(Q2) If yes, can the convergence be made effective?

These questions arise in particular in light of Benoist–Quint's Theorem B stated in the Introduction. Answers are known only in special cases: Breuillard [**22**] established (Q1) for certain measures supported on unipotent subgroups, and more recently Buenger [**23**] was able to positively answer (Q1) and (Q2) for some sparse solvable measures. In this chapter, we add to this list the class of aperiodic spread out measures.

Definition 3.0.1. Let μ be a probability measure on G.

- The measure μ is called *spread out* if for some $n_0 \in \mathbb{N}$ the convolution power μ^{*n_0} is not singular with respect to Haar measure on G.
- Let $\mathcal{G}$ denote the closed subgroup of G generated by $\operatorname{supp}(\mu)$. Then we call μ *aperiodic* if μ is not supported on a coset of a proper normal open subgroup of $\mathcal{G}$ containing the commutator subgroup $[\mathcal{G},\mathcal{G}]$.

As we shall see, the qualitative behavior of spread out random walks on finite volume homogeneous spaces can be understood in great detail, and in fact for a much larger class of groups than (semisimple) real Lie groups. In particular, no connectedness assumption needs to be imposed, so that e.g. discrete or p-adic groups are naturally included in our setup.

THEOREM 3.0.2. *Let $\Lambda < G$ be a lattice and μ an aperiodic spread out probability measure on G. Then for every $x_0 \in X$ the orbit $\mathcal{G}x_0$ is clopen in X and we have*

$$\|\mu^{*n} * \delta_{x_0} - m_{\mathcal{G}x_0}\| \longrightarrow 0 \tag{3.0.2}$$

as $n \to \infty$, where $m_{\mathcal{G}x_0}$ denotes the normalized Haar measure on $\mathcal{G}x_0$ and $\|\cdot\|$ is the total variation norm. If the random walk additionally admits a continuous and everywhere finite Lyapunov function (see §3.3.2), then there is a constant $\kappa > 0$ such that for every compact subset $K \subset X$ and $n \in \mathbb{N}$ we have

$$\sup_{x \in K} \|\mu^{*n} * \delta_x - m_{\mathcal{G}x}\| \ll_K \mathrm{e}^{-\kappa n}.$$

For example, the latter holds when G is a connected semisimple real algebraic group without compact factors and μ has compact support.

For a statement without the aperiodicity assumption we refer the reader to the discussion in §3.3.

In two special cases, the above result takes a particularly simple form. One of them is when X is connected, the other when μ is adapted.

DEFINITION 3.0.3. A probability measure μ on G is called *adapted* if the closed subgroup $\mathcal{G}$ generated by $\operatorname{supp}(\mu)$ coincides with G.

COROLLARY 3.0.4. *Let $\Lambda < G$ be a lattice and μ a spread out probability measure on G. Suppose that X is connected or that μ is additionally adapted and aperiodic. Then for every $x_0 \in X$ we have*

$$\|\mu^{*n} * \delta_{x_0} - m_X\| \longrightarrow 0$$

as $n \to \infty$, where m_X denotes the normalized Haar measure on X.

REMARK 3.0.5. In the literature on spread out random walks it has been customary to restrict attention to adapted measures μ ([**54, 63, 108, 121, 122**]). This is indeed often justified, since one can replace G by $\mathcal{G} = \overline{\langle \operatorname{supp}(\mu) \rangle}$ (see Lemma 3.2.1). However, as a consequence one must also replace X by an orbit $\mathcal{G}x$, which is not always desirable. Hence, we emphasize that in the case of a connected space X, adaptedness (or aperiodicity) of μ are not needed as assumptions in the above corollary, distinguishing this result from the existing literature. ◇

Our approach is to analyze the random walk given by a spread out measure μ from the viewpoint of general state space Markov chain theory. The key observation is that it is a positive Harris recurrent T-chain on every $\mathcal{G}$-orbit in X. A connectedness assumption can then be used to establish transitivity (i.e. $\mathcal{G}x = X$) and rule out periodic behavior. Feeding all of this into the general theory, we obtain our results.

As a matter of fact, exploring the extent to which Markov chain theory can be of use in the study of random walks on finite volume homogeneous spaces has been one of the motivations for the present work. As they note, already Benoist–Quint's approach was inspired by Markov chain methods ([**8**, p. 702]); however, they could not directly apply available results, since the key assumption of ψ-irreducibility was not satisfied in the applications they had in mind ([**8**, p. 703]). A natural question is when this assumption is satisfied. As part of our discussion, we show that this is the case precisely for spread out measures (see Proposition 3.2.5 and Corollary 3.2.6).

3.0.2. Infinite Volume Spaces. Most of the qualitative analysis underlying Theorem 3.0.2 can also be carried out in the infinite volume case. For the upgrade to quantitative information though, one has to deal with an additional issue: recurrence of the random walk. The following dichotomy theorem of Hennion–Roynette describes the situations that can occur for spread out random walks. We write $\mathbb{P}_x$ for a probability measure under which the random walk (3.0.1) starts at $x \in X$ and $\mathbb{E}_x$ for the associated expectation (see §3.1.1).

THEOREM 3.0.6 (Hennion–Roynette [**63**]). *Let μ be an adapted spread out probability measure on G. Suppose that X admits a Haar measure m_X. Then either*

(i) *all states $x \in X$ are* topologically Harris recurrent*, meaning that*

$$\mathbb{P}_x[\Phi_n \in B \textit{ infinitely often}] = 1$$

for all neighborhoods B of x, or

(ii) *all states $x \in X$ are* topologically transient*, meaning that for some neighborhood B of x*

$$\mathbb{E}_x\left[\sum_{n=1}^{\infty} \mathbb{1}_{\Phi_n \in B}\right] < \infty.$$

Accordingly, the random walk on X given by μ is called topologically Harris recurrent *or* topologically transient.

It is not difficult to see that spread out random walks on finite volume spaces are topologically Harris recurrent. Indeed, Kakutani's random ergodic theorem ([**67**], see also [**48**]) implies that m_X-a.e. point satisfies the condition in (i). In general, what spaces X admit topologically Harris recurrent spread out random walks is a difficult question, extensively studied by Schott [**54, 108, 120, 121, 122**], which turns out to be intimately linked to the growth of the space.

DEFINITION 3.0.7. Suppose that X admits a Haar measure m_X. Then X is said to have *polynomial growth of degree at most d* if there exists a generating relatively compact neighborhood B of the identity in G and $x \in X$ such that

$$\limsup_{n\to\infty} \frac{m_X(B^n x)}{n^d} < \infty.$$

In this case, it can be shown that the above holds for all choices of x and B ([**54**]). When $d = 2$ we say the growth is *at most quadratic*.

Analogous to the more classical case of random walks on groups (for which see e.g. [**56**] and the references therein), the "quadratic growth conjecture" states that the homogeneous space $X = G/\Lambda$ admits topologically Harris recurrent spread out random walks if and only if it is of at most quadratic growth. For example, this is known to hold if G is a connected real Lie group of polynomial growth (Hebisch–Saloff-Coste [**61**, §10]) or a p-adic algebraic group of polynomial growth (Raja–Schott [**108**]). In this chapter, we show that one implication holds in general.

THEOREM 3.0.8. *Suppose that X admits a Haar measure and has at most quadratic growth. Let μ be an adapted symmetric spread out probability measure on G with compact support. Then the random walk on X given by μ is topologically Harris recurrent.*

Here the requirement of μ being "symmetric" means that $\mu(A) = \mu(A^{-1})$ for all measurable $A \subset G$.

Once Harris recurrence is established, we have an analogue of (3.0.2) in the form of a ratio limit theorem.

THEOREM 3.0.9. *Let μ be an adapted spread out probability measure on G. Suppose that X admits a Haar measure m_X and that the random walk on X given by μ is topologically Harris recurrent. Then for any $x_1, x_2 \in X$ and two bounded measurable functions f_1, f_2 on X with compact support such that $f_2 \geq 0$ and $\int_X f_2 \,\mathrm{d}m_X \neq 0$ we have*

$$\frac{\sum_{j=0}^{n} \int_X f_1 \,\mathrm{d}(\mu^{*j} * \delta_{x_1})}{\sum_{j=0}^{n} \int_X f_2 \,\mathrm{d}(\mu^{*j} * \delta_{x_2})} \longrightarrow \frac{\int_X f_1 \,\mathrm{d}m_X}{\int_X f_2 \,\mathrm{d}m_X}$$

as $n \to \infty$. If μ is additionally symmetric and aperiodic, then

$$\frac{\int_X f_1 \,\mathrm{d}(\mu^{*n} * \nu_1)}{\int_X f_2 \,\mathrm{d}(\mu^{*n} * \nu_2)} \longrightarrow \frac{\int_X f_1 \,\mathrm{d}m_X}{\int_X f_2 \,\mathrm{d}m_X} \tag{3.0.3}$$

as $n \to \infty$ for any two probability measures $\nu_1, \nu_2 \ll m_X$ with bounded density.

REMARK 3.0.10. We conjecture that (3.0.3) also holds with Dirac measures $\delta_{x_1}, \delta_{x_2}$ in place of ν_1, ν_2 for arbitrary $x_1, x_2 \in X$. Unfortunately, we can only prove this under the additional condition

$$\limsup_{n\to\infty} \frac{(\mu^{*n} * \delta_{x_i})(A)}{(\mu^{*n} * m_A)(A)} \leq 1 \tag{3.0.4}$$

for $i = 1, 2$, where A is a certain "small" subset of X (see the proof of Theorem 3.0.9 in §3.3.5) and $m_A = m_X(A)^{-1} m_X|_A$ is the normalized restriction of m_X to A. ◇

A standard example to which the previous results apply is the following.

EXAMPLE 3.0.11 (Covering spaces). Let G be a connected real Lie group, $\Lambda' < G$ a cocompact lattice and $\Lambda < \Lambda'$ a normal subgroup. Then $X = G/\Lambda$ is a Λ'/Λ-cover of G/Λ', so that X has at most quadratic growth if this is the case for the discrete group Λ'/Λ. ◇

For simple non-compact Lie groups of real rank 1, symmetric finitely supported measures μ, and $\Lambda'/\Lambda \cong \mathbb{Z}$ or $\mathbb{Z}^2$, recurrence in the above example has been known (Conze–Guivarc'h [**28**, Proposition 4.5]). The corresponding recurrence result under our conditions is new.

3.0.3. Examples of Spread Out Measures. We conclude this introduction by shedding some more light on the nature of spread out measures. Naturally, the first examples coming to mind are measures absolutely continuous with respect to Haar measure on G. However, the class of spread out measures is much larger and also contains many interesting singular measures, as the following examples aim to illustrate.

Example 3.0.12 (Affine random walks on the torus). An affine transformation on the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ is a map of the form

$$\mathbb{T}^d \ni x \mapsto gx + v, \tag{3.0.5}$$

where $g \in \mathrm{SL}_d(\mathbb{Z})$ is a unimodular integer matrix and $v \in \mathbb{R}^d$ is a translation vector. They fit into our setup in the following way: The group G is the semidirect product $\mathrm{SL}_d(\mathbb{Z}) \ltimes \mathbb{R}^d$ with group law $(g,v)(h,w) = (gh, gw+v)$ and the lattice is given by $\Lambda = \mathrm{SL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d$. Then $\mathbb{T}^d \cong X = G/\Lambda$, an element $(g,v) \in G$ acts on $x \in X$ precisely by (3.0.5), and an affine random walk on the torus is described by a measure μ on G.

We shall now explain when such a measure μ is spread out in two cases. Let us write λ_v for the pushforward of a measure λ on $\mathbb{R}$ to a line $\mathbb{R}v \subset \mathbb{R}^d$ via $t \mapsto tv$.

(i) The simplest case is when the linear part of the random walk is deterministic, given by a single matrix $a \in \mathrm{SL}_d(\mathbb{Z})$. For the measure μ, this means that $\mu = \delta_a \otimes \mu_{\mathrm{trans}}$ for some probability measure μ_{trans} on $\mathbb{R}^d$ giving the distribution of the translational part. When μ_{trans} has d-dimensional density, already μ is not singular with respect to Haar measure $m_G = m_{\mathrm{count}} \otimes m_{\mathbb{R}^d}$ on G, and so in particular spread out. However, we can do much better than that: It often suffices for μ_{trans} to have density in only one direction. More precisely, let λ be a probability measure on $\mathbb{R}$ that is not singular with respect to Lebesgue measure, $v \in \mathbb{R}^d$ a unit vector, and $\mu_{\mathrm{trans}} = \lambda_v$. Then $\mu = \delta_a \otimes \lambda_v$ is spread out if and only if $\{v, av, \dots, a^{d-1}v\}$ spans $\mathbb{R}^d$.

(ii) A similar characterization is possible when the linear and translational parts of μ are only assumed to be independent, i.e. if $\mu = \mu_{\mathrm{lin}} \otimes \mu_{\mathrm{trans}}$ for some probability measures μ_{lin} on $\mathrm{SL}_d(\mathbb{Z})$ and μ_{trans} on $\mathbb{R}^d$. Aiming to introduce as little density as possible, we again suppose $\mu_{\mathrm{trans}} = \lambda_v$ for some λ non-singular with respect to Lebesgue measure on $\mathbb{R}$ and a unit vector $v \in \mathbb{R}^d$. Then $\mu = \mu_{\mathrm{lin}} \otimes \lambda_v$ is spread out if and only if v is not contained in a proper $\mathrm{supp}(\mu_{\mathrm{lin}})$-invariant subspace of $\mathbb{R}^d$. For example, this is automatically the case under the common assumption that the semigroup S generated by $\mathrm{supp}(\mu_{\mathrm{lin}})$ acts irreducibly on $\mathbb{R}^d$.

The justification of the claims in the two points above is the following observation: If η is a measure on a subspace $V \subset \mathbb{R}^d$ non-singular with respect to Lebesgue measure on that subspace, then by definition of the group law on G we have

$$\mu * (\delta_s \otimes \eta) \gg \delta_{as} \otimes \eta'$$

for any $a \in \mathrm{supp}(\mu_{\mathrm{lin}})$ and $s \in \mathrm{SL}_d(\mathbb{Z})$, where η' is supported on $V' = \mathbb{R}v + aV$ and again non-singular with respect to Lebesgue measure on that space. In other words, in each convolution step we can pass from a density on V to a density on $V' = \mathbb{R}v + aV$ for any $a \in \operatorname{supp}(\mu_{\mathrm{lin}})$. Starting from $\eta = \lambda_v$ and $V = \mathbb{R}v$, the question of whether μ is spread out is thus equivalent to asking if it is possible to reach $V' = \mathbb{R}^d$ in finitely many such steps. With a little work, this yields the stated conditions. $\diamond$

EXAMPLE 3.0.13. Let $G = \mathrm{SL}_2(\mathbb{R})$ and

$$U = \left\{ u_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \,\middle|\, s \in \mathbb{R} \right\} \cong \mathbb{R}$$

be the upper unipotent subgroup. Furthermore, let $f\colon U \to [0,\infty)$ be any continuous density with $f(u_0) > 0$ and $\int_U f \,\mathrm{d}s = 1$, set $\mathrm{d}\mu_U = f\,\mathrm{d}s$ and $u_- = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Then for the probability measure $\mu = \frac{1}{2}(\mu_U + \delta_{u_-})$, the fifth convolution power μ^{*5} has a non-trivial absolutely continuous component, as a calculation shows. (For example, observe that in a neighborhood of the origin, $(a,b,c) \mapsto u_a u_- u_b u_- u_c$ is a smooth chart of a neighborhood of u_-^2 inside G.) Hence, μ is singular with respect to Haar measure, yet spread out. $\diamond$

3.1. Markov Chain Theory for Random Walks

In this section, we lay the foundations for all following discussions. We review the relevant concepts and results from general state space Markov chain theory in §3.1.1, and make the connection to spread out random walks in §3.1.2. Throughout, an important reference is going to be Meyn and Tweedie's comprehensive book [**90**].

3.1.1. Preliminaries. We begin with preliminaries from general state space Markov chain theory. Readers familiar with the subject may skip this subsection and only consult it for notation, when necessary.

Even though large parts of the theory are valid under the mere assumption that the state space is a measurable space endowed with a countably generated σ-algebra, for us it is not going to be a restriction to assume that X is a σ-compact locally compact metrizable space endowed with its Borel σ-algebra $\mathcal{B}$.

The first notion to introduce is that of a *transition kernel* on X: This is a map $P\colon X \times \mathcal{B} \to [0,\infty]$ such that $P(x,\cdot)$ is a Borel measure on X for every $x \in X$ and $x \mapsto P(x,A)$ is measurable for every $A \in \mathcal{B}$. It acts on functions f on X from the left and on measures ν on X from the right by virtue of

$$Pf(x) = \int_X P(x,\mathrm{d}y)f(y) \quad \text{and} \quad \nu P(A) = \int_X \nu(\mathrm{d}x)P(x,A)$$

for $x \in X$ and $A \in \mathcal{B}$. A transition kernel is called *stochastic* if every $P(x,\cdot)$ is a probability measure, and *substochastic* if $P(x,X) \le 1$ for every $x \in X$. A σ-finite measure ν on X is called *P-subinvariant* if $\nu P \le \nu$ and *P-invariant* if $\nu P = \nu$. When the transition kernel is clear from context, we just speak of *(sub)invariant* measures. The powers P^n of P are defined inductively by $P^0(x,\cdot) = \delta_x$ and $P^n(x,A) = \int_X P^{n-1}(x,\mathrm{d}y)P(y,A)$ for $n \in \mathbb{N}$, which generalizes to the "Chapman–Kolmogorov equations"

$$P^{m+n}(x,A) = \int_X P^m(x,\mathrm{d}y)P^n(y,A)$$

for $x \in X$, $A \in \mathcal{B}$, and $m,n \in \mathbb{N}$.

A *Markov chain* on X is an X-valued stochastic process $\Phi = (\Phi_n)_{n\in\mathbb{N}_0}$ whose steps are governed by a stochastic transition kernel. Formally, this means that there exists a starting distribution ν on X and a stochastic transition kernel P on X such that

$$\mathbb{P}[\Phi_0 \in A_0, \dots, \Phi_n \in A_n] = \int_{x_0\in A_0} \cdots \int_{x_{n-1}\in A_{n-1}} \nu(\mathrm{d}x_0) P(x_0, \mathrm{d}x_1) \cdots P(x_{n-1}, A_n)$$

for every $n \in \mathbb{N}_0$ and $A_0, \dots, A_n \in \mathcal{B}$. This formula (specifically, the absence of the variables $x_0, \dots, x_{k-1}$ in the term $P(x_k, \mathrm{d}x_{k+1})$) captures the quintessential idea behind a Markov chain that the distribution of the following state Φ_{n+1} depends only on the current state Φ_n via the transition kernel P. In terms of conditional distributions, this dependence can be expressed as

$$\mathcal{L}(\Phi_{n+1}|\Phi_n = x, \Phi_{n-1}, \dots, \Phi_0) = \mathcal{L}(\Phi_{n+1}|\Phi_n = x) = P(x, \cdot).$$

It can be shown that a Markov chain on X exists for every fixed starting distribution ν and stochastic transition kernel P ([**90**, Theorem 3.4.1]). In fact Φ may always be assumed to be the canonical coordinate process on $X^{\mathbb{N}_0}$; only the probability measure $\mathbb{P}$ on $X^{\mathbb{N}_0}$ needs to be chosen accordingly. It is customary to regard the starting distribution as variable and think of a Markov chain on X as being defined by the transition kernel P alone. The probability measure on $X^{\mathbb{N}_0}$ making the canonical process into a Markov chain with starting distribution ν is then denoted by $\mathbb{P}_\nu$. When $\nu = \delta_x$ is the Dirac mass at some $x \in X$, one simply writes $\mathbb{P}_x$. The associated expectations are denoted $\mathbb{E}_\nu$ and $\mathbb{E}_x$, respectively.

EXAMPLE 3.1.1. The random walk on $X = G/\Lambda$ given by a probability measure μ on G is a Markov chain with transition kernel

$$P(x, \cdot) = \mu * \delta_x.$$

Its powers are given by $P^n(x, \cdot) = \mu^{*n} * \delta_x$, where μ^{*n} is the n-th convolution power of μ, defined inductively by $\mu^{*0} = \delta_e$, where $e \in G$ is the identity element, and $\mu^{*n} = \int_G g_*\mu^{*(n-1)}\, \mathrm{d}\mu(g)$ for $n \in \mathbb{N}$. Equivalently, μ^{*n} is the law of a product $Y_n \cdots Y_1$ of i.i.d. random variables $Y_1, \dots, Y_n$ in G with common distribution μ. If $\mathcal{L}_x$ denotes the law under $\mathbb{P}_x$ for some $x \in X$, we thus have

$$\mathcal{L}_x(\Phi_n) = P^n(x, \cdot) = \delta_x P^n = \mu^{*n} * \delta_x,$$

and, more generally, for a starting distribution ν on X,

$$\mathcal{L}_\nu(\Phi_n) = \nu P^n = \mu^{*n} * \nu. \qquad \diamond$$

Let us next introduce a few important quantities associated to a Markov chain. The *first return time* τ_A and *occupation time* η_A of a set $A \in \mathcal{B}$ are defined by

$$\tau_A = \min\{n \geq 1 \mid \Phi_n \in A\}, \quad \eta_A = \sum_{n=1}^{\infty} \mathbb{1}_{\Phi_n \in A},$$

and the *return probability* and *expected number of visits* to A starting from x are

$$L(x, A) = \mathbb{P}_x[\tau_A < \infty], \quad U(x, A) = \mathbb{E}_x[\eta_A] = \sum_{n=1}^{\infty} P^n(x, A),$$

respectively. Note that $U\colon X \times \mathcal{B} \to [0, \infty]$ is a transition kernel on X.

We now address the notion of ψ-irreducibility, which was already mentioned in the introduction to this chapter. A σ-finite measure φ on X is called an *irreducibility measure* for a Markov chain on X if for every $A \in \mathcal{B}$ with $\varphi(A) > 0$ we have $L(x, A) > 0$ for all $x \in X$. In other words, this means that any φ-positive set can be reached from everywhere with positive probability. The Markov chain is called *ψ-irreducible* if it admits a non-trivial irreducibility measure. In this case, it can be shown that there exists a *maximal irreducibility measure*, that is, an irreducibility measure ψ with the property that every other irreducibility measure is absolutely continuous with respect to ψ ([**90**, Proposition 4.2.2]). Without loss of generality one may assume ψ to be a probability measure. By definition, the measure class of a maximal irreducibility measure is uniquely determined by the Markov chain (i.e. by its defining transition kernel P). This justifies the implicit understanding (and slight abuse of notation) common in the literature that, given a ψ-irreducible Markov chain, ψ always denotes an associated maximal irreducibility measure.

For ψ-irreducible chains there is a recurrence/transience dichotomy similar to the classical discrete theory. To state it, we call a set $A \subset X$ *uniformly transient* if the expected number of returns to A is bounded on A, i.e. if $\sup_{x \in A} U(x, A) < \infty$, and *recurrent* if the expected number of returns is infinite on all of A, i.e. if $U(x, A) = \infty$ for all $x \in A$.

THEOREM 3.1.2 ([**90**, Theorem 8.0.1]). *Suppose Φ is ψ-irreducible. Then either*

(i) *every ψ-positive set is recurrent, in which case Φ is called* recurrent*, or*
(ii) *the state space X can be covered by countably many uniformly transient sets, in which case Φ is called* transient*.*

We emphasize that ψ-irreducibility is included in these definitions of recurrence and transience. For recurrent chains, one has the following conclusion about invariant measures.

THEOREM 3.1.3 ([**90**, Theorem 10.4.9]). *Suppose Φ is recurrent. Then there exists a non-trivial σ-finite invariant measure π, which is unique up to scalar multiples. Moreover, π is a maximal irreducibility measure.*

As in the classical theory, a further refinement of recurrence is possible: The chain is called *positive* if it is ψ-irreducible and admits a non-trivial finite invariant measure. This forces the chain to be recurrent.

PROPOSITION 3.1.4 ([**90**, Proposition 10.1.1]). *A positive chain is recurrent. In particular, a positive chain admits a unique invariant probability measure, which is a maximal irreducibility measure.*

For this reason, positive chains are also called “positive recurrent”.

In the general theory, there is one more important notion of recurrence that does not appear in the discrete theory. Namely, in the latter, a recurrent state x always satisfies $\mathbb{P}_x[\tau_x < \infty] = 1$, and hence by the Markov property also $\mathbb{P}_x[\eta_x = \infty] = 1$. Since in more general spaces there might be no returns to the precise starting point, such conclusions can no longer be drawn. Let us write

$$Q(x, A) = \mathbb{P}_x[\eta_A = \infty]$$

for $x \in X$ and $A \in \mathcal{B}$, call the set A *Harris recurrent* if $Q(x, A) = 1$ for every state $x \in A$, and the whole chain Φ *Harris recurrent* if it is ψ-irreducible and every ψ-positive set is Harris recurrent. Clearly, Harris recurrence implies recurrence. We call Φ *positive Harris recurrent* if it is positive and Harris recurrent.

The final notion we need to introduce is that of aperiodicity, which naturally plays a role in questions of convergence to a stable distribution.

THEOREM 3.1.5 ([**90**, Theorem 5.4.4]). *Let Φ be ψ-irreducible. Then there exists a maximal positive integer d, called the* period *of Φ, with the property that there are pairwise disjoint sets $D_0, \dots, D_{d-1} \in \mathcal{B}$ such that $P(x, D_{i+1 \bmod d}) = 1$ for each $x \in D_i$ and $i = 0, \dots, d-1$ and such that the union $\bigcup_{i=0}^{d-1} D_i$ is ψ-full.*

A collection of measurable sets $D_0, \dots, D_{d-1}$ as in the above theorem is referred to as a *d-cycle* for Φ. A ψ-irreducible chain with period 1 is called *aperiodic*.

3.1.2. T-Chains. As already pointed out, the notions of recurrence for ψ-irreducible chains require certain properties of returns to ψ-positive measurable sets from arbitrary starting points, not taking into account topological properties of the state space. Of course this makes sense, as the topology did not feature in any of the definitions up to this point. In order to connect the chain to the topology, one thus needs an additional concept. Several notions accomplishing this appear in the literature; the one best suited for the study of random walks is that of a "T-chain" introduced by Tuominen–Tweedie [**130**]. Its definition involves the "sampling" of a transition kernel P: Given a probability distribution a on $\mathbb{N}_0$, the *sampled transition kernel K_a* with *sampling distribution a* is defined by

$$K_a = \sum_{n=0}^{\infty} a(n) P^n.$$

DEFINITION 3.1.6. A Markov chain Φ on X given by a transition kernel P is called a *T-chain* if there exists a sampling distribution a on $\mathbb{N}_0$ and a substochastic transition kernel T on X with

(i) $K_a(x, A) \geq T(x, A)$ for all $x \in X$ and $A \in \mathcal{B}$,
(ii) $T(x, X) > 0$ for all $x \in X$, and such that
(iii) $T(\cdot, A)$ is lower semicontinuous for all $A \in \mathcal{B}$.

We call T a *continuous component of P*.

Let us describe the links the T-property establishes between recurrence and topology. We call a state $x_0 \in X$ *reachable* if $L(x, B) > 0$ for every $x \in X$ and neighborhood B of x_0, *topologically Harris recurrent* if $Q(x_0, B) = 1$ for each neighborhood B of x_0, and *topologically recurrent* if $U(x_0, B) = \infty$ for each neighborhood of x_0. If x_0 is not topologically recurrent, it is called *topologically transient*. The first result we shall need infers ψ-irreducibility from the existence of a reachable state.

Proposition 3.1.7 ([**90**, Proposition 6.2.1]). *If a T-chain admits a reachable state, it is ψ-irreducible.*

The second one is a strong decomposition statement, allowing the splitting of the state space X into a Harris recurrent and a transient part.

Theorem 3.1.8 ([**90**, Theorem 9.3.6]). *For a ψ-irreducible T-chain, the state space X admits a decomposition*

$$X = H \sqcup N$$

into a Harris set H (*meaning that* $P(x, H) = 1$ *for each* $x \in H$ *and the restriction of the chain to* H *is Harris recurrent*) *and a set* N *consisting of topologically transient states.*

The following result is the motivation for introducing Markov chain methods in the study of spread out random walks. For random walks on groups it is due to Tuominen–Tweedie [**130**, Theorem 5.1(i)]; the case of a homogeneous space X is not much more complicated.

Proposition 3.1.9. *Let G be a σ-compact locally compact metrizable group, $\Lambda < G$ a discrete subgroup, and X the homogeneous space G/Λ. Then the random walk on X given by a probability measure μ on G is a T-chain if and only if μ is spread out. In this case, the sampling distribution a and the continuous component T may be chosen such that*

- *$a = \delta_{n_0}$ for some $n_0 \in \mathbb{N}$,*
- *$T(\cdot, X)$ is constant, and*
- *Tf is continuous for every bounded measurable function f on X.*

For convenience we include a proof, which adapts that of [**90**, Proposition 6.3.2] to the setting at hand.

Proof. Denote by $\mathrm{pr}\colon G \to X$, $g \mapsto g\Lambda$ the canonical projection. Recalling Example 3.1.1, we see that the powers of the transition kernel P of the random walk on X are given by

$$P^n(x, A) = \int_G h_*\delta_{g\Lambda}(A)\,\mathrm{d}\mu^{*n}(h) = \mu^{*n}(\mathrm{pr}^{-1}(A)g^{-1}) \qquad (3.1.1)$$

for $n \in \mathbb{N}$, $x = g\Lambda \in X$, and $A \subset X$. Let m_G denote a left Haar measure on G.

Assume first that the random walk is a T-chain. If every convolution power μ^{*n} for $n \in \mathbb{N}$ is singular with respect to m_G, we find a set $E_G \subset G$ with $\mu^{*n}(E_G) = 1$ for all $n \in \mathbb{N}$ and $m_G(E_G) = 0$. Enlarging E_G if necessary, we may assume that the identity $e \in G$ belongs to E_G and that E_G is right-Λ-invariant. Write $E = \mathrm{pr}(E_G)$ and let a be the sampling distribution associated to the continuous component T of the random walk. Then

$$T(e\Lambda, E^c) \le K_a(e\Lambda, E^c) = \sum_{n=0}^{\infty} a(n)\underbrace{P^n(e\Lambda, E^c)}_{=\mu^{*n}(E_G^c)=0} = 0,$$

where we used (3.1.1), that $\mathrm{pr}^{-1}(E^c) = E_G^c$ by the assumed right-Λ-invariance and $e \in E_G$ for $n = 0$. Properties (ii) and (iii) in the definition of a T-chain thus produce $\delta > 0$ and a neighborhood B of $e\Lambda \in X$ with $T(x,E) \geq \delta$, and hence also

$$K_a(x,E) \geq \delta$$

for all $x \in B$. But by translation invariance of m_G and Fubini's theorem, we find

$$\begin{aligned} m_G(E_G) &= \int_G m_G(g^{-1}E_G)\,\mathrm{d}\mu^{*n}(g) \\ &= \int_G \mu^{*n}(E_G h^{-1})\,\mathrm{d}m_G(h) \\ &= \int_G P^n(h\Lambda, E)\,\mathrm{d}m_G(h), \end{aligned}$$

which, after summing with the weights $a(n)$, yields the contradiction

$$\begin{aligned} m_G(E_G) &= \int_G K_a(h\Lambda, E)\,\mathrm{d}m_G(h) \\ &\geq \int_{\mathrm{pr}^{-1}(B)} K_a(h\Lambda, E)\,\mathrm{d}m_G(h) \\ &\geq \delta m_G(\mathrm{pr}^{-1}(B)) > 0. \end{aligned}$$

For the converse, suppose that μ^{*n_0} is not singular with respect to m_G for some $n_0 \in \mathbb{N}$. Then there exists a non-negative m_G-integrable function $p\colon G \to \mathbb{R}$ with $\int p\,\mathrm{d}m_G > 0$ and $\mathrm{d}\mu^{*n_0} \geq p\,\mathrm{d}m_G$. Denoting by Δ the modular character of G, we obtain for $x = g\Lambda \in X$ and $A \subset X$

$$P^{n_0}(x,A) \geq \int_{\mathrm{pr}^{-1}(A)g^{-1}} p\,\mathrm{d}m_G = \Delta(g)^{-1} \int_{\mathrm{pr}^{-1}(A)} p(g'g^{-1})\,\mathrm{d}m_G(g') =: T(x,A).$$

The sampling distribution $a = \delta_{n_0}$ together with this T are then seen to possess all claimed properties. □

3.2. Spread Out Random Walks

This section is the central part of the chapter, aiming to give a complete picture of the qualitative behavior of spread out random walks on homogeneous spaces.

In what follows, we are not going to assume that Λ is a lattice or that G is unimodular, so that there will in general be no G-invariant measure on the quotient $X = G/\Lambda$. However, for every continuous character $\chi\colon G \to \mathbb{R}_{>0}$ extending the restriction $\Delta|_\Lambda$ of the modular character Δ of G to Λ, there exists a non-trivial Radon measure $m_{X,\chi}$ on X that is χ-*quasi-invariant* in the sense that

$$g_* m_{X,\chi} = \chi(g) m_{X,\chi}$$

for all $g \in G$. Such a measure is unique up to scalars. Two important cases of this construction are $\chi = \Delta$, the choice of which is always possible, and $\chi = \mathbb{1}$, which is a possible choice whenever $\Delta(\gamma) = 1$ for all $\gamma \in \Lambda$. In the latter case, $m_X = m_{X,\mathbb{1}}$ is a Haar measure on X. All $m_{X,\chi}$ belong to the same measure class, which we refer to as the *Haar measure class* on X. This terminology is justified by the fact that $m_{X,\Delta}$ can be identified with the restriction of a right Haar measure on G to a fundamental domain for Λ. We refer to [**19**, Ch.VII§2] for details.

Slightly abusing notation, we are going to denote the Haar measure class on X by $[m_X]$, and for a measure ν on X write $\nu \ll [m_X]$, $\nu \sim [m_X]$, $[m_X] \ll \nu$ to express that ν is absolutely continuous with respect to $[m_X]$, contained in $[m_X]$, or that $[m_X]$ is absolutely continuous with respect to ν, respectively.

Standing Assumptions & Notation. Let us summarize at this point the standing assumptions and notations that will be in effect for the remainder of the chapter when nothing else is specified: μ is a probability measure on a locally compact σ-compact metrizable group G; $\mathcal{S}$ and $\mathcal{G}$ are the closed subsemigroup and subgroup of G generated by $\operatorname{supp}(\mu)$, respectively; $\Lambda < G$ is a discrete subgroup; X is the homogeneous space G/Λ; $[m_X]$ is the Haar measure class on X and m_X a Haar measure (when one exists); and P is the transition kernel of the random walk on X induced by μ.

3.2.1. Transitivity & ψ-Irreducibility. Let $x \in X$ be the starting point for our random walk. Then, in some sense, everything outside the closed subgroup $\mathcal{G}$ of G generated by $\operatorname{supp}(\mu)$ and outside the orbit $\mathcal{G}x \subset X$ is irrelevant for the study of the random walk. The following simple lemma shows how such redundancy can be removed.

Lemma 3.2.1. *Let μ be a spread out probability measure on G. Then $\mathcal{G}$ is an open subgroup of G. For every $x = g\Lambda \in X$ the orbit $\mathcal{G}x$ is a clopen subset of X satisfying $\mathcal{G}/(\mathcal{G} \cap g\Lambda g^{-1}) \cong \mathcal{G}x$. If X has finite volume, then so does $\mathcal{G}/(\mathcal{G} \cap g\Lambda g^{-1})$.*

Proof. From the formula

$$\overline{\operatorname{supp}(\mu^{*m})\operatorname{supp}(\mu^{*n})} = \operatorname{supp}(\mu^{*(m+n)}) \tag{3.2.1}$$

for $m, n \in \mathbb{N}$ we see that $\operatorname{supp}(\mu^{*n}) \subset \mathcal{G}$ for every $n \in \mathbb{N}$. Since μ is spread out and the convolution of bounded integrable functions on G is continuous, some convolution power μ^{*n_0} has a component with continuous density with respect to Haar measure on G. Thus $\mathcal{G} \supset \operatorname{supp}(\mu^{*n_0})$ has non-empty interior, and consequently $\mathcal{G}$ is open. Since the action map $G \ni g \mapsto gx \in X$ is a local homeomorphism, this implies that also $\mathcal{G}x$ is open. But then X is a disjoint union of such open $\mathcal{G}$-orbits, so that all of them must also be closed. Writing $x = g\Lambda$, the isomorphism $\mathcal{G}/(\mathcal{G} \cap g\Lambda g^{-1}) \cong \mathcal{G}x$ of $\mathcal{G}$-spaces follows, since $\mathcal{G} \cap g\Lambda g^{-1} = \operatorname{Stab}_{\mathcal{G}}(x)$. When X has finite volume, this quotient supports a finite invariant measure inherited from the restriction of Haar measure on X to $\mathcal{G}x$, so that $\mathcal{G} \cap g\Lambda g^{-1}$ is a lattice in $\mathcal{G}$. $\square$

In other words, at the price of replacing X by $\mathcal{G}x$, we are free to assume that μ is adapted. In view of this, we will formulate most of the following results only for adapted measures.

Preparing for the proof of ψ-irreducibility of spread out random walks, our next objective is to find a more efficient description of an orbit $\mathcal{G}x$. We will use the notation $\Lambda_g = g\Lambda g^{-1}$ for $g \in G$.

Lemma 3.2.2. *The set $S = \bigcup_{n=1}^{\infty} \operatorname{supp}(\mu^{*n})$ is a subsemigroup of G with $\overline{S} = \mathcal{S}$. If μ is spread out and adapted and $\mathcal{S}\Lambda_g = \{s\gamma \,|\, s \in \mathcal{S}, \gamma \in \Lambda_g\}$ equals G for all $g \in G$, then S acts transitively on X.*

PROOF. That S is a semigroup with $\operatorname{supp}(\mu) \subset S \subset \mathcal{S}$ follows from (3.2.1). Since $\mathcal{S}$ is by definition the smallest closed subsemigroup of G containing $\operatorname{supp}(\mu)$, we must have $\overline{S} = \mathcal{S}$.

Let us now show transitivity of the S-action on X under the stated assumptions. To this end, note first that $S\Lambda_g$ is dense in G for every $g \in G$, since

$$\overline{S\Lambda_g} = \overline{\bigcup_{\gamma\in\Lambda_g} S\gamma} \supset \bigcup_{\gamma\in\Lambda_g} \overline{S\gamma} = \bigcup_{\gamma\in\Lambda_g} \mathcal{S}\gamma = \mathcal{S}\Lambda_g = G.$$

Now let $x, y \in X$ be arbitrary. We need to find an element of S sending x to y. Choose $g \in G$ with $gx = y$ and write $x = h\Lambda$ for some $h \in G$. Using that S has non-empty interior (by the same argument as in Lemma 3.2.1), we can find a non-empty open subset U of G contained in S. By density of $S\Lambda_h$ in G, it follows that $U^{-1}g$ intersects $S\Lambda_h$ non-trivially, say $u^{-1}g = sh\gamma h^{-1}$ for some $u \in U \subset S$, $s \in S$ and $\gamma \in \Lambda$. Recalling that $x = h\Lambda$, we conclude that

$$usx = gh\gamma^{-1}h^{-1}x = gh\Lambda = gx = y,$$

so that the element $us \in S$ has the required property. □

The conclusion of the previous lemma will be important for many of the following results. Let us therefore give a name to its set of assumptions.

DEFINITION 3.2.3. We say that a probability measure μ on G is Λ-*adapted* if μ is adapted and $\mathcal{S}\Lambda_g = G$ for all $g \in G$, where $\Lambda_g = g\Lambda g^{-1}$.

For spread out random walks on finite volume spaces, the second requirement in the above definition is redundant.

PROPOSITION 3.2.4. *Let μ be spread out and adapted and suppose that X has finite volume. Then μ is Λ-adapted.*

PROOF. We claim that for every $x \in X$, the orbit $A = \mathcal{S}x$ equals X. This will imply that for every $g, g' \in G$ there exists $s \in \mathcal{S}$ with $sg\Lambda = g'g\Lambda$, which is the desired conclusion.

To prove the claim, observe that A satisfies $sA \subset A \subset s^{-1}A$ for every $s \in \mathcal{S}$. By invariance of m_X we also know that the m_X-measures of these three sets coincide, so it follows that the characteristic function $\mathbb{1}_A$ is m_X-a.s. invariant under each element of $\mathcal{S} \cup \mathcal{S}^{-1}$ (individually). We conclude that $\mathbb{1}_A$ is m_X-a.s. invariant under each element of a dense subset of G, hence under all of G by continuity of the regular representation on $L^1(X)$. But as $\mathcal{S}$ has non-empty interior, we know that A has positive measure, so that G-invariance forces $m_X(A) = 1$. But then, if there was some $y \in X \setminus A$, we would have a set $\mathcal{S}^{-1}y$ disjoint from A which also has positive measure (since also $\mathcal{S}^{-1}$ has non-empty interior), which is a contradiction. □

We can now relate the property of a probability measure being spread out to ψ-irreducibility of the induced random walk. Recall the convention that when speaking about ψ-irreducibility, ψ always denotes a maximal irreducibility measure.

PROPOSITION 3.2.5. *Let μ be a probability measure on G. If the random walk on X given by μ is ψ-irreducible, then μ is spread out and $\psi \ll [m_X]$.*

Conversely, if μ is spread out and Λ-adapted, then the random walk on X is ψ-irreducible with $[m_X] \sim \psi$.

PROOF. Let us suppose first that the random walk on X given by μ is ψ-irreducible and let $m_{X,\chi}$ be a quasi-invariant measure on X for some character χ of G. Then for every measurable subset $A \subset X$ we find, using Fubini's theorem,

$$\int_X P(x, A)\, \mathrm{d}m_{X,\chi}(x) = \int_G m_{X,\chi}(h^{-1}A)\, \mathrm{d}\mu(h) = \int_G \chi\, \mathrm{d}\mu \cdot m_{X,\chi}(A). \quad (3.2.2)$$

In other words, we have $m_{X,\chi}P = c(\chi, \mu)m_{X,\chi}$ for the constant $c(\chi, \mu) = \int_G \chi\, \mathrm{d}\mu$. Consider the sampled transition kernel $K_a = \sum_{n\geq 0} a(n)P^n$ with $a(n) = 2^{-(n+1)}$ for $n \in \mathbb{N}_0$. By definition of an irreducibility measure, for every ψ-positive set $A \subset X$ it satisfies $K_a(x, A) > 0$ for all $x \in X$. If A were an $[m_X]$–null set, it would follow that

$$0 < m_{X,\chi}K_a(A) = \sum_{n=0}^{\infty} a(n)m_{X,\chi}P^n(A) = \sum_{n=0}^{\infty} a(n)c(\chi, \mu)^n m_{X,\chi}(A) = 0,$$

which is a contradiction. We have thus shown that $\psi \ll [m_X]$. If μ is not spread out, then as in the proof of Proposition 3.1.9 there exists a right-Λ-invariant measurable set $e \in E_G \subset G$ with $\mu^{*n}(E_G) = 1$ for all $n \in \mathbb{N}$ and $m_G(E_G) = 0$, where m_G denotes a left Haar measure on G. The set $E = \mathrm{pr}(E_G)$, where $\mathrm{pr}\colon G \to X$, $g \mapsto g\Lambda$ denotes the projection, is then an $[m_X]$–null set. Using $\psi \ll [m_X]$ it follows that $\psi(E^c) > 0$; yet we have $P^n(e\Lambda, E^c) = \mu^{*n}(E_G^c) = 0$ for all $n \in \mathbb{N}_0$. This contradicts ψ-irreducibility, hence μ must be spread out.

For the converse, recall from Proposition 3.1.9 that the random walk on X induced by a spread out measure μ is a T-chain. By Proposition 3.1.7, ψ-irreducibility can be established by proving existence of a reachable state. But from Lemma 3.2.2 it in fact follows that every $x_0 \in X$ is reachable: Given any other point $x \in X$, it can be written as $x_0 = sx$ for some $s \in \mathrm{supp}(\mu^{*n})$, and we conclude for any neighborhood B of the identity in G that

$$L(x, Bx_0) \geq P^n(x, Bx_0) = P^n(x, Bsx) \geq \mu^{*n}(Bs) > 0.$$

Hence, the random walk is ψ-irreducible. The first part of the proposition thus yields $\psi \ll [m_X]$. To also obtain $[m_X] \ll \psi$, it suffices to show that members of the Haar measure class are irreducibility measures. Let therefore $A \subset X$ be an $[m_X]$-positive set and define $A_G = \mathrm{pr}^{-1}(A)$. Then also $m_G(A_G) > 0$. By Proposition 3.1.9 and its proof, for some $n_0 \in \mathbb{N}$ the kernel P^{n_0} has a continuous component T given by an absolutely continuous measure $p\, \mathrm{d}m_G$ on G, where p is an m_G-integrable function on G with $\int_G p\, \mathrm{d}m_G > 0$. In particular, we know $m_G(p^{-1}((0, \infty))) > 0$, so that by a standard fact of measure theory also $m_G(A_G g^{-1} \cap p^{-1}((0, \infty))) > 0$ for some $g \in G$. It follows that

$$T(g\Lambda, A) = \int_{A_G g^{-1}} p\, \mathrm{d}m_G > 0. \quad (3.2.3)$$

But as $g\Lambda$ is reachable, [**90**, Proposition 6.2.1] implies that $T(g\Lambda, \cdot)$ is an irreducibility measure, so that (3.2.3) entails $L(x, A) > 0$ for all $x \in X$. This completes the proof. □

COROLLARY 3.2.6. *Let μ be spread out and adapted and suppose that X has finite volume. Then the random walk on X given by μ is ψ-irreducible with $\psi \sim [m_X]$.*

PROOF. Combine Propositions 3.2.4 and 3.2.5. □

One may wonder whether in the first statement of the previous proposition, ψ must even belong to the Haar measure class on X. In view of the second conclusion this is true when μ is additionally Λ-adapted. In general however, it does not hold, as the following example demonstrates.

EXAMPLE 3.2.7. Let $G = \mathbb{R}_{>0} \ltimes \mathbb{R}$ be the $ax+b$-group of affine transformations of $\mathbb{R}$, with group law given by

$$(a,b)(a',b') = (aa', ab' + b)$$

for $a, a' \in \mathbb{R}_{>0}$ and $b, b' \in \mathbb{R}$, and consider the discrete subgroup Λ of G given by $\Lambda = \{2^n \mid n \in \mathbb{Z}\} \times \{0\}$. We decompose G into

$$G^+ = \{(a,b) \in G \mid b \geq 0\} \text{ and } G^- = \{(a,b) \in G \mid b < 0\}$$

and define $X^\pm = \mathrm{pr}(G^\pm)$, where $\mathrm{pr}\colon G \to X = G/\Lambda$ denotes the projection. Our goal is to construct a ψ-irreducible random walk on X which never moves from X^+ to X^-. Then it will follow that any irreducibility measure for this random walk must have support inside X^+ and thus cannot belong to the Haar measure class.

The following construction achieves this goal. Let μ be a probability measure on G absolutely continuous with respect to a right Haar measure m_G with a density that is strictly positive on $(0,1) \times \mathbb{R}_{\geq 0}$ and 0 otherwise. For example, one may choose $\mathrm{d}\mu(a,b) = \mathbb{1}_{(0,1)\times\mathbb{R}_{\geq 0}}(a,b)\mathrm{e}^{-b}\,\mathrm{d}a\,\mathrm{d}b$. Let $m_{X,\Delta}$ be the quasi-invariant measure on X coming from the modular character Δ of G. Then given a starting point $x \in X$, the law $\mathcal{L}_x(\Phi_1) = \mu * \delta_x$ after the first step of the random walk is absolutely continuous with respect to $m_{X,\Delta}$. By choice of μ and definition of the group operation, the corresponding density $p_x = \mathrm{d}(\mu * \delta_x)/\,\mathrm{d}m_{X,\Delta}$ is seen to have the following properties:

- Irrespective of the location of the starting point x, this density p_x is strictly positive almost everywhere on X^+.
- For $x \in X^+$, p_x is 0 on X^-.

Indeed, both properties follow from the geometry of the left action of G on X, which can be understood e.g. by identifying X with the fundamental domain $F = [1,2) \times \mathbb{R}$ for Λ inside G; see Figure 1. We deduce that $L(x, X^-) = 0$ for all $x \in X^+$, and $L(x, A) > 0$ for all $A \subset X$ intersecting X^+ in a positive measure set and all $x \in X$. Hence, the random walk is ψ-irreducible, with maximal irreducibility measure being given e.g. by $\psi = m_{X,\Delta}|_{X^+}$. ◇

Let us record a situation in which we do not need adaptedness to guarantee that the random walk is ψ-irreducible on all of X.

COROLLARY 3.2.8. *Let μ be spread out. Suppose that X is connected and that*

- *X has finite volume, or that*
- *$\mathcal{G} = \mathcal{S}$.*

Then the random walk on X given by μ is ψ-irreducible with $\psi \sim [m_X]$ and the semigroup S from Lemma 3.2.2 acts transitively on X.

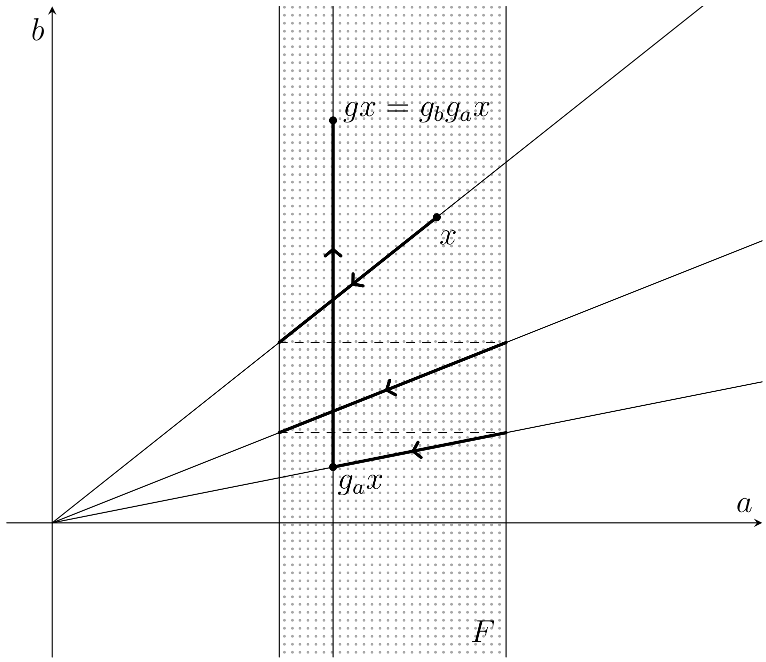

FIGURE 1. Illustration of the left action of G on X in the fundamental domain F. The acting element $g = (a, b) \in (0, 1) \times \mathbb{R}_{\geq 0}$ is decomposed as $g = g_b g_a$ for $g_a = (a, 0)$ and $g_b = (1, b)$. The dashed lines indicate identifications using the right action of Λ.

PROOF. Lemma 3.2.1 implies that $\mathcal{G}x$ is clopen, hence equal to X by connectedness. The same lemma thus allows us to assume that μ is adapted without changing X. Then, using Proposition 3.2.4 in the finite volume case, we see that μ is Λ-adapted. Lemma 3.2.2 and Proposition 3.2.5 now give all conclusions. $\square$

3.2.2. Periodicity. Proposition 3.2.5 states that for (reasonably nice) spread out random walks we have at our disposal the whole theory of ψ-irreducible Markov chains from §3.1. In particular, Theorem 3.1.5 tells us that they have a well-defined period $d \in \mathbb{N}$. Let us look at the sets D_i in a corresponding d-cycle in more detail.

PROPOSITION 3.2.9. *Let μ be a probability measure on G. Suppose that the random walk on X given by μ is ψ-irreducible with $\psi \sim [m_X]$ and let $d \in \mathbb{N}$ be its period. Then there exist subsets $D_0, \dots, D_{d-1}$ of X with the following properties:*

(i) *The D_i are clopen, non-empty, and form a partition of X,*
(ii) *we have $P(x, D_{i+1 \bmod d}) = 1$ for every $x \in D_i$ and $gD_i = D_{i+1 \bmod d}$ for every $g \in \operatorname{supp}(\mu)$,*
(iii) *if $\mathcal{G}_d$ denotes the closed subgroup of G generated by $\operatorname{supp}(\mu^{*d})$, then for every $x \in D_i$ we have $D_i = \mathcal{G}_d x$, and*
(iv) *the d-step random walk on each D_i is ψ-irreducible and aperiodic,*

where always $i = 0, \dots, d-1$.

In other words, a general spread out random walk governed by μ splits up into d aperiodic spread out random walks governed by the d-fold convolution power μ^{*d}.

Proof. Throughout the proof, the terms "null set" or "full measure set" are understood with respect to the Haar measure class on X, to which ψ belongs by assumption. We shall make repeated use of the fact that open null sets are empty. Moreover, for ease of notation, we will drop the specifier "$\bmod d$" from the indices, implicitly viewing them as elements of $\mathbb{Z}/d\mathbb{Z}$.

Let $D'_0, \ldots, D'_{d-1} \subset X$ be a d-cycle as in Theorem 3.1.5. Proposition 3.2.5 shows that μ is spread out. Thus, by Proposition 3.1.9, for some $n_0 \in \mathbb{N}$ there is a continuous component T of P^{n_0} with the property that $T\mathbb{1}_X$ is constant, say $T\mathbb{1}_X \equiv \alpha \in (0,1]$, and $T\mathbb{1}_{D'_i}\colon X \to [0,\alpha]$ is continuous for $i = 0, \ldots, d-1$. By the properties of a d-cycle there exists a cyclic permutation σ of $\{0, \ldots, d-1\}$ such that $P^{n_0}\mathbb{1}_{D'_{\sigma(i)}}$ is 1 on D'_i and 0 on $\bigcup_{j\neq i} D'_j$. Together with the above this implies that $f_i = T\mathbb{1}_{D'_{\sigma(i)}}$ is α on D'_i and 0 on $\bigcup_{j\neq i} D'_j$. The claim is that the sets

$$D_i = f_i^{-1}(\{\alpha\})$$

have the desired properties. Indeed, by construction we know that on each fixed set D'_i the function f_i is α and all other f_j are 0. In particular, the sets $f_i^{-1}((0,\alpha))$ are contained in the complement of the full measure set $\bigcup_{j=0}^{d-1} D'_j$. Being open by continuity, they must thus be empty. This means that the f_i are in fact continuous maps from X to the discrete space $\{0,\alpha\}$. The sets $P_v = \{f_0 = v_0, \ldots, f_{d-1} = v_{d-1}\}$ defined by value tuples $v = (v_0, \ldots, v_{d-1}) \in \{0,\alpha\}^d$ thus form a partition of X consisting of clopen sets. However, for every such tuple v not having precisely one entry α we know that the corresponding set P_v is again contained in the complement of $\bigcup_{j=0}^{d-1} D'_j$, so that $P_v = \emptyset$ by the same logic as above. Altogether, this shows that the non-empty sets in the so-constructed partition are precisely the D_i, proving (i).

To show that $P(x, D_{i+1}) = 1$ for each $x \in D_i$, note first that by definition of a d-cycle we know $P(x, D_{i+1}) \geq P(x, D'_{i+1}) = 1$ whenever $x \in D'_i$. To extend this to $x \in D_i$, we claim that

$$D_i = \overline{D'_i}.$$

Indeed, the inclusion "$\supset$" follows from the D_i being clopen and the differences $D_i \setminus \overline{D'_i}$ are empty because they are open sets contained in a null set. Thus, we may choose a sequence $(x_n)_n$ in D'_i converging to a given $x \in D_i$. Writing $x = g\Lambda$, $x_n = g_n\Lambda$, and $\mathrm{pr}\colon G \to X$ for the canonical projection, we find

$$P(x, D_{i+1}) = \mu(\mathrm{pr}^{-1}(D_{i+1})g^{-1}) = \lim_{n\to\infty} \mu(\mathrm{pr}^{-1}(D_{i+1})g_n^{-1}) = 1$$

by dominated convergence, since by clopenness the indicator functions of the sets $\mathrm{pr}^{-1}(D_{i+1})g_n^{-1}$ converge pointwise to that of $\mathrm{pr}^{-1}(D_{i+1})g^{-1}$.

Next, take $g \in \mathrm{supp}(\mu)$ and $x \in D_i$. Then for any neighborhood B of $g \in G$ we have $P(x, Bx) > 0$. If $gx \notin D_{i+1}$, this would contradict $P(x, D_{i+1}) = 1$ by choosing B small enough. Hence, $gD_i \subset D_{i+1}$. But the same argument applied to $x \in g^{-1}D_{i+1}$ shows that such an x needs to lie in D_i, so that also $g^{-1}D_{i+1} \subset D_i$. This proves (ii).

For (iii), note that $D_i \supset \mathcal{G}_d x$ follows by combining (3.2.1), part (ii) above, and clopenness of D_i. The set $D_i \setminus \mathcal{G}_d x$ is open, since both D_i and $\mathcal{G}_d x$ are clopen (the latter by Lemma 3.2.1, using that also μ^{*d} is spread out) and

$$P^n(x, D_i \setminus \mathcal{G}_d x) = 0$$

for all $n \in \mathbb{N}$: Indeed, if $d \mid n$ we have $P^n(x, \mathcal{G}_d x) = 1$ and if $d \nmid n$ then $P^n(x, D_j) = 1$ for some $j \neq i$. The assumed ψ-irreducibility therefore forces $D_i \setminus \mathcal{G}_d x = \emptyset$, giving (iii).

It remains to prove (iv). Knowing from (ii) that the random walk cycles through the sets $D_0, \ldots, D_{d-1}$, ψ-irreducibility of the d-step random walk on every D_i follows from ψ-irreducibility of the whole random walk. From [**90**, Proposition 5.4.6] we know that the d-step random walk on the full measure subset D_i' of D_i is aperiodic. Suppose that the d-step random walk in D_i has a period strictly larger than 1. Then we can apply what we have already proved and deduce that D_i splits into a non-trivial cycle of clopen subsets. By the second statement in (ii), none of the sets in such a cycle can be null sets. Restricting to D_i' would thus produce a non-trivial cycle inside D_i', which is a contradiction. □

It is natural to ask when the particularly desirable aperiodic case $d = 1$ occurs.

Definition 3.2.10. If μ has the property that the induced random walk on $\mathcal{G}x$ is ψ-irreducible and aperiodic for every $x \in X$, we call μ *aperiodic on* X.

Proposition 3.2.11. *Let μ be spread out. Suppose that*

- *X has finite volume, or that*
- *$\mathcal{G} = \mathcal{S}$.*

Then either one of the following conditions is sufficient for μ to be aperiodic on X:

(i) *X is connected,*
(ii) *μ is aperiodic in the sense of Definition* 3.0.1.

Proof. We first replace the pair (G, X) by $(\mathcal{G}, \mathcal{G}x)$ using Lemma 3.2.1. Then μ is Λ-adapted (in the finite volume case by Proposition 3.2.4), so that by Proposition 3.2.5 the random walk on X is ψ-irreducible with ψ in the Haar measure class.

Sufficiency of (i) is then evident from Proposition 3.2.9, since it shows that the sets in a d-cycle may be chosen to be clopen.

For (ii), we argue by contradiction and assume that the period d of the random walk on X is at least 2. Let us partition $X = \mathcal{G}x$ into clopen sets $D_0, \ldots, D_{d-1}$ as in Proposition 3.2.9. Its part (ii) implies that all elements of $\mathrm{supp}(\mu)$ act on the D_i by the cyclic permutation $D_0 \mapsto D_1 \mapsto \ldots \mapsto D_{d-1} \mapsto D_0$. Since $\mathrm{supp}(\mu)$ generates G topologically and the D_i are clopen, this yields a continuous homomorphism φ from $G = \mathcal{G}$ into the symmetric group of $\{D_0, \ldots, D_{d-1}\}$ with image $\varphi(G) \cong \mathbb{Z}/d\mathbb{Z}$. But then, the kernel $N = \ker(\varphi)$ is a normal open subgroup of G, which contains $[G, G]$ since the quotient $G/N \cong \mathbb{Z}/d\mathbb{Z}$ is abelian, and such that $\mathrm{supp}(\mu)$ is contained in a non-identity–coset of N, since $\varphi(\mathrm{supp}(\mu)) = \{1 + d\mathbb{Z}\}$ under the identification $\varphi(G) \cong \mathbb{Z}/d\mathbb{Z}$. Hence, (ii) does not hold. □

In particular, adapted spread out probability measures are automatically aperiodic on any finite volume quotient when G is connected or a *perfect* group, i.e. one with $G = [G, G]$. An example of the latter case not covered by the first is $G = \mathrm{SL}_d(\mathbb{Q}_p)$. This is an instance of a more general fact.

COROLLARY 3.2.12. *Let k be the field $\mathbb{R}$ of real numbers or the field $\mathbb{Q}_p$ of p-adic numbers for a prime p. Suppose that $G = \mathbf{G}(k)$ for a Zariski connected, simply connected, semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{Q}$ such that G has no compact factors, and let μ be a spread out probability measure on G. Then μ is aperiodic on $X = G/\Lambda$ in both of the following cases:*

(i) *μ is adapted and X has finite volume,*
(ii) *μ is* strongly adapted, *meaning that $\mathcal{S} = G$.*

PROOF. By [**85**, Corollary 2.3.2(b)] G is perfect, so Proposition 3.2.11(ii) applies. □

3.2.3. Harris Recurrence. As final part of our qualitative analysis, we establish Harris recurrence of spread out random walks for homogeneous spaces X with at most quadratic growth. As warm-up, let us show how recurrence can be deduced from what we have already proved in the finite volume case.

PROPOSITION 3.2.13. *Suppose $\Lambda < G$ is a lattice and that the random walk on X induced by μ is ψ-irreducible. Then this random walk is positive Harris recurrent.*

PROOF. Positive recurrence follows from Proposition 3.1.4, since m_X is an invariant probability measure. In order to upgrade this to Harris recurrence, we will show that the set N in the decomposition $X = H \sqcup N$ from Theorem 3.1.8 must be empty. (This theorem can be applied since we know from Proposition 3.2.5 that μ must be spread out, so that the random walk is a T-chain by Proposition 3.1.9.) It thus only remains to show that there are no topologically transient points. But this is easily seen: Proposition 3.1.4 also implies that m_X is equivalent to ψ, so that every non-empty open subset of X is ψ-positive. Recalling the definition of recurrence from Theorem 3.1.2, it follows that $U(x, B) = \infty$ for every neighborhood B of any point $x \in X$. This precisely means that every point of X is topologically recurrent. We thus conclude that $N = \emptyset$, finishing the proof. □

The remainder of this section is dedicated to the proof of Theorem 3.0.8. The following proposition contains the essential lower bound.

PROPOSITION 3.2.14. *Suppose that the homogeneous space X admits a Haar measure m_X. Let B be a symmetric relatively compact neighborhood of the identity in G, $A \subset B\Lambda \subset X$ a positive measure set, and μ a symmetric probability measure on G with* $\operatorname{supp}(\mu) \subset B$. *Then for $n, \ell \in \mathbb{N}$ satisfying*

$$\ell \geq \sqrt{n \log \frac{16 m_X(B^{n+1}\Lambda)}{m_X(A)}}$$

we have

$$\langle P^{2n} \mathbb{1}_A, \mathbb{1}_A \rangle \geq \frac{m_X(A)^2}{4 m_X(B^\ell \Lambda)},$$

where $\langle\cdot,\cdot\rangle$ *denotes the pairing* $\langle\varphi,\psi\rangle = \int_X \varphi\psi \,\mathrm{d}m_X$ *for two measurable functions* φ,ψ *on* X*.*

The proof is adapted from Lust-Piquard [**83**] and contains ideas going back to Carne [**24**].

PROOF. From the defining property $P\varphi(x) = \int_G \varphi(gx)\,\mathrm{d}\mu(g)$ of the action of P on measurable functions and invariance of m_X we get that P is a well-defined operator from $L^1(m_X)$ to itself as well as from $L^\infty(m_X)$ to itself, with operator norm bounded by 1 in both cases. By interpolation, the same is true for all L^p-spaces. Symmetry of μ implies that P is self-adjoint in the sense that $\langle P\varphi,\psi\rangle = \langle\varphi,P\psi\rangle$ whenever these pairings are defined. Using the Cauchy–Schwarz inequality we thus find

$$\langle P^{2n}\mathbb{1}_A,\mathbb{1}_A\rangle^{1/2} = \|P^n\mathbb{1}_A\|_{L^2(m_X)} \geq \frac{\langle P^n\mathbb{1}_A,\mathbb{1}_{B^\ell\Lambda}\rangle}{m_X(B^\ell\Lambda)^{1/2}}.$$

Writing

$$\begin{aligned}\langle P^n\mathbb{1}_A,\mathbb{1}_{B^\ell\Lambda}\rangle &= \langle P^n\mathbb{1}_A,\mathbb{1}\rangle - \langle P^n\mathbb{1}_A,\mathbb{1}_{(B^\ell\Lambda)^c}\rangle = \langle\mathbb{1}_A,\underbrace{P^n\mathbb{1}}_{=\mathbb{1}}\rangle - \langle P^n\mathbb{1}_A,\mathbb{1}_{(B^\ell\Lambda)^c}\rangle\\ &= m_X(A) - \langle P^n\mathbb{1}_A,\mathbb{1}_{(B^\ell\Lambda)^c}\rangle,\end{aligned}$$

we see that it remains to show

$$\langle P^n\mathbb{1}_A,\mathbb{1}_{(B^\ell\Lambda)^c}\rangle \leq \frac{m_X(A)}{2}. \tag{3.2.4}$$

We remark that all pairings above are defined since $P^n\mathbb{1}_A$ has compact support. In fact, by positivity of P we know $0 \leq P^n\mathbb{1}_A \leq P^n\mathbb{1}_{B\Lambda}$, and the latter function has support in $B^{n+1}\Lambda$ since $\operatorname{supp}(\mu)\subset B$.

To prove (3.2.4), we shall use an argument due to Carne [**24**] (see also [**83**, Lemma 1]): The operator P^n can be written as

$$P^n = \sum_{0\leq k\leq n} \alpha_{k,n} Q_k(P),$$

where $\alpha_{k,n} = 0$ if $n-k$ is odd and otherwise $\alpha_{k,n} = 2^{-n+1}\binom{n}{(k+n)/2}$ for $k>0$ and $\alpha_{0,n} = 2^{-n}\binom{n}{n/2}$, and Q_k is the k-th Chebychev polynomial. As the operator P considered on $L^2(m_X)$ is self-adjoint with spectrum contained in $[-1,1]$, the same is true for the operators $Q_k(P)$, since the Chebychev polynomials are real-valued and bounded by 1 on $[-1,1]$. Moreover, Q_k is of degree k so that $Q_k(P)\mathbb{1}_A$ is supported in $B^{k+1}\Lambda$ (using the corresponding property of $P^k\mathbb{1}_A$ established above). Combining these facts we find

$$\begin{aligned}\langle P^n\mathbb{1}_A,\mathbb{1}_{(B^\ell\Lambda)^c}\rangle &= \sum_{\ell\leq k\leq n}\alpha_{k,n}\int_{(B^\ell\Lambda)^c} Q_k(P)\mathbb{1}_A\,\mathrm{d}x\\ &\leq \sum_{\ell\leq k\leq n}\alpha_{k,n}\langle|Q_k(P)\mathbb{1}_A|,\mathbb{1}_{B^{n+1}\Lambda}\rangle\\ &\leq m_X(A)^{1/2}m_X(B^{n+1}\Lambda)^{1/2}\sum_{\ell\leq k\leq n}\alpha_{k,n}\\ &\leq 2m_X(A)^{1/2}m_X(B^{n+1}\Lambda)^{1/2}\mathrm{e}^{-\ell^2/(2n)},\end{aligned}$$

where the last inequality uses a well-known escape estimate for the symmetric random walk on $\mathbb{Z}$ starting at 0 (cf. e.g. [**24**]). Plugging in the inequality for ℓ

from the statement of the proposition, we obtain precisely (3.2.4) and the proof is complete. □

Proof of Theorem 3.0.8. Take a symmetric relatively compact neighborhood B of the identity in G containing $\operatorname{supp}(\mu)$. If the random walk on X given by μ is topologically transient, then by [**63**, Theorem 1] the potential $\sum_{n=1}^{\infty} P^n \mathbb{1}_{B\Lambda}$ is uniformly bounded on X. In particular,

$$\sum_{n=1}^{\infty} \langle P^n \mathbb{1}_{B\Lambda}, \mathbb{1}_{B\Lambda} \rangle < \infty. \tag{3.2.5}$$

Since X has at most quadratic growth, we can apply Proposition 3.2.14 with $A = B\Lambda$ and $\ell = O(\sqrt{n \log n})$ and find for n large enough that

$$\langle P^{2n} \mathbb{1}_{B\Lambda}, \mathbb{1}_{B\Lambda} \rangle \geq \frac{m_X(B\Lambda)^2}{4 m_X(B^\ell \Lambda)} \geq \frac{C}{n \log n},$$

where $C > 0$ is a fixed constant. However, the latter contradicts (3.2.5), because $\sum_n 1/(n \log n) = \infty$. □

In the introduction, Theorem 3.0.8 was stated for topological Harris recurrence, since the concept of Harris recurrence was only introduced in §3.1. Using the following fact contained in [**63**, Theorem 1], one also obtains Harris recurrence.

Proposition 3.2.15 ([**63**]). *Let μ be an adapted spread out probability measure on G. Suppose that there exists a quasi-invariant measure $m_{X,\chi}$ on $X = G/\Lambda$ that is P-subinvariant and that the random walk on X induced by μ is topologically Harris recurrent. Then this random walk is ψ-irreducible with $\psi \sim [m_X]$ and Harris recurrent.*

We point out that, in view of (3.2.2), the measure $m_{X,\chi}$ is P-subinvariant if and only if the character χ satisfies $\int_G \chi \, \mathrm{d}\mu \leq 1$. Therefore, the first condition in the above proposition is satisfied in particular when X admits a Haar measure m_X.

3.3. Consequences

In this final section of the chapter we reap the rewards of the preceding work. In the finite volume setting, we will establish total variation norm convergence of the laws $\mathcal{L}_x(\Phi_n)$ in §3.3.1, see how existence of Lyapunov functions makes this convergence exponentially fast in §3.3.2 and §3.3.3, and present versions of some classical limit theorems in §3.3.4. We end the chapter with the proof of the Ratio Limit Theorem 3.0.9 in §3.3.5.

The standing assumptions from the beginning of §3.2 are still considered to be in effect. Let us quickly review the definition of the total variation norm: Given a finite signed measure ν on X it is defined by

$$\|\nu\| = \sup_{|f| \leq 1} \left| \int_X f \, \mathrm{d}\nu \right|,$$

where the supremum is over all measurable functions $f\colon X \to \mathbb{C}$ bounded by 1. With this definition we have

$$\sup_{\substack{A\subset X\\ \text{measurable}}} |\nu_1(A) - \nu_2(A)| = \tfrac{1}{2}\|\nu_1 - \nu_2\| \tag{3.3.1}$$

for two probability measures ν_1, ν_2 on X. We remark that some authors use the left-hand side above as definition for the total variation distance. Due to the factor of 2 in (3.3.1), some care needs to be taken when consulting the literature when concerned with the precise value of constants. Given a measurable function $V\colon X \to [1, \infty)$, we also define the *$V$-norm* of a finite signed measure ν as

$$\|\nu\|_V = \sup_{|f|\le V}\left|\int_X f \,\mathrm{d}\nu\right|.$$

Note that $\|\cdot\| = \|\cdot\|_1 \le \|\cdot\|_V$.

3.3.1. Convergence of the Laws. Using the results from §3.2, we can now easily prove convergence to equilibrium of the n-step distributions $\mathcal{L}_x(\Phi_n)$, which is sometimes referred to as "mixing" of the random walk.

Recall from the discussion in §3.2.1 that spread out random walks on finite volume spaces are automatically ψ-irreducible on each orbit $\mathcal{G}x$ with ψ equivalent to $m_{\mathcal{G}x}$, so that the concept of periodicity treated in Theorem 3.1.5 and Proposition 3.2.9 is available.

Theorem 3.3.1. *Suppose that $\Lambda < G$ is a lattice. Let μ be spread out and $d \in \mathbb{N}$ be the period of the induced random walk on $\mathcal{G}x$ for some $x \in X$. Then for any starting distribution ν on $\mathcal{G}x$ we have*

$$\left\|\frac{1}{d}\sum_{j=0}^{d-1}\mu^{*(n+j)} * \nu - m_{\mathcal{G}x}\right\| \longrightarrow 0$$

as $n \to \infty$.

Proof. By Lemma 3.2.1 we may assume without loss of generality that $X = \mathcal{G}x$ and that μ is adapted. Then Corollary 3.2.6 and Proposition 3.2.13 together imply that the random walk on X is positive Harris recurrent. Its unique invariant probability is m_X. In the aperiodic case $d = 1$ the result thus is a direct consequence of [**90**, Theorem 13.3.3].

We will now reduce the general case to the aperiodic one. Let $D_0, \dots, D_{d-1}$ be a d-cycle in X with the properties from Proposition 3.2.9. Writing ν as a convex combination and using the triangle inequality, we may assume that ν is supported on one of the D_i. It will be enough to prove the result for n tending to ∞ inside each one of the arithmetic progressions $r + d\mathbb{N}$ for $r = 0, \dots, d-1$. So let us fix one such r and replace n by $nd+r$ in the claimed statement. After renumbering the D_i we may assume that $\mu^{*r} * \nu$ is supported inside D_0. Setting $\nu_i = \mu^{*(i+r)} * \nu$ for $i = 0, \dots, d-1$ we have that ν_i is supported inside D_i and are left to show that

$$\left\|\frac{1}{d}\sum_{i=0}^{d-1}\mu^{*nd} * \nu_i - m_X\right\| \longrightarrow 0$$

as $n \to \infty$. However, in view of part (iv) of Proposition 3.2.9, this follows from the aperiodic case, after writing $m_X = \frac{1}{d}(m_{D_0} + \cdots + m_{D_{d-1}})$ and applying the triangle inequality once more. $\square$

3.3.2. Lyapunov Functions and Effective Mixing. In the literature, functions enjoying certain contraction properties under a transition kernel are known as "Foster–Lyapunov functions" or simply "Lyapunov functions" and have played a major role in questions of recurrence of dynamical systems since their introduction. In our setup, they will produce an exponential rate for the conclusion of Theorem 3.3.1.

We shall say that a (not necessarily continuous) function $f\colon X \to [0,\infty]$ is *proper* if for every $R \in [0,\infty)$ the preimage $f^{-1}([0,R])$ is relatively compact.

DEFINITION 3.3.2. A proper Borel function $V\colon X \to [0,\infty]$ is called a *Lyapunov function* for a Markov chain on X given by a transition kernel P if there exist constants $\alpha < 1$, $\beta \geq 0$ such that $PV \leq \alpha V + \beta$.

Such a function should be thought of as directing the dynamics of the Markov chain towards the "center" of the space, where the function value of V is below some threshold.

REMARK 3.3.3. Let us collect some immediate observations about Lyapunov functions.

(i) If V is a Lyapunov function, then so are cV and $V+c$ for any constant $c > 0$. In particular, one may impose an arbitrary lower bound on V. This will be relevant at some points, where we want V to take values ≥ 1.

(ii) Given a function $V'\colon X \to [0,\infty]$ as in the definition of a Lyapunov function, except that V' is contracted by some power P^{n_0} instead of P, one can construct a Lyapunov function V by setting

$$V = \sum_{k=0}^{n_0-1} \alpha^{(n_0-1-k)/n_0} P^k V'.$$

(iii) By enlarging α and using properness, the contraction inequality in the definition of a Lyapunov function V may be replaced by

$$PV \leq \alpha V + \beta \mathbb{1}_K$$

for some compact $K \subset X$ (cf. [**90**, Lemma 15.2.8]). $\diamond$

The constant function $V \equiv \infty$ always is a Lyapunov function, though one of little use. Of greater interest is the existence of Lyapunov functions that are finite on prescribed parts of the space, or even finite everywhere.

DEFINITION 3.3.4. We say that a subset $A \subset X$ is *Lyapunov small* for a random walk on X given by μ if the random walk admits a Lyapunov function $V_A\colon X \to [0,\infty]$ that is bounded on A. We say the random walk satisfies the *contraction hypothesis* if every compact subset $K \subset X$ is Lyapunov small.

Constructions of Lyapunov functions on quotients of semisimple Lie groups were given by Eskin–Margulis [**40**] and Benoist–Quint [**7**]. We record the consequences for spread out random walks in the example below. Recall that a measure μ on a Lie group G with Lie algebra $\mathfrak{g}$ is said to have *finite exponential moments in* $\mathfrak{g}$ if for sufficiently small $\delta > 0$

$$\int_G \mathrm{N}_a(g)^\delta \,\mathrm{d}\mu(g) < \infty,$$

where $\mathrm{N}_a(g) = \max(\|\mathrm{Ad}(g)\|, \|\mathrm{Ad}(g)^{-1}\|)$.

Example 3.3.5.

(i) ([**7**]) Let G be a real Lie group and μ an adapted spread out probability measure on G with finite exponential moments in $\mathfrak{g}$. Suppose that the Zariski closure of $\mathrm{Ad}(G)$ in $\mathrm{Aut}(\mathfrak{g})$ is Zariski connected and semisimple. Then the random walk on $X = G/\Lambda$ given by μ satisfies the contraction hypothesis. Using the setup in [**7**, Section 7], a similar statement can also be made about p-adic Lie groups.

(ii) ([**40**]) Let $G = \mathbf{G}(\mathbb{R})$ be the group of real points of a Zariski connected semisimple algebraic group $\mathbf{G}$ defined over $\mathbb{R}$ such that G has no compact factors and let μ be a spread out probability measure on G with finite exponential moments in $\mathfrak{g}$. Then the random walk on $X = G/\Lambda$ admits a continuous and everywhere finite Lyapunov function. $\diamond$

Equipped with these concepts, we can now explain how Lyapunov functions make mixing of spread out random walks effective. For the sake of simplicity, we only state the result in the adapted and aperiodic case. We have repeatedly seen that the former is no restriction, and the corresponding statements in the periodic case can be obtained by employing similar reductions as in the proof of Theorem 3.3.1.

Theorem 3.3.6. *Suppose that $\Lambda < G$ is a lattice. Let μ be an adapted spread out probability measure on G such that the random walk on X given by μ is aperiodic.*

(i) *For every Lyapunov small subset $A \subset X$ there exists a constant $\kappa = \kappa(A) > 0$ such that for every $n \in \mathbb{N}$*

$$\sup_{x\in A} \|\mu^{*n} * \delta_x - m_X\| \ll_A \mathrm{e}^{-\kappa n}.$$

In particular, this holds for all compact subsets $A = K$ of X if the random walk satisfies the contraction hypothesis.

(ii) *If the random walk admits an everywhere finite Lyapunov function $V \geq 1$, then there is a constant $\kappa > 0$ such that for all $n \in \mathbb{N}$*

$$\sup_{x\in X} \frac{1}{V(x)} \|\mu^{*n} * \delta_x - m_X\|_V \ll \mathrm{e}^{-\kappa n}.$$

Proof. As in the proof of Theorem 3.3.1 we see that the random walk on X is positive Harris recurrent with unique invariant probability m_X. To establish (i), recall that Remark 3.3.3 allows us to assume that V_A is bounded below by 1 and that $PV_A \leq \alpha V_A + \beta \mathbb{1}_L$ for some compact $L \subset X$. Since compact sets are petite for T-chains ([**90**, Theorem 6.2.5], see [**90**, p. 117] for the definition of petite sets), V_A satisfies the condition in (iii) of [**90**, Theorem 15.0.1]. Since V_A is bounded on A, the claim follows from the last statement of that theorem. With the same arguments, (ii) follows from [**90**, Theorem 16.1.2]. $\square$

On compact spaces, one may always choose $V = 1$ as Lyapunov function. This immediately gives the following corollary.

Corollary 3.3.7. *Suppose in addition to the assumptions in Theorem* 3.3.6 *that X is compact. Then there exists $\kappa > 0$ such that for all $n \in \mathbb{N}$*

$$\sup_{x \in X} \|\mu^{*n} * \delta_x - m_X\| \ll \mathrm{e}^{-\kappa n}. \qquad \square$$

Proof of Theorem 3.0.2. That $\mathcal{G}x_0$ is clopen was part of Lemma 3.2.1. In view of Proposition 3.2.11, (3.0.2) follows from Theorem 3.3.1.

To obtain the statement about effective mixing, first ensure that the Lyapunov function is bounded below by 1 using Remark 3.3.3(i) and then apply Theorem 3.3.6(ii) to each of the finitely many $\mathcal{G}$-orbits intersecting the compact set K. The conclusion follows, since $\|\cdot\| \leq \|\cdot\|_V$ and V is bounded on K by the assumed continuity.

The final remark about existence of Lyapunov functions is Example 3.3.5(ii). $\square$

Proof of Corollary 3.0.4. In both cases of the corollary

- μ is aperiodic on X by Proposition 3.2.11, and
- we have $\mathcal{G}x_0 = X$ for all $x_0 \in X$, using Corollary 3.2.8 or adaptedness of μ, respectively.

The statement now follows from Theorem 3.3.1. $\square$

3.3.3. Small Sets and Mixing Rates. Given the existence of Lyapunov functions or compactness of the state space, we know from §3.3.2 that the convergence

$$\mu^{*n} * \delta_x \stackrel{n\to\infty}{\longrightarrow} m_X$$

happens with exponential speed. As long as the value of the exponent and the implicitly appearing constants are unknown, this does not yet give any information about the actual variation distance between $\mu^{*n} * \delta_x$ and m_X for any given $n \in \mathbb{N}$. In this subsection, we will address this issue. The crucial concept is the following.

Definition 3.3.8. Let P be a transition kernel on X. A set $A \subset X$ is called (n, ε)-*small* for an integer $n \in \mathbb{N}$ and $\varepsilon > 0$ if there exists a probability measure λ on X such that

$$P^n(x, \cdot) \geq \varepsilon\lambda$$

for all $x \in A$. If A is (n, ε)-small for some $n \in \mathbb{N}$ and $\varepsilon > 0$, A is called *small*.

Small sets are in fact one of the central notions on which the whole theory of general state space Markov chains is built. Their significance lies in the fact that they provide the Markov chain with a regenerative structure: After each return to A, there is a positive probability of taking the next step according to the fixed measure λ. This structure also plays an important role when trying to establish bounds on the speed of convergence. The simplest result in this direction assumes that the whole state space is small, which is known as the "Doeblin condition".

THEOREM 3.3.9 ([**90**, Theorem 16.2.4]). *Suppose the whole state space X is (n_0, ε)-small for a Markov chain with transition kernel P and invariant probability π. Then for all $n \in \mathbb{N}$ and any starting distribution ν on X we have*

$$\|\nu P^n - \pi\| \leq 2(1-\varepsilon)^{\lfloor n/n_0 \rfloor}.$$

When the state space is not small, a rate of convergence as simple as above may not be available. We shall use the following result due to Rosenthal [**115**].

THEOREM 3.3.10 ([**115**, Theorem 5]). *Given a transition kernel P on X, denote the product kernel by $\overline{P}((x,y),\cdot) = P(x,\cdot) \otimes P(y,\cdot)$. Let π be a P-invariant probability measure on X. Suppose there exists an (n_0, ε)-small set A and a measurable function $h \geq 1$ on $X \times X$ together with a constant $\overline{\alpha} < 1$ such that*

$$\overline{P}h(x,y) \leq \overline{\alpha} h(x,y)$$

for all $(x,y) \notin A \times A$. Then, with $R = \sup_{(x,y)\in A\times A} \overline{P}^{n_0} h(x,y)$, we have for all $j, n \in \mathbb{N}$ and any starting distribution ν on X

$$\|\nu P^n - \pi\| \leq 2(1-\varepsilon)^{\lfloor j/n_0 \rfloor} + 2\overline{\alpha}^{n-jn_0+1} R^{j-1} \int_{X\times X} h \,\mathrm{d}(\nu \otimes \pi).$$

In order to apply these theorems, we see that it is important to identify small sets for spread out random walks. From a qualitative point of view, this task is not too difficult.

PROPOSITION 3.3.11. *Let μ be a probability measure on G. Suppose that the random walk on $X = G/\Lambda$ is ψ-irreducible and aperiodic. Then every compact subset $K \subset X$ is small.*

PROOF. Combining Propositions 3.2.5 and 3.1.9 we see that the random walk on X is a T-chain. Compact sets are thus petite by [**90**, Theorem 6.2.5]. By aperiodicity and [**90**, Theorem 5.5.7], they are also small. □

In the case of a compact state space, we therefore immediately get the following.

THEOREM 3.3.12. *Let μ be adapted and spread out. Suppose that X is compact and that the random walk on X given by μ is aperiodic. Then X is (n_0, ε)-small for some $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ and for any starting distribution ν on X we have*

$$\|\mu^{*n} * \nu - m_X\| \leq 2(1-\varepsilon)^{\lfloor n/n_0 \rfloor}$$

for every $n \in \mathbb{N}$.

PROOF. Note that the random walk is ψ-irreducible by Corollary 3.2.6. Then X is small by Proposition 3.3.11 and Theorem 3.3.9 gives the result. □

Unfortunately, so far we still have no information about the value of n_0 and ε. We shall now outline a hands-on approach to find them. The idea is the following: Denote by $f_n\colon G \to [0,\infty)$ the density of the part of μ^{*n} absolutely continuous with respect to a right Haar measure on G and endow X with the quasi-invariant measure $m_{X,\Delta}$ coming from the modular character Δ of G.

Then the probability of going from $x \in X$ to $y \in X$ in n steps (using only the continuous part) is represented by the quantity

$$\sum_{\gamma \in \Lambda} f_n(y\gamma x^{-1}), \tag{3.3.2}$$

which is a function on $X \times X$. Note that for fixed x, the sum is finite for a.e. y, since f_n is integrable. Hence, the minorization condition in the definition of small sets is certainly satisfied on A for the right-hand side

$$\inf_{x \in A} \sum_{\gamma \in \Lambda} f_n(y\gamma x^{-1}) \, \mathrm{d}m_{X,\Delta}(y), \tag{3.3.3}$$

which can be thought of as the lower envelope of the shifts by elements of A of the density (3.3.2). The remaining question is whether this measure is non-trivial. Intuitively, the spread out assumption should guarantee this for large n, at least when the shifting set A is not too large, say compact. That this is indeed true is the content of the next lemma.

LEMMA 3.3.13. *Let μ be spread out and Λ-adapted. Suppose that the induced random walk on X is aperiodic. Then for every compact subset $K \subset X$ there exists an integer $n_0 \in \mathbb{N}$ such that the measure* (3.3.3) *has positive mass $\varepsilon > 0$. In particular, K is (n_0, ε)-small.*

PROOF. By Proposition 3.2.5 the random walk is ψ-irreducible with $\psi \sim [m_X]$. Enlarging K if necessary, we may assume that K is $[m_X]$-positive. Proposition 3.3.11 implies that K is small. Choose $n_1, \varepsilon_1, \lambda$ as in the definition of a small set. From [**90**, Proposition 5.5.4(ii)] it then follows that $\lambda \ll [m_X]$. Hence, there exists an $[m_X]$-positive set A such that $\lambda|_A$ belongs to the Haar measure class restricted to A. Let us now split the transition kernels P^n into the absolutely continuous and singular parts with respect to $[m_X]$. As explained before the statement of the lemma, the absolutely continuous part can then be written as $T_n(x, \mathrm{d}y) = p_n(x, y) \, \mathrm{d}m_{X,\Delta}(y)$ with $p_n \colon X \times X \to [0, \infty]$ given by (3.3.2). According to [**98**, Proposition 1.2] (cf. also [**90**, Theorem 5.2.1] and its proof)

- after modifying them on null sets if necessary, the densities p_n can be assumed to satisfy

$$p_{m+n}(x, z) \geq \int_X P^m(x, \mathrm{d}y) p_n(y, z) \tag{3.3.4}$$

  for all $x, z \in X$ and $m, n \in \mathbb{N}$, and
- there exists an $[m_X]$-positive set $C \subset A$ and $n_2 \in \mathbb{N}$ such that

$$p_{n_2}(x, y) \geq \delta \tag{3.3.5}$$

  for all $x, y \in C$ and some fixed $\delta > 0$.

By construction of A we then know $\lambda(C) > 0$, so that for all $x \in K$

$$P^{n_1}(x, C) \geq \varepsilon_1 \lambda(C) > 0. \tag{3.3.6}$$

Combining (3.3.4), (3.3.5) and (3.3.6), we find for $x \in K$, $z \in C$ and $n_0 = n_1 + n_2$ that

$$p_{n_0}(x, z) \geq \int_C P^{n_1}(x, \mathrm{d}y) p_{n_2}(y, z) \geq \delta P^{n_1}(x, C) \geq \delta \varepsilon_1 \lambda(C).$$

Hence, the mass of (3.3.3) is at least

$$\int_C \inf_{x\in A} p_{n_0}(x,z)\,\mathrm{d}m_{X,\Delta}(z) \geq \delta\varepsilon_1\lambda(C)m_{X,\Delta}(C) > 0,$$

which is the claim. □

EXAMPLE 3.3.14. Let us illustrate the method above by calculating a rate of convergence in a concrete instance of Example 3.0.12. We let $n=2$, define $a = \left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)$, $b = \left(\begin{smallmatrix}1&1\\1&2\end{smallmatrix}\right)$, $\mu_{\mathrm{lin}} = \frac{1}{2}(\delta_a+\delta_b)$, $v = e_1 = (1,0)^t$, and assume that λ has a component with a density f bounded below by $\delta > 0$ on $[0,1]$. Let us see how we need to choose n_0. We certainly cannot use $n_0 = 1$, since $\mu = \mu_{\mathrm{lin}} \otimes \lambda_{e_1}$ is singular with respect to Haar measure. If we denote the first two displacements by the random variables D_1, D_2, the possible two-step transformations are

$$a^2\colon x \mapsto a(ax+D_1v)+D_2v = a^2x + D_1ae_1 + D_2e_1 = a^2x + \begin{pmatrix}2D_1+D_2\\D_1\end{pmatrix},$$

$$ab\colon x \mapsto a(bx+D_1v)+D_2v = abx + D_1ae_1 + D_2e_1 = abx + \begin{pmatrix}2D_1+D_2\\D_1\end{pmatrix},$$

$$ba\colon x \mapsto b(ax+D_1v)+D_2v = bax + D_1be_1 + D_2e_1 = bax + \begin{pmatrix}D_1+D_2\\D_1\end{pmatrix},$$

$$b^2\colon x \mapsto b(bx+D_1v)+D_2v = b^2x + D_1be_1 + D_2e_1 = b^2x + \begin{pmatrix}D_1+D_2\\D_1\end{pmatrix}.$$

Since D_1, D_2 are i.i.d. with density f, the densities of the above displacements are

$$g_{a^2}(s,t) = g_{ab}(s,t) = f(t)f(t-2s),\ g_{ba}(s,t) = g_{b^2}(s,t) = f(t)f(t-s)$$

for $s,t \in \mathbb{R}^2$, which, by our assumption, are all bounded below by δ^2 on a fundamental domain for $\mathbb{T}^2$. Hence, the mass of the measure (3.3.3) for $n_0 = 2$ and $A = \mathbb{T}^2$ is at least δ^2, so that Theorem 3.3.12 produces the bound

$$\|\mu^{*n} * \nu - m_X\| \leq 2(1-\delta^2)^{\lfloor n/2\rfloor}$$

for all $n \in \mathbb{N}$, where ν is an arbitrary starting distribution. (Note that aperiodicity is guaranteed here in view of Proposition 3.2.11.) ◇

We now turn our attention to the case of a non-compact finite-volume space X. Here we shall assume that the random walk on X admits a Lyapunov function V and apply Theorem 3.3.10 in a similar way as in the proof of [**115**, Theorem 12].

The set A from Theorem 3.3.10 is going to be the sublevel set

$$A = \{x \in X \mid V(x) \leq d\}$$

for some $d > 1$, and h is going to be defined as

$$h(x,y) = 1 + V(x) + V(y)$$

for $x, y \in X$. Note that A is relatively compact since V is proper and thus a small set by Proposition 3.3.11. Let now α, β be the constants associated to the Lyapunov function V and $(x, y) \notin A \times A$. Then $h(x, y) > 1 + d$, and thus we find

$$\begin{aligned}\overline{P}h(x,y) &= 1 + PV(x) + PV(y)\\ &\le 1 - \alpha + \alpha h(x,y) + 2\beta\\ &\le \Big(\frac{1-\alpha+2\beta}{1+d} + \alpha\Big)h(x,y)\\ &= \frac{1+\alpha d + 2\beta}{1+d}h(x,y).\end{aligned}$$

Choosing $\overline{\alpha} = (1 + \alpha d + 2\beta)/(1 + d)$, this will be the contraction condition in Theorem 3.3.10. In order for $\overline{\alpha}$ to be less than 1, d needs to be chosen so that

$$d > \frac{2\beta}{1-\alpha}.$$

This choice of d determines the set A.

By iterating the Lyapunov property of V and using the definition of A, the value of R in Theorem 3.3.10 can be estimated as

$$\begin{aligned}R &= 1 + 2\sup_{x\in A} P^{n_0}V(x)\\ &\le 1 + 2\Big(\alpha^{n_0}\sup_{x\in A} V(x) + \beta\frac{1-\alpha^{n_0}}{1-\alpha}\Big)\\ &\le 1 + 2\Big(\alpha d + \frac{\beta}{1-\alpha}\Big).\end{aligned}$$

For the integral of h, note first that V is necessarily m_X-integrable by the equivalence of (i) and (iii) in [**90**, Theorem 14.0.1] (use $f = (1-\alpha)V$), so that P-invariance of m_X and the contraction property of V yield $\int_X V\,\mathrm{d}m_X \le \beta/(1-\alpha)$. It follows that

$$\begin{aligned}\int_{X\times X} h\,\mathrm{d}(\nu\otimes m_X) &\le 1 + \int_X V\,\mathrm{d}\nu + \int_X V\,\mathrm{d}m_X\\ &\le 1 + \int_X V\,\mathrm{d}\nu + \frac{\beta}{1-\alpha}.\end{aligned}$$

Putting everything together, we arrive at the following theorem.

THEOREM 3.3.15. *Let $\Lambda < G$ be a lattice and μ be an adapted spread out probability measure on G. Suppose that the random walk on X given by μ is aperiodic and admits a Lyapunov function V with $PV \le \alpha V + \beta$ for some $\alpha < 1$, $\beta \ge 0$. Let $d > 2\beta/(1-\alpha)$ and set $A = \{x \in X \mid V(x) \le d\}$. Then A is (n_0, ε)-small for some $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ and for any starting distribution ν on X with $\int_X V\,\mathrm{d}\nu < \infty$ we have for all $j, n \in \mathbb{N}$*

$$\|\mu^{*n} * \nu - m_X\| \le 2(1-\varepsilon)^{\lfloor j/n_0\rfloor} + 2\overline{\alpha}^{n-jn_0+1}R^{j-1}\Big(1 + \int_X V\,\mathrm{d}\nu + \frac{\beta}{1-\alpha}\Big),$$

where $\overline{\alpha} = (1 + \alpha d + 2\beta)/(1 + d) < 1$ and $R = 1 + 2(\alpha d + \beta/(1-\alpha))$. □

Note that by introducing the relationship $j = \lfloor n/k\rfloor$ for some $k \in \mathbb{N}$ for which $\overline{\alpha}^{k-n_0}R < 1$, the right-hand side above decays exponentially in n, and moreover that all the constants are given explicitly in terms of the starting distribution ν, the Lyapunov function V together with its parameters, and the measure μ.

3.3.4. Limit Theorems. Recall that Benoist–Quint's Theorem B not only makes a statement about convergence in law, but also about the distribution of typical trajectories: For every $x \in X$ and $\mu^{\otimes \mathbb{N}}$-a.e. $(g_n)_n \in G^{\mathbb{N}}$ it holds that

$$\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} \longrightarrow \nu_x$$

as $n \to \infty$ in the weak* topology. In the notation of this chapter, for every $f \in C_c(X)$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1} f(\Phi_k) \longrightarrow \int_X f \, \mathrm{d}\nu_x \quad \mathbb{P}_x\text{-a.s.}$$

as $n \to \infty$, where, as before, Φ_k is given by (3.0.1) and stands for the location after k-steps of the random walk. Until now, we have not yet touched upon the validity of such a "Strong Law of Large Numbers" in the spread out case; an omission that will be corrected now.

To fix the terminology, let us quickly review three of the classical limit theorems in the context of Markov chains. For brevity, we shall use the notation

$$\Sigma_n(f) = \sum_{k=0}^{n-1} f(\Phi_k)$$

for a function f on X.

Definition 3.3.16. Consider the random walk on X given by a probability measure μ on G. Let $f\colon X \to \mathbb{R}$ be a real-valued m_X-integrable function on X. We say

- that the *Strong Law of Large Numbers* (*SLLN*) holds for the function f if for every $x \in X$

$$\lim_{n\to\infty} \frac{\Sigma_n(f)}{n} = \int_X f \, \mathrm{d}m_X \quad \mathbb{P}_x\text{-a.s.},$$

- that the *Central Limit Theorem* (*CLT*) holds for f if there exists a non-negative number $\gamma_f \in [0,\infty)$ such that for the centered function $f_0 = f - \int_X f \, \mathrm{d}m_X$ and under each $\mathbb{P}_x$ we have convergence in distribution

$$\frac{\Sigma_n(f_0)}{\sqrt{n}} \xrightarrow{\mathrm{d}} N(0,\gamma_f^2),$$

where $N(0,\gamma_f^2)$ denotes the normal distribution with mean 0 and variance γ_f^2 (to be understood as the Dirac distribution at 0 in the degenerate case $\gamma_f = 0$), and
- that the *Law of the Iterated Logarithm* (*LIL*) holds for f if for the number γ_f in the CLT and every $x \in X$

$$\limsup_{n\to\infty} \frac{\Sigma_n(f_0)}{\sqrt{2n\log\log(n)}} = \gamma_f \quad \mathbb{P}_x\text{-a.s.}$$

The remarkable fact is that spread out random walks always satisfy the SLLN, and satisfy the CLT and LIL as soon as they admit an everywhere finite Lyapunov function.

THEOREM 3.3.17. *Let $\Lambda < G$ be a lattice and μ be spread out and adapted. Then:*

(i) *The SLLN holds for every m_X-integrable function on X. In particular, for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_n)_n \in G^{\mathbb{N}}$ we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}\delta_{g_k\cdots g_1 x_0} \longrightarrow m_X$$

as $n \to \infty$ in the weak topology.*

(ii) *Suppose the random walk admits an everywhere finite Lyapunov function $V\colon X \to [1,\infty)$ and let $f\colon X \to \mathbb{R}$ be measurable and satisfy $f^2 \leq V$. Then for the centered function $f_0 = f - \int_X f \,\mathrm{d}m_X$, the asymptotic variance*

$$\gamma_f^2 = \lim_{n\to\infty}\tfrac{1}{n}\mathbb{E}_{m_X}[\Sigma_n(f_0)^2]$$

exists and is finite, and the CLT and LIL hold for f and this number γ_f.

PROOF. Combining Corollary 3.2.6 and Proposition 3.2.13 we know that the random walk on X is a positive Harris recurrent Markov chain with invariant probability m_X. Part (i) thus follows from [**90**, Theorem 17.1.7], noting for the second claim that $C_c(X)$ is separable. Under the assumptions of (ii), Theorem 3.3.6(ii) ensures that the conditions of [**90**, Theorem 17.0.1] are satisfied, and everything follows from that theorem. □

3.3.5. Proof of the Ratio Limit Theorem. It remains to prove the ratio limit theorem for the infinite volume case.

PROOF OF THEOREM 3.0.9. For both statements, it suffices to consider the case in which also $f_1 \geq 0$. By Proposition 3.2.15, the random walk on X given by μ is Harris recurrent with invariant measure m_X. The first statement of the theorem thus immediately follows by combining [**114**, Corollary 8.4.3] and [**114**, Theorem 6.6.5].

It remains to prove (3.0.3) under the additional assumptions that μ is symmetric and aperiodic. In view of Proposition 3.2.11, aperiodicity of μ implies aperiodicity of the random walk. Let $A \subset X$ be a small set with positive and finite m_X-measure, say with $P^k(x,\cdot) \geq \varepsilon\lambda$ for all $x \in A$. In view of [**90**, Proposition 5.2.4(iii)] we may assume that $\lambda(A) > 0$, and the discussion in [**90**, §5.4.3] shows that we may take k to be even. With similar arguments as in the proof of Lemma 3.3.13, after shrinking A and ε we may even assume that $\lambda = m_A = m_X(A)^{-1}m_X|_A$ is the normalized restriction of m_X to A (cf. also Orey's C-set theorem [**98**, Theorem 2.1]). Then [**80**, Theorem 2.1] implies

$$\lim_{m\to\infty}\frac{\lambda P^{2m}(A)}{\lambda P^{2(m-1)}(A)} = 1.$$

Applying this $k/2$ times, we see that also

$$\lim_{m\to\infty}\frac{\lambda P^{km}(A)}{\lambda P^{k(m-1)}(A)} = \lim_{m\to\infty}\frac{\lambda P^{km}(A)}{\lambda P^{km-2}(A)}\frac{\lambda P^{km-2}(A)}{\lambda P^{km-4}(A)}\cdots\frac{\lambda P^{km-k+2}(A)}{\lambda P^{km-k}(A)} = 1.$$

This shows that all the assumptions in [**95**] are satisfied.

In view of [**95**, Theorem 1(ii)], it remains to argue that compactly supported bounded measurable functions on X and compactly supported probability measures on X with bounded density with respect to m_X are "small" in Nummelin's sense. Identifying such measures with their density and noting that by symmetry of μ the action of P on functions and measures respects this identification, we see that it suffices to show this claim for functions. For this, by [**95**, Corollary 2.4], we need only show that for every compact subset $K \subset X$ there exists $N \in \mathbb{N}$ such that $\sum_{n=0}^{N} P^n \mathbb{1}_A$ is bounded away from 0 on K. However, since the random walk is a T-chain and compact sets are petite for T-chains ([**90**, Theorem 6.2.5(ii)]), the latter follows from [**90**, Proposition 5.5.5(i)] and [**90**, Proposition 5.5.6(i)], as $m_X(A) > 0$. $\square$

Invoking [**95**, Theorem 1(i)], the proof above also justifies the claim in Remark 3.0.10: Taking for A the same small set as above, we see that (3.0.4) implies

$$\frac{\int_X f_i \,\mathrm{d}(\mu^{*n} * \delta_{x_i})}{(\mu^{*n} * m_A)(A)} \longrightarrow \frac{\int_X f_i \,\mathrm{d}m_X}{m_X(A)}$$

as $n \to \infty$ for $i = 1, 2$. Taking the quotient yields (3.0.3) with δ_{x_i} in place of ν_i.

CHAPTER 4

Expanding Measures: Random Walks and Rigidity on Homogeneous Spaces

Joint with Çağrı Sert and Ronggang Shi

We have already seen in Chapter 2 that expansion conditions on a random walk can serve as replacement for Benoist–Quint's assumption of Zariski density in a semisimple group. There, this observation was motivated by the study of random walks driven by general stochastic processes, such as Markov random walks, and we essentially proved results only for the basic case of homogeneous spaces of simple Lie groups, where the only possible limits are finite orbit measures and the Haar measure on the whole space.

In this final chapter of the thesis, we systematically study random walks with expansion properties in the fully general setting. We will introduce and study a new class of measures μ supported on a connected semisimple subgroup $H \leqslant G$ without compact factors and with finite center that we call H-expanding measures. These are defined by an expansion condition in non-trivial irreducible finite-dimensional representations of H resembling the conclusion of the fundamental result of Furstenberg on the positivity of the top Lyapunov exponent. In particular, this class contains the Zariski dense measures underlying the work of Benoist–Quint. After deducing a measure classification result analogous to Benoist–Quint's Theorem A in the Introduction based on the progress by Eskin–Lindenstrauss [**39**], we will prove orbit closure descriptions, as well as recurrence and equidistribution results for the random walk on G/Λ given by an H-expanding probability measure μ. In particular, we also obtain a full analogue of Theorem B. Finally, taking advantage of the generality of H-expanding measures, these main results will be used to establish new equidistribution statements for diagonalizable flows, which in turn have implications for Diophantine approximation problems on fractals.

To introduce the notion of H-expansion, we say that a Borel probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ is *uniformly expanding* if for every nonzero $v \in \mathbb{R}^d$, we have

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|g_n \cdots g_1 v\| > 0$$

for $\mu^{\otimes\mathbb{N}}$-almost every (a.e.) sequence $(g_i)_i$. A probability measure μ on H is said to be *H-expanding* if for every finite-dimensional representation (V, ρ) of H without nonzero H-fixed vectors, the measure $\rho_*\mu$ is uniformly expanding, where $\rho_*\mu$ denotes the pushforward of μ by ρ. We are going to elaborate on this definition and give non-trivially equivalent formulations in §4.1.

Ranging over all finite-dimensional representations, the H-expansion property of a probability measure μ on H is a universal condition and as such ensures validity of our results for an arbitrary embedding $H \hookrightarrow G$ and any lattice $\Lambda < G$. This universality notwithstanding, the class of H-expanding measures contains an abundance of interesting examples:

- Zariski dense measures (§4.2.1): If the closed subgroup Γ_μ of H generated by the support of μ has Zariski dense image in $\mathrm{Ad}(H)$ and μ satisfies a moment condition, then μ is H-expanding as a consequence of Furstenberg's theorem on positivity of the top Lyapunov exponent.
- Measures on parabolic groups (§4.2.2): We give a general criterion for H-expansion of a measure μ on a parabolic subgroup of H and, using the notion of expanding cone introduced by Shi [**127**], explicitly exhibit a class of examples of such measures. For the sake of concreteness, let us mention here that, for example, our results directly imply that any probability measure on $H = \mathrm{SL}_4(\mathbb{R})$ with support consisting of the five matrices
$$\begin{pmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1/4 \end{pmatrix}, \begin{pmatrix} 2 & 1 & & \\ 1 & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & \\ 1 & 2 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$$
is H-expanding.
- Epimorphic subgroups (§4.2.3): The closed subgroup Γ_μ generated by the support of μ is necessarily an epimorphic subgroup of H when μ is H-expanding. Conversely, thanks to the work of Bien–Borel [**14**] and its subsequent developments, we will see that many epimorphic subgroups of H support H-expanding measures. For example, any $\mathbb{R}$-split simple group H admits distinguished three-dimensional epimorphic subgroups for which this is the case, showing that H-expanding measures may live on subgroups which are very small compared to H itself. See also Corollary 4.3.8.

Under various weaker assumptions than H-expansion, not all of our conclusions hold in full strength. For instance, requiring uniform expansion only in the adjoint representation, homogeneity of stationary measures can fail, as an example at the end of [**39**, §1.2] shows. For unipotent random walks, recurrence is not always guaranteed [**21**, §10.2.1]. On the other hand, in the particular case of measures on parabolic subgroups, slightly weaker expansion properties were first used in the work of Simmons–Weiss [**129**] and subsequently in [**105**] (Chapter 2 of this thesis) to prove measure rigidity and equidistribution results in a setting corresponding to the case $H = G$ in the framework of this chapter. See also Remark 4.0.3.

We next recall the terminology necessary to state our main results. Given a continuous action of a locally compact second countable group G on a locally compact second countable metrizable space X, a probability measure ν on X is said to be *μ-stationary* if $\nu = \mu * \nu$, where the convolution is defined by
$$\int_X f \,\mathrm{d}(\mu * \nu) = \int_X \int_G f(gx) \,\mathrm{d}\mu(g) \,\mathrm{d}\nu(x)$$
for non-negative Borel functions f on X. A μ-stationary probability measure ν is said to be *μ-ergodic* if it is extremal in the convex set of μ-stationary probability measures.

Now let G be a real Lie group, $\Lambda < G$ a discrete subgroup and $X = G/\Lambda$. A probability measure ν on X is said to be *homogeneous* if there exists $x \in X$ and a closed subgroup N of G preserving ν such that $\nu(Nx) = 1$. In this case, the orbit Nx is automatically closed and is called a *homogeneous subspace* of X. It is equivalent to require that ν assigns full measure to an orbit of its stabilizer group

$$\mathrm{Stab}_G(\nu) = \{g \in G \mid g_*\nu = \nu\}.$$

This gives a one-to-one correspondence between homogeneous measures on X and homogeneous subspaces of X. For a closed subgroup Γ of G, a homogeneous subspace Y of X is said to be Γ-*ergodic* if Γ preserves the corresponding homogeneous probability measure ν_Y and the action of Γ on (Y, ν_Y) is ergodic.

Finally, for $g \in \mathrm{GL}_d(\mathbb{R})$ we set $\mathrm{N}(g) = \max(\|g\|, \|g^{-1}\|)$ for some choice of operator norm on $\mathbb{R}^{d\times d}$. A probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ is said to have a *finite first moment* if

$$\int \log \mathrm{N}(g)\, \mathrm{d}\mu(g) < \infty,$$

and to have *finite exponential moments* if

$$\int \mathrm{N}(g)^{\delta}\, \mathrm{d}\mu(g) < \infty$$

for $\delta > 0$ sufficiently small. These definitions are independent of the choice of operator norm. We say that a probability measure μ on a connected semisimple Lie group H with finite center has a finite first moment or finite exponential moments if its image in a finite-dimensional representation of H with finite kernel has the corresponding property. This does not depend on the choice of such a linear representation (see Lemma 4.1.9). Both moment conditions are automatically satisfied, for example, if μ has compact support.

4.0.1. Measure Rigidity. We start with the classification of stationary measures. Recall that given a measure μ on H, we denote by Γ_μ the closed subgroup generated by the support of μ.

THEOREM 4.0.1. *Let G be a real Lie group and $\Lambda < G$ a discrete subgroup. Let $H \leqslant G$ be a connected semisimple subgroup without compact factors and with finite center. Let μ be a probability measure on H that is H-expanding and has a finite first moment. Then any μ-ergodic μ-stationary probability measure ν on G/Λ is Γ_μ-invariant and homogeneous. Moreover, the connected component of* $\mathrm{Stab}_G(\nu)$ *is normalized by H.*

Using the properties of H-expanding measures, the above theorem is deduced by an iterative application of the recent measure classification results of Eskin–Lindenstrauss [**39**]; see §4.3.1. The proof is similar to the argument Eskin–Lindenstrauss use to show that their result implies Benoist–Quint's measure classification.

In certain cases, the last conclusion of Theorem 4.0.1 allows us to show that ν is actually H-invariant; see Proposition 4.7.2 and also the corollary below. For its statement, recall that a discrete subgroup Λ is said to be a *lattice* in G if $X = G/\Lambda$ admits a G-invariant probability measure m_X. In this case, we refer to m_X as the *Haar measure* on X. A lattice Λ in a connected semisimple

Lie group G without compact factors is said to be *irreducible* if $\Lambda \cap S$ is not a lattice in S for every non-trivial proper connected normal subgroup S of G. Equivalently, $S\Lambda$ is dense in G for every such S.

COROLLARY 4.0.2. *Let G be a connected semisimple Lie group without compact factors and with finite center and $\Lambda < G$ an irreducible lattice. Let H be a connected normal subgroup of G of positive dimension and let μ be an H-expanding probability measure on H with finite first moment.*

(i) *If $H \neq G$, then the Haar measure m_X on $X = G/\Lambda$ is the unique μ-stationary probability measure on X.*
(ii) *If $H = G$, then the only μ-ergodic μ-stationary probability measures on X are uniform measures on finite Γ_μ-orbits and the Haar measure m_X on X. Moreover, m_X is the only non-atomic μ-stationary probability measure on X.*

We note that finite Γ_μ-orbits do only occur when Γ_μ is virtually contained in a conjugate of Λ. The proof of part (i) of the corollary above relies on Margulis' arithmeticity theorem and a careful analysis of stationary measures charging an orbit of the centralizer of Γ_μ, which is carried out in §4.3.2. The last statement in part (ii) additionally requires countability of finite Γ_μ-orbits, which follows from a general countability result for homogeneous subspaces in §4.4.

REMARK 4.0.3. As mentioned before, the H-expansion condition is universal so that all our results hold for an arbitrary embedding $H \hookrightarrow G$. For a fixed Lie group G, it suffices to impose uniform expansion on $\rho_*\mu$ only for a finite collection of representations (V, ρ) of H (which depends on G), as the proofs show. In §4.3.3 we track which representations are needed in the case of measure classification; see Theorem 4.3.7 for the precise statement. Our countability result (Proposition 4.4.1) will also be phrased using only this finite collection of representations, allowing us to prove it without an assumption of compact generation (cf. [**9**, Proposition 2.1]). $\diamond$

4.0.2. Recurrence and Lyapunov Functions. Now we assume in addition that Λ is a lattice and that μ has finite exponential moments. Under certain assumptions including semisimplicity of the non-compact part of the Zariski closure of Γ_μ, Eskin–Margulis [**40**] and later Benoist–Quint [**7**] have shown that the random walk on $X = G/\Lambda$ given by μ satisfies strong recurrence properties. If δ_x denotes the Dirac measure at $x \in X$ and μ^{*n} is the n-fold convolution power of μ, these recurrence statements take the general form that $\mu^{*n} * \delta_x(M)$ is close to 1 for large n, where $M \subset X$ is a certain compact set. We obtain analogous results for H-expanding measures.

THEOREM 4.0.4. *Let Λ be a lattice in a real Lie group G. Let $H \leqslant G$ be a connected semisimple subgroup without compact factors and with finite center. Let μ be an H-expanding probability measure with finite exponential moments on H. Let Y be a Γ_μ-ergodic homogeneous subspace of $X = G/\Lambda$ or the empty set. Finally, let K_L be any compact subset of the centralizer L of Γ_μ in G, and set $\mathcal{N} = K_L Y$. Then for any compact subset $Z \subset X \setminus \mathcal{N}$ and $\delta > 0$ there exists* *a compact subset $M_{Z,\delta}$ of $X \setminus \mathcal{N}$ such that*

$$\mu^{*n} * \delta_x(M_{Z,\delta}) \geq 1 - \delta$$

for every $n \geq 0$ and $x \in Z$.

This result will be proved in §4.6.1 using height functions on $X = G/\Lambda$ satisfying a contraction property with respect to the averaging operator (or convolution operator) $\pi(\mu)$ defined by

$$\pi(\mu)f(x) = \int_G f(gx)\,\mathrm{d}\mu(g)$$

for non-negative Borel functions f on X. Heuristically, if β is a function on X with values in $[0, \infty]$ such that

$$\pi(\mu)\beta \leq a\beta + b \tag{4.0.1}$$

for constants $a \in (0,1)$ and $b \geq 0$, then, with high probability, the dynamics of the random walk are driven towards the part of the space where β takes values below a certain threshold, and $X_\infty = \beta^{-1}(\{\infty\})$ acts as a repeller. Putting this heuristic into quantitative terms yields strong recurrence properties of the random walk away from X_∞, which play a key role not only in the proof of Theorem 4.0.4, but also for orbit closure and equidistribution results to be described in what follows.

Ideas of this kind have a rich history in the theory of stochastic processes and dynamical systems and trace back to the work of Foster [**45**] and Lyapunov [**81**] (see also [**90**, §15]). In homogeneous dynamics, they first appear in Eskin–Margulis–Mozes' work on a quantitative version of the Oppenheim conjecture [**41**]. In the study of random walks on homogeneous spaces, height functions were first systematically used by Eskin–Margulis [**40**] to establish recurrence properties. Functions satisfying the contraction property (4.0.1) are therefore often referred to either as "Lyapunov functions" or "Margulis functions".

To obtain our results, we will need to construct two types of Lyapunov functions.

- Height functions with respect to the cusps (§4.5.1): First, corresponding to the case $Y = \emptyset$ in Theorem 4.0.4, we require a Lyapunov function β_∞ that stays bounded on a prescribed compact subset Z of X and tends to infinity when leaving compact parts of the space into the cusps of X. Its role is to rule out escape of mass, i.e. ensure that the random walk does not escape to infinity. For this case, we will show that we can use the height function constructed by Benoist–Quint [**7**]. Indeed, as it turns out, the algebraic condition that is imposed in their paper on the Zariski closure of Γ_μ is only crucially used to ensure an expansion property in representations of H, so that the proof also goes through under our H-expansion assumption.
- Height functions with respect to singular subspaces (§4.5.2): Secondly, corresponding to the case of a lower-dimensional homogeneous subspace Y in Theorem 4.0.4, we also need Lyapunov functions which blow up near the singular subspace Y. These are used to ensure that

random walk trajectories do not accumulate near Y when starting outside of it. Here, we give a construction inspired by the work of Eskin–Mirzakhani–Mohammadi [**43**] for random walks on moduli space. This will allow us to avoid the use of the first return cocycles and operators appearing in [**8, 9**], and to obtain a height function $\beta_{\mathcal{N}}$ which satisfies the contraction property (4.0.1) with respect to $\pi(\mu)$ itself.

We remark that the finite exponential moments assumption is essential in our method to obtain contraction properties for various averaging operators, e.g. in Lemma 4.5.2.

4.0.3. Orbit Closures and Equidistribution. Measure classification and recurrence properties at hand, the next step is the question of equidistribution of random walks with respect to a homogeneous probability measure, which, once established, yields orbit closure descriptions analogous to Ratner's theorems in unipotent dynamics.

Let Γ_μ^+ be the closed semigroup generated by the support of μ. If Γ_μ has Zariski dense image in $\mathrm{Ad}(H)$, then Theorem B in the Introduction asserts that the orbit closure $\overline{\Gamma_\mu^+ x}$ is a homogeneous subspace of X inside which the random walk equidistributes. Our next result is a generalization of this and other rigidity results for the random trajectory of points proved in [**9, 129**] and also in Chapter 2.

THEOREM 4.0.5. *Let Λ be a lattice in a real Lie group G. Let $H \leqslant G$ be a connected semisimple subgroup without compact factors and with finite center. Let μ be an H-expanding probability measure with finite exponential moments on H. Then for every $x \in X = G/\Lambda$ there is a Γ_μ-ergodic homogeneous subspace $Y_x \subset X$ with corresponding homogeneous probability measure ν_x such that the following hold:*

(i) *The orbit closure $\overline{\Gamma_\mu^+ x}$ equals Y_x.*

(ii) *One has*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_x = \nu_x.$$

(iii) *For $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in H^{\mathbb{N}}$ one has*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k\cdots g_1 x} = \nu_x.$$

In statements (ii) and (iii) of the theorem above, convergence is understood with respect to the weak* topology, where weak* convergence of a sequence of probability measures ν_n on X to a finite measure ν on X is defined to mean that

$$\lim_{n\to\infty} \int_X f \,\mathrm{d}\nu_n = \int_X f \,\mathrm{d}\nu \tag{4.0.2}$$

for every compactly supported continuous test function f on X. In case the limit measure ν is a probability measure, weak* convergence $\nu_n \to \nu$ implies that (4.0.2) holds for any bounded continuous function f on X.

Theorem 4.0.5 will be proved in §4.6.2. It has the non-trivial topological consequence that any infinite Γ_μ^+-orbit in X is dense in a homogeneous subspace of positive dimension. In the G-expanding case with an irreducible lattice Λ in G, this means that every infinite Γ_μ^+-orbit in $X = G/\Lambda$ is dense.

Remark 4.0.6. Using auxiliary constructions, our results can be applied in certain cases where the connected semisimple group H is invisible. For example, they cover random walks by automorphisms on a compact nilmanifold N/Λ' by considering $G = \mathrm{Zcl}(\mathrm{Aut}(\Lambda')) \ltimes N$ and $\Lambda = \mathrm{Aut}(\Lambda') \ltimes \Lambda'$, where $\mathrm{Zcl}(\mathrm{Aut}(\Lambda'))$ denotes the Zariski closure of $\mathrm{Aut}(\Lambda')$ inside $\mathrm{Aut}(N)$; see §4.6.4. $\diamond$

4.0.4. The Space of Homogeneous Measures. Given a closed subgroup Γ of the Lie group G, we consider

$$\mathcal{S}(\Gamma) = \{\Gamma\text{-invariant } \Gamma\text{-ergodic homogeneous subspaces } Y \subset X\},$$

where, as before, $X = G/\Lambda$ is the quotient of G by a lattice Λ. By definition, associated to each $Y \in \mathcal{S}(\Gamma)$ is a Γ-invariant and ergodic probability measure ν_Y with support Y. This defines an embedding of $\mathcal{S}(\Gamma)$ into the space of probability measures on X, which we use to endow $\mathcal{S}(\Gamma)$ with the weak* topology. In the unipotent case, Mozes–Shah [**94**] proved that convergence of homogeneous subspaces in this topology behaves in a very rigid way. Benoist–Quint [**9**, §1.3] later obtained a version of this result for a subgroup Γ that is Zariski dense in a semisimple group. Following their strategy, we obtain similar results in our setup.

Given a subset Z of X, let us write $\mathcal{S}_Z(\Gamma) = \{Y \in \mathcal{S}(\Gamma) \mid Y \cap Z \neq \emptyset\}$ and denote by δ_∞ the Dirac measure at ∞ in the one-point compactification $\overline{X} = X \cup \{\infty\}$ of X.

Proposition 4.0.7. *Retain the notation and assumptions of Theorem* 4.0.5. *Then we have:*

(i) *For every compact subset $Z \subset X$, $\mathcal{S}_Z(\Gamma_\mu)$ is compact, and $\mathcal{S}_{HZ}(\Gamma_\mu)$ is relatively compact inside $\mathcal{S}(\Gamma_\mu)$. Moreover, the set $\mathcal{S}(\Gamma_\mu) \cup \{\delta_\infty\}$ is compact.*

(ii) *If $Y_n \to Y_\infty$ in $\mathcal{S}(\Gamma_\mu)$, then there exists a sequence $l_n \in C_G(\Gamma_\mu)$ with $l_n \to e$ and $Y_n \subset l_n Y_\infty$ for every $n \in \mathbb{N}$ large enough.*

Similar remarks as in [**9**, §1.3] apply: When there exists a compact subset Z of X such that $HZ = X$, the set $\mathcal{S}(\Gamma_\mu)$ is compact. Otherwise, the only additional thing that can happen is escape of mass to infinity.

This proposition is a manifestation of strong rigidity of the Γ_μ-invariant and ergodic homogeneous subspaces. For example, given a compact subset Z of X and $Y_\infty \in \mathcal{S}(\Gamma_\mu)$ with $Z^\circ \cap Y_\infty \neq \emptyset$, if for a sequence $Y_n \in \mathcal{S}(\Gamma_\mu)$ we have $Y_n \cap Z \to Y_\infty \cap Z$ in the Hausdorff metric, then one can conclude that $Y_n \to Y_\infty$ in $\mathcal{S}(\Gamma_\mu)$. In particular, the weak* topology on $\mathcal{S}(\Gamma_\mu)$ coincides with the restriction to $\mathcal{S}(\Gamma_\mu)$ of the Fell topology on closed subsets of X.

Another consequence of Proposition 4.0.7 is the following equidistribution result for sequences of homogeneous subspaces in the case that Γ_μ has discrete centralizer in G.

Corollary 4.0.8. *Retain the notation and assumptions of Theorem* 4.0.5 *and assume in addition that the centralizer $C_G(\Gamma_\mu)$ of Γ_μ in G is discrete. Let $Y_\infty \in \mathcal{S}(\Gamma_\mu)$ and consider the set*

$$\mathcal{S}(\Gamma_\mu, Y_\infty) = \{Y \in \mathcal{S}(\Gamma_\mu) \mid Y \subset Y_\infty\}$$

of ergodic homogeneous subspaces of Y_∞. *Suppose that* $(Y_n)_n$ *is a sequence in* $\mathcal{S}(\Gamma_\mu, Y_\infty)$ *such that for every fixed* $Y \in \mathcal{S}(\Gamma_\mu, Y_\infty) \setminus \{Y_\infty\}$ *one has* $Y_n \not\subset Y$ *for all but finitely many* n, *and such that no subsequence of* $(Y_n)_n$ *escapes to infinity. Then* $Y_n \to Y_\infty$ *in* $\mathcal{S}(\Gamma_\mu)$.

Here, by "escape to infinity" we mean weak* convergence towards the Dirac measure δ_∞ at infinity.

The proofs of both statements above will be given in §4.6.3.

4.0.5. Birkhoff Genericity. We still assume that Λ is a lattice in the Lie group G. Let $(a(t))_{t\in\mathbb{R}}$ be a one-parameter Ad-diagonalizable subgroup of H and ν a probability measure on $X = G/\Lambda$ invariant under $a(t)$ for every $t \in \mathbb{R}$. We say that a Radon measure η on H is $a(t)$-*Birkhoff generic at* $x \in X$ *with respect to* ν if

$$\frac{1}{T}\int_0^T \delta_{a(t)hx}\,\mathrm{d}t \longrightarrow \nu$$

in the weak* topology as $T \to \infty$ for η-almost every $h \in H$. It was first noticed by Simmons–Weiss [**129**] that, in certain situations, pathwise equidistribution of random walks as in Theorem 4.0.5(iii) can be used to deduce Birkhoff genericity of fractal measures η on unipotent subgroups of H with respect to the Haar measure on X, which has consequences in Diophantine approximation thanks to the Dani correspondence principle. Recently, more results were obtained in this direction in [**105**] (Chapter 2 of this thesis). Both of these papers only deal with cases corresponding to $H = G$ in our setup. We are going to extend the existing results by removing this restriction. Even in the case where $H = G$, we obtain Birkhoff genericity for more general one-parameter subgroups and fractal measures, which will also give new results on Diophantine approximation (see §4.0.6).

The one-parameter subgroups to which our results apply are required to satisfy certain expansion condition with respect to a unipotent subgroup of H. To phrase it, we use the concept of an a-expanding subgroup of H introduced in [**126**]. Namely, given an Ad-diagonalizable element $a \in H$, a connected Ad-unipotent subgroup U of H normalized by a is said to be a-*expanding* if for any non-trivial irreducible representation of H on a finite-dimensional real vector space V, the subspace V^U of U-fixed vectors is expanded by a, i.e. $\lim_{n\to\infty} a^{-n}v = 0$ for any $v \in V^U$. If the projection of a to each simple factor of H is non-trivial, then certain horospherical subgroups of H are a-expanding. For example, this holds for the unstable horospherical subgroup

$$H_a^+ \coloneqq \{h \in H \mid \lim_{n\to\infty} a^{-n}ha^n = 1_H\},$$

of a; see §4.2.2.

Now let U be an $a(1)$-expanding subgroup contained in the unstable horospherical subgroup $H^+_{a(1)}$ of $a(1)$. We wish to introduce a family of measures on U which are generated by random walks, in a sense to be made precise in what follows. Let $A' = \{a(t) \mid t \in \mathbb{R}\}$, K be a maximal compact subgroup of H, and $K' = C_K(A') \cap N_H(U)$, where $C_K(A')$ denotes the centralizer of A' in K and $N_H(U)$ the normalizer of U in H. We set $P \coloneqq K'A'U \subset H$ and denote by λ the function which associates to $g \in P$ the real parameter of its A' component in its $K'A'U$ factorization; that is, $\lambda(g) = t \in \mathbb{R}$ for $g = ka(t)u \in K'A'U$.

Finally, given $\omega = (g_i)_i \in P^{\mathbb{N}}$ and $n \in \mathbb{N}$, let $k_{\omega,n} \in K'$, $a_{\omega,n} \in A'$ and $u_{\omega,n} \in U$ be such that

$$g_n \cdots g_1 = k_{\omega,n} a_{\omega,n} u_{\omega,n}.$$

With this notation, we are ready to define the class of measures on U we shall be interested in.

DEFINITION 4.0.9. Let $(a(t))_{t\in\mathbb{R}} \leqslant H$ be a one-parameter Ad-diagonalizable subgroup of H and U an $a(1)$-expanding subgroup of H contained in $H^+_{a(1)}$. A probability measure η on U is said to be *generated by $a(1)$-expanding random walks* if there is a probability measure μ on H with finite exponential moments satisfying the following properties:

(1) $\mu(P) = 1$ and $\int_P \lambda(g)\,\mathrm{d}\mu(g) > 0$,
(2) the Zariski closure of the image of Γ_μ in $\mathrm{Ad}(H)$ contains $\mathrm{Ad}(U)$, and
(3) η is equivalent to the pushforward of $\mu^{\otimes\mathbb{N}}$ by the map

$$P^{\mathbb{N}} \to U,\ \omega \mapsto u_\omega \coloneqq \lim_{n\to\infty} u_{\omega,n}.$$

The almost sure existence of the limit in point (3) above is the content of Lemma 4.7.1. Moreover, we will see as part of our discussion in §4.7 that conditions (1) and (2) imply that μ is H-expanding, which will allow us to employ our main measure classification and equidistribution results discussed before.

For the statement of our result on Birkhoff genericity, recall that by Ratner's theorems the orbit closure $\overline{Hx}$ is homogeneous for any $x \in X$. We denote the homogeneous probability measure corresponding to $\overline{Hx}$ by $\nu_{\overline{Hx}}$.

THEOREM 4.0.10. *Let Λ be a lattice in a real Lie group G and let $H \leqslant G$ be a connected semisimple subgroup without compact factors and with finite center. Let $(a(t))_{t\in\mathbb{R}}$ be a one-parameter* Ad*-diagonalizable subgroup of H and U an $a(1)$-expanding subgroup of H contained in $H^+_{a(1)}$. Suppose that η is a probability measure on U generated by $a(1)$-expanding random walks. Then for every $x \in X$, η is $a(t)$-Birkhoff generic at x with respect to $\nu_{\overline{Hx}}$.*

Theorem 4.0.10 extends the main results of [**126**], which used the method of Chaika–Eskin [**25**] developed for the Teichmüller geodesic flow to prove Birkhoff genericity for the Haar measure on U. The same method was employed in [**47**] to obtain Birkhoff genericity for volume measures on curves. The proof of Theorem 4.0.10 will be given in §4.7, using the connection to random walks observed in [**129**].

Probability measures generated by expanding random walks include a piece of Haar measure on U and, under irreducibility conditions, self-similar measures on $\mathbb{R}^m$ as well as natural self-affine measures on Bedford–McMullen carpets. The latter example is crucial for our application to Diophantine approximation problems on fractals described next. In §4.8.2 we will also discuss a more general class of fractal measures covered by Definition 4.0.9.

4.0.6. Diophantine Approximation. By virtue of a correspondence principle going back to the work of Dani [**30**] and Kleinbock [**75**], Theorem 4.0.10 on Birkhoff genericity has consequences for problems in Diophantine approximation, which we shall now describe.

Let $m \in \mathbb{N}$ be a positive integer, $\mathbf{v} = (v_1, \dots, v_m)^t$ a (column) vector in $\mathbb{R}^m$, and $\mathbf{r} = (r_1, \dots, r_m) \in (0, 1]^m$ such that $\sum_{i=1}^m r_i = 1$. The vector $\mathbf{v}$ is called $\mathbf{r}$-*badly approximable* if there exists a constant $C > 0$ such that

$$\max_{1 \le i \le m} |v_i q - p_i|^{1/r_i} \cdot |q| \ge C \tag{4.0.3}$$

for every $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ and $q \in \mathbb{Z} \setminus \{0\}$. When $r_i = 1/m$ for every $i = 1, \dots, m$, such a vector is simply called *badly approximable*. In the case $m = 1$, the latter corresponds to the classical definition of a badly approximable number. It is easily seen by Dirichlet's principle that for any vector $\mathbf{v} \in \mathbb{R}^m$, the left-hand side of (4.0.3) is ≤ 1 for infinitely many pairs $(\mathbf{p}, q) \in \mathbb{Z}^m \times (\mathbb{Z} \setminus \{0\})$.

The existence of badly approximable vectors was observed by Perron [**101**] a century ago. It follows from Schmidt's results [**119**] that such vectors constitute a subset of $\mathbb{R}^m$ of everywhere-full Hausdorff dimension. This was strengthened in more recent works [**71, 79**] to the statement that badly approximable vectors contained in a sufficiently regular fractal $\mathcal{K}$ form a subset of full Hausdorff dimension in $\mathcal{K}$. For a general weight $\mathbf{r}$, the results of [**73, 79, 102**] imply that $\mathbf{r}$-badly approximable vectors have everywhere-full Hausdorff dimension in $\mathbb{R}^m$. For $\mathbf{r}$-badly approximable vectors on a fractal set $\mathcal{K}$, the full-dimension statement is known to hold when $\mathcal{K}$ has a certain product structure (see [**71**, Theorem 8.4], [**79**, Theorems 11,13]).

The results outlined above can be summarized by saying that ($\mathbf{r}$-)badly approximable vectors are abundant from the viewpoint of Hausdorff dimension. On the Lebesgue measure side, however, Khintchine's theorem [**70**] implies that badly approximable vectors have zero Lebesgue measure. Using a generalization of Khintchine's theorem [**118**], the same is seen to be true for $\mathbf{r}$-badly approximable vectors. The question whether badly approximable vectors on a given fractal $\mathcal{K}$ also form a null set with respect to a natural measure on the fractal proved to be rather more delicate. The first results in this direction are due to Einsiedler–Fishman–Shapira [**37**], who proved that badly approximable vectors have zero Hausdorff measure on certain fractals invariant under toral endomorphisms (in case the dimension is $m = 1$) or toral automorphisms (in case $m = 2$). For example, their results apply to the middle-third Cantor set. This was vastly generalized by Simmons–Weiss [**129**], who established the same statement for general self-similar fractals satisfying a separation condition. To the best of our knowledge, for general weights $\mathbf{r}$ or on fractals which are not strictly self-similar, the question of the measure of badly approximable vectors is open. Our methods allow us to make an initial contribution in this direction. For simplicity, here in the introduction we will describe only the special case of "Bedford–McMullen carpets"; see §4.8 for the discussion in full generality.

Bedford–McMullen carpets are two-dimensional self-affine fractals, introduced independently by Bedford [**2**] and McMullen [**89**], which admit a particularly simple construction. Let $a, b \ge 2$ be distinct integers and divide the unit square $[0, 1]^2$ into an $a \times b$-grid parallel to the coordinate axes. Choose an arbitrary subcollection S of the ab rectangles created and discard the remaining ones. Iteratively proceed in the same way for each of the rectangles that remain, using the same pattern S. The points remaining after infinite iteration form a *Bedford–McMullen carpet* $\mathcal{K}$. If $(c_i, d_i)_{i=1}^k$ denote the coordinates of the bottom-left corners of the rectangles kept in the first construction step and we define the affine maps $\phi_i\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\phi_i(x,y) = \begin{pmatrix} a^{-1} & \\ & b^{-1} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c_i \\ d_i \end{pmatrix},$$

then $\mathcal{K}$ is the unique non-empty compact subset of $\mathbb{R}^2$ with the property that $\bigcup_{i=1}^k \phi_i(\mathcal{K}) = \mathcal{K}$. The Hausdorff dimension of fractals of this type was explicity calculated by Bedford and McMullen. Except for special cases, it turns out that their Hausdorff measure in the correct dimension is infinite [**100**]. However, there exists another natural measure $\nu_{\mathcal{K}}$ on $\mathcal{K}$, known as the *McMullen measure*: It is the unique T-invariant ergodic probability measure on $\mathcal{K}$ of full Hausdorff dimension, where T is the toral endomorphism corresponding to $\left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$ [**68, 89**]. For further background on the fractal geometry of Bedford–McMullen carpets, we refer to the survey article [**46**].

The following is a specialization of our Theorem 4.8.3 to the case of weighted badly approximable vectors on Bedford–McMullen carpets (see Corollary 4.8.5).

Theorem 4.0.11. *Let a, b be positive integers with $\min(a^2, b^2) > \max(a,b)$ and let $\mathcal{K} \subset \mathbb{R}^2$ be a Bedford–McMullen carpet invariant under the toral endomorphism $T = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$. Suppose that $\mathcal{K}$ is not contained in any straight line. Then for the choice of weights*

$$\mathbf{r} = \Big(\frac{2\log a - \log b}{\log a + \log b}, \frac{2\log b - \log a}{\log a + \log b}\Big),$$

the set of $\mathbf{r}$-badly approximable vectors on $\mathcal{K}$ has measure zero with respect to the McMullen measure $\nu_{\mathcal{K}}$ on $\mathcal{K}$.

The requirement above that $\mathcal{K}$ is not contained in any straight line plays the role of an irreducibility condition. It is satisfied when, in the construction of the Bedford–McMullen carpet described above, the kept rectangles in the pattern S do not all belong to a single line or column in the $a \times b$-grid.

As mentioned before its statement, the above theorem will follow from a much more general result about Diophantine properties of "$(\mathbf{r}, \mathbf{s})$-matrix sponges" (Theorem 4.8.3)—a class of fractals that we will introduce in §4.8.2.3. In fact, the latter result will imply a version of Theorem 4.0.11 for higher-dimensional analogues of Bedford–McMullen carpets, which are called "self-affine Sierpiński sponges" in [**68**]; see Corollary 4.8.5.

4.1. H-Expansion: Definition and Basic Properties

We start by properly stating the definition of uniform expansion and giving alternative formulations thereof.

Definition 4.1.1. Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$. A vector v in $\mathbb{R}^d$ is said to be *μ-expanded* if

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|g_n \cdots g_1 v\| > 0 \tag{4.1.1}$$

for $\mu^{\otimes\mathbb{N}}$-almost every sequence $(g_i)_i$ of elements of $\mathrm{GL}_d(\mathbb{R})$. The measure μ is said to be *uniformly expanding* if every nonzero $v \in \mathbb{R}^d$ is μ-expanded. If (4.1.1) holds with $\geq$ in place of $>$ for every nonzero $v \in \mathbb{R}^d$, we call μ *non-contracting.*

The above definition is the most general, but it can be hard to verify in practice. The characterization in the following proposition is often simpler to check. Moreover, it will also play an important role in the height function constructions in §4.5. Recall that a probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ is said to have a finite first moment if $\int \log \mathrm{N}(g)\,\mathrm{d}\mu(g) < \infty$, where $\mathrm{N}(g) = \max(\|g\|, \|g^{-1}\|)$.

PROPOSITION 4.1.2 (Proposition 2.1.4, [**39**, Lemma 1.5]). *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment. Then μ is uniformly expanding if and only if there exists $N \in \mathbb{N}$ and a constant $C > 0$ such that for every nonzero $v \in \mathbb{R}^d$*

$$\int_{\mathrm{GL}_d(\mathbb{R})} \log \frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu^{*N}(g) \geq C.$$

Uniform expansion can also be conveniently understood in light of the following theorem of Furstenberg–Kifer and Hennion. Recall that given a probability measure μ on a Lie group G, we denote by Γ_μ the closed subgroup generated by the support of μ.

THEOREM 4.1.3 (Furstenberg–Kifer [**50**], Hennion [**62**]). *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment. Then there exists a partial flag $\mathbb{R}^d = F_1 \supset F_2 \supset \cdots \supset F_k \supset F_{k+1} = \{0\}$ of Γ_μ-invariant subspaces and a collection of real numbers $\beta_1(\mu) > \cdots > \beta_k(\mu)$ such that for every $v \in F_i \setminus F_{i+1}$, we have $\mu^{\otimes \mathbb{N}}$-a.s.*

$$\lim_{n\to\infty} \tfrac{1}{n} \log\|g_n \cdots g_1 v\| = \beta_i(\mu).$$

Moreover, the $\beta_i(\mu)$ are the values of

$$\alpha(\nu) := \int_{\mathbb{P}(\mathbb{R}^d)} \int_{\mathrm{GL}_d(\mathbb{R})} \log \frac{\|gv\|}{\|v\|}\,\mathrm{d}\mu(g)\,\mathrm{d}\nu(\mathbb{R}v)$$

that occur when ν ranges over μ-ergodic μ-stationary probability measures on the projective space $\mathbb{P}(\mathbb{R}^d)$.

In this result, the set of exponents $\{\beta_1(\mu), \ldots, \beta_k(\mu)\}$ is contained in the set of Lyapunov exponents of μ and $\beta_1(\mu)$ coincides with the top Lyapunov exponent.

Uniform expansion can now be rephrased as follows.

LEMMA 4.1.4. *A probability measure μ on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment is uniformly expanding if and only if $\beta_k(\mu) > 0$, where $\beta_k(\mu)$ is the smallest exponent appearing in Theorem 4.1.3.* □

Furstenberg–Kifer's theorem can also be used to see that, in fact, almost sure divergence is enough to get uniform expansion. It will be useful to denote by $F^{\leqslant 0}$ the largest subspace among $F_1, \ldots, F_{k+1}$ with non-positive exponent.

PROPOSITION 4.1.5. *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ with finite first moment. Then μ is uniformly expanding if and only if for every nonzero vector $v \in \mathbb{R}^d$ we have*

$$\lim_{n\to\infty} \|g_n \cdots g_1 v\| = \infty \tag{4.1.2}$$

for $\mu^{\otimes \mathbb{N}}$-a.e. sequence $(g_i)_i$ of elements of $\mathrm{GL}_d(\mathbb{R})$.

PROOF. We only need to show that (4.1.2) implies uniform expansion. We apply Theorem 4.1.3 and consider the space $F^{\leqslant 0}$ defined before the statement of the proposition. This space is Γ_μ-invariant. If it is nonzero, its projectivization thus supports an ergodic μ-stationary probability measure ν. Using the assumed almost sure divergence and Atkinson/Kesten's lemma (see e.g. [**18**, Lemma II.2.2]), it follows that $\alpha(\nu) > 0$, where $\alpha(\nu)$ is as defined in Theorem 4.1.3, a contradiction. □

For later use, let us also record at this point an immediate restriction that the presence of expansion puts on μ-stationary measures on finite-dimensional vector spaces.

LEMMA 4.1.6. *Let μ be a probability measure on $\mathrm{GL}_d(\mathbb{R})$ and E a measurable subset of $\mathbb{R}^d$ such that every $v \in E$ is μ-expanded. Then every μ-stationary probability measure ν on $\mathbb{R}^d$ satisfies $\nu(E) = 0$.*

In particular, if μ has a finite first moment, then any μ-stationary probability measure ν on $\mathbb{R}^d$ is supported on the Furstenberg–Kifer subspace $F^{\leqslant 0}$ of subexponential expansion. Together with a similar argument for vectors that are contracted instead of expanded, one can more generally show that $\nu((F^{\leqslant 0} \setminus F^{<0}) \cup \{0\}) = 1$, where $F^{<0}$ is defined in a way analogous to $F^{\leqslant 0}$.

PROOF. Write $G = \mathrm{GL}_d(\mathbb{R})$ and $V = \mathbb{R}^d$. By [**10**, Proposition 2.14], the forward dynamical system $(G^{\mathbb{N}} \times V, \mu^{\otimes \mathbb{N}} \otimes \nu, T^V)$ is measure-preserving, where

$$T^V((g_i)_i, v) = ((g_{i+1})_i, g_1 v).$$

Let K be a compact subset of V. Then by the Poincaré recurrence theorem applied to $G^{\mathbb{N}} \times K$, we know that $\nu(K \cap E) = 0$, and the conclusion follows. □

Now we come to the central concept of this article: H-expansion.

DEFINITION 4.1.7. Let H be a connected semisimple Lie group with finite center and μ a probability measure on H. Given a representation (V, ρ) of H we say that μ is *uniformly expanding in* (V, ρ) if $\rho_*\mu$ is uniformly expanding. We say that μ is *H-expanding* if μ is uniformly expanding in every representation of H without nonzero H-fixed vectors, or equivalently, in every non-trivial irreducible representation of H.

Here and everywhere else, by a "representation" we always mean a continuous homomorphism into the group of invertible linear transformations of a finite-dimensional real vector space. It is well known that such representations are automatically smooth. For notational simplicity, we are going to simply write $h \cdot v$ for $\rho(h)v$ for $h \in H$ and $v \in V$ when the representation (V, ρ) is clear from context. In this case, we also just say that μ is uniformly expanding on V to mean that μ is uniformly expanding in (V, ρ).

We next explain what the moment conditions mean for a probability measure on a semisimple group that is not necessarily linear.

DEFINITION 4.1.8. Let H be a connected semisimple Lie group with finite center. Let μ be a probability measure on H. Then μ is said to have a *finite first moment* (resp. *finite exponential moments*) if $\rho_*\mu$ has a finite first moment (resp. finite exponential moments) for some representation ρ of H with finite kernel.

Of course, these moment conditions are automatically satisfied when μ has compact support.

LEMMA 4.1.9 ([**10**, Lemmas 10.6, 10.7]). *Let H and μ be as in Definition 4.1.8 and suppose that μ has a finite first moment (resp. finite exponential moments). Then $\rho_*\mu$ has a finite first moment (resp. finite exponential moments) for any representation ρ of H.*

We remark that even though in [**10**], the above lemma is proved for algebraic groups, the given proof also works for a connected semisimple group H with finite center. Indeed, the argument relies only on a reformulation of the moment condition into an integrability condition on the Cartan projection $\kappa\colon H \to \mathfrak{a}^+$, which is related to representations of H by the formula $\|\rho(h)\| = \mathrm{e}^{\chi(\kappa(h))}$ for $h \in H$, where (V, ρ) is an irreducible representation of H with highest weight χ and $\|\cdot\|$ is the operator norm associated to a Euclidean norm on V invariant under the maximal compact subgroup K of H used to define κ.

In the proposition below we collect some first facts about H-expansion.

PROPOSITION 4.1.10. *Let H be a connected semisimple Lie group with finite center and μ a probability measure on H. Then:*

(i) *Given a representation (V, ρ) of H, the following are equivalent:*
   - *Any vector $v \in V$ that is not $\rho_*\mu$-expanded is H-fixed.*
   - *The measure μ is uniformly expanding on the quotient V/V^H.*

(ii) *If μ is H-expanding, then H has no compact factors.*

(iii) *If μ is H-expanding and $\psi\colon H \to G'$ is a non-trivial continuous homomorphism into a real Lie group G', then $H' = \psi(H)$ is a connected, closed, semisimple subgroup of G' with finite center and $\psi_*\mu$ is H'-expanding.*

(iv) *Suppose H is an almost direct product of connected normal subgroups H_1 and H_2 and let μ_i be probability measures on H_i with finite first moments, $i = 1, 2$. If μ_i is H_i-expanding for $i = 1, 2$ and μ is the pushforward of $\mu_1 \otimes \mu_2$ by multiplication, then μ is H-expanding.*

PROOF. For (i), note that by semisimplicity of H, the quotient V/V^H identifies with an H-invariant complement V^+ of V^H in V. Thus we only need to prove that uniform expansion of μ on V^+ implies the statement in the first bullet point. Let $p_+\colon V \to V^+$ be the projection and take $v \in V$ which is not $\rho_*\mu$-expanded. Then also $p_+(v)$ is not $\rho_*\mu$-expanded, so that uniform expansion on V^+ implies $p_+(v) = 0$. Hence, v is H-fixed.

For (ii), suppose H has a compact factor K. Then μ cannot be uniformly expanding in the representation of H obtained by composing the projection onto K with the adjoint representation of K. Thus, μ is not H-expanding.

As H is semisimple and has finite center, H' is a connected and semisimple immersed Lie subgroup of G' with finite center in the setting of (iii). As representations of H' induce representations of H by precomposition with ψ, the H'-expansion condition is immediate. It only remains to argue that H' is closed in G'. As this is in fact a more general statement, we drop the accents and simply show that a semisimple immersed Lie subgroup H of a Lie group G must be closed when H has finite center. For this, it suffices to show that if a sequence $(h_n)_n$ in H converges to the identity e in the topology of G, then this convergence holds also in the topology of H. Notice that $\mathrm{Ad}_G(h_n)$ considered as elements of $\mathrm{Aut}(\mathfrak{h})$ converges to the identity map when $\mathrm{Aut}(\mathfrak{h})$ is endowed with the subspace topology inherited from $\mathrm{Aut}(\mathfrak{g})$. However, as linear semisimple Lie algebras are algebraic (see [**64**, Theorem VIII.3.2]), this subspace topology coincides with the usual topology of $\mathrm{Aut}(\mathfrak{h})$. Since near the identity, Ad_H is a local isomorphism from H to $\mathrm{Aut}(\mathfrak{h})$, we thus find a sequence $(h'_n)_n$ converging to e in H such that $\mathrm{Ad}_H(h_n) = \mathrm{Ad}_H(h'_n)$ for all n. This implies that $h_n^{-1}h'_n$ is contained in the center of H and converges to e. As the center is finite, we have $h_n = h'_n$ for all n large enough. We conclude that, indeed, $h_n \to e$ as $n \to \infty$ holds also in the topology of H.

Finally, to prove (iv), let (V, ρ) be a non-trivial irreducible representation of H. Since H_1 and H_2 commute, for every $n \in \mathbb{N}$, μ^{*n} is the pushforward by multiplication of $\mu_1^{*n} \otimes \mu_2^{*n}$, and the subspaces V^{H_i} of H_i-fixed vectors in V are H-invariant. By irreducibility, they are trivial or all of V. It follows that one of V^{H_1}, V^{H_2} is zero. We assume without loss of generality that $V^{H_1} = \{0\}$.

Note that both $\rho_*\mu_1$ and $\rho_*\mu_2$ have a finite first moment by Lemma 4.1.9. This readily implies that $\rho_*\mu$ has a finite first moment. By Proposition 4.1.2, it suffices to show that for N large enough and $v \neq 0$, the quantity

$$\begin{aligned}
&\int_{H_1\times H_2} \log \frac{\|h_1h_2 \cdot v\|}{\|v\|} \, \mathrm{d}\mu_1^{*N}(h_1) \, \mathrm{d}\mu_2^{*N}(h_2) \\
&\qquad = \int_{H_2}\int_{H_1} \log \frac{\|h_1h_2 \cdot v\|}{\|h_2 \cdot v\|} \, \mathrm{d}\mu_1^{*N}(h_1) \, \mathrm{d}\mu_2^{*N}(h_2) + \int_{H_2} \log \frac{\|h_2 \cdot v\|}{\|v\|} \, \mathrm{d}\mu_2^{*N}(h_2)
\end{aligned}$$

is uniformly bounded from below by some $C > 0$. As $\rho_*\mu_1$ is uniformly expanding, Proposition 4.1.2 gives this lower bound for the first integral above for N large enough. By the same argument, the second term is either equal to 0 or also bounded below by some $C > 0$, according to whether V^{H_2} is V or $\{0\}$, respectively. $\square$

Remark 4.1.11. We point out that in part (iii) of the previous proposition, if the target G' of the homomorphism ψ is a real algebraic group, then the conclusion can be strengthened to the statement that the semisimple group $H' = \psi(H)$ is *almost algebraic*, meaning that it has finite index in a real algebraic subgroup of G'. Indeed, as already exploited in the proof above, the point is that linear semisimple Lie algebras are algebraic. In particular, this applies when ψ is a representation (V, ρ) of H. This fact is useful to keep in mind. $\diamond$

Combining Proposition 4.1.10(i) with Lemma 4.1.6, we immediately obtain the following corollary about μ-stationary measures on vector spaces.

Corollary 4.1.12. *Let (V, ρ) be a representation of H and suppose that μ is uniformly expanding on V/V^H. Then any μ-stationary probability measure on V is supported on the subspace V^H of H-fixed vectors.* $\square$

4.2. Examples of H-Expanding Measures

In this section, we exhibit classes of probability measures on semisimple Lie groups that satisfy the H-expansion property.

4.2.1. Zariski Dense Measures. As already mentioned in the introduction to this chapter, the first class of examples of H-expanding measures consists of those whose support generates a Zariski dense subgroup of H. This is the class of measures considered by Benoist–Quint [**5, 8, 9**].

PROPOSITION 4.2.1. *Let H be a connected semisimple Lie group without compact factors and with finite center. Let μ be a probability measure on H with finite first moment. Suppose that* $\mathrm{Ad}(\Gamma_\mu)$ *is Zariski dense in* $\mathrm{Ad}(H)$. *Then μ is H-expanding.*

For the proof we need the following lemma, which is used to extend the Zariski density assumption to arbitrary representations.

LEMMA 4.2.2. *Let Γ be a subsemigroup of H and S a connected subgroup of H. Suppose that the Zariski closure of* $\mathrm{Ad}(\Gamma)$ *contains* $\mathrm{Ad}(S)$. *Then for every representation (V,ρ) of H, $\rho(S)$ is contained in* $\mathrm{Zcl}(\rho(\Gamma))$.

PROOF. We consider the product representation $\rho' = \mathrm{Ad}\times\rho$. Let $\mathcal{H}'$ be the Zariski closure of $\rho'(H)$ inside $\mathrm{GL}(\mathfrak{h})\times\mathrm{GL}(V)$. Then both Ad and ρ factor through $\mathcal{H}'$. As noted in Remark 4.1.11, $\rho'(H)$ has finite index in $\mathcal{H}'$. The same holds for the Zariski closure $\mathcal{H}$ of $\mathrm{Ad}(H)$, so that both $\mathcal{H}$ and $\mathcal{H}'$ are Zariski connected real algebraic groups of dimension $\dim(H)$. Thus, projection to the first factor of $\mathrm{GL}(\mathfrak{h})\times\mathrm{GL}(V)$ gives an isogeny $p\colon \mathcal{H}'\to\mathcal{H}$, and we know that $\mathrm{Zcl}(\rho'(\Gamma))$ has finite index in $p^{-1}(\mathrm{Zcl}(\mathrm{Ad}(\Gamma)))$. Since $\rho'(S)$ is connected and $\mathrm{Ad}(S)$ is contained in $\mathrm{Zcl}(\mathrm{Ad}(\Gamma))$ by assumption, it follows that $\rho'(S)$ is contained in $\mathrm{Zcl}(\rho'(\Gamma))$. By projecting to the second factor, we conclude that $\rho(S)$ is contained in $\mathrm{Zcl}(\rho(\Gamma))$. □

PROOF OF PROPOSITION 4.2.1. Let (V,ρ) be a representation of H without nonzero H-fixed vectors. By Lemma 4.2.2, $\rho(\Gamma_\mu)$ is Zariski dense in $\rho(H)$. Now uniform expansion in (V,ρ) follows directly from Furstenberg's theorem on positivity of the top Lyapunov exponent (see [**51**, Theorem 8.6]). To see that the assumptions of Furstenberg's theorem are satisfied, note that by Lemma 4.1.9 we know that $\rho_*\mu$ has a finite first moment, and using Zariski density of $\rho(\Gamma_\mu)$ together with complete reducibility one may assume that $\rho(\Gamma_\mu)$ acts irreducibly, which implies strong irreducibility in view of Zariski connectedness of $\rho(H)$. Finally, since the ground field is $\mathbb{R}$, the fact that the Zariski closure of $\rho(\Gamma_\mu)$ is non-compact implies that $\rho(\Gamma_\mu)$ is not relatively compact, finishing the proof. □

4.2.2. Measures on Parabolic Groups. Our next goal is to exhibit probability measures supported on proper parabolic subgroups of H which are H-expanding. Combining general criteria with the notion of the expanding cone, which was introduced by Shi in [**127**] and which traces back to the works of Shah and Weiss [**123, 124, 133**], we will obtain another easy-to-verify sufficient condition for H-expansion.

We start by explaining our general setup. Let H be a connected semisimple real Lie group without compact factors and with finite center and let a be an Ad-diagonalizable element of H. Then given a representation (V,ρ) of H, we have a direct sum decomposition

$$V = V_a^+ \oplus V_a^0 \oplus V_a^-,$$

where V_a^+, V_a^0, V_a^- are the sums of the eigenspaces of $\rho(a)$ with eigenvalues $>$, $=$ or $<$ 1, respectively. Let U be a connected Ad-unipotent subgroup of H normalized by a. Following [**126**], we say that U is *a-expanding* if for every non-trivial irreducible representation (V, ρ) of H, the subspace V^U of U-fixed vectors is contained in V_a^+. It is equivalent ([**126**, Lemma A.1]) to require that in any irreducible representation of (V, ρ) of H and for any nonzero vector $v \in V$, the $\rho(U)$-orbit of v is not contained in $V_a^0 \oplus V_a^-$. For example, if a has a non-trivial projection to every simple factor of H, then the unstable horospherical subgroup $H_a^+ = \{h \in H \mid \lim_{n\to\infty} a^{-n} h a^n = 1_H\}$ is a-expanding ([**123**, Lemma 5.2]). In fact, it can be shown that U is a-expanding if and only if $U \cap H_a^+$ is ([**126**, Lemma A.2]).

Now let $Q \leqslant H$ be a parabolic subgroup with maximal connected $\mathbb{R}$-split torus A. Using the above, we will give two criteria for a measure on Q to be H-expanding. To state the first, write $Q = MA_cN$ for the Langlands decomposition of Q. In particular, this means that N is the unipotent radical of Q, $MA_c = C_H(A_c)$ is a (reductive) Levi subgroup of Q, and A_c is a maximal central connected $\mathbb{R}$-split torus in MA_c (see e.g. [**76**, §VII.7]). We may assume that $A_c \leqslant A$. Given a probability measure μ on Q, by using the diffeomorphism $Q \cong M \times A_c \times N$ given by multiplication and projecting to some of the factors, we obtain associated probability measures μ_M, μ_{A_c}, μ_{MA_c} etc. Finally, we denote by $\lambda_c \colon Q \to \mathfrak{a}$ the composition of the projection to A_c with the logarithm map $\log\colon A \to \mathfrak{a}$, where $\mathfrak{a}$ is the Lie algebra of A.

Proposition 4.2.3 (H-expanding measures (1)). *Let μ be a probability measure on H with finite first moment such that $\mu(Q) = 1$ for some parabolic subgroup $Q = MA_cN$ of H. Denote by $a_{c,\mathrm{avg}}(\mu) = \exp(\int \lambda_c(g)\, \mathrm{d}\mu(g)) \in A_c$ the A_c-average of μ. Let U be a connected Lie subgroup N and suppose the following:*

(1) $\mathrm{supp}(\mu) \subset MA_cU \cap N_H(U)$ *and the Zariski closure of* $\mathrm{Ad}(\Gamma_\mu)$ *contains* $\mathrm{Ad}(U)$,
(2) *U is $a_{c,\mathrm{avg}}(\mu)$-expanding, and*
(3) *μ_M is non-contracting in every representation of H.*

Then μ is H-expanding.

Before proceeding with the preparations for the proof of the above proposition, let us provide a few brief comments on its hypotheses.

Remark 4.2.4 (On the hypotheses of Proposition 4.2.3).

- In fact, there is no freedom in the choice of U: Condition (1) implies that it needs to be the Zariski closure of the projection of Γ_μ to N.
- When $U = N$ and the parabolic group Q is absolutely proper, condition (2) can conveniently be checked using the notion of expanding cone to be discussed in §4.2.2.1.
- The non-contraction requirement on μ_M in condition (3) is satisfied, for instance, when the identity component of the Zariski closure of $\mathrm{Ad}(\Gamma_{\mu_M})$ is reductive with compact center (for example, the identity component of $\mathrm{Ad}(M)$ itself). Indeed, in this case similar arguments as in the proof of Lemma 4.2.2 can be used to show that Γ_{μ_M} acts completely reducibly and by transformations of determinant ± 1 in

every representation (V,ρ) of H. Then the Lyapunov exponents of μ_M in any Γ_{μ_M}-irreducible subspace of V sum to 0 and one concludes using Theorem 4.1.3.

- Another useful fact for the verification of condition (3) is that the connected component M° of M is the almost direct product of its semisimple part $S = [M^\circ, M^\circ]$ and a compact center. Provided μ_M is supported on M°, one can thus project to the non-compact part S^{nc} and is only left checking non-contraction for $\mu_{S^{nc}}$. The latter could follow from Zariski density (Proposition 4.2.1), or by a recursive application of Proposition 4.2.3 above to $H = S^{nc}$. In the general case, one can obtain from μ_M a probability measure μ_M° on M° defined as the law of the first return to M° of the random walk on M induced by μ_M; see [**10**, §5.2]. Using [**10**, Proposition 5.9] and Theorem 4.1.3, one sees that the non-contraction property of μ_M° implies that of μ_M.

$\diamond$

For the proof of Proposition 4.2.3 we require the following lemma, which reduces checking expansion to vectors fixed by some unipotent subgroup of the image of the algebraic group generated by $\mathrm{supp}(\mu)$.

LEMMA 4.2.5 (A criterion for expansion). *Let V be a finite-dimensional real vector space and μ' a probability measure on $\mathrm{GL}(V)$ with finite first moment. Denote by Q' the Zariski closure of $\Gamma_{\mu'}$ and let U' be a unipotent subgroup of Q'. Suppose that every nonzero vector $v \in V^{U'}$ is μ'-expanded, where $V^{U'}$ denotes the subspace of U'-fixed vectors. Then μ' is uniformly expanding.*

PROOF. Suppose for a contradiction that μ' is not uniformly expanding. Then there exists a vector $v \in V \setminus \{0\}$ with $\liminf_{n\to\infty} \frac{1}{n}\log\|g_n \cdots g_1 v\| \le 0$ for a positive measure subset of $(g_i)_i \in (Q')^{\mathbb{N}}$ with respect to $(\mu')^{\otimes\mathbb{N}}$. By Theorem 4.1.3, there exists a non-trivial $\Gamma_{\mu'}$-invariant subspace $W \leqslant V$ such that for every $w \in W$, we have $\lim_{n\to\infty} \frac{1}{n}\log\|g_n \cdots g_1 w\| \le 0$ for $(\mu')^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in (Q')^{\mathbb{N}}$. Since Q' is the Zariski closure of $\Gamma_{\mu'}$, the subspace W is stabilized by Q' and hence, by U'. By the Lie–Kolchin theorem, we have $W^{U'} \ne \{0\}$. This implies that for any nonzero $w \in W^{U'} \leqslant V^{U'}$, we have $\lim_{n\to\infty} \frac{1}{n}\log\|g_n \cdots g_1 w\| \le 0$ for $(\mu')^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in (Q')^{\mathbb{N}}$, contradicting expansion on $V^{U'}$. $\square$

PROOF OF PROPOSITION 4.2.3. Let (V,ρ) be a non-trivial irreducible representation of H. By Lemma 4.1.9, the measure $\rho_*\mu$ has a finite first moment, and Lemma 4.2.2 implies that $\rho(U)$ is a unipotent subgroup of the Zariski closure of $\rho(\Gamma_\mu)$. In view of Lemma 4.2.5, to prove uniform expansion of $\rho_*\mu$ it suffices to show that for every nonzero $v \in V^U$, we have

$$\liminf_{n\to\infty} \tfrac{1}{n}\log\|g_n \cdots g_1 \cdot v\| > 0$$

for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in H^{\mathbb{N}}$. Since condition (1) implies that $\Gamma_\mu \subset MA_cU$ and v is U-fixed, it suffices to prove the above for μ_{MA_c}-a.e. $(g_i)_i \in H^{\mathbb{N}}$, where μ_{MA_c} is the MA_c-projection of μ. Writing $g_i = m_i a_i$ for the MA_c-factorization of

$g_i \in MA_c$ and using that M and A_c commute, we see that

$$\tfrac{1}{n}\log\|g_n\cdots g_1\cdot v\| = \tfrac{1}{n}\log\frac{\|a_n\cdots a_1 m_n\cdots m_1\cdot v\|}{\|m_n\cdots m_1\cdot v\|} + \tfrac{1}{n}\log\|m_n\cdots m_1\cdot v\|. \tag{4.2.1}$$

The second term above is almost surely non-negative in the limit, by the assumed non-contraction property of μ_M.

To deal with the first term, let $\Phi(A_c,\rho)$ be the set of weights of A_c for the representation (V,ρ). Let $\{\chi_1,\dots,\chi_t\}$ be the subcollection of those weights χ in $\Phi(A_c,\rho)$ with $\chi(a_{c,\mathrm{avg}}(\mu)) > 1$ and denote the corresponding weight spaces by $V_1,\dots,V_t$. Then by the assumption on U, we have $V^U \subset \bigoplus_{j=1}^t V_j =: W$. Since A_c and M commute, W is M-invariant. Lemma 4.2.6 below applied to the space W and $\mu' = \mu_{A_c}$ with $v_n = m_n\cdots m_1\cdot v$ thus implies that the first term in (4.2.1) has strictly positive limit inferior $\mu_{MA_c}^{\otimes\mathbb{N}}$-almost surely. This finishes the proof. □

Lemma 4.2.6. *Let V be a finite-dimensional real vector space and A' a closed connected diagonalizable subgroup of* $\mathrm{GL}(V)$ *with Lie algebra $\mathfrak{a}$. Write $V = \bigoplus_{\chi\in\Phi(A')} V^\chi$ for the weight space decomposition of V with respect to A', where $V^\chi = \{v \in V \mid av = \chi(a)v \text{ for all } a \in A'\}$ and $\Phi(A')$ is the set of characters χ of A' such that $V^\chi \neq \{0\}$. Let μ' be a probability measure on A' with finite first moment and denote $a_{\mathrm{avg}}(\mu') = \exp(\int \log(a)\,\mathrm{d}\mu'(a))$. Suppose that $\chi(a_{\mathrm{avg}}(\mu')) > 1$ for every $\chi \in \Phi(A')$. Then for $(\mu')^{\otimes\mathbb{N}}$-a.e. $(a_i)_i \in (A')^{\mathbb{N}}$ we have*

$$\liminf_{n\to\infty} \tfrac{1}{n}\log\frac{\|a_n\cdots a_1 v_n\|}{\|v_n\|} > 0$$

for every choice of nonzero vectors $v_n \in V$.

Proof. For convenience, we assume the norm $\|\cdot\|$ on V is Euclidean and that the distinct weight spaces are orthogonal. Given a nonzero $v \in V$, write $v = \sum_{\chi\in\Phi(A')} v^\chi(v)$ for the corresponding weight space decomposition, where $v^\chi(v) \in V^\chi$. Then for any $a_1,\dots,a_n \in A'$ and nonzero $v_n \in V$ we have

$$a_n\cdots a_1 v_n = \sum_{\chi\in\Phi(A')} \chi(a_n\cdots a_1) v^\chi(v_n).$$

Choosing for every $n \in \mathbb{N}$ a character χ_n with $\|v^{\chi_n}(v_n)\| \geq \dim(V)^{-1/2}\|v_n\|$ and recalling that $\chi(a_{\mathrm{avg}}(\mu')) > 1$ for all $\chi \in \Phi(A')$ by assumption, we conclude that

$$\begin{aligned}\tfrac{1}{n}\log\frac{\|a_n\cdots a_1 v_n\|}{\|v_n\|} &\geq o(1) + \tfrac{1}{n}\log\chi_n(a_n\cdots a_1) \geq o(1) + \min_{\chi\in\Phi(A')}\frac{1}{n}\sum_{i=1}^n \log\chi(a_i)\\ &\overset{n\to\infty}{\longrightarrow} \min_{\chi\in\Phi(A')}\log\chi(a_{\mathrm{avg}}(\mu')) > 0,\end{aligned}$$

where the last convergence holds $(\mu')^{\otimes\mathbb{N}}$-almost surely by the classical law of large numbers. □

One drawback of Proposition 4.2.3 is that, in some sense, it requires the M- and A_c-parts of μ to both exhibit expansion (or at least non-contraction) individually. It would be natural to only ask the combination of both to be expanding, a behavior which should be reflected in the A-average of μ. When

μ does not charge M in a too complicated way, we can also prove H-expansion in this case.

To state this second criterion, let $U \leqslant H$ be any connected Ad-unipotent subgroup. Then there exists a parabolic subgroup Q of H containing U in its unipotent radical such that also $N_H(U) \leqslant Q$ [**17**]. As before, let $A \leqslant Q$ be a maximal $\mathbb{R}$-split torus and denote by K a maximal compact subgroup of Q. Given a non-trivial subtorus $A' \leqslant A$ normalizing U, set $K' = C_K(A') \cap N_H(U)$ and let P be the closed subgroup $K'A'U$ of Q. We write $\lambda\colon P \to \mathfrak{a}$ for the morphism given by $\lambda(kau) = \log a$.

Proposition 4.2.7 (H-expanding measures (2)). *Retain the notation from the paragraph above and let μ be a probability measure on H with finite first moment such that $\mu(P) = 1$. Denote by $a_{\mathrm{avg}}(\mu) = \exp(\int \lambda(g)\,\mathrm{d}\mu(g)) \in A$ the A-average of μ. Suppose that:*

1. *The Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ contains $\mathrm{Ad}(U)$, and*
2. *U is $a_{\mathrm{avg}}(\mu)$-expanding.*

Then μ is H-expanding.

We emphasize that, in contrast to Proposition 4.2.3, here the A-average is considering also the part of the torus A inside M, if $Q = MA_cN$ is the Langlands decomposition of Q.

Proof. Exactly as in the proof of Proposition 4.2.3, given a non-trivial irreducible representation (V,ρ) of H, it suffices to prove that

$$\liminf_{n\to\infty} \tfrac{1}{n} \log\|g_n \cdots g_1 \cdot v\| > 0$$

for $\mu_{K'A'}^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in H^{\mathbb{N}}$ and every $v \in V^U$, where $\mu_{K'A'}$ is the pushforward of μ by the map $K'A'U \to K'A'$, $kau \mapsto ka$. As K' is compact and commutes with A', we can disregard the K'-component and consider only $\mu_{A'}$, defined in the analogous way. Now the statement follows from Lemma 4.2.6. $\square$

4.2.2.1. *Expanding Cone.* Now we present a construction which can be used to ensure the expansion condition on U with respect to the A- or A_c-average of μ in the criteria above (condition (2) in Propositions 4.2.3 and 4.2.7) in the case that U is the unipotent radical of an absolutely proper parabolic subgroup Q of H, where "absolutely proper" means that the projection of Q to each simple factor of H is non-surjective. As before, we let A be a maximal connected $\mathbb{R}$-split torus of Q.

The *expanding cone* of U in A is defined to be

$$A_U^+ = \{a \in A \mid U \text{ is } a\text{-expanding}\}.$$

It is proved in [**127**, Theorem 1.2] that A_U^+ only depends on the Lie algebras $\mathfrak{h} \coloneqq \mathrm{Lie}(H)$ and $\mathfrak{u} \coloneqq \mathrm{Lie}(U)$, and that it can be described explicitly as follows. Let $\mathfrak{a}$ be the Lie algebra of A and let $\Sigma(\mathfrak{h},\mathfrak{a}) \subset \mathfrak{a}^* \coloneqq \mathrm{Hom}(\mathfrak{a},\mathbb{R})$ be the restricted root system of $(\mathfrak{h},\mathfrak{a})$. Denote by $\Sigma(\mathfrak{u}) \subset \Sigma(\mathfrak{h},\mathfrak{a})$ the subset of roots whose eigenvectors lie in $\mathfrak{u}$. By semisimplicity, the Killing form $\langle\cdot,\cdot\rangle$ of $\mathfrak{h}$ is positive definite on $\mathfrak{a}$. So for each $\alpha \in \mathfrak{a}^*$ we can associate $s_\alpha \in \mathfrak{a}$ by $\langle s_\alpha, v\rangle = \alpha(v)$ for every $v \in \mathfrak{a}$. Using this isomorphism, we associate to $\Sigma(\mathfrak{u})$ the following convex cone in $\mathfrak{a}$:

$$\mathfrak{a}_{\mathfrak{u}}^{+} := \Big\{ \sum_{\alpha \in \Sigma(\mathfrak{u})} t_\alpha s_\alpha \,\Big|\, t_\alpha > 0 \Big\}.$$

The expanding cone A_U^+ of U is then given by $A_U^+ = \exp \mathfrak{a}_{\mathfrak{u}}^{+}$. By abuse of language, we shall sometimes also refer to $\mathfrak{a}_{\mathfrak{u}}^{+}$ as the expanding cone of U.

Using these notions, we get the following immediate corollary of Proposition 4.2.7.

Corollary 4.2.8. *Let U be the unipotent radical of an absolutely proper parabolic subgroup Q of H, $A \leqslant Q$ a maximal connected $\mathbb{R}$-split torus and $A' \leqslant A$ a non-trivial subtorus. Moreover, let K be a maximal compact subgroup of H, $K' = C_K(A') \cap Q$, set $P = K'A'U$ and let μ be a probability measure on H with finite first moment such that $\mu(P) = 1$. Suppose that the Zariski closure of* $\mathrm{Ad}(\Gamma_\mu)$ *contains* $\mathrm{Ad}(U)$ *and that* $\int \lambda(g)\,\mathrm{d}\mu(g) \in \mathfrak{a}_{\mathfrak{u}}^{+}$. *Then μ is H-expanding.* □

4.2.2.2. *Explicit Examples.* We end this subsection by giving two explicit examples where the criteria developed so far are applicable.

The first of them is the prototypical example of an expanding cone. Although simple, it turns out to be of significant importance to Diophantine approximation problems on fractals. We will take up this point and elaborate on the connection in §4.8.

Example 4.2.9. Let $H = \mathrm{SL}_{m+n}(\mathbb{R})$, and

$$Q = \Big\{ \begin{pmatrix} p_{11} & p_{12} \\ 0 & p_{22} \end{pmatrix} \in H \,\Big|\, p_{11} \in \mathrm{GL}_m(\mathbb{R}), p_{22} \in \mathrm{GL}_n(\mathbb{R}), p_{12} \in \mathbb{R}^{m \times n} \Big\},$$

$$U = \Big\{ \begin{pmatrix} \mathbf{1}_m & p_{12} \\ 0 & \mathbf{1}_n \end{pmatrix} \in H \,\Big|\, p_{12} \in \mathbb{R}^{m \times n} \Big\},$$

where we denote by $\mathbf{1}_d$ the $d \times d$-identity matrix. The group A consists of diagonal matrices in H with positive entries, and we have

$$A_U^+ = \{\mathrm{diag}(\mathrm{e}^{r_1}, \dots, \mathrm{e}^{r_m}, \mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_n}) \in H \mid r_i, s_j > 0\}$$

(see [**127**, Example 1.1]).

For concreteness, we exemplify a class of H-expanding measures on Q: Fix a Borel subset B_U of U not contained in a proper vector subspace of $U \cong \mathbb{R}^{mn}$. For example, B_U can be taken to be a non-degenerate curve in U or a collection of $k \geq mn$ points in $U \cong \mathbb{R}^{mn}$ that linearly spans U. Let μ be a compactly supported probability measure on AU such that

- its support contains an element of A_U^+,
- the set of unipotent parts u_g of elements $g = a_g u_g$ in $\mathrm{supp}(\mu) \subset AU$ contains B_U, and
- its A-average lies in the expanding cone of U, i.e. $\int \lambda(g)\,\mathrm{d}\mu(g) \in \mathfrak{a}_{\mathfrak{u}}^{+}$.

Then μ can be seen to be H-expanding by Corollary 4.2.8. Indeed, as we will see in §4.8 on Diophantine approximation on fractals, the first two points above imply that the Zariski closure of Γ_μ contains U (see the proof of Theorem 4.8.3). ◇

Note that the above example covers in particular Example 2.1.8 in Chapter 2. We also point out that, in Example 4.2.9, the assumption that $\mathrm{supp}(\mu)$ contains an element of A_U^+ is not strictly necessary. The first two bullet points could be replaced by a certain "irreducibility condition" of an affine action of the group generated by the support of μ (which is what we will do in §4.8), or, alternatively, by the assumption that the commutator group $[\Gamma_\mu, \Gamma_\mu]$ is Zariski dense in U.

The second example is one where the reductive group M in the Langlands decomposition of Q contributes to expansion in a non-trivial way.

EXAMPLE 4.2.10. Let Q be the standard parabolic subgroup of $\mathrm{SL}_4(\mathbb{R})$ given by

$$Q = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & & * \end{pmatrix} \leqslant \mathrm{SL}_4(\mathbb{R}).$$

The maximal connected $\mathbb{R}$-split torus A consists of diagonal matrices with positive entries. In the Langlands decomposition $Q = MA_cN$ we have

$$A_c = \left\{ d_{\alpha,\beta} \coloneqq \mathrm{diag}((\alpha\beta)^{-1/2}, (\alpha\beta)^{-1/2}, \alpha, \beta) \,\middle|\, \alpha, \beta > 0 \right\},$$

$$M = \begin{pmatrix} \mathrm{SL}_2(\mathbb{R}) & \\ & \mathbf{1}_2 \end{pmatrix}, \text{ and } N = \begin{pmatrix} 1 & & * & * \\ & 1 & * & * \\ & & 1 & * \\ & & & 1 \end{pmatrix}.$$

Using the explicit description of the expanding cone in §4.2.2.1, one can calculate directly that the intersection of the expanding cone of $U = N$ in A with A_c is given by

$$A_c \cap A_U^+ = \{ d_{\alpha,\beta} \mid \beta < 1, \alpha\beta < 1 \}.$$

For $i, j \in \{1, 2, 3, 4\}$ let $u_{i,j}$ be the unipotent element whose only nonzero off-diagonal term is 1 at the (i, j)-entry. Let $g = \left(\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right)$ and consider the element s of Q given by the block diagonal matrix $s = (g, \mathbf{1}_2)$. Now let μ be any compactly supported probability measure on Q whose support is given by the union of $\{s, s^\top, u_{2,3}, u_{3,4}\}$ and some diagonal matrices $d_{\alpha,\beta}$ in Q. It is not difficult to see that $U \leqslant \mathrm{Zcl}(\Gamma_\mu)$ and the A_c-part μ_{A_c} of μ consists of the latter diagonal matrices. Moreover, M is semisimple and the M-part of μ is Zariski dense in M. So, in view of Propositions 4.2.3 and 4.2.1, provided that the integral $\int (\log \alpha, \log \beta) \,\mathrm{d}\mu_{A_c}(d_{\alpha,\beta})$ is in the cone in $\mathbb{R}^2$ defined by the inequalities $y < 0$ and $x + y < 0$, the measure μ is $\mathrm{SL}_4(\mathbb{R})$-expanding. $\diamond$

4.2.3. Split Solvable Epimorphic Subgroups. The goal of this part is to discuss a further class of H-expanding measures. They will be supported on solvable epimorphic subgroups $F = A'U$ of semisimple real algebraic groups H, where A' is a one-dimensional algebraic $\mathbb{R}$-split torus and U is unipotent. The arguments rely on Proposition 4.2.7, ideas going back to Weiss [**133**] and Shah–Weiss [**124**], and the work of Bien–Borel [**14, 16**].

We start with a brief discussion of epimorphic subgroups, which have close connections to the notion of H-expanding measures.

4.2.3.1. *Epimorphic Subgroups.* The concept of epimorphic subgroups of algebraic groups was introduced by Bien–Borel [**14, 15**]. In [**127**], this notion was adapted to subgroups of connected semisimple Lie groups without compact factors.

Definition 4.2.11. A subgroup F of H is said to be *epimorphic* in H if for every representation of H, the vectors fixed by F are also fixed by H.

It can be shown that if H is almost algebraic in the sense of Remark 4.1.11 and $F \leqslant H$ is a connected Lie subgroup or a Zariski connected algebraic subgroup, it suffices to check the epimorphic property of F in real algebraic representations of H (see Proposition A.3). Consequently, in the algebraic category the above definition coincides with that of Bien–Borel. Moreover, it follows that a connected Lie subgroup F is epimorphic in H if and only if its Zariski closure $\mathrm{Zcl}(F)$ is.

Mozes [**93**] proved that an F-invariant probability measure on G/Λ is already invariant under H (and thus homogeneous by Ratner's theorem) in the case where all of F, H, G are real algebraic groups. This measure rigidity result was later generalized by Shah–Weiss [**124**, Theorem 1.8] to actions of connected epimorphic Lie subgroups which are not necessarily algebraic.

Examples of epimorphic subgroups include parabolic subgroups of H and Zariski dense subgroups, in case H is almost algebraic. One may notice that these classes of subgroups also prominently featured in the previous parts of this section, where we gave our first examples of H-expanding measures. That this is not a coincidence becomes clear with the following observation.

Proposition 4.2.12. *If μ is H-expanding, then the closed subgroup Γ_μ generated by the support of μ is epimorphic in H.*

Proof. In any given representation (V, ρ) of H, a Γ_μ-fixed vector $v \in V$ cannot be $\rho_*\mu$-expanded. In view of Proposition 4.1.10(i), it follows that v is H-fixed. $\square$

On the other hand, there exist connected epimorphic subgroups of H which do not support any H-expanding probability measure.

Example 4.2.13. We take $H = \mathrm{SL}_3(\mathbb{R})$,

$$A' = \left\{\mathrm{diag}(\mathrm{e}^t, \mathrm{e}^{-\sqrt{2}t}, \mathrm{e}^{(\sqrt{2}-1)t}) \,\middle|\, t \in \mathbb{R}\right\}$$

and let U be as in Example 4.2.9 for $m = 2, n = 1$. The Zariski closure of $A'U$ contains AU where $A \leqslant H$ is the diagonal subgroup with positive entries. It follows that $A'U$ is an epimorphic subgroup of H, since AU is. On the other hand, A' has empty intersection with the expanding cone A_U^+ which is described explicitly in Example 4.2.9. Therefore, for any probability measure μ on $A'U$ with finite first moment, we have

$$a \coloneqq a_{\mathrm{avg}}(\mu) \notin A_U^+,$$

where a_{avg} is as in Proposition 4.2.7. It follows from the definition of the expanding cone that there is a non-trivial irreducible representation V of H such that $V^U \cap (V_a^- \oplus V_a^0) \neq \{0\}$. Therefore, μ is not H-expanding. $\diamond$

We point out that the phenomenon in the above example crucially depends on the one-dimensional torus A' not being algebraic, as the discussion in the upcoming part will show.

4.2.3.2. *Expanding Rays in One-Dimensional Algebraic Tori.* We now state an observation (Lemma 4.2.14) ensuring the expansion of the unipotent part of a split solvable group with respect to its one dimensional torus. Based on this observation, in §4.2.3.3 we will outline two constructions due to Bien–Borel–Kollár [**16**], which, thanks to Proposition 4.2.7, yield further classes of H-expanding measures with small support on a semisimple group H.

Let H be a connected almost algebraic semisimple real Lie group without compact factors and F a connected epimorphic subgroup of H of the form $F = A'U$ where A' is a connected algebraic $\mathbb{R}$-split torus and U is a unipotent subgroup of H normalized by A'. It is known that any connected algebraic epimorphic subgroup of H contains an epimorphic subgroup of this form [**14**, §10, Theorem 2].

The following lemma can be proved in a similar way as Lemma 4.2.6 using additionally [**133**, Lemma 1]. We omit the routine details of the proof for brevity.

Lemma 4.2.14. *Let H and $F = A'U$ be as above and suppose that A' is one-dimensional. Then there exists a parametrization $A' = (a(t))_{t\in\mathbb{R}}$ as one-parameter subgroup such that for every representation (V, ρ) of H and U-fixed vector $v \in V^U$, either v is H-fixed or $\lim_{t\to\infty}\|\rho(a(t))v\| = \infty$. For such a parametrization, U is $a(t)$-expanding in the sense of* §4.2.2 *for every $t > 0$.* ☐

4.2.3.3. *Examples.* Let H be a connected almost algebraic semisimple real Lie group and denote its Lie algebra by $\mathfrak{h}$. Let Z be a one-parameter unipotent subgroup of H and z a generator of the Lie algebra of Z. By the Jacobson–Morozov theorem z is part of an $\mathfrak{sl}_2$-triple (a, z, z_-). Let $\mathfrak{s}$ be the Lie algebra spanned by this triple and S the corresponding connected subgroup of H. Let A' be the one-parameter diagonalizable subgroup with Lie algebra spanned by a. Via the adjoint representation, write $\mathfrak{h}$ as direct sum of the centralizer $\mathfrak{z}_o$ of $\mathfrak{s}$ and of non-trivial irreducible $\mathfrak{s}$-submodules $\mathfrak{m}_1 = \mathfrak{s}, \mathfrak{m}_2, \ldots, \mathfrak{m}_k$.

Example 4.2.15 ([**16**, Proposition 4.5]). Retain the notation from the paragraph above and suppose that z has non-trivial projections to each of the simple factors of $\mathfrak{h}$. Let z_i be highest weight vectors of the irreducible $\mathfrak{s}$-modules $\mathfrak{m}_i$, with $z_1 = z$. Write $\mathfrak{u}$ for the direct sum of their $\mathbb{R}$-spans. Denoting by U the corresponding unipotent subgroup of H, it follows that $F = A'U$ is a split solvable algebraic subgroup of H, which can be seen to be epimorphic in H thanks to [**16**, Proposition 4.5]. Therefore, by virtue of Proposition 4.2.7, we see that any probability measure μ on F whose A'-average lies in the expanding ray given by Lemma 4.2.14 is H-expanding. ◇

Example 4.2.16 ([**16**, §4.6]). Retain the notation from above. Suppose that H is an $\mathbb{R}$-split simple real algebraic group and that the one-parameter unipotent subgroup Z of H contains "regular" unipotent elements. For example, the generator z can be taken as sum of eigenvectors for all simple roots of $\mathfrak{h}$. Then the subgroup S whose Lie algebra is spanned by the $\mathfrak{sl}_2$-triple (a, z, z_-) is a "principal TDS" (three-dimensional subgroup) in H. It is known that either S is properly contained in exactly one proper connected subgroup R of H, or S is maximal among proper connected subgroups of H, in which case we set $R = S$. See Kostant [**77**] for a treatment of the notions used here. Choose $\mathfrak{m}_j$ so that it does not intersect the Lie algebra $\mathfrak{r}$ of R and let Z_j be the subgroup of H whose Lie algebra is generated by a highest weight vector of $\mathfrak{m}_j$. Then, as discussed in [**16**, §4.6], $F = A'ZZ_j$ is a three-dimensional split solvable algebraic epimorphic subgroup of H. Therefore, as in the previous example, three-dimensional solvable subgroups obtained by this construction support many H-expanding measures thanks to Proposition 4.2.7 and Lemma 4.2.14. ◇

We end this section by mentioning an ensuing question, which was also posed to us by Barak Weiss.

Question. Let H be a semisimple real algebraic group without compact factors. Is it true that every algebraic epimorphic subgroup $F \leqslant H$ supports an H-expanding probability measure?

The answer to the above question is negative if we do not require F to be epimorphic (Proposition 4.2.12) or to be algebraic (Example 4.2.13).

On the other hand, let $F = A'U$ be an $\mathbb{R}$-split solvable epimorphic subgroup of F, where U is a unipotent group and A' is an $\mathbb{R}$-split algebraic torus normalizing U. Then [**14**, §7, Lemma (iii)] provides a sufficient condition (in terms of finite-generation of a monoid generated by certain characters of A') for F to contain an $\mathbb{R}$-split solvable epimorphic subgroup $F_0 = A'_0U$ with one-dimensional $\mathbb{R}$-split algebraic torus $A'_0 < A'$. In view of Lemma 4.2.14, any such subgroup F_0 supports H-expanding probability measures. However, we do not know whether the hypothesis of the aforementioned lemma of Bien–Borel is always satisfied in the context of the question above, or whether a different construction can be used to obtain H-expanding probability measures on F in case it is not.

4.3. Measure Rigidity

This section is dedicated to the statements outlined in §4.0.1. In §4.3.1, we first prove our general measure rigidity result (Theorem 4.0.1), followed by a discussion of stationary measures charging an orbit of the centralizer in §4.3.2, which leads to the proof of Corollary 4.0.2. Finally, we more closely analyze in §4.3.3 expansion in which representations is necessary to obtain the conclusion of Theorem 4.0.1. This will yield a finite criterion weaker than H-expansion for measure rigidity to hold when the ambient Lie group G is fixed.

4.3.1. Rigidity for Expanding Measures. Let Λ be a discrete subgroup of a real Lie group G and $X = G/\Lambda$. Moreover, we let $H \leqslant G$ be a connected semisimple subgroup without compact factors and with finite center and μ a probability measure on H. For the proof of Theorem 4.0.1, we will follow the strategy in the proof of [**39**, Theorem 1.3]. The argument is based on the following measure classification result of Eskin–Lindenstrauss.

Definition 4.3.1 ([**39**, Definition 1.6]). Let Z be a connected Lie subgroup of G. A probability measure μ on G is said to be *uniformly expanding mod Z* if the following hold:

(a) Z is normalized by Γ_μ,

(b) the conjugation action of Γ_μ on Z factors through the action of a compact subgroup of $\mathrm{Aut}(Z)$, and
(c) there is a Γ_μ-invariant direct sum decomposition $\mathfrak{g} = \mathrm{Lie}(Z) \oplus V$ such that μ is uniformly expanding on V.

Theorem 4.3.2 (Eskin–Lindenstrauss [**39**, Theorem 1.7]). *Let G be a real Lie group and $\Lambda < G$ a discrete subgroup. Suppose that μ is a probability measure on G with finite first moment for which there exists a connected Lie subgroup Z of G such that μ is uniformly expanding mod Z. Let ν be any ergodic μ-stationary probability measure on G/Λ. Then one of the following holds:*

(a) *There exists a closed subgroup $N \leqslant G$ with* $\dim(N) > 0$*, an N-homogeneous probability measure ν_0 on G/Λ, and a μ-stationary probability measure η on G/N such that*
$$\nu = \int_{G/N} g_*\nu_0 \,\mathrm{d}\eta(g).$$
(b) *The measure ν is Γ_μ-invariant and supported on a finite union of compact subsets of Z-orbits.*

The following two lemmas will go into the proof of Theorem 4.0.1.

Lemma 4.3.3. *Suppose that μ is H-expanding. Then the Lie algebra $\mathfrak{g}$ of G admits an H-invariant direct sum decomposition $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{v}$, where $\mathfrak{l}$ is the Lie algebra of the centralizer L of Γ_μ in G and $\mathfrak{v} \subset \mathfrak{g}$ is a subspace on which μ is uniformly expanding. In particular, μ is uniformly expanding mod L° in the sense of Definition* 4.3.1.

Proof. Since, by Proposition 4.2.12, Γ_μ is epimorphic in H, $\mathfrak{l}$ is the space of H-fixed vectors in the adjoint representation of G. Semisimplicity thus implies the existence of an H-invariant complementary subspace $\mathfrak{v}$. Now the claim follows directly from the definition of H-expansion. □

The second lemma concerns μ-stationary measures assigning positive mass to centralizer orbits.

Lemma 4.3.4 ([**8**, Lemma 7.6]). *Suppose that ν is an ergodic μ-stationary probability measure on X such that ν assigns positive mass to some L-orbit in X, where $L = C_G(\Gamma_\mu)$. Let L_0 be any open subgroup of $L \cap \mathrm{Stab}_G(\nu)$. Then ν is homogeneous under the closed subgroup $\Gamma_\mu L_0$ and L_0 is open in* $\mathrm{Stab}_G(\nu)$.

We point out that the last claim in the statement above follows from the proof of [**8**, Lemma 7.6], where it is shown that the support of ν is a finite union of closed L_0-orbits which are transitively permuted by Γ_μ. In fact, even more conclusions can be drawn in the context of this lemma; see Proposition 4.3.5.

Proof of Theorem 4.0.1. Our main tool is Theorem 4.3.2. Its assumptions are satisfied, since by Lemma 4.3.3, μ is uniformly expanding mod L°, where L denotes the centralizer of Γ_μ in G. If Theorem 4.3.2(b) holds, then by Lemma 4.3.4, ν is homogeneous and the connected component of $\mathrm{Stab}_G(\nu)$

is contained in L. By the epimorphic property of Γ_μ in H from Proposition 4.2.12 applied to the adjoint representation of G, the connected components of $C_G(\Gamma_\mu)$ and $C_G(H)$ coincide. Thus, it follows that the connected component of $\mathrm{Stab}_G(\nu)$ is centralized by H.

If Theorem 4.3.2(a) holds, then there exists a closed subgroup N of G with $\dim(N) > 0$, an N-homogeneous probability measure ν_0 on $X = G/\Lambda$, and a μ-stationary probability measure η on G/N such that

$$\nu = \int_{G/N} g_*\nu_0 \,\mathrm{d}\eta(g). \tag{4.3.1}$$

We may assume that η is μ-ergodic. Indeed, if $\eta = \int_Y \eta_y \,\mathrm{d}y$ is a μ-ergodic decomposition of η, then

$$\nu = \int_Y \Big(\int_{G/N} g_*\nu_0 \,\mathrm{d}\eta_y(g) \Big) \,\mathrm{d}y$$

is a convex decomposition of ν into μ-stationary measures. Since ν is μ-ergodic, we must have $\nu = \int_{G/N} g_*\nu_0 \,\mathrm{d}\eta_y(g)$ for almost every y. Thus, we can replace η by one of the η_y, if necessary. We consider N such that $\dim(N)$ is maximal among possible representations of ν of the form (4.3.1).

Now consider the adjoint action of G on $S^2(\mathfrak{g}^{\wedge \dim(N)})$, where S^2 denotes the symmetric square representation, and let ω correspond to a vector given by a basis of the Lie algebra of N. Let P be the stabilizer of ω in G. Since N admits a lattice, it is unimodular, implying that $N \leqslant P$. Let η' be the pushforward of η by the canonical projection $G/N \to G/P$. The measure η' can be thought of as an ergodic μ-stationary measure on $S^2(\mathfrak{g}^{\wedge \dim(N)})$. By Corollary 4.1.12, η' must concentrate on the subspace of H-fixed vectors. Then by ergodicity, η' is a Dirac measure. After replacing N and P by their conjugates, we may assume without loss of generality that η' is the Dirac measure on the coset P. It follows that ω is H-fixed. Hence $H \leqslant P$ and $H \cap N^\circ$ is a normal subgroup of H. If $H \leqslant N^\circ$, then the action of H on P/N is trivial, so that by ergodicity of η we have $\nu = g_*\nu_0$ for an element $g \in P$ with $\mathrm{supp}(\eta) = \{gN\}$ and we are done.

So let us now assume that H is not contained in N°. In this case, we consider the action of $(H/(H \cap N^\circ), \mu')$ on $P/N \cong (P/N^\circ)/(N/N^\circ)$ with the μ'-stationary measure η, where μ' is the pushforward of μ by the canonical projection $H \to H/(H \cap N^\circ)$. Since μ is H-expanding and H is not contained in N°, μ' is $H/(H \cap N^\circ)$-expanding in view of Proposition 4.1.10(iii). Now, in view of Lemma 4.3.3, we are in a position to apply Theorem 4.3.2 again. We claim that thanks to the choice of N as having maximal dimension in (4.3.1), the case (a) in Theorem 4.3.2 does not occur. Suppose it does. This means that there exist a closed subgroup $M \leqslant P/N^\circ$ of positive dimension, an M-homogeneous probability measure ν_0' on P/N and a μ'-stationary probability measure η' on $(P/N^\circ)/M$ such that we have

$$\eta = \int_{(P/N^\circ)/M} g_*\nu_0' \,\mathrm{d}\eta'(g). \tag{4.3.2}$$

Denote by $\hat{M}$ the preimage of M under the canonical projection $P \to P/N^\circ$ so that we can identify $(P/N^\circ)/M$ with $P/\hat{M}$. By combining (4.3.1) and (4.3.2), we deduce that

$$\nu = \int_{P/N}\int_{P/\hat{M}} (gh)_*\nu_0 \, \mathrm{d}\eta'(g) d\nu_0'(h) = \int_{P/\hat{M}} g_*\Big(\int_{P/N} h_*\nu_0 \, \mathrm{d}\nu_0'(h)\Big)\, \mathrm{d}\eta'(g).$$

Now it is easily observed that the probability measure $\Psi = \int_{P/N} h_*\nu_0 \, \mathrm{d}\nu_0'(h)$ on X is $\hat{M}$-invariant and supported on finitely many $\hat{M}$-orbits. By μ-ergodicity of ν, for every $\hat{M}$-ergodic component Ψ_y of Ψ, we have

$$\nu = \int_{P/\hat{M}} g_*\Psi_y \, \mathrm{d}\eta'(g).$$

Take such a component Ψ_y which assigns positive mass to an $\hat{M}$-orbit. Then Ψ_y is $\hat{M}$-homogeneous and the fact that $\dim(\hat{M}) > \dim(N)$ yields a contradiction to the maximality of $\dim(N)$ in (4.3.1).

Therefore we can conclude by case (b) of Theorem 4.3.2 that η is $\Gamma_{\mu'}$-invariant and supported on finitely many compact subsets of $C_{P/N^\circ}(\Gamma_{\mu'})$-orbits. By Lemma 4.3.4, η is M-homogeneous for a closed subgroup $M \leqslant P/N^\circ$. In particular, η can be written in the form (4.3.2) with $\nu_0' = \eta$ and η' the Dirac mass at the identity coset, the latter being μ'-stationary since η is $\Gamma_{\mu'}$-invariant. As we have argued above, this cannot happen if the support of η has positive dimension. Thus, η is a finite periodic orbit measure, and using (4.3.1) it directly follows that ν is homogeneous. The connected component of $\mathrm{Stab}_G(\nu)$ is N°, which is normalized by H, as we already established above. Hence, the proof is complete. ☐

4.3.2. Stationary Measures Charging an Orbit of the Centralizer. The following proposition gives additional information about the measure ν in the setting of Lemma 4.3.4, or more generally, in the setting of [**8**, §7.3]. It will be used below to deduce Corollary 4.0.2(i) from Theorem 4.0.1.

The general setting is as follows: G is a locally compact second countable group, Λ a discrete subgroup of G, μ is a probability measure on G, L denotes the centralizer of Γ_μ in G, and ν is a μ-ergodic μ-stationary probability measure on $X = G/\Lambda$ assigning positive mass to some L-orbit. Finally, L_0 is any open subgroup of $L \cap \mathrm{Stab}_G(\nu)$.

PROPOSITION 4.3.5. *Retain the notation and assumptions above and fix a point $x = g\Lambda \in \mathrm{supp}(\nu)$. Let ν_0 be the restriction of ν to L_0x, Γ_0 the stabilizer of ν_0 in Γ_μ and*

$$\Gamma_0^L = \{l \in L_0 \mid \text{there exists } h \in \Gamma_0 \text{ such that } hl \in g\Lambda g^{-1}\}.$$

Then in addition to the conclusion of Lemma 4.3.4, the following holds:

(i) *Γ_0 has finite index in Γ_μ,*
(ii) *Γ_0^L is a dense subgroup of L_0 with $\Gamma_0 x = \Gamma_0^L x$, and*
(iii) *$L_0 \cap g\Lambda g^{-1}$ is a cocompact normal subgroup of L_0.*

In particular, ν is compactly supported and is the unique ergodic μ-stationary probability measure on X assigning positive measure to $\mathrm{supp}(\nu)$.

PROOF. By [**8**, Lemma 7.6] and its proof, we know that ν is the homogeneous measure on $\Gamma_\mu L_0 x$ and that $\mathrm{supp}(\nu)$ consists of finitely many closed L_0-orbits which are transitively permuted by Γ_μ. In particular, $\nu(L_0x) > 0$. It follows that Γ_0 has finite index in Γ_μ. Moreover, since Γ_μ preserves ν and acts ergodically, the group Γ_0 acts ergodically with respect to ν_0. This implies that we can find $l_0 \in L_0$ such that $\Gamma_0 l_0 x$ is dense in $L_0 x$. As l_0 commutes with Γ_0, it immediately follows that $\Gamma_0 x$ is dense in $L_0 x$. Since Γ_0^L is precisely defined for $\Gamma_0 x = \Gamma_0^L x$ to hold, we conclude that $\Gamma_0^L = \Gamma_0^L(L_0 \cap g\Lambda g^{-1})$ is dense in L_0.

We next prove that $L_0 \cap g\Lambda g^{-1}$ is a cocompact normal subgroup of L_0. Since we have already shown that Γ_0^L is dense in L_0, it suffices to show that $L_0 \cap g\Lambda g^{-1}$ is normal in Γ_0^L. To see this, taking an arbitrary $l \in \Gamma_0^L$ and choosing $h \in \Gamma_0$ with $hl \in g\Lambda g^{-1}$, we calculate

$$l(L_0 \cap g\Lambda g^{-1})l^{-1} = hl(L_0 \cap g\Lambda g^{-1})(hl)^{-1} = L_0 \cap g\Lambda g^{-1},$$

where we used again that Γ_μ and L_0 commute. Since there is a finite L_0-invariant measure on $L_0/(L_0 \cap g\Lambda g^{-1})$, the latter quotient group must be compact.

It remains to prove the uniqueness of ν. Let ν' be an arbitrary ergodic μ-stationary probability measure on X with $\nu'(\mathrm{supp}(\nu)) > 0$. Take $x \in \mathrm{supp}(\nu) \cap \mathrm{supp}(\nu')$. Then by what we have shown above, ν' is homogeneous and $\mathrm{supp}(\nu) = \overline{\Gamma_\mu x} = \mathrm{supp}(\nu')$. Hence, $\nu = \nu'$ by homogeneity. $\square$

To keep the continuity, we now prove the corollary on measure rigidity on quotients of a semisimple group G by an irreducible lattice Λ, even though one part of the statement relies on the countability result for homogeneous subspaces to be established in §4.4.

Proof of Corollary 4.0.2. Let ν be an ergodic μ-stationary probability measure on $X = G/\Lambda$. By Theorem 4.0.1 we know that ν is homogeneous and $\mathrm{Stab}_G(\nu)^\circ$ is normalized by H. By conjugating if necessary, we may assume the identity coset Λ is in the support of ν. If $\mathrm{Stab}_G(\nu) \cap H$ is non-discrete, then $\mathrm{Stab}_G(\nu)$ must contain a normal subgroup of H of positive dimension. Since Λ is irreducible, this implies that ν is G-invariant. Indeed, $\mathrm{Stab}_G(\nu)\Lambda$ is closed since the stabilizer intersects Λ in a lattice ([**107**, Theorem 1.13]), and also dense by irreducibility of Λ if $\mathrm{Stab}_G(\nu)$ contains a simple factor of G.

Let us now assume that $\mathrm{Stab}_G(\nu) \cap H$ is discrete and $H \neq G$ and use this to derive a contradiction. Since $\mathrm{Stab}_G(\nu)^\circ$ is normalized by H, we may view its Lie algebra as H-submodule of $\mathfrak{g} = \mathrm{Lie}(G)$. As every non-trivial H-isotypic component of $\mathfrak{g}$ is contained in $\mathrm{Lie}(H)$, it follows from the discreteness assumption that we must have $\mathrm{Stab}_G(\nu)^\circ \leqslant C_G(H) \leqslant C_G(\Gamma_\mu)$. This puts us in the setting of Proposition 4.3.5, namely, the homogeneous measure ν gives positive mass to an orbit of the centralizer L of Γ_μ in G. We apply this proposition with $x = \Lambda$ and L_0 the connected component of $\mathrm{Stab}_G(\nu) \cap L$ and let Γ_0 and Γ_0^L be as defined there. Then $L_0 \cap \Lambda$ is central by irreducibility of Λ ([**107**, Corollary 5.21]), hence finite, which by part (iii) of the proposition implies that L_0 is compact.

We now invoke Margulis' arithmeticity theorem in [**85**]. The conclusion is that we may assume that $G = \prod_{\sigma \in S} \mathbf{G}^\sigma(k_\sigma)$, where $\mathbf{G}$ is a Zariski connected absolutely simple linear algebraic group defined over a number field k, $k_\sigma \in \{\mathbb{R}, \mathbb{C}\}$ denotes the completion of $\sigma(k)$ for a field embedding $\sigma\colon k \to \mathbb{C}$, and S is a finite set of inequivalent such embeddings with the property that $\mathbf{G}^\sigma(k_\sigma)$ is non-compact if and only if σ or $\overline{\sigma}$ is in S. The lattice Λ is given as the diagonal embedding of $\mathbf{G}(\mathcal{O}_k)$ in G via $k \ni z \mapsto (\sigma(z))_{\sigma \in S}$, where $\mathcal{O}_k$ is the ring of integers of k. As $H \neq G$ is a connected normal subgroup of G of positive dimension, there is a non-empty proper subset $S_1 \subset S$ such that $H = \prod_{\sigma \in S_1} \mathbf{G}^\sigma(k_\sigma)$. We remark that in the above, strictly speaking, G and H should be the analytic identity components of the appearing groups, but we ignore this point for ease of notation and without loss of generality. We write $S_2 = S \setminus S_1$ and also assume without loss of generality that the identity map is contained in S_1.

Let Γ_1 be the projection of Γ_0 to $\mathbf{G}$, and for every $\sigma \in S_2$ let Γ_σ be the projection of Γ_0^L to $\mathbf{G}^\sigma$. Then part (ii) of Proposition 4.3.5 implies that $\Gamma_1 \leqslant \mathbf{G}(\mathcal{O}_k)$ and $\Gamma_\sigma = \sigma(\Gamma_1)$. From Proposition 4.2.12 it follows that the Zariski closure of Γ_μ is an epimorphic subgroup of H in the category of real algebraic groups. As Γ_0 has finite index in Γ_μ, also $\mathrm{Zcl}(\Gamma_0)$ is epimorphic in H. We claim that $\mathrm{Zcl}(\Gamma_0)$ is also reductive. Otherwise, its projection to one of the simple factors of H is not reductive. Without loss of generality assume that this holds for $\mathbf{F} = \mathrm{Zcl}(\Gamma_1)$. Then we get a contradiction since $\mathbf{F}^\sigma = \mathrm{Zcl}(\Gamma_\sigma)$ is reductive because Γ_σ is dense in a compact group. So $\mathrm{Zcl}(\Gamma_0)$ is a reductive epimorphic subgroup of H, which implies that Γ_0 is Zariski dense in H.

Now consider the Zariski closure G' of the diagonal embedding of Γ_1 in $\mathrm{Res}_{k/\mathbb{Q}} \mathbf{G}(\mathbb{R}) = \prod_\sigma \mathbf{G}^\sigma(k_\sigma)$, where the product runs over all inequivalent field embeddings σ of k into $\mathbb{C}$. Then G' is an algebraic $\mathbb{Q}$-group without non-trivial $\mathbb{Q}$-characters, so that by Borel–Harish-Chandra's theorem the intersection of G' with $\mathbf{G}(\mathcal{O}_k)$ is a lattice in G'. As $\Gamma_\sigma = \sigma(\Gamma_1)$ is relatively compact in $\mathbf{G}^\sigma(k_\sigma)$ for each $\sigma \in S_2$ and $\mathbf{G}^\sigma(k_\sigma)$ is compact for $\sigma \notin S$, G' has compact projections to all $\mathbf{G}^\sigma(k_\sigma)$ with $\sigma \notin S_1$. Moreover, the projection from G' to H is onto since Γ_0 is Zariski dense in H. Thus, the projection Λ_0 of $G' \cap \mathbf{G}(\mathcal{O}_k)$ to H is a lattice in H. Let Λ_1 be the projection of Λ_0 to $\mathbf{G}$ and let k' be the subfield of k generated by the set $\mathrm{Tr}(\mathrm{Ad}(\Lambda_1))$. Then $\mathrm{Ad}(\Lambda_1)$ is definable over k' (see [**85**, IX.1.8]). So we may and will assume that $\mathbf{G}$ is defined over k' and $\Lambda_1 \leqslant \mathbf{G}(k')$. The group $\mathrm{Res}_{k'/\mathbb{Q}} \mathbf{G}(\mathbb{R}) = \prod_{\tau\colon k' \to \mathbb{C}} \mathbf{G}^\tau(k'_\tau)$ is naturally embedded in $\mathrm{Res}_{k/\mathbb{Q}} \mathbf{G}(\mathbb{R}) = \prod_\sigma \mathbf{G}^\sigma(k_\sigma)$ as a real algebraic subgroup, by identifying $\mathbf{G}^\tau(k'_\tau)$ with its diagonal embedding in $\prod_{\sigma\colon \sigma|_{k'} = \tau} \mathbf{G}^\sigma(k_\sigma)$. Under this identification, we have $G' \leqslant \mathrm{Res}_{k'/\mathbb{Q}} \mathbf{G}(\mathbb{R})$. We deduce the following facts:

(a) The embeddings $\sigma|_{k'}$ of k' for $\sigma \in S_1$ are pairwise distinct and $\sigma \in S_1$ is a complex embedding if and only if $\sigma|_{k'}$ is, since the diagonally embedded $\mathbf{G}(k')$ is Zariski dense in H because it contains the lattice Λ_0.

(b) If τ admits an extension $\sigma\colon k \to \mathbb{C}$ not in S_1, then $\mathbf{G}^\tau(k'_\tau)$ is compact. Indeed, since $\tau(\Lambda_1)$ is Zariski dense in $\mathbf{G}^\tau$, the Zariski closure of $\tau(\Lambda_1)$ in $\mathrm{Res}_{k'_\tau/\mathbb{R}} \mathbf{G}^\tau(\mathbb{R})$ is either $\mathbf{G}^\tau(\mathbb{C})$ or a real form of it. In the latter case, the field $\tau(k')$ generated by $\mathrm{Tr}(\mathrm{Ad}(\tau(\Lambda_1)))$ is contained in $\mathbb{R}$, so $k'_\tau = \mathbb{R}$. Therefore, $\tau(\Lambda_1)$ is Zariski dense in $\mathrm{Res}_{k'_\tau/\mathbb{R}} \mathbf{G}^\tau(\mathbb{R})$, and as $\sigma \notin S_1$ we also know that $\tau(\Lambda_1)$ is relatively compact.

But these facts yield a contradiction: As S_2 is non-empty we cannot have $k = k'$ due to (b), so that the identity embedding $k' \to \mathbb{C}$ must admit a non-trivial extension $\sigma\colon k \to \mathbb{C}$. This embedding σ cannot be contained in S_1 due to point (a) above, but also cannot lie outside of S_1, by combining points (a) and (b). This finishes the proof of (i).

In the case $H = G$ of part (ii), the arguments at the beginning of the proof show that either $\nu = m_X$ or $\mathrm{Stab}_G(\nu)$ is discrete. In the latter case, ν must be the uniform probability measure on a finite Γ_μ-orbit (see [**5**, Lemma 8.3]). Moreover, in this case we have that $C_G(\Gamma_\mu)$ is discrete by the epimorphic property of Γ_μ in G from Proposition 4.2.12. Proposition 4.4.1 thus implies that there are only countably many distinct finite Γ_μ-orbits in X. Hence, if ν is any non-atomic μ-stationary probability measure on X, $\nu = m_X$ follows by considering an ergodic decomposition of ν. This completes the proof. ☐

4.3.3. Expansion on Grassmannians. The H-expansion condition on μ is a universal requirement in the sense that all our results (including the measure classification theorem) hold for any embedding $H \hookrightarrow G$ and any discrete subgroup Λ in G. Having fixed $H \leqslant G$, however, close inspection of the proof of Theorem 4.0.1 reveals that it is sufficient to have uniform expansion on the quotient of each exterior power of $\mathfrak{g}$ by the corresponding H-fixed subspace.

DEFINITION 4.3.6. Let G be a real Lie group with Lie algebra $\mathfrak{g}$ and $H \leqslant G$ a connected semisimple subgroup with finite center. We say that a probability measure μ on H is *H-expanding relative to G* if μ is uniformly expanding on the quotient of the exterior power representation $(\mathfrak{g}^{\wedge k}, \mathrm{Ad}^{\wedge k})$ by the corresponding H-fixed subspace for every $1 \leq k \leq \dim(G) - 1$.

We point out that a related notion was already studied in Chapter 2, under the name "uniform expansion on Grassmannians".

THEOREM 4.3.7. *Let G be a real Lie group, $\Lambda \leqslant G$ a discrete subgroup, and H a connected semisimple subgroup of G with finite center. Let μ be an H-expanding probability measure relative to G with finite first moment. Then the conclusions of Theorem* 4.0.1 *hold for every ergodic μ-stationary probability measure ν on G/Λ.*

PROOF. We analyze the applications of the H-expansion property in the proof of Theorem 4.0.1, so we retain the notation used there.

- The first application of Lemma 4.3.3 is possible without problems.
- Next, expansion is used for the representation $S^2(\mathfrak{g}^{\wedge \dim(N)})$. In case $\dim(N) = \dim(G)$, the probability measure η in (4.3.1) is finitely supported and Γ_μ-invariant by [**5**, Lemma 8.3], so all claims follow. Otherwise, we know that the measure η' on $S^2(\mathfrak{g}^{\wedge \dim(N)})$ is supported on $\{v \otimes v \mid v \in \mathfrak{g}^{\wedge \dim(N)}\}$ by construction. Using that $\|v \otimes v\| = \|v\|^2$ and the assumed expansion in $\mathfrak{g}^{\wedge \dim(N)}$, we can again draw the desired conclusion that η' is supported on the set of H-fixed vectors.
- Finally, expansion is needed to reapply Theorem 4.3.2 in the quotient by N°. The assumption there implies that $H/(H \cap N^\circ)$ is still a semisimple group, so that $\dim(N) \leq \dim(G) - 3$. Let $v \in \mathfrak{g}^{\wedge \dim(N)}$ correspond to a basis of the Lie algebra $\mathfrak{n}$ of N. Then a norm on $\mathfrak{g}/\mathfrak{n}$ is given by $\|w + \mathfrak{n}\| = \|w \wedge v\|$ for $w \in \mathfrak{g}$. Since H fixes the vector $\omega = v \otimes v$ in $S^2(\mathfrak{g}^{\wedge \dim(N)})$ and H is connected, v is fixed by H. Thus, for every $h \in H$ and $w \in \mathfrak{g}$ we have
$$\|h \cdot (w + \mathfrak{n})\| = \|h \cdot w \wedge v\| = \|h \cdot (w \wedge v)\|. \tag{4.3.3}$$

Hence, we again obtain expansion for every vector in $\mathfrak{g}/\mathfrak{n}$ that is not H-fixed. This justifies the application of Lemma 4.3.3 in the quotient. $\square$

Combining the above with some properties of epimorphic subgroups, we obtain the following.

COROLLARY 4.3.8. *Let G be a real algebraic group, $\Lambda < G$ a lattice, and $H \leqslant G$ a Zariski connected semisimple algebraic subgroup without compact factors. Then any Zariski connected real algebraic epimorphic subgroup F of H supports probability measures μ for which the conclusions of Theorem* 4.0.1 *hold.*

PROOF. It is known that the epimorphic subgroup F contains a split solvable algebraic subgroup $A'U$, where A' is an algebraic $\mathbb{R}$-split torus and U is unipotent and normalized by A', that is still epimorphic in H (see [**14**, §10, Theorem 2]). Thus we may assume $F = A'U$ is of this form to begin with. By [**133**, Lemma 1] there is a non-empty open cone A'_+ in A' such that $\chi(a) > 1$ for all $a \in A'_+$ and all characters of A' having an eigenvector in one of the U-fixed subspaces V_k^U of the finitely many representations $V_1, \dots, V_r$ appearing in the statement of Theorem 4.3.7. Then any probability measure μ on F with finite first moment whose A'-average $a_{\mathrm{avg}}(\mu)$ lies in A'_+ and for which the Zariski closure of Γ_μ contains U is uniformly expanding in all of the representations V_k. Indeed, this follows directly by combining Lemmas 4.2.5 and 4.2.6. Theorem 4.3.7 thus applies to all measures μ satisfying these conditions. $\square$

4.4. Countability of Homogeneous Subspaces

Let Γ be a closed subsemigroup of G and $\Lambda < G$ a lattice. A homogeneous subspace $Y \subset X = G/\Lambda$ is said to be Γ-invariant if Γ preserves the associated homogeneous probability measure η on Y. It is called Γ-ergodic if Γ acts ergodically on (Y, η). Define

$$\mathcal{S}(\Gamma) = \{\Gamma\text{-invariant } \Gamma\text{-ergodic homogeneous subspaces } Y \subset X\}.$$

A key input to the proof of Theorem 4.0.5 is countability of $\mathcal{S}(\Gamma_\mu)$ modulo the centralizer of H. Our strategy to prove this result closely follows the approach in [**9**], where this result is proved under the assumption that the Zariski closure of $\mathrm{Ad}(\Gamma_\mu)$ is semisimple and has no compact factors. The goal of this subsection is therefore to prove the following analogue of [**9**, Proposition 2.1].

PROPOSITION 4.4.1. *Let G be a real Lie group, $H \leqslant G$ a connected semisimple subgroup with finite center, and $\Gamma < H$ a subsemigroup that supports a probability measure with finite first moment that is H-expanding relative to G. Denote by L the centralizer of Γ in G. Then there exists a countable subset $\mathcal{Y}$ of $\mathcal{S}(\Gamma)$ such that*

$$\mathcal{S}(\Gamma) = \{lY \mid l \in L, Y \in \mathcal{Y}\}. \tag{4.4.1}$$

Note that the set $\mathcal{S}(\Gamma)$ remains the same if we replace the semigroup Γ by the closed group that it generates. Therefore, in the proof of the previous result, we can suppose that Γ is a closed subgroup of H.

The key ingredient of the proof of this proposition is Lemma 4.4.3 below, which will imply countability of the closed subgroups of G that arise as the stabilizer of homogeneous subspaces in $\mathcal{S}(\Gamma)$. To this end, we introduce the following definition, which, in view of Theorem 4.0.1, is the appropriate replacement of [**9**, Definition 2.4].

DEFINITION 4.4.2. Let $\Delta \subset \Sigma$ be discrete subgroups of a real Lie group G. The set $\mathcal{T}(G, \Delta, \Sigma)$ is defined to be the set of closed subgroups N of G such that

(i) Σ is contained in N and is a lattice in N,
(ii) $\Delta = \Sigma \cap N^\circ$, where N° is the connected component of N,
(iii) there exist a connected semisimple Lie group $H_N \leqslant G$ and a subgroup $\Gamma \leqslant H_N \cap N$ which acts ergodically on N/Σ and which supports an H_N-expanding probability measure relative to G.

LEMMA 4.4.3. *Let G be a real Lie group and $\Delta \subset \Sigma$ finitely generated discrete subgroups of G. Then the set $\mathcal{T}(G, \Delta, \Sigma)$ is countable.*

For the proof we require the following strengthening of [**9**, Lemma 2.6].

LEMMA 4.4.4. *Let G be a real Lie group, $\mathfrak{g}$ its Lie algebra, and $\Delta \subset \Sigma$ discrete subgroups of G. Let N belong to $\mathcal{T}(G, \Delta, \Sigma)$, H_N be any connected semisimple subgroup of G as in* (iii) *of Definition* 4.4.2*, and let M be a unimodular Lie subgroup of G containing Σ. Let $\omega \in S^2(\mathfrak{g}^{\wedge \dim(M)})$ correspond to a basis of the Lie algebra of M. Then ω is fixed by N and H_N, and hence M° is normalized by N and H_N. In particular, this holds whenever $M \in \mathcal{T}(G, \Delta, \Sigma)$.*

In this lemma, $S^2(\mathfrak{g}^{\wedge \dim(M)})$ denotes the symmetric square of $\mathfrak{g}^{\wedge \dim(M)}$. If $v \in \mathfrak{g}^{\wedge \dim(M)}$ corresponds to a basis of the Lie algebra of N, the appearing vector ω is given by $\omega = v \otimes v$.

PROOF. If $\dim(M) = \dim(G)$, then $M^\circ = G^\circ$ and the statement is clear. So we assume that $\dim(M) < \dim(G)$. Since M is unimodular and contains Σ, the action of Σ fixes ω. Therefore, the map

$$N \to S^2(\mathfrak{g}^{\wedge \dim(M)}),\ h \mapsto h \cdot \omega$$

descends to a map $N/\Sigma \to S^2(\mathfrak{g}^{\wedge \dim(M)})$. Denote by η the pushforward of the Haar probability measure on N/Σ to $S^2(\mathfrak{g}^{\wedge \dim(M)})$ by this map and let $\Gamma \leqslant N \cap H_N$ be as in (iii) of the definition of $\mathcal{T}(G, \Delta, \Sigma)$. Then η is an ergodic Γ-invariant probability measure supported on the set $\{v \otimes v \,|\, v \in \mathfrak{g}^{\wedge \dim(M)}\}$. Since Γ supports an H_N-expanding probability measure relative to G and $\|v \otimes v\| = \|v\|^2$, Lemma 4.1.6 implies that η is concentrated on the subspace of H_N-fixed vectors. The ergodicity forces η to be the Dirac mass at ω. Hence, ω is N- and H_N-fixed, as required. $\square$

We can now prove Lemma 4.4.3. The argument is basically the same as in the proof of [**9**, Lemma 2.5], but we need to handle an additional difficulty coming from the fact that Γ is not necessarily Zariski dense in a H_N, but only carries a probability measure that is H_N-expanding relative to G.

PROOF OF LEMMA 4.4.3. For every $N \in \mathcal{T}(G, \Delta, \Sigma)$, we fix a connected semisimple group H_N as in (iii) of Definition 4.4.2. Considering the closure of the group generated by the set $\bigcup_{N\in\mathcal{T}(G,\Delta,\Sigma)} H_N N$, we can assume that this set generates a dense subgroup of G. By Lemma 4.4.4,

$$M \coloneqq \bigcap_{N\in\mathcal{T}(G,\Delta,\Sigma)} N^\circ$$

is a normal subgroup of G. Let $\mathrm{pr}\colon G \to G/M$ be the canonical projection.

We argue next that $\iota\colon N \mapsto \mathrm{pr}(N)$ gives an injection of $\mathcal{T}(G,\Delta,\Sigma)\setminus\{\Sigma M\}$ into $\mathcal{T}(G/M,\{e\},\mathrm{pr}(\Sigma))$. First, note that $N \mapsto \mathrm{pr}(N)$ is an injective map from $\mathcal{T}(G,\Delta,\Sigma)$ into the set of closed subgroups of G/M. Since $\Sigma\cap M = \Delta$ is a lattice in M, ΣM is closed in G by [**107**, Theorem 1.13], which implies that $\mathrm{pr}(\Sigma)$ is discrete. As there is an equivariant projection $N/\Sigma \to \mathrm{pr}(N)/\mathrm{pr}(\Sigma)$, $\mathrm{pr}(\Sigma)$ is a lattice in $\mathrm{pr}(N)$. If $\mathrm{pr}(n) \in \mathrm{pr}(\Sigma)$ for some $n \in N^\circ$, then $n = \sigma m$ for some $m \in M$ and $\sigma \in \Sigma$. Since $M \subset N^\circ$, it follows that $\sigma \in \Sigma\cap N^\circ = \Delta \subset M$, which proves that $\mathrm{pr}(N)^\circ\cap\mathrm{pr}(\Sigma) = \{e\}$ is the trivial group. So we have verified conditions (i) and (ii) of Definition 4.4.2 for any element $\mathrm{pr}(N)$ in the image of ι. To also verify condition (iii), let $H_N \leqslant G$ be the connected semisimple subgroup from condition (iii) for N and Γ a subgroup of $H_N \cap N$ that acts ergodically on N/Σ and carries an H_N-expanding probability measure μ relative to G. Then it is clear that $\mathrm{pr}(\Gamma)$ acts ergodically on $\mathrm{pr}(N)/\mathrm{pr}(\Sigma)$. Now, if $H_N \leqslant M$, then ergodicity of this action forces $N = \Sigma M$. Otherwise, $\mathrm{pr}(H_N)$ is a connected semisimple Lie group. By Lemma 4.4.4 and connectedness, H_N fixes a vector $v \in \mathfrak{g}^{\wedge\dim(M)}$ corresponding to a basis of the Lie algebra $\mathfrak{m}$ of M. For $1 \le k \le \dim(G/M)-1$, we may use a norm on $(\mathfrak{g}/\mathfrak{m})^{\wedge k}$ with the property that $\|[w]\| = \|w\wedge v\|$ for every $w \in \mathfrak{g}^{\wedge k}$, where $[w]$ denotes the projection of w to $(\mathfrak{g}/\mathfrak{m})^{\wedge k}$. Then the same calculation as in (4.3.3) shows that $\mathrm{pr}_*\mu$ is $\mathrm{pr}(H_N)$-expanding relative to G/M. So also condition (iii) of Definition 4.4.2 holds for $\mathrm{pr}(N)$.

Thus, it suffices to prove the lemma under the assumption that $\Delta = \{e\}$ is the trivial group and that for every $N \in \mathcal{T}(G,\{e\},\Sigma)$, the connected component N° is normal in G. In view of condition (ii), this implies that N° is a compact normal subgroup of G. By [**9**, Lemma 2.7], there are only countably many such N°. Similar to the first reduction step above, after fixing N° and replacing G by G/N° and Σ by $\Sigma N^\circ/N^\circ$, we are left to show that the set $\mathcal{V}(G,\Sigma)$ of discrete subgroups N containing Σ as a finite index subgroup such that (iii) of Definition 4.4.2 holds is countable. For each $N \in \mathcal{V}(G,\Sigma)$, there is a finite index subgroup $\Sigma' \leqslant \Sigma$ such that Σ' is normal in N. Recall that by assumption Σ is finitely generated, so that it admits only finitely many homomorphisms to any fixed finite group. It follows that there are countably many such Σ'. Therefore, it suffices to show that, for Σ' fixed, the set $\mathcal{V}(G,\Sigma',\Sigma)$ of $N \in \mathcal{V}(G,\Sigma)$ with Σ' normal in N is countable. Let S be the closed subgroup generated by $\bigcup_{N\in\mathcal{V}(G,\Sigma',\Sigma)} N$. Then Σ' is a discrete normal subgroup of S. For any $g \in \Sigma'$, the set $\{sgs^{-1} \mid s \in S^\circ\}$ is a connected subset of Σ', so it has to be $\{g\}$. It follows that Σ' centralizes S°.

Given $N \in \mathcal{V}(G,\Sigma',\Sigma)$, let Γ be a subgroup of $H_N \cap N$ acting ergodically on N/Σ as in (iii) of Definition 4.4.2. By ergodicity, we have $N = \Gamma\Sigma$ and since $\Gamma\Sigma = \Gamma(\Sigma'\Sigma) = (\Gamma\Sigma')\Sigma$, N is uniquely determined by the discrete group $\Gamma\Sigma'$. So it suffices to show that the set of subgroups $\Gamma\Sigma'$ appearing in this way is countable. The finite index subgroup $\Gamma\cap\Sigma'$ of Γ centralizes S°

and Γ normalizes S°. It follows that the conjugation action of Γ on S° factors through a finite group. Now, according to (iii) of Definition 4.4.2, there exists a probability measure on Γ that is H_N-expanding relative to G. By part (i) of Proposition 4.1.10 applied to the adjoint representation of H_N on $\mathfrak{g}$, we conclude that every element of the Lie algebra of S is fixed by H_N. This implies that $\Gamma \leqslant H_N$ centralizes S°. Therefore $\Gamma\Sigma'/\Sigma'$ is a finite subgroup of S/Σ' centralizing $S^\circ\Sigma'/\Sigma'$. By [**9**, Lemma 2.8], the set of compact subgroups of S/Σ' centralizing $S^\circ\Sigma'/\Sigma'$ is countable. This gives the required countability and hence completes the proof. $\square$

We also need the following version of [**9**, Lemma 2.2].

LEMMA 4.4.5. *Let G be a real Lie group, H a connected semisimple subgroup of G, and Γ a subgroup of H that supports an H-expanding probability measure relative to G. Moreover, let L be the centralizer of Γ in G and N a closed unimodular subgroup of G. Then the set of Γ-fixed points in $Y = G/N$ is a countable union of L-orbits.*

PROOF. It is enough to consider the case $\dim(N) < \dim(G)$. Denote by Y^Γ the set of Γ-fixed points in Y. Then it suffices to show that every L-orbit Ly in Y^Γ is open in Y^Γ. After a conjugation we may assume $y = eN$ is the identity coset. In particular, we then have $\Gamma \leqslant N$. Let $\mathfrak{l}$ denote the Lie algebra of L. By finite-dimensionality, we can find $\gamma_1, \ldots, \gamma_r \in \Gamma$ such that

$$\mathfrak{l} = \{v \in \mathfrak{g} \mid \mathrm{Ad}(\gamma_i)v = v \text{ for } 1 \le i \le r\}.$$

In view of unimodularity of N, considering a vector in $S^2(\mathfrak{g}^{\wedge \dim(N)})$ corresponding to a basis of the Lie algebra $\mathfrak{n}$ of N and arguing as in Lemma 4.4.4 yields that $\mathfrak{n}$ is H-invariant. Thanks to the expansion in the adjoint representation, it moreover follows that $\mathfrak{l}$ coincides with the space of H-fixed vectors in $\mathfrak{g}$. We choose an H-invariant complement $\mathfrak{v}$ of $\mathfrak{n} + \mathfrak{l}$ in $\mathfrak{g}$. Then for any $v \in \mathfrak{v}$ sufficiently small, if $\exp(v)y$ is Γ-fixed, then for all $1 \le i \le r$ we have

$$\exp(\mathrm{Ad}(\gamma_i)v)y = \gamma_i \exp(v)y = \exp(v)y,$$

which implies $\mathrm{Ad}(\gamma_i)v = v$ and thus $v \in \mathfrak{l} \cap \mathfrak{v} = \{0\}$. This shows that Ly is open in Y^Γ and hence finishes the proof that Y^Γ is a countable union of L-orbits. $\square$

Finally, we can prove the main result of this subsection. We adapt the proof of [**9**, Proposition 2.1] by substituting Lemmas 4.4.3 and 4.4.5 for the corresponding results, and extend it to cover semigroups that are not compactly generated.

PROOF OF PROPOSITION 4.4.1. We first establish the statement assuming additionally that Γ is compactly generated. Let $Y \in \mathcal{S}(\Gamma)$ and denote by G_Y the stabilizer of the homogeneous probability measure ν corresponding to Y. Let μ be a probability measure on Γ that is H-expanding relative to G. Choose $g \in G$ such that $g\Lambda \in Y$ and consider the closed subgroup $N = g^{-1}\Gamma G_Y^\circ g$ of G. Now, the discrete groups $\Delta = N^\circ \cap \Lambda$ and $\Sigma = N \cap \Lambda$ are lattices in N° and N, respectively. Being a lattice in a connected Lie group, Δ is finitely generated (see [**107**, 6.18]). As $N = g^{-1}\Gamma G_Y^\circ g$ and Γ is compactly generated, N/N° is finitely generated. Since Σ/Δ has finite index in N/N°, also

Σ is finitely generated. As Λ admits only countably many finitely generated subgroups, one may assume that Δ and Σ are fixed. We claim that N belongs to $\mathcal{T}(G, \Delta, \Sigma)$. To see this, we first record that (i) and (ii) in Definition 4.4.2 are immediate. Considering $H_N = g^{-1}Hg$, its subgroup $g^{-1}\Gamma g$ and the image of μ by conjugation by g^{-1}, also (iii) is seen to hold. Consequently, we can also assume N to be fixed by virtue of Lemma 4.4.3. As the point $gN \in G/N$ is Γ-invariant, by Lemma 4.4.5 one may further assume the L-orbit $LgN \subset G/N$ is fixed. It only remains to note that for $l \in L$, the orbit $lgN\Lambda \subset X = G/\Lambda$ is precisely the translate lY of Y.

To treat the general case without the compact generation assumption, given an arbitrary probability measure μ' on Γ with finite first moment that is H-expanding relative to G, we consider the probability measure μ given as the normalized restriction of μ' to a sufficiently large compact ball B around the identity. By choosing B large enough, we can guarantee that the integral characterization of uniform expansion from Proposition 4.1.2 still holds for the finite collection of representations in Definition 4.3.6. In view of expansion in the adjoint representation, the connected components of the centralizers $L = C_G(\Gamma)$ and $L_\mu = C_G(\Gamma_\mu)$ coincide. Therefore, applying the above to the compactly generated subgroup Γ_μ, we can find a countable collection $\mathcal{Y}_\mu \subset \mathcal{S}(\Gamma_\mu)$ such that $\mathcal{S}(\Gamma_\mu) = \{lY_\mu \mid l \in L, Y_\mu \in \mathcal{Y}_\mu\}$. We claim that

$$\mathcal{Y} = \{\overline{\Gamma Y_\mu} \mid Y_\mu \in \mathcal{Y}_\mu\} \cap \mathcal{S}(\Gamma)$$

satisfies the conclusion of the proposition. To see this, let $Y \in \mathcal{S}(\Gamma)$ be arbitrary and ν_Y be the associated Γ-invariant Γ-ergodic homogeneous measure. By Theorem 4.3.7 we know that every Γ_μ-ergodic component of ν_Y is an element of $\mathcal{S}(\Gamma_\mu)$. By Fubini's theorem and Γ-ergodicity of ν_Y, we can thus find some $Y'_\mu \in \mathcal{S}(\Gamma_\mu)$ such that almost every point $x \in Y'_\mu$ with respect to the homogeneous measure on Y'_μ satisfies $Y = \overline{\Gamma x}$. We also know that $Y'_\mu = lY_\mu$ for some $Y_\mu \in \mathcal{Y}_\mu$ and $l \in L = C_G(\Gamma)$. We can thus conclude that $Y = \overline{\Gamma Y'_\mu} = l\overline{\Gamma Y_\mu}$, which shows that $\overline{\Gamma Y_\mu} \in \mathcal{Y}$ and completes the proof. $\square$

4.5. Height Functions With Contraction Properties

A Markov chain on a standard Borel space X is defined by a measurable map $X \ni x \mapsto P_x$ from X to the space of Borel probability measures on X, specifying the transition probabilities at each point of X. The associated Markov operator P is defined by

$$Pf(x) = \int_X f \, \mathrm{d}P_x$$

for a non-negative Borel function f on X and $x \in X$. If G is a locally compact second countable group with a Borel G-action on X, then a choice of a probability measure μ on G induces a Markov chain on X with transition probabilities $P_x = \mu * \delta_x$, which can be thought of as the formalization of the concept of the random walk on X given by μ. We denote the associated Markov operator by $\pi(\mu)$, which is given in this context by the explicit formula

$$\pi(\mu)f(x) = \int_G f(gx) \, \mathrm{d}\mu(g).$$

We also refer to $\pi(\mu)$ as the "averaging operator" or "convolution operator" associated to μ. See §3.1 in Chapter 3 for background on Markov chains on general state spaces, and [**9**, §3] and [**10**, §2] more specifically for discussions of Markov operators in the context of the study of random walks.

Coming back to our setting, recall that Λ denotes a lattice in a Lie group G and H a connected semisimple subgroup of G without compact factors and with finite center, and μ is an H-expanding probability measure on H.

The goal of this section is to construct height functions on $X = G/\Lambda$ that are contracted by the averaging operator $\pi(\mu)$ (also known as "Lyapunov functions" or sometimes "Margulis functions"), which will yield the recurrence properties of the random walk on X necessary for the proof of our main theorems. As already explained in §4.0.2, two types of height functions are required. First, one needs a height function that is proper but stays bounded on prescribed compact subsets of the space X, which prevents the random walk from escaping to infinity. Secondly, in order to ensure equidistribution towards a homogeneous measure sitting on the orbit closure, we will need to construct height functions which are unbounded near lower dimensional homogeneous subspaces. These ensure that the random walk does not accumulate near such "singular subspaces", i.e. does not spend too much time in their vicinity.

4.5.1. Height Function With Respect to the Cusps. We first present the construction of the height functions responsible for ruling out escape of mass.

THEOREM 4.5.1 (Exponential μ-unstability of the cusps, [**7**]). *Let μ be an H-expanding probability measure with finite exponential moments. For any compact subset Z of $X = G/\Lambda$, there exist constants $m \in \mathbb{N}$, $a \in (0,1)$, $b > 0$, and a lower semicontinuous function $\beta_\infty \colon X \to [1,\infty]$ uniformly bounded on Z such that for every $x \in X$ we have*

$$\pi(\mu)^m \beta_\infty(x) \leq a\beta_\infty(x) + b. \tag{4.5.1}$$

Moreover,

(i) *for every $\ell > 1$, the set $\beta_\infty^{-1}([1,\ell])$ is compact,*
(ii) *the set $\beta_\infty^{-1}(\{\infty\})$ is H-invariant, and*
(iii) *there exists a constant $\kappa > 0$ such that for every $h \in H$ and $x \in G/\Lambda$ we have $\beta_\infty(hx) \leq \mathrm{N}(\mathrm{Ad}\, h)^\kappa \beta_\infty(x)$.*

In what follows, we will sometimes say that a height function is "proper" to refer to property (i) above.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{r}$ the largest amenable ideal of $\mathfrak{g}$ and $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$. A Lyapunov function as in the above theorem is constructed in [**7**] in the case the non-compact part of the Zariski closure of the support of the probability measure $(\mathrm{Ad}_\mathfrak{s})_*\mu$ is semisimple. However, as it turns out, this Zariski density assumption in a semisimple group without compact factors is only critically used, via Furstenberg's result of positivity of the top Lyapunov exponent, to ensure (4.5.2) below, which is also guaranteed by our dynamical H-expansion assumption. Therefore, Benoist–Quint's proof goes through in our setting with minor adaptations. We now explain this in more detail.

A version of the following elementary but key lemma was already used in [**40**] (see also [**8**, Lemma 6.12]). In our case, it holds true thanks to the characterization of uniform expansion expressed in Proposition 4.1.2.

LEMMA 4.5.2. *Let μ be an H-expanding probability measure on H with finite exponential moments and (V, ρ) be a representation of H without nonzero H-fixed vectors. Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and $c \in (0,1)$, for every $n \in \mathbb{N}$ large enough, we have*

$$\int_H \frac{1}{\|h \cdot v\|^\delta} \,\mathrm{d}\mu^{*n}(h) \leq \frac{c}{\|v\|^\delta} \tag{4.5.2}$$

for every $v \in V \setminus \{0\}$.

PROOF. Using the elementary fact that for every $\varepsilon \in (0,1)$, $x \in (0, \varepsilon)$ and $a > 0$, we have $a^x = 1 + x \log a + (x/\varepsilon)^2 R_a(x)$ with $|R_a(x)| \leq \mathrm{e}^{\varepsilon|\log a|}$ together with $|\log(\|v\|/\|gv\|)| \leq \log \mathrm{N}(g)$ for every $g \in \mathrm{GL}(V)$, we see that for every $n \in \mathbb{N}$, $\varepsilon \in (0,1)$ and $\delta \in (0, \varepsilon)$

$$\int_H \frac{\|v\|^\delta}{\|h \cdot v\|^\delta} \,\mathrm{d}\mu^{*n}(h) \leq 1 + \delta \int_H \log \frac{\|v\|}{\|h \cdot v\|} \,\mathrm{d}\mu^{*n}(h) + \left(\frac{\delta}{\varepsilon}\right)^2 \int_H \mathrm{N}(\rho(h))^\varepsilon \,\mathrm{d}\mu^{*n}(h). \tag{4.5.3}$$

By Proposition 4.1.2, there exists $N \in \mathbb{N}$ and $C > 0$ such that for all $v \in V \setminus \{0\}$, we have

$$\int_H \log \frac{\|v\|}{\|h \cdot v\|} \,\mathrm{d}\mu^{*N}(h) \leq -C. \tag{4.5.4}$$

Since $\rho_*\mu$ has finite exponential moments by Lemma 4.1.9, we can choose $\varepsilon_0 > 0$ such that $\int_H \mathrm{N}(\rho(h))^{\varepsilon_0} \,\mathrm{d}\mu^{*n}(h) < \infty$ for every $n \in \mathbb{N}$. Now applying (4.5.3) with $n = N$, $\varepsilon = \varepsilon_0 > 0$ and using (4.5.4), we get that for every $\delta > 0$ smaller than some $\delta_0 > 0$, there exists $c' \in (0,1)$ such that we have

$$\int_H \frac{1}{\|h \cdot v\|^\delta} \,\mathrm{d}\mu^{*N}(h) \leq \frac{c'}{\|v\|^\delta} \tag{4.5.5}$$

for every $v \in V \setminus \{0\}$. Writing an arbitrary $n \in \mathbb{N}$ as $n = mN + k$ with $m, k \in \mathbb{N}$ and $k < N$, using the facts that $\mu^{*n} = \mu^{*mN} * \mu^{*k}$, $\|h \cdot v\|^{-1} \leq \mathrm{N}(\rho(h))\|v\|^{-1}$ and the existence of finite exponential moments, iterating (4.5.5) now yields

$$\int_H \frac{1}{\|h \cdot v\|^\delta} \,\mathrm{d}\mu^{*n}(h) \leq \frac{(c')^m}{\|v\|^\delta} \left(\int_H \mathrm{N}(\rho(h))^\delta \,\mathrm{d}\mu(h)\right)^k,$$

the right-hand side of which can be made to be smaller than $c/\|v\|^\delta$ by requiring m to be large enough. □

PROOF OF THEOREM 4.5.1. We start the proof with a few general remarks on Lyapunov functions and their construction.

(1) It suffices to construct the function β_∞ with values in $[0, \infty]$. Indeed, in the end one can simply add 1, if necessary, to ensure values in $[1, \infty]$.
(2) The conclusion of the theorem is unaffected when replacing Λ by a commensurable lattice Λ', that is, a lattice such that the intersection $\Lambda_0 = \Lambda \cap \Lambda'$ has finite index in both Λ and Λ'. Indeed, given a Lyapunov function $G/\Lambda \to [0, \infty]$, one can just precompose it with

the projection $G/\Lambda_0 \to G/\Lambda$, and, conversely, starting with a function $\beta\colon G/\Lambda_0 \to [0,\infty]$, one can define the function β_∞ on G/Λ by setting

$$\beta_\infty(g\Lambda) = \sum_{\gamma\in\Lambda/\Lambda_0} \beta(g\gamma\Lambda_0)$$

for $g \in G$, which is easily seen to have the correct properties.

(3) We may always assume that the lattice Λ is non-uniform, i.e. that $X = G/\Gamma$ is non-compact. For on a compact quotient, the constant function 1 already has all required properties.

(4) In the construction, we may without loss of generality replace G by any open subgroup G_0. Indeed, X is the disjoint union of G_0-orbits, and these are Γ_μ-invariant since H is connected. Thus, one can translate a function β_∞ on $G_0/(G_0 \cap \Lambda)$ to other G_0-orbits.

From now on, we assume G is connected and prove the existence of the height function β_∞ with the required properties. The proof proceeds in several steps.

Case 1: $G = \mathrm{SL}_d(\mathbb{R})$ and $X = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$. We show that the Benoist–Quint height function in [**7**] has the required properties. We endow $E = \mathbb{R}^d$ with a Euclidean structure invariant by some maximal compact subgroup of H. We endow the vector space $\bigwedge^* E = \bigoplus_{i=0}^d \bigwedge^i E$ with the induced Euclidean structure and use $\|\cdot\|$ to denote the corresponding norm on E and on $\bigwedge^* E$. For $0 \le i \le d$, we fix constants $\delta_i = (d-i)i$; they satisfy

$$\delta_{r+s} + \delta_{r+t} \ge \delta_r + \delta_{r+s+t} + 1 \tag{4.5.6}$$

for every $r, s, t \in \mathbb{N}$ with $s > 0$ and $t > 0$.

We fix a maximal split torus A of H. Let $\mathfrak{a}$ and $\mathfrak{h}$ be the Lie algebras of A and H, respectively. Let $\Sigma(\mathfrak{h},\mathfrak{a})$ be the set of restricted roots. We fix a positive system in $\Sigma(\mathfrak{h},\mathfrak{a})$. Let $\mathcal{W} \subset \mathfrak{a}^*$ be the set of restricted weights appearing in finite-dimensional representations of H. We define a partial order on $\mathcal{W}$ by

$$\lambda \le \eta \iff \eta - \lambda \text{ is a sum of positive roots.} \tag{4.5.7}$$

Recall that any representation of a connected semisimple real Lie group is completely reducible and each irreducible representation has a unique highest weight. We denote by $\mathcal{W}^+ \subset \mathcal{W}$ the set of highest weights and let $\mathcal{S} \subset \mathcal{W}^+$ be the set of nonzero highest weights corresponding to the non-trivial irreducible representations of H appearing as direct summands in $\bigwedge^* E$, where the representation of H on E is just the restriction of the standard representation of G. So the action of H on $\bigwedge^* E$ decomposes into a direct sum

$$\bigwedge\nolimits^* E = E_*^H \oplus \bigoplus_{\lambda\in\mathcal{S}} E_*^\lambda,$$

where E_*^H is the space of H-fixed vectors in $\bigwedge^* E$ and E_*^λ is the sum of all the irreducible subspaces of $\bigwedge^* E$ with highest weight λ (i.e. the isotypic component of λ). We fix $s_0 \in \mathfrak{a}$ in the interior of the positive Weyl chamber and define $\delta_\lambda = \lambda(s_0)$ for $\lambda \in \mathcal{W}^+$, so that the δ_λ satisfy $\lambda \le \mu$ if and only if $\delta_\lambda \le \delta_\mu$ and $\delta_\lambda = 0$ if and only if $\lambda = 0$ for all $\lambda, \mu \in \mathcal{W}^+$. For $\lambda \in \mathcal{S}$, we use q_λ (resp. q_0) to denote the H-equivariant projection from $\bigwedge^* E$ to E_*^λ (resp. E_*^H). For any

$\varepsilon > 0$ and $v \in \bigwedge^i E$ with $0 < i < d$, define

$$\varphi_\varepsilon(v) = \begin{cases} \min_{\lambda\in\mathcal{S}} \varepsilon^{\delta_i/\delta_\lambda} \|q_\lambda(v)\|^{-1/\delta_\lambda}, & \text{if } \|q_0(v)\| < \varepsilon^{\delta_i}, \\ 0, & \text{otherwise,} \end{cases}$$

with the convention $\min\emptyset = \infty$. Using Lemma 4.5.2 and H-equivariance of the projections q_λ, one readily observes (cf. [**7**, Lemma 4.3]) that for every $\delta > 0$ small enough, for every $c \in (0,1)$, there exists $n \in \mathbb{N}$ such that for every $i = 1,\ldots,d$ and $v \in \bigwedge^i E$ it holds that

$$\pi(\mu)^n \varphi_\varepsilon^\delta(v) \le a\varphi_\varepsilon^\delta(v) \tag{4.5.8}$$

for every $\varepsilon > 0$. For $\varepsilon > 0$, let the function $\beta_{\varepsilon,\infty}$ on G/Λ be defined by

$$\beta_{\varepsilon,\infty}(x) = \max \varphi_\varepsilon(v),$$

where, writing $x = g\Lambda$, the maximum is taken over all $0 < i < d$ and nonzero vectors $v \in \bigwedge^i E$ that can be written as $v = v_1 \wedge \cdots \wedge v_i$ with $v_j \in \Lambda_x = g\mathbb{Z}^d$ for $j = 1,\ldots,i$ (following [**7**], such pure wedge products v will be called "x-integral monomials").

Note that by construction we have $\beta_{\varepsilon,\infty}(x) = \infty$ if and only if there exists a nonzero H-fixed x-integral monomial $v \in \bigwedge^i E$ whose norm is less than ε^{δ_i}. Therefore, the set $\beta_{\varepsilon,\infty}^{-1}(\{\infty\})$ is H-invariant. Moreover, for every $\varepsilon > 0$, the function $\beta_{\varepsilon,\infty}$ is proper and lower semicontinuous (see [**7**, Remark 5.2]). Setting $\kappa' = \max_{\lambda\in\mathcal{S}} \delta_\lambda^{-1}$, it is also readily verified that for every $h \in H$ we have $\beta_{\varepsilon,\infty}(hx) \le \mathrm{N}(h)^{d\kappa'}\beta_{\varepsilon,\infty}(x)$.

Now it follows precisely in the same way as in [**7**, Proposition 5.3], by simply replacing [**7**, Lemma 4.3] by (4.5.8), that for every $\delta > 0$ and $\varepsilon > 0$ small enough, there exist $n \in \mathbb{N}$, $a \in (0,1)$ and $b > 0$ such that

$$\pi(\mu)^n \beta_{\varepsilon,\infty}^\delta \le a\beta_{\varepsilon,\infty}^\delta + b.$$

For brevity and to avoid mere repetition, we will not reproduce this part of the proof here. We note however that this passage is the part where the crucial "Mother inequality" [**7**, §3] and the convexity assumptions (4.5.6) and (4.5.7) are used.

Finally, given a compact set Z as in the statement, by Mahler's compactness criterion, we can choose $\varepsilon > 0$ and $\delta > 0$ small enough so that the function $\beta_\infty \coloneqq \beta_{\varepsilon,\infty}^\delta$ is uniformly bounded on Z. By the discussion above, this function has all desired properties.

Case 2: G is closed subgroup of $\mathrm{SL}_d(\mathbb{R})$ and $\Lambda = G \cap \mathrm{SL}_d(\mathbb{Z})$. Then $X = G/\Lambda$ is a closed subset of $X_0 = \mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$ by [**107**, Theorem 1.13]. Thus, we can use the height function from Case 1 above.

Case 3: $G = H$ is a connected real rank one simple Lie group. We assume $X = G/\Lambda$ is noncompact. Let $V = \bigwedge^r \mathfrak{g}$ endowed with a norm $\|\cdot\|$, where r is the dimension of the unipotent radical of a minimal parabolic subgroup of G. Let $v_0 \in V$ be a nonzero vector which corresponds to the Lie algebra of such a unipotent radical. It follows from [**55**] (cf. [**74**, Proposition 3.1] and [**7**, p. 54]) that there exist $g_1,\ldots,g_r \in G$ such that for $i = 1,\ldots,r$ the vectors $v_i = g_i \cdot v_0$ in V have the following properties:

(a) Λv_i is closed and hence discrete in V for $1 \le i \le r$.

(b) For any subset $F \subset G$, the set $F\Lambda \subset G/\Lambda$ is relatively compact if and only if there exists $a > 0$ such that $\|g\gamma \cdot v_i\| > a$ for any $\gamma \in \Lambda$, $g \in F$ and $1 \leq i \leq r$.

(c) There exists $a_0 > 0$ such that for any $g \in G$ there exists at most one $v \in \bigcup_{1\leq i\leq r} \Lambda \cdot v_i$ such that $\|g \cdot v\| < a_0$.

Let V' be the H-invariant subspace complementary to V^H. In view of property (b), we know that $v_0 \in V'$. By Lemma 4.5.2, for every $\delta > 0$ small enough, for every $c > 0$, we have that for every $n \in \mathbb{N}$ large enough

$$\int_H \|h \cdot v\|^{-\delta} \,\mathrm{d}\mu^{*n}(h) < c\|v\|^{-\delta} \tag{4.5.9}$$

holds for all nonzero $v \in V'$. Using properties (a)–(c) and (4.5.9) it is straightforward to check that

$$\beta_\infty(g\Lambda) = \max_{1\leq i\leq r} \max_{\gamma\in\Lambda} \|g\gamma \cdot v_i\|^{-\delta}$$

is continuous, proper and satisfies (4.5.1) when $\delta > 0$ is small enough. It is also readily checked that $\beta_\infty(hx) \leq \mathrm{N}(\mathrm{Ad}\, h)^{\kappa'\delta}\beta_\infty(x)$ for some κ' depending only on G.

Case 4: $G = \mathrm{Aut}(\mathfrak{g})$ for $\mathfrak{g}$ semisimple without compact ideals. In view of (4) at the begining of the proof, we may assume that G is connected. As G is of adjoint type, it is center-free. By [**107**, Theorem 5.22], after replacing Λ by a finite index subgroup, there is a collection of semisimple factors G_i of G such that $G = \prod_i G_i$ and $\Lambda_i = G_i \cap \Lambda$ is an irreducible lattice in G_i. Then we have $G/\Lambda = \prod_i G_i/\Lambda_i$. Thus, if we can construct functions with the desired properties on all spaces G_i/Λ_i, then their sum is a Lyapunov function on $X = G/\Lambda$ with the same properties (possibly with different constants). In other words, we have further reduced to the case where the lattice Λ in G is irreducible. We can also assume that Λ is non-uniform in view of (3) at the beginning of the proof.

Case 3 handles the case of G with real rank one. Thus, we may additionally assume that the rank is at least two. Then Margulis' arithmeticity theorem says that Λ is arithmetic. In our setting, this implies that there is an isomorphism $\sigma\colon G \to G'$ where G' is the connected component of a semisimple real algebraic subgroup of $\mathrm{SL}_{d'}(\mathbb{R})$ defined over $\mathbb{Q}$ such that $\sigma(\Lambda)$ and $\Lambda' = G' \cap \mathrm{SL}_{d'}(\mathbb{Z})$ are commensurable (see [**136**, Corollary 6.1.10]). Then by Proposition 4.1.10(iii), $\sigma_*\mu$ is $\sigma(H)$-expanding, and we conclude using Case 2 and the comment (2) on commensurability at the start of the proof.

Case 5: General case. Let $\mathfrak{r}$ be the maximal amenable ideal of $\mathfrak{g}$, set $\mathfrak{s} = \mathfrak{g}/\mathfrak{r}$ and $R = \ker(\mathrm{Ad}_\mathfrak{s})$. Then $\mathfrak{s}$ is the largest semisimple quotient of $\mathfrak{g}$ without compact ideals and, by semisimplicity, G/R identifies with a finite index subgroup S of $\mathrm{Aut}(\mathfrak{s})$. From [**7**, Lemma 6.1] we know that $\Lambda \cap R$ is a cocompact lattice in R and the image $\Lambda_S = \mathrm{Ad}_\mathfrak{s}(\Lambda)$ is a lattice in S. In particular, the projection $G/\Lambda \to S/\Lambda_S$ is proper. Setting $H_S = \mathrm{Ad}_\mathfrak{s}(H)$, we moreover have that $(\mathrm{Ad}_\mathfrak{s})_*\mu$ is H_S-expanding by Proposition 4.1.10(iii). By Case 4 above, the theorem holds for S/Λ_S. Precomposing the obtained Lyapunov function with the projection $G/\Lambda \to S/\Lambda_S$ produces the desired function β_∞ on X. Properties (i)–(iii) carry over from the subcases, using for the latter property that the norm in the adjoint representation controls the norms in any other representation after taking a suitable power. □

Before moving on, we make a simple remark that will be of use in the next part.

Remark 4.5.3. Notice that by considering a small power of β_∞, at the cost of increasing the constants $a \in (0,1)$ and b, one can modify $\kappa > 0$ that satisfies property (iii) in Theorem 4.5.1. Indeed, given $\delta \in (0, \kappa)$, using Jensen's inequality, the function $\beta_\infty^{\delta/\kappa}$ is seen to also satisfy the contraction condition (4.5.1) with the same $m \in \mathbb{N}$ and possibly different constants $a \in (0,1)$ and $b > 0$. Moreover, $\beta_\infty^{\delta/\kappa}(hx) \leq \mathrm{N}(\operatorname{Ad} h)^\delta \beta_\infty^{\delta/\kappa}(x)$. ◇

4.5.2. Height Function With Respect to Singular Subspaces. In this section we construct a height function with respect to a relatively compact subset of a lower-dimensional homogeneous subspace of $X = G/\Lambda$. In contrast to the height function used in [**8**], which satisfies a contraction property with respect to a first return Markov operator, our height function will satisfy a contraction property with respect to $\pi(\mu)$ itself. Our construction is inspired by the work of Eskin–Mirzakhani–Mohammadi [**43**] on random walks on moduli space.

To state the main result of this subsection, we start by recalling some notation and fixing some data. Let G be a Lie group and $\Lambda < G$ a lattice. Let $H \leqslant G$ be a connected semisimple Lie subgroup with finite center and no compact factors. Let μ be an H-expanding probability measure on H with finite exponential moments. Since μ has finite exponential moments, we can fix $\delta_0 \in (0,1)$ such that $\int_H \mathrm{N}(\operatorname{Ad}(h))^{\delta_0}\,\mathrm{d}\mu(h) < \infty$. Fix an arbitrary compact subset Z of G/Λ and let $\beta_\infty \colon G/\Lambda \to [1,\infty]$ be the proper lower semicontinuous function given by Theorem 4.5.1. By passing to a small enough power, we will suppose that β_∞ satisfies $\beta_\infty(hx) \leq \mathrm{N}(\operatorname{Ad}(h))^{\delta_0}\beta_\infty(x)$ for every $h \in H$ and $x \in G/\Lambda$ (see Remark 4.5.3). Moreover, given $\varepsilon > 0$, we define

$$X_\varepsilon = \{x \in G/\Lambda \mid \beta_\infty(x) \leq \varepsilon^{-1}\}.$$

Since β_∞ is lower semicontinuous and proper, X_ε is a compact subset of X. Here is the result we aim to prove.

Theorem 4.5.4. *Given $\varepsilon > 0$ sufficiently small, for any sufficiently small open neighborhood O of the identity in $C_G(\Gamma_\mu)$ and for any $Y \in \mathcal{S}(\Gamma_\mu)$ there exists a height function $\beta_\mathcal{N} \colon HX_\varepsilon \to [1,\infty]$ together with constants $n \in \mathbb{N}$, $a_0 \in (0,1)$ and $b_0 > 0$ such that for any $x \in HX_\varepsilon$ we have*

$$\pi(\mu)^n \beta_\mathcal{N}(x) \leq a_0 \beta_\mathcal{N}(x) + b_0,$$

and such that, denoting $\mathcal{N} = OY$,

(i) *$\beta_\mathcal{N}(x) = \infty$ if and only if $x \in \mathcal{N} \cap HX_\varepsilon$,*
(ii) *$\beta_\mathcal{N}$ is bounded on compact subsets of $X_\varepsilon \setminus \overline{O}Y$,*
(iii) *for any $\ell \geq 1$, the set $\beta_\mathcal{N}^{-1}([1,\ell])$ is a compact subset of X.*

The rest of this subsection is devoted to the proof of this result, which will require two preliminary lemmas. We fix an inner product on $\mathfrak{g}$, denote by $\|\cdot\|$ the associated operator norm on $\operatorname{End}(\mathfrak{g})$, and to ease the notation, we set

$$\mathrm{N}_a(h) := \mathrm{N}(\operatorname{Ad}(h)) = \max(\|\operatorname{Ad}(h)\|, \|\operatorname{Ad}(h^{-1})\|),$$

where Ad denotes the adjoint action of H on the Lie algebra $\mathfrak{g}$ of G.

LEMMA 4.5.5. *There exist constants $C \geq 1$, $k \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and any $x \in HX_\varepsilon$ there exists $h \in \Gamma_\mu$ with $\mathrm{N}_a(h) \leq C\beta_\infty(x)^k$ such that $hx \in X_\varepsilon$.*

PROOF. Set $M \coloneqq \int \mathrm{N}_a(h)^{\delta_0}\,\mathrm{d}\mu(h) < \infty$ and let a positive

$$\varepsilon < \min\left(\frac{1}{4}, \left(\frac{1-a}{1-a+b}\right)^2\right) =: \varepsilon_0$$

be given, where $a \in (0,1)$ and $b > 0$ are the constants given by Theorem 4.5.1. Let $x \in HX_\varepsilon$. Since $\beta_\infty^{-1}(\{\infty\})$ is H-invariant, we have $\beta_\infty(x) < \infty$, so that we may define $n_x \geq 1$ to be the smallest integer such that $a^{n_x}\beta_\infty(x) \leq 1$. It follows that

$$\pi(\mu)^{mn_x}\beta_\infty(x) \leq a^{n_x}\beta_\infty(x) + \frac{b}{1-a} \leq \frac{1}{\sqrt{\varepsilon}},$$

where $m \in \mathbb{N}$ is as in Theorem 4.5.1.

Now decompose μ^{*mn_x} as a sum of two non-negative measures $\mu_1 + \mu_2$ where μ_2 is the restriction of μ^{*mn_x} to the set $\{\mathrm{N}_a(\cdot) \geq R_x\}$ for $R_x = 2^{1/\delta_0}M^{mn_x/\delta_0}$. By submultiplicativity of N_a we have $\int \mathrm{N}_a(h)^{\delta_0}\,\mathrm{d}\mu^{*mn_x}(h) \leq M^{mn_x}$. Using this bound together with the Markov inequality, we deduce that $\mu_2(H) \leq 1/2$ and hence $\mu_1(H) \geq 1/2 \geq \sqrt{\varepsilon}$. On the other hand, we know

$$\int_H \beta_\infty(hx)\,\mathrm{d}\mu_1(h) \leq \pi(\mu)^{mn_x}\beta_\infty(x) \leq \frac{1}{\sqrt{\varepsilon}}.$$

Now, considering the probability measure $\hat{\mu}_1 = \mu_1(H)^{-1}\mu_1$, we deduce that $\pi(\hat{\mu}_1)\beta_\infty(x) \leq \varepsilon^{-1}$. This means that there exists $h \in \mathrm{supp}(\hat{\mu}_1) \subset \Gamma_\mu$ such that $\beta_\infty(hx) \leq \varepsilon^{-1}$. Finally, since by construction $n_x \leq 1 + (\log\beta_\infty(x))/(-\log a)$, we also obtain

$$\mathrm{N}_a(h) \leq R_x = 2^{1/\delta_0}M^{mn_x/\delta_0} \leq 2^{1/\delta_0}M^{m/\delta_0}\beta_\infty(x)^{(m\log M)/(-\delta_0\log a)}.$$

This shows that the statement of the lemma holds with $C = 2^{1/\delta_0}M^{m/\delta_0}$ and $k = m\lceil(\log M)/(-\delta_0\log a)\rceil$. □

Let Y be a homogeneous space in $\mathcal{S}(\Gamma_\mu)$ and denote by N its stabilizer group. Recall that this means that $N \geqslant \Gamma_\mu$ is a closed subgroup of G, Y is given by Nx for some $x \in G/\Lambda$, and there is an N-invariant probability measure on Nx which is invariant and ergodic with respect to Γ_μ. By Theorem 4.0.1, the Lie algebra $\mathfrak{n}$ of N is H-invariant with respect to the adjoint action. We write $\mathfrak{g}$ as a direct sum of $\mathrm{Ad}(H)$-invariant subspaces

$$\mathfrak{g} = (\mathfrak{n} + \mathfrak{l}) \oplus \mathfrak{v}, \tag{4.5.10}$$

where $\mathfrak{l}$ is the centralizer of $\mathfrak{h}$ and $\mathfrak{v}$ is a complementary H-invariant subspace of $\mathfrak{n} + \mathfrak{l}$. Recall that by the epimorphic property of Γ_μ in H, $\mathfrak{l}$ is also the Lie algebra of $C_G(\Gamma_\mu)$.

LEMMA 4.5.6. *Let $Y \in \mathcal{S}(\Gamma_\mu)$ and retain the notation of the previous paragraph. Given a compact set $K \subset X = G/\Lambda$, there exist an open neighborhood O*

of the identity in $C_G(\Gamma_\mu)$ *and* $r \in (0,1)$ *with the property that for any* $x \in K$, *there is at most one* $v \in \mathfrak{v}$ *such that*

$$\exp(v)x \in OY \quad \textit{and} \quad \|v\| < r. \tag{4.5.11}$$

Moreover, the set E *of* $x \in X$ *for which* $v \in \mathfrak{v}$ *with* (4.5.11) *exists is open in* X *and the map* $E \cap K \to \mathfrak{v}$, $x \mapsto v$ *is continuous.*

Proof. Let K' be a compact neighborhood of K. In view of (4.5.10), we can choose O, r and a neighborhood U of the identity in G so that all of the following hold:

(a) we have $UK \subset K'$,
(b) the natural map $U \to Uy$ is injective for all $y \in K'$,
(c) for every $y \in Y \cap K'$ we have $Uy \cap Y = (U \cap N)y$,
(d) the map $B_r(\mathfrak{v}) \times (U \cap C_G(\Gamma_\mu)N) \to G$, $(v,g) \mapsto \exp(v)g$ is a diffeomorphism onto an open neighborhood of the identity in G, where $B_r(\mathfrak{v})$ denotes the open r-ball in $\mathfrak{v}$, and
(e) we have $o_2^{-1}\exp(v_2)\exp(-v_1)o_1 \in U$ for every $v_1, v_2 \in \mathfrak{g}$ with $\|v_i\| < r$, $i = 1,2$, and $o_1, o_2 \in O$.

Now let $x \in K$ and $v_1, v_2 \in \mathfrak{v}$ satisfy (4.5.11), say $\exp(v_i)x = o_i y_i$ with $o_i \in O$ and $y_i \in Y$ for $i = 1,2$. Using properties (a) and (e) we know $y_1 \in K'$. Moreover, $y_2 = o_2^{-1}\exp(v_2)\exp(-v_1)o_1 y_1$. Applying properties (b), (c) and (e), we deduce that

$$o_2^{-1}\exp(v_2)\exp(-v_1)o_1 = n \in U \cap N,$$

which means that

$$\exp(-v_1)o_1 = \exp(-v_2)o_2 n.$$

Using (e) once more, we see that $o_1, o_2 n \in U \cap C_G(\Gamma_\mu)N$. Hence, property (d) implies that $v_1 = v_2$, giving uniqueness. Since $O \subset U$, the final claims of the lemma also follows from (d). $\square$

Proof of Theorem 4.5.4. Since there is a substantial amount of relevant notation and auxiliary objects, let us start the proof by recalling the initial data. The probability measure μ on H is H-expanding with finite exponential moments, Z is a compact subset of $X = G/\Lambda$ and $\beta_\infty\colon G/\Lambda \to [1,\infty]$ is as given by Theorem 4.5.1. By the latter (and Remark 4.5.3), the function β_∞ satisfies (4.5.1) with some $m \in \mathbb{N}$, $a \in (0,1)$ and $b > 0$ and $\beta_\infty(hx) \le \mathrm{N}_a(h)^{\delta_0}\beta_\infty(x)$ for every $x \in G/\Lambda$ and $h \in H$, where $\delta_0 \in (0,1)$ is chosen so that $\int_H \mathrm{N}_a(h)^{\delta_0}\,\mathrm{d}\mu(h) < \infty$. Let $\varepsilon_0 > 0$, $k \in \mathbb{N}$ and $C \ge 1$ be given by Lemma 4.5.5 and fix $\varepsilon \in (0,\varepsilon_0)$. Let O be a relatively compact open neighborhood of the identity in $C_G(\Gamma_\mu)$ and $r \in (0,1)$ such that the conclusion of Lemma 4.5.6 holds with a compact neighborhood K of

$$X_\varepsilon = \{x \in X \mid \beta_\infty(x) \le \varepsilon^{-1}\}.$$

Let $Y \in \mathcal{S}(\Gamma_\mu)$, denote by N its stabilizer group, by $\mathfrak{n}$ its Lie algebra, and set $\mathcal{N} = OY$. Finally, let $\mathfrak{l}$ be the Lie algebra of $C_G(\Gamma_\mu)$ and choose an $\mathrm{Ad}(H)$-invariant complementary space $\mathfrak{v}$ so that (4.5.10) holds.

Since μ is H-expanding with finite exponential moments and $\mathfrak{v}$ has no nonzero H-fixed vectors, by Lemma 4.5.2 there exists

$$0 < \theta < \min(\delta_0, 1/k) \tag{4.5.12}$$

such that for every $a' \in (0,1)$ we have, for all $n \in \mathbb{N}$ large enough,

$$\int_H \|\mathrm{Ad}(h)v\|^{-\theta} \, \mathrm{d}\mu^{*n}(h) \le a' \|v\|^{-\theta} \tag{4.5.13}$$

for any nonzero $v \in \mathfrak{v}$. We fix such $n \in \mathbb{N}$ that is a positive multiple of $m \in \mathbb{N}$. Without loss of generality, we assume $a' > a$ and let $\varepsilon' > 0$ be such that $a' = (1+\varepsilon')a$. Since $m|n$, (4.5.1) implies that

$$\int_H \beta_\infty(hx) \, \mathrm{d}\mu^{*n}(h) \le a\beta_\infty(x) + \frac{b}{1-a}. \tag{4.5.14}$$

For $x \in HX_\varepsilon$, we define

$$r_x = rC^{-1}\beta_\infty(x)^{-k}.$$

We claim that for every $x \in HX_\varepsilon$, there exists at most one $v \in \mathfrak{v}$ such that

$$\exp(v)x \in \mathcal{N} \quad \text{and} \quad \|v\| < r_x. \tag{4.5.15}$$

Indeed, by Lemma 4.5.5, there exists $h \in \Gamma_\mu$ with $\mathrm{N}_a(h) \le C\beta_\infty(x)^k$ such that $hx \in X_\varepsilon$. Since $\mathcal{N}$ is Γ_μ-invariant, we have

$$\exp(v)x \in \mathcal{N} \quad \text{if and only if} \quad h\exp(v)x = \exp(\mathrm{Ad}(h)v)hx \in \mathcal{N}.$$

Since $\|\mathrm{Ad}(h)v\| \le \mathrm{N}_a(h)\|v\| \le r$, if such an $v \in \mathfrak{v}$ exists, it is unique thanks to Lemma 4.5.6 (applied to $hx \in X_\varepsilon$) and the choice of $r > 0$, where we are using that $\mathfrak{v}$ is H-invariant.

Using the claim above, we may define $\alpha\colon HX_\varepsilon \to [1,\infty]$ by

$$\alpha(x) = \begin{cases} \|v\|^{-\theta}, & \text{if there exists } v \in \mathfrak{v} \text{ satisfying (4.5.15)}, \\ r_x^{-\theta}, & \text{otherwise.} \end{cases}$$

Using the corresponding property for β_∞ and the choice of θ in (4.5.12), it is readily checked that for every $x \in HX_\varepsilon$ and $h \in \Gamma_\mu$, we have the inequality $\alpha(hx) \le \mathrm{N}_a(h)^{\delta_0}\alpha(x)$. We shall show that

$$\beta_{\mathcal{N}} = \beta_\infty(x) + \alpha(x).$$

satisfies all requirements of the theorem.

To proceed, we start by decomposing μ^{*n} as a sum $\mu_1 + \mu_2$ of two non-negative measures with μ_1 of compact support and μ_2 satisfying

$$\int_H \mathrm{N}_a(h)^{\delta_0} \, \mathrm{d}\mu_2(h) < \frac{1-a'}{2}.$$

It follows that

$$\int_H \alpha(hx) \, \mathrm{d}\mu_2(h) \le \alpha(x) \int_H \mathrm{N}_a(h)^{\delta_0} \, \mathrm{d}\mu_2(h) \le \alpha(x)\frac{1-a'}{2}. \tag{4.5.16}$$

Denote by D the constant $r^{-1}CM^k$, where $M = \sup\{\mathrm{N}_a(h) \mid h \in \mathrm{supp}(\mu_1)\}$. Then $D > M^k \ge 1$ by choice of r, and for any element

$$h \in S_\pm \coloneqq \mathrm{supp}(\mu_1) \cup \mathrm{supp}(\mu_1)^{-1}$$

we have

$$\beta_\infty(hx) \le M\beta_\infty(x) \text{ and hence } r_x \le D r_{hx}. \tag{4.5.17}$$

We are now going to establish the contraction property for $\beta_{\mathcal{N}}$ by distinguishing several cases based upon the size of $\alpha(x)$.

If $\alpha(x) > D^2 r_x^{-\theta}$, then there exists a uniquely determined $v \in \mathfrak{v}$ so that (4.5.15) holds and $\alpha(x) = \|v\|^{-\theta}$. In particular,

$$\|v\| < D^{-2/\theta} r_x < D^{-2} r_x.$$

Together with (4.5.17), the previous inequality implies that for $h \in S_\pm$, we have

$$\|\mathrm{Ad}(h)v\| \le \mathrm{N}_a(h) \cdot \|v\| < D \cdot D^{-2} r_x = D^{-1} r_x \le r_{hx}. \tag{4.5.18}$$

Since $\exp(v)x$ belongs to the Γ_μ-invariant set $\mathcal{N}$, we have $\exp(\mathrm{Ad}(h)v)hx \in \mathcal{N}$. In view of (4.5.18) and the definition of α it follows that $\alpha(hx) = \|\mathrm{Ad}(h)v\|^{-\theta}$. By (4.5.13),

$$\int_H \alpha(hx)\,\mathrm{d}\mu_1(h) = \int_H \|\mathrm{Ad}(h)v\|^{-\theta}\,\mathrm{d}\mu_1(h) \le \int_H \|\mathrm{Ad}(h)v\|^{-\theta}\,\mathrm{d}\mu^{*n}(h) \le a'\alpha(x).$$

Combining with (4.5.16), we get

$$\int_H \alpha(hx)\,\mathrm{d}\mu^{*n}(h) = \int_H \alpha(hx)\,\mathrm{d}(\mu_1+\mu_2)(h) \le \frac{1+a'}{2}\alpha(x).$$

Together with (4.5.14), the previous inequality yields

$$\int_H \beta_{\mathcal{N}}(hx)\,\mathrm{d}\mu^{*n}(h) \le \frac{1+a'}{2}\beta_{\mathcal{N}}(x) + \frac{b}{1-a}.$$

Therefore, we proved the contraction property of $\beta_{\mathcal{N}}$ for $x \in HX_\varepsilon$ satisfying $\alpha(x) > D^2 r_x^{-\theta}$.

Now let $x \in HX_\varepsilon$ be such that $\alpha(x) \le D^2 r_x^{-\theta}$. In this case, we have

$$\alpha(x) \le D^2 r_x^{-\theta} = D^2 r^{-\theta} C^\theta \beta_\infty^{k\theta}(x) \le D^3 \beta_\infty(x). \tag{4.5.19}$$

We claim that for any $h \in S_\pm$, we have

$$\alpha(hx) \le D^4 r_{hx}^{-\theta}. \tag{4.5.20}$$

If not, then using (4.5.17) and the fact that $\alpha(hx) \le M\alpha(x) \le D\alpha(x)$, we find

$$\alpha(x) \ge D^{-1}\alpha(hx) > D^{-1}\cdot D^4 r_{hx}^{-\theta} = D^3 r_{hx}^{-\theta} \ge D^{3-\theta} r_x^{-\theta},$$

which contradicts the first inequality in (4.5.19) since $\theta \in (0,1)$ and $D > 1$. By (4.5.20) and (4.5.17)

$$\alpha(hx) \le D^4 r_{hx}^{-\theta} = D^4 r^{-\theta} C^\theta \cdot \beta_\infty^{k\theta}(hx) \le D^5 \beta_\infty^{k\theta}(x) = D^5 \beta_\infty^{k\theta-1}(x)\cdot\beta_\infty(x).$$

Since $k\theta < 1$, if $\beta_\infty(x)$ is larger than some constant depending only on $\varepsilon' a, k\theta$ and D, we will have

$$D^5 \beta_\infty^{k\theta-1}(x) < \varepsilon' a.$$

In view of (4.5.19), we know that $\beta_\infty(x)$ is sufficiently large provided that $\alpha(x)$ is (depending on D). Therefore, there exists $b' > 0$ (depending on $\varepsilon' a, k\theta, D$) so that if

$$b' \le \alpha(x) \le D^2 r_x^{-\theta}, \tag{4.5.21}$$

then for any $h \in S_\pm$

$$\alpha(hx) \leq \varepsilon' a \beta_\infty(x). \tag{4.5.22}$$

So in the case where (4.5.21) holds, combining (4.5.14), (4.5.16) and (4.5.22), we deduce

$$\int_H \beta_{\mathcal{N}}(hx)\,\mathrm{d}\mu^{*n}(h) \leq \frac{1+a'}{2}\beta_{\mathcal{N}}(x) + \frac{b}{1-a},$$

proving the required contraction property.

To treat the last remaining case, suppose now that $x \in HX_\varepsilon$ is such that $\alpha(x) \leq \min(b', D^2 r_x^{-\theta})$. We claim that this implies $\alpha(hx) \leq D^3 b'$ for all $h \in S_\pm$. Supposing the contrary, we would have

$$\alpha(hx) > D^3 b' \geq D^3 \alpha(x) \geq D^3 r_x^{-\theta}.$$

From this, using the inequality $\alpha(hx) \leq D\alpha(x)$, it follows that

$$\alpha(x) \geq D^{-1}\alpha(hx) > D^2 r_x^{-\theta},$$

a contradiction. Therefore, recalling (4.5.14) and (4.5.16), we obtain

$$\begin{aligned}
\int_H \beta_{\mathcal{N}}(hx)\,\mathrm{d}\mu^{*n}(h) &= \int_H \alpha(hx)\,\mathrm{d}\mu^{*n}(h) + \int_H \beta_\infty(hx)\,\mathrm{d}\mu^{*n}(h)\\
&\leq D^3 b' + \frac{1-a'}{2}\alpha(x) + a\beta_\infty(x) + \frac{b}{1-a}\\
&\leq \frac{1+a'}{2}\beta_{\mathcal{N}}(x) + D^3 b' + \frac{b}{1-a}.
\end{aligned}$$

We have thus concluded the proof of the contraction property for $a_0 = (1+a')/2$ and the additive constant $b_0 = D^3 b' + b/(1-a)$.

It remains to prove the claims (i)–(iii). Since β_∞ is finite on HX_ε, (i) is directly seen to hold by definition of $\beta_{\mathcal{N}}$. Property (ii) is also immediate from the definition of $\beta_{\mathcal{N}}$, since β_∞ is bounded on X_ε and any compact subset not intersecting $\overline{O}Y$ has positive distance to $\mathcal{N}$. To prove (iii), let $(x_j)_j$ be a sequence in HX_ε with $\beta_{\mathcal{N}}(x_j) \leq \ell$ for all $j \in \mathbb{N}$ for some $\ell \in \mathbb{R}$. Since $\beta_{\mathcal{N}} = \beta_\infty + \alpha$ with $\alpha \geq 0$, we also have $\beta_\infty(x_j) \leq \ell$ for all j. Since β_∞ is proper, we may suppose that $\lim_{j\to\infty} x_j = x$ for some point $x \in X$. We need to prove that $x \in HX_\varepsilon$ and $\beta_{\mathcal{N}}(x) \leq \ell$.

We first show that $x \in HX_\varepsilon$. It follows from Lemma 4.5.5 that there is a compact subset K_ℓ of Γ_μ such that for any $j \in \mathbb{N}$, there exists $h_j \in K_\ell$ so that $h_j x_j \in X_\varepsilon$. Since X_ε is compact, by possibly passing to a subsequence, we may assume that $h_j x_j$ converges to some $y \in X_\varepsilon$ and h_j converges to some $h \in \Gamma_\mu$. So we have

$$\lim_{j\to\infty} h_j x_j = hx = y,$$

which implies $x = h^{-1}y \in HX_\varepsilon$.

Finally, we show that

$$\alpha(x) \leq \liminf_{j\to\infty} \alpha(x_j), \tag{4.5.23}$$

which will complete the proof in view of the lower semicontinuity of β_∞ and the definition of $\beta_{\mathcal{N}}$. First, let us pass to a subsequence so that the liminf in (4.5.23) is a limit, say $\liminf_{j\to\infty}\alpha(x_j) = \lim_{j\to\infty}\alpha(x_j) =: \alpha_1$. If $\alpha(x) = r_x^{-\theta}$, then (4.5.23) follows from the definition of r_x and lower semicontinuity of β_∞. Suppose therefore that $\alpha(x) > r_x^{-\theta}$. This implies that there exists a unique $v \in \mathfrak{v}$ such that $\exp(v)x \in \mathcal{N}$ and $\|v\| < r_x$. Using Lemma 4.5.5, choose $h \in \Gamma_\mu$ with $\mathrm{N}_a(h) \le C\beta_\infty(x)^k$ such that $hx \in X_\varepsilon$. Then $\|\mathrm{Ad}(h)v\| < r$ and $\exp(\mathrm{Ad}(h)v)hx \in \mathcal{N}$. Now, since the points hx_j converge to hx, for large j they lie in the neighborhood K of X_ε to which we applied Lemma 4.5.6. Thus, the last claim in this lemma imply that there exist $v_j \in \mathfrak{v}$ with $v_j \to v$ such that $\exp(v_j)x_j \in \mathcal{N}$. Note that since the values $r_{x_j}^{-\theta}$ are contained in $[0, \ell]$, up to passing to a further subsequence, we may suppose that they converge to α_2. Clearly, $\alpha_1 \ge \alpha_2$. If $\alpha_1 > \alpha_2$, then for large j we have $\alpha(x_j) \ge \|v_j\|^{-\theta}$ and it follows that (4.5.23) holds since $\|v_j\|^{-\theta} \to \|v\|^{-\theta} = \alpha(x)$. On the other hand, in case $\alpha_1 = \alpha_2$ we know that for every $\varepsilon > 0$, for $j \in \mathbb{N}$ large enough, we have $\|v_j\| + \varepsilon > r_{x_j}$. But since $v_j \to v$ and $\varepsilon > 0$ is arbitrary, this implies that $\alpha(x) = \|v\|^{-\theta} \le \lim_{j\to\infty} r_{x_j}^{-\theta} = \alpha_2 = \alpha_1$, as desired. □

4.6. Recurrence, Equidistribution, Topology of Homogeneous Measures

Using the ingredients from §§4.3–4.5, we can now give the proofs of our results on recurrence, orbit closures, equidistribution, and topology of $\mathcal{S}(\Gamma_\mu)$. The following lemma is used to extract the necessary information from the height functions constructed in the previous section.

LEMMA 4.6.1. *Let H be a locally compact σ-compact metrizable group and X a locally compact σ-compact metrizable space endowed with a continuous H-action. Let μ be a Borel probability measure on H and $\beta\colon X \to [1, \infty]$ be a lower semicontinuous function such that there exist $m \in \mathbb{N}$, $a \in (0, 1)$ and $b > 0$ such that*

$$\pi(\mu)^m \beta(x) \le a\beta(x) + b \tag{4.6.1}$$

for all $x \in X$. Suppose that for every $\varepsilon > 0$ the set $X_\varepsilon = \beta^{-1}([0, \varepsilon^{-1}])$ is compact and that the set $X_\infty = \beta^{-1}(\{\infty\})$ is Γ_μ-invariant. Then the following holds:

(i) *For any $\delta > 0$ there exists a compact subset $R_\delta \subset X \setminus X_\infty$ such that for any $x \in X$ with $\beta(x) < \infty$ there exists $n_x \in \mathbb{N}$ with $n_x = O(\log \beta(x))$ such that*

$$\mu^{*n} * \delta_x(R_\delta) \ge 1 - \delta$$

for every $n \ge n_x$.

(ii) *For every $x \in X$ with $\beta(x) < \infty$, for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in \Gamma_\mu^{\mathbb{N}}$, every weak* limit ν of the sequence $(\frac{1}{n}\sum_{k=0}^{n-1} \delta_{g_k\cdots g_1 x})_n$ of empirical measures satisfies $\nu(X \setminus X_\infty) = 1$.*

The techniques going into the first part of the lemma are by now standard. The second part is basically [**9**, Proposition 3.9]. Related ideas also appear in Markov chain theory (see e.g. [**90**, Theorem 18.5.2] and the references given there). We include a brief proof for convenience.

Proof. Let $x \in X$ be such that $\beta(x) < \infty$. Iterating (4.6.1), we find for every $\varepsilon > 0$ and $n \in \mathbb{N}$

$$\mu^{*mn} * \delta_x(X_\varepsilon^c) \leq \varepsilon \int_H \beta(hx)\,\mathrm{d}\mu^{*mn}(h) \leq \varepsilon\Big(a^n\beta(x) + \frac{b}{1-a}\Big).$$

For the proof of (i), given $\delta > 0$, we set $\varepsilon = \delta(1-a)/(2b+2)$. Then the above estimate implies that for every $n \geq n_{0,x} := \lceil(\log\beta(x))/(-\log a)\rceil$, we have $\mu^{*mn} * \delta_x(X_\varepsilon) \geq 1 - \delta/2$. Moreover, we may choose a compact subset F of Γ_μ such that $\mu^{*l}(F) \geq 1 - \delta/2$ for all $0 \leq l < m$. Now setting R_δ to be the compact set FX_ε which, since $X \setminus X_\infty$ is Γ_μ-invariant, is contained in $X \setminus X_\infty$, we find

$$\mu^{*n} * \delta_x(R_\delta) \geq 1 - \delta$$

for all $n \geq n_x := mn_{0,x}$.

For (ii), we appeal to [**9**, Proposition 3.9], which implies that for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in \Gamma_\mu^{\mathbb{N}}$, for every $\delta > 0$ there exists a compact subset $K \subset X \setminus X_\infty$ we have

$$\liminf_{n\to\infty} \tfrac{1}{n}|\{0 \leq k < n \mid g_{km}\cdots g_1 x \in K\}| \geq 1 - \delta/2.$$

Moreover, by the law of large numbers, by choosing a large enough compact set $F \subset \Gamma_\mu$ we can ensure that for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in \Gamma_\mu^{\mathbb{N}}$

$$\liminf_{n\to\infty} \tfrac{1}{n}|\{0 \leq k < n \mid g_{km+l}\cdots g_{km+1} \in F \text{ for } 0 \leq l < m\}| \geq 1 - \delta/2.$$

Combining the above, it follows that for the compact subset $R = FK$ of $X \setminus X_\infty$ we have

$$\liminf_{n\to\infty} \tfrac{1}{n}|\{0 \leq k < n \mid g_k\cdots g_1 x \in R\}| \geq 1 - \delta$$

for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in \Gamma_\mu^{\mathbb{N}}$, and we conclude using a version of the Portmanteau lemma. $\square$

4.6.1. Recurrence. We first prove our results about recurrence properties of H-expanding random walks.

Proof of Theorem 4.0.4. Let Z be a compact subset of $X \setminus \mathcal{N}$, where we recall that $\mathcal{N} = K_LY$ for a compact subset K_L of $L = C_G(\Gamma_\mu)$, and let β_∞ be a height function coming from Theorem 4.5.1 such that β_∞ is bounded on Z, say $Z \subset X_\varepsilon = \{x \in X \mid \beta_\infty(x) \leq \varepsilon^{-1}\}$ for some $\varepsilon > 0$. If $Y = \emptyset$, we set $\beta = \beta_\infty$. Otherwise, we apply Theorem 4.5.4 to $Y_l = lY$ for finitely many points $l \in L$ such that the associated neighborhoods O_l of the identity in L coming out of the theorem satisfy $\overline{O_l}lY \cap Z = \emptyset$ and $K_L \subset \bigcup_l O_l l$. The associated height functions β_l take the value ∞ on $O_l lY$ and are bounded on Z. We set $\beta = \sum_l \beta_l$.

In both cases, we now apply Lemma 4.6.1(i) to the height function β. The set R_δ coming out of the lemma is a compact subset of $X \setminus \mathcal{N}$ such that for every $x \in X$ with $\beta(x) < \infty$, for $n \geq n_x$ with $n_x = O(\log\beta(x))$, we have $\mu^{*n} * \delta_x(R_\delta) \geq 1 - \delta$. Since β is bounded on Z by construction, this estimate holds for all $n \geq n_0$ for all $x \in Z$. If F is a compact subset of Γ_μ such that $\mu^{*n}(F) \geq 1 - \delta$ for all $0 \leq n < n_0$, it follows that $\mu^{*n} * \delta_x(M_{Z,\delta}) \geq 1 - \delta$ for all $n \geq 0$ and all $x \in Z$ for the compact subset $M_{Z,\delta} := R_\delta \cup FZ$ of $X \setminus \mathcal{N}$, where we used for the last containment that $\beta^{-1}(\{\infty\})$ is Γ_μ-invariant. $\square$

REMARK 4.6.2. For $Y = \emptyset$, the recurrence property in Theorem 4.0.4 is referred to as (R1) in [**7, 40**]. In the case of a random walk given by a G-expanding probability measure on the quotient of G by an irreducible lattice, a slightly stronger, "uniform" recurrence property (referred to as (R2)) can be established by using some results of [**40**]. $\diamond$

4.6.2. Orbit Closures and Equidistribution. We now turn to the proof of Theorem 4.0.5. It is similar to the proofs of the main results in [**9**].

PROOF OF THEOREM 4.0.5. By Lemma 0.3.1, it suffices to establish statement (iii) for a Γ_μ-ergodic homogeneous subspace Y_x containing x. For $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in H^{\mathbb{N}}$, every weak* limit ν of the sequence $(\frac{1}{n}\sum_{k=0}^{n-1}\delta_{g_k\cdots g_1 x})_n$ of empirical measures is μ-stationary by the Breiman law of large numbers (see [**9**, Corollary 3.3]). By Theorem 4.5.1 and Lemma 4.6.1(ii), for $\mu^{\otimes\mathbb{N}}$-a.e. $(g_i)_i \in H^{\mathbb{N}}$ every such weak* limit is a probability measure on X. We restrict to a full measure set of $(g_i)_i$ where both these conclusions hold and let ν be a weak* limit of the sequence of empirical measures.

Let Y_0 be a Γ_μ-invariant homogeneous subspace of X containing x of minimal dimension. By Theorem 4.0.1 every ergodic component of ν is the homogeneous probability measure associated to an element of

$$\mathcal{S}(\Gamma_\mu, Y_0) := \{Y \in \mathcal{S}(\Gamma_\mu) \mid Y \subset Y_0\}.$$

Let $Y \in \mathcal{S}(\Gamma_\mu, Y_0)$ be such that Y is not open in Y_0. Then by minimality of $\dim(Y_0)$ we know that $x \notin lY$ for any $l \in L := C_G(\Gamma_\mu)$.

Let Z be an arbitrary compact subset of X, take a height function β_∞ as in Theorem 4.5.1, and recall that $X_\varepsilon = \beta_\infty^{-1}([1, \varepsilon^{-1}])$. By Theorem 4.5.4, for sufficiently small $\varepsilon > 0$, there is an open neighborhood O of the identity in L and a height function $\beta_{\mathcal{N}}\colon HX_\varepsilon \to [1, \infty]$ satisfying the contraction property (4.6.1) and such that

- for $x \in HX_\varepsilon$, $\beta_{\mathcal{N}}(x) = \infty$ if and only if $x \in OY$,
- for every $\ell \geq 1$, $\beta_{\mathcal{N}}^{-1}([1, \ell])$ is a compact subset of X.

We extend $\beta_{\mathcal{N}}$ to all of X with the value ∞ outside of HX_ε. Then the extension satisfies the assumptions of Lemma 4.6.1. Write $X_{\infty,\mathcal{N}}$ for the set $\beta_{\mathcal{N}}^{-1}(\{\infty\})$, so that $HX_\varepsilon \cap OY \subset X_{\infty,\mathcal{N}}$. After further restricting to a full measure set of $(g_i)_i$ so that Lemma 4.6.1(ii) holds, we thus find $\nu(HX_\varepsilon \cap OY) = 0$. When ε is small enough, this implies $\nu(Z \cap OY) = 0$. We repeat this process for the homogeneous subspaces lY for countably many $l \in L$ such that the translations Ol of the associated neighborhoods O cover L. This gives $\nu(Z \cap LY) = 0$. Repeating again for countably many compact subsets Z covering X, it follows that $\nu(LY) = 0$.

Hence, in view of the countability statement in Proposition 4.4.1, we deduce that $\nu(LY) = 0$ holds for every $Y \in \mathcal{S}(\Gamma_\mu, Y_0)$ that is not open in Y_0 (to be precise, after once more restricting to a countable intersection of full measure sets of $(g_i)_i \in H^{\mathbb{N}}$, once for each Y in a countable set of representatives in (4.4.1)). It follows that each ergodic component of ν must be a homogeneous measure of some $Y \in \mathcal{S}(\Gamma_\mu, Y_0)$ that is open in Y_0. By [**9**, Lemma 2.9], these Y are pairwise disjoint, so that there are only countably many of them. This means that for some $Y \in \mathcal{S}(\Gamma_\mu, Y_0)$ open in Y_0 we must have $\nu(Y) > 0$. Then necessarily $x \in Y$. By construction of ν and Γ_μ-invariance of Y it follows that $\nu(Y') = 0$ for any $Y' \in \mathcal{S}(\Gamma_\mu, Y_0)$ distinct from Y. Hence, all ergodic components of ν are in fact equal to the homogeneous probability measure on Y, which finishes the proof. $\square$

Remark 4.6.3 (Non-averaged convergence in law). It is a natural question, already posed by Benoist–Quint at the end of their survey [**6**], whether, or under what conditions, the Cesàro average in Theorem 4.0.5(ii) can be removed. Unfortunately, in the generality of our results, this question of convergence of $\mu^{*n} * \delta_x$ towards ν_x seems to be out of reach with current methods. Answers are available only in certain special cases where additional structure can be exploited. For example, in the setting of toral automorphisms, the harmonic analytic approach used by Bourgain–Furman–Lindenstrauss–Mozes [**20**] allows them to obtain the convergence of $\mu^{*n} * \delta_x$ together with a speed depending on Diophantine properties of the starting point x. Their approach was recently refined and generalized to some nilmanifolds in the works [**58, 59, 60**] of He–de Saxcé and He–Lakrec–Lindenstrauss. Outside the realm of nilmanifolds, quantitative results on the convergence of $\mu^{*n} * \delta_x$ include the work of Buenger [**23**, §3] and Khalil–Luethi [**69**], who consider some classes of measures supported on compact-by-solvable groups, and the author's work on spread out measures in Chapter 3. $\diamond$

4.6.3. Topology of Homogeneous Measures. Here we prove the Mozes–Shah type results regarding the weak* topology on the set of ergodic homogeneous subspaces of X.

Let $G, H, \Lambda, X, \mu, \Gamma_\mu$ be as in Theorem 4.0.5 and recall that $\mathcal{S}(\Gamma_\mu)$ denotes the set of all Γ_μ-invariant Γ_μ-ergodic homogeneous subspaces Y of X. Each element Y of $\mathcal{S}(\Gamma_\mu)$ carries an associated Γ_μ-invariant and ergodic homogeneous probability measure ν_Y. Using this, we embed $\mathcal{S}(\Gamma_\mu)$ into the space $\mathcal{P}(X)$ of Borel probability measures on X and endow $\mathcal{S}(\Gamma_\mu)$ with the weak* topology induced from $\mathcal{P}(X)$. Also recall that given a subset $Z \subset X$, we set $\mathcal{S}_Z(\Gamma) = \{Y \in \mathcal{S}(\Gamma) \mid Y \cap Z \neq \emptyset\}$.

The following lemma will be useful for the proof of Proposition 4.0.7. In the statement, given $Y \in \mathcal{S}(\Gamma_\mu)$, we shall say that a point $y \in Y$ is *Y-generic* if the conclusion of Theorem 4.0.5(ii) holds, i.e. if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mu^{*k} * \delta_y = \nu_Y$ in the weak* topology. Note that ν_Y-a.e. point is Y-generic by the Chacon–Ornstein ergodic theorem.

Lemma 4.6.4. *Let $(\nu_j)_j$ be a sequence of ergodic homogeneous measures associated to subspaces $Y_j \in \mathcal{S}(\Gamma_\mu)$ converging to a finite measure ν on X in the weak* topology. Let β be a height function on X satisfying the assumptions of Lemma* 4.6.1 *and denote $X_\infty = \beta^{-1}(\{\infty\})$. Suppose that there is a sequence of Y_j-generic points $y_j \in Y_j$ such that $y_j \notin X_\infty$ for infinitely many j. Then $\nu(X \setminus X_\infty) = 1$.*

Proof. We may assume $y_j \notin X_\infty$ for all j. Let $\delta > 0$. By Lemma 4.6.1(i) there exists a compact subset $R_\delta \subset X \setminus X_\infty$ such that $\mu^{*n} * \delta_{y_j}(R_\delta) \geq 1 - \delta$ for all $n \geq n_{y_j}$. Passing to the limit in the Y_j-genericity, this implies $\nu_j(R_\delta) \geq 1-\delta$. Letting $j \to \infty$, it follows that also $\nu(R_\delta) \geq 1-\delta$. The conclusion follows, since $R_\delta \subset X \setminus X_\infty$ and $\delta > 0$ was arbitrary. $\square$

Proof of Proposition 4.0.7. We first prove (ii). Let $(\nu_j)_j$ be a sequence of ergodic homogeneous probability measures associated to homogeneous subspaces Y_j in $\mathcal{S}(\Gamma_\mu)$ converging to the homogeneous measure ν_∞ associated to $Y_\infty \in \mathcal{S}(\Gamma_\mu)$. Take a sequence of Y_j-generic points $y_j \in Y_j$ such that $Z = \overline{\{y_1, y_2, \ldots\}}$ is compact. Let β_∞ be a height function from Theorem 4.5.1 that is finite on Z, say with $Z \subset X_\varepsilon$ for some $\varepsilon > 0$ sufficiently small. Let O be a small neighborhood of the identity in $L = C_G(\Gamma_\mu)$ and $\beta_{\mathcal{N}}$ a height function from Theorem 4.5.4 taking the value ∞ on $HX_\varepsilon \cap OY_\infty$. Extending $\beta_{\mathcal{N}}$ from HX_ε to X using the value ∞, we are in the setting of Lemma 4.6.4 and know $\nu_\infty(X_{\infty,\mathcal{N}}) = 1$, where $X_{\infty,\mathcal{N}} = \beta_{\mathcal{N}}^{-1}(\{\infty\})$. Thus, the lemma implies $\beta_{\mathcal{N}}(y_j) = \infty$ for all large j, which means that $y_j \in OY_\infty$ since $y_j \in Z \subset X_\varepsilon$. Since O can be chosen arbitrarily small, (ii) is proved.

Now let us establish (i). Note that (ii) implies that for $Z \subset X$ compact, $\mathcal{S}_Z(\Gamma_\mu)$ is closed in $\mathcal{S}(\Gamma_\mu)$. So we only have to exhibit a limit point in $\mathcal{S}(\Gamma_\mu)$ of a given sequence $(Y_j)_j$ in $\mathcal{S}_Z(\Gamma_\mu)$. Thus, we may replace Z by a compact neighborhood and assume that the homogeneous measures ν_j associated to the Y_j all satisfy $\nu_j(Z) > 0$. Then we can find Y_j-generic points $y_j \in Z$. Letting β_∞ be a height function from Theorem 4.5.1 that is finite on Z, say again with $Z \subset X_\varepsilon$, Lemma 4.6.4 thus implies that any limit point ν of $(\nu_j)_j$ is a probability measure on X. By passing to a subsequence we may assume that $\nu_j \to \nu$. Then ν is a Γ_μ-invariant probability measure on X. By Proposition 4.4.1, there exists $Y \in \mathcal{S}(\Gamma_\mu)$ and a relatively compact neighborhood O of the identity in L such that $\nu(OY) > 0$. We suppose that the dimension of Y is minimal so that the latter holds. As in the first part of the proof, using a height function $\beta_{\mathcal{N}}$ and Lemma 4.6.4, this implies that $y_j \in OY$ for all large j. After passing to a subsequence, we have that $Y_j \subset l_j Y_\infty$ for some $l_j \in C_G(\Gamma_\mu)$ converging to the identity and $Y_\infty = lY$ for some $l \in C_G(\Gamma_\mu)$. Then all ergodic components of the limit measure ν are homogeneous probability measures associated to some ergodic homogeneous subspace $Y' \subset Y_\infty$. If subspaces $Y' \subsetneq Y_\infty$ were to feature in the ergodic decomposition with positive weight, then another application of Proposition 4.4.1 would imply that $\nu(LY') > 0$ for some $Y' \in \mathcal{S}(\Gamma_\mu)$ of lower dimension, contradicting the choice of Y. Hence, we have established convergence of ν_j to the homogeneous probability measure associated to Y_∞, proving compactness of $\mathcal{S}_Z(\Gamma_\mu)$.

To obtain relative compactness of $\mathcal{S}_{HZ}(\Gamma_\mu)$, note that by H-invariance of $\beta_\infty^{-1}(\{\infty\})$ for the height functions β_∞ coming out of Theorem 4.5.1, we know that $\beta_\infty(x) < \infty$ for every $x \in HZ$ if β_∞ is chosen to be finite on Z. Thus, Lemma 4.6.1(i) implies that there exists a compact subset $R_{1/2}$ of X such that $\mathcal{S}_{HZ}(\Gamma_\mu) \subset \mathcal{S}_{R_{1/2}}(\Gamma_\mu)$, and the latter set is compact, as shown above.

Finally, if a limit point of a sequence of probability measures in $\mathcal{S}(\Gamma_\mu) \cup \{\delta_\infty\}$ has a point $x \in X$ in its support, then a subsequence is contained in $\mathcal{S}_Z(\Gamma_\mu)$ for some compact neighborhood Z of x, proving compactness of $\mathcal{S}(\Gamma_\mu) \cup \{\delta_\infty\}$. $\square$

Proof of Corollary 4.0.8. Clearly, $\mathcal{S}(\Gamma_\mu, Y_\infty)$ is closed in $\mathcal{S}(\Gamma_\mu)$. In view of the last statement in Proposition 4.0.7(i), we only have to show that the only possible limit point of $(Y_n)_n$ inside $\mathcal{S}(\Gamma_\mu, Y_\infty)$ is Y_∞. Let Y be such a limit point. By Proposition 4.0.7, since $C_G(\Gamma_\mu)$ is assumed discrete, it follows that $Y_n \subset Y$ for infinitely many n. By assumption, this forces $Y = Y_\infty$, and we are done. $\square$

4.6.4. Application to Nilmanifolds. Let Λ' be a lattice in a connected simply connected nilpotent Lie group N and X the compact nilmanifold N/Λ'. The automorphism group $\mathrm{Aut}(\Lambda')$ of Λ' is defined to be the subset of automorphisms of N preserving Λ'. It is well known that any abstract automorphism of Λ' extends to an automorphism of N, therefore defines an element of $\mathrm{Aut}(\Lambda')$ (see e.g. [**107**, §II]).

A probability measure μ on $\mathrm{Aut}(\Lambda')$ defines a random walk on $X = N/\Lambda'$ by nilmanifold automorphisms. Our results have the following immediate corollaries for such random walks. Under an *affine submanifold* of X we understand a closed subset of X that is the translate of the image in X of a closed subgroup of N.

COROLLARY 4.6.5. *Let $X = N/\Lambda'$ be a compact nilmanifold and μ a probability measure on* $\mathrm{Aut}(\Lambda')$ *with finite first moment such that the Zariski closure H of Γ_μ in* $\mathrm{Aut}(N)$ *is a connected semisimple group without compact factors. Then every μ-ergodic μ-stationary probability measure on X is Γ_μ-invariant, homogeneous, and supported on a finite union of affine submanifolds.* $\square$

COROLLARY 4.6.6. *Let $X = N/\Lambda'$ be a compact nilmanifold and μ a probability measure on* $\mathrm{Aut}(\Lambda')$ *with finite exponential moments such that the Zariski closure H of Γ_μ in* $\mathrm{Aut}(N)$ *is a connected semisimple group without compact factors. Then:*

(i) *Every Γ_μ-orbit closure in X is a finite union of affine submanifolds.*
(ii) *For every $x \in X$, for $\mu^{\otimes \mathbb{N}}$-a.e. $(g_i)_i$ one has*
$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \delta_{g_k \cdots g_1 x} = \nu_x,$$
where ν_x is the homogeneous probability measure on $\overline{\Gamma_\mu x}$.
(iii) *The set $\mathcal{S}(\Gamma_\mu)$ is compact. If $Y_n \to Y_\infty$ in $\mathcal{S}(\Gamma_\mu)$, then there exists a sequence $(l_n)_n$ of Γ_μ-invariant elements in N converging to the identity such that $Y_n \subset l_n Y_\infty$ for all large n.* $\square$

The above corollaries are slight extensions of [**8**, Corollary 1.3] and [**9**, Corollary 1.10], respectively, removing the assumption that the probability measure μ is finitely supported.

To deduce these corollaries from our general theorems, one needs to exhibit an embedding $X \hookrightarrow G/\Lambda$ into the quotient of a real Lie group G containing $\mathrm{Aut}(\Lambda')$ by a lattice $\Lambda < G$. In the classical case of toral automorphisms, one has $\mathrm{Aut}(\Lambda') = \mathrm{GL}_d(\mathbb{Z})$, and we may simply choose $G = \mathrm{SL}_{d+1}(\mathbb{R})$ with its lattice $\Lambda = \mathrm{SL}_{d+1}(\mathbb{Z})$ admitting the embedding
$$X = (\mathrm{GL}_d(\mathbb{Z}) \ltimes \mathbb{R}^d)/(\mathrm{GL}_d(\mathbb{Z}) \ltimes \mathbb{Z}^d) \hookrightarrow G/\Lambda.$$
More generally, we can define $G = \mathrm{Zcl}(\mathrm{Aut}(\Lambda')) \ltimes N$ and $\Lambda = \mathrm{Aut}(\Lambda') \ltimes \Lambda'$, where $\mathrm{Zcl}(\mathrm{Aut}(\Lambda'))$ denotes the Zariski closure of $\mathrm{Aut}(\Lambda')$ inside $\mathrm{Aut}(N)$. Then Λ is a lattice in G by Borel–Harish-Chandra, since $\mathrm{Aut}(\Lambda')$ is commensurable to the subgroup of integer points of $\mathrm{Zcl}(\mathrm{Aut}(\Lambda'))$ for a suitable $\mathbb{Q}$-structure on

$\mathrm{Aut}(N)$ (see [**107**, Theorem 2.12] and its discussion). Hence, our results apply with $H = \mathrm{Zcl}(\Gamma_\mu)$ in view of Proposition 4.2.1.

4.7. Birkhoff Genericity

The aim of this section is to prove Theorem 4.0.10. Recall that H is a connected semisimple Lie group without compact factors and with finite center, $A' = \{a(t) \mid t \in \mathbb{R}\}$ is a one-parameter Ad-diagonalizable subgroup of H, and U an $a(1)$-expanding subgroup of H contained in $H^+_{a(1)}$. In particular, U is connected, Ad-unipotent, and normalized by A'. Moreover, having fixed a maximal compact subgroup K of H, K' is defined to be the compact group $C_K(A') \cap N_H(U)$, and μ is a probability measure on $K'A'U =: P \leqslant H$ with finite exponential moments satisfying $\int_P \lambda(g)\,\mathrm{d}\mu(g) > 0$, where λ is defined by the $K'A'U$-factorization $g = ka(\lambda(g))u$ for $g \in P$. Recall also that for $\omega = (g_i)_i \in P^{\mathbb{N}}$ and $n \in \mathbb{N}$, we write

$$g_{\omega,n} \coloneqq g_n \cdots g_1 = k_{\omega,n} a_{\omega,n} u_{\omega,n}$$

for the $K'A'U$-factorization of $g_n \cdots g_1$. All these notations and assumptions will be understood to be in place until the end of this section.

The first lemma we prove ensures that the limit in condition (3) of Definition 4.0.9 exists almost surely.

LEMMA 4.7.1. *For $\mu^{\otimes\mathbb{N}}$-almost every $\omega \in P^{\mathbb{N}}$ the sequence $(u_{\omega,n})_n$ converges to some $u_\omega \in U$.*

PROOF. Since U does not intersect the (finite) center of H, the restriction $\mathrm{Ad}_H\colon U \to \mathrm{Ad}(U)$ is a Lie group isomorphism. To prove the claimed convergence, we may thus assume that H is a linear group. Let $\omega = (g_i)_i \in P^{\mathbb{N}}$. For $n \in \mathbb{N}$, write $g_n = k_n a_n u_n$ its (unique) factorization into K', A' and U components. We also set $p_n = k_n a_n$. One readily observes that the term $u_{\omega,n}$ is equal to the product

$$u_n^{p_{n-1}\cdots p_1} \cdots u_3^{p_2 p_1} u_2^{p_1} u_1, \tag{4.7.1}$$

where we use the shorthand $g^h = h^{-1}gh$. In the product (4.7.1), a term $u_k^{p_{k-1}\cdots p_1}$ is equivalently expressed as $\exp(\mathrm{Ad}((p_{k-1}\cdots p_1)^{-1})(\log u_k))$. Here, the log map is well-defined since U being a unipotent linear group implies that the exponential map is a diffeomorphism from $\mathfrak{u} = \mathrm{Lie}(U)$ onto U. Moreover, since the Lie algebra $\mathfrak{u}$ is nilpotent, we know that $\exp\colon \mathfrak{u} \to U$ is given by $v \mapsto \mathbf{1} + vq(v)$, where q is a polynomial map. Therefore, to show that the product (4.7.1) converges for $\mu^{\otimes\mathbb{N}}$-almost every ω, by a general convergence criterion for infinite matrix products (see e.g. [**132**, §8.10]), it suffices to show that for $\mu^{\otimes\mathbb{N}}$-a.e. ω,

$$\sum_{k\geq 1} \|\mathrm{Ad}((a_{k-1}\cdots a_1)^{-1})(\log u_k)\|$$

converges, where $\|\cdot\|$ is an arbitrary matrix norm on $\mathfrak{u}$. We now prove this convergence. We start by observing that for $u \in U$ the logarithm $\log u$ is a polynomial in u. Hence, the random nilpotent elements $(\log u_k)_{k\geq 1}$ are i.i.d. and their distribution has a finite first moment. By the law of large numbers, it follows that almost surely $\|\log u_k\| = o(k)$. Almost surely, we thus obtain the bound

$$\begin{aligned}\|\mathrm{Ad}((a_{k-1}\cdots a_1)^{-1})(\log u_k)\| &\le o(k)\max_{\alpha\in\Pi}\prod_{i=1}^{k-1}\exp(-\alpha\lambda(a_i)) \\ &= o(k)\max_{\alpha\in\Pi}\exp\Big(-\alpha\sum_{i=1}^{k-1}\lambda(a_i)\Big),\end{aligned} \tag{4.7.2}$$

where

$$\Pi = \{\alpha\in\mathbb{R} \mid \mathrm{Ad}(a(t))v = e^{\alpha t}v \text{ for all } t\in\mathbb{R} \text{ for some nonzero } v\in\mathfrak{u}\}$$

is the finite set of real numbers corresponding to the weights of A' on $\mathfrak{u}$. Since U is contained in $H^+_{a(1)}$, we have $\Pi\subset(0,\infty)$. Together with $\int_P \lambda(g)\,\mathrm{d}\mu(g) > 0$, it thus follows from the Birkhoff ergodic theorem that, $\mu^{\otimes\mathbb{N}}$-almost surely, the last term in (4.7.2) decays exponentially. This gives the summability claimed above and hence the lemma. □

Proposition 4.7.2. *Suppose that the Zariski closure of* $\mathrm{Ad}(\Gamma_\mu)$ *contains* $\mathrm{Ad}(U)$. *Then the probability measure* μ *is* H*-expanding. For a discrete subgroup* Λ *of a real Lie group* G *containing* H*, any ergodic* μ*-stationary probability measure on* G/Λ *is* H*-invariant. If* Λ *is a lattice in* G*, then the conclusion of Theorem* 4.0.5 *holds with* $Y_x = \overline{Hx}$ *and* $\nu_x = \nu_{\overline{Hx}}$.

The following observations will be useful in the proof of the previous proposition. We denote by $A'_+ = \{a(t) \mid t > 0\}$ the positive ray in A'.

Lemma 4.7.3. *Let* Γ *be a subsemigroup of* P *such that* $\Gamma\cap K'A'_+U\neq\emptyset$. *Then there exists* $u\in U$ *such that* $u\Gamma u^{-1}\cap K'A'_+\neq\emptyset$.

Proof. By hypothesis there exists an element $\gamma_0\in K'A'_+U\cap\Gamma$. Factorize $\gamma_0 = p_0u_0$ with $p_0\in K'A'_+$ and $u_0\in U$. Endow $\mathfrak{u}$ with some Euclidean structure. As in the proof of Lemma 4.7.1, the linear map $\mathrm{Ad}(p_0^{-1})$ preserves the Lie algebra $\mathfrak{u}$ and any large power of it acts on $\mathfrak{u}$ as a contraction. Moreover, since U is connected and simply connected, as a consequence of the Baker–Campbell–Hausdorff formula (see e.g. [**29**, §1.2]), for every $u\in U$, the map $q_u\colon\mathfrak{u}\to\mathfrak{u}$ defined by $X\mapsto\log(\exp(X)u)$ is a polynomial map whose degree depends only on U and whose coefficients depend continuously on u.

Using the same notation and reasoning as in the proof of Lemma 4.7.1, we observe that for every $n\ge 1$, we have $\gamma_0^n = p_0^n u_0^{p_0^{n-1}}\cdots u_0^{p_0}u_0$, with the term $u(\gamma_0^n) \coloneqq u_0^{p_0^{n-1}}\cdots u_0^{p_0}u_0$ converging in U as $n\to\infty$. From these facts, one deduces that there exists a ball B in $\mathfrak{u}$ around $0\in\mathfrak{u}$ such that for every $n\in\mathbb{N}$ large enough, the continuous map $f_n\colon\mathfrak{u}\to\mathfrak{u}$ defined by

$$f_n(X) = q_{u(\gamma_0^n)}(\mathrm{Ad}(p_0^{-n})X) = \log(\exp(\mathrm{Ad}(p_0^{-n})X)u(\gamma_0^n))$$

satisfies $f_n(B)\subset B$. It follows from the Brouwer fixed point theorem that f_n has a fixed point $X\in\mathfrak{u}$. We claim that $u=\exp(X)\in U$ is the desired element. Indeed, since $\exp(\mathrm{Ad}(p_0^{-n})X) = p_0^{-n}\exp(X)p_0^n$, we have $p_0^{-n}up_0^nu(\gamma_0^n) = u$ and hence $u\gamma_0^nu^{-1} = up_0^nu(\gamma_0^n)u^{-1} = p_0^n\in K'A'_+$. □

Given $g\in P$, we write $g = k_ga_gu_g$ for its $K'A'U$-factorization.

LEMMA 4.7.4. *For a subset $C \subset P$, let $U_C = \{u_g \mid g \in C\}$ be the set of its U-parts. If the Zariski closure of* $\mathrm{Ad}(C)$ *contains* $\mathrm{Ad}(U)$, *then* $\mathrm{Ad}(U_C)$ *is Zariski dense in* $\mathrm{Ad}(U)$.

PROOF. Denote by Q the Zariski closure of $\mathrm{Ad}(P)$, and observe that $\mathrm{Ad}(U)$ is contained in the unipotent radical $R_u(Q)$ of Q. Since $\mathrm{Ad}(K'A')$ is a linearly reductive subgroup of Q, there is a Levi factor L of Q containing $\mathrm{Ad}(K'A')$ (see [**64**, Theorem VIII.4.3]). Then we have $Q = L \ltimes R_u(Q)$ as algebraic groups. This implies

$$\begin{aligned}\mathrm{Ad}(U) \subset \mathrm{Zcl}(\mathrm{Ad}(C)) &\subset \mathrm{Zcl}(\mathrm{Ad}(K'A')\,\mathrm{Ad}(U_C)) \\ &= \underbrace{\mathrm{Zcl}(\mathrm{Ad}(K'A'))}_{\subset L}\,\underbrace{\mathrm{Zcl}(\mathrm{Ad}(U_C))}_{\subset R_u(Q)}.\end{aligned}$$

We conclude that $\mathrm{Ad}(U) \subset \mathrm{Zcl}(\mathrm{Ad}(U_C))$, which is what we needed to show. □

PROOF OF PROPOSITION 4.7.2. The assumptions of Proposition 4.2.7 are satisfied for any maximal connected $\mathbb{R}$-split torus A in H containing A'. Thus, μ is H-expanding. Now let ν be an ergodic μ-stationary probability measure on $X = G/\Lambda$. By Theorem 4.0.1, ν is Γ_μ-invariant and homogeneous, and the connected component N of $\mathrm{Stab}_G(\nu)$ is normalized by H.

In order to prove the statement about H-invariance, we can assume without loss of generality that Γ_μ contains an element in $K'A'_+$. Indeed, suppose that the conclusion is true for such measures; call them *special*. Given an arbitrary measure μ as in the statement, by Lemma 4.7.3 we can find an element $u \in U$ such that $(\tau_u)_*\mu$ is special, where τ_u denotes conjugation by u. The properties in Definition 4.0.9 are preserved by this conjugation. Then $u_*\nu$ is $(\tau_u)_*\mu$-ergodic and stationary and hence it is H-invariant. But since $u \in U \leqslant H$, this implies that ν itself is H-invariant.

So let us take $g_0 = k_0 a_0 \in \Gamma_\mu \cap K'A'_+$. Then, given an arbitrary $g \in \Gamma_\mu$ written as $g = k_g a_g u_g$ in its $K'A'U$ factorization, by considering a sequence n_k such that $k_0^{n_k} \to e$ as $k \to \infty$, we get that the conjugates $g_0^{-n_k} g g_0^{n_k}$ converge to $k_g a_g$. This implies that $k_g a_g$ and thus also u_g belongs to Γ_μ. In other words, Γ_μ contains all of its U-parts.

We next claim that for any proper connected normal subgroup $S \leqslant H$, there exists $g \in \Gamma_\mu$ whose U-part u_g does not belong to S. To see this, by way of contradiction, let us suppose that all U-parts of elements of Γ_μ belong to some proper normal subgroup S. Using Lemma 4.7.4, we deduce from this that $\mathrm{Ad}(U) \leqslant \mathrm{Ad}(S)$, which entails that U acts trivially in the adjoint representation of H on $\mathfrak{h}/\mathfrak{s}$. On the other hand, the image of $a(1)$ in this representation has determinant one, so that it cannot expand all nonzero elements of $\mathfrak{h}/\mathfrak{s}$, contradicting $a(1)$-expansion of U.

Assuming that H is not contained in N, we can apply the above with $S = (N \cap H)^\circ$. Take $g = k_g a_g u_g \in \Gamma_\mu$ with $u_g \notin (N \cap H)^\circ$. By normality, also the U-parts of $g_0^{-n_k} g g_0^{n_k}$ do not belong to $(N \cap H)^\circ$. On the other hand, as observed above, these U-parts lie in $\Gamma_\mu \leqslant H \cap \mathrm{Stab}_G(\nu)$ and converge to the identity, which is impossible. This contradiction shows that $H \leqslant N$, and hence that any ergodic μ-stationary probability measure ν is H-invariant.

Finally, applying the H-invariance statement to the homogeneous measure ν_x from Theorem 4.0.5, we see that the conclusions of that theorem hold with $Y_x = \overline{Hx}$. □

The following elementary but key equivariance property is the final ingredient required for the proof of Theorem 4.0.10.

LEMMA 4.7.5. *For $\mu^{\otimes \mathbb{N}}$-almost every $\omega = (g_i)_i \in P^{\mathbb{N}}$ and every $n \in \mathbb{N}$, we have*

$$a_{\omega,n} u_\omega = k_{\omega,n}^{-1} u_{T^n\omega} g_{\omega,n},$$

where $T\colon P^{\mathbb{N}} \to P^{\mathbb{N}}, (g_i)_i \mapsto (g_{i+1})_i$ denotes the shift map.

PROOF. By Lemma 4.7.1, there exists a set Ω of full $\mu^{\otimes\mathbb{N}}$-measure such that for every $\omega \in \Omega$, the sequence $u_{\omega,n}$ converges (to the limit u_ω). Replacing Ω by $\bigcap_{i\geq 0} T^{-i}\Omega$ if necessary, we may assume that $T\Omega \subset \Omega$. Let $\omega = (g_i)_i \in \Omega$ and $n \in \mathbb{N}$. Writing $g_i = k_i a_i u_i$ in its $K'A'U$ factorization, a straightforward computation shows that $u_{\omega,n} = a_1^{-1} k_1^{-1} u_{T\omega,n} k_1 a_1 u_1$. Passing to the limit as $n \to \infty$, we obtain $u_\omega = a_1^{-1} k_1^{-1} u_{T\omega} g_1$. The lemma now follows by iterating the latter equality, using that A' and K' commute. □

PROOF OF THEOREM 4.0.10. Suppose the measure η is generated by the probability measure μ supported on $P = K'A'U$ as in Definition 4.0.9. By Theorem 4.0.5 and Proposition 4.7.2, we know that for every $x \in X$, for $\mu^{\otimes\mathbb{N}}$-almost every $\omega = (g_i)_i \in P^{\mathbb{N}}$, the sequence of points

$$(g_{\omega,n}x)_n$$

is equidistributed with respect to $\nu = \nu_{\overline{Hx}}$.

Replacing K' by a subgroup, we may assume without loss of generality that $\mathrm{pr}_{K'}(\Gamma_\mu)$ is dense in K', where $\mathrm{pr}_{K'}\colon P \to K'$ is the projection map. So the action of $\mathrm{pr}_{K'}(\Gamma_\mu)$ on $(K', m_{K'})$ by left translation is ergodic, where $m_{K'}$ is the Haar probability measure on K'. By a version of Moore's ergodicity theorem (see [**4**, Theorem III.2.5(i)]) applied to the regular representation on the Hilbert space $L^2_0(X,\nu)$ of square integrable functions with mean zero, the action of Γ_μ on (X,ν) is weakly mixing. Therefore, the action of Γ_μ on $(X \times K', \nu \otimes m_{K'})$ given by $g(y,k) = (gy, \mathrm{pr}_{K'}(g)k)$ is ergodic (cf. e.g. [**117**, Proposition 2.2]). Thus it follows from [**129**, Corollary 5.5] that for almost every $\omega = (g_i)_i \in P^{\mathbb{N}}$, the sequence

$$(g_{\omega,n}x, k_{\omega,n})_n$$

is equidistributed with respect to $\nu \otimes m_{K'}$. Next, applying [**129**, Proposition 5.1], this can be upgraded to almost sure equidistribution of

$$(g_{\omega,n}x, k_{\omega,n}, T^n\omega)_n \tag{4.7.3}$$

with respect to $\nu \otimes m_{K'} \otimes \mu^{\otimes\mathbb{N}}$, where $T\colon P^{\mathbb{N}} \to P^{\mathbb{N}}$ denotes the shift map. We caution here that when the support of μ is non-compact, the above equidistribution takes place in a non-locally compact space, so that the class of test functions to consider is that of bounded continuous functions. The proof of [**129**, Proposition 5.1], however, only needs minor amending to accommodate this issue; see Lemma 2.2.9 in Chapter 2 and the short discussion before its proof.

Applying the map $\omega = (g_i)_i \mapsto (u_\omega, g_1)$ to the equidistribution in (4.7.3), we conclude that, for almost every $\omega = (g_i)_i \in P^{\mathbb{N}}$, the sequence

$$(g_{\omega,n}x, k_{\omega,n}, u_{T^n\omega}, g_{n+1})_n \tag{4.7.4}$$

is equidistributed with respect to $\nu \otimes m_{K'} \otimes \tilde{\eta}$, where $\tilde{\eta}$ is a probability measure on $U \times P$ that projects to μ in the second coordinate. Again, some caution is needed at this step, since $\omega \mapsto u_\omega$ is not necessarily continuous. However, also this can be dealt with by considering Lusin sets and continuous extensions coming from Tietze's theorem as in the proof of [**129**, Proposition 5.2].

The rest of the proof is the same as in [**129**, §12]; we reproduce it for the convenience of the reader. Given $f \in C_c(X)$, one considers the bounded continuous function φ on $X \times K' \times U \times P$ defined by

$$\varphi(x, k, u, g) = \int_0^{\lambda(g)} f(a(t)k^{-1}ux)\,\mathrm{d}t,$$

where $g = k_g a(\lambda(g))u_g$ is the decomposition according to $P = K'A'U$. A direct calculation using the invariance of ν under H shows that

$$\int \varphi \,\mathrm{d}(\nu \otimes m_{K'} \otimes \tilde{\eta}) = \int_P \lambda(g)\,\mathrm{d}\mu(g) \int_X f\,\mathrm{d}\nu. \tag{4.7.5}$$

Suppose $\omega = (g_i)_i$ is a generic point with respect to the equidistribution of (4.7.4) for which also Lemma 4.7.5 holds for every n. Using only the last factor P in the equidistribution, it follows that

$$\lim_{n\to\infty} \frac{\lambda(g_{\omega,n})}{n} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \lambda(g_i) = \int_P \lambda(g)\,\mathrm{d}\mu(g) > 0. \tag{4.7.6}$$

We thus obtain, by the equidistribution (4.7.4),

$$\begin{aligned}
\int \varphi\,\mathrm{d}(\nu \otimes m_{K'} \otimes \tilde{\eta}) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \varphi(g_{\omega,i}x, k_{\omega,i}, u_{T^i\omega}, g_{i+1}) \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_0^{\lambda(g_{i+1})} f(a(t)k_{\omega,i}^{-1}u_{T^i\omega}g_{\omega,i}x)\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_0^{\lambda(g_{i+1})} f(a(t)a_{\omega,i}u_\omega x)\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \int_{\lambda(g_{\omega,i})}^{\lambda(g_{\omega,i+1})} f(a(t)u_\omega x)\,\mathrm{d}t \\
&= \lim_{n\to\infty} \frac{\lambda(g_{\omega,n})}{n} \frac{1}{\lambda(g_{\omega,n})} \int_0^{\lambda(g_{\omega,n})} f(a(t)u_\omega x)\,\mathrm{d}t \\
&= \int_P \lambda(g)\,\mathrm{d}\mu(g) \lim_{n\to\infty} \frac{1}{\lambda(g_{\omega,n})} \int_0^{\lambda(g_{\omega,n})} f(a(t)u_\omega x)\,\mathrm{d}t,
\end{aligned}$$

where we used Lemma 4.7.5 in the third equality and the fact that $\lambda(g_{\omega,i+1}) = \lambda(g_{\omega,i}) + \lambda(g_{i+1})$ in the fourth. Together with (4.7.5), this implies

$$\lim_{n\to\infty} \frac{1}{\lambda(g_{\omega,n})} \int_0^{\lambda(g_{\omega,n})} f(a(t)u_\omega x)\,\mathrm{d}t = \int f\,\mathrm{d}\nu. \tag{4.7.7}$$

Finally, notice that since the random variables $\lambda(g_{\omega,n}) - \lambda(g_{\omega,n-1}) = \lambda(g_n)$ are i.i.d. with a distribution that has a finite first moment, it follows from the law of large numbers that almost surely

$$\lambda(g_{\omega,n}) - \lambda(g_{\omega,n-1}) = o(n). \tag{4.7.8}$$

Now (4.7.6), (4.7.7) and (4.7.8) together imply the Birkhoff genericity of $u_\omega x$ with respect to $(a(t))_{t\geq 0}$ and ν. $\square$

4.8. Connections to Diophantine Approximation on Fractals

The goal of this section is to explain the connection between random walks and Diophantine approximation on affine fractals, prove a general result (Theorem 4.8.3) which will imply Theorem 4.0.11 on Diophantine properties of Bedford–McMullen carpets, and mention some further directions.

4.8.1. Weighted Diophantine Approximation and Dani–Kleinbock Flow. To begin with, we recall basic notions in Diophantine approximation of matrices and the connection to homogeneous dynamics.

4.8.1.1. *Badly Approximable Matrices and Dirichlet Improvability.* Let m, n be positive integers, $\mathbf{r} = (r_1, \ldots, r_m) \in (0,1]^m$ and $\mathbf{s} = (s_1, \ldots, s_n) \in (0,1]^n$ be such that $\sum_{i=1}^m r_i = \sum_{j=1}^n s_j = 1$ and $M \in \mathbb{R}^{m\times n}$ a matrix with rows $M_1, \ldots, M_m$. Then M is called $(\mathbf{r}, \mathbf{s})$-*badly approximable* or *badly approximable for the weights* $(\mathbf{r}, \mathbf{s})$ if there exists a constant $C > 0$ such that

$$\max_{1\leq i\leq m} |M_i\mathbf{q} - p_i|^{1/r_i} \cdot \max_{1\leq j\leq n} |q_j|^{1/s_j} \geq C \tag{4.8.1}$$

for every $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{0\})$. Otherwise, M is called $(\mathbf{r}, \mathbf{s})$-*well approximable.*

One can see by Dirichlet's principle, or by Blichfeldt and Minkowski's convex body results, that for every matrix $M \in \mathbb{R}^{m\times n}$, there exist infinitely many pairs $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{0\})$ such that the left-hand side of (4.8.1) is bounded above by 1. As a consequence of a general form of Khintchine's theorem [**118**], the set of $(\mathbf{r}, \mathbf{s})$-badly approximable matrices is a Lebesgue null set. However, it has everywhere-full Hausdorff dimension; see [**72**, Corollary 4.5] and [**73**, §5.4].

Given weights $(\mathbf{r}, \mathbf{s})$, an equivalent way to express the aforementioned consequence of the Dirichlet principle is to say that for every matrix $M \in \mathbb{R}^{m\times n}$ and for every $t > 0$, the following system of inequalities has a solution in $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{0\})$:

$$|M_i\mathbf{q} - p_i| \leq \mathrm{e}^{-tr_i} \quad \text{and} \quad |q_j| \leq \mathrm{e}^{ts_j} \qquad (1 \leq i \leq m, 1 \leq j \leq n).$$

One says that the matrix $M \in \mathbb{R}^{m\times n}$ is $(\mathbf{r}, \mathbf{s})$-*Dirichlet improvable* if there exists $\varepsilon \in (0,1)$ such that for every $t \geq 0$ large enough, the following system of inequalities has a solution in $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^m \times (\mathbb{Z}^n \setminus \{0\})$:

$$|M_i\mathbf{q} - p_i| \leq \varepsilon\mathrm{e}^{-tr_i} \quad \text{and} \quad |q_j| \leq \varepsilon\mathrm{e}^{ts_j} \qquad (1 \leq i \leq m, 1 \leq j \leq n).$$

In the special case where the weights $(\mathbf{r}, \mathbf{s})$ are given by $(\mathbf{m}, \mathbf{n})$—by which we mean that $r_i = 1/m$ and $s_j = 1/n$ for all i, j—the notion of Dirichlet improvability was introduced and studied by Davenport–Schmidt, who showed that the set of $(\mathbf{m}, \mathbf{n})$-Dirichlet improvable matrices has zero Lebesgue measure [**31**] and that every $(\mathbf{m}, \mathbf{n})$-badly approximable matrix is $(\mathbf{m}, \mathbf{n})$-Dirichlet improvable [**32**]. The former result was generalized to arbitrary weights $(\mathbf{r},\mathbf{s})$ by Kleinbock–Weiss [**72**].

4.8.1.2. *Dani–Kleinbock Flow.* Let $G=\mathrm{PGL}_d(\mathbb{R})$, $\Lambda=\mathrm{PGL}_d(\mathbb{Z})$, and set $X=G/\Lambda$. It is easy to see that the homogeneous space X can alternatively be written as $\mathrm{SL}_d(\mathbb{R})/\mathrm{SL}_d(\mathbb{Z})$, which can be identified with the space of unimodular lattices in $\mathbb{R}^d$ via $g\,\mathrm{SL}_d(\mathbb{Z})\leftrightarrow g\mathbb{Z}^d$. For every $\varepsilon>0$, we define

$$K_\varepsilon \coloneqq \Big\{g\Lambda\in X \,\Big|\, g\in\mathrm{SL}_d(\mathbb{R}),\ \max_{i=1,\dots,d}|(g\mathbf{v})_i|\geq\varepsilon \text{ for every } v\in\mathbb{Z}^d\setminus\{0\}\Big\}.$$

Viewing X as space of unimodular lattices in $\mathbb{R}^d$, K_ε is nothing but the subset of lattices all of whose nonzero vectors have length at least ε in the supremum norm. The collection of sets K_ε is clearly decreasing in ε. For $\varepsilon<1$ the set K_ε has non-empty interior, and for $\varepsilon>1$ one has $K_\varepsilon=\emptyset$, as can be seen by Minkowski's convex body theorem from geometry of numbers. Moreover, Mahler's compactness criterion states that the sets $K_\varepsilon\subset X$ for $\varepsilon>0$ are compact and that a subset of X is relatively compact if and only if it is contained in one of the K_ε.

Now let $d=m+n$ and denote by x_0 the identity coset in $X=G/\Lambda$. The Dani–Kleinbock correspondence principle—observed first by Dani [**30**] and developed further, among others, by Kleinbock [**75**] and later Kleinbock–Weiss [**72**]—states that, loosely speaking, the Diophantine properties of a matrix $M\in\mathbb{R}^{m\times n}$ are encoded in the behavior of the trajectory of the point u_Mx_0 inside X under suitable one-parameter diagonal subgroups of G, where $u_M\coloneqq\left(\begin{smallmatrix}\mathbf{1}_m & -M\\ 0 & \mathbf{1}_n\end{smallmatrix}\right)$. We are going to use this principle in the form of the following proposition. Given weights $(\mathbf{r},\mathbf{s})\in(0,1]^m\times(0,1]^n$ as before, let $(a(t))_{t\in\mathbb{R}}$ be the one-parameter subgroup of G corresponding to the diagonal matrix $a(1)=\mathrm{diag}(\mathrm{e}^{r_1},\dots,\mathrm{e}^{r_m},\mathrm{e}^{-s_1},\dots,\mathrm{e}^{-s_n})$.

PROPOSITION 4.8.1 (Dani–Kleinbock correspondence). *A real matrix M is*

- ([**75**]) $(\mathbf{r},\mathbf{s})$-*badly approximable if and only if the forward-orbit*
$$\{a(t)u_Mx_0 \mid t\geq 0\}$$
is relatively compact in X, and
- ([**72**]) $(\mathbf{r},\mathbf{s})$-*Dirichlet improvable if and only if there exists $\varepsilon\in(0,1)$ such that $a(t)u_Mx_0\notin K_\varepsilon$ for every $t\geq 0$ large enough.*

An obvious consequence of this proposition is that given weights $(\mathbf{r},\mathbf{s})$, if the forward orbit $\{a(t)u_Mx_0\,|\,t\geq 0\}$ associated to a matrix $M\in\mathbb{R}^{m\times n}$ is dense in X, then M is $(\mathbf{r},\mathbf{s})$-well approximable and not $(\mathbf{r},\mathbf{s})$-Dirichlet improvable.

In fact, the ergodic theoretic approach that we adopt will allow us to establish the following finer Diophantine property.

DEFINITION 4.8.2. Given weights $(\mathbf{r},\mathbf{s})$ and the associated one-parameter diagonal group $(a(t))_{t\in\mathbb{R}}$, a matrix $M\in\mathbb{R}^{m\times n}$ is said to be of $(\mathbf{r},\mathbf{s})$-*generic type* if the forward-orbit $(a(t)u_Mx_0)_{t\geq 0}$ equidistributes to the Haar measure m_X on X.

4.8.2. Matrix Sponges and Self-Affine Measures. Here we briefly describe the iterated function system (IFS) construction of affine fractals and introduce the subfamily of affine fractals (matrix sponges) and self-affine measures whose Diophantine properties will be studied in the subsequent part.

4.8.2.1. *Affine Fractals.* Let ϕ be an affine transformation of $\mathbb{R}^D$ given by $\phi(x) = Ax + b$ where $A \in \mathrm{GL}_D(\mathbb{R})$ and $b \in \mathbb{R}^D$. It is called *contracting* if the operator norm of its linear part A with respect to the standard Euclidean structure of $\mathbb{R}^D$ satisfies $\|A\| < 1$. We shall refer to a finite set $\Phi = \{\phi_1, \dots, \phi_k\}$ of contracting affine transformations ϕ_i of $\mathbb{R}^D$ as a *contracting affine IFS*. Given such an IFS Φ, there exists a unique non-empty compact subset $\mathcal{K}$ of $\mathbb{R}^D$ satisfying $\mathcal{K} = \bigcup_{i=1}^k \phi_i(\mathcal{K})$, referred to as the *attractor* of Φ. Putting less emphasis on the IFS, $\mathcal{K}$ is also called an *affine fractal* or *self-affine set*. In the particular case where all the ϕ_i are similarities, the attractor $\mathcal{K}$ is also called a *self-similar set*.

The *natural projection* Π associated to a contracting affine IFS Φ is the map defined by

$$\Pi\colon \Phi^{\mathbb{N}} \to \mathbb{R}^D,\ (\phi_{i_j})_j \mapsto \lim_{n\to\infty} \phi_{i_1} \cdots \phi_{i_n}(x) \tag{4.8.2}$$

for some $x \in \mathbb{R}^D$. The limit in the definition is independent of x. The image of the natural projection Π is precisely the affine fractal $\mathcal{K}$, and we have the following equivariance property with respect to the shift map T on $\Phi^{\mathbb{N}}$:

$$\Pi((\phi_{i_j})_j) = \phi_{i_1}\Pi((\phi_{i_{j+1}})_j) = \phi_{i_1}\Pi(T(\phi_{i_j})_j). \tag{4.8.3}$$

Our results on random walks on homogeneous spaces also allow us to study a more general situation where the IFS is not required to be finite and where one can allow contraction to only take place on average. To describe this, let Φ be a compact subset of the group $\mathrm{GL}_D(\mathbb{R}) \ltimes \mathbb{R}^D$ of invertible affine transformations of $\mathbb{R}^D$. Then Φ is said to be a *compact affine IFS*. Given a probability measure μ with $\mathrm{supp}(\mu) = \Phi$, we shall refer to the couple (Φ, μ) as a *contracting-on-average compact affine IFS* if there exists $N \in \mathbb{N}$ such that

$$\int \log\|A_{\phi_N} \cdots A_{\phi_1}\| \,\mathrm{d}\mu^{\otimes N}(\phi_1, \dots, \phi_N) < 0, \tag{4.8.4}$$

where A_ϕ denotes the linear part of an affine transformation ϕ. This definition does not depend on the choice of operator norm.

Using only boundedness of the translation parts, it is not hard to see that the limit $\lim_{n\to\infty} \phi_1 \cdots \phi_n(x)$ exists and does not depend on $x \in \mathbb{R}^D$ whenever the sequence $(\|A_{\phi_1} \cdots A_{\phi_n}\|)_{n\geq 1}$ decays fast enough (e.g. exponentially). Under the contraction-on-average assumption, this holds for $\mu^{\otimes\mathbb{N}}$-almost every sequence $(\phi_j)_j$, as one can see using submultiplicativity of the operator norm and Kingman's subadditive ergodic theorem. Thus, in this case, we again obtain a measurable map $\Pi\colon (\phi_j)_j \mapsto \lim_{n\to\infty} \phi_1 \cdots \phi_n(x)$ defined $\mu^{\otimes\mathbb{N}}$-almost everywhere on $\Phi^{\mathbb{N}}$ that we refer to as the *natural projection* of (Φ, μ). Note that the subset Ω of elements of $\Phi^{\mathbb{N}}$ for which the limit in the definition of Π exists satisfies $T\Omega \subset \Omega$ and that on this set we have the same equivariance relation as in (4.8.3). In the case of a compact affine IFS Φ comprising only contractions, the natural projection Π is a continuous map defined everywhere on $\Phi^{\mathbb{N}}$ and its image $\mathcal{K}$ coincides with the support of $\Pi_*\mu^{\otimes\mathbb{N}}$ for any probability measure μ on Φ with full support.

Finally, we shall say that a compact IFS Φ of affine transformations of $\mathbb{R}^D$ is *irreducible* if there does not exist a proper affine subspace W of $\mathbb{R}^D$ such that $\phi(W) = W$ for every $\phi \in \Phi$.

4.8.2.2. *Self-Affine Measures.* Let (Φ, μ) be a contracting-on-average compact affine IFS and Π the associated natural projection. Then the probability measure $\nu_\mu = \Pi_* \mu^{\otimes \mathbb{N}}$ on $\mathbb{R}^D$ is called the associated *self-affine measure* (or *self-similar measure* if the IFS comprises only similarities). It is with respect to these self-affine measures that we will study the typical Diophantine behavior of vectors in $\mathbb{R}^D$ or more generally matrices in $\mathbb{R}^{m\times n}$. The measure ν_μ is the unique stationary probability measure for the random walk on $\mathbb{R}^D$ given by the IFS; see [**33**]. In the case of a finite IFS $\Phi = \{\phi_1, \dots, \phi_k\}$, this just means that ν_μ is the unique probability measure on $\mathbb{R}^D$ satisfying $\nu_\mu = \sum_{i=1}^k \mu(\{\phi_i\})(\phi_i)_* \nu_\mu$.

For a finite contracting IFS Φ consisting of similarities of $\mathbb{R}^D$, under a separation condition (see [**65**]), the Hausdorff measure on the attractor $\mathcal{K}$ is given by a self-similar measure which is also the unique measure on $\mathcal{K}$ whose pointwise dimension matches the Hausdorff dimension of the similarity fractal. For genuinely self-affine fractals, the situation is considerably more complicated (see e.g. [**1, 66, 91, 92**] and the references therein). On the other hand, for the Bedford–McMullen carpets introduced in §4.0.6 and their higher-dimensional generalizations, there exists a unique ergodic shift-invariant probability measure on $\Phi^{\mathbb{N}}$ whose pushforward ν by the natural projection has full Hausdorff dimension [**68**]. This measure ν is self-affine. In dimension 2, it was already explicitly constructed and used by McMullen [**89**], and is referred to as the *McMullen measure* in the literature.

4.8.2.3. *Matrix Sponges.* We now describe the family of affine fractals and self-affine measures that will be of interest to us. Let $\mathbf{r} = (r_1, \dots, r_m) \in (0,1]^m$ and $\mathbf{s} = (s_1, \dots, s_n) \in (0,1]^n$ be such that $\sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j$. Consider the diagonalizable one-parameter groups $A'_{\mathbf{r}} \subset \mathrm{GL}_m(\mathbb{R})$ and $A'_{\mathbf{s}} \subset \mathrm{GL}_n(\mathbb{R})$ given by $\{a_{\mathbf{r}}(t) \coloneqq \mathrm{diag}(\mathrm{e}^{tr_1}, \dots, \mathrm{e}^{tr_m}) \,|\, t \in \mathbb{R}\}$ and $\{a_{\mathbf{s}}(t) \coloneqq \mathrm{diag}(\mathrm{e}^{ts_1}, \dots, \mathrm{e}^{ts_n}) \,|\, t \in \mathbb{R}\}$ respectively. Denote by $K_{\mathbf{r}}$ the compact group $C_{\mathrm{GL}_m(\mathbb{R})}(A'_{\mathbf{r}}) \cap \mathrm{O}_m(\mathbb{R})$ and similarly for $K_{\mathbf{s}}$ substituting $\mathbf{s}$ for $\mathbf{r}$ and n for m.

We identify the matrix space $\mathbb{R}^{m\times n}$ with $\mathbb{R}^{mn}$ and consider affinities ϕ of $\mathbb{R}^{m\times n}$ of the type

$$M \mapsto A_1 M A_2 + B, \tag{4.8.5}$$

where $B \in \mathbb{R}^{m\times n}$, $A_1 \in \mathrm{GL}_m(\mathbb{R})$ and $A_2 \in \mathrm{GL}_n(\mathbb{R})$. We will refer to affinities of this form as *matrix affinities* and use the notation (A_1, A_2, B) to denote such a map. If a matrix affinity ϕ can be written as $\phi = (A_1, A_2, B)$ with $A_1 \in a_{\mathbf{r}}(t)K_{\mathbf{r}}$ and $A_2 \in a_{\mathbf{s}}(t)K_{\mathbf{s}}$ for some $t \in \mathbb{R}$, then we call it an $(\mathbf{r}, \mathbf{s})$-*matrix sponge affinity.*

Given a contracting-on-average compact IFS (Φ, μ) of $(\mathbf{r}, \mathbf{s})$-matrix sponge affinities, we call the associated attractor $\mathcal{K}$ an $(\mathbf{r}, \mathbf{s})$-*matrix sponge.*

A cautionary remark is in order about our terminology. In the literature, the terms "carpet" (in dimension 2) or "sponge" (in general dimension) are used to describe self-affine fractals associated to IFS's whose linear parts are simultaneously diagonalizable with non-trivial (i.e. non-scalar) diagonals. However, the matrix sponge affinities that we just described also comprise many similarities of $\mathbb{R}^{mn}$. Similarities of $\mathbb{R}^{mn}$ of this form are called "algebraic similarities" by Simmons–Weiss [**129**, §8.4], which thus form a strict subclass of matrix sponge affinities. For example, specializing to $n = 1$ we can record that the class of $(\mathbf{m}, 1)$-matrix sponges contains all self-similar fractals in $\mathbb{R}^m$ and the class of $(\mathbf{r}, 1)$-matrix sponges contains many examples of Bedford–McMullen carpets and their higher-dimensional analogues—the self-affine Sierpiński sponges—for suitably chosen weight vectors $\mathbf{r}$.

4.8.3. Relation With Random Walks and Consequences. Here we first adapt the constructions of Simmons–Weiss [**129**] relating algebraic similarities with elements of $\mathrm{PGL}_d(\mathbb{R})$ to the more general setting of matrix affinities. Then, we state and prove the main result of this section (Theorem 4.8.3) on Diophantine properties of matrix sponges.

4.8.3.1. *Embedding Matrix Sponge Affinities Into* $\mathrm{PGL}_d(\mathbb{R})$*.* Let $d = m+n$. Given a matrix affinity $\phi = (A_1, A_2, B)$ of $\mathbb{R}^{m\times n}$ with $B \in \mathbb{R}^{m\times n}$, $A_1 \in \mathrm{GL}_m(\mathbb{R})$ and $A_2 \in \mathrm{GL}_n(\mathbb{R})$, we consider the element $\hat{A}_\phi$ of $\mathrm{PGL}_d(\mathbb{R})$ corresponding to the matrix

$$\hat{A}_\phi = \begin{pmatrix} A_1 & 0 \\ 0 & A_2^{-1} \end{pmatrix}.$$

The following basic relation in $\mathrm{PGL}_d(\mathbb{R})$, which is readily verified, plays a key role in transferring the results on random walks on homogeneous spaces to the study of Diophantine properties of matrix sponges: For $M \in \mathbb{R}^{m\times n}$, we have

$$\hat{A}_\phi u_M \hat{A}_\phi^{-1} u_B = u_{\phi(M)}, \tag{4.8.6}$$

where, as before, $u_M = \left(\begin{smallmatrix} \mathbf{1}_m & -M \\ 0 & \mathbf{1}_n \end{smallmatrix}\right)$. We set $g_\phi := \hat{A}_\phi^{-1} u_B \in \mathrm{PGL}_d(\mathbb{R})$. Given matrix affinities $\phi_1, \ldots, \phi_n$, iterating (4.8.6) yields

$$g_{\phi_n} \cdots g_{\phi_1} = \hat{A}_{\phi_n}^{-1} \cdots \hat{A}_{\phi_1}^{-1} u_{\phi_1 \cdots \phi_n(0)}. \tag{4.8.7}$$

4.8.3.2. *Genericity of Typical Points on Matrix Sponges.* To state the following main result of this section, recall that given a contracting-on-average compact affine IFS (Φ, μ), we denote by Π the associated natural projection and by ν_μ the pushforward of the Bernoulli measure $\beta = \mu^{\otimes \mathbb{N}}$ by Π.

THEOREM 4.8.3. *Let* (Φ, μ) *be an irreducible contracting-on-average compact IFS consisting of* $(\mathbf{r}, \mathbf{s})$*-matrix sponge affinities. Then* ν_μ*-almost every point of* $\mathbb{R}^{mn}$ *is of* $(\mathbf{r}, \mathbf{s})$*-generic type; in particular,* $(\mathbf{r}, \mathbf{s})$*-well approximable and not* $(\mathbf{r}, \mathbf{s})$*-Dirichlet improvable.*

In the classical case where the weights are given by $(\mathbf{r}, \mathbf{s}) = (\mathbf{m}, \mathbf{n})$, this result corresponds to Simmons–Weiss' [**129**, Theorem 8.11], which implies one of the main results of that article ([**129**, Theorem 1.2]). We are going to see in the proof that the contracting-on-average assumption in the theorem above amounts to asking that the μ-average of the t-parameters associated to the $(\mathbf{r}, \mathbf{s})$-matrix sponge affinities ϕ in the IFS is negative. This allows for easy checking of this condition.

REMARK 4.8.4. The conclusion of Theorem 4.8.3 also holds for any measure $\tilde{\nu}_\mu$ obtained as pushforward of ν_μ by an affine transformation of the linear space $\mathbb{R}^{m\times n}$ of the form $M \mapsto \alpha M \beta + \gamma$, where $\alpha \in \mathrm{GL}_m(\mathbb{R})$ commutes with the diagonal group $A'_{\mathbf{r}}$, $\beta \in \mathrm{GL}_n(\mathbb{R})$ commutes with $A'_{\mathbf{s}}$ and $\gamma \in \mathbb{R}^{m\times n}$. In particular, these Diophantine properties of ν_μ are invariant under translation of ν_μ. ◇

We will deduce the theorem above by combining Theorem 4.0.10, Dani–Kleinbock correspondence and the introduced constructions.

Proof of Theorem 4.8.3. Recall that $\mathbf{r} = (r_1, \dots, r_m) \in (0,1]^m$ and $\mathbf{s} = (s_1, \dots, s_n) \in (0,1]^n$ are such that $\sum_{i=1}^m r_i = 1 = \sum_{j=1}^n s_j$, where m and n are positive integers. Let $d = m+n$ and set $G = H = \mathrm{PGL}_d(\mathbb{R})$ and $\Lambda = \mathrm{PGL}_d(\mathbb{Z})$. Moreover, we let $A' = \{a(t) \mid t \in \mathbb{R}\}$ be the one-parameter diagonalizable subgroup of G containing $a(1) = \mathrm{diag}(\mathrm{e}^{r_1}, \dots, \mathrm{e}^{r_m}, \mathrm{e}^{-s_1}, \dots, \mathrm{e}^{-s_n})$, and denote by A'_+ its positive ray $\{a(t) \mid t > 0\}$. Take U to be the unipotent subgroup of G given by the image of $\mathbb{R}^{m \times n}$ under the map $M \mapsto u_M$. It is $a(1)$-expanding (see Example 4.2.9). In view of Dani–Kleinbock correspondence and Theorem 4.0.10, all we need to check is that the pushforward η_0 of the self-affine measure ν_μ by the map $M \mapsto u_M$ is generated by $a(1)$-expanding random walks in the sense of Definition 4.0.9.

We begin by defining the probability measure μ_0 on G. Given a matrix affinity $\phi = (A_1, A_2, B)$, recall the notation $g_\phi = \hat{A}_\phi^{-1} u_B \in \mathrm{PGL}_d(\mathbb{R})$ introduced in §4.8.3.1. We take

$$\mu_0 \coloneqq c_* \mu, \tag{4.8.8}$$

the pushforward of μ by the map $c\colon \phi \mapsto g_\phi$. Then it follows from our constructions that $\mu_0(P) = 1$, where $P = K'A'U$ is defined as before Definition 4.0.9. Moreover, we claim that the contraction-on-average assumption implies that $\int_P \lambda(g)\, \mathrm{d}\mu_0(g) > 0$. To see this, endow $\mathbb{R}^{m\times n} \cong \mathbb{R}^{mn}$ with the standard Euclidean structure and denote by $\|\cdot\|$ the associated operator norm on $\mathrm{End}(\mathbb{R}^{m \times n})$. Given an $(\mathbf{r}, \mathbf{s})$-matrix sponge affinity ϕ, let us denote by $A_\phi \in \mathrm{End}(\mathbb{R}^{m\times n})$ its linear part. By definition, we may write $\phi = (A_1, A_2, B)$ as in (4.8.5) with $A_1 \in a_{\mathbf{r}}(t) K_{\mathbf{r}}$ and $A_2 \in a_{\mathbf{s}}(t) K_{\mathbf{s}}$ for some $t \in \mathbb{R}$. Observe that by construction, the t-parameter is given by $t = -\lambda(g_\phi)$. This implies that

$$\|A_\phi\| \geq \mathrm{e}^{\kappa t} = \mathrm{e}^{-\kappa \lambda(g_\phi)},$$

where $\kappa \coloneqq \min_{i,j}(r_i + s_j) > 0$. Plugging this inequality into the contraction-on-average property (4.8.4) and observing that $\lambda(g_{\phi_N \cdots \phi_1}) = \lambda(g_{\phi_N}) + \cdots + \lambda(g_{\phi_1})$ yields $\int_P \lambda(g)\, \mathrm{d}\mu_0(g) = \int \lambda(g_\phi)\, \mathrm{d}\mu(\phi) > 0$, hence the claim.

We now show that the irreducibility assumption entails that $U \leqslant \mathrm{Zcl}(\Gamma_{\mu_0})$. As in the proof of Proposition 4.7.2, we will first reduce to the case of *special* measures μ_0 for which Γ_{μ_0} contains an element of $K'A'_+$. Indeed, given a general μ_0 as in (4.8.8), using that $\int_P \lambda(g)\, \mathrm{d}\mu_0(g) > 0$ and Lemma 4.7.3, it follows that there exists $u_0 \in U$ such that the pushforward by conjugation $(\tau_{u_0})_* \mu_0$ is special. The closed group generated by the support of $(\tau_{u_0})_*\mu_0$ is $u_0 \Gamma_{\mu_0} u_0^{-1}$ and if the Zariski closure of this group contains U, then that of Γ_{μ_0} also contains U. Moreover, this conjugation corresponds to conjugating the IFS by a translation, so that also irreducibility is preserved. So we now suppose that μ_0 is special. Then as in the proof of Proposition 4.7.2, for every $g \in \Gamma_{\mu_0}$ written $g = k_g a_g u_g$ in its $K'A'U$-factorization, we know that also $k_g a_g$ and u_g belong to Γ_{μ_0}. It follows that for every $g \in \Gamma_{\mu_0}$, the one-parameter unipotent subgroup of U containing u_g is contained in the Zariski closure of Γ_{μ_0}. Now consider the connected unipotent group $V = \mathrm{Zcl}(\Gamma_{\mu_0}) \cap U$ and let W_V be the corresponding subspace of $\mathbb{R}^{mn}$ under (the inverse of) the identification $M \mapsto u_M$. We claim that the subspace W_V is invariant by the IFS of matrix sponge affinities. Indeed, by construction, for any $\phi = (A_1, A_2, B)$ in the IFS, the unipotent part u_B of the associated element g_ϕ belongs to V and hence $B \in W_V$. Moreover, for any $g \in \Gamma_{\mu_0}$, its $K'A'$-component $k_g a_g$ normalizes V. In view of (4.8.6), this translates to the statement that for any ϕ of the IFS, the linear part of ϕ leaves the subspace W_V invariant. It follows that the subspace W_V of $\mathbb{R}^{mn}$ is invariant by the IFS. Hence, by the irreducibility hypothesis, we have $W_V = \mathbb{R}^{mn}$, or equivalently, $V = U$.

It remains to check that the measure η_0 coincides with the image of $\mu_0^{\otimes \mathbb{N}}$ under the map $\omega \mapsto u_\omega$ defined by Lemma 4.7.1. To do this, let $\omega = (g_{\phi_j})_j$. By definition of the natural projection (4.8.2) and the map $\omega \mapsto u_\omega$, it suffices to observe that for every $n \in \mathbb{N}$, factorizing $g_{\phi_n} \cdots g_{\phi_1}$ as $k_{\omega,n} a_{\omega,n} u_{\omega,n}$ with $k_{\omega,n} \in K'$, $a_{\omega,n} \in A'$ and $u_{\omega,n} \in U$, we have $u_{\omega,n} = u_{\phi_1 \cdots \phi_n(0)}$; see (4.8.7). This finishes the proof. $\square$

Finally, we state and prove the corollary of the previous theorem regarding the higher-dimensional analogues of Bedford–McMullen carpets, which was announced at the end of §4.0.6. These higher-dimensional fractals are constructed by the exact analogue in $\mathbb{R}^m$ of the procedure for Bedford–McMullen carpets described before Theorem 4.0.11, now using pairwise distinct integers $a_1, \dots, a_m \geq 2$ and a division of $[0,1]^m$ into an $a_1 \times \cdots \times a_m$-grid. The obtained fractals are called *self-affine Sierpiński sponges* (see Kenyon–Peres [**68**]).

Corollary 4.8.5. *Let $m \geq 2$ and $a_1, \dots, a_m \geq 2$ be pairwise distinct integers satisfying*

$$\frac{1}{m} \sum_{j \neq i} \log a_j < \log a_i < \frac{2}{m-1} \sum_{j \neq i} \log a_j \tag{4.8.9}$$

for $i = 1, \dots, m$. Let $\mathcal{K} \subset \mathbb{R}^m$ be a self-affine Sierpiński sponge invariant under the toral endomorphism T corresponding to the matrix $A = \mathrm{diag}(a_1, \dots, a_m)$ such that $\mathcal{K}$ is not contained in any affine hyperplane. Then for the choice of weights

$$\mathbf{r} = \left(\frac{m \log a_i - \sum_{j \neq i} \log a_j}{\sum_j \log a_j} \right)_{1 \leq i \leq m}, \tag{4.8.10}$$

the set of $\mathbf{r}$-badly approximable vectors on $\mathcal{K}$ has measure zero with respect to the unique T-invariant ergodic probability measure $\nu_{\mathcal{K}}$ of full Hausdorff dimension on $\mathcal{K}$.

This corollary directly implies Theorem 4.0.11.

Proof. We start by noting that $\mathcal{K}$ is the attractor of a finite contracting affine IFS $\Phi = \{\phi_1, \dots, \phi_k\}$, where $\phi_i \colon x \mapsto A^{-1}x + b_i$ with translation vectors b_i contained in $\prod_{j=1}^m \{0, 1/a_j, \dots, (a_j - 1)/a_j\}$. Denoting by $\Pi \colon \Phi^{\mathbb{N}} \to \mathbb{R}^m$ the natural projection, the proof of [**68**, Theorem 1.2] shows that $\nu_{\mathcal{K}} = \nu_\mu = \Pi_* \mu^{\otimes \mathbb{N}}$ for some probability measure μ on Φ of full support. Then the assumption that $\mathcal{K}$ is not contained in any affine hyperplane implies that the IFS Φ is irreducible. We wish to arrange that the ϕ_i can be seen as $(\mathbf{r}, 1)$-matrix sponge affinities. By definition, this means that we have to write the common linear part $A^{-1} = \mathrm{diag}(a_1^{-1}, \dots, a_m^{-1})$ as $\mathrm{e}^t a_{\mathbf{r}}(t)$ for some $t \in \mathbb{R}$, where $a_{\mathbf{r}}(t) = \mathrm{diag}(\mathrm{e}^{tr_1}, \dots, \mathrm{e}^{tr_m})$. Solving the resulting system of equations under the constraint $r_1 + \cdots + r_m = 1$

yields the weights specified by (4.8.10). The condition (4.8.9) ensures that $\mathbf{r} \in (0,1)^m$. Hence, Theorem 4.8.3 applies and gives the desired conclusion. $\square$

We end our discussion of Diophantine approximation by mentioning that our approach has serious limitations when trying to tackle the general problem of understanding the measure-theoretic size of badly approximable vectors or matrices—weighted or not—in general self-affine fractals. Even seemingly tractable cases—e.g. $\mathbf{r}$-badly approximable vectors on an affine fractal for which $\mathbf{r}$ represents the average contraction ratio—require a further understanding of diagonal flows and, frustratingly, remain open.

APPENDIX A

Epimorphic Subgroups and Subalgebras

In category theory, an epimorphism is by definition a morphism $f\colon A \to B$ satisfying the right cancellation property: $g \circ f = h \circ f$ implies $g = h$ for any two morphisms g, h from B to another object of the category. In categories where morphisms are maps with certain properties between underlying sets, the epimorphism property is equivalent to the question whether the values on the image of f uniquely determine morphisms from B to other objects. In this case, surjective morphisms are clearly epimorphisms. In many familiar categories, the converse, i.e. that only surjective morphisms can be epimorphisms, is also true. For example, this holds in the categories of C^*-algebras, groups, finite groups, all Lie algebras over a field k, and finite-dimensional Lie algebras over a field k of positive characteristic; see [**12, 113**]. However, there are notable exceptions. These include the categories of finite-dimensional Lie algebras over a field of characteristic 0 and that of algebraic groups, which are our main interest. The corresponding lines of study were initiated by Bergman [**12**] and Bien–Borel [**14, 15**], respectively, who proved the following.

Proposition A.1.

(i) ([**12**, Corollary 3.2]) *Let $\mathfrak{f} \subset \mathfrak{g}$ be finite-dimensional Lie algebras over a field k. Then the inclusion $\mathfrak{f} \hookrightarrow \mathfrak{g}$ is an epimorphism if and only if in every finite-dimensional representation of $\mathfrak{g}$, the subspaces annihilated by $\mathfrak{f}$ and $\mathfrak{g}$ coincide.*

(ii) ([**14**, Theorem 1]) *Let $\mathbf{G}$ be a Zariski connected linear algebraic group over an algebraically closed field k, and $\mathbf{F} \leqslant \mathbf{G}$ an algebraic subgroup. Then the inclusion $\mathbf{F} \hookrightarrow \mathbf{G}$ is an epimorphism if and only if in every finite-dimensional algebraic representation of $\mathbf{G}$, the subspaces of $\mathbf{F}$- and $\mathbf{G}$-fixed vectors coincide.*

We take this representation-theoretic characterization as the defining property of an epimorphic subgroup of a semisimple real Lie group.

Definition A.2.

(i) Let $\mathfrak{f}$ be a subalgebra of a finite-dimensional real Lie algebra $\mathfrak{g}$. We say that $\mathfrak{f}$ is *epimorphic* in $\mathfrak{g}$ if for any finite-dimensional real representation of $\mathfrak{g}$, the subspaces annihilated by $\mathfrak{f}$ and $\mathfrak{g}$ coincide.

(ii) Let G be a connected semisimple real Lie group. A subgroup F of G is said to be *epimorphic* in G if for every finite-dimensional representation of G, the vectors fixed by F are also fixed by G.

In the literature, it has been common to only introduce and study the concept of epimorphic subgroups for algebraic groups. Let us therefore check that our definition coincides with the usual one when the groups involved are algebraic.

Proposition A.3. *Let G be a Zariski connected semisimple real algebraic group and F a Lie subgroup of G such that F° is Zariski dense in F. Suppose that F is epimorphic in G in the category of real algebraic groups, meaning that in every finite-dimensional real algebraic representation of G, the vectors fixed by F are also fixed by G. Then F° is epimorphic in G° in the sense of Definition* A.2.

To be precise, by G being a real algebraic group we mean that $G = \mathbf{G}(\mathbb{R})$ is the group of real points of an underlying complex algebraic group $\mathbf{G}$ defined over $\mathbb{R}$, and a real algebraic representation is the restriction to real points of an algebraic representation of $\mathbf{G}$ defined over $\mathbb{R}$. Moreover, F° and G° denote the connected components of F and G, respectively, in the Lie group topology. It is easy to see that the converse of the proposition is also true. Finally, we remark that F is epimorphic in G in the category of real algebraic groups if and only if F is epimorphic in $\mathbf{G}$ in the category of complex algebraic groups.

The idea of the proof of the proposition above is to pass to the Lie algebra level, where all representations are algebraic thanks to semisimplicity. The following two lemmas enable this step.

Lemma A.4. *Let G be a connected semisimple Lie group and F a closed subgroup of G. If $\mathfrak{f} = \mathrm{Lie}(F)$ is an epimorphic subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$, then F is epimorphic in G.*

Proof. A representation of G naturally induces a representation of its Lie algebra. A vector that is F-fixed on the Lie group level is then $\mathfrak{f}$-annihilated on the Lie algebra level. Therefore, such vectors are annihilated by $\mathfrak{g}$ and hence fixed by G, since G is connected. □

Lemma A.5. *Let F and G be as in Proposition* A.3. *Then $\mathfrak{f} = \mathrm{Lie}(F)$ is an epimorphic subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$.*

Proof. If $\mathfrak{f}$ is not an epimorphic subalgebra of $\mathfrak{g}$, then using complete reducibility of $\mathfrak{g}$-representations, we can find a non-trivial irreducible representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ such that the subspace

$$V_0 = V^{\mathfrak{f}} = \{v \in V \mid \rho(f)v = 0 \text{ for all } f \in \mathfrak{f}\}$$

is nonzero. Let $\mathfrak{g}_{\mathbb{C}}$ and $V_{\mathbb{C}}$ be the complexifications of $\mathfrak{g}$ and V, respectively. It follows from the discussion in [**97**, §8] (Theorem 1 and Corollary 1) that either (1) $\mathfrak{g}_{\mathbb{C}}$ acts irreducibly on $V_{\mathbb{C}}$, or (2) V has a complex structure and $\mathfrak{g}$ acts by $\mathbb{C}$-linear transformations. In both cases, we thus obtain an irreducible complex representation of $\mathfrak{g}_{\mathbb{C}}$ (either on $V_{\mathbb{C}}$ or on V), which we denote by $\rho_{\mathbb{C}}$. We also set $k = \mathbb{R}$ in the first case and $k = \mathbb{C}$ in the second, and record that since $\mathfrak{g}$ acts k-linearly, the subspace V_0 is k-invariant.

We claim that there exists $n \in \mathbb{N}$ such that the tensor product representation $\rho^{\otimes_k n}$ of $\mathfrak{g}$ lifts to a real algebraic representation of G. Assuming the claim and using that F° is Zariski dense in F, we find that $V_0^{\otimes_k n}$ is a nonzero F-fixed subspace of $V^{\otimes_k n}$. Since F is an epimorphic subgroup of G in the algebraic category, the space $V_0^{\otimes_k n}$ is G-fixed. It follows that $\mathfrak{g}$ annihilates $V_0^{\otimes_k n}$, hence $\mathfrak{g}$ annihilates V_0. This contradicts the assumption that (ρ, V) is a non-trivial irreducible representation, and thus establishes the statement of the lemma.

It remains to prove the claim. Let $\mathbf{G}$ be a Zariski connected semisimple complex algebraic group defined over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})$. Then $\mathfrak{g}_{\mathbb{C}}$ is the Lie algebra of $\mathbf{G}$. By [**27**, Corollary A.4.11] there is a simply connected algebraic cover $\tilde{\mathbf{G}}$ of $\mathbf{G}$ defined over $\mathbb{R}$.

In case (1), since the representation $\rho_{\mathbb{C}} \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$ is algebraic by semisimplicity, it lifts to an irreducible algebraic representation $\tilde{\mathbf{G}} \to \mathrm{GL}(V_{\mathbb{C}})$ defined over $\mathbb{R}$ (with respect to the real structure on $V_{\mathbb{C}}$ given by V). The kernel $\mathbf{N}$ of the covering map $\tilde{\mathbf{G}} \to \mathbf{G}$ is finite and central. By Schur's lemma and irreducibility, $\mathbf{N}$ thus acts on $V_{\mathbb{C}}$ by scalar multiplication by roots of unity. Therefore, there exists $n \in \mathbb{N}$ such that $\mathbf{N}$ acts trivially on $V_{\mathbb{C}}^{\otimes_{\mathbb{C}} n}$. Since the representation of $\tilde{\mathbf{G}}$ on $V_{\mathbb{C}}^{\otimes_{\mathbb{C}} n}$ is defined over $\mathbb{R}$, we deduce that it induces a real algebraic representation of G on $V^{\otimes_k n} = V^{\otimes_{\mathbb{R}} n}$.

In case (2), $\rho_{\mathbb{C}} \colon \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V)$ lifts to an irreducible algebraic representation $\tilde{\mathbf{G}} \to \mathrm{GL}(V)$. By the same argument as in the first case, for some $n \in \mathbb{N}$ the kernel $\mathbf{N}$ of the covering map acts trivially on $V^{\otimes_{\mathbb{C}} n}$. Hence, the action of $\tilde{\mathbf{G}}$ on $V^{\otimes_k n} = V^{\otimes_{\mathbb{C}} n}$ factors through an algebraic representation of $\mathbf{G}$. By restriction of scalars, we can view $\mathbf{G}$ and $\mathrm{GL}(V^{\otimes_{\mathbb{C}} n})$ as groups of real points of algebraic groups defined over $\mathbb{R}$. Composing the map $G \to \mathbf{G}$ with the representation of $\mathbf{G}$ on $V^{\otimes_{\mathbb{C}} n}$ we obtain the desired lift of $\rho^{\otimes_k n}$. $\square$

Proof of Proposition A.3. By Lemma A.5, $\mathfrak{f} = \mathrm{Lie}(F)$ is an epimorphic subalgebra of $\mathfrak{g} = \mathrm{Lie}(G)$. Then Lemma A.4 implies that F° is epimorphic in G° in the sense of Definition A.2(ii). $\square$

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