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Analytical Triple-2D Leakage Inductance Model of Cone Winding Matrix Transformers

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Transformer, Modelling, Design, Pulsed power, Magnetic Device

Abstract
The transformer leakage inductance is one of the limiting factors for pulse shape quality in high voltage pulsed power (HVPP) applications such as cancer treatment, particle accelerators, and free electron lasers. Cone winding matrix (CWM) transformers are commonly used in HVPP applications as they offer low leakage inductance, low parasitic capacitance, high power density, and high insulation distance. This paper proposes an analytical Triple-2D leakage inductance model for CWM transformers. The model is based on a 2D model applicable to tilted cone windings which is derived by analytically integrating the magnetic potential. The Triple-2D modelling concept enables high accuracy and versatility. The model is verified with 2D FEM simulations and measurements on an existing pulse transformer for the compact linear collider at CERN. The analytical model is not only accurate and generally applicable but also rapidly executable enabling its time-efficient integration in optimisations.

1 Introduction
High voltage pulsed power (HVPP) converters are a key element in applications such as cancer treatment [1], particle accelerators [2, 3], and X-ray free electron lasers for material science [4]. The leakage inductance of the HVPP converter’s transformer is one of the limiting parasitics that affect pulse shape parameters such as rise time and overshoot [5–7]. In HVPP converters, matrix transformers with cone-shaped high voltage windings as shown in Fig. 1a are commonly used. The matrix transformer concept describes a transformer with multiple cores and primary windings wound on each core leg whereas two secondary windings enclose all core legs in a row. This ensures high power density and a low product of leakage inductance and parasitic capacitance, resulting in low pulse rise time and overshoot. [2, 5]. Cone windings are used because they offer a favourable trade-off between low leakage inductance and dielectric insulation distance [5, 8]. This is because typically both coils are grounded as shown in Fig. 1b. Therefore, the top end of the HV-coil reaches substantially high electric potentials with respect to...

Fig. 1: a) Matrix transformer with two cores and cone-shaped secondary windings. b) Grounding of transformer coils in high voltage pulsed power (HVPP) transformers. c) Inside-window (IW) cross section of cone winding transformer. d) Outside-window (OW) cross section of cone winding transformer.
ground resulting in field strengths that are critical for dielectric breakdowns. Hence, the top end of the HV-coil is tilted outwards to maximise the distance to all neighbouring conducting elements, i.e. core and LV-winding [2].

In the design process of HVPP converters, the operating and design parameters are often determined with an optimisation procedure before the system is built to save time and costs [6, 9, 10]. In such optimisation procedures, the system parameters are typically recalculated several thousand times (see e.g. [11]). Consequently, the employed models have to be fast as well as accurate to deliver results within a reasonable amount of time. Analytical models are well suited for this purpose as they are typically faster executable than reluctance network models and numerical methods such as the finite element method (FEM) [12–14].

Transformer leakage inductance is one of the central system parameter subject to optimisation. Analytically modelling the leakage inductance of cone winding transformers is more complicated than for parallel winding-transformers as the tilted cone winding is not parallel to the other winding and the core edges. This complicates finding a closed form analytical expression of the magnetic field. Existing cone winding transformer leakage inductance models are based on rather strong simplifications and approximations. In [5] and [7], the magnetic field is assumed to be axial (1D-field). This first-order assumption leads to a very compact expression but remains a rather rough approximation of the real magnetic field leading to substantial errors. The error is about 35 % for a typical cone winding geometry as examined in this paper (see section 4.2). The exact error however, depends on the geometry. In [8], each conductor is mirrored separately and the leakage inductance is calculated with the method of mean geometric distances (MGD). This method is more accurate but mirroring each conductor individually is computationally quite expensive. Furthermore, this method assumes filamentary currents and neglects the spatial distribution of the current density, which limits accuracy [12]. Also, infinite core permeability is assumed. The existing models are based on only one or two transformer cross sections: the inside-window (IW) and the outside-window (OW) cross section shown in Figs. 1c&d. These cross sections can also be identified for regular shell-type transformers [12–17]. This modelling concept is often referred to as “Double-2D” concept. However, for accurately and reliably modelling the transformer leakage inductance, it is necessary to take all elementary cross sections into account. In the special case of matrix transformers, even a third cross section can be identified: the primary double mirror (PDM) cross section shown in Fig. 2c. All in all, an accurate & fast leakage inductance model that takes into account all relevant cross sections of cone winding matrix (CWM) transformers for converter optimisations is still missing.

This paper proposes an accurate leakage inductance model of CWM transformers that takes into account all three elementary cross sections, leading to a Triple-2D concept. The key element of the model consists of a 2D model that unifies each winding to a rectangular block, which is a common assumption in leakage inductance calculations [12]. The resulting 2D-field is calculated based on the magnetic vector potential of a rectangular conductor proposed in [18] and used for leakage inductance calculation in [19, 20]. This approach results in a more accurate magnetic field and also requires only a modest amount of computational effort as the windings are mirrored as a block instead of mirroring each wire separately. As second step, the leakage inductance is calculated by integrating the magnetic energy density. This integration is carried out over the tilted cone winding by using an analytically derived antiderivative. The proposed model is verified by a measurement on an existing benchmark transformer [2] that is used in the compact linear collider of the European Council for Nuclear Research (CERN).

The paper is organised as follows. Section 2 elaborates general facts about matrix transformers and the Triple-2D modelling concept. Section 3 presents the equations of the proposed model. Finally, the model is verified by 2D FEM and measurements in section 4.

2 Leakage Inductance Modelling of Matrix Transformers

2.1 Matrix Transformers – Basic Concept

Matrix transformers consist of multiple cores (typically U-cores [2]) with primary windings on each core leg whereas two secondary windings enclose all core legs in a row cf. Fig. 1a. Usually, these secondary windings are connected in parallel to halve the leakage inductance. With the matrix transformer winding arrangement, both leakage inductance and parasitic capacitance are minimal because volume between primary and secondary windings is saved [5]. For generating pulses with lower power, the transformer
Fig. 2: a) Matrix transformer with two cores and the three elementary cross section planes. b) Top view of matrix transformer with partial leakage lengths and elementary cross sections. c) Cross section specific for matrix transformers: primary double mirror (PDM). d) Mathematical equivalent to c). Mirror factor \( m = \frac{\mu_{r,\text{core}} - 1}{\mu_{r,\text{core}} + 1} \). Further details are given in section 3.1.1

could also feature only one single secondary winding with primary windings only inside this single secondary winding. [2].

2.2 Analytical Triple-2D modelling concept

This section elaborates the Triple-2D modelling concept applied to matrix transformers. As mentioned in the introduction, there are three elementary cross sections when looking at matrix transformers: Inside-window (IW – Fig. 1c), outside-window (OW – Fig. 1d), and primary double mirror (PDM – Fig. 2c). The first two are basic cross sections of any shell-type transformer [12–17], whereas the PDM cross section is specific to matrix transformers. Figs. 2a&b show a matrix transformer with two cores but the number of cores \( N_{\text{cores}} \) is arbitrary and can be even higher (e.g. [4]).

In analytical models, usually the leakage inductance per unit length \( L'_\sigma \) is calculated for a 2D cross section and then scaled with the corresponding partial leakage length \( l_p \) [12, 13]. In the Triple-2D modelling concept, three leakage inductances per unit length are calculated for the elementary cross sections IW, OW, and PDM. These inductances per unit length need to be scaled with their corresponding partial leakage lengths shown in Fig. 2b. Finally, the leakage inductance of matrix transformers according to the Triple-2D concept can be calculated with (1). All in all, six central quantities in the bracket of (1) occur in the Triple-2D modelling concept.

\[
L_\sigma = c_w \left( L'_{\sigma,\text{IW}} l_p,\text{IW} + L'_{\sigma,\text{OW}} l_p,\text{OW} + L'_{\sigma,\text{PDM}} l_p,\text{PDM} \right) \tag{1}
\]

where \( c_w \) is a factor that takes into account how the windings are electrically connected. In this paper, the leakage inductance is referred to the secondary HV-side as shown in Fig. 1b. When referring the leakage inductance of a matrix transformer to the secondary side (2) applies.

\[
c_w = \frac{1}{2} \ldots \text{ for parallel connection of two secondary windings} \\
\]
\[
c_w = 2 \ldots \text{ for serial connection of two secondary windings} \\
\]
\[
c_w = 1 \ldots \text{ for only one secondary winding} \tag{2}
\]

The conductors are usually unified to rectangular blocks in analytical leakage inductance models [12]. This simplification comes with a negligible decrease in accuracy while drastically reducing the geometrical complexity of the problem. The decrease in accuracy is low because magnetic energy is a macroscopic quantity, i.e. the value is received by integration over the total computational domain. Hence, it is only slightly affected by minor local variations of the magnetic field. Hence, this simplification is also used in the model presented in this paper. Note that for all leakage inductance calculations, the magnetomotive forces of the windings are equal in magnitude but opposite in direction. Further details on modelling of leakage inductance can be found in [21, 22].
3 Triple-2D Leakage Inductance Model of Cone Winding Matrix Transformers

This section presents all necessary equations, derivations, and definitions for the models of the six required quantities introduced in (1).

3.1 Cone Winding Leakage Inductance Per Unit Length Model

The key element of the model is the analytical leakage inductance per unit length model applicable to tilted cone windings. The mathematical challenge here is that the secondary cone windings are not parallel to the primary windings and the core edges as they are tilted outwards as explained in the introduction. In this section, the model is presented in detail.

3.1.1 Fundamentals of Leakage Inductance Per Unit Length Model

The proposed model is based on the analytical expression of the magnetic vector potential published in [18] that is also used in the leakage inductance model presented by Margueron et al. [19, 20]. The advantages and disadvantages of Margueron’s model are summarised in [12] and the equations of Margueron’s model are listed in the supplementary document of [12]. Therefore, only the most essential equations of the model are recaptured here.

The vector potential of an infinitely long conductor $k$ with rectangular cross section in a homogeneous medium with permeability $\mu_0$ can be expressed as (3) [18–20].

$$A_{c,k}(x,y) = \frac{-\mu_0}{4\pi} J_k \left[ F \left( x - x_{c,k} - \frac{w_h}{2}, y - y_{c,k} - \frac{h_c}{2} \right) - F \left( x - x_{c,k} + \frac{w_h}{2}, y - y_{c,k} + \frac{h_c}{2} \right) ight]$$

$$- F \left( x - x_{c,k} - \frac{w_h}{2}, y - y_{c,k} + \frac{h_c}{2} \right) + F \left( x - x_{c,k} + \frac{w_h}{2}, y - y_{c,k} - \frac{h_c}{2} \right) \right] \right)$$

The geometrical parameters of the winding are defined in Fig. 3a and the number of turns of the winding are taken into account in the current density $J_k = \frac{N_{I_k}}{w \cdot h_l}$. The function $F(X, Y)$ from (3) is defined according to (4).

$$F(X, Y) = X Y \ln(X^2 + Y^2) + X^2 \arctan \left( \frac{Y}{X} \right) + Y^2 \arctan \left( \frac{X}{Y} \right)$$

The influence of the core is taken into account by mirroring the windings across the edges of the core and replacing the core material with air [23]. The current density of the image currents is $J_{image} = m J_{original}$ where $m = \frac{\mu_{core} - 1}{\mu_{core} + 1}$ [19, 20, 23]. The OW cross section leads to only one image winding pair [12] as shown in Fig. 3b. The PDM cross section features two parallel image planes, which leads theoretically to an infinite number of images in $x$-direction as indicated in Fig. 2d. The IW cross section also results in an infinite amount of mirror images but in $x$- and $y$-direction as shown in Fig. 3c. However, one image layer already results in a good accuracy while keeping the computational effort within a reasonable range [12].

Fig. 3: a) Geometric winding parameters. b) Mirror image of the outside-window cross section. c) Mirror images of the inside-window cross section. d) Parametrisation of tilted winding $l$. 
The magnetic energy per unit length is calculated according to (5).

\[
W_{\text{mag}}' = \frac{1}{2} \sum_{k=1}^{D} \oint_{\text{original windings}} A_{z,k} J_z \, dA
\]  
(5)

where \( D \) is the number of all windings including the image windings. Note that the integral is evaluated solely over the two original windings. Hence, the model’s central mathematical operation is the integration of the potential of each winding (index \( k \)) over the two original windings (index \( i \)). The corresponding basic mathematical operation of this is given in (6).

\[
\oint_{\text{winding } i} F(x-x_{c,k} \pm \frac{w}{2}, y-y_{c,k} \pm \frac{h}{2}) \, dA = \oint_{\text{winding } i} F_k(x,y) \, dA
\]  
(6)

For parallel windings, Margeron [20] found an analytical solution to (6) which is given in (7).

\[
\oint_{\text{winding } i} F(x-x_{c,k} \pm \frac{w}{2}, y-y_{c,k} \pm \frac{h}{2}) \, dA = \int_{h_i}^{a_i} \int_{a_i}^{h_i} F(x-x_{c,k} \pm \frac{w}{2}, y-y_{c,k} \pm \frac{h}{2}) \, dx \, dy =
\]

\[
= G(a_i^+ - x_{c,k} \pm \frac{w}{2}, h_i^+ - y_{c,k} \pm \frac{h}{2}) - G(a_i^- - x_{c,k} \pm \frac{w}{2}, h_i^- - y_{c,k} \pm \frac{h}{2})
\]

\[
- G(a_i^+ - x_{c,k} \pm \frac{w}{2}, h_i^- - y_{c,k} \pm \frac{h}{2}) + G(a_i^- - x_{c,k} \pm \frac{w}{2}, h_i^+ - y_{c,k} \pm \frac{h}{2})
\]

with \( G(X,Y) \) as the antiderivative according to (8).

\[
G(X,Y) = -\frac{1}{4} (X^4 - 6X^2Y^2 + Y^4) \ln(X^2 + Y^2) + \frac{1}{4} XY [X^2 \arctan \left( \frac{X}{Y} \right) + Y^2 \arctan \left( \frac{Y}{X} \right)] - \frac{3}{8} X^2 Y^2
\]  
(8)

Finally, the leakage inductance per unit length is calculated with (9).

\[
L'_a = \frac{2W_{\text{mag}}'}{I_{\text{ref}}^2}
\]  
(9)

where \( W_{\text{mag}}' \) is the magnetic energy per unit length and \( I_{\text{ref}} \) is the current through the excited winding. As mentioned in sec. 2.2, the leakage inductance is referred to the secondary side in this paper, i.e. \( I_{\text{ref}} = I_2 \).

The equations above summarise how to calculate the leakage inductance of parallel windings. The following section extends this model to tilted cone windings.

### 3.1.2 Integration over Tilted Cone Winding

If the integral (6) is evaluated over a winding that is tilted with respect to the winding causing the magnetic field and tilted with respect to the coordinate system, (7) can no longer be used as solution. This is because the integral limits are no pure coordinate limits anymore. Instead, the integration domain (tilted winding) needs to be split into three subdomains I, II, and III with the inner integral limits describing the boundaries of the particular subdomain. The resulting three subdomains are shown in Fig. 3d. Overall, the integral (6) of the field caused by winding \( k \) computed over a tilted winding \( i \) can be expressed as (10)

\[
\oint_{\text{tilted winding } i} F_k(x,y) \, dA = \int_{y_1}^{y_2} F_k(x,y) \, dy + \int_{y_3}^{y_4} F_k(x,y) \, dy + \int_{y_5}^{y_6} F_k(x,y) \, dy
\]  
(10)

The integral limits are visualised in Fig. 3d. The boundary functions \( u_k(y) \), \( u_t(y) \), \( v_h(y) \), and \( v_t(y) \) describe the boundaries of the winding and can be expressed as \( f(y) = ky + d \) where \( k \) is the slope and \( d \) is the intercept on the x-axis. These boundary functions are parameterised according to (11) in a coordinate system where the winding is tilted by the angle \( \gamma \) in clockwise direction as shown in Fig. 3d.
The limits of the outer integrals in (10) are given in (11).

\[
\begin{align*}
 u_b(y) &= -\cot(\gamma) y + x_{\text{ref}} + \cot(\gamma) y_{\text{ref}} \\
v_b(y) &= \tan(\gamma) y + x_{\text{ref}} - \tan(\gamma) y_{\text{ref}} + \frac{w_i}{\cos(\gamma)} \\
u_i(y) &= \tan(\gamma) y + x_{\text{ref}} + \cot(\gamma) y_{\text{ref}} + \frac{h_i}{\sin(\gamma)} \\
y_1 &= y_{\text{ref}} - w_i \sin(\gamma) \\
y_3 &= y_{\text{ref}} + h_i \cos(\gamma) \\
y_4 &= y_{\text{ref}} + h_i \cos(\gamma)
\end{align*}
\]

In case of tilted windings, the elementary mathematical problem boils down to integral (12).

\[
\int_{y_3}^{y_4} \int_{y_1}^{y_2} F(x-x_{c,k} \pm \frac{w_i}{2}, y-y_{c,k} \pm \frac{h_i}{2}) \, dx \, dy
\]

Integral (12) is solved by analytical integration and results in (13)

\[
\int_{y_3}^{y_4} \int_{y_1}^{y_2} F_k(x+x_k, y+y_k) \, dx \, dy = G_{\text{tilt}}(y_1, y_2, d_1, \ldots, y_3, y_4) - G_{\text{tilt}}(y_1, y_2, d_1, \ldots, y_3, y_4) + G_{\text{tilt}}(y_1, y_2, d_1, \ldots, y_3, y_4)
\]

The antiderivative \(G_{\text{tilt}}\) can be expressed according to (14)

\[
G_{\text{tilt}} = \ln \left( \left( d + k y \right)^2 \right) \left[ \left( \frac{k^2}{d^2} - \frac{1}{4} \right) y^4 + \left( \frac{4}{d^2} + \frac{4}{d} - \frac{1}{4} \right) y^3 + \left( \frac{1}{2} + \frac{k}{d} - \frac{1}{4} \right) y^2 + \frac{(3k^2 - 3)}{d^2} y \right]
\]

\[
- \frac{1}{192} \left( 7k^3 + 14d^2 k - 13d^3 \right) y^4 - \frac{1}{1152} \ln \left( 4d^2 + 4d y + 4d^2 k^2 + 4k^4 \right)
\]

The overall leakage inductance per unit length is calculated via the magnetic energy per unit length according to (9) just as in Margueron’s model.

**Simplifications for Efficient Implementation**

In general, the integral of the potential of winding \(k\) integrated over winding \(i\) equals the integral of the potential of winding \(i\) over winding \(k\). With this, one out of four integrals belonging to a winding pair can be omitted.

\[
\int \int \text{winding } i \, A_i \, dA = \int \int \text{winding } k \, A_i \, dA \Rightarrow \int \int \text{winding } i \, F_k(x,y) \, dA = \int \int \text{winding } k \, F_i(x,y) \, dA
\]

**Implementation of Mirror Image Scenarios**

A precondition to execute integral (10) is that the coordinate system has to be parallel to the winding that creates the magnetic field. Considering all the possible mirror images of the windings, three different coordinate systems are required as indicated in Fig. 4a.

- The original \(x,y\)-coordinate system is parallel to the transformer core, the inner winding and all its mirror images.
- The \(x_{\text{ccw}}\)-system is parallel to the original outer winding and is therefore rotated by \(\gamma\) in clockwise direction in relation to the original \(x,y\)-system.
- The \(x_{\text{cw}}\)-system is rotated by \(\gamma\) in counterclockwise direction in relation to the original \(x,y\)-system.

Note that the integral limits (11) always have to be adapted according to the coordinate system and the winding \(i\) over which the integration is executed. This can be done by inserting the appropriate rotation
angle for $\gamma$ in (11). The appropriate rotation angle is the angle between the field creating winding $k$ and the integration-domain winding $i$. The fixed integration limits $y_1, y_{\text{ref}}, y_3, y_4$ in (11) can be transformed with a rotational coordinate transformation according to (16)

$$\begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

(16)

where $r$ and $s$ are the new coordinates, $\beta$ is the rotation angle (positive for counterclockwise rotation) and $x$ and $y$ are the original coordinates. In the implementation for this paper, the mirror image windings are assumed as field creating windings and integrated over the original windings. Due to (15), the other way round would be possible as well.

### 3.2 PDM Leakage Inductance Per Unit Length model

This section introduces the leakage inductance per unit length model that is used for the PDM cross section shown in Fig. 2c. As mentioned in sec. 3.1.1, the PDM cross section theoretically leads to an infinite amount of mirror images in $x$-direction. However, there are two geometric circumstances that can be used in this cross section:

1. The arrangement is symmetrical;
2. The windings are typically thin and tall (see also sec. 4.1).

These two geometric circumstances lead to the fact that the field is axial to a substantial degree. Hence, Kapp’s most basic equation of leakage inductance per unit length [24] is used. The equation assumes axial 1D-field and is given in (17).

$$L'_L = \mu_0 N_{\text{ref}}^2 \left( \frac{2a_1}{3} + d_{\text{PDM}} \right) \frac{1}{h_1}$$

(17)

where $d_{\text{PDM}} = d_{\text{cores}} - 2a_1 - 2d_{x,i}$. As mentioned in section 2.2, the leakage inductance is referred to the secondary side in this paper. Hence, $N_{\text{ref}} = N_2$.

### 3.3 Partial Leakage Length Models

The partial leakage lengths of all three elementary cross sections are visualised in Fig. 2b and can be calculated according to (18), (19), and (20).

$$l_{p,\text{IW}} = N_{\text{cores}} \frac{d_c}{2} + (N_{\text{cores}} - 1) d_{\text{cores}}$$

(18)

$$l_{p,\text{PDM}} = (N_{\text{cores}} - 1) (b_{\text{leg}} + 2d_{x,i})$$

(19)

$$l_{p,\text{OW}} = l_{\text{corners}} + 2b_{\text{leg}} + N_{\text{cores}} \frac{d_c}{2} + (N_{\text{cores}} - 1) d_{\text{cores}}$$

(20)

where $N_{\text{cores}}$ is the number of cores and all other relevant parameters are given in Fig. 2b. The equation for the length relating to the corner sections $l_{\text{corners}} = 8 (d_{x,i} + a_1 + d + \frac{1}{2} h_2 \sin(\gamma))$ is taken from [2].
4  Model Verification

This section verifies the proposed Triple-2D cone winding matrix (CWM) transformer model. First, the newly derived cone winding leakage inductance per unit length model from sec. 3.1 is verified. Second, the accuracy of the trivial model used for the PDM cross section is shown. Finally, the accuracy of the overall model is evaluated.

4.1  Benchmark Transformer

The CWM transformer shown in Fig. 4b is taken from [2] as benchmark to verify the models. This transformer has been designed for the compact linear collider in the European Organization for Nuclear Research (CERN). The benchmark transformer has geometric characteristics that are typical of CWM transformers in HVPP applications. In particular, the step-up ratio in HVPP applications is very high to achieve the required high output voltages. Typically, the primary LV-side features foil windings. For the secondary HV-side, solid round wires arranged in a single layer are used to achieve the high step-up ratio. This leads to the fact that the windings are typically thin and tall in CWM transformers. The geometrical parameters of the benchmark transformer are listed in Tabs. I and II.

Table I: Geometry of benchmark transformer [2]. Parameters see Figs. 1c,&2b. All distances in mm.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_1$</td>
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</tr>
<tr>
<td>$a_2$</td>
<td>2</td>
</tr>
<tr>
<td>$d$</td>
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</tr>
<tr>
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<tr>
<td>$d_{c,i}$</td>
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</tr>
<tr>
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<tr>
<td>$b_{leg}$</td>
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<tr>
<td>$d_c$</td>
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<tr>
<td>$d_{cores}$</td>
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<tr>
<td>$b_{leg}$</td>
<td>50</td>
</tr>
</tbody>
</table>

Table II: Geometry of benchmark transformer [2]. Parameters see Figs. 1c,&2b, and (2).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
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<td>$\gamma$ (°)</td>
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<td>$N_1$</td>
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<tr>
<td>$N_2$</td>
<td>124</td>
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<tr>
<td>$N_{cores}$</td>
<td>2</td>
</tr>
<tr>
<td>$c_w$</td>
<td>$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

4.2  Verification of 2D Cone Winding Leakage Inductance per Unit Length Model

2D FEM models of the benchmark transformer are set up to obtain benchmark values of the leakage inductance per unit length of the cross sections with tilted cone windings. Three scenarios are considered for model verification: outside-window, inside-window, and free space. The latter is a scenario with windings surrounded by air and no core in vicinity – this is used to verify the model without the influence of the mirroring method. The results of the proposed model are listed in Tab. III and compared with the corresponding FEM simulation. The table shows that the analytical model agrees very well with the results of the 2D FEM simulations. For the free space and the outside-window scenario, the model error is close to zero. For the inside-window cross section, a comparably small error in the range of 0.5% results as it is approximated with one image layer (8 mirror images) instead of an infinite amount of image layers. Overall, the error is very low, especially if the model is compared to an existing model that assumes 1D-field and energy stored only between the windings [5].

Table III: Results of analytical cone winding leakage inductance per unit length model from sec. 3.1 compared to 1D-model proposed in [5].

<table>
<thead>
<tr>
<th>Scenario</th>
<th>2D-FEM $\frac{\text{mH}}{\text{m}}$</th>
<th>Proposed model</th>
<th>1D-model [5]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Free space</td>
<td>1.258 $\frac{\text{mH}}{\text{m}}$</td>
<td>0.110%</td>
<td>-</td>
</tr>
<tr>
<td>Outside-window</td>
<td>1.361 $\frac{\text{mH}}{\text{m}}$</td>
<td>-0.028%</td>
<td>-</td>
</tr>
<tr>
<td>Inside-window</td>
<td>1.442 $\frac{\text{mH}}{\text{m}}$</td>
<td>0.456%*</td>
<td>-35.3%</td>
</tr>
</tbody>
</table>

*1 image layer, i.e. 8 mirror images
4.3 Verification of 2D PDM model

For calculating the leakage inductance per unit length of the primary double mirror (PDM) cross section in Fig. 2c, Kapp’s basic leakage inductance per unit length equation (17) is used. The error of the model is shown in Tab. IV.

<table>
<thead>
<tr>
<th>Scenario</th>
<th>2D-FEM $L'_\sigma$ mH</th>
<th>Kapp’s equation (17) $L'_\sigma$ mH</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDM</td>
<td>1.71</td>
<td>1.77</td>
<td>3.4%</td>
</tr>
</tbody>
</table>

Tab. IV shows that the error of the 1D-model of Kapp is only 3.4%. This is because the symmetry and the tall thin windings lead to a magnetic field that consists mainly of a $y$-component. Hence, it is axial to a substantial degree. The very low computational effort and the low error of this model favour the use of this simplified model in the PDM cross section.

4.4 Verification of Triple-2D model

The overall accuracy of the Triple-2D model (1) is evaluated by comparing the calculated leakage inductance with the measurement of the benchmark transformer. Tab. V shows that the error of the proposed Triple-2D model is as low as 7.6% for the benchmark transformer from [2]. This accuracy is in a very satisfactory range because geometrical and measurement inaccuracies can lead to substantially higher errors.

The computation time of the model is approximately 1 ms, which is also shown in Tab. V. The computation time was extracted from a standard notebook with Windows 10 64bit, an intel i7-8550U processor, and 16 GB RAM. The function time was meared with the `timeit()`-function of Matlab. At the time of execution, Matlab was the only program active on the PC. The small amount of computation time makes the model well suitable for using it in optimisations.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1.325 mH</td>
<td>1.426 mH</td>
<td>7.6%</td>
<td>1 ms</td>
</tr>
</tbody>
</table>

5 Conclusion

A novel analytical Triple-2D leakage inductance model for matrix transformers with cone windings was proposed. The key element of the model is a leakage inductance per unit length model based on the model of Margueron et al. of the magnetic vector potential of rectangular windings. In this paper, the integration of the magnetic energy density is extended to the tilted cone winding by applying an analytically derived antiderivative. For maximum accuracy, three elementary cross sections are considered for matrix-type transformers: inside-window, outside-window, and primary double mirror. To obtain the leakage inductance, the leakage inductances per unit length of the elementary cross sections are scaled with their particular geometrical partial leakage lengths and the products are summed up. This concept is referred to as the Triple-2D modelling concept.

The model is verified by measurements on an existing cone winding matrix transformer that is used in the compact linear collider at CERN. The analytical leakage inductance per unit length model for tilted cone windings features an error below 0.5% compared with 2D FEM simulations of the benchmark transformer. The overall error of the Triple-2D model is 7.6% compared with measurements on the original transformer.

As the model is based on analytical derivations, the Triple-2D model is scalable to any cone winding matrix transformer with arbitrary geometry. Furthermore, the model is rapidly executable with a computation time of about 1 ms on a standard notebook and therefore well suitable its use in optimisations.
References


