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Numerical Models and Controller Design Parameters for the Balancing Cube

Technical Report

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Abstract

The Balancing Cube is a dynamic sculpture that can balance on any of its edges or corners through the action of six rotating arms mounted on its inner faces. The arms are self-contained units, each equipped with sensors, actuation, and a computer. They exchange data over a network and coordinate to stabilize the cube. The system is used as a testbed for distributed estimation and control. The design, modeling, and control of the system is presented in [1]. This report complements [1]; it states the numerical values for the linear dynamic models of the cube in [1] and the parameters of the state-feedback control design in [1].

1 Introduction

The Balancing Cube is a dynamic sculpture and a testbed for distributed estimation and control. Six rotating arms (called *modules*) that are mounted on the cube's inner faces are the actuation mechanisms. The modules are self-contained control agents carrying sensors, a motor, a computer, and a communication unit allowing the exchange of data with each other. For a video of the system, refer to the project website [2]. The design, modeling, and control of the Balancing Cube is described in [1].

This technical report complements the article [1] by presenting the numerical values for the linear models of the system dynamics and the parameters used in the controller design. The presentation herein is kept to the essentials in order not to repeat [1]. In particular, this report does not motivate, explain, and derive the models and controllers. The corresponding sections in [1] are referenced instead.

2 Linearized Models of the System Dynamics

Two balancing configuration are considered (see [1, Sec. "Modeling"]): (i) the cube stands upright on one of its corners, and all modules point downward (called *corner balancing*); (ii) the cube stands on one of its edges, and the modules are rotated as shown in the figure titled "Edge balancing" in [1] (called *edge balancing*).

2.1 Edge Balancing

The dynamics of the Balancing Cube about the edge balancing equilibrium are captured by the linear, time-invariant, continuous-time, state-space model

$$\dot{x}(t) = A_{\mathrm{e}}\, x(t) + B_{\mathrm{e}}\, u(t), \tag{1}$$

where the state vector $x(t) \in \mathbb{R}^{14}$ includes the angles of the modules relative to the cube body ($x_{1:6} = [x_1, \ldots, x_6]^T$), the angular rates of the modules ($x_{7:12}$), the cube pitch angle (x_{13}), and the cube pitch angular rate (x_{14}); and $u(t) \in \mathbb{R}^6$ are the torques applied at the modules. The state-space matrices in (1) are:

$$A_{\mathrm{e}} = \begin{bmatrix} \mathit{0} & \mathit{I} & \mathit{0} \\ A^{\mathrm{e}}_{21} & A^{\mathrm{e}}_{22} & A^{\mathrm{e}}_{23} \\ A^{\mathrm{e}}_{31} & A^{\mathrm{e}}_{32} & A^{\mathrm{e}}_{33} \end{bmatrix} \quad \text{and} \quad B_{\mathrm{e}} = \begin{bmatrix} \mathit{0} \\ B^{\mathrm{e}}_{2} \\ B^{\mathrm{e}}_{3} \end{bmatrix}$$

with

$$A^{\mathrm{e}}_{21} = \begin{bmatrix} -9.7 & 0.158 & 0.0484 & 0.0289 & -0.079 & -0.0289 \\ 0.247 & -20.1 & -0.247 & -0.148 & 0.404 & 0.147 \\ 0.0484 & -0.158 & -9.7 & -0.0289 & 0.079 & 0.0289 \\ 0.497 & -1.62 & -0.497 & -0.296 & 0.811 & 0.296 \\ -0.249 & 0.814 & 0.249 & 0.149 & -17.5 & -0.149 \\ -0.497 & 1.62 & 0.497 & 0.296 & -0.811 & -0.296 \end{bmatrix}$$

$$A^{\mathrm{e}}_{22} = \begin{bmatrix} -0.589 & 0.00231 & 0.000453 & 0.00465 & -0.00233 & -0.00465 \\ 0.00231 & -0.6 & -0.00231 & -0.0238 & 0.0119 & 0.0238 \\ 0.000453 & -0.00231 & -0.589 & -0.00465 & 0.00233 & 0.00465 \\ 0.00465 & -0.0238 & -0.00465 & -1.09 & 0.024 & 0.0477 \\ -0.00233 & 0.0119 & 0.00233 & 0.024 & -1.06 & -0.024 \\ -0.00465 & 0.0238 & 0.00465 & 0.0477 & -0.024 & -1.09 \end{bmatrix}$$

$$A_{23}^{\mathrm{e}} = \begin{bmatrix} 8 & 0 \\ -10.9 & 0 \\ -8 & 0 \\ 4.88 & 0 \\ 8.57 & 0 \\ -4.88 & 0 \end{bmatrix}$$

$$A_{31}^{\mathrm{e}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.267 & -0.87 & -0.267 & -0.159 & 0.435 & 0.159 \end{bmatrix}$$

$$A_{32}^{\mathrm{e}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.0025 & -0.0128 & -0.0025 & -0.0256 & 0.0129 & 0.0256 \end{bmatrix}$$

$$A_{33}^{\mathrm{e}} = \begin{bmatrix} 0 & 1 \\ 9.11 & 0 \end{bmatrix}$$

$$B_{2}^{\mathrm{e}} = \begin{bmatrix} 1.18 & -0.00463 & -0.000906 & -0.0093 & 0.00467 & 0.0093 \\ -0.00463 & 1.2 & 0.00463 & 0.0475 & -0.0238 & -0.0475 \\ -0.000906 & 0.00463 & 1.18 & 0.0093 & -0.00467 & -0.0093 \\ -0.0093 & 0.0475 & 0.0093 & 2.18 & -0.0479 & -0.0954 \\ 0.00467 & -0.0238 & -0.00467 & -0.0479 & 2.11 & 0.0479 \\ 0.0093 & -0.0475 & -0.0093 & -0.0954 & 0.0479 & 2.18 \end{bmatrix}$$

$$B_{3}^{\mathrm{e}} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.00499 & 0.0255 & 0.00499 & 0.0513 & -0.0257 & -0.0513 \end{bmatrix}.$$

Equation (1) corresponds to the state-space model presented in Sec. “Modeling/Linear dynamics model” of [1] for edge balancing.

2.2 Corner Balancing

The dynamics about the corner balancing equilibrium are captured by

$$\dot{x}(t) = A_{\mathrm{c}}\, x(t) + B_{\mathrm{c}}\, u(t), \tag{2}$$

where the input $u(t) \in \mathbb{R}^6$ is the same as in (1), and the state vector $x(t) \in \mathbb{R}^{16}$ has the same elements as in (1), but, in addition, includes the cube roll angle (x_{15}) and roll rate (x_{16}). The matrices are:

$$A_{\mathrm{c}} = \begin{bmatrix} \mathit{0} & I & \mathit{0} \\ A_{21}^{\mathrm{c}} & A_{22}^{\mathrm{c}} & A_{23}^{\mathrm{c}} \\ A_{31}^{\mathrm{c}} & A_{32}^{\mathrm{c}} & A_{33}^{\mathrm{c}} \end{bmatrix} \quad \text{and} \quad B_{\mathrm{c}} = \begin{bmatrix} \mathit{0} \\ B_{2}^{\mathrm{c}} \\ B_{3}^{\mathrm{c}} \end{bmatrix}$$

with

$$A_{21}^{\mathrm{c}} = \begin{bmatrix} -16.4 & 0.103 & 0.103 & 0.138 & -0.411 & -0.411 \\ 0.103 & -16.4 & 0.103 & -0.411 & 0.138 & -0.411 \\ 0.103 & 0.103 & -16.4 & -0.411 & -0.411 & 0.138 \\ 0.58 & -0.976 & -0.976 & -15.4 & -0.315 & -0.315 \\ -0.976 & 0.58 & -0.976 & -0.315 & -15.4 & -0.315 \\ -0.976 & -0.976 & 0.58 & -0.315 & -0.315 & -15.4 \end{bmatrix}$$

$$A_{22}^{\mathrm{c}} = \begin{bmatrix} -0.602 & -0.00162 & -0.00162 & -0.000628 & -0.0253 & -0.0253 \\ -0.00162 & -0.602 & -0.00162 & -0.0253 & -0.000628 & -0.0253 \\ -0.00162 & -0.00162 & -0.602 & -0.0253 & -0.0253 & -0.000628 \\ -0.000628 & -0.0253 & -0.0253 & -1.13 & -0.0347 & -0.0347 \\ -0.0253 & -0.000628 & -0.0253 & -0.0347 & -1.13 & -0.0347 \\ -0.0253 & -0.0253 & -0.000628 & -0.0347 & -0.0347 & -1.13 \end{bmatrix}$$

$$A_{23}^{c} = \begin{bmatrix} 10.4 & 0 & 4.9 & 0 \\ -10.4 & 0 & 4.9 & 0 \\ 0 & 0 & -9.79 & 0 \\ -1.93 & 0 & -0.911 & 0 \\ 1.93 & 0 & -0.911 & 0 \\ 0 & 0 & 1.82 & 0 \end{bmatrix}$$

$$A_{31}^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.489 & -0.489 & 0 & -0.35 & 0.35 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.346 & 0.346 & -0.691 & -0.248 & -0.248 & 0.495 \end{bmatrix}$$

$$A_{32}^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0.00775 & -0.00775 & 0 & -0.0157 & 0.0157 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0.00548 & 0.00548 & -0.011 & -0.0111 & -0.0111 & 0.0223 \end{bmatrix}$$

$$A_{33}^{c} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 8.08 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 8.08 & 0 \end{bmatrix}$$

$$B_{2}^{c} = \begin{bmatrix} 1.2 & 0.00324 & 0.00324 & 0.00126 & 0.0506 & 0.0506 \\ 0.00324 & 1.2 & 0.00324 & 0.0506 & 0.00126 & 0.0506 \\ 0.00324 & 0.00324 & 1.2 & 0.0506 & 0.0506 & 0.00126 \\ 0.00126 & 0.0506 & 0.0506 & 2.26 & 0.0695 & 0.0695 \\ 0.0506 & 0.00126 & 0.0506 & 0.0695 & 2.26 & 0.0695 \\ 0.0506 & 0.0506 & 0.00126 & 0.0695 & 0.0695 & 2.26 \end{bmatrix}$$

$$B_{3}^{c} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -0.0155 & 0.0155 & 0 & 0.0315 & -0.0315 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.011 & -0.011 & 0.0219 & 0.0223 & 0.0223 & -0.0445 \end{bmatrix}.$$

Equation (2) corresponds to the state-space model presented in Sec. "Modeling/Linear dynamics model" of [1] for corner balancing.

3 Linear Models Incorporating Local Velocity Feedback

The input signal in the models (1) and (2) is torque applied at the modules. In usual operation of the cube, a local velocity feedback loop is implemented on the motor unit of each module: velocity commands are tracked by a local controller, which receives measurements of the motor shaft velocity and computes the torque applied to the module. Hence, for the design of an outer-loop controller that computes the velocity commands, a model is needed that incorporates the linear dynamics (1) and (2), and the effects of the local velocity feedback. These models are given below for edge and corner balancing. They are discrete-time models with a sampling rate equal to the rate at which the velocity commands are updated.

For motivation and a more detailed discussion of the cascaded control architecture, refer to [1, Sec. "Control"]. The procedure for obtaining the models below (which incorporate the effect of the local feedback loops) from the models (1) and (2) is explained in [1, "Time Scale Separation Algorithm"].

3.1 Edge Balancing

The dynamics of the cube about the edge balancing equilibrium that include the effect of the velocity feedback on each module are described by the discrete-time model

$$x[k+1] = \tilde{A}_\mathrm{e}\, x[k] + \tilde{B}_\mathrm{e}\, v[k], \tag{3}$$

with sampling time $T_\mathrm{s} = 10\,\mathrm{ms}$; and where $x[k] := x(kT_\mathrm{s})$ denotes the signal $x(t)$ sampled at the rate T_s. The discrete-time signal $v[k] \in \mathbb{R}^6$ are the velocity commands changing at the rate T_s. The matrices of the state-space model (3) are:

$$\tilde{A}_\mathrm{e} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{A}^\mathrm{e}_{31} & \tilde{A}^\mathrm{e}_{32} & \tilde{A}^\mathrm{e}_{33} \end{bmatrix} \quad \text{and} \quad \tilde{B}_\mathrm{e} = \begin{bmatrix} T_\mathrm{s}\, I \\ I \\ \tilde{B}^\mathrm{e}_3 \end{bmatrix}$$

with

$$\begin{aligned}
\tilde{A}^\mathrm{e}_{31} &= \begin{bmatrix} \text{1e-5} & -\text{2.01e-5} & -\text{1e-5} & -\text{7.08e-6} & \text{1e-5} & \text{7.08e-6} \\ 0.00201 & -0.00402 & -0.00201 & -0.00142 & 0.002 & 0.00142 \end{bmatrix} \\
\tilde{A}^\mathrm{e}_{32} &= \begin{bmatrix} \text{3.78e-5} & -0.000193 & -\text{3.78e-5} & -0.000218 & 0.00011 & 0.000218 \\ 0.00378 & -0.0193 & -0.00378 & -0.0218 & 0.011 & 0.0218 \end{bmatrix} \\
\tilde{A}^\mathrm{e}_{33} &= \begin{bmatrix} 1 & 0.01 \\ 0.0926 & 1 \end{bmatrix} \\
\tilde{B}^\mathrm{e}_{3} &= \begin{bmatrix} -\text{3.77e-5} & 0.000193 & \text{3.77e-5} & 0.000218 & -0.000109 & -0.000218 \\ -0.00377 & 0.0193 & 0.00377 & 0.0218 & -0.0109 & -0.0218 \end{bmatrix}.
\end{aligned}$$

Equation (3) corresponds to the state-space model presented in Sec. "Control/Simplified model incorporating local feedback loops from time scale separation" of [1] for edge balancing.

3.2 Corner Balancing

The dynamics about the corner balancing equilibrium including the effect of the velocity feedback on each module are given by

$$x[k+1] = \tilde{A}_\mathrm{c}\, x[k] + \tilde{B}_\mathrm{c}\, v[k], \tag{4}$$

with sampling time $T_\mathrm{s} = 10\,\mathrm{ms}$, $v[k]$ as in (3), and

$$\tilde{A}_\mathrm{c} = \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{A}^\mathrm{c}_{31} & \tilde{A}^\mathrm{c}_{32} & \tilde{A}^\mathrm{c}_{33} \end{bmatrix} \quad \text{and} \quad \tilde{B}_\mathrm{c} = \begin{bmatrix} T_\mathrm{s}\, I \\ I \\ \tilde{B}^\mathrm{c}_3 \end{bmatrix}$$

with

$$\begin{aligned}
\tilde{A}^\mathrm{c}_{31} &= \begin{bmatrix} \text{1.32e-5} & -\text{1.32e-5} & 0 & -\text{6.55e-6} & \text{6.55e-6} & 0 \\ 0.00263 & -0.00263 & 0 & -0.00131 & 0.00131 & 0 \\ \text{9.3e-6} & \text{9.3e-6} & -\text{1.86e-5} & -\text{4.63e-6} & -\text{4.63e-6} & \text{9.27e-6} \\ 0.00186 & 0.00186 & -0.00372 & -0.000927 & -0.000927 & 0.00185 \end{bmatrix} \\
\tilde{A}^\mathrm{c}_{32} &= \begin{bmatrix} 0.000123 & -0.000123 & 0 & -0.000141 & 0.000141 & 0 \\ 0.0123 & -0.0123 & 0 & -0.0141 & 0.0141 & 0 \\ \text{8.72e-5} & \text{8.72e-5} & -0.000174 & -\text{9.97e-5} & -\text{9.97e-5} & 0.000199 \\ 0.00872 & 0.00872 & -0.0174 & -0.00998 & -0.00998 & 0.02 \end{bmatrix}
\end{aligned}$$

$$\tilde{A}^{\mathrm{c}}_{33} = \begin{bmatrix} 1 & 0.01 & 0 & 0 \\ 0.0839 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0.01 \\ 0 & 0 & 0.0839 & 1 \end{bmatrix}$$

$$\tilde{B}^{\mathrm{c}}_{3} = \begin{bmatrix} -0.000123 & 0.000123 & 0 & 0.000141 & -0.000141 & 0 \\ -0.0123 & 0.0123 & 0 & 0.0141 & -0.0141 & 0 \\ -8.72\text{e-}5 & -8.72\text{e-}5 & 0.000174 & 9.97\text{e-}5 & 9.97\text{e-}5 & -0.000199 \\ -0.00871 & -0.00871 & 0.0174 & 0.00997 & 0.00997 & -0.0199 \end{bmatrix}.$$

Equation (4) corresponds to the state-space model presented in Sec. "Control/Simplified model incorporating local feedback loops from time scale separation" of [1] for corner balancing.

4 Controller Design Parameters

A stabilizing state-feedback controller for the Balancing Cube is obtained in [1, Sec. "Control"] using an LQR design based on the models (3) and (4), augmented with integrator states on the module angles $x_{1:6}$. The state vector of the augmented system is

$$\tilde{x}[k] = (x_{7:12}[k], x_{1:6}[k], x_{13:n}[k], x_{\mathrm{int}}[k]),$$

where $n = 14$ for edge balancing, $n = 16$ for corner balancing, and x_{int} are the augmented integrator states. The cost function that is used for the LQR design is

$$\sum_{k=0}^{\infty} \tilde{x}^T[k] \begin{bmatrix} \Theta & 0 \\ 0 & Q \end{bmatrix} \tilde{x}[k] + v^T[k]\,(R + \Theta)\,v[k] + 2\tilde{x}^T[k] \begin{bmatrix} -\Theta \\ 0 \end{bmatrix} v[k]. \tag{5}$$

(See [1, Sec. "Control"] for an explanation of the special block structure of the weighting matrices.)

4.1 Edge Balancing

For the controller design presented in [1, Sec. "Control"], the following weights are used in the cost function (5) for edge balancing:

$$\begin{aligned} Q_{\mathrm{e}} &= \mathrm{diag}\big([1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0.2\ 0.2\ 0.2\ 0.2\ 0.2\ 0.2]\big) \\ R_{\mathrm{e}} &= \mathrm{diag}\big([0.035\ 0.035\ 0.035\ 0.035\ 0.035\ 0.035]\big) \\ \Theta_{\mathrm{e}} &= \mathrm{diag}\big([4\ 4\ 4\ 4\ 4\ 4]\big), \end{aligned}$$

where diag(·) denotes the diagonal matrix with the vector argument being the diagonal.

4.2 Corner Balancing

The following weights are used in the cost function (5) for corner balancing:

$$\begin{aligned} Q_{\mathrm{c}} &= \mathrm{diag}\big([1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0.2\ 0.2\ 0.2\ 0.2\ 0.2\ 0.2]\big) \\ R_{\mathrm{c}} &= \mathrm{diag}\big([0.035\ 0.035\ 0.035\ 0.035\ 0.035\ 0.035]\big) \\ \Theta_{\mathrm{c}} &= \mathrm{diag}\big([4\ 4\ 4\ 4\ 4\ 4]\big). \end{aligned}$$

References

[1] S. Trimpe and R. D'Andrea, "The Balancing Cube – a dynamic scuplture as testbed for distributed estimation and control," *IEEE Control Systems Magazine*, Dec. 2012, accepted.

[2] "Balancing Cube website," [accessed 13.08.2012]. [Online]. Available: http://www.cube.ethz.ch