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System-level, input–output and new parameterizations of stabilizing controllers, and their numerical computation[☆]



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ABSTRACT

It is known that the set of internally stabilizing controller C_{stab} is non-convex, but it admits convex characterizations using certain closed-loop maps: a classical result is the Youla parameterization, and two recent notions are the system-level parameterization (SLP) and the input–output parameterization (IOP). In this paper, we address the existence of new convex parameterizations and discuss potential tradeoffs of each parameterization in different scenarios. Our main contributions are: (1) We reveal that only four groups of stable closed-loop transfer matrices are equivalent to internal stability: one of them is used in the SLP, another one is used in the IOP, and the other two are new, leading to two new convex parameterizations of C_{stab} . (2) We investigate the properties of these parameterizations after imposing the finite impulse response (FIR) approximation, revealing that the IOP has the best ability of approximating C_{stab} given FIR constraints. (3) These four parameterizations require *a priori* doubly-coprime factorization of the plant, but impose a set of equality constraints. However, these equality constraints will never be satisfied exactly in floating-point arithmetic computation and/or implementation. We prove that the IOP is numerically robust for open-loop stable plants, in the sense that small mismatches in the equality constraints do not compromise the closed-loop stability; but a direct IOP implementation will fail to stabilize open-loop unstable systems in practice. The SLP is known to enjoy numerical robustness in the state feedback case; here, we show that numerical robustness of the four-block SLP controller requires case-by-case analysis even when the plant is open-loop stable.

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1. Introduction

Feedback systems must be stable in some appropriate sense for practical deployment, and thus one fundamental problem in control theory is to design a feedback controller that stabilizes a given dynamical system (Zhou, Doyle, Glover, et al., 1996). Indeed, many control synthesis problems include stability as a constraint while optimizing some performance (Dullerud & Paganini, 2013). However, it is well-known that the set of stabilizing controllers is

non-convex, and hence, hard to search directly over. One standard approach is to parameterize all stabilizing controllers and the corresponding closed-loop responses in a convex way, and then to optimize the performance over the new parameter(s) using convex optimization (Boyd & Barratt, 1991).

A classical method for parameterizing the set of all internally stabilizing controllers is based on the celebrated *Youla parameterization* (Youla, Jabr, & Bongiorno, 1976) which relies on a doubly-coprime factorization of the system. It is shown in Boyd and Barratt (1991) that many performance specifications on the closed-loop system can be expressed in the Youla parameterization framework via convex optimization. Moreover, the foundational results of robust and optimal control are built on the Youla parameterization (Francis, 1987; Zhou et al., 1996). Recently, a *system-level parameterization* (SLP) (Wang, Matni, & Doyle, 2019) and an *input–output parameterization* (IOP) (Furieri, Zheng, Papachristodoulou, & Kamgarpour, 2019) were proposed to characterize the set of internally stabilizing controllers, without relying on the doubly-coprime factorization technique.

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In principle, Youla, the SLP, and the IOP all directly treat certain closed-loop responses as design parameters and thus shift the controller synthesis from the design of a controller to the design of the closed-loop responses. We note that an open-source Python-based implementation for the SLP and the IOP is available (Tseng & Shuang, 2020).

Besides the classical control synthesis problems (Francis, 1987; Zhou et al., 1996), closed-loop parameterizations are powerful tools in other areas, including distributed optimal control (Lessard & Lall, 2015; Matni & Chandrasekaran, 2016; Qi, Salapaka, Voulgaris, & Khammash, 2004; Rotkowitz & Lall, 2006; Sabău & Martins, 2014; Shah & Parrilo, 2013) and quantifying the performance of learning in control (Dean, Mania, Matni, Recht, & Tu, 2017, 2018; Furieri, Zheng and Kamgarpour, 2020; Lale, Aziz-zadenesheli, Hassibi, & Anandkumar, 2020; Simchowitz, Singh, & Hazan, 2020; Zheng, Furieri, Kamgarpour, & Li, 2021). In distributed control, the goal is to design sub-controllers relying on locally available information, which is crucial for many cyber-physical systems. Enforcing these information constraints, however, may make the problems computationally intractable (Tsitsiklis & Athans, 1985; Witsenhausen, 1968). Nevertheless, it is well-known that a notion of *quadratic invariance* (QI) (Lessard & Lall, 2015; Rotkowitz & Lall, 2006; Sabău & Martins, 2014) allows equivalently translating information constraints on the controller to convex constraints on the Youla parameter, thus preserving the convexity of distributed controller synthesis. The QI notion can also be integrated with the SLP and the IOP, resulting in equivalent convex reformulations (Furieri et al., 2019; Wang et al., 2019). Together with a recent notion of *Sparsity Invariance* (Furieri, Zheng, Papachristodoulou and Kamgarpour, 2020), these closed-loop parameterizations enable deriving convex approximations for problems with general sparsity constraints beyond QI; see Zheng, Furieri, Papachristodoulou, Li, and Kamgarpour (2020, Remark 5) for example.

For learning in control, the SLP was integrated within a *Coarse-ID* control procedure to derive a sample complexity bound for learning the classical linear quadratic regulator (LQR) (Dean et al., 2017). This procedure was exploited in Dean et al. (2018) to derive high probability guarantees of sub-linear regret using an adaptive LQR control architecture. In Lale et al. (2020) and Simchowitz et al. (2020), based on the Youla parameterization, an online gradient descent algorithm was proposed to achieve sub-linear regret for learning the linear quadratic gaussian (LQG) controller. In Furieri, Zheng and Kamgarpour (2020), the Youla framework was used to derive a sample complexity bound on learning the *globally optimal* distributed controller subject to QI constraints. The IOP was also used to derive an end-to-end sample complexity bound on learning LQG controllers for stable systems (Zheng et al., 2021). The results in Furieri, Zheng and Kamgarpour (2020), Lale et al. (2020), Simchowitz et al. (2020) and Zheng et al. (2021) motivate the shift from static controllers to dynamic ones in complex learning-based control tasks.

Youla (Youla et al., 1976), the SLP (Wang et al., 2019), and the IOP (Furieri et al., 2019) are fundamental building blocks for distributed controller synthesis and learning-based control applications. Nevertheless, a few critical issues have been left unexplored. First, while it is known that Youla, the SLP, and the IOP are all equivalent (Zheng et al., 2020), it remains unclear whether there exist other equivalent parameterizations beyond them. Second, the decision variables in these closed-loop parameterizations are, in general, infinite-dimensional. The works (Anderson, Doyle, Low, & Matni, 2019; Furieri et al., 2019; Wang et al., 2019) enforce *finite impulse response* (FIR) constraints on the decision variables to enable formulation as finite-dimensional convex optimization problems. However, these works do not characterize the conservatism introduced by the FIR approximation using different

parameterizations. Third, unlike Youla, the SLP and the IOP do not need to know a doubly-coprime factorization *a priori*, and instead introduce a set of equality constraints for achievable closed-loop responses. A fact that is not investigated in the SLP (Anderson et al., 2019; Wang et al., 2019), the IOP (Furieri et al., 2019) or the recent work (Zheng et al., 2020) is that the set of equality constraints can never be satisfied exactly in numerical computation, potentially affecting the closed-loop stability.

1.1. Contributions

This paper aims to investigate the issues raised above and provide a complete understanding of closed-loop parameterizations. We introduce new parameterizations beyond SLP/IOP and discuss tradeoffs among these parameterizations in different scenarios. Note that our previous work (Zheng et al., 2020) established explicit affine mappings among Youla, SLP, and IOP parameters, but it provides no investigation of the issues mentioned above. Specifically, the contributions of this paper are as follows.

First, we examine all possible parameterizations for the set of internally stabilizing controllers C_{stab} using closed-loop responses from the disturbances $(\delta_x, \delta_y, \delta_u)$ to state, output, control signals $(\mathbf{x}, \mathbf{y}, \mathbf{u})$; see Fig. 1 for illustration. Our strategy is to examine the cases where the stability of external transfer matrices is equivalent to internal stability. We reveal that only four groups of stable disturbance-to-signal maps can guarantee internal stability (see Theorem 1): one of them is used in the SLP (Wang et al., 2019), another one is a classical result and is used in the IOP (Furieri et al., 2019), and the other two have not been discussed before and thus can be used to derive two new parameterizations (Propositions 3 and 4). Our results are *complete and exclusive*, in the sense that there are no other parameterizations for C_{stab} using closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$.

Second, we investigate the impact of imposing FIR constraints on the closed-loop parameterizations. We show that the IOP provides the tightest approximation of C_{stab} after imposing FIR constraints (Theorem 2). This result is enabled by the fact that the IOP directly deals with the maps from inputs to outputs without passing through the system state, while the SLP and the two new parameterizations explicitly involve the system state or disturbances on the system state. Motivated by Anderson and Matni (2017), we characterize state-space realizations for the controllers in closed-loop parameterizations after imposing FIR approximations (Theorem 3). The state-space realizations provide easily implementable controllers for practical deployment.

Third, we quantify the numerical robustness of closed-loop parameterizations due to floating-point arithmetic in both numerical computation and controller implementation. We prove that in the IOP, small numerical mismatches in the equality constraints do not compromise closed-loop stability for open-loop stable plants, but will destabilize the closed-loop system for unstable plants (Theorem 4). This result holds similarly for the two new closed-loop parameterizations. We also show that, in general, the four-block SLP controller in the output-feedback case is vulnerable to destabilization due to small mismatches in the equality constraints (Theorem 5), no matter whether the plant is open-loop stable or unstable. This issue arises irrespective of which SLP controller implementation is used (Anderson et al., 2019; Wang et al., 2019). These results suggest that it is unwise to directly apply the IOP or related parameterizations for open-loop unstable systems; instead, utilizing a pre-stabilizing controller is more appealing since it avoids the destabilization issue.

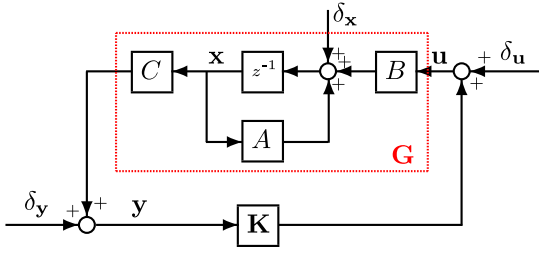


Fig. 1. Interconnection of the plant G and the controller K .

1.2. Paper structure

The rest of this paper is organized as follows. We state the problem in Section 2. The relationship between the stability of external transfer matrices and internal stability is revealed in Section 3. Four parameterizations of stabilizing controllers using closed-loop responses, including the SLP and the IOP, are presented in Section 4. Numerical computation using the FIR constraints and controller implementation are discussed in Section 5. We investigate the numerical robustness of closed-loop parameterizations in Section 6. A numerical application is shown in Section 7, and we conclude the paper in Section 8. Some auxiliary results and proofs are provided in an extended arXiv version (Zheng, Furieri, Kamgarpour, & Li, 2019).

1.3. Notation

The symbols \mathbb{R} and \mathbb{N} refer to the set of real and integer numbers, respectively. We use lower and upper case letters (e.g. x and A) to denote vectors and matrices, respectively. Lower and upper case boldface letters (e.g. \mathbf{x} and \mathbf{G}) are used to denote signals and transfer matrices, respectively. We denote the set of real-rational proper stable¹ transfer matrices as \mathcal{RH}_∞ . We use $\mathbf{G} \in \frac{1}{2}\mathcal{RH}_\infty$ to denote that \mathbf{G} is stable and strictly proper. Given $\mathbf{G} \in \mathcal{RH}_\infty$, we denote its \mathcal{H}_∞ norm by $\|\mathbf{G}\|_\infty := \sup_\omega \sigma_{\max}(\mathbf{G}(e^{j\omega}))$, where $\sigma_{\max}(\cdot)$ denotes the maximum singular value. Given a stable transfer matrix $\mathbf{G}(z)$, the square of its \mathcal{H}_2 norm is $\|\mathbf{G}\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Trace}(\mathbf{G}^*(e^{j\omega})\mathbf{G}(e^{j\omega})) d\omega$. For simplicity, we omit the dimension of transfer matrices, which shall be clear in the context. Also, we use I (resp. 0) to denote the identity matrix (resp. zero matrix) of compatible dimension. In Section 5.2, to avoid ambiguity, we explicitly write the matrix dimension and use I_p to denote the identity matrix of dimension p .

2. Problem statement

2.1. System model

We consider strictly proper discrete-time linear time-invariant (LTI) plants of the form

$$\begin{aligned} x[t+1] &= Ax[t] + Bu[t] + \delta_x[t], \\ y[t] &= Cx[t] + \delta_y[t], \end{aligned} \quad (1)$$

where $x[t] \in \mathbb{R}^n$, $u[t] \in \mathbb{R}^m$, $y[t] \in \mathbb{R}^p$ are the state vector, control action, and measurement vector at time t , respectively; $\delta_x[t] \in \mathbb{R}^n$ and $\delta_y[t] \in \mathbb{R}^p$ are external disturbances on the state and measurement vectors at time t , respectively. The transfer

¹ Throughout the paper, “stable” means “asymptotically stable”, i.e., all eigenvalues/poles have strictly negative real parts in continuous time (magnitudes less than 1 in discrete time).

matrix from u to y is $\mathbf{G} = C(zI - A)^{-1}B$. Consider an LTI dynamical controller

$$u = \mathbf{K}y + \delta_u, \quad (2)$$

where δ_u is the external disturbance on the control action. A state-space realization of (2) is

$$\begin{aligned} \xi[t+1] &= A_k \xi[t] + B_k y[t], \\ u[t] &= C_k \xi[t] + D_k y[t] + \delta_u[t], \end{aligned} \quad (3)$$

where $\xi[t] \in \mathbb{R}^q$ is the internal state of the controller at time t . The formulation (3) reduces to a static controller when $(A_k, B_k, C_k, D_k) = (0, 0, 0, K)$ for some $K \in \mathbb{R}^{m \times p}$. In this paper, we make the following standard assumption.

Assumption 1. Both the plant and controller realizations are stabilizable and detectable, i.e., (A, B) and (A_k, B_k) are stabilizable, and (A, C) and (A_k, C_k) are detectable.

Applying the controller (2) to the plant (1) leads to a closed-loop system shown in Fig. 1. Since the plant is strictly proper, the closed-loop system is always well-posed (Zhou et al., 1996, Lemma 5.1).

2.2. Internal stability

Internal stability is defined as follows (Zhou et al., 1996, Chapter 5.3):

Definition 1. The system in Fig. 1 is *internally stable* if it is well-posed, and the states $(x[t], \xi[t])$ converge to zero as $t \rightarrow \infty$ for all initial states $(x[0], \xi[0])$ when $\delta_x[t] = 0$, $\delta_y[t] = 0$, $\delta_u[t] = 0$, $\forall t$.

We say the controller \mathbf{K} *internally stabilizes* the plant \mathbf{G} if the closed-loop system in Fig. 1 is internally stable. The set of all LTI internally stabilizing controllers is defined as

$$\mathcal{C}_{\text{stab}} := \{\mathbf{K} \mid \mathbf{K} \text{ internally stabilizes } \mathbf{G}\}. \quad (4)$$

Note that when an infinite time-horizon is considered, a feedback system must at least be stable, and any controller synthesis will implicitly or explicitly involve a constraint $\mathbf{K} \in \mathcal{C}_{\text{stab}}$. Therefore, it is of fundamental importance to characterize $\mathcal{C}_{\text{stab}}$. Indeed, it is well-known that $\mathcal{C}_{\text{stab}}$ is non-convex, and it is not difficult to find explicit examples where $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{C}_{\text{stab}}$ and $\frac{1}{2}(\mathbf{K}_1 + \mathbf{K}_2) \notin \mathcal{C}_{\text{stab}}$. Accordingly, it is not easy to directly search over $\mathbf{K} \in \mathcal{C}_{\text{stab}}$ for control synthesis, and a suitable change of variables is used in many controller synthesis procedures (Boyd & Barratt, 1991; Dullerud & Paganini, 2013; Francis, 1987; Furieri et al., 2019; Wang et al., 2019; Youla et al., 1976; Zheng et al., 2020; Zhou et al., 1996).

A standard state-space characterization of internal stabilization is as follows.

Lemma 1 (Zhou et al., 1996, Lemma 5.2). *Under Assumption 1, \mathbf{K} internally stabilizes \mathbf{G} if and only if*

$$A_{cl} := \begin{bmatrix} A + BD_k C & BC_k \\ B_k C & A_k \end{bmatrix} \quad (5)$$

is stable.

Note that the result in Lemma 1 is a simplified version of Zhou et al. (1996, Lemma 5.2) because we focus on strictly proper plants for simplicity. Lemma 1 leads to an explicit state-space characterization of the set $\mathcal{C}_{\text{stab}}$ as follows:

$$\mathcal{C}_{\text{stab}} = \left\{ \mathbf{K} \mid A_{cl} = \begin{bmatrix} A + BD_k C & BC_k \\ B_k C & A_k \end{bmatrix} \text{ is stable} \right\}, \quad (6)$$

where $\mathbf{K} = C_k(zI - A_k)^{-1}B_k + D_k$. Unfortunately, the stability condition on A_{cl} in (6) is still non-convex in terms of the parameters A_k, B_k, C_k, D_k .

2.3. Characterizations based on transfer functions

Unlike the state–space parameterization (6), there are several frequency-domain characterizations for C_{stab} , where only convex constraints are involved in certain transfer functions. A classical approach is the celebrated *Youla parameterization* (Youla et al., 1976), where a doubly-coprime factorization of the plant is used.

Definition 2. A collection of stable transfer matrices, $\mathbf{U}_l, \mathbf{V}_l, \mathbf{N}_l, \mathbf{M}_l, \mathbf{U}_r, \mathbf{V}_r, \mathbf{N}_r, \mathbf{M}_r \in \mathcal{RH}_\infty$ is called a doubly-coprime factorization of \mathbf{G} if $\mathbf{G} = \mathbf{N}_r \mathbf{M}_r^{-1} = \mathbf{M}_l^{-1} \mathbf{N}_l$ and

$$\begin{bmatrix} \mathbf{U}_l & -\mathbf{V}_l \\ -\mathbf{N}_l & \mathbf{M}_l \end{bmatrix} \begin{bmatrix} \mathbf{M}_r & \mathbf{V}_r \\ \mathbf{N}_r & \mathbf{U}_r \end{bmatrix} = \mathbf{I}.$$

Such doubly-coprime factorization can always be computed if the state–space realization of \mathbf{G} is stabilizable and detectable (Nett, Jacobson, & Balas, 1984). We have the following equivalence (Youla et al., 1976)

$$C_{\text{stab}} = \{\mathbf{K} = (\mathbf{V}_r - \mathbf{M}_r \mathbf{Q})(\mathbf{U}_r - \mathbf{N}_r \mathbf{Q})^{-1} \mid \mathbf{Q} \in \mathcal{RH}_\infty\}, \quad (7)$$

where \mathbf{Q} is denoted as the *Youla parameter*. Note that the Youla parameter \mathbf{Q} can be freely chosen in \mathcal{RH}_∞ . We refer the interested reader to Francis (1987), Youla et al. (1976) and Zhou et al. (1996) for more details on the Youla parameterization.

Two recent approaches are the *system-level parameterization* (SLP) (Wang et al., 2019) and the *input–output parameterization* (IOP) (Furieri et al., 2019), where no doubly-coprime factorization is required *a priori*. Both the SLP and the IOP use certain closed-loop responses for parameterizing C_{stab} with the addition of a set of equality constraints. In particular, the SLP utilizes the closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{u}) , and the IOP relies on the closed-loop responses from (δ_y, δ_u) to (\mathbf{y}, \mathbf{u}) . It is known that Youla parameterization, the SLP, and the IOP are equivalent to each other (Zheng et al., 2020). Inspired by these results (Furieri et al., 2019; Wang et al., 2019; Zheng et al., 2020), we aim to investigate the following questions:

- What are all the possible parameterizations for C_{stab} using closed-loop responses, beyond the SLP and the IOP? We present a complete and exclusive answer by examining all closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ (Sections 3 and 4).
- All the closed-loop parameterizations for C_{stab} are equivalent in theory. When applying FIR approximations for practical computation, are they still equivalent? What are the corresponding state–space realizations for controller implementation? Both of them are answered in Section 5.
- Finally, floating-point arithmetic computation and/or implementation always introduce small numerical mismatches for equality constraints. What are the impact of these numerical mismatches for closed-loop parameterizations? In Section 6, we investigate a property termed as *numerical robustness* of closed-loop parameterizations.

2.4. Optimal controller synthesis

As a particular application, we consider the following optimal control problem

$$\begin{aligned} \min_{\mathbf{K}} \quad & \lim_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \sum_{t=0}^T (y[t]^T \mathbf{Q} y[t] + u[t]^T \mathbf{R} u[t]) \right] \\ \text{subject to} \quad & x[t+1] = \mathbf{A}x[t] + \mathbf{B}(u[t] + \delta_u[t]), \\ & y[t] = \mathbf{C}x[t] + \delta_y[t], \\ & \mathbf{u} = \mathbf{K}y, \end{aligned} \quad (8)$$

where $\delta_u[t] \sim \mathcal{N}(0, I)$, $\delta_y[t] \sim \mathcal{N}(0, I)$, $\mathbf{Q} \succ 0$ and $\mathbf{R} \succ 0$ are performance-weight matrices with compatible dimensions. Problem (8) can be reformulated into a problem in the frequency domain (see e.g., Zheng et al., 2021, Appendix G)

$$\min_{\mathbf{K}} \left\| \begin{bmatrix} \mathbf{Q}^{\frac{1}{2}} & \\ & \mathbf{R}^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} (\mathbf{I} - \mathbf{G}\mathbf{K})^{-1} & (\mathbf{I} - \mathbf{G}\mathbf{K})^{-1} \mathbf{G} \\ \mathbf{K}(\mathbf{I} - \mathbf{G}\mathbf{K})^{-1} & (\mathbf{I} - \mathbf{K}\mathbf{G})^{-1} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \quad (9)$$

subject to $\mathbf{K} \in C_{\text{stab}}$,

where $\mathbf{G} = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B}$. It is easy to see that the optimal synthesis problem (9) is non-convex in terms of \mathbf{K} since both the cost function and constraint are non-convex. Instead of optimizing over the controller directly, we will establish a variety of equivalent convex reformulations based upon optimizing over closed-loop responses. Our results will reveal computational properties and numerical robustness of such reformulations based on closed-loop parameterizations.

We conclude this section by stating the following classical result, which will be frequently used.

Lemma 2 (Zhou et al., 1996, Chapter 3). *Given a transfer matrix $\mathbf{T}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1} \mathbf{B} + \mathbf{D}$, we have*

- If $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ is detectable and stabilizable, then $\mathbf{T}(z) \in \mathcal{RH}_\infty$ if and only if \mathbf{A} is stable;
- If (\mathbf{A}, \mathbf{B}) is not stabilizable, or (\mathbf{A}, \mathbf{C}) is not detectable, then the stability of \mathbf{A} is sufficient but not necessary for $\mathbf{T}(z) \in \mathcal{RH}_\infty$.

3. External transfer matrix characterization of internal stability

In this section, we revisit the external transfer matrix characterization of internal stability, which will be applied to characterize C_{stab} in the next section.

Combining (1) with (2), we can write the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ as

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} & \Phi_{xu} \\ \Phi_{yx} & \Phi_{yy} & \Phi_{yu} \\ \Phi_{ux} & \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \\ \delta_u \end{bmatrix}, \quad (10)$$

where we have $\Phi_{xx} = (z\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{K}\mathbf{C})^{-1}$ and

$$\begin{aligned} \Phi_{xy} &= \Phi_{xx} \mathbf{B}\mathbf{K}, & \Phi_{xu} &= \Phi_{xx} \mathbf{B}, \\ \Phi_{yx} &= \mathbf{C}\Phi_{xx}, & \Phi_{yy} &= \mathbf{C}\Phi_{xx} \mathbf{B}\mathbf{K} + \mathbf{I}, \\ \Phi_{yu} &= \mathbf{C}\Phi_{xx} \mathbf{B}, & \Phi_{ux} &= \mathbf{K}\mathbf{C}\Phi_{xx}, \\ \Phi_{uy} &= \mathbf{K}(\mathbf{C}\Phi_{xx} \mathbf{B}\mathbf{K} + \mathbf{I}), & \Phi_{uu} &= \mathbf{K}\mathbf{C}\Phi_{xx} \mathbf{B} + \mathbf{I}. \end{aligned} \quad (11)$$

We define the closed-loop response transfer matrix as

$$\Phi := \begin{bmatrix} \Phi_{xx} & \Phi_{xy} & \Phi_{xu} \\ \Phi_{yx} & \Phi_{yy} & \Phi_{yu} \\ \Phi_{ux} & \Phi_{uy} & \Phi_{uu} \end{bmatrix}. \quad (12)$$

A notion of external transfer matrix stability is defined as follows.

Definition 3 (Zhou et al., 1996, Chapter 5). The closed-loop system is *disturbance-to-signal stable* if the closed-loop responses from $(\delta_x, \delta_y, \delta_u)$ to $(\mathbf{x}, \mathbf{y}, \mathbf{u})$ are all stable, i.e., $\Phi \in \mathcal{RH}_\infty$.

3.1. General plant

Under Assumption 1, it is known that the internal stability in Definition 1 and the disturbance-to-signal stability in Definition 3 are equivalent (Zhou et al., 1996, Chapter 5), i.e., we have

$$C_{\text{stab}} = \{\mathbf{K} \mid \Phi \in \mathcal{RH}_\infty, \text{ with } \Phi \text{ defined in (12)}\}. \quad (13)$$

In fact, it is sufficient to enforce a subset of elements in Φ to be stable, as shown in Zhou et al. (1996, Lemma 5.3).

Lemma 3. Under Assumption 1, \mathbf{K} internally stabilizes \mathbf{G} if and only if the closed-loop responses from (δ_y, δ_u) to (\mathbf{y}, \mathbf{u}) are stable, i.e.,

$$\begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \in \mathcal{RH}_\infty.$$

For notational simplicity, we denote

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) := \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix}.$$

The result in Lemma 3 motivates the question of whether we can select different minimal sets of elements in Φ for internal stability. For example, if the closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{y}) are stable, i.e.,

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) := \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{yx} & \Phi_{yy} \end{bmatrix} \in \mathcal{RH}_\infty,$$

can we guarantee that the closed-loop system is internally stable? The answer is negative, as proved in Theorem 1 below.

In particular, we consider all possible combinations of four closed-loop responses that may guarantee internal stability. When choosing two disturbances and two outputs from (10), we have in total $\binom{2}{2} \times \binom{2}{2} = 9$ choices, i.e.,

$$\begin{aligned} & \left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right), \\ & \left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right), \\ & \left(\begin{bmatrix} \delta_x \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_x \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right), \left(\begin{bmatrix} \delta_x \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right). \end{aligned} \quad (14)$$

Note that it is in general not sufficient to select less than four closed-loop responses since there are two dynamical parts in system (1) and controller (3). One main result of this section shows that the stability of any of the groups of four closed-loop responses in the top-right corner of (14), highlighted in blue, is equivalent to internal stability.

Theorem 1. Consider the LTI system (1), evolving under a dynamic control policy (3). Under Assumption 1, the following statements are equivalent:

- (1) \mathbf{K} internally stabilizes \mathbf{G} ;
- (2) $\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_\infty$;
- (3) $\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_\infty$;
- (4) $\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_\infty$;
- (5) $\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_\infty$.

Moreover, the stability of any other group of four closed-loop responses in (14) is not sufficient for internal stability.

Proof. The idea of our proof is to use a state-space representation of the closed-loop system, which is motivated by Zhou et al. (1996, Lemma 5.3). From (1) and (3), we have

$$\begin{bmatrix} x[t+1] \\ \xi[t+1] \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A_k \end{bmatrix} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & B_k \end{bmatrix} \begin{bmatrix} u[t] \\ y[t] \end{bmatrix} + \begin{bmatrix} \delta_x[t] \\ 0 \end{bmatrix}, \quad (15)$$

and

$$\begin{bmatrix} I & -D_k \\ 0 & I \end{bmatrix} \begin{bmatrix} u[t] \\ y[t] \end{bmatrix} = \begin{bmatrix} 0 & C_k \\ C & 0 \end{bmatrix} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} \delta_u[t] \\ \delta_y[t] \end{bmatrix}. \quad (16)$$

Substituting (16) into (15) leads to

$$\begin{bmatrix} x[t+1] \\ \xi[t+1] \end{bmatrix} = A_{cl} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} B & BD_k \\ 0 & B_k \end{bmatrix} \begin{bmatrix} \delta_u[t] \\ \delta_y[t] \end{bmatrix} + \begin{bmatrix} \delta_x[t] \\ 0 \end{bmatrix},$$

i.e., we have

$$\begin{bmatrix} x[t+1] \\ \xi[t+1] \end{bmatrix} = A_{cl} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} I & BD_k & B \\ 0 & B_k & 0 \end{bmatrix} \begin{bmatrix} \delta_x[t] \\ \delta_y[t] \\ \delta_u[t] \end{bmatrix}, \quad (17)$$

$$\begin{bmatrix} x[t] \\ y[t] \\ u[t] \end{bmatrix} = \begin{bmatrix} I & 0 \\ C & 0 \\ D_k C & C_k \end{bmatrix} \begin{bmatrix} x[t] \\ \xi[t] \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & D_k & I \end{bmatrix} \begin{bmatrix} \delta_x[t] \\ \delta_y[t] \\ \delta_u[t] \end{bmatrix}.$$

Therefore, the closed-loop responses from $(\delta_x, \delta_y, \delta_u) \rightarrow (\mathbf{x}, \mathbf{y}, \mathbf{u})$ are

$$\begin{bmatrix} I & 0 \\ C & 0 \\ D_k C & C_k \end{bmatrix} (zI - A_{cl})^{-1} \begin{bmatrix} I & BD_k & B \\ 0 & B_k & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & D_k & I \end{bmatrix}, \quad (18)$$

from which, we get state-space realizations of the following closed-loop responses

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_1 (zI - A_{cl})^{-1} \hat{B}_1 + \begin{bmatrix} 0 & I \\ 0 & D_k \end{bmatrix}, \quad (19a)$$

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_2 (zI - A_{cl})^{-1} \hat{B}_1 + \begin{bmatrix} 0 & 0 \\ 0 & D_k \end{bmatrix}, \quad (19b)$$

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_1 (zI - A_{cl})^{-1} \hat{B}_2 + \begin{bmatrix} I & 0 \\ D_k & I \end{bmatrix}, \quad (19c)$$

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) = \hat{C}_2 (zI - A_{cl})^{-1} \hat{B}_2 + \begin{bmatrix} 0 & 0 \\ D_k & I \end{bmatrix}, \quad (19d)$$

where

$$\begin{aligned} \hat{B}_1 &= \begin{bmatrix} I & BD_k \\ 0 & B_k \end{bmatrix}, & \hat{B}_2 &= \begin{bmatrix} BD_k & B \\ B_k & 0 \end{bmatrix}, \\ \hat{C}_1 &= \begin{bmatrix} C & 0 \\ D_k C & C_k \end{bmatrix}, & \hat{C}_2 &= \begin{bmatrix} I & 0 \\ D_k C & C_k \end{bmatrix}. \end{aligned} \quad (20)$$

By Lemma 1, we know that \mathbf{K} internally stabilizes \mathbf{G} if and only if the closed-loop matrix A_{cl} defined in (5) is stable. It is obvious true that (1) \Rightarrow (2), (1) \Rightarrow (3), (1) \Rightarrow (4), and (1) \Rightarrow (5).

Next, we prove if anyone of (2) – (5) is true, the matrix A_{cl} is stable. According to Lemma 2, it remains to prove that the state-space realizations (19a)–(19d) are all stabilizable and detectable. This is equivalent to showing that (A_{cl}, \hat{B}_1) , (A_{cl}, \hat{B}_2) are stabilizable and that (A_{cl}, \hat{C}_1) , (A_{cl}, \hat{C}_2) are detectable. We let $\hat{F}_1 = \begin{bmatrix} F & 0 \\ -C & F_k \end{bmatrix}$ where F and F_k are chosen such that $A + F$ and $A_k + B_k F_k$ are stable (since (A, B_k) is stabilizable). Then, we have that

$$A_{cl} + \hat{B}_1 \hat{F}_1 = \begin{bmatrix} A + F & BC_k + BD_k F_k \\ 0 & A_k + B_k F_k \end{bmatrix}$$

is stable, and thus (A_{cl}, \hat{B}_1) is stabilizable. Similar arguments show that (A_{cl}, \hat{B}_2) is stabilizable, and (A_{cl}, \hat{C}_1) , (A_{cl}, \hat{C}_2) are detectable.

For the second part of Theorem 1, we first prove that the stability of $\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right)$ is not sufficient for internal stability. From (18), a state-space realization of the closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{y}) is

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right) = \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix} (zI - A_{cl})^{-1} \hat{B}_1 + \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Since $(A_{cl}, \begin{bmatrix} I & 0 \\ C & 0 \end{bmatrix})$ is not detectable in general, the stability of

$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \right)$ cannot guarantee the stability of A_{cl} . Therefore,

it is not sufficient for internal stability either. The other claims can be proved in a similar way: the corresponding state–space realization of the closed-loop transfer matrix is not stabilizable and/or detectable. \square

As shown in [Theorem 1](#), to guarantee internal stability for general plants, it is always required to select δ_y as an input and \mathbf{u} as an output, leading to four possible groups of closed-loop responses. The groups of closed-loop responses in (14), except those in blue, do not have either δ_y or \mathbf{u} , and thus fail to guarantee internal stability. Note that [Theorem 1](#) is exclusive in the sense that there exist no other combinations of stable closed-loop responses that are equivalent to internal stability, and [Lemma 3](#) is included as the equivalence between (1) and (4) in [Theorem 1](#). Thus, [Theorem 1](#) offers a complete picture between internal stability and stable closed-loop responses.

3.2. Two special cases: Stable plants and state feedback

Here, we show that the transfer matrix characterization of internal stability can be simplified for special cases: (1) open-loop stable plants; (2) the state feedback case. To guarantee internal stability, instead of considering four closed-loop responses in [Theorem 1](#), the stability of one particular closed-loop response is sufficient in the case of open-loop stable plants, and the stability of two particular closed-loop responses is sufficient in the state feedback case.

The following result is classical, which is the same as [Zhou et al. \(1996, Corollary 5.5\)](#). For completeness, we provide a proof from a state–space perspective.

Corollary 1. *Consider the LTI system (1), evolving under a dynamic control policy (3). If the LTI system is open-loop stable (i.e., A is stable), then $\mathbf{K} \in \mathcal{C}_{stab}$ if and only if $(\delta_y \rightarrow \mathbf{u}) := \Phi_{uy} \in \mathcal{RH}_\infty$.*

Proof. The “only if” direction is true by definition. We now prove the sufficiency. From (18), we have

$$\Phi_{uy} = [D_k C \quad C_k] (zI - A_{cl})^{-1} \begin{bmatrix} BD_k \\ B_k \end{bmatrix} + D_k.$$

Considering the fact that the following matrix

$$A_{cl} + \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \begin{bmatrix} -C & F_k \end{bmatrix} = \begin{bmatrix} A & BC_k + BD_k F_k \\ 0 & A_k + B_k F_k \end{bmatrix},$$

is stable when A and $A_k + B_k F_k$ are stable, we know that $\left(A_{cl}, \begin{bmatrix} BD_k \\ B_k \end{bmatrix} \right)$ is stabilizable. Similarly, we can show that $(A_{cl}, [D_k C \quad C_k])$ is detectable. Therefore, if $\Phi_{uy} \in \mathcal{RH}_\infty$, we have A_{cl} is stable, meaning that $\mathbf{K} \in \mathcal{C}_{stab}$. \square

In the state feedback case, we have the following result.

Corollary 2. *Consider the LTI system (1), evolving under a dynamic control policy (3). If $C = I$, then $\mathbf{K} \in \mathcal{C}_{stab}$ if and only if $(\delta_x \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}) := \begin{bmatrix} \Phi_{xx} \\ \Phi_{ux} \end{bmatrix} \in \mathcal{RH}_\infty$.*

Proof. When $C = I$, from (18), we have

$$\begin{bmatrix} \Phi_{xx} \\ \Phi_{ux} \end{bmatrix} = \begin{bmatrix} I & 0 \\ D_k & C_k \end{bmatrix} \left(zI - \begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix} \right)^{-1} \begin{bmatrix} I \\ 0 \end{bmatrix},$$

From the proof of [Theorem 1](#), we know

$$\left(\begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix}, \begin{bmatrix} I & 0 \\ D_k & C_k \end{bmatrix} \right)$$

is detectable. It is not difficult to prove that

$$\left(\begin{bmatrix} A + BD_k & BC_k \\ B_k & A_k \end{bmatrix}, \begin{bmatrix} I \\ 0 \end{bmatrix} \right) \quad (21)$$

is stabilizable (see Appendix C of our technical report [Zheng et al., 2019](#) for proof details). Therefore, if $\Phi_{xx} \in \mathcal{RH}_\infty$, $\Phi_{ux} \in \mathcal{RH}_\infty$, we have A_{cl} is stable, meaning that $\mathbf{K} \in \mathcal{C}_{stab}$. \square

The result in [Corollary 2](#) has been used in the state feedback case of the system-level parameterization ([Wang et al., 2019](#)). The proof in [Wang et al. \(2019\)](#) used a frequency-based method. Here, we provided an alternative proof from a state–space perspective, which is consistent with the proofs for [Theorem 1](#) and [Corollary 1](#).

4. Parameterizations of stabilizing controllers

The results in [Theorem 1](#) can be used to parameterize the set of internally stabilizing controllers \mathcal{C}_{stab} , leading to four equivalent parameterizations. One of them corresponds to the SLP ([Wang et al., 2019](#)), and another one is the IOP ([Furiere et al., 2019](#)). The remaining two parameterizations are new and, to the best of the authors’ knowledge, have not been characterized before. The results in [Corollaries 1](#) and [2](#) can also be used to parameterize \mathcal{C}_{stab} in a simplified way.

4.1. Four equivalent parameterizations for general plants

The closed-loop responses from (δ_x, δ_y) to (\mathbf{x}, \mathbf{u}) have been utilized in the SLP ([Wang et al., 2019](#)). Specifically, consider

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}. \quad (22)$$

We have the following system-level parameterization (SLP).

Proposition 1 (SLP [Wang et al., 2019, Theorem 2](#)). *Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:*

- (1) For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses (22) are in the following affine subspace

$$\begin{aligned} [zI - A \quad -B] \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} &= [I \quad 0], \\ \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} &= \begin{bmatrix} I \\ 0 \end{bmatrix}, \\ \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} &\in \mathcal{RH}_\infty. \end{aligned} \quad (23)$$

- (2) For any transfer matrices $\Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy}$ satisfying (23), $\mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy} \in \mathcal{C}_{stab}$.

We refer to $\mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy}$ as the four-block SLP controller. Also, the closed-loop responses from (δ_y, δ_u) to (\mathbf{y}, \mathbf{u}) have been used in the IOP ([Furiere et al., 2019](#)). Specifically, consider

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}. \quad (24)$$

We have the following input–output parameterization (IOP).

Proposition 2 (IOP [Furiere et al., 2019, Theorem 1](#)). *Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:*

(1) For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses (24) are in the following affine subspace

$$\begin{aligned} [I \quad -\mathbf{G}] \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} &= [I \quad 0], \\ \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} &= \begin{bmatrix} 0 \\ I \end{bmatrix}, \end{aligned} \quad (25)$$

$\Phi_{yy}, \Phi_{uy}, \Phi_{yu}, \Phi_{uu} \in \mathcal{RH}_\infty$.

(2) For any transfer matrices $\Phi_{yy}, \Phi_{uy}, \Phi_{yu}, \Phi_{uu}$ satisfying (25), $\mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \in \mathcal{C}_{stab}$.

Next, we consider the following closed-loop responses

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix}. \quad (26)$$

We have a new parameterization of \mathcal{C}_{stab} .

Proposition 3 (Mixed I). Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses (26) are in the following affine subspace

$$\begin{aligned} [I \quad -\mathbf{G}] \begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} &= [C(zI - A)^{-1} \quad I], \\ \begin{bmatrix} \Phi_{yx} & \Phi_{yy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} &= 0, \end{aligned} \quad (27)$$

$\Phi_{yx}, \Phi_{ux}, \Phi_{yy}, \Phi_{uy} \in \mathcal{RH}_\infty$.

(2) For any transfer matrices $\Phi_{yx}, \Phi_{ux}, \Phi_{yy}, \Phi_{uy}$ satisfying (27), $\mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \in \mathcal{C}_{stab}$.

The proof is provided in Appendix A. Finally, we consider the case

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \Phi_{xy} & \Phi_{xu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}. \quad (28)$$

The following result mirrors Proposition 3 for an additional new parameterization of \mathcal{C}_{stab} .

Proposition 4 (Mixed II). Consider the LTI system (1), evolving under a dynamic control policy (3). The following statements are true:

(1) For any $\mathbf{K} \in \mathcal{C}_{stab}$, the resulting closed-loop responses (28) are in the following affine subspace

$$\begin{aligned} [zI - A \quad -B] \begin{bmatrix} \Phi_{xy} & \Phi_{xu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} &= 0, \\ \begin{bmatrix} \Phi_{xy} & \Phi_{xu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} &= \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \end{aligned} \quad (29)$$

$\Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu} \in \mathcal{RH}_\infty$.

(2) For any transfer matrices $\Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu}$ satisfying (29), $\mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy} \in \mathcal{C}_{stab}$.

The proof of Proposition 4 is similar to that of Proposition 3, which is provided in Appendix D of Zheng et al. (2019).

To summarize, Propositions 1–4 establish four equivalent methods to parameterize the set of internally stabilizing controllers using closed-loop responses:

$$\mathcal{C}_{stab} = \{ \mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy} \mid \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \text{ are in the affine subspace (23)} \},$$

$$\mathcal{C}_{stab} = \{ \mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \mid \Phi_{yy}, \Phi_{uy}, \Phi_{yu}, \Phi_{uu} \text{ are in the affine subspace (25)} \},$$

$$\mathcal{C}_{stab} = \{ \mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \mid \Phi_{yx}, \Phi_{ux}, \Phi_{yy}, \Phi_{uy} \text{ are in the affine subspace (27)} \},$$

$$\mathcal{C}_{stab} = \{ \mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy} \mid \Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu} \text{ are in the affine subspace (29)} \}.$$

Unlike the state-space characterization (6), the constraints (23), (25), (27), and (29) are all affine in the new parameters. Based on (23), (25), (27), and (29), convex optimization problems can be derived for the classical optimal controller synthesis; see Furieri et al. (2019), Wang et al. (2019) and Zheng et al. (2020) for details. We will present a case study in Section 7.

Remark 1 (Equivalence with Youla). The explicit equivalence between Propositions 1 & 2 and the Youla parameterization has been derived in Zheng et al. (2020). It is not difficult to derive the explicit relationship between Propositions 3 & 4 and the Youla parameterization (7) using the approach of Furieri et al. (2019) and Zheng et al. (2020). While there are four parameters in (23), (25), (27), or (29), there is only one freedom due to the affine constraints. This is consistent with the Youla parameterization, where only one parameter is involved with no explicit affine constraints. In Proposition 6, we will show that any doubly-coprime factorization of the plant can exactly eliminate the affine constraints (23), (25), (27), and (29).

Remark 2 (Numerical Computation). We note that while being convex, the decision variables in (23), (25), (27), and (29) are infinite-dimensional. Thus, finite-dimensional approximations are usually needed for numerical computations, which will be discussed in Section 5. However, the affine constraints (23), (25), (27), and (29) can never be exactly satisfied in numerical computation. Section 6 will formally discuss the issue of numerical robustness.

4.2. Two special cases: stable plants and state feedback

The results in Corollaries 1 and 2 can be exploited to derive simplified versions of Propositions 1–4. We will later show that these simplified parametrizations enjoy provable numerical robustness. When the plant is open-loop stable, the IOP (Proposition 2) and the Mixed I (Proposition 3) are simplified as follows.

Corollary 3. Consider the LTI system (1), evolving under a dynamic controller policy (3). If the LTI system is open-loop stable, then we have

$$\mathcal{C}_{stab} = \left\{ \mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \mid [I \quad -\mathbf{G}] \begin{bmatrix} \Phi_{yy} \\ \Phi_{uy} \end{bmatrix} = I, \Phi_{uy} \in \mathcal{RH}_\infty \right\}.$$

This result is consistent with the classical one in Zhou et al. (1996, Theorem 12.7). Note that for open-loop stable plants, the transfer matrix Φ_{uy} from the measurement disturbance δ_y to the control input \mathbf{u} is the same as the Youla parameter \mathbf{Q} . Under the condition in Corollary 3, the Mixed II (Proposition 4) can be simplified as well:

$$\mathcal{C}_{stab} = \left\{ \mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy} \mid \begin{bmatrix} \Phi_{uy} & \Phi_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} = I, \Phi_{uy} \in \mathcal{RH}_\infty \right\}.$$

If the state is directly measurable for control, i.e., $C = I$, [Corollary 2](#) leads to the following simplified version of SLP.

Corollary 4 ([Wang et al., 2019, Theorem 1](#)). Consider the LTI system (1), evolving under a dynamic controller policy (3). If $C = I$, then we have

$$C_{stab} = \left\{ \mathbf{K} = \Phi_{ux} \Phi_{xx}^{-1} (zI - A) \Phi_{xx} - B \Phi_{ux} = I, \right. \\ \left. \Phi_{ux}, \Phi_{xx} \in \frac{1}{z} \mathcal{RH}_\infty \right\}.$$

Note that the simplified IOP/Mixed I/Mixed II requires the stability of only one parameter, while the simplified SLP requires the stability of two parameters. The proofs for [Corollaries 3](#) and [4](#) are similar to that of [Proposition 3](#).

5. Numerical computation and controller implementation

This section investigates the numerical computation and controller implementation using the closed-loop parameterization for C_{stab} . As noted in [Remark 2](#), since the decision variables in the affine constraints (23), (25), (27), and (29) are infinite-dimensional, it is not immediately obvious to derive efficient numerical computation to search over the feasible region. One practical method is to apply the finite impulse response (FIR) approximation, which is extensively used in [Furieri et al. \(2019\)](#) and [Wang et al. \(2019\)](#). As we will see, the SLP, the IOP and the two new mixed parametrizations are not equivalent to each other after imposing FIR constraints. In this section, we also present standard state-space realizations (3) for the controllers using closed-loop responses.

5.1. Numerical computation via FIR

We denote the space of finite impulse response (FIR) transfer matrices with horizon T as

$$\mathcal{F}_T := \left\{ \mathbf{H} \in \mathcal{RH}_\infty \mid \mathbf{H} = \sum_{k=0}^T \frac{1}{z^k} H_k \right\},$$

where H_k denotes the i th spectral component of the FIR transfer matrix \mathbf{H} . It is known that on letting the FIR length T go to infinity, \mathcal{F}_T converges to the space \mathcal{RH}_∞ ([Pohl & Boche, 2009, Theorem 4.7](#)). It is not difficult to check that after imposing the decision variables to be FIR transfer matrices of horizon T , the constraints (23), (25), (27), and (29) all become finite-dimensional affine constraints in terms of the spectral components of the closed-loop responses. Specifically, the constraints are obtained by matching the coefficients associated with the terms z^{-k} . Thus, searching for an internally stabilizing controller only requires solving a linear program (LP) under the FIR assumption.²

Here, we show that imposing the FIR assumption has different effects depending on the chosen closed-loop parameterization.

Theorem 2. Given the LTI system (1), evolving under a dynamic control policy (3), we consider the statements:

$$(i) \ \Phi \in \mathcal{F}_T; \\ (ii) \ \left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T; \quad (SLP) \\ (iii) \ \left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T; \quad (Mixed I)$$

² Depending on the choice of the cost function, optimal controller synthesis may be cast as a quadratic program (QP) under the FIR assumption; see a case study in Section 7.

$$(iv) \ \left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T; \quad (Mixed II) \\ (v) \ \left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{F}_T. \quad (IOP)$$

If (A, B, C) and (A_k, B_k, C_k) are both stabilizable and detectable, we have (i) \Leftrightarrow (ii) \Rightarrow (iii) \Rightarrow (v) and (i) \Leftrightarrow (ii) \Rightarrow (iv) \Rightarrow (v). In addition, if (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we have (i) \Leftrightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv) \Leftrightarrow (v).

The proof is not mathematically involved, and we provide it in [Appendix B](#). Note that minimal state-space realizations of the plant and the controller deserve more investigations in distributed control when particular structures are required; see [Rantzer \(2019\)](#) and [Vamsi and Elia \(2015\)](#) for details.

Upon defining the following sets

$$C_{SLP} = \{ \mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy} \mid \Phi_{xx}, \Phi_{ux}, \Phi_{xy}, \Phi_{uy} \in \mathcal{F}_T \\ \text{are in the affine subspace (23)} \},$$

$$C_{M1} = \{ \mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \mid \Phi_{yx}, \Phi_{yy}, \Phi_{ux}, \Phi_{uy} \in \mathcal{F}_T \\ \text{are in the affine subspace (27)} \},$$

$$C_{M2} = \{ \mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy} \mid \Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu} \in \mathcal{F}_T \\ \text{are in the affine subspace (29)} \},$$

$$C_{IOP} = \{ \mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1} \mid \Phi_{yy}, \Phi_{uy}, \Phi_{yu}, \Phi_{uu} \in \mathcal{F}_T \\ \text{are in the affine subspace (25)} \},$$

it is easy to derive the following corollary.

Corollary 5. If (A, B, C) and (A_k, B_k, C_k) are both stabilizable and detectable, we have $C_{SLP} \subset C_{M1} \subset C_{IOP} \subset C_{stab}$ and $C_{SLP} \subset C_{M2} \subset C_{IOP} \subset C_{stab}$. If (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we have $C_{SLP} = C_{M1} = C_{M2} = C_{IOP} \subset C_{stab}$.

Theoretically, the closed-loop parameterizations in [Propositions 1–4](#) are equivalent to each other. However, after imposing the FIR approximation on the decision variables, [Corollary 5](#) shows that the IOP ([Furieri et al., 2019](#)) in [Proposition 2](#) has the best ability to approximate the set of stabilizing controllers C_{stab} , as it exclusively deals with the maps from inputs to outputs without passing through the system state; see [Fig. 2](#) for illustration. Precisely, when there are some stable uncontrollable and/or unobservable modes in (1), these modes cannot be changed by any feedback controller and will be reflected in the closed-loop responses involving the state \mathbf{x} . Therefore, for systems with stable uncontrollable and/or unobservable modes, the parameters in the SLP ([Wang et al., 2019](#)), or the new parameterization in [Proposition 3/4](#) (Mixed I/II), cannot be made FIR by definition, since these parameterizations involve the state \mathbf{x} and/or the disturbance on the state δ_x explicitly. For example, consider an LTI system (1) with matrices as

$$A = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [0 \quad 1].$$

There is one uncontrollable and unobservable mode $z = 0.5$, and this mode is stable. The affine constraints (23), (27), (29) are all infeasible for any FIR approximation with finite horizon T since the mode $z = 0.5$ cannot be represented by FIR exactly, while the IOP in [Proposition 2](#) is feasible as long as the horizon $T \geq 1$.

Remark 3. Note that if there are some stable uncontrollable and/or unobservable modes in (1), one may perform a model reduction to get an equivalent state-space realization that is controllable and observable. Then, all the closed-loop parameterizations in [Propositions 1–4](#) have the same ability for approximating C_{stab} when imposing the FIR assumption. We note that model reduction generally destroys the underlying sparsity structure in the original system (1), which may be unfavorable for distributed controller synthesis ([Wang, Matni, & Doyle, 2018](#)).

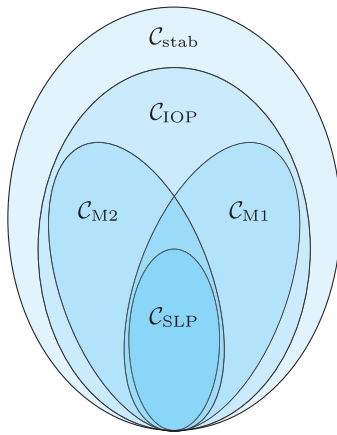


Fig. 2. The IOP provides the best inner approximation of C_{stab} using FIR approximations: $C_{SLP} \subset C_{M1} \subset C_{IOP} \subset C_{stab}$, and $C_{SLP} \subset C_{M2} \subset C_{IOP} \subset C_{stab}$.

5.2. Controller implementation via state-space realization

In Propositions 1–4, to get the controller \mathbf{K} , we need to compute the inverse of some transfer matrix as well as the product of transfer matrices. For the SLP in Wang et al. (2019), the authors proposed the following implementation of the controller $\mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy}$ from the system responses matrices $\Phi_{xx}, \Phi_{xy}, \Phi_{ux}, \Phi_{uy}$:

$$\begin{aligned} z\beta &= z(I - z\Phi_{xx})\beta - z\Phi_{xy}\mathbf{y}, \\ \mathbf{u} &= z\Phi_{ux}\beta + \Phi_{uy}\mathbf{y}. \end{aligned} \quad (30)$$

The implementation (30) avoids the explicit computation of matrix inverse and matrix product. We note that a few other realizations have been discussed in Jensen and Bamieh (2021) and Rantzer (2019). However, the controller matrices in (30) still contain transfer matrices. Motivated by Anderson and Matni (2017), this subsection provides a standard state-space realization (3) for the controller in closed-loop parameterizations after imposing the FIR approximation.

We consider the controller $\mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1}$ in Propositions 2 and 3 (IOP and Mixed I). We assume that the system response Φ_{uy} and Φ_{yy} are FIR with horizon T , i.e.,

$$\Phi_{uy} = \sum_{t=0}^T U_t \frac{1}{z^t} \in \mathcal{RH}_\infty, \quad \Phi_{yy} = \sum_{t=0}^T Y_t \frac{1}{z^t} \in \mathcal{RH}_\infty. \quad (31)$$

Upon defining the following real matrices

$$\begin{aligned} \hat{U} &= [U_1 \ U_2 \ \dots \ U_T] \in \mathbb{R}^{m \times pT}, \\ \hat{Y} &= [Y_1 \ Y_2 \ \dots \ Y_T] \in \mathbb{R}^{p \times pT}, \end{aligned} \quad (32)$$

and $Z_p \in \mathbb{R}^{pT \times pT}$ as the down shift operator with sub-diagonal containing identity matrices of dimension $p \times p$ and $\mathcal{I}_p = [I_p, 0, \dots, 0]^T \in \mathbb{R}^{pT \times p}$, we have the following result.

Theorem 3. Suppose that Φ_{uy} and Φ_{yy} are FIR transfer matrices with horizon T in (31). A state-space realization for the output feedback controller $\mathbf{K} = \Phi_{uy} \Phi_{yy}^{-1}$ is given by

$$\mathbf{K} = \left[\begin{array}{c|c} Z_p - \mathcal{I}_p \hat{Y} & -\mathcal{I}_p \\ \hline U_0 \hat{Y} - \hat{U} & U_0 \end{array} \right]. \quad (33)$$

A state-space realization for the controller $\mathbf{K} = \Phi_{uu}^{-1} \Phi_{uy}$ in Proposition 4 (Mixed II) can be developed similarly. The proof of Theorem 3 is motivated by Anderson and Matni (2017), and is based on some standard operations on dynamical systems (see,

e.g., Zhou et al., 1996, Chapter 3.6). We provide the proof in Appendix F of Zheng et al. (2019). In Zheng et al. (2019), we also provide a state-space realization for the SLP controller $\mathbf{K} = \Phi_{uy} - \Phi_{ux} \Phi_{xx}^{-1} \Phi_{xy}$. Finally, we note that the state-space realization in (33) is in general not minimal.

6. Numerical robustness of closed-loop parameterizations

The previous sections highlighted the benefits of closed-loop parameterizations: the set of internally stabilizing controllers can be fully characterized by a set of affine constraints on certain closed-loop responses, leading to finite-dimensional convex optimization problems for controller synthesis after imposing the FIR constraints. However, numerical solutions computed via arbitrarily precise floating point arithmetic can never solve the affine constraints exactly. This phenomenon is further exacerbated by the finite stopping criteria used in common solvers, like SeDuMi (Sturm, 1999) and Mosek (Andersen & Andersen, 2000). Moreover, any controller implementation that uses floating-point arithmetic also introduce errors. Therefore, numerical mismatches in solving the affine constraints in Propositions 1–4 always exist.

This section investigates how the numerical mismatches in the affine constraints affect the stability of the closed-loop system, a property termed as *numerical robustness* of closed-loop parameterizations. An overview of the results in this section is presented in Table 1.

6.1. Robustness results for the IOP and the SLP

We begin with the IOP in Proposition 2. The transfer matrices $\hat{\Phi}_{yy}, \hat{\Phi}_{uy}, \hat{\Phi}_{yu}, \hat{\Phi}_{uu}$ only approximately satisfy the affine constraint (25), i.e., we have

$$\begin{aligned} [I \quad -\mathbf{G}] \begin{bmatrix} \hat{\Phi}_{yy} & \hat{\Phi}_{yu} \\ \hat{\Phi}_{uy} & \hat{\Phi}_{uu} \end{bmatrix} &= [I + \Delta_1 \quad \Delta_2], \\ \begin{bmatrix} \hat{\Phi}_{yy} & \hat{\Phi}_{yu} \\ \hat{\Phi}_{uy} & \hat{\Phi}_{uu} \end{bmatrix} \begin{bmatrix} -\mathbf{G} \\ I \end{bmatrix} &= \begin{bmatrix} \Delta_3 \\ I + \Delta_4 \end{bmatrix}, \end{aligned} \quad (34)$$

$$\hat{\Phi}_{yy}, \hat{\Phi}_{uy}, \hat{\Phi}_{yu}, \hat{\Phi}_{uu} \in \mathcal{RH}_\infty,$$

where the residuals are $\Delta_1 = \hat{\Phi}_{yy} - \mathbf{G} \hat{\Phi}_{uy} - I$, $\Delta_2 = \hat{\Phi}_{yu} - \mathbf{G} \hat{\Phi}_{uu}$, $\Delta_3 = -\hat{\Phi}_{yy} \mathbf{G} + \hat{\Phi}_{yu}$, $\Delta_4 = -\hat{\Phi}_{uy} \mathbf{G} + \hat{\Phi}_{uu} - I$.

Remark 4. We note that the residuals $\Delta_i (i = 1, \dots, 4)$ can come from (1) numerical computation when enforcing the equality constraints numerically, and (2) the controller implementation that uses floating-point arithmetic. We use $\Delta_i (i = 1, \dots, 4)$ to denote such mismatches from the equality constraints. The results in this section work regardless of the phenomenon generating $\Delta_i (i = 1, \dots, 4)$.

Theorem 4. Let $\hat{\Phi}_{yy}, \hat{\Phi}_{uy}, \hat{\Phi}_{yu}, \hat{\Phi}_{uu}$ satisfy (34). Then, we have the following statements.

- (1) In the case of $\mathbf{G} \in \mathcal{RH}_\infty$, the controller $\mathbf{K} = \hat{\Phi}_{uy} \hat{\Phi}_{yy}^{-1}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \Delta_1)^{-1}$ is stable.
- (2) In the case of $\mathbf{G} \notin \mathcal{RH}_\infty$, the controller $\mathbf{K} = \hat{\Phi}_{uy} \hat{\Phi}_{yy}^{-1}$ fails to internally stabilize the closed-loop system due to non-zero residuals Δ_i .

Proof. Given a controller \mathbf{K} , the closed-loop responses from (δ_y, δ_u) to (\mathbf{y}, \mathbf{u}) are

$$\begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} (I - \mathbf{GK})^{-1} & (I - \mathbf{GK})^{-1} \mathbf{G} \\ \mathbf{K}(I - \mathbf{GK})^{-1} & I + \mathbf{K}(I - \mathbf{GK})^{-1} \mathbf{G} \end{bmatrix} \begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix}.$$

Table 1
Comparison of numerical robustness among different closed-loop parameterizations.

	Coprime factorization	Equality constraints	Controller recovery \mathbf{K}	Open-loop stable plants	Open-loop unstable plants	Pre-stabilizing the plant ³
SLP (Wang et al., 2019)	No	Yes	$\Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy}$ $\Phi_{uy}(I + C\Phi_{xy})^{-1}$ $\Phi_{ux}\Phi_{xx}^{-1}$ (when $C = I$) ^a	*	*	*
IOP (Furiere et al., 2019)	No	Yes	$\Phi_{uy}\Phi_{yy}^{-1}$	✓	✗	✓
Mixed I	No	Yes	$\Phi_{uy}\Phi_{yy}^{-1}$	✓	✗	✓
Mixed II	No	Yes	$\Phi_{uu}^{-1}\Phi_{uy}$	✓	✗	✓
Youla (Youla et al., 1976)	Yes	No	(7)	✓	✓	✓

*: The situation requires care-by-case analysis; see Theorem 5 and Section 6.2 for details.

✓: The parameterization is numerically robust (see Corollary 6).

✗: The parameterization cannot guarantee the closed-loop stability if small numerical mismatches in the equality constraints exist.

^aThis only works for the state feedback case, i.e., $C = I$.

^bThis applies an initial stabilizing controller that is stable itself (see Proposition 5).

Considering $\mathbf{K} = \hat{\Phi}_{uy}\hat{\Phi}_{yy}^{-1}$, where $\hat{\Phi}_{yy}, \hat{\Phi}_{uy}, \hat{\Phi}_{yu}, \hat{\Phi}_{uu}$ satisfy (34), we can verify the following identities:

$$\begin{aligned} (I - \mathbf{G}\mathbf{K})^{-1} &= (I - \mathbf{G}\hat{\Phi}_{uy}\hat{\Phi}_{yy}^{-1})^{-1} \\ &= \hat{\Phi}_{yy}(I + \Delta_1)^{-1}, \\ (I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} &= \hat{\Phi}_{yy}(I + \Delta_1)^{-1}\mathbf{G}, \end{aligned} \tag{35}$$

$$\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} = \hat{\Phi}_{uy}(I + \Delta_1)^{-1},$$

and $I + \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = I + \hat{\Phi}_{uy}(I + \Delta_1)^{-1}\mathbf{G}$.

Proof of Statement 1: Suppose that $\mathbf{G} \in \mathcal{RH}_\infty$. If $(I + \Delta_1)^{-1}$ is stable, it is easy to verify that all transfer matrices in (35) are stable. This means that

$$\left(\begin{bmatrix} \delta_y \\ \delta_u \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix} \right) \in \mathcal{RH}_\infty.$$

By Theorem 1, we know $\mathbf{K} = \hat{\Phi}_{uy}\hat{\Phi}_{yy}^{-1}$ internally stabilizes the plant \mathbf{G} . If $(I + \Delta_1)^{-1}$ is unstable, then the closed-loop response from δ_y to \mathbf{y} will be unstable in general, and thus the controller does not internally stabilize the system.

Proof of Statement 2: If \mathbf{G} is unstable, the transfer matrices in (35) cannot be guaranteed to be stable if $\Delta_1 \neq 0$. When $\Delta_1 = 0$, we have $(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = \hat{\Phi}_{yy}\mathbf{G} = \hat{\Phi}_{yu} - \Delta_3$ and $I + \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = I + \hat{\Phi}_{uy}\mathbf{G} = \hat{\Phi}_{uu} - \Delta_4$. Note that the residuals Δ_3, Δ_4 in (34) can be unstable if \mathbf{G} is unstable (since the product of an unstable transfer matrix and a stable one can be unstable). Thus, the controller fails to guarantee the internal stability of the closed-loop system unless $\Delta_1 = 0, \Delta_2 = 0, \Delta_3 = 0, \Delta_4 = 0$. □

We now focus on the SLP in Proposition 1. The transfer matrices $\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}$ only approximately satisfy the affine constraint (23), i.e., we have

$$\begin{aligned} [zI - A \quad -B] \begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix} &= [I + \hat{\Delta}_1 \quad \hat{\Delta}_2], \\ \begin{bmatrix} \hat{\Phi}_{xx} & \hat{\Phi}_{xy} \\ \hat{\Phi}_{ux} & \hat{\Phi}_{uy} \end{bmatrix} \begin{bmatrix} zI - A \\ -C \end{bmatrix} &= \begin{bmatrix} I + \hat{\Delta}_3 \\ \hat{\Delta}_4 \end{bmatrix}, \end{aligned} \tag{36}$$

$$\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy} \in \mathcal{RH}_\infty,$$

where the residuals are

$$\hat{\Delta}_1 = (zI - A)\hat{\Phi}_{xx} - B\hat{\Phi}_{ux} - I,$$

$$\hat{\Delta}_2 = (zI - A)\hat{\Phi}_{xy} - B\hat{\Phi}_{uy},$$

$$\hat{\Delta}_3 = \hat{\Phi}_{xx}(zI - A) - \hat{\Phi}_{xy}C - I$$

$$\hat{\Delta}_4 = \hat{\Phi}_{ux}(zI - A) - \hat{\Phi}_{uy}C.$$

Note that there are multiple ways to recover the controller \mathbf{K} in the SLP framework. The SLP controller can also be recovered in another way as $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ (Wang et al., 2019). Now, we have the following result.

Theorem 5. Let $\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}$ satisfy (36). We have the following statements.

- (1) In the state feedback case, i.e., $C = I$, the controller $\mathbf{K} = \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$ internally stabilizes the plant \mathbf{G} if and only if $(I + \hat{\Delta}_1)^{-1}$ is stable.
- (2) The four-block SLP controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ cannot guarantee the internal stability of the closed-loop system if $(I + \hat{\Delta})^{-1}$ is unstable, where³

$$\hat{\Delta} := \hat{\Delta}_3 + \hat{\Phi}_{xx}(I + \hat{\Delta}_1)^{-1}(B\hat{\Delta}_4 - (zI - A)\hat{\Delta}_3). \tag{37}$$

- (3) For the controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$,

- (a) if $\mathbf{G} \in \mathcal{RH}_\infty$, \mathbf{K} internally stabilizes the plant \mathbf{G} if and only if $(I + C(zI - A)^{-1}\Delta_2)^{-1}$ is stable.
- (b) if $\mathbf{G} \notin \mathcal{RH}_\infty$, \mathbf{K} fails to internally stabilize the closed-loop system due to non-zero residuals $\hat{\Delta}_i, i = 1, \dots, 4$.

Proof. The proof of Statement 1 is presented in Anderson et al. (2019, Theorem 4.3). We prove the second statement here. Given $\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}$ satisfying (36) and the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$, we consider the closed-loop response from δ_x to \mathbf{x} . After some tedious algebra (see Appendix G of Zheng et al., 2019), we derive

$$(zI - A - B\mathbf{K}C)^{-1} = (I + \hat{\Delta})^{-1}\hat{\Phi}_{xx}(I + \hat{\Delta}_1)^{-1}, \tag{38}$$

with $\hat{\Delta}$ defined in (37). If $(I + \hat{\Delta})^{-1}$ is unstable, there is no guarantee that the closed-loop response from δ_x to \mathbf{x} is stable. In this case, the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ cannot internally stabilize the plant.

For Statement 3, considering Corollary 1, we only need to check the closed-loop response from δ_y to \mathbf{u} , which is

$$\begin{aligned} &\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} \\ &= \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}(I - \mathbf{G}\hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1})^{-1} \\ &= \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy} - C(zI - A)^{-1}B\hat{\Phi}_{uy})^{-1} \\ &= \hat{\Phi}_{uy}(I + C(zI - A)^{-1}\hat{\Delta}_2)^{-1}. \end{aligned}$$

The rest of the proof is similar to Theorem 4. □

Theorem 5 quantifies the numerical robustness of different controller recovery in the SLP due to the mismatches of the associated equality constraints. The mismatches come from floating-point arithmetic in either numerical computation or controller implementation. This is irrespective of whether the FIR approximation is used for the closed-loop responses. Similar robustness results can be derived for the Mixed I/II parameterizations

³ Note that because $\|\hat{\Delta}\|_\infty$ may large than 1, there is no guarantee that $(I + \hat{\Delta})^{-1}$ is always stable. See Example (39).

(Propositions 3/4); see Appendix F of Zheng et al. (2019) for details. Theorems 4 and 5 can now be combined with the small gain theorem (Zhou et al., 1996, Theorem 9.1) to provide simple sufficient conditions for numerical robustness.

Corollary 6. Let $\hat{\Phi}_{yy}, \hat{\Phi}_{uy}, \hat{\Phi}_{yu}, \hat{\Phi}_{uu}$ satisfy (34). Then

- for open-loop stable plants, the IOP controller $\mathbf{K} = \hat{\Phi}_{uy}\hat{\Phi}_{yy}^{-1}$ internally stabilizes the plant if $\|\hat{\Delta}_1\|_\infty < 1$.

Let $\hat{\Phi}_{xx}, \hat{\Phi}_{ux}, \hat{\Phi}_{xy}, \hat{\Phi}_{uy}$ satisfy (36). Then

- for the state feedback case, the SLP controller $\mathbf{K} = \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$ internally stabilizes the plant if $\|\hat{\Delta}_1\|_\infty < 1$.
- for open-loop stable plants, the SLP controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ internally stabilizes the plant if

$$\|\hat{\Delta}_2\|_\infty \leq \frac{1}{\|C(zI - A)\|_\infty}.$$

The sufficient condition for robustness of the SLP state feedback case first appeared in Matni, Wang, and Anderson (2017), which is one key result in the recent learning-based control applications (Dean et al., 2017, 2018).

Remark 5. The controller recovery in IOP/Mixed I/II, the two-block state-feedback SLP controller, and the SLP controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ only involve two parameters explicitly. Thus, their robustness analysis is straightforward. The corresponding analysis for the four-block SLP controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$, instead, does not provide a numerical robustness result. As shown in Theorem 5, the residuals $\hat{\Delta}_i, i = 1, \dots, 4$ play a more complex role in the resulting closed-loop responses, irrespective of state- or output-feedback, or open-loop stability of the plant. Since the closed-loop responses from $\delta_x, \delta_y, \delta_u$ to $\mathbf{x}, \mathbf{y}, \mathbf{u}$ can be computed using these residuals $\hat{\Delta}_i$, one may find sophisticated sufficient conditions on the residuals $\hat{\Delta}_i$ to ensure the internal stability of the closed-loop system. These conditions might relate these residuals with numerical solutions such as $\hat{\Phi}_{xx}$. Deriving such conditions and finding tractable ways to enforce these conditions are beyond the scope of this paper and left as future work.

6.2. Implications in numerical computation and controller implementation

Here, we discuss the implication of Theorems 4, 5 and Corollary 6. In practice, the residuals $\hat{\Delta}_1 = \hat{\Phi}_{yy} - \mathbf{G}\hat{\Phi}_{yy} - I$ (when \mathbf{G} is stable) and $\hat{\Delta}_1 = (zI - A)\hat{\Phi}_{xx} - B\hat{\Phi}_{xy} - I$ are very small numerically. It is fairly safe to say that $\|\hat{\Delta}_1\|_\infty < 1$ and $\|\hat{\Delta}_1\|_\infty < 1$ in floating-point implementation and numerical computation using any common interior-point solvers, such as SeDuMi (Sturm, 1999) and Mosek (Andersen & Andersen, 2000). Similar statements are true for the Mixed I/II parameterizations. This observation leads to the following summary (see Table 1 for an overview).

Numerical Robustness. Consider closed-loop parameterizations (SLP, IOP, Mixed I/II) in numerical computation. We have

- (i) the SLP with controller $\mathbf{K} = \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$ is numerically robust in the state feedback case;
- (ii) the IOP, Mixed I/II, and SLP with controller $\hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ are numerically robust for open-loop stable plants.

On the other hand, we have

- (I) the IOP, Mixed I/II, and SLP with controller $\hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ are not numerically robust for open-loop unstable plants, irrespective of having state- or output-feedback;

- (II) the SLP with controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ is not numerically robust in general, irrespective of open-loop stability of the plant.

The statements (i), (ii) and (I) are easy to see from the previous section. The statement (II) comes from Theorem 5 but requires more attention. Although the computational residuals $\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3, \hat{\Delta}_4$ in (36) are typically very small element-wise by interior-point solvers, we still cannot guarantee that $\|\hat{\Delta}\|_\infty < 1$ (where $\hat{\Delta}$ is defined in (37)), since $\hat{\Delta}$ involves $\hat{\Phi}_{xx}$ explicitly. Consequently, it is possible that $(I + \hat{\Delta})^{-1}$ is unstable in numerical computation. Therefore, one may argue that the four-block SLP controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$ is not numerically robust in general.⁴ Further, we notice that the controller implementation (30) proposed in Wang et al. (2019) also suffers the issue of numerical instability, as the right-hand-side of (38) represents the corresponding closed-loop response using (30).

Example 1. To understand the role of the residuals, we present a simple example. Consider a stable LTI system (1) with $A = 0, B = 1, C = 1$. It can be verified that the following transfer functions

$$\begin{aligned} \hat{\Phi}_{xx} &= \frac{1}{z} + \frac{(z-5)(z+6)^2}{z^5}, \\ \hat{\Phi}_{ux} &= \frac{(z-5)(z+6)^2}{z^4}, \\ \hat{\Phi}_{xy} &= \frac{(z-5)(z+6)^2}{z^4} - \frac{1}{1000} \frac{(z+2)^2}{z^3}, \\ \hat{\Phi}_{uy} &= \frac{(z-5)(z+6)^2}{z^3}, \end{aligned} \tag{39}$$

satisfy (36) with residuals

$$\hat{\Delta}_1 = 0, \hat{\Delta}_2 = -\frac{(z+2)^2}{1000z^2}, \hat{\Delta}_3 = \frac{(z+2)^2}{1000z^3}, \hat{\Delta}_4 = 0.$$

For this example, we verify that $(I + \hat{\Delta})^{-1}$ has a pair of unstable poles $z = 0.9522 \pm 0.5226i$, despite the norm $\|\hat{\Delta}_3\|_\infty = 9 \times 10^{-3}$ being very small, and that this pair of unstable poles also appears in the closed-loop system $(zI - A - B\mathbf{K})^{-1}$ using the controller $\mathbf{K} = \hat{\Phi}_{uy} - \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}\hat{\Phi}_{xy}$. This example is open-loop stable and it is also in state feedback form. Nonetheless, a small residual can destabilize the closed-loop using the four-block SLP controller.

We remark that in Example 1, since $A = 0$, the optimal LQR controller will be $\mathbf{K} = 0$ for any weight matrices Q and R . Thus, any sensible formulation of optimal control problems using the SLP will not lead to the highly suboptimal solution (39). However, we emphasize that the numerical residuals play a complex role in the closed-loop system using the four-block SLP controller, and residuals with a small norm may lead to an undesirable destabilization situation. Indeed, numerical instability is tightly linked to the specific controller recovery. Unlike the four-block SLP controller, from Corollary 6, the state feedback SLP controller $\mathbf{K} = \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$ is numerically robust as long as $\|\hat{\Delta}_1\|_\infty < 1$. Since $\hat{\Delta}_1 = 0$ in Example 1, the closed-loop system $(zI - A - B\mathbf{K})^{-1}$ has all zero eigenvalues using $\mathbf{K} = \hat{\Phi}_{ux}\hat{\Phi}_{xx}^{-1}$. Meanwhile, we can verify that $\|C(zI - A)^{-1}\hat{\Delta}_2\|_\infty = 0.009 < 1$, thus it is guaranteed that the controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$ internally stabilizes the plant (the largest norm of the closed-loop eigenvalues is 0.1675).

The question remains whether the phenomenon highlighted in Example 1 may lead to numerical instability when solving optimal control formulation in practice. We observed several cases

⁴ As discussed in Remark 5, sufficient conditions could exist to ensure internal stability, and they will depend on $\hat{\Phi}_{xx}$.

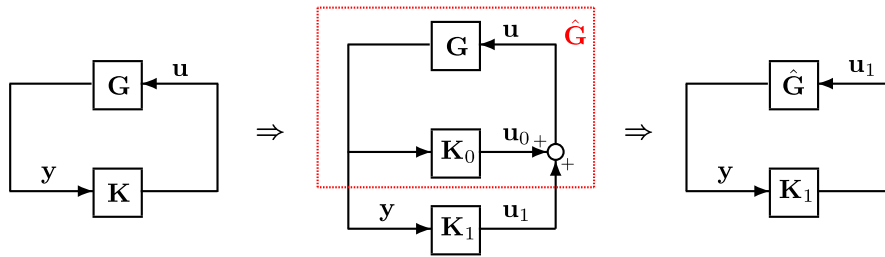


Fig. 3. Given an initial controller $\mathbf{K}_0 \in \mathcal{C}_{stab} \cap \mathcal{RH}_\infty$, we search for \mathbf{K}_1 to stabilize the new stable plant $\hat{\mathbf{G}} := (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$.

where the four-block SLP controller failed to stabilize the plant even using the default setting (high precision) in Mosek (Andersen & Andersen, 2000) for numerical computation.⁵ This is likely due to $\|\hat{\Phi}_{xx}\|_\infty$ being high, despite solving an optimal control formulation. How to avoid this issue requires more investigations, which is left for future work.

6.3. Open-loop unstable plants and relation with the Youla parameterization

To characterize the set of internally stabilizing controllers \mathcal{C}_{stab} , the closed-loop parameterizations in Propositions 1–4 can avoid computing the doubly co-prime factorization of the plant *a priori*, but they all need to impose a set of affine constraints for achievable closed-loop responses. As discussed above, any small mismatch in the additional affine constraints can make the resulting controller unimplementable when the plant is open-loop unstable (IOP, Mixed I/II), and the four-block SLP controller requires a case-by-case investigation.

For the case of open-loop unstable plants, there exists a valid remedy by pre-stabilizing the plant. Suppose that \mathbf{G} is unstable, and that a *stable* and *stabilizing* controller \mathbf{K}_0 is known *a priori*. We can split the control signal as $\mathbf{u} = \mathbf{K}_0\mathbf{y} + \mathbf{u}_1$, and design \mathbf{u}_1 . This is equivalent to applying the closed-loop parameterization to the new stable plant $\hat{\mathbf{G}} := (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$ (see Fig. 3 for illustration). Upon defining

$$\hat{\mathcal{C}}_{stab} := \{\mathbf{K}_0 + \mathbf{K}_1 \mid \mathbf{K}_1 \text{ internally stabilizes } \hat{\mathbf{G}}\},$$

we have the following result.

Proposition 5. *Given an initial controller $\mathbf{K}_0 \in \mathcal{C}_{stab} \cap \mathcal{RH}_\infty$, we have $\mathcal{C}_{stab} = \hat{\mathcal{C}}_{stab}$.*

The proof is based on algebra verification; see Appendix I of Zheng et al. (2019). Proposition 5 shows that searching over $\hat{\mathcal{C}}_{stab}$ has no conservatism. The new plant $\hat{\mathbf{G}} = (I - \mathbf{G}\mathbf{K}_0)^{-1}\mathbf{G}$ is stable, and thus any closed-loop parameterization in Propositions 1–4 for this plant has good numerical robustness.⁶ As shown in Rotkowitz and Lall (2006, Theorem 17), giving $\mathbf{K}_0 \in \mathcal{C}_{stab} \cap \mathcal{RH}_\infty$, the Youla parameterization (7) has a simple form as well, since one can choose an explicit doubly-coprime factorization as

$$\begin{aligned} \mathbf{M}_l &= (I - \mathbf{G}\mathbf{K}_0)^{-1}, & \mathbf{M}_r &= -(I - \mathbf{K}_0\mathbf{G})^{-1} \\ \mathbf{N}_l &= \mathbf{G}(I - \mathbf{K}_0\mathbf{G})^{-1}, & \mathbf{N}_r &= -\mathbf{G}(I - \mathbf{K}_0\mathbf{G})^{-1}, \\ \mathbf{U}_l &= -I, & \mathbf{V}_l &= -\mathbf{K}_0, & \mathbf{U}_r &= I, & \mathbf{V}_r &= \mathbf{K}_0. \end{aligned}$$

If the plant is open-loop stable, we can choose $\mathbf{K}_0 = 0$.

Unlike the closed-loop parameterizations in Propositions 1–4, the Youla parameterization (7) allows the parameter \mathbf{Q} to

⁵ See the examples at https://github.com/zhengy09/h2_clp, where the system matrices $A \in \mathbb{R}^{3 \times 3}, B \in \mathbb{R}^{3 \times 1}, C \in \mathbb{R}^{1 \times 3}$ have integer elements randomly generated from -5 to 5 , and the weight matrices are chosen $Q = I, R = I$ in (8).

⁶ For the SLP, we use the controller $\mathbf{K} = \hat{\Phi}_{uy}(I + C\hat{\Phi}_{xy})^{-1}$.

be freely chosen in \mathcal{RH}_∞ with no equality constraints. Indeed, any doubly-coprime factorization of the plant can be used to eliminate the affine constraints in Propositions 1–4 exactly, as shown below.

Proposition 6. *Let $\mathbf{U}_r, \mathbf{V}_r, \mathbf{U}_l, \mathbf{V}_l, \mathbf{M}_r, \mathbf{M}_l, \mathbf{N}_r, \mathbf{N}_l$ be any doubly-coprime factorization of \mathbf{G} . For any $\mathbf{Q} \in \mathcal{RH}_\infty$, the following transfer matrices*

$$\begin{aligned} \Phi_{yy} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{M}_l, \\ \Phi_{uy} &= (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{M}_l, \\ \Phi_{yu} &= (\mathbf{U}_r - \mathbf{N}_r\mathbf{Q})\mathbf{N}_l, \\ \Phi_{uu} &= I + (\mathbf{V}_r - \mathbf{M}_r\mathbf{Q})\mathbf{N}_l, \end{aligned} \tag{40}$$

and $\Phi_{xx} = (zI - A)^{-1} + (zI - A)^{-1}B\Phi_{uy}C(zI - A)^{-1}$,

$$\begin{aligned} \Phi_{ux} &= \Phi_{uy}C(zI - A)^{-1}, \\ \Phi_{xy} &= (zI - A)^{-1}B\Phi_{uy}, \\ \Phi_{xu} &= (zI - A)^{-1}B\Phi_{uu}, \\ \Phi_{yx} &= \Phi_{uu}C(zI - A)^{-1}, \end{aligned} \tag{41}$$

satisfy the affine constraints (23), (25), (27), (29).

The proof is based on direct verification, which is omitted here; see Zheng et al. (2020) for further discussions on the equivalence of the Youla parameterization, the IOP, and the SLP. We note that a doubly-coprime factorization can be found in the state-space domain (Nett et al., 1984), and this pre-process might introduce numerical issues that affect closed-loop stability, which is beyond the scope of this paper.

7. Case studies

In this section, we present a case study of optimal controller synthesis for *open-loop stable* plants using Propositions 1–4. We show that the optimal controller synthesis problem can be cast into a quadratic program (QP) after imposing the FIR constraint.⁷

7.1. Application to optimal controller synthesis

Consider the optimal controller synthesis (9). Using a change of variables, as suggested in Propositions 1–4, it is equivalent to replace $\mathbf{K} \in \mathcal{C}_{stab}$ with the affine constraints (23), (25), (27), or (29). It remains to reformulate the cost function in terms of these new variables. Simple algebra shows that

$$\begin{aligned} (I - \mathbf{G}\mathbf{K})^{-1} &= \Phi_{yy} = C\Phi_{xy} + I, \\ \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} &= \Phi_{uy}, \\ (I - \mathbf{K}\mathbf{G})^{-1} &= \Phi_{uu} = \Phi_{ux}B + I, \end{aligned}$$

and $(I - \mathbf{G}\mathbf{K})^{-1}\mathbf{G} = \Phi_{yu} = C\Phi_{xx}B = C\Phi_{xu} = \Phi_{yx}B$.

⁷ Code is available at https://github.com/soc-ucsd/h2_clp.

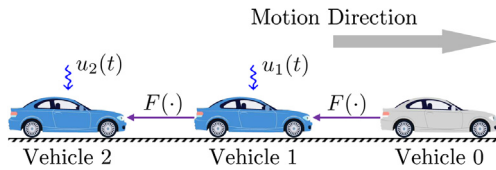


Fig. 4. Each vehicle has a pre-existing car-following dynamics $F(\cdot)$ and the goal is to design an additional input $u_i(t)$, $i = 1, 2$ to improve the car-following performance.

Therefore, problem (9) is equivalent to any of the following convex optimization problems (42)–(45) corresponding to Propositions 1–4, respectively.

$$\min \left\| \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C\Phi_{xy} + I & C\Phi_{xx}B \\ \Phi_{uy} & \Phi_{ux}B + I \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \quad (42)$$

subject to $\Phi_{xx}, \Phi_{xy}, \Phi_{ux}, \Phi_{uy}$ satisfy (23).

$$\min \left\| \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \Phi_{yy} & \Phi_{yu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \quad (43)$$

subject to $\Phi_{yy}, \Phi_{yu}, \Phi_{uy}, \Phi_{uu}$ satisfy (25).

$$\min \left\| \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \Phi_{yy} & \Phi_{yx}B \\ \Phi_{uy} & \Phi_{ux}B + I \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \quad (44)$$

subject to $\Phi_{yy}, \Phi_{uy}, \Phi_{yx}, \Phi_{ux}$ satisfy (27).

$$\min \left\| \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & R^{\frac{1}{2}} \end{bmatrix} \begin{bmatrix} C\Phi_{xy} + I & C\Phi_{xu} \\ \Phi_{uy} & \Phi_{uu} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \quad (45)$$

subject to $\Phi_{xy}, \Phi_{uy}, \Phi_{xu}, \Phi_{uu}$ satisfy (29).

Note that the \mathcal{H}_2 norm of an FIR transfer matrix $\mathbf{H} = \sum_{k=1}^T \frac{1}{z^k} H_k$ admits the following expression $\|\mathbf{H}\|_{\mathcal{H}_2}^2 = \sum_{k=1}^T \text{Trace}(H_k^T H_k)$. Thus, after imposing the FIR constraint on the decision variables, problems (42)–(45) can be reformulated into QPs, for which very efficient solvers exist.

7.2. Numerical experiments

Here, we use a car-following control scenario (Zheng, Li, Wang, Cao, & Li, 2015) (see Fig. 4 for illustration) to demonstrate the numerical performance of the parameterizations in Propositions 1–4.

Modeling: We denote the position and velocity of vehicle i as p_i and v_i . The spacing of vehicle i , i.e., its relative distance from vehicle $i - 1$, is defined as $s_i = p_{i-1} - p_i$, $i = 1, 2$. Without loss of generality, the vehicle length is ignored. It is assumed that the leading vehicle 0 runs at a constant velocity v_0 . Each vehicle has pre-existing car-following dynamics, and we aim to design an additional control signal $u_i(t)$ to improve the car-following performance, i.e.,

$$\dot{v}_i(t) = F(s_i(t), \dot{s}_i(t), v_i(t)) + u_i(t), \quad (46)$$

where $\dot{s}_i(t) = v_{i-1}(t) - v_i(t)$, and $F(\cdot)$ characterizes the driver's natural car-following behavior (see Orosz, Wilson, & Stépán, 2010 for details). In an equilibrium car-following state, each vehicle moves with the same equilibrium velocity, i.e., $v_i(t) = v_0$, $\dot{s}_i(t) = 0$, for $i = 1, 2$. Assuming that each vehicle has a small perturbation from the equilibrium state (s_i^*, v^*), we define the error state between actual and equilibrium state of vehicle i as $x_i(t) = [\tilde{s}_i(t), \tilde{v}_i(t)]^T = [s_i(t) - s_i^*, v_i(t) - v^*]^T$. Applying the first-order

Table 2

\mathcal{H}_2 norm for different FIR lengths when solving the car-following problem.							
FIR T	10	15	20	25	30	50	75
\mathcal{H}_2 norm	54.20	17.41	7.56	4.08	2.49	2.03	2.02

[‡]: The \mathcal{H}_2 norms from (42)–(45) have no difference up to four significant figures.
[†]: The true \mathcal{H}_2 norm from h2syn in MATLAB is 2.02.

Taylor expansion to (46), we can derive a linearized model for each vehicle ($i = 1, 2$)

$$\begin{cases} \dot{\tilde{s}}_i(t) = \tilde{v}_{i-1}(t) - \tilde{v}_i(t), \\ \dot{\tilde{v}}_i(t) = \alpha_1 \tilde{s}_i(t) - \alpha_2 \tilde{v}_i(t) + \alpha_3 \tilde{v}_{i-1}(t) + u_i(t), \end{cases}$$

with $\alpha_1 = \frac{\partial F}{\partial s_i}$, $\alpha_2 = \frac{\partial F}{\partial \tilde{s}_i} - \frac{\partial F}{\partial v_i}$, $\alpha_3 = \frac{\partial F}{\partial \tilde{s}_i}$ evaluated at the equilibrium state. Assuming that we can measure the relative spacing, we arrive at the following state–space model

$$\dot{x} = \begin{bmatrix} P_1 & 0 \\ P_2 & P_1 \end{bmatrix} x + \begin{bmatrix} B_1 & 0 \\ 0 & B_1 \end{bmatrix} (u + \delta_u), \quad (47)$$

$$y = \begin{bmatrix} C_1 & 0 \\ 0 & C_1 \end{bmatrix} x + \delta_y,$$

where $x = [x_1^T \ x_2^T]^T$, $u = [u_1 \ u_2]^T$, $y = [\tilde{s}_1(t) \ \tilde{s}_2(t)]^T$, δ_u and δ_y are control input noise and measurement noise, respectively, and

$$P_1 = \begin{bmatrix} 0 & -1 \\ \alpha_1 & -\alpha_2 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 1 \\ 0 & \alpha_3 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1 = [1 \ 0].$$

The objective is to design $u_i(t)$ to regulate the spacing error $s_i(t) - s_i^*$ and velocity error $v_i(t) - v^*$ based on the output information $y(t)$. This problem can be formulated into (8) in the discrete-time domain (see Table 2).

Numerical results: In our numerical simulations, the car-following parameters $\alpha_1 = 0.94$, $\alpha_2 = 1.5$, $\alpha_3 = 0.9$ are chosen according to the setup in Zheng, Wang, and Li (2020), and the open-loop system is stable. Using a forward Euler-discretization of (47) with a sampling time of $dT = 0.1$ s, we formulate the corresponding optimal controller synthesis problem (8) in discrete-time with $Q = I$ and $R = I$. This can be solved via any of the convex problems (42)–(45). We varied the FIR length T from 10 to 75, and the results are listed in Table 2. As expected, when increasing the FIR length, the optimal cost from (42)–(45) converges to the true value returned by the standard synthesis h2syn in MATLAB. Given an initial state $x_0 = [3, 0, -2, 0]^T$, Fig. 5 shows the time-domain responses⁸ of the closed-loop system using the resulting controllers from (42)–(45) when the FIR length is $T = 30$. By design, the closed-loop system converges to the equilibrium state within 3 s. For the same initial state, Fig. 6 shows the time-domain responses of the closed-loop system when the FIR length is $T = 75$, where the system converges the equilibrium state within 7.5 s with lower peak values during the transient process compared to the case $T = 30$.

8. Conclusions

In this paper, we have characterized all possible parameterizations for the set of stabilizing controllers using closed-loop maps. We have revealed two other parallel choices beyond the recent notions of SLP (Wang et al., 2019) and IOP (Furieri et al., 2019). In fact, our analysis allows to treat the SLP (Wang et al., 2019)

⁸ The responses from (42)–(45) have no visible difference.

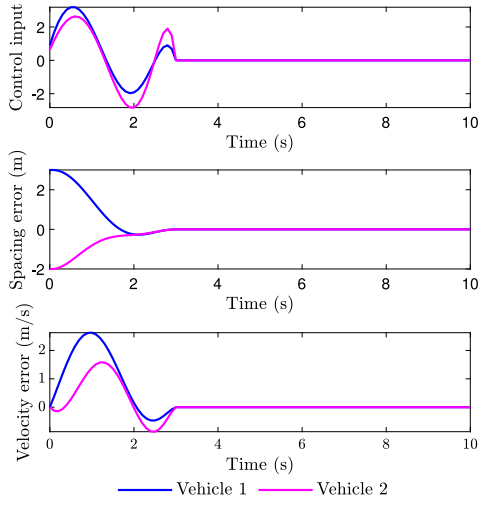


Fig. 5. Responses using the controllers from (42)–(45) with FIR length $T = 30$.

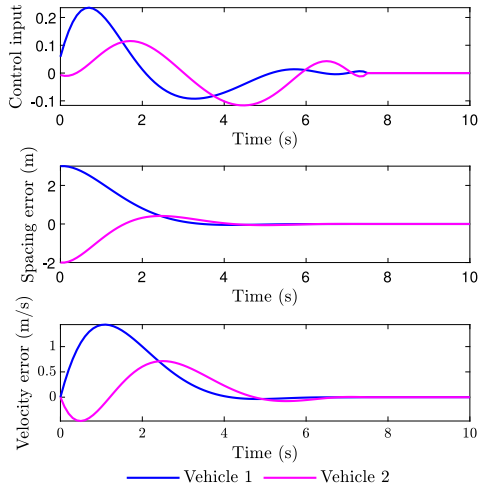


Fig. 6. Responses using the controllers from (42)–(45) with FIR length $T = 75$.

and the IOP (Furieri et al., 2019) in a unified way. After imposing the FIR approximation, the ability of the four parameterizations for encoding $\mathcal{C}_{\text{stab}}$ becomes different, and the IOP enjoys the best approximation ability. These closed-loop parameterizations can avoid computing the doubly co-prime factorization of the plant *a priori*, but instead require imposing a set of affine constraints for achievable closed-loop responses. We have discussed two numerically robust scenarios: the SLP in the state feedback case and the IOP for open-loop stable plants.

One future direction is to address decentralized control, e.g., the notion of quadratic invariance (QI) (Rotkowitz & Lall, 2006) and sparsity invariance (SI) (Furieri, Zheng, Papachristodoulou et al., 2020), using different parameterizations. Also, similar to SLP (Dean et al., 2017, 2018) and Youla (Furieri, Zheng and Kamgarpour, 2020; Lale et al., 2020; Simchowitz et al., 2020), it will be extremely interesting to investigate the features of different parameterizations in robust synthesis for uncertain systems and their applications in learning-based control. Finally, we have established that closed-loop parameterizations are more subtle in practice for open-loop unstable plants with output feedback, and further investigation is needed to unravel a more precise and thorough understanding of related aspects.

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Appendix A. Proof of Proposition 3

Statement 1: Given any $\mathbf{K} \in \mathcal{C}_{\text{stab}}$, it is easy to derive that the closed-loop responses (26) are

$$\Phi_{yx} = (I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1},$$

$$\Phi_{yy} = (I - \mathbf{G}\mathbf{K})^{-1},$$

$$\Phi_{ux} = \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1},$$

$$\Phi_{uy} = \mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1},$$

which are all stable by definition. Then, it is not difficult to verify that

$$\Phi_{yx} - \mathbf{G}\Phi_{ux} = C(zI - A)^{-1},$$

$$\Phi_{yy} - \mathbf{G}\Phi_{uy} = (I - \mathbf{G}\mathbf{K})^{-1} - \mathbf{G}\mathbf{K}(I - \mathbf{G}\mathbf{K})^{-1} = I,$$

and that

$$\Phi_{yx}(zI - A) - \Phi_{yy}C = 0,$$

$$\Phi_{ux}(zI - A) - \Phi_{uy}C = 0.$$

Therefore, the closed-loop responses Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfy (27).

Statement 2: Consider any Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfying (27). Since $\Phi_{yy} = I + \mathbf{G}\Phi_{uy}$ and \mathbf{G} is strictly proper, we know that Φ_{yy} is always invertible. Let $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1}$. We now verify the resulting closed-loop responses in (26) are all stable. In particular, we have

$$\mathbf{y} = (I - \mathbf{G}\mathbf{K})^{-1}C(zI - A)^{-1}\delta_x,$$

and with $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1}$, we have

$$\begin{aligned} & (I - \mathbf{G}\Phi_{uy}\Phi_{yy}^{-1})^{-1}C(zI - A)^{-1} \\ &= \Phi_{yy}(\Phi_{yy} - \mathbf{G}\Phi_{uy})^{-1}C(zI - A)^{-1} \\ &= \Phi_{yy}C(zI - A)^{-1} \\ &= \Phi_{yx} \in \mathcal{RH}_{\infty}, \end{aligned}$$

where the equalities follow from the fact that Φ_{yx} , Φ_{yy} , Φ_{ux} , Φ_{uy} satisfy (27). Also, we have that

$$\mathbf{y} = (I - \mathbf{G}\Phi_{uy}\Phi_{yy}^{-1})^{-1}\delta_y = \Phi_{yy}\delta_y.$$

Similarly, we can show that

$$\mathbf{u} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}C_2(zI - A)^{-1}\delta_x = \Phi_{ux}\delta_x,$$

$$\mathbf{u} = \mathbf{K}(I - \mathbf{P}_{22}\mathbf{K})^{-1}\delta_y = \Phi_{uy}\delta_y.$$

Therefore, we have proved that $\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{y} \\ \mathbf{u} \end{bmatrix}\right) \in \mathcal{RH}_{\infty}$, using the controller $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1}$. By Theorem 1, we know $\mathbf{K} = \Phi_{uy}\Phi_{yy}^{-1} \in \mathcal{C}_{\text{stab}}$.

Appendix B. Proof of Theorem 2

The direction (i) \Rightarrow (ii), (iii), (iv), (v) is true by definition. For any controller \mathbf{K} , the closed-loop responses are given in (10) and (11).

We now prove (ii) \Rightarrow (i). Suppose we have

$$\left(\begin{bmatrix} \delta_x \\ \delta_y \end{bmatrix} \rightarrow \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix}\right) = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \in \mathcal{F}_T.$$

From (11), it is not difficult to check $\Phi_{xu} = \Phi_{xx}B \in \mathcal{F}_T$, and

$$\begin{aligned}\Phi_{yx} &= C\Phi_{xx} \in \mathcal{F}_T, & \Phi_{yy} &= C\Phi_{xy} + I \in \mathcal{F}_T \\ \Phi_{yu} &= C\Phi_{xx}B \in \mathcal{F}_T, & \Phi_{uu} &= \Phi_{ux}B + I \in \mathcal{F}_T.\end{aligned}$$

This means that the statement (i) is true. Similar arguments can prove (iii) \Rightarrow (v) and (iv) \Rightarrow (v).

Finally, if (A, B, C) and (A_k, B_k, C_k) are both controllable and observable, we prove that (v) \Rightarrow (i). According to (19), we have the following state-space realization

$$\begin{pmatrix} \delta_y \\ \delta_u \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{y} \\ \mathbf{u} \end{pmatrix} = \hat{C}_1(zI - A_{cl})^{-1}\hat{B}_2 + \begin{bmatrix} I & 0 \\ D_k & I \end{bmatrix}, \quad (\text{B.1})$$

where \hat{C}_1 and \hat{B}_2 are defined in (20). We can show that $(A_{cl}, \hat{B}_2, \hat{C}_1)$ is controllable and observable (see Appendix C of Zheng et al., 2019 for details). This means that the eigenvalues of A_{cl} are the same as the poles of the transfer matrices (Zhou et al., 1996, Chapter 3). Therefore, if the statement (v) is true, then the closed-loop matrix A_{cl} only has zero eigenvalues and no eigenvalues of A_{cl} is hidden from the input-output behavior. This completes the proof.

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