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Half-integral weight Eisenstein series, double Dirichlet series and equidistribution

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Abstract

In this thesis we first define weight $\frac{1}{2}$ Eisenstein series and the corresponding double Dirichlet series, and revisit their most important properties. We show a connection between the double Dirichlet series and some Rankin-Selberg type integral, and use the Maass-Selberg relations in order to prove a convexity bound for the double Dirichlet series.

Subsequently, we are considering the measures defined by the weight $\frac{1}{2}$ Eisenstein series truncated at height $|t|$. We construct microlocal lifts of the Eisenstein series to an appropriate metaplectic group. We prove that any weak*-limit of the sequence of measures defined by these microlocal lifts is invariant under the geodesic flow, satisfies Hecke p-recurrence for some prime p, and that every ergodic component has positive entropy. Using Lindenstrauss measure classification theorem, we conclude that any such weak*-limit is a constant multiple of the Haar-measure. Hence, provided that there is no escape of mass, the sequence of measures defined by the Eisenstein series equidistributes.

Finally, we use the equidistribution result to show a weak subconvexity bound for the double Dirichlet series.

Zusammenfassung

In dieser Arbeit definieren wir Eisenstein Reihen von Gewicht $\frac{1}{2}$, die zugehörigen doppelten Dirichlet-Reihen, und untersuchen ihre wichtigsten Eigenschaften. Wir zeigen einen Zusammenhang zwischen den doppelten Dirichlet-Reihen und Integrale von Rankin-Selbergscher Art, und benutzen die Maass-Selberg Beziehungen, um eine Konvexitäts-Schranke für die doppelten Dirichlet-Reihen zu zeigen.

Anschliessend betrachten wir eine Folge von Massen, die durch die Gewicht $\frac{1}{2}$ Eisenstein Reihen, die auf Höhe $|t|$ abgeschnitten sind, definiert sind. Wir heben die Eisenstein Reihen auf eine geeignete metaplektische Gruppe. Wir beweisen, dass jeder Grenzwert - in der schwach* Topologie - der Folge von Massen, die durch die gehobenen Eisenstein Reihen definiert sind, folgende drei Eigenschaften erfüllt: Er ist invariant unter dem geodesischen Fluss, erfüllt Hecke p-Rückkehr für eine Primzahl p, und jede ergodische Komponente hat positive Entropie. Mithilfe eines bekannten Satzes von Lindenstrauss schliessen wir, dass jeder solche Grenzwert ein konstantes Vielfaches des Haar-Masses ist. Vorausgesetzt es gibt kein Entschwinden von Masse ins Unendliche, ist die Folge der Masse, die durch die Eisenstein Reihen definiert sind, somit gleichverteilt.

Zuletzt benutzen wir das Gleichverteilungs-Resultat, um eine schwache Subkonvexitäts-Schranke für die doppelten Dirichlet-Reihen zu zeigen.

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Contents

Introduction

An important issue in quantum chaos is the behavior of eigenfunctions φ_j of the Laplacian with eigenvalues λ_j going to infinity as $j \to \infty$. Assume that the φ_j are Laplace eigenfunctions on $\Gamma\backslash\mathbb{H}$ for some discrete co-compact subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{R})$ which are normalized such that $\int_{\Gamma\backslash\mathbb{H}} |\varphi(z)|^2 \frac{dxdy}{y^2} = 1$. Consider the probability measures $d\mu_j = |\varphi_j(z)|^2 \,\mathrm{dvol}$ which describe the probability density of finding a particle in the state φ_j at the point z. Works by Schnirelman [26], Zelditch [32] and Colin de Verdiere [5] showed in the 70's, 80's the existence of a density one subsequence converging to the volume measure, which is called quantum ergodicity. In 1994, Rudnick and Sarnak [24] conjectured that $d\mu_j \overset{\text{weak}^*}{\longrightarrow} \mathrm{dvol}$ as $j \to \infty$ which is called the quantum unique ergodicity conjecture. If this conjecture holds, it means that there is only one possible quantum limit while the classical analog (i.e. uniqueness of the invariant measure for the Hamiltonian flow) is never satisfied for chaotic systems. So, this conjecture was quite surprising and it was not clear whether it actually holds.

A major step towards the credibility of the QUE-conjecture was done by Luo and Sarnak [17] in 1995, where they discovered a surprising relation between QUE and Dirichlet series: They showed that the known subconvexity bound of some single Dirichlet series implies quantum unique ergodicity for weight 0 Eisenstein series. Later, this was generalized to integral weight by Jakobson [14]. A natural question is whether the same method applies for Eisenstein series of fractional weight. Recently, Petridis, Raulf and Risager [21] showed that QUE of weight $\frac{1}{2}$ Eisenstein series would follow from a subconvexity bound for the double Dirichlet series

$$Z(\tfrac{1}{2} + it, \tfrac{1}{2} - it, \chi, \chi') = \zeta^{(2)}(1 + 4it) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \frac{\chi(n) t_n L^*(\frac{1}{2} - 2it, n, \chi')}{n^{\frac{1}{2}+2it}}$$

for two Dirichlet characters χ and χ' modulo 8. Here, L^* is the product of a certain L-function with second factor removed and a so called correction polynomial. These definitions will be given in detail in chapter 2, Definition 2.2.5. By $\zeta^{(2)}$ we denote the zeta function with the second factor removed and t_n are the Fourier coefficients coming from a certain cusp form.

Theorem 1.0.1. *(Petridis, Raulf, Risager, [21]) Suppose that for all Dirichlet characters χ and χ' modulo 8, and t_n being either Fourier coefficients as above or $t_n = \tau(n)$ the divisor* *function, the double Dirichlet series* $Z(\frac{1}{2}+it,\frac{1}{2}-it,\chi,\chi')$ *satisfies a subconvex bound, that is*

$$Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') = O_\psi\left(q(t)^{\frac{1}{4}-\delta}\right) \quad \text{as } |t|\to\infty$$

for some $\delta>0$*, where* $q(t):=|t|^2$ *is the conductor. Then, for any two compact Jordan measurable subsets* A *and* B *of* $\Gamma\backslash\mathbb{H}$*, we have*

$$\frac{\int_A \left|E(z,\frac{1}{2}+it,\frac{1}{2})\right|^2 d\mu(z)}{\int_B \left|E(z,\frac{1}{2}+it,\frac{1}{2})\right|^2 d\mu(z)} \longrightarrow \frac{\mathrm{vol}(A)}{\mathrm{vol}(B)} \qquad \text{as} \quad |t|\to\infty.$$

Even though subconvexity is well-known for single Dirichlet series, for double Dirichlet series this seems to be a very difficult problem. In fact, the best unconditional bound which was known so far for double Dirichlet series is the "trivial bound" (cf [21]), i.e.

$$Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') = O_\psi\left(q(t)^{\frac{1}{2}+\epsilon}\right). \tag{1.0.1}$$

This is very far away from the desired subconvexity bound, in contrast to the case of integral weight, where the well-known convexity bound gives an estimate at the threshold of subconvexity, analogous here to

$$Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') = O_\psi\left(q(t)^{\frac{1}{4}+\epsilon}\right). \tag{1.0.2}$$

The thesis is structured as follows: In chapter 2 we are showing an unconditional convexity bound for the double Dirichlet series. In chapter 3 we are proving an unconditional quantum unique ergodicity result for the weight $\frac{1}{2}$ Eisenstein series. In the last chapter we deduce a weak subconvexity bound for the double Dirichlet series.

Petridis, Raulf and Risager [21] give a conditional proof of (1.0.2) assuming some strong hypotheses which would also imply the generalized Lindelöf hypothesis. Moreover, they prove unconditionally that

$$Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') + bZ(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi') = O_\psi\left(q(t)^{\frac{1}{4}+\epsilon}\right), \tag{1.0.3}$$

where b is some sign.

In the second chapter of this thesis, we give an unconditional proof of the convexity bound (1.0.2), namely of

Theorem 1.0.2. *Let* χ *and* χ' *be two Dirichlet characters modulo* 2^l*,* $l\geq 2$*, and let* $\{t_n\}_{n\in\mathbb{N}}$ *be the Fourier coefficients of the normalized L-function of a self-dual* GL_2 *automorphic form. The double Dirichlet series satisfies the convexity bound*

$$\begin{aligned} Z\left(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi'\right) &= O\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right)\\ &= O\left(q(t)^{\frac{1}{4}+\epsilon}\right) \quad \text{for every } \epsilon>0 \text{ as } |t|\to\infty, \end{aligned}$$

where $q(t):=|t|^2$*.*

The proof uses the same arithmetic tools as in [21] to prove (1.0.3) but we analyze in more detail the asymptotic behavior of some integral transform of products of Whittaker functions of $\frac{1}{4}$-integral index, see section 2.3. In particular, the proof uses only very mild tools which are comparible to the Phragmen-Lindelöf principle. This is also why we call this bound the "convexity bound".

For simplicity we only consider the case where t_n are Fourier coefficients of a cusp form. However, by more technical manipulations, we expect the same statement to hold as well for $t_n = \tau(n)$ the divisor function.

For congruence lattice over $\mathbb{Q}$, a powerful tool for studying quantum unique ergodicity is the following measure classification theorem:

Theorem 1.0.3. *(Lindenstrauss, [15]) Let Γ be a congruence lattice over $\mathbb{Q}$, let $X = \Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ and let μ be a probability measure satisfying the following properties:*

[I] *μ is invariant under the geodesic flow,*

$[R]_p$ *μ is Hecke p-recurrent for a prime p, and*

[E] *the entropy of every ergodic component of μ is positive for the geodesic flow.*

Then, $\mu = m_X$ is the Haar measure on X.

So, to show that a sequence of probability measures on $\Gamma\backslash\mathbb{H}$, where Γ is a congruence lattice over $\mathbb{Q}$, distributes uniformly one can lift the measures to $\Gamma\backslash \mathrm{SL}_2(\mathbb{R})$ and use Lindenstrauss' measure classification theorem to show that the lifted sequence of measures converges to the Haar measure. Hence, the original sequence on $\Gamma\backslash\mathbb{H}$ distributes uniformly. This method goes back to Lindenstrauss [16], [15], Bourgain-Lindenstrauss [2] and many others. It was used e.g. to show quantum unique ergodicity for Hecke Maass cusp forms by Lindenstrauss [15] and Soundararajan [30], where the latter ruled out escape of mass.

Theorem 1.0.4. *(Lindenstrauss [15], Soundararajan [30]) Let Γ be a congruence lattice over $\mathbb{Q}$ and $M = \Gamma\backslash\mathbb{H}$. Every sequence (φ_i) of L^2-normalized Hecke Maass cusp forms with Laplace eigenvalues $\lambda_i \to \infty$ as $i \to \infty$ satisfies*

$$|\varphi_i|^2 \mathrm{dvol_M} \overset{\text{weak}^*}{\longrightarrow} \mathrm{dvol_M} \quad \textit{as } \mathrm{i} \to \infty,$$

i.e. the measures distribute uniformly.

However, as far as we know, there is no literature about applying this method for half-integral weight functions. In this thesis we are doing this for weight $\frac{1}{2}$ Eisenstein series. Compared to Hecke Maass cusp forms there are two issues appearing in this setting: First, the Eisenstein series is not L^2-integrable, hence the corresponding measures $|E|^2\,\mathrm{dvol}_M$ are infinite. Second, for half-integral weight there is a non-trivial multiplier system, wherefore it is not possible to construct the microlocal lift on $\mathrm{SL}_2(\mathbb{R})$. The first issue can be solved by considering the measures μ_t obtained by truncating the constant term of the Eisenstein series at height $|t|$. Indeed, the measures $\mu_t = \frac{1}{c\log|t|}|E^{|t|}(z,\frac{1}{2}+it)|^2 \,\mathrm{dvol}_M$ with a normalizing scalar c (see Definition 3.1.6) are finite by the Maass-Selberg relations. Moreover, integrating a compactly supported function against the truncated Eisenstein series is equivalent to integrate it against the normal Eisenstein series but only over the truncated space. While the argument of Hecke recurrence and positive entropy works completely analogous to the well-known proofs, one has to be a bit careful when showing invariance. In particular, integration by parts doesn't work as nicely on the truncated space, since there is a boundary term. The second issue can be resolved by not lifting the Eisenstein series to $\mathrm{SL}_2(\mathbb{R})$ but instead to a metaplectic group that consists of several copies of the double cover of $\mathrm{SL}_2(\mathbb{R})$. However, we will see that the absolute value of the microlocal lift, and hence the sequence of measure constructed by this lift can be considered as functions on $\mathrm{SL}_2(\mathbb{R})$. Hence, the measure classification theorem still applies.

In chapter 3 we are proving that the weight $\frac{1}{2}$ Eisenstein series satisfy quantum unique ergodicity in the following sense:

Theorem 1.0.5. *Let (μ_t) be the sequence of measures on $M = \Gamma_0(2^l)\backslash\mathbb{H}$ with $l \geq 2$ defined by the truncated Eisenstein series. Then, there exists a constant $0 \leq \lambda \leq 1$ with*

$$\mu_t \overset{weak^*}{\longrightarrow} \lambda\,\mathrm{vol_M} \quad as \quad |\mathrm{t}| \to \infty.$$

If $\lambda = 1$, then this theorem is the quantum unique ergodicity statement. If $\lambda < 1$ then there is escape of mass, compare with the works of Lindenstrauss [15] and Soundararajan [30] in weight 0. So, provided that there is no escape of mass, we prove QUE for weight $\frac{1}{2}$ Eisenstein series unconditionally, in particular without using the unknown subconvexity bound of the double Dirichlet series. In particular, this solves the motivating problem of Petridis, Raulf and Risager [21].

In the last chapter we are combining the method from chapter 2 with the quantum unique ergodicity result from chapter 3 in order to show some weak subconvexity bound for the double Dirichlet series:

Theorem 1.0.6. *Let χ and χ' be two Dirichlet characters modulo 2^l, $l \geq 2$, and let $\{t_n\}_{n\in\mathbb{N}}$ be the Fourier coefficients of the normalized L-function of a self-dual GL_2 automorphic form. The double Dirichlet series satisfies a weak subconvexity bound*

$$Z\left(\tfrac{1}{2}+it, \tfrac{1}{2}-it, \chi, \chi'\right) = o\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right) \qquad as\ |t| \to \infty.$$

The Convexity Bound

In this chapter we are defining weight $\frac{1}{2}$ Eisenstein series and the corresponding double Dirichlet series. In contrast to L-functions (i.e. single Dirichlet series), the double Dirichlet series is a series where each summand has again something like an L-function in the numerator. Therefore the expression “double”. While the theory of L-functions is well-established, this is not the case for double Dirichlet series.

In 2014, Petridis, Raulf and Risager [21] showed a convexity bound for a certain linear combination of double Dirichlet series. In this chapter we are proving that the convexity bound holds not only for linear combinations but for the double Dirichlet series itself. We are using the same mild tools as in [21]. The improvement comes from a more detailed analysis of the asymptotic behaviour of an integral of two Whittaker function that is appearing in the calculations. We also want to emphasize that we are not restricting to Dirichlet characters modulo 8 but prove everything for Dirichlet characters modulo powers of 2. In particular, it is not necessary to assume the characters to be quadratic.

2.1 Basic Definitions

In this section we are introducing the standard notation used later. Moreover, we are repeating in detail some basic theory which can also be found e.g. in [11], [12] and [13].

We denote by $\mathbb{H} := \{z \in \mathbb{C} \mid \Im(z) > 0\}$ the upper half plane. Invertible real 2×2-matrices act on $\mathbb{H} \cup \{i\infty\}$ by Möbius transformation i.e.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d}$$

with the convention that $\frac{a}{0} = \infty$ for $a \neq 0$. By “arg” we denote the principal argument, i.e. $\arg(z) \in (-\pi, \pi]$. For matrices $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $z \in \mathbb{H}$ we define

$$j(\gamma, z) := cz + d \quad \text{and} \quad j_\gamma(z) := \frac{cz+d}{|cz+d|}.$$

For any $k \in \mathbb{R}$ we define

$$j_\gamma(z)^k := e^{ik\arg(cz+d)}.$$

A straightforward calculation shows that

$$j(\gamma_1\gamma_2, z) = j(\gamma_1, \gamma_2 z)j(\gamma_2, z) \quad \text{for } \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R}) \text{ and } z \in \mathbb{H}. \tag{2.1.1}$$

Throughout the paper we will use the abbreviation $e(x) = e^{2\pi i x}$.

Definition 2.1.1. *A factor system of weight $k \in \mathbb{R}$ is defined as follows: For $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$ let*

$$\omega(\gamma_1, \gamma_2) := \frac{1}{2\pi}\left(\arg j(\gamma_1, \gamma_2 z) + \arg j(\gamma_2, z) - \arg j(\gamma_1\gamma_2, z)\right),$$

which is an integer independent of the choice of $z \in \mathbb{H}$ by (2.1.1). To be more precise, $\omega(\gamma_1, \gamma_2)$ will take values only in $\{-1, 0, 1\}$. Now,

$$w_k(\gamma_1, \gamma_2) := e\left(k\omega(\gamma_1, \gamma_2)\right) = e^{2\pi i k\omega(\gamma_1, \gamma_2)},$$

is the factor system of weight k.

It is a short calculation to show that the factor system of weight k satisfies the following, very useful property:

$$w_k(\gamma_1, \gamma_2)j_{\gamma_1\gamma_2}(z)^k = j_{\gamma_1}(\gamma_2 z)^k j_{\gamma_2}(z)^k \text{ for } \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R}) \text{ and } z \in \mathbb{H}. \tag{2.1.2}$$

The following properties will are very useful in order to work with multiplier systems:

Proposition 2.1.2. *Let $D = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B, C \in \mathrm{SL}_2(\mathbb{R})$. Then, we have*

(i) $\omega(D, A) = \omega(A, D) = 0$,

(ii) $\omega(A, A^{-1}) = \begin{cases} 1 & \text{if } c = 0 \text{ and } d < 0, \\ 0 & \text{otherwise,} \end{cases}$

(iii) $\omega(AB, C) + \omega(A, B) = \omega(A, BC) + \omega(B, C)$,

(iv) $\omega(A, B) = \omega(B, A)$ *if* $AB = BA$,

(v) $\omega(A^{-1}DA, B) + \omega(A, A^{-1}DAB) = \omega(A, B)$,

(vi) $\omega(AD, A^{-1}) = \omega(A, DA^{-1}) = \omega(A, A^{-1})$,

(vii) $\omega(ADA^{-1}, A) = \omega(A, A^{-1}DA) = 0$,

(viii) $\omega(-D, A) = \omega(-\mathrm{id}, A)$.

Proof. For proofs of these properties we refer to [12], [20] and [22]. Here, we only prove the last property as it does not appear in this form in the references:

$$\begin{aligned}
\omega(-D, A) - \omega(-\mathrm{id}, A) &= \frac{1}{2\pi}[\arg j(-D, A\cdot z) + \arg j(A, z) - \arg j(-DA, z) \\
&\qquad - \arg j(-\mathrm{id}, A\cdot z) - \arg j(A, z) + \arg j(-A, z)] \\
&= \frac{1}{2\pi}[\arg j(-D, Az) - \arg j(-DA, z) - \arg j(-\mathrm{id}, Az) + \arg j(-A, z)] \\
&= \frac{1}{2\pi}[\arg(-1) - \arg(-cz-d) - \arg(-1) + \arg(-cz-d)] \\
&= 0.
\end{aligned}$$

□

For the special case of weight $k = \frac{1}{2}$ the following Proposition simplifies the calculations with multiplier systems:

Proposition 2.1.3. *For* $\gamma_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ *with* $j = 1, 2$ *we have*

$$w_{\frac{1}{2}}(\gamma_1, \gamma_2) = \begin{cases} 1 & \text{if } \arg(c_1 i + d_1) + \arg(c_2 i + a_2) \in (-\pi, \pi], \\ -1 & \text{otherwise.} \end{cases}$$

Proof. Take $z = \gamma_2^{-1}\cdot i$. Then, we have

$$j(\gamma_1, \gamma_2 z) = j(\gamma_1, i) = c_1 i + d_1$$

and

$$\begin{aligned}
j(\gamma_2, z) = j(\gamma_2, \gamma_2^{-1} i) &= j\left(\gamma_2, \begin{pmatrix} d_2 & -b_2 \\ -c_2 & a_2 \end{pmatrix}\cdot i\right) = j\left(\gamma_2, \frac{d_2 i - b_2}{-c_2 i + a_2}\right) = c_2\frac{d_2 i - b_2}{-c_2 i + a_2} + d_2 \\
&= \frac{c_2 d_2 i - b_2 c_2 - c_2 d_2 i + a_2 d_2}{-c_2 i + a_2} = \frac{1}{a_2 - c_2 i} = \frac{a_2 + c_2 i}{a_2^2 + c_2^2}.
\end{aligned}$$

So, we have

$$\begin{aligned}
\omega(\gamma_1, \gamma_2) &= \begin{cases} 1 & \text{if } \arg j(\gamma_1, \gamma_2 z) + \arg j(\gamma_2, z) \in (\pi, 2\pi] \\ 0 & \text{if } \arg j(\gamma_1, \gamma_2 z) + \arg j(\gamma_2, z) \in (-\pi, \pi] \\ -1 & \text{if } \arg j(\gamma_1, \gamma_2 z) + \arg j(\gamma_2, z) \in (-2\pi, -\pi] \end{cases} \\
&= \begin{cases} 1 & \text{if } \arg(c_1 i + d_1) + \arg(a_2 + c_2 i) \in (\pi, 2\pi], \\ 0 & \text{if } \arg(c_1 i + d_1) + \arg(a_2 + c_2 i) \in (-\pi, \pi], \\ -1 & \text{if } \arg(c_1 i + d_1) + \arg(a_2 + c_2 i) \in (-2\pi, -\pi]. \end{cases}
\end{aligned}$$

Thus,

$$w_{\frac{1}{2}}(\gamma_1,\gamma_2) = e^{\pi i \omega(\gamma_1,\gamma_2)} = \begin{cases} 1 & \text{if } \arg(c_1 i + d_1) + \arg(c_2 i + a_2) \in (-\pi, \pi], \\ -1 & \text{otherwise.} \end{cases}$$

□

Remark. Note that for any $k = \frac{1}{2} + n \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ we have

$$w_k(\gamma_1,\gamma_2) = e^{2\pi i k \omega(\gamma_1,\gamma_2)} = e^{2\pi i (\frac{1}{2}+n)\omega(\gamma_1,\gamma_2)} = e^{\pi i \omega(\gamma_1,\gamma_2)} = w_{\frac{1}{2}}(\gamma_1,\gamma_2)$$

for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R}$.

Definition 2.1.4. *Let $\Gamma < \mathrm{SL}_2(\mathbb{R})$ be a cofinite subgroup. A weight k multiplier system on Γ is a function*

$$\nu : \Gamma \longrightarrow \mathbb{C}$$

satisfying

(i) $|\nu(\gamma)| = 1$ *for all* $\gamma \in \Gamma$

(ii) $\nu(\gamma_1\gamma_2) = w_k(\gamma_1,\gamma_2)\nu(\gamma_1)\nu(\gamma_2)$ *for all* $\gamma_1, \gamma_2 \in \Gamma$

(iii)' $\nu(-\mathrm{id}) = e\left(-\frac{k}{2}\right) = e^{-k\pi i}$ *if* $-\mathrm{id} \in \Gamma$.

Remark. The last condition is the so-called "consistency condition". Since $\nu(\mathrm{id}) = \nu(\mathrm{id} \cdot \mathrm{id}) = w_k(\mathrm{id},\mathrm{id})\nu(\mathrm{id})\nu(\mathrm{id}) = \nu(\mathrm{id})^2$ we have $\nu(\mathrm{id}) = 1$. But also, $\nu(\mathrm{id}) = \nu((-\mathrm{id})(-\mathrm{id})) = w_k(-\mathrm{id},-\mathrm{id})\nu(-\mathrm{id})\nu(-\mathrm{id}) = e^{2\pi i k}\nu(-\mathrm{id})^2$ by Proposition 2.1.2, (ii). Hence, $\nu(-\mathrm{id}) = \pm e^{-k\pi i}$. If $-\mathrm{id} \in \Gamma$ and $\nu(-\mathrm{id}) = -e^{-k\pi i}$ then the only function f which satisfies $f(\gamma \cdot z) = \nu(\gamma) j_\gamma(z)^k f(z)$ for every $\gamma \in \Gamma$ is the zero function. Indeed, if $-\mathrm{id} \in \Gamma$ then $f(z) = f(-\mathrm{id} \cdot z) \overset{!}{=} \nu(-\mathrm{id})(-1)^k f(z) = -e^{-k\pi i}(-1)^k f(z) = -f(z)$. Hence, the consistency condition is often added in order to avoid having the trivial automorphic form only.

Example. Given a character η modulo 2^l for $l \geq 2$ the function

$$\nu\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) := \eta(d)\left(\frac{c}{d}\right)\epsilon_d^{-1} \tag{2.1.3}$$

is a weight $\frac{1}{2}$ multiplier system on $\Gamma = \Gamma_0(2^l) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}) \;\middle|\; c \equiv 0 \mod 2^l \right\}$. Here,

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \mod 4, \\ i & \text{if } d \equiv 3 \mod 4 \end{cases}$$

and $\left(\frac{\cdot}{\cdot}\right)$ is the extended Jacobi symbol (cf. page 442 in [28]), i.e. for d odd it is defined by

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{d}\right) & \text{if } c \neq 0, d > 0, \\ \operatorname{sgn}(c)\left(\frac{c}{|d|}\right) & \text{if } c \neq 0, d < 0, \\ 1 & \text{if } c = 0 \text{ and } d = \pm 1, \\ 0 & \text{if } c = 0 \text{ and } d \neq \pm 1, \end{cases}$$

where $\left(\frac{c}{d}\right)$ for $d > 0$ odd is the Legendre symbol. First, note that ν is well-define. Indeed, for every $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2^l)$ the entry d has to be odd, so ϵ_d is defined. We check the properties of a multiplier system:

1. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2^l)$ is invertible, $(c, d) = 1$ and the Jacobi symbol is non-zero. Hence, every factor in $\nu(\gamma) = \eta(d)\left(\frac{c}{d}\right)\epsilon_d^{-1}$ has absolute value 1.

2. By equation (2.73) in [12] the theta function satisfies

$$\theta(\gamma z) = \epsilon_d^{-1}\left(\frac{c}{d}\right) j_\gamma(z)^{\frac{1}{2}}\theta(z) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4), \tag{2.1.4}$$

   hence also for $\gamma \in \Gamma_0(2^l)$. Denote by d_γ the lower right entry of $\gamma \in \Gamma_0(2^l)$. For all $\gamma_1, \gamma_2 \in \Gamma_0(2^l)$ we have $d_{\gamma_1\gamma_2} \equiv d_{\gamma_1} d_{\gamma_2} \mod 2^l$. Using equation (2.1.4) we get

$$\begin{aligned} \nu(\gamma_1\gamma_2) j_{\gamma_1\gamma_2}(z)^{\frac{1}{2}}\theta(z) &= \eta(d_{\gamma_1\gamma_2})\theta((\gamma_1\gamma_2)z) = \eta(d_{\gamma_1})\eta(d_{\gamma_2})\theta(\gamma_1(\gamma_2 z)) \\ &= \eta(d_{\gamma_2})\nu(\gamma_1) j_{\gamma_1}(\gamma_2 z)^{\frac{1}{2}}\theta(\gamma_2 z) \\ &= \nu(\gamma_1) j_{\gamma_1}(\gamma_2 z)^{\frac{1}{2}}\nu(\gamma_2) j_{\gamma_2}(z)^{\frac{1}{2}}\theta(z) \end{aligned}$$

   for every $z \in \mathbb{H}$. Hence,

$$\nu(\gamma_1\gamma_2) = j_{\gamma_1}(\gamma_2 z)^{\frac{1}{2}} j_{\gamma_2}(z)^{\frac{1}{2}} j_{\gamma_1\gamma_2}(z)^{-\frac{1}{2}}\nu(\gamma_1)\nu(\gamma_2) = w_{\frac{1}{2}}(\gamma_1, \gamma_2)\nu(\gamma_1)\nu(\gamma_2)$$

   by equation (2.1.2).

3.' We have $\nu(-\mathrm{id}) = \eta(-1)\left(\frac{0}{-1}\right)\epsilon_{-1}^{-1} = \eta(-1)(-i) = \eta(-1)e^{-\pi i/2}$. So, the consistency condition is only satisfied if $\eta(-1) = 1$. (This means that if we want to avoid multiplier systems which lead only to trivial automorphic forms, then we have to assume that η is an even character.)

For a cusp $\mathfrak{a}$ of $\Gamma < \mathrm{SL}_2(\mathbb{R})$ we denote by

$$\Gamma_{\mathfrak{a}} := \{\gamma \in \Gamma \mid \gamma \cdot \mathfrak{a} = \mathfrak{a}\} = \langle \pm\gamma_{\mathfrak{a}} \rangle$$

its stabilizer. (Here, the $-\gamma_{\mathfrak{a}}$ occurs only if $-\mathrm{id} \in \Gamma$.) In particular, at the cusp ∞ we have

$$\Gamma_\infty = \left(\left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \cup \left\{ \begin{pmatrix} -1 & x \\ 0 & -1 \end{pmatrix} \,\middle|\, x \in \mathbb{R} \right\} \right) \cap \Gamma.$$

(Here, the second set occurs only if $-\mathrm{id} \in \Gamma$.) In particular, for $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ or $\Gamma_0(M)$ the stabilizer at infinity is

$$\Gamma_\infty = \Big\langle \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}}_{=:\gamma_\infty}, \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix} \Big\rangle.$$

Let $\sigma_{\mathfrak{a}} \in \mathrm{SL}_2(\mathbb{R})$ be the scaling matrix of the cusp $\mathfrak{a}$ i.e.

$$\sigma_{\mathfrak{a}} \cdot \infty = \mathfrak{a} \text{ and } \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \gamma_\infty, \text{ hence also } \sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}^r = \gamma_\infty^r \sigma_{\mathfrak{a}}^{-1} \text{ for all } r \in \mathbb{Z}.$$

Note that for $\Gamma = \Gamma_0(M)$ we have

$$\Gamma_0 = \langle \pm\gamma_0 \rangle \quad \text{with } \gamma_0 = \begin{pmatrix} 1 & 0 \\ -M & 1 \end{pmatrix} \quad \text{and scaling matrix} \quad \sigma_0 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{M}} \\ \sqrt{M} & 0 \end{pmatrix}.$$

A cusp $\mathfrak{a}$ of $\Gamma < \mathrm{SL}_2(\mathbb{R})$ is called *open* with respect to the multiplier system ν, if $\nu(\gamma_{\mathfrak{a}}) = 1$.

Proposition 2.1.5. *For every open cusp $\mathfrak{a}$ we have $\nu(\gamma_{\mathfrak{a}}^r) = 1$ for every $r \in \mathbb{Z}$.*

Proof. We have $\nu(\gamma_{\mathfrak{a}}^0) = \nu(\mathrm{id}) = 1$ by the definition of a multiplier system. For $r \in \mathbb{Z}_{\geq 1}$ we prove the statement by induction on r: Since $\mathfrak{a}$ is an open cusp, we have $\nu(\gamma_{\mathfrak{a}}) = 1$. Let now $r \geq 2$ and assume that $\nu(\gamma_{\mathfrak{a}}^n) = 1$ for all $0 \leq n \leq r-1$. Using Proposition 2.1.2, part (v) and (vii) we get

$$\begin{aligned} \nu(\gamma_{\mathfrak{a}}^r) = \nu(\gamma_{\mathfrak{a}})\nu(\gamma_{\mathfrak{a}}^{r-1})w_k(\gamma_{\mathfrak{a}}, \gamma_{\mathfrak{a}}^{r-1}) = w_k(\gamma_{\mathfrak{a}}, \gamma_{\mathfrak{a}}^{r-1}) &= w_k(\underbrace{\sigma_{\mathfrak{a}}}_{=:A^{-1}} \underbrace{\gamma_\infty}_{=:D} \underbrace{\sigma_{\mathfrak{a}}^{-1}}_{=:A}, \underbrace{\sigma_{\mathfrak{a}}\gamma_\infty^{r-1}\sigma_{\mathfrak{a}}^{-1}}_{=:B}) \\ &\overset{(v)}{=} w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma_\infty^{r-1}\sigma_{\mathfrak{a}}^{-1})w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma_\infty^{r}\sigma_{\mathfrak{a}}^{-1})^{-1} \overset{(vii)}{=} 1. \end{aligned}$$

For $r \in \mathbb{Z}_{<0}$ we have (using again Proposition 2.1.2, parts (v), (vii) and (i))

$$\begin{aligned} \nu(\gamma_{\mathfrak{a}}^r)\nu(\gamma_{\mathfrak{a}}^{-r}) = \nu(\mathrm{id})w_k(\gamma_{\mathfrak{a}}^r, \gamma_{\mathfrak{a}}^{-r})^{-1} &= w_k(\underbrace{\sigma_{\mathfrak{a}}}_{=:A^{-1}} \underbrace{\gamma_\infty^r}_{=:D} \underbrace{\sigma_{\mathfrak{a}}^{-1}}_{=:A}, \underbrace{\sigma_{\mathfrak{a}}\gamma_\infty^{-r}\sigma_{\mathfrak{a}}^{-1}}_{=:B})^{-1} \\ &\overset{(v)}{=} w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma_\infty^{-r}\sigma_{\mathfrak{a}}^{-1}))^{-1}w_k(\sigma_{\mathfrak{a}}^{-1}, \mathrm{id}) \overset{(i),(vii)}{=} 1. \end{aligned}$$

□

Remark. Note that $\mathfrak{a}$ being an open cusp does not imply that $\nu|_{\Gamma_\mathfrak{a}} = 1$. (This is misleadingly stated in [12].) Indeed, if $-\mathrm{id} \in \Gamma$ and assuming the consistency condition, we have

$$\nu(-\gamma_\mathfrak{a}^r) = \nu(-\mathrm{id})\nu(\gamma_\mathfrak{a}^r)w_k(-\mathrm{id}, \gamma_\mathfrak{a}^r) = e^{-\pi ik} w_k(-\mathrm{id}, \gamma_\mathfrak{a}^r)$$

which e.g. for $\mathfrak{a} = \infty$ equals $e^{-\pi ik}$ and this is $\neq 1$ for general $k \in \mathbb{R}$.

Definition 2.1.6. *Given a multiplier system ν on Γ and open cusps $\mathfrak{a}$ and $\mathfrak{b}$ we define*

$$\nu_{\mathfrak{a}\mathfrak{b}}(\gamma) := \nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}^{-1}, \sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\gamma\sigma_\mathfrak{b}^{-1}, \sigma_\mathfrak{b})$$

which is a function on $\sigma_\mathfrak{a}^{-1}\Gamma\sigma_\mathfrak{b}$.

Remark. If $\mathfrak{a} = \mathfrak{b}$ i.e. $\sigma_\mathfrak{a} = \sigma_\mathfrak{b} =: \sigma$ this defines a multiplier system on $\sigma^{-1}\Gamma\sigma$. Indeed, since a factor system w_k of weight k is by definition of absolute value 1, we have $|\nu_{\mathfrak{a}\mathfrak{a}}| = 1$. For $\gamma_1, \gamma_2 \in \sigma^{-1}\Gamma\sigma$ we have

$$\begin{aligned}
\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1\gamma_2) &= \nu(\sigma\gamma_1\sigma^{-1}\sigma\gamma_2\sigma^{-1})w_k(\sigma^{-1}, \sigma\gamma_1\gamma_2\sigma^{-1})w_k(\gamma_1\gamma_2\sigma^{-1}, \sigma) \\
&= \nu(\sigma\gamma_1\sigma^{-1})\nu(\sigma\gamma_2\sigma^{-1})w_k(\sigma\gamma_1\sigma^{-1}, \sigma\gamma_2\sigma^{-1})w_k(\sigma^{-1}, \sigma\gamma_1\gamma_2\sigma^{-1})w_k(\gamma_1\gamma_2\sigma^{-1}, \sigma) \\
&= \nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1)w_k(\sigma^{-1}, \sigma\gamma_1\sigma^{-1})^{-1}w_k(\gamma_1\sigma^{-1}, \sigma)^{-1}\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_2)w_k(\sigma^{-1}, \sigma\gamma_2\sigma^{-1})^{-1}w_k(\gamma_2\sigma^{-1}, \sigma)^{-1} \\
&\qquad \cdot w_k(\sigma\gamma_1\sigma^{-1}, \sigma\gamma_2\sigma^{-1})w_k(\sigma^{-1}, \sigma\gamma_1\gamma_2\sigma^{-1})w_k(\gamma_1\gamma_2\sigma^{-1}, \sigma).
\end{aligned}$$

Applying Proposition 2.1.2, part (iii), with $A = \sigma^{-1}$, $B = \sigma\gamma_1\sigma^{-1}$ and $C = \sigma\gamma_2\sigma^{-1}$ we have

$$w_k(\sigma^{-1}, \sigma\gamma_1\sigma^{-1})w_k(\sigma\gamma_1\sigma^{-1}, \sigma\gamma_2\sigma^{-1})w_k(\sigma^{-1}, \sigma\gamma_1\gamma_2\sigma^{-1}) = w_k(\gamma_1\sigma^{-1}, \sigma\gamma_2\sigma^{-1}).$$

Hence,

$$\begin{aligned}
\nu_{\mathfrak{a},\mathfrak{a}}(\gamma_1\gamma_2) &= \nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1)w_k(\gamma_1\sigma^{-1}, \sigma)^{-1}\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_2)w_k(\sigma^{-1}, \sigma\gamma_2\sigma^{-1})^{-1}w_k(\gamma_2\sigma^{-1}, \sigma)^{-1} \\
&\qquad \cdot w_k(\gamma_1\sigma^{-1}, \sigma\gamma_2\sigma^{-1})w_k(\gamma_1\gamma_2\sigma^{-1}, \sigma) \\
&= \nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1)w_k(\gamma_1\sigma^{-1}, \sigma)^{-1}\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_2)w_k(\sigma^{-1}, \sigma\gamma_2\sigma^{-1})^{-1}w_k(\gamma_2\sigma^{-1}, \sigma)^{-1} \\
&\qquad \cdot w_k(\sigma\gamma_2\sigma^{-1}, \sigma)w_k(\gamma_1\sigma^{-1}, \sigma\gamma_2),
\end{aligned}$$

where we used the same property, but now with $A = \gamma_1\sigma^{-1}$, $B = \sigma\gamma_2\sigma^{-1}$ and $C = \sigma$. Using part (i) and (iii) of Proposition 2.1.2 again and again, we can rewrite the appearing w_k-terms as follows:

$$\begin{aligned}
w_k(\gamma_1\sigma^{-1}, \sigma)^{-1} &= w_k(\gamma_1, \sigma^{-1})w_k(\gamma_1, \mathrm{id})^{-1}w_k(\sigma^{-1}, \sigma)^{-1} = w_k(\gamma_1, \sigma^{-1})w_k(\sigma^{-1}, \sigma)^{-1} \\
w_k(\sigma^{-1}, \sigma\gamma_2\sigma^{-1})^{-1} &= w_k(\sigma, \gamma_2\sigma^{-1})w_k(\mathrm{id}, \gamma_2\sigma^{-1})^{-1}w_k(\sigma^{-1}, \sigma)^{-1} \\
&= w_k(\sigma\gamma_2, \sigma^{-1})w_k(\sigma, \gamma_2)w_k(\gamma_2, \sigma^{-1})^{-1}w_k(\sigma^{-1}, \sigma)^{-1} \\
w_k(\gamma_2\sigma^{-1}, \sigma)^{-1} &= w_k(\gamma_2, \sigma^{-1})w_k(\gamma_2, \mathrm{id})^{-1}w_k(\sigma^{-1}, \sigma)^{-1} = w_k(\gamma_2, \sigma^{-1})w_k(\sigma^{-1}, \sigma)^{-1} \\
w_k(\sigma\gamma_2\sigma^{-1}, \sigma) &= w_k(\sigma\gamma_2, \mathrm{id})w_k(\sigma^{-1}, \sigma)w_k(\sigma\gamma_2, \sigma^{-1})^{-1} = w_k(\sigma^{-1}, \sigma)w_k(\sigma\gamma_2, \sigma^{-1})^{-1} \\
w_k(\gamma_1\sigma^{-1}, \sigma\gamma_2) &= w_k(\gamma_1, \gamma_2)w_k(\sigma^{-1}, \sigma\gamma_2)w_k(\gamma_1, \sigma^{-1})^{-1} \\
&= w_k(\gamma_1, \gamma_2)w_k(\sigma^{-1}, \sigma)w_k(\sigma, \gamma_2)^{-1}w_k(\gamma_1, \sigma^{-1})^{-1}.
\end{aligned}$$

The only w_k-terms that do not cancel are $w_k(\gamma_1,\gamma_2)w_k(\sigma^{-1},\sigma)^{-1}$. However, the scaling matrix σ cannot be of the form $\left(\begin{smallmatrix} * & * \\ 0 & d \end{smallmatrix}\right)$ with $d<0$ as in that case $\sigma\cdot\infty=-\infty$ is not a cusp. Hence, by part (ii) of Proposition 2.1.2, $w_k(\sigma^{-1},\sigma)=1$. So, we indeed have

$$\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1\gamma_2)=\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_1)\nu_{\mathfrak{a}\mathfrak{a}}(\gamma_2)w_k(\gamma_1,\gamma_2).$$

Proposition 2.1.7. *For every $\gamma\in\Gamma$ and every weight k we have*

$$\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})w_k(\sigma_\mathfrak{a},\gamma)^{-1}.$$

Proof. Using Proposition 2.1.2, part (iii) once with $A=\sigma_\mathfrak{a}^{-1}$, $B=\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1}$ and $C=\sigma_\mathfrak{b}$, and once with $A'=\sigma_\mathfrak{a}^{-1}$, $B'=\sigma_\mathfrak{a}$ and $C'=\gamma$ we have

$$\begin{aligned}\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)&=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\gamma\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})\\&=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma)w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})\\&=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})\underbrace{w_k(\mathrm{id},\gamma)}_{=1}\underbrace{w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a})}_{=1}w_k(\sigma_\mathfrak{a},\gamma)^{-1}w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})\\&=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a},\gamma)^{-1}w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b}).\end{aligned}$$

In the last step we used part (i) and (ii) of Proposition 2.1.2 and the fact that a scaling matrix is never of the form $\left(\begin{smallmatrix} * & * \\ 0 & d \end{smallmatrix}\right)$ with $d<0$. □

Proposition 2.1.8. *For every $\gamma\in\Gamma$ and $l\in\mathbb{Z}$ we have $\nu_{\mathfrak{a}\mathfrak{b}}(\gamma\gamma_\infty^l)=\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)$.*

Proof. Recall that we have $\sigma_\mathfrak{b}^{-1}\gamma_\mathfrak{b}^l=\gamma_\infty^l\sigma_\mathfrak{b}^{-1}$ and $\nu(\gamma_\mathfrak{b}^l)=1$ by Proposition 2.1.5. Hence, we have

$$\nu(\sigma_\mathfrak{a}\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1})=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1}\gamma_\mathfrak{b}^l)=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})\nu(\gamma_\mathfrak{b}^l)w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l)=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l).$$

Using part (iii) from Proposition 2.1.2 we have

$$w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1})=w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1}\gamma_\mathfrak{b}^l)=w_k(\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l)w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l)^{-1}$$

and

$$w_k(\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})=w_k(\gamma\sigma_\mathfrak{b}^{-1}\gamma_\mathfrak{b}^l,\sigma_\mathfrak{b})=w_k(\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l\sigma_\mathfrak{b})w_k(\gamma_\mathfrak{b}^l,\sigma_\mathfrak{b})w_k(\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l)^{-1}.$$

Hence, we have

$$\begin{aligned}\nu_{\mathfrak{a}\mathfrak{b}}(\gamma\gamma_\infty^l)&=\nu(\sigma_\mathfrak{a}\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1})w_k(\gamma\gamma_\infty^l\sigma_\mathfrak{b}^{-1},\sigma_\mathfrak{b})\\&=\nu(\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\sigma_\mathfrak{a}^{-1},\sigma_\mathfrak{a}\gamma\sigma_\mathfrak{b}^{-1})w_k(\gamma\sigma_\mathfrak{b}^{-1},\gamma_\mathfrak{b}^l\sigma_\mathfrak{b})w_k(\gamma_\mathfrak{b}^l,\sigma_\mathfrak{b}).\end{aligned}$$

In order to have this equal to

$$\nu_{\mathfrak{a},\mathfrak{b}}(\gamma) = \nu(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}) w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}) w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}),$$

it remains to show that $w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \gamma_{\mathfrak{b}}^l\sigma_{\mathfrak{b}}) w_k(\gamma_{\mathfrak{b}}^l, \sigma_{\mathfrak{b}}) = w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}})$. This is indeed true:

$$\begin{aligned} w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \gamma_{\mathfrak{b}}^l\sigma_{\mathfrak{b}}) w_k(\gamma_{\mathfrak{b}}^l, \sigma_{\mathfrak{b}}) &= w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}\gamma_{\infty}^l) w_k(\gamma_{\mathfrak{b}}^l, \sigma_{\mathfrak{b}}) \\ &= \underbrace{w_k(\gamma, \gamma_{\infty}^l)}_{=1} w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}) \underbrace{w_k(\sigma_{\mathfrak{b}}, \gamma_{\infty}^l)^{-1}}_{=1} w_k(\gamma_{\mathfrak{b}}^l, \sigma_{\mathfrak{b}}) \\ &= w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}) \underbrace{w_k(\sigma_{\mathfrak{b}}\gamma_{\infty}^l\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}})}_{=1 \text{ by Prop 2.1.2, part (vii)}} = w_k(\gamma\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}}). \end{aligned}$$

□

2.2 Half-integral weight Eisenstein series

In this section we define half-integral weight Eisenstein series and revisit some of their properties, in particular its Fourier expansion. We also define the correction polynomial and some L-function that we will use later to define the double Dirichlet series. Finally, we prove some identities involving this L-function.

Definition 2.2.1. *The weight k (for $k \in \frac{1}{2}\mathbb{Z}$) Eisenstein series for a cofinite subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ at an open cusp $\mathfrak{a}$ with respect to the multiplier system ν is defined to be*

$$E_{\mathfrak{a}}(z,s,k) := \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\nu(\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s \quad \textit{for } z\in\mathbb{H} \textit{ and } s\in\mathbb{C}, \Re(s) > 1.$$

Remark. For η a character modulo 2^l we denote by $E_{\mathfrak{a},\eta}(z,s,\frac{1}{2})$ the weight $\frac{1}{2}$ Eisenstein series for $\Gamma_0(2^l)$ with respect to the multiplier system $\nu\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \eta(d)\left(\frac{c}{d}\right)\epsilon_d^{-1}$.

Remark. Note that this definition is well-defined, i.e. it is independent of the choice of representative in $\Gamma_{\mathfrak{a}}\backslash\Gamma$. Indeed, let γ and $\gamma' = \xi\gamma$ with $\xi\in\Gamma_{\mathfrak{a}}$ be two representatives of $\Gamma_{\mathfrak{a}}\backslash\Gamma$. We need to show that

$$\underbrace{\overline{\nu(\gamma')w_k(\sigma_{\mathfrak{a}}^{-1},\gamma')}}_{=:①'} \underbrace{j_{\sigma_{\mathfrak{a}}^{-1}\gamma'}(z)^{-k}}_{=:②'} \underbrace{\Im(\sigma_{\mathfrak{a}}^{-1}\gamma' z)^s}_{=:③'} = \underbrace{\overline{\nu(\gamma)w_k(\sigma_{\mathfrak{a}}^{-1},\gamma)}}_{=:①} \underbrace{j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k}}_{=:②} \underbrace{\Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s}_{=:③}.$$

We will treat the three parts separately. Note that $\xi\in\Gamma_{\mathfrak{a}} = \langle\pm\gamma_{\mathfrak{a}}\rangle$ i.e. $\xi = \pm\gamma_{\mathfrak{a}}^r$ for some $r\in\mathbb{Z}$. We have

$$\begin{aligned} ③' &= \Im(\pm\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}^r\gamma z)^s = \Im(\pm\gamma_{\infty}^r\sigma_{\mathfrak{a}}^{-1}\gamma z)^s = \Im\left(\begin{pmatrix} \pm1 & \pm r \\ 0 & \pm1 \end{pmatrix}\cdot(\sigma_{\mathfrak{a}}^{-1}\gamma z)\right)^s \\ &= \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z + r)^s = \Im(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s = ③, \end{aligned}$$

so the parts ③′ and ③ coincide. From the second part we get an extra factor as follows:

$$
\begin{aligned}
②' &= j_{\pm\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}^r\gamma}(z)^{-k} = j_{\pm\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \overset{(2.1.2)}{=} w_k(\pm\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) j_{\pm\gamma_\infty^r}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{-k} j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(z)^{-k} \\
&= w_k(\pm\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) j_{\pm\gamma_\infty^r}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{-k} \cdot ②,
\end{aligned}
$$

where the extra factor is

$$
\begin{aligned}
&w_k(\pm\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) j_{\pm\gamma_\infty^r}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^{-k} \\
&\qquad = \begin{cases} 1 & \text{if } \xi = +\gamma_{\mathfrak{a}}^r, \\ w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) e^{-ik\arg(-1)} = w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) e^{-\pi ik} & \text{if } \xi = -\gamma_{\mathfrak{a}}^r. \end{cases}
\end{aligned}
$$

It remains to show that this extra factor cancels with the one coming from the first part.

$$
①' = \overline{\nu(\pm\gamma_{\mathfrak{a}}^r\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \pm\gamma_{\mathfrak{a}}^r\gamma)} = \overline{\nu(\pm\gamma_{\mathfrak{a}}^r)\nu(\gamma) w_k(\pm\gamma_{\mathfrak{a}}^r, \gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \pm\gamma_{\mathfrak{a}}^r\gamma)}.
$$

Recall from Proposition 2.1.5 that $\nu(\gamma_{\mathfrak{a}}^r) = 1$.

- Case $\xi = +\gamma_{\mathfrak{a}}^r$: Using $\gamma_{\mathfrak{a}}^r = \sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}$ and part (v) of Proposition 2.1.2 we get

$$
①' = \overline{\nu(\gamma) w_k(\sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}, \gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}\gamma)} \overset{\text{Prop 2.1.2,(5)}}{=} \overline{\nu(\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} = ①.
$$

- Case $\xi = -\gamma_{\mathfrak{a}}^r$: Note that $\nu(-\gamma_{\mathfrak{a}}^r) = \nu(-\mathrm{id})\nu(\gamma_{\mathfrak{a}}^r) w_k(-\mathrm{id}, \gamma_{\mathfrak{a}}^r) = e^{-\pi ik} w_k(-\mathrm{id}, \gamma_{\mathfrak{a}}^r)$. Using several parts from Proposition 2.1.2 we calculate

$$
\begin{aligned}
①' &= e^{\pi ik}\overline{w_k(-\mathrm{id}, \gamma_{\mathfrak{a}}^r)\nu(\gamma) w_k(-\gamma_{\mathfrak{a}}^r, \gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, -\gamma_{\mathfrak{a}}^r\gamma)} \\
&\overset{(iii)}{=} e^{\pi ik}\overline{w_k(-\mathrm{id}, \gamma_{\mathfrak{a}}^r)\nu(\gamma) w_k(-\sigma_{\mathfrak{a}}^{-1}\gamma_{\mathfrak{a}}^r, \gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, -\gamma_{\mathfrak{a}}^r)} \\
&= e^{\pi ik}\overline{w_k(-\mathrm{id}, \sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})\nu(\gamma) w_k(-\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}, \gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, -\sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})} \\
&\overset{(iii)}{=} e^{\pi ik}\overline{w_k(-\mathrm{id}, \sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})\nu(\gamma) w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}) \overline{w_k(\sigma_{\mathfrak{a}}^{-1}, -\sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})} \\
&= e^{\pi ik}\overline{w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma)\nu(\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}) \underbrace{\overline{w_k(-\mathrm{id}, \sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})}}_{\overset{(iv)}{=}\overline{w_k(\sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1}, -\mathrm{id})}} \overline{w_k(\sigma_{\mathfrak{a}}^{-1}, -\sigma_{\mathfrak{a}}\gamma_\infty^r\sigma_{\mathfrak{a}}^{-1})} \\
&\overset{(v)}{=} e^{\pi ik}\overline{w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma)\nu(\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}) \underbrace{\overline{w_k(\sigma_{\mathfrak{a}}^{-1}, -\mathrm{id})}}_{\overset{(iv)}{=}\overline{w_k(-\mathrm{id}, \sigma_{\mathfrak{a}}^{-1})}} \\
&\overset{(viii)}{=} e^{\pi ik}\overline{w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma)\nu(\gamma) w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)} = e^{\pi ik}\overline{w_k(-\gamma_\infty^r, \sigma_{\mathfrak{a}}^{-1}\gamma)} \cdot ①.
\end{aligned}
$$

This shows, that the expression is indeed independent of the choice of representative.

Proposition 2.2.2. *The weight k Eisenstein series for a cofinite subgroup $\Gamma < \mathrm{SL}_2(\mathbb{R})$ has the following properties:*

(a) It is an automorphic form for Γ of weight k and multiplier system ν, i.e.

$$E_{\mathfrak{a}}(\gamma \cdot z, s, k) = \nu(\gamma) j_\gamma(z)^k E_{\mathfrak{a}}(z, s, k)$$

for every cusp $\mathfrak{a}$ of Γ, $\gamma \in \Gamma$, $z \in \mathbb{H}$, $s \in \mathbb{C}$ with $\Re(s) > 1$ and every $k \in \mathbb{R}$.

(b) The Fourier coefficients of $E_{\mathfrak{a}}(z, s, k)$ at an open cusp $\mathfrak{b}$ are

$$c_0(y) = \delta_{\mathfrak{a},\mathfrak{b}} y^s + \sum_{\mathrm{id} \neq \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty} \mathrm{sgn}(c)^k \frac{\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}}{c^{2s}} \pi 4^{1-s} e^{-ik\frac{\pi}{2}} \frac{\Gamma(2s-1) y^{1-s}}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})}$$

and

$$c_n(y) = \sum_{\mathrm{id} \neq \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty} \mathrm{sgn}(c)^k \frac{\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}}{c^{2s}} e(n\tfrac{d}{c}) \pi^s e^{-ik\frac{\pi}{2}} \frac{|n|^{s-1}}{\Gamma(s+\frac{kn}{2|n|})} W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y)$$

for $n \neq 0$. In particular, the Fourier expansion of $E_{\mathfrak{a}}(z, s, k)$ at an open cusp $\mathfrak{b}$ is

$$j_{\sigma_{\mathfrak{b}}}(z)^{-k} E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, k) = \sum_{n \in \mathbb{Z}} c_n(y) e(nx).$$

(c) $E_{\mathfrak{a}}(z, \cdot, k)$ admits a meromorphic continuation to $\mathbb{C}$.

(d) It is an eigenfunction of the weight k Laplacian with eigenvalue $s(1-s)$, i.e.

$$\Delta_k E_{\mathfrak{a}}(z, s, k) = s(1-s) E_{\mathfrak{a}}(z, s, k),$$

where $\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}$ is the weight k Laplacian.

Proof. (a) For every cusp $\mathfrak{a}$ of Γ and all $\gamma \in \Gamma, z \in \mathbb{H}, s \in \mathbb{C}$ with $\Re(s) > 1$ and every $k \in \mathbb{R}$ we have

$$\begin{aligned}
E_{\mathfrak{a}}(\gamma z, s, k) &= \sum_{\eta \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\nu(\eta) w_k(\sigma_{\mathfrak{a}}^{-1}, \eta)} j_{\sigma_{\mathfrak{a}}^{-1}\eta}(\gamma z)^{-k} \Im(\sigma_{\mathfrak{a}}^{-1} \eta \gamma z)^s \\
&\overset{\mu = \eta\gamma}{=} \sum_{\mu \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{\nu(\mu\gamma^{-1}) w_k(\sigma_{\mathfrak{a}}^{-1}, \mu\gamma^{-1})} j_{\sigma_{\mathfrak{a}}^{-1}\mu\gamma^{-1}}(\gamma z)^{-k} \Im(\sigma_{\mathfrak{a}}^{-1} \mu z)^s \\
&= \sum_{\mu \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \overline{w_k(\mu, \gamma^{-1}) \nu(\mu) \nu(\gamma^{-1}) w_k(\sigma_{\mathfrak{a}}^{-1}, \mu\gamma^{-1})} \Im(\sigma_{\mathfrak{a}}^{-1} \mu z)^s \\
&\qquad \cdot j_\gamma(z)^k j_{\sigma_{\mathfrak{a}}^{-1}\mu\gamma^{-1}\gamma}(z)^{-k} w_k(\sigma_{\mathfrak{a}}^{-1}\mu\gamma^{-1}, \gamma)^{-1},
\end{aligned}$$

where we used in the first step that whenever η runs over a complete set of cosets, then so does $\eta\gamma$ for $\gamma \in \Gamma$. Note that $1 = \nu(\mathrm{id}) = \nu(\gamma^{-1}\gamma) = w_k(\gamma^{-1}, \gamma)\nu(\gamma^{-1})\nu(\gamma)$, hence

$$\overline{\nu(\gamma^{-1})} = \nu(\gamma^{-1})^{-1} = w_k(\gamma^{-1}, \gamma)\nu(\gamma).$$

Moreover, using several parts of Proposition 2.1.2 we ca simplify the w_k-factors as follows:

$$\begin{aligned}\overline{w_k(\mu, \gamma^{-1})w_k(\sigma_{\mathfrak{a}}^{-1}, \mu\gamma^{-1})}w_k(\sigma_{\mathfrak{a}}^{-1}\mu\gamma^{-1}, \gamma)^{-1} &\overset{(iii)}{=} \overline{w_k(\sigma_{\mathfrak{a}}^{-1}, \mu)w_k(\sigma_{\mathfrak{a}}^{-1}\mu, \gamma^{-1})w_k(\sigma_{\mathfrak{a}}^{-1}\mu\gamma^{-1}, \gamma)} \\ &\overset{(iii)}{=} \overline{w_k(\sigma_{\mathfrak{a}}^{-1}, \mu)w_k(\gamma^{-1}, \gamma)w_k(\sigma_{\mathfrak{a}}^{-1}\mu, \mathrm{id})} \\ &\overset{(i)}{=} \overline{w_k(\sigma_{\mathfrak{a}}^{-1}, \mu)w_k(\gamma^{-1}, \gamma)}.\end{aligned}$$

Hence,

$$\begin{aligned}E_{\mathfrak{a}}(\gamma z, s, k) &= \nu(\gamma)j_\gamma(z)^k \sum_{\mu\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\nu(\mu)w_k(\sigma_{\mathfrak{a}}^{-1}, \mu)}j_{\sigma_{\mathfrak{a}}^{-1}\eta}(z)^{-k}\Im(\sigma_{\mathfrak{a}}^{-1}\mu z)^s \\ &= \nu(\gamma)j_\gamma(z)^k E_{\mathfrak{a}}(z, s, k)\end{aligned}$$

(b) (cf. [21]) Let $\mathfrak{b}$ be an open cusp of Γ, i.e. $\nu(\gamma_{\mathfrak{b}}) = 1$. By definition of Fourier coefficients at a cusp we have $j_{\sigma_{\mathfrak{b}}}(z)^{-k}E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} \cdot z, s, k) = \sum_{m\in\mathbb{Z}} c_m(y)e(mx)$ with $z = x + iy$. So, the Fourier coefficients are $c_n(y) = \int_0^1 j_{\sigma_{\mathfrak{b}}}(z)^{-k}E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} \cdot z, s, k)e(-nx)dx$. First, note that using property (2.1.2), we get

$$\begin{aligned}j_{\sigma_{\mathfrak{b}}}(z)^{-k}E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} \cdot z, s, k) &= j_{\sigma_{\mathfrak{b}}}(z)^{-k} \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\nu(\gamma)w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)}j_{\sigma_{\mathfrak{a}}^{-1}\gamma}(\sigma_{\mathfrak{b}} \cdot z)^{-k}\Im(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^s \\ &= \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} \overline{\nu(\gamma)w_k(\sigma_{\mathfrak{a}}^{-1}, \gamma)w_k(\sigma_{\mathfrak{a}}^{-1}\gamma, \sigma_{\mathfrak{b}})}j_{\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}}(z)^{-k}\Im(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}z)^s \\ &= \sum_{\gamma'\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \overline{\nu(\sigma_{\mathfrak{a}}\gamma'\sigma_{\mathfrak{b}}^{-1})w_k(\sigma_{\mathfrak{a}}^{-1}, \sigma_{\mathfrak{a}}\gamma'\sigma_{\mathfrak{b}}^{-1})w_k(\gamma'\sigma_{\mathfrak{b}}^{-1}, \sigma_{\mathfrak{b}})}j_{\gamma'}(z)^{-k}\Im(\gamma' z)^s \\ &= \sum_{\gamma'\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}} \overline{\nu_{\mathfrak{ab}}(\gamma')}j_{\gamma'}(z)^{-k}\Im(\gamma' z)^s.\end{aligned}$$

In the third equality we used that if γ is running over a set of representatives for $\Gamma_{\mathfrak{a}}\backslash\Gamma$, then $\gamma' = \sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}}$ is running over a set of representatives for $\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$. Note that $\mathrm{id} \in \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ only if $\mathfrak{a} = \mathfrak{b}$. Hence, we continue our calculation as follows:

$$
\begin{aligned}
j_{\sigma_{\mathfrak{b}}}(z)^{-k}E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}\cdot z,s,k) &= \delta_{\mathfrak{a},\mathfrak{b}}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\mathrm{id})}j_{\mathrm{id}}(z)^{-k}\Im(z)^s + \sum_{\gamma\in\Gamma_\infty\backslash(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}-\Gamma_\infty)} \overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}j_\gamma(z)^{-k}\Im(\gamma z)^s \\
&= \delta_{\mathfrak{a},\mathfrak{b}}(\Im z)^s + \sum_{\gamma\in\Gamma_\infty\backslash(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}-\Gamma_\infty)/\Gamma_\infty}\sum_{l\in\mathbb{Z}}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma\gamma_\infty^l)}j_{\gamma\gamma_\infty^l}(z)^{-k}\Im(\gamma\gamma_\infty^l z)^s \\
&\overset{\text{Prop 2.1.8}}{=} \delta_{\mathfrak{a},\mathfrak{b}}y^s + \sum_{\gamma\in\Gamma_\infty\backslash(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}-\Gamma_\infty)/\Gamma_\infty}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}\sum_{l\in\mathbb{Z}}j_{\gamma\gamma_\infty^l}(z)^{-k}\Im(\gamma\gamma_\infty^l z)^s.
\end{aligned}
$$

Hence, the Fourier coefficients are

$$
\begin{aligned}
c_n(y) &= \int_0^1 j_{\sigma_{\mathfrak{b}}}(z)^{-k}E_{\mathfrak{a}}(\sigma_{\mathfrak{b}}\cdot z,s,k)e(-nx)dx \\
&= \int_0^1\left(\delta_{\mathfrak{a},\mathfrak{b}}y^s + \sum_{\gamma\in\Gamma_\infty\backslash(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}-\Gamma_\infty)/\Gamma_\infty}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}\sum_{l\in\mathbb{Z}}j_{\gamma\gamma_\infty^l}(z)^{-k}\Im(\gamma\gamma_\infty^l z)^s\right)e(-nx)dx \\
&= \delta_{\mathfrak{a},\mathfrak{b}}y^s\int_0^1 e(-nx)dx + \sum_{\substack{\mathrm{id}\neq\gamma\in\\ \Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty}}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}\sum_{l\in\mathbb{Z}}\int_0^1 j_\gamma(z+l)^{-k}\Im(\gamma(z+l))^s e(-nx)dx \\
&= \delta_{n=0}\delta_{\mathfrak{a},\mathfrak{b}}y^s + \sum_{\mathrm{id}\neq\gamma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty}\overline{\nu_{\mathfrak{a}\mathfrak{b}}(\gamma)}\int_{-\infty}^{\infty}j_\gamma(z)^{-k}\Im(\gamma z)^s e(-nx)dx.
\end{aligned}
$$

The integral can be rewritten as

$$
\begin{aligned}
\int_{-\infty}^{\infty}j_\gamma(z)^{-k}\Im(\gamma z)^s e(-nx)dx &= \int_{-\infty}^{\infty}\left(\frac{cz+d}{|cz+d|}\right)^{-k}\left(\frac{y}{|cz+d|^2}\right)^s e(-nx)dx \\
&\overset{\substack{x'=x+\frac{d}{c}\\ z'=x'+iy}}{=} \int_{-\infty}^{\infty}\left(\frac{cz'}{|cz'|}\right)^{-k}\left(\frac{y}{|cz'|^2}\right)^s e\left(-nx'\right)e(n\tfrac{d}{c})\,dx' \\
&= e\left(n\tfrac{d}{c}\right)c^{-2s}\mathrm{sgn}(c)^{-k}y^s\int_{-\infty}^{\infty}\left(\frac{z'}{|z'|}\right)^{-k}\frac{1}{|z'|^{2s}}e(-nx')dx' \\
&\overset{t=\frac{x'}{y}}{=} e\left(n\tfrac{d}{c}\right)c^{-2s}\mathrm{sgn}(c)^{k}y^s\int_{-\infty}^{\infty}\left(\frac{yt+iy}{|yt+iy|}\right)^{-k}\frac{1}{|yt+iy|^{2s}}e(-nyt)ydt \\
&= e\left(n\tfrac{d}{c}\right)c^{-2s}\mathrm{sgn}(c)^{k}y^{1-s}\int_{-\infty}^{\infty}\left(\frac{t+i}{|t+i|}\right)^{-k}\frac{1}{|t+i|^{2s}}e(-nty)dt \\
&= e\left(n\tfrac{d}{c}\right)c^{-2s}\mathrm{sgn}(c)^{k}y^{1-s}\int_{-\infty}^{\infty}(t+i)^{-\frac{k}{2}-s}(t-i)^{\frac{k}{2}-s}e(-nty)dt \\
&= e\left(n\tfrac{d}{c}\right)c^{-2s}\mathrm{sgn}(c)^{k}y^{1-s}e^{-ik\frac{\pi}{2}}\int_{-\infty}^{\infty}(1-it)^{-\frac{k}{2}-s}(1+it)^{\frac{k}{2}-s}e(-nty)dt.
\end{aligned}
$$

Here, we used several times that $y > 0$ as we are working with $z \in \mathbb{H}$. From formula 3.384(9) in Gradshteyn and Ryzhik [10] we know that

$$\int_{-\infty}^{\infty} (\beta + ix)^{-2\mu}(\gamma - ix)^{-2\nu} e^{-ipx} dx$$
$$= \begin{cases} 2\pi(\beta+\gamma)^{-\mu-\nu} \frac{p^{\mu+\nu-1}}{\Gamma(2\nu)} e^{\frac{\beta-\gamma}{2}p} W_{\nu-\mu, \frac{1}{2}-\nu-\mu}(\beta p + \gamma p) & \text{for } p > 0, \\ 2\pi(\beta+\gamma)^{-\mu-\nu} \frac{(-p)^{\mu+\nu-1}}{\Gamma(2\mu)} e^{\frac{\beta-\gamma}{2}p} W_{\mu-\nu, \frac{1}{2}-\nu-\mu}(-\beta p - \gamma p) & \text{for } p < 0, \end{cases}$$

where $\Re(\beta), \Re(\gamma) > 0$ and $\Re(\mu + \nu) > \frac{1}{2}$. Using this formula with $\beta = \gamma = 1$, $\mu = \frac{s}{2} - \frac{k}{4}, \nu = \frac{s}{2} + \frac{k}{4}$ and $p = 2\pi ny$ we get

$$\int_{-\infty}^{\infty} (1 + ix)^{-s+\frac{k}{2}}(1 - ix)^{-s-\frac{k}{2}} e(-nyx) dx$$
$$= \begin{cases} 2\pi 2^{-s} \frac{(2\pi ny)^{s-1}}{\Gamma(s+\frac{k}{2})} W_{\frac{k}{2}, \frac{1}{2}-s}(4\pi ny) & \text{for } n > 0 \\ 2\pi 2^{-s} \frac{(-2\pi ny)^{s-1}}{\Gamma(s-\frac{k}{2})} W_{-\frac{k}{2}, \frac{1}{2}-s}(-4\pi ny) & \text{for } n < 0 \end{cases}$$
$$= \pi^s \frac{|n|^{s-1}}{\Gamma(s + \frac{kn}{2|n|})} y^{s-1} W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y) \quad \text{for } n \neq 0,$$

where we used the symmetry of the Whittaker function, namely $W_{a,-b} = W_{a,b}$. So, for $n \neq 0$ we found the Fourier coefficients

$$c_n(y) = \sum_{\substack{\mathrm{id} \neq \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \\ \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty}} \frac{\overline{\nu_{\mathfrak{a},\mathfrak{b}}(\gamma)}}{c^{2s}} e\left(n\tfrac{d}{c}\right) \operatorname{sgn}(c)^k e^{-ik\frac{\pi}{2}} \pi^s \frac{|n|^{s-1}}{\Gamma(s + \frac{kn}{2|n|})} W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y).$$

In order to simplify our expression for the 0-th Fourier coefficient we use the following formula, which may be found on page 84-85 in Shimura [29]:

$$\int_{-\infty}^{\infty} z^{-\alpha} \bar{z}^{-\beta} dx = (2\pi)^{\alpha+\beta} i^{\beta-\alpha} \Gamma(\alpha)^{-1} \Gamma(\beta)^{-1} e^{2\pi i y} \Gamma(\alpha + \beta - 1)(4\pi y)^{1-\alpha-\beta},$$

if $\Re(\alpha), \Re(\beta) > 0$ and $\Re(\alpha + \beta) > 1$. Using this formula with $\alpha = s + \frac{k}{2}$, $\beta = s - \frac{k}{2}$ and $z = t + i$ we get

$$\int_{-\infty}^{\infty} (t + i)^{-s-\frac{k}{2}}(t - i)^{-s+\frac{k}{2}} dt = (2\pi)^{2s} i^{-k} \Gamma(s + \tfrac{k}{2})^{-1} \Gamma(s - \tfrac{k}{2})^{-1} e^{2\pi i} \Gamma(2s - 1)(4\pi)^{1-2s}$$
$$= 4^{1-s} \pi i^{-k} \frac{\Gamma(2s - 1)}{\Gamma(s + \frac{k}{2})\Gamma(s - \frac{k}{2})}.$$

Hence, the 0-th Fourier coefficient is

$$c_0(y) = \delta_{\mathfrak{a},\mathfrak{b}} y^s + \sum_{\mathrm{id} \neq \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty} \frac{\overline{\nu_{\mathfrak{a},\mathfrak{b}}(\gamma)}}{c^{2s}} \operatorname{sgn}(c)^k e^{-ik\frac{\pi}{2}} 4^{1-s} \pi \frac{\Gamma(2s-1)}{\Gamma(s+\frac{k}{2})\Gamma(s-\frac{k}{2})} y^{1-s}.$$

(c) All summands in the Fourier expansion of the Eisenstein series have meromorphic continuation to $\mathbb{C}$. Moreover, since the Whittaker function is of exponential decay, the sum converges. This proves the meromorphic continuation of the Eisenstein series to $\mathbb{C}$.

(d) By definition, the Whittaker function satisfies the following differential equation:

$$\frac{1}{16\pi^2 n^2} \frac{\partial^2}{\partial y^2} W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y) = \left(\frac{1}{4} - \frac{\frac{kn}{2|n|}}{4\pi|n|y} - \frac{\frac{1}{4} - \left(s - \frac{1}{2}\right)^2}{16\pi^2 n^2 y^2} \right) W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y),$$

i.e.

$$\frac{\partial^2}{\partial y^2} W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y) = \left(4\pi^2 n^2 - \frac{2\pi kn}{y} + \frac{s^2 - s}{y^2} \right) W_{\frac{kn}{2|n|}, s-\frac{1}{2}}(4\pi|n|y).$$

Hence, the n-th Fourier coefficient ($n \neq 0$) of the Eisenstein series satisfies

$$\frac{\partial^2}{\partial y^2} c_n(y) = \left(4\pi^2 n^2 - \frac{2\pi kn}{y} + \frac{s^2 - s}{y^2} \right) c_n(y).$$

Using the Fourier expansion of the Eisenstein series at the open cusp ∞ we have

$$\begin{aligned} \Delta_k E_{\mathfrak{a}}(z,s,k) &= \Delta_k c_0(y) + \sum_{n \in \mathbb{Z} \backslash \{0\}} \Delta_k \left(c_n(y) e(nx) \right) \\ &= -y^2 \frac{\partial^2}{\partial y^2} c_0(y) + \sum_{n \in \mathbb{Z} \backslash \{0\}} \Bigg(c_n(y) \left(-y^2 \frac{\partial^2}{\partial x^2} + iky \frac{\partial}{\partial x} \right) e(nx) \\ &\qquad\qquad - y^2 e(nx) \frac{\partial^2}{\partial y^2} c_n(y) \Bigg) \\ &= s(1-s) c_0(y) + \sum_{n \in \mathbb{Z} \backslash \{0\}} \Bigg(c_n(y) \left(4\pi^2 n^2 y^2 - 2\pi kny \right) e(nx) \\ &\qquad\qquad - y^2 e(nx) \left(4\pi^2 n^2 - \frac{2\pi kn}{y} + \frac{s^2 - s}{y^2} \right) c_n(y) \Bigg) \\ &= s(1-s) c_0(y) + \sum_{n \in \mathbb{Z} \backslash \{0\}} (s - s^2) c_n(y) e(nx) \\ &= s(1-s) \sum_{n \in \mathbb{Z}} c_n(y) e(nx) = s(1-s) E_{\mathfrak{a}}(z,s,k). \end{aligned}$$

□

Definition 2.2.3. *For $n \in \mathbb{Z}$ and odd $c \in \mathbb{Z}$ the Gauss sum is defined as*

$$G_n(c) := \sum_{m \mod c} \left(\frac{m}{c}\right) e\left(\frac{nm}{c}\right).$$

We also denote

$$H_n(c) := \epsilon_c^{-1} G_n(c).$$

Remark. For $c \in \mathbb{Z}$ odd and square-free, the first Gauss sum is $G_1(c) = \epsilon_c \sqrt{c}$.

Proposition 2.2.4. *(Proposition 2.1 in [21]) The function $H_n(c)$ satisfies the following properties:*

(a) If $n \in \mathbb{Z}$ and $c_1, c_2 \in \mathbb{Z}_{>0}$ are odd and coprime, then $H_n(c_1c_2) = H_n(c_1)H_n(c_2)$.

(b) If $(n_1, c) = 1$, then $H_{n_1n_2}(c) = \left(\frac{n_1}{c}\right) H_{n_2}(c)$.

(c) If $\alpha, \beta \in \mathbb{Z}_{\geq 0}$ and $p \neq 2$ is a prime, then

$$H_{p^\alpha}(p^\beta) = \begin{cases} \varphi(p^\beta) & \text{if } \alpha \geq \beta, \beta \text{ even}, \\ p^{\beta - \frac{1}{2}} \delta_{\beta \text{ odd}} - p^{\beta-1} \delta_{\beta \text{ even}} & \text{if } \alpha = \beta - 1, \\ 0 & \text{otherwise}, \end{cases}$$

where φ is Euler's phi function, i.e. $\varphi(n) = |\{1 \leq k \leq n \mid \gcd(n, k) = 1\}|$.

Proof. (a) By quadratic reciprocity we have

$$\left(\frac{c_1}{c_2}\right)\left(\frac{c_2}{c_1}\right) = (-1)^{\frac{c_1-1}{2}\frac{c_2-1}{2}} = \begin{cases} 1 & \text{if } c_1 \equiv 1 \mod 4 \text{ or } c_2 \equiv 1 \mod 4 \\ -1 & \text{if } c_1 \equiv c_2 \equiv 3 \mod 4 \end{cases} = \frac{\epsilon_{c_1}\epsilon_{c_2}}{\epsilon_{c_1c_2}}.$$

Using the Chinese remainder theorem, we get

$$\begin{aligned} H_n(c_1c_2) &= \epsilon_{c_1c_2}^{-1} G_n(c_1c_2) = \epsilon_{c_1}^{-1}\epsilon_{c_2}^{-1} \left(\frac{c_1}{c_2}\right)\left(\frac{c_2}{c_1}\right) \sum_{m \mod c_2c_2} \underbrace{\left(\frac{m}{c_1c_2}\right)}_{=\left(\frac{m}{c_1}\right)\left(\frac{m}{2c_2}\right)} e\left(\frac{nm}{c_1c_2}\right) \\ &\overset{m=l+kc_1}{=} \epsilon_{c_1}^{-1}\epsilon_{c_2}^{-1} \sum_{l \mod c_1} \sum_{k \mod c_2} \underbrace{\left(\frac{c_2(l+kc_1)}{c_1}\right)}_{=\left(\frac{c_2 l}{c_1}\right)} \left(\frac{c_1(l+kc_1)}{c_2}\right) e\left(\frac{n(l+kc_1)}{c_1c_2}\right) \\ &\overset{l=c_2j}{=} \epsilon_{c_1}^{-1}\epsilon_{c_2}^{-1} \sum_{j \mod c_1} \sum_{k \mod c_2} \left(\frac{j}{c_1}\right)\left(\frac{k}{c_2}\right) e\left(\frac{n(jc_2+kc_1)}{c_1c_2}\right) \\ &= \epsilon_{c_1}^{-1} \sum_{j \mod c_1} \left(\frac{j}{c_1}\right) e\left(\frac{nj}{c_1}\right) \epsilon_{c_2}^{-1} \sum_{k \mod c_2} \left(\frac{k}{c_2}\right) e\left(\frac{nk}{c_2}\right) \\ &= \epsilon_{c_1}^{-1} G_n(c_1) \epsilon_{c_2}^{-1} G_n(c_2) = H_n(c_1)H_n(c_2). \end{aligned}$$

(b) Since $(n_1, c) = 1$, whenever m runs through a set of representatives mod c, so does $n_1 m$. Hence,

$$\begin{aligned} H_{n_1 n_2}(c) &= \epsilon_c^{-1} G_{n_1 n_2}(c) = \epsilon_c^{-1} \sum_{m \mod c} \left(\frac{m}{c}\right) e\left(\frac{n_1 n_2 m}{c}\right) \\ &\overset{l = n_1 m}{=} \epsilon_c^{-1} \sum_{l \mod c} \left(\frac{\overline{n_1} l}{c}\right) e\left(\frac{n_2 l}{c}\right) = \left(\frac{\overline{n_1}}{c}\right) H_{n_2}(c) = \left(\frac{n_1}{c}\right) H_{n_2}(c). \end{aligned}$$

(c) First, note that

$$\epsilon_{p^\beta} = \begin{cases} 1 & \text{if } \beta \text{ is even,} \\ \epsilon_p & \text{if } \beta \text{ is odd.} \end{cases}$$

Indeed, with $p = 2n + 1$ and $k \in \mathbb{Z}$ we have

$$p^{2k} = (2n+1)^{2k} = \sum_{r=0}^{2k} \binom{2k}{r} (2n)^r = 1 + 4kn + 4 \sum_{r=2}^{2k} \binom{2k}{r} 2^{r-2} n^r \equiv 1 \mod 4.$$

Hence,

$$\epsilon_{p^\beta} = \begin{cases} \epsilon_{p^{2k}} = \epsilon_1 = 1 & \text{if } \beta = 2k \text{ is even,} \\ \epsilon_{p^{2k+1}} = \epsilon_{p^{2k} p} = \epsilon_p & \text{if } \beta = 2k+1 \text{ is odd.} \end{cases}$$

Now, we distinguish different cases:

- Case 1: $\beta = 0$. Then, we have $H_{p^\alpha}(1) = 1 = \varphi(1)$ as claimed.
- Case 2: $r := \alpha - \beta \geq 0$ and $\beta > 0$ even. We have

$$H_{p^\alpha}(p^\beta) = \underbrace{\epsilon_{p^\beta}^{-1}}_{=1} \sum_{m \mod p^\beta} \left(\tfrac{m}{p^\beta}\right) e\left(\frac{p^\alpha m}{p^\beta}\right) = \sum_{m \mod p^\beta} \underbrace{\left(\tfrac{m}{p}\right)^\beta}_{=1} \underbrace{e(p^r m)}_{=1} = \varphi(p^\beta).$$

- Case 3: $r := \alpha - \beta \geq 0$ and $\beta > 0$ odd. We use $\sum_{n=0}^{p-1} \left(\frac{n}{p}\right) = 0$ to get

$$\begin{aligned} H_{p^\alpha}(p^\beta) &= \epsilon_{p^\beta}^{-1} \sum_{m \mod p^\beta} \left(\tfrac{m}{p}\right)^\beta e(p^r m) = \epsilon_p^{-1} \sum_{m=0}^{p^\beta - 1} \left(\tfrac{m}{p}\right) \\ &= \epsilon_p^{-1} \sum_{a=0}^{p^{\beta-1}-1} \sum_{n=0}^{p-1} \left(\tfrac{n+ap}{p}\right) = \epsilon_p^{-1} \sum_{a=0}^{p^{\beta-1}-1} \sum_{n=0}^{p-1} \left(\tfrac{n}{p}\right) = 0. \end{aligned}$$

- Case 4: $r := \alpha - \beta = -1$ and β even. We have

$$H_{p^\alpha}(p^\beta) = \underbrace{\epsilon_{p^\beta}^{-1}}_{=1} \sum_{m \mod p^\beta} \underbrace{\left(\frac{m}{p}\right)^\beta}_{=\delta_{(m,p)=1}} e(\tfrac{m}{p}) = \sum_{m=0}^{p^\beta-1} e(\tfrac{m}{p}) - \sum_{\substack{m=0\\p|m}}^{p^\beta-1} \underbrace{e(\tfrac{m}{p})}_{=1}$$
$$= \frac{1-e(p^{\beta-1})}{1-e(p^{-1})} - |\{0 \le m \le p^\beta - 1 \mid p|m\}| = 0 - p^{\beta-1} = -p^{\beta-1}.$$

- Case 5: $r := \alpha - \beta = -1$ and β odd. We have

$$H_{p^\alpha}(p^\beta) = \epsilon_{p^\beta}^{-1} \sum_{m \mod p^\beta} \left(\frac{m}{p}\right)^\beta e(\tfrac{m}{p}) = \epsilon_p^{-1} \sum_{m=0}^{p^\beta-1} \left(\frac{m}{p}\right) e(\tfrac{m}{p})$$
$$= \epsilon_p^{-1} \sum_{a=0}^{p^{\beta-1}-1} \sum_{n=0}^{p-1} \underbrace{\left(\frac{n+ap}{p}\right)}_{=\left(\frac{n}{p}\right)} \underbrace{e(\tfrac{n+ap}{p})}_{e(\frac{n}{p})} = \epsilon_p^{-1} p^{\beta-1} G_1(p) = p^{\beta-\frac{1}{2}}.$$

- Case 6: $r := \alpha - \beta \le -2$. We have

$$|H_{p^\alpha}(p^\beta)| \le \underbrace{|\epsilon_{p^\beta}^{-1}|}_{=1} \sum_{m \mod p^\beta} \underbrace{\left|\left(\frac{m}{p}\right)\right|^\beta}_{=\delta_{p\nmid m}} e(mp^{\alpha-\beta}) = \sum_{\substack{m \mod p^\beta\\ p\nmid m}} e(mp^{\alpha-\beta})$$
$$= \sum_{m=0}^{p^\beta-1} e(mp^{\alpha-\beta}) - \sum_{n=0}^{p^{\beta-1}-1} e(np \cdot p^{\alpha-\beta}) = \frac{1-e(p^\alpha)}{1-e(p^{\alpha-\beta})} - \frac{1-e(p^\alpha)}{1-e(p^{\alpha-\beta+1})} = 0.$$

□

We will use the following notation for the remaining of the chapter: For $n \in \mathbb{Z}\backslash\{0\}$ we denote by $n_0 \in \mathbb{Z}$ square-free and $n_1 \in \mathbb{Z}_{\ge 1}$ the unique integers such that $n = n_0 n_1^2$. Also we write $\chi_m(p) := \left(\frac{m}{p}\right)$.

Definition 2.2.5. *For $a \in \mathbb{C}$, $n \in \mathbb{Z}\backslash\{0\}$ and a Dirichlet character η modulo 2^l for some $l \in \mathbb{Z}_{\ge 0}$ we define so-called correction polynomials*

$$q(a,n,\eta) := \prod_{2 \ne p | n_1} \left(1 - (1 - \chi_{n_0}(p)^2)\eta(p)^2 p^{-2a}\right) \frac{1 - \chi_{n_0}(p)^{2^r} p^{-2^r a}}{(1-p^{-2a}) \prod_{j=1}^{r-1} \left(1 + (\chi_{n_0}(p)\eta(p)p^{-s})^{2^j}\right)}$$
$$\cdot \sum_{\beta=0}^{\nu_p(n_1)} \eta(p)^{2\beta} \frac{1 - \delta_{\{\beta < \nu_p(n_1)\}} \chi_{n_0}(p)\eta(p)p^{-a}}{p^{2\beta(a-\frac{1}{2})}}$$
$$= \prod_{2 \ne p | n_1} \frac{1 - \eta(p)^{2(1-\chi_{n_0}(p)^2)} p^{-2a(1+\chi_{n_0}(p)^2)^{r-1}}}{(1-p^{-2a}) \prod_{j=1}^{r-1} \left(1 + (\chi_{n_0}(p)\eta(p)p^{-s})^{2^j}\right)} \sum_{\beta=0}^{\nu_p(n_1)} \eta(p)^{2\beta} \frac{1 - \delta_{\{\beta < \nu_p(n_1)\}} \chi_{n_0}(p)\eta(p)p^{-a}}{p^{2\beta(a-\frac{1}{2})}},$$

where $r = r(p,\eta)$ for $p \neq 2$ is the smallest non-negative integer such that $\eta(p)^{2^r} = 1$. We denote

$$L^*(a,n,\eta) := q(a,n,\eta)L^{(2)}(a,\chi_{n_0}\eta),$$

where $L^{(2)}(a,\chi_{n_0}\eta) = \prod_{p\neq 2}\sum_{\lambda=0}^{\infty}\frac{(\chi_{n_0}\eta)(p^\lambda)}{p^{a\lambda}}$ is the L-function with second factor removed.

Note that if η is a character modulo 1, 2, 4 or 8, then $r = 1$ and this coincides with the definition of the correction polynomial $q(a,n,\eta)$ in [21].

Proposition 2.2.6.

$$L^*(a,-n,\eta) = L^*(a,n,\chi_4\eta),$$

where χ_4 is the Dirichlet character modulo 4 given by $\chi_4(m) = \left(\frac{-1}{m}\right) = \begin{cases} 1 & \text{if } m \equiv 1 \mod 4 \\ -1 & \text{if } m \equiv 3 \mod 4 \end{cases}$ for m odd and positive.

Proof. Note that

$$\chi_{-n_0}(p) = \left(\frac{-n_0}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{n_0}{p}\right) = \left(\frac{-1}{p}\right)\chi_{n_0}(p) = \chi_4(p)\chi_{n_0}(p).$$

Hence, $L^{(2)}(a,\chi_{-n_0}\eta) = L^{(2)}(a,\chi_4\chi_{n_0}\eta)$. Note that $-n = -(n_0n_1^2) = (-n_0)n_1^2$ and n_0 appears in the definition of the correction polynomial only in $\chi_{n_0}(p)$. By considering the two places where $\chi_{n_0}(p)$ is not squared, we see that also $q(a,-n,\eta) = q(a,n,\chi_4\eta)$. Hence,

$$L^*(a,-n,\eta) = q(a,-n,\eta)L^{(2)}(a,\chi_{-n_0}\eta) = q(a,n,\chi_4\eta)L^{(2)}(a,\chi_{n_0}\chi_4\eta) = L^*(a,n,\chi_4\eta)$$

as claimed. □

Lemma 2.2.7. *Let χ be a Dirichlet character modulo 2^l for some $l \in \mathbb{Z}_{\geq 0}$. For every $n \in \mathbb{Z}\backslash\{0\}$ we have*

$$\sum_{\substack{c=1\\ \text{odd}}}^{\infty}\frac{\chi(c)H_n(c)}{c^{2w}} = \frac{L^*(2w-\frac{1}{2},n,\chi)}{\zeta^{(2)}(4w-1)}.$$

(For $l = 0,1,2,3$ this is Lemma 2.2 in [21].)

Proof. First, note that the summands are multiplicative, hence it reduces to show the equation for the local factors, i.e.

$$\sum_{\beta=0}^{\infty}\frac{\chi(p)^\beta H_n(p^\beta)}{p^{2\beta w}} = \frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)}q_p(2w-\tfrac{1}{2},n,\chi) \tag{2.2.1}$$

for every odd prime p. We will use the following:

Claim. For every $r \in \mathbb{Z}_{\geq 1}$ we have

$$\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)} = (1-p^{1-4w})\prod_{j=0}^{r-1}\left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)\sum_{\lambda=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^r\lambda}.$$

Proof of Claim. This can be shown by induction on $r \geq 1$. We have

$$\begin{aligned}\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)} &= \frac{\sum_{j=0}^{\infty}\frac{(\chi_{n_0}\chi)(p)^j}{p^{j(2w-\frac{1}{2})}}}{\sum_{b=0}^{\infty}\frac{1}{p^{b(4w-1)}}} = (1-p^{1-4w})\left(\sum_{\substack{j=0\\ \text{even}}}^{\infty}\frac{(\chi_{n_0}\chi)(p)^j}{p^{j(2w-\frac{1}{2})}}+\sum_{\substack{j=0\\ \text{odd}}}^{\infty}\frac{(\chi_{n_0}\chi)(p)^j}{p^{j(2w-\frac{1}{2})}}\right)\\ &= (1-p^{1-4w})\left(\sum_{\lambda=0}^{\infty}(\chi_{n_0}\chi)(p)^{2\lambda}p^{2\lambda(\frac{1}{2}-2w)}+\sum_{\lambda=0}^{\infty}(\chi_{n_0}\chi)(p)^{2\lambda+1}p^{(2\lambda+1)(\frac{1}{2}-2w)}\right)\\ &= (1-p^{1-4w})(1+\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})\sum_{\lambda=0}^{\infty}(\chi_{n_0}\chi)(p)^{2\lambda}p^{2\lambda(\frac{1}{2}-2w)},\end{aligned}$$

which proves the claim for $r = 1$. Let $r \geq 2$ and assume that the claimed equation holds for $r-1$. Since

$$\begin{aligned}\sum_{\lambda=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}\lambda} &= \left(\sum_{\substack{\lambda=0\\ \text{even}}}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}\lambda}+\sum_{\substack{\lambda=0\\ \text{odd}}}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}\lambda}\right)\\ &= \left(\sum_{j=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}\cdot 2j}+\sum_{j=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}(2j+1)}\right)\\ &= \left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}}\right)\sum_{j=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^r j},\end{aligned}$$

it follows by the induction hypothesis that

$$\begin{aligned}\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)} &= (1-p^{1-4w})\prod_{j=0}^{r-2}\left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)\sum_{\lambda=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^{r-1}\lambda}\\ &= (1-p^{1-4w})\prod_{j=0}^{r-1}\left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)\sum_{\lambda=0}^{\infty}(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^r\lambda},\end{aligned}$$

which finishes the proof of the claim.

Now, fix a prime $p \neq 2$ and write $n = n'p^\alpha$ with $(n', p) = 1$. The left hand side of equation (2.2.1) is

$$\begin{aligned}
R_p(w) &:= \sum_{\beta=0}^{\infty} \frac{\chi(p)^\beta H_n(p^\beta)}{p^{2\beta w}} \\
&\overset{\text{Prop 2.2.4}}{=} \sum_{\beta=0}^{\infty} \frac{\chi(p)^\beta}{p^{2\beta w}} \left(\frac{n'}{p}\right)^\beta H_{p^\alpha}(p^\beta) \\
&\overset{\text{Prop 2.2.4}}{=} \sum_{\substack{\beta=0 \\ \text{even}}}^{\alpha} \frac{\chi(p)^\beta}{p^{2\beta w}} \left(\frac{n'}{p}\right)^\beta \varphi(p^\beta) + \frac{\chi(p)^{\alpha+1}}{p^{2(\alpha+1)w}} \left(\frac{n'}{p}\right)^{\alpha+1} \left(p^{\alpha+\frac{1}{2}}\delta_{\alpha \text{ even}} - p^\alpha \delta_{\alpha \text{ odd}}\right) \\
&= 1 + \sum_{\substack{\beta=2 \\ \text{even}}}^{\alpha} \frac{\chi(p)^\beta}{p^{2\beta w}} (p^\beta - p^{\beta-1}) + \frac{\chi(p)^{\alpha+1}}{p^{2(\alpha+1)w}} \left(\frac{n'}{p}\right)^{\alpha+1} \left(p^{\alpha+\frac{1}{2}}\delta_{\alpha \text{ even}} - p^\alpha \delta_{\alpha \text{ odd}}\right).
\end{aligned}$$

On the other hand, the p-th factor of the correction polynomial is

$$\begin{aligned}
q^{(p)}(2w - \tfrac{1}{2}, n, \chi) = \left(1 - (1 - \chi_{n_0}(p)^2)\chi(p)^2 p^{1-4w}\right) \frac{1 - \chi_{n_0}(p)^{2^r} p^{2^r(\frac{1}{2}-2w)}}{(1 - p^{1-4w}) \prod_{j=1}^{r-1} \left(1 + (\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)} \\
\cdot \sum_{\beta=0}^{\nu_p(n_1)} \chi(p)^{2\beta} \frac{1 - \delta_{\beta < \nu_p(n_1)} \chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w}}{p^{2\beta(2w-1)}},
\end{aligned}$$

where $r = r(p, \chi)$ is the smallest non-negative integer such that $\chi(p)^{2^r} = 1$. We distinguish between α even and α odd. Also, we write $n = n_0 n_1^2$ as usual.

- α even: In this case we have $\alpha = 2\nu_p(n_1)$ and $\left(\frac{n'}{p}\right) = \left(\frac{n_0}{p}\right) = \chi_{n_0}(p) \in \{\pm 1\}$. Hence,

$$\begin{aligned}
R_p(w) &= 1 + \sum_{\substack{\beta=2\\ \text{even}}}^{\alpha} \frac{\chi(p)^\beta}{p^{2\beta w}}(p^\beta - p^{\beta-1}) + \frac{\chi(p)^{\alpha+1}}{p^{2(\alpha+1)w}}\chi_{n_0}(p)p^{\alpha+\frac{1}{2}} \\
&= 1 + \sum_{\substack{\beta=2\\ \text{even}}}^{\alpha} \chi(p)^\beta p^{\beta-2\beta w} - \sum_{\substack{\beta=2\\ \text{even}}}^{\alpha} \chi(p)^\beta p^{\beta-1-2\beta w} + \chi_{n_0}(p)\chi(p)^{\alpha+1}p^{\alpha+\frac{1}{2}-2(\alpha+1)w} \\
&= \sum_{\substack{\beta=0\\ \text{even}}}^{\alpha} \chi(p)^\beta p^{\beta-2\beta w} - \chi(p)^2 \sum_{\substack{\lambda=0\\ \text{even}}}^{\alpha-2} \chi(p)^\lambda p^{\lambda+1-2\lambda w-4w} + \chi_{n_0}(p)\chi(p)^{\alpha+1}p^{\alpha(1-2w)+\frac{1}{2}-2w} \\
&= \sum_{\substack{\lambda=0\\ \text{even}}}^{\alpha} \chi(p)^\lambda p^{\lambda(1-2w)} - \chi(p)^2 p^{1-4w} \sum_{\substack{\lambda=0\\ \text{even}}}^{\alpha-2} \chi(p)^\lambda p^{\lambda(1-2w)} + \chi_{n_0}(p)\chi(p)^{\alpha+1}p^{\frac{1}{2}-2w}p^{\alpha(1-2w)} \\
&= (1+\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})\Bigg(\sum_{\substack{\lambda=0\\ \text{even}}}^{\alpha} \chi(p)^\lambda p^{\lambda(1-2w)} \\
&\qquad\qquad - \chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w} \sum_{\substack{\lambda=0\\ \text{even}}}^{\alpha-2} \chi(p)^\lambda p^{\lambda(1-2w)}\Bigg) \\
&= (1+\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})\Bigg(\sum_{\beta=0}^{\nu_p(n_1)} \chi(p)^{2\beta} p^{2\beta(1-2w)} \\
&\qquad\qquad - \chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w} \sum_{\beta=0}^{\nu_p(n_1)-1} \chi(p)^{2\beta} p^{2\beta(1-2w)}\Bigg) \\
&= (1+\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w}) \sum_{\beta=0}^{\nu_p(n_1)} \chi(p)^{2\beta} \frac{1-\delta_{\{\beta<\nu_p(n_1)\}}\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w}}{p^{2\beta(2w-1)}}
\end{aligned}$$

and

$$\begin{aligned}
q_p(2w-\tfrac{1}{2}, n, \chi) = \frac{1-\chi_{n_0}(p)^{2^r}p^{2^r(\frac{1}{2}-2w)}}{(1-p^{1-4w})\prod_{j=1}^{r-1}\left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)} \\
\cdot \sum_{\beta=0}^{\nu_p(n_1)} \chi(p)^{2\beta} \frac{1-\delta_{\{\beta<\nu_p(n_1)\}}\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w}}{p^{2\beta(2w-1)}}.
\end{aligned}$$

Using the claim with r the smallest non-negative integer such that $\chi(p)^{2^r}=1$, we have

$$\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)}=(1-p^{1-4w})\prod_{j=0}^{r-1}\left(1+(\chi_{n_0}(p)\chi(p)p^{\frac{1}{2}-2w})^{2^j}\right)\frac{1}{1-\chi_{n_0}(p)^{2^r}p^{2^r(\frac{1}{2}-2w)}}.$$

Hence, we indeed have $R_p(w)=\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)}q_p(2w-\frac{1}{2},n,\chi)$ for every prime $p\neq 2$ such that α in $n=n'p^\alpha$ is even.

- α odd: In this case we have $\alpha=2\nu_p(n_1)+1$ and $\chi_{n_0}(p)=0$ as $p\mid n_0$. Hence,

$$\begin{aligned}
R_p(w)&=1+\sum_{\substack{\beta=2\\\text{even}}}^{\alpha}\frac{\chi(p)^\beta}{p^{2\beta w}}(p^\beta-p^{\beta-1})-\frac{\chi(p)^{\alpha+1}}{p^{2(\alpha+1)w}}p^\alpha\\
&=1+\sum_{\substack{\beta=2\\\text{even}}}^{\alpha}\chi(p)^\beta p^{\beta(1-2w)}-\sum_{\substack{\beta=2\\\text{even}}}^{\alpha}\chi(p)^\beta p^{\beta-1-2\beta w}-\chi(p)^{\alpha+1}p^{\alpha(1-2w)-2w}\\
&=\sum_{\substack{\beta=0\\\text{even}}}^{\alpha}\chi(p)^\beta p^{\beta(1-2w)}-\chi(p)^2p^{1-4w}\sum_{\substack{\lambda=0\\\text{even}}}^{\alpha-2}\chi(p)^\lambda p^{\lambda(1-2w)}-\chi(p)^{\alpha+1}p^{(\alpha-1)(1-2w)}p^{1-4w}\\
&=\sum_{\substack{\beta=0\\\text{even}}}^{\alpha}\chi(p)^\beta p^{\beta(1-2w)}-\chi(p)^2p^{1-4w}\sum_{\substack{\lambda=0\\\text{even}}}^{\alpha-1}\chi(p)^\lambda p^{\lambda(1-2w)}\\
&=\left(1-\chi(p)^2p^{1-4w}\right)\sum_{\substack{\lambda=0\\\text{even}}}^{\alpha-1}\chi(p)^\lambda p^{\lambda(1-2w)}\\
&=\left(1-\chi(p)^2p^{1-4w}\right)\sum_{\beta=0}^{\nu_p(n_1)}\chi(p)^{2\beta}p^{2\beta(1-2w)}
\end{aligned}$$

and

$$q_p(2w-\tfrac{1}{2},n,\chi)=\left(1-\chi(p)^2p^{1-4w}\right)\frac{1}{(1-p^{1-4w})}\sum_{\beta=0}^{\nu_p(n_1)}\chi(p)^{2\beta}\frac{1}{p^{2\beta(2w-1)}}.$$

Also, using again $\chi_{n_0}(p)=0$, we have

$$\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)}=(1-p^{1-4w}).$$

Hence, we indeed have $R_p(w)=\frac{L_p(2w-\frac{1}{2},\chi_{n_0}\chi)}{\zeta_p(4w-1)}q_p(2w-\frac{1}{2},n,\chi)$ for every prime $p\neq 2$ such that α in $n=n'p^\alpha$ is odd.

So, we proved equation (2.2.1) for every odd prime p which finishes the proof of the Lemma. □

Lemma 2.2.8. *(For χ' a quadratic Dirichlet character, this is equation (3-11) in [21]) Let $\Gamma = \Gamma_0(M)$ with $M = 2^l$, $l \in \mathbb{Z}_{\geq 2}$. Let χ' be a Dirichlet character modulo 2^l (not necessarily primitive) and consider the weight $\frac{1}{2}$ multiplier system $\nu(\gamma) = \chi'(d)\left(\frac{c}{d}\right)\epsilon_d^{-1}$ for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2^l)$. Then, for every $n \in \mathbb{Z}\backslash\{0\}$ we have*

$$\sum_{\mathrm{id} \neq \gamma \in \Gamma_\infty \backslash \sigma_0^{-1}\Gamma_0(M)/\Gamma_\infty} \operatorname{sgn}(c)^{\frac{1}{2}} \frac{\overline{\nu_{0\infty}(\gamma)}}{c^{2w}} e(n\tfrac{d}{c}) = \frac{i\chi'(-1)}{M^w} \frac{L^*(2w - \frac{1}{2}, n, \chi')}{\zeta^{(2)}(4w-1)}.$$

Proof. Recall (see e.g. [12]) that the double coset decomposition for $\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}$ is

$$\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} = \begin{cases} \delta_{\mathfrak{a},\mathfrak{b}}\Gamma_\infty \cup \bigcup_{c>0} \bigcup_{\substack{d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty & \text{if } -\mathrm{id} \in \Gamma, \\ \\ \delta_{\mathfrak{a},\mathfrak{b}}\Gamma_\infty \cup \bigcup_{c\neq 0} \bigcup_{\substack{d \bmod c \\ \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}}} \Gamma_\infty \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_\infty & \text{if } -\mathrm{id} \notin \Gamma. \end{cases}$$

Using this double coset decomposition for $\sigma_0^{-1}\Gamma_0(M)\sigma_\infty$ we have

$$\sum_{\mathrm{id} \neq \gamma \in \Gamma_\infty \backslash \sigma_0^{-1}\Gamma_0(M)/\Gamma_\infty} \operatorname{sgn}(c)^{\frac{1}{2}} \frac{\overline{\nu_{0\infty}(\gamma)}}{c^{2w}} e(n\tfrac{d}{c}) = \sum_{\substack{c>0 \\ d \bmod c \\ \gamma = \left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \sigma_0^{-1}\Gamma_0(M)}} \frac{\overline{\nu_{0\infty}(\gamma)}}{c^{2w}} e(n\tfrac{d}{c}).$$

By Propsition 2.1.7 we have $\nu_{0\infty}(\gamma) = \nu(\sigma_0\gamma)w_{\frac{1}{2}}(\sigma_0\gamma, \mathrm{id})w_{\frac{1}{2}}(\sigma_0, \gamma)^{-1} = \nu(\sigma_0\gamma)w_{\frac{1}{2}}(\sigma_0, \gamma)^{-1}$. Hence, we have $\overline{\nu_{0\infty}(\gamma)} = \overline{\nu(\sigma_0\gamma)}w_{\frac{1}{2}}(\sigma_0, \gamma)$. Recall that the scaling matrix at cusp ∞ is $\sigma_0 = \left(\begin{smallmatrix} 0 & -\frac{1}{\sqrt{M}} \\ \sqrt{M} & 0 \end{smallmatrix}\right)$. For $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have

$$\sigma_0\gamma = \begin{pmatrix} 0 & -\frac{1}{\sqrt{M}} \\ \sqrt{M} & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -\frac{c}{\sqrt{M}} & -\frac{d}{\sqrt{M}} \\ a\sqrt{M} & b\sqrt{M} \end{pmatrix} \overset{!}{\in} \Gamma_0(M)$$

if and only if

$$a \in \sqrt{M}\mathbb{Z} \quad , \quad b \in \frac{1}{\sqrt{M}}\mathbb{Z} \quad , \quad c, d \in \sqrt{M}\mathbb{Z}.$$

To make notation slightly easier, we will write $a' := a\sqrt{M}$, $b' = b\sqrt{M}$, $c' = \frac{c}{\sqrt{M}}$ and $d' = \frac{d}{\sqrt{M}}$ so that

$$\sigma_0\gamma = \begin{pmatrix} -c' & -d' \\ a' & b' \end{pmatrix} \in \Gamma_0(M).$$

Since $\arg(\sqrt{M}i+0)+\arg(ci+a)=\frac{\pi}{2}+\arg(ci+a)$ and $c>0$ we know by Proposition 2.1.3 that $w_{\frac{1}{2}}(\sigma_0,\gamma)=\operatorname{sgn}(a)=\operatorname{sgn}(a')$. Moreover,

$$\nu(\sigma_0\gamma)=\nu\left(\begin{pmatrix}-c' & -d'\\ a' & b'\end{pmatrix}\right)=\chi'(b')\left(\frac{a'}{b'}\right)\epsilon_{b'}^{-1}.$$

So, we have

$$\overline{\nu_{0\infty}(\gamma)}=\overline{\nu(\sigma_0\gamma)}w_{\frac{1}{2}}(\sigma_0,\gamma)=\overline{\chi'(b')}\epsilon_{b'}\operatorname{sgn}(a')\left(\frac{a'}{b'}\right).$$

Note that $c>0$ implies also $c'>0$. Since the sum is taken over $d \mod c$, we may assume that d and hence also d' are positive. In order to show that $\operatorname{sgn}(a')\left(\frac{a'}{b'}\right)=\left(\frac{d'}{c'}\right)$ we distinguish the following three cases:

1. $a'>0$: Since $a'\equiv 0 \mod M$ it follows that $a'\geq M$, so $b'c'=a'd'-1>0$ which implies that, since $c'>0$, also $b'>0$.

2. $a'<0$: Since $d'>0$ it follows that $b'c'=a'd'-1<0$. Hence, $b'<0$.

3. $a'=0$: We have $b'c'=a'd'-1=-1$, hence $c'=1$ and $b'=-1$.

So, we have

1. $a'>0$ and $b'>0$:

$$\underbrace{\operatorname{sgn}(a')}_{=1}\left(\frac{a'}{b'}\right)=\left(\frac{a'd'}{b'}\right)\left(\frac{d'}{b'}\right)=\left(\frac{1+b'c'}{b'}\right)\left(\frac{d'}{b'}\right)=\left(\frac{d'}{b'}\right)=\left(\frac{d'}{c'}\right)\left(\frac{d'}{b'c'}\right).$$

   With $d'=2^nd''$ with d'' odd and $n\geq 0$, we get $\left(\frac{d'}{b'c'}\right)=\left(\frac{2}{b'c'}\right)^n\left(\frac{d''}{b'c'}\right)$. Since $a'\equiv 0 \mod M$ and $4\mid M$ it follows that $4\mid a'$. Hence, we have

$$b'c'=a'd'-1\equiv -1 \mod 8 \quad \text{if } d' \text{ is even, i.e. if } n\geq 1.$$

   Hence, we have $\left(\frac{2}{b'c'}\right)^n=1$ for every $n\geq 0$. For the remaining term we can apply quadratic reciprocity and use that $b'c'\equiv -1 \mod 4$ to get

$$\left(\frac{d''}{b'c'}\right)=\left(\frac{b'c'}{d''}\right)(-1)^{\frac{d''-1}{2}\frac{b'c'-1}{2}}=\left(\frac{b'c'}{d''}\right)\left(\frac{-1}{d''}\right)=\left(\frac{-b'c'}{d''}\right)=\left(\frac{1-a'd'}{d''}\right)=1.$$

   Hence, in this first case we have $\operatorname{sgn}(a')\left(\frac{a'}{b'}\right)=\left(\frac{d'}{c'}\right)$.

2. $a' < 0$ and $b' < 0$: Using the definition of the Jacobi symbol, we have

$$\operatorname{sgn}(a')\left(\frac{a'}{b'}\right) = \operatorname{sgn}(a')(-1)\left(\frac{a'}{-b'}\right) = \left(\frac{a'}{-b'}\right).$$

Now, we do the exactly same steps as in case 1 to get

$$= \left(\frac{d'}{-b'}\right) = \left(\frac{d'}{c'}\right)\left(\frac{d'}{-b'c'}\right) = \left(\frac{d'}{c'}\right)\left(\frac{2}{-b'c'}\right)^n\left(\frac{d''}{-b'c'}\right) = \left(\frac{d'}{c'}\right)\left(\frac{d''}{-b'c'}\right).$$

Using quadratic reciprocity on the second term and $-b'c' = 1 - a'd' \equiv 1 \mod 4$, we get further

$$= \left(\frac{d'}{c'}\right)\left(\frac{-b'c'}{d''}\right) = \left(\frac{d'}{c'}\right)\left(\frac{1 - a'd'}{d''}\right) = \left(\frac{d'}{c'}\right).$$

3. $a' = 0$ and $b' = -1$, $c' = 1$: By definition we have $\left(\frac{a'}{b'}\right) = \left(\frac{0}{-1}\right) = 1$ and $\left(\frac{d'}{c'}\right) = \left(\frac{d'}{1}\right) = 1$. Hence, also in this case we have $\operatorname{sgn}(a')\left(\frac{a'}{b'}\right) = \left(\frac{d'}{c'}\right)$.

Moreover, we have $b'c' = a'd' - 1 \equiv -1 \mod M$, hence $b' \equiv -\overline{c'} \mod M$, hence also mod 4. Thus, we have

$$\chi'(b') = \chi'(-\overline{c'}) = \chi'(-1)\chi'(\overline{c'}) = \chi'(-1)\chi'(c')^{-1} \quad \text{and} \quad \epsilon_{b'} = \epsilon_{-\overline{c'}} = \epsilon_{-c'} = i\epsilon_{c'}^{-1}.$$

Indeed, since χ' is modulo M, we have $1 = \chi'(1) = \chi'(c'\overline{c'}) = \chi'(c')\chi'(\overline{c'})$. And, since $c'\overline{c'} \equiv 1 \mod 4$, we have $\overline{c'} \equiv 1 \mod 4$ if and only if $c' \equiv 1 \mod 4$. Hence, we have

$$\overline{\nu_{0\infty}(\gamma)} = \chi'(-1)\chi'(c')i\epsilon_{c'}^{-1}\left(\frac{d'}{c'}\right).$$

So, for $n \in \mathbb{Z}\backslash\{0\}$ we have

$$\begin{aligned}
\sum_{\mathrm{id}\neq\gamma\in\Gamma_\infty\backslash\sigma_0^{-1}\Gamma_0(M)/\Gamma_\infty} \mathrm{sgn}(c)^{\frac{1}{2}}\frac{\overline{\nu_{0\infty}(\gamma)}}{c^{2w}}e(n\tfrac{d}{c}) &= \sum_{\substack{c>0 \\ d \mod c \\ \gamma=\left(\begin{smallmatrix} * & * \\ c & d\end{smallmatrix}\right)\in\sigma_0^{-1}\Gamma_0(M)}} \frac{\overline{\nu_{0\infty}(\gamma)}}{c^{2w}}e(n\tfrac{d}{c}) \\
&= \sum_{\substack{c'>0 \\ d' \mod c' \\ \left(\begin{smallmatrix} -c' & -d' \\ a' & b'\end{smallmatrix}\right)\in\Gamma_0(M)}} \frac{\chi'(-1)\chi'(c')i\epsilon_{c'}^{-1}\left(\frac{d'}{c'}\right)}{(\sqrt{M}c')^{2w}}e(n\tfrac{d'}{c'}) \\
&= \frac{i\chi'(-1)}{M^w}\sum_{c'=1}^{\infty}\frac{\chi'(c')}{c'^{2w}}\epsilon_{c'}^{-1}\sum_{d' \mod c'}\left(\frac{d'}{c'}\right)e(n\tfrac{d'}{c'}) \\
&= \frac{i\chi'(-1)}{M^w}\sum_{\substack{c'=1 \\ \text{odd}}}^{\infty}\frac{\chi'(c')H_n(c')}{c'^{2w}} \\
&\overset{\text{Lemma 2.2.7}}{=} \frac{i\chi'(-1)}{M^w}\frac{L^*(2w-\frac{1}{2},n,\chi')}{\zeta^{(2)}(4w-1)}.
\end{aligned}$$

□

2.3 Double Dirichlet series and integral of two Whittaker functions

In this section we define the double Dirichlet series. Moreover, we are repeating some well-known facts about hypergeometric series in order to analyze the asymptotic behaviour of the integral of two specific Whittaker function. Via this integral we can relate the double Dirichlet series to a Rankin-Selberg type integral.

We recall the following well-known fact which we prove for completeness:

Proposition 2.3.1. *Let f be a Maass cusp newform of weight 0 with trivial multiplier system on $\Gamma_0(q)$ and with Laplace eigenvalue $s_0(1-s_0)$. Then, f has Fourier expansion at ∞ given by*

$$f(z) = \sum_{n\in\mathbb{Z}\backslash\{0\}} b_n W_{0,s_0-\frac{1}{2}}(4\pi|n|y)e(nx) \qquad \text{for } z = x+iy \in \mathbb{H},$$

with some coefficients $b_n \in \mathbb{C}$.

Proof. Since $\gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(q)$, the function f is 1-periodic. Hence, it has a Fourier expansion $f(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n(y)e(nx)$. Since f is an eigenfunction of the Laplacian $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ with eigenvalue $s_0(1-s_0)$ we get

$$-y^2 \sum_{n\in\mathbb{Z}\setminus\{0\}} \left(c_n''(y) - 4\pi^2 n^2 c_n(y)\right) e(nx) = s_0(1-s_0) \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n(y)e(nx) \quad \text{for all } x \in \mathbb{R}, y \in \mathbb{R}_{>0}$$

$$\Leftrightarrow \qquad -y^2(c_n''(y) - 4\pi^2 n^2 c_n(y)) = s_0(1-s_0)c_n(y) \quad \text{for all } y \in \mathbb{R}_{>0}$$

$$\Leftrightarrow \qquad \frac{\partial^2}{\partial y^2} c_n(y) = \left(4\pi^2 n^2 - \frac{s_0(1-s_0)}{y^2}\right) c_n(y) \quad \text{for all } y \in \mathbb{R}_{>0}$$

$$\Leftrightarrow \qquad \frac{\partial^2}{\partial (4\pi|n|y)^2} c_n(y) = \left(\frac{1}{4} - \frac{\frac{1}{4} - (s_0 - \frac{1}{2})^2}{(4\pi|n|y)^2}\right) c_n(y) \quad \text{for all } y \in \mathbb{R}_{>0}.$$

A solution of this differential equation is the Whittaker function $W_{0,s_0-\frac{1}{2}}(4\pi|n|y)$. Since every constant multiple is also a solution, we set $c_n(y) = b_n W_{0,s_0-\frac{1}{2}}(4\pi|n|y)$ for $b_n \in \mathbb{C}$. This gives indeed the Fourier expansion

$$f(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} b_n W_{0,s_0-\frac{1}{2}}(4\pi|n|y)e(nx).$$

□

Let ψ be a cuspidal Hecke newform of weight 0 with real Hecke eigenvalue t_n and trivial multiplier system on $\Gamma_0(2^r)$, $r \in \mathbb{Z}_{\geq 0}$, with Laplace eigenvalue $s_0(1-s_0)$. It has Fourier expansion

$$\psi(z) = \sum_{n\in\mathbb{Z}\setminus\{0\}} b_n W_{0,s_0-\frac{1}{2}}(4\pi|n|y)e(nx)$$

by Proposition 2.3.1. By an appropriate normalization we may assume that

$$b_1 = 1 \quad , \quad b_{-1} \in \{-1, 1\}$$
$$b_n = b_{\frac{n}{|n|}} |n|^{-\frac{1}{2}} t_{|n|} \in \mathbb{R} \text{ for every } n \in \mathbb{Z}.$$

Definition 2.3.2. *For $s, w \in \mathbb{C}$, a cusp form ψ as above and χ, η two Dirichlet characters modulo 2^m and 2^l, respectively, we define the double Dirichlet series to be*

$$Z_\psi(s, w, \chi, \eta) := \zeta^{(2)}(4s-1) \sum_{\substack{n=1 \\ odd}}^{\infty} \frac{\chi(n) t_n L^*(2w - \frac{1}{2}, n, \eta)}{n^{s-w+\frac{1}{2}}}.$$

The double Dirichlet series is initially defined only for $\Re(2w - \frac{1}{2})$ and $\Re(s - w + \frac{1}{2})$ big enough, where the series converges absolutely and locally uniformly. It can be shown that there is a meromorphic continuation to $\mathbb{C}^2$ as it is proved in [21] (section 2.4).

Let χ be a Dirichlet character modulo 2^m. Then, $\psi \otimes \chi$ is a weight zero cusp form on $\Gamma_0(M)$ and trivial character χ_0^M for some integer M with $M \mid \operatorname{lcm}(2^r, 2^{2m})$. (Indeed, by Theorem 7.4 in Iwaniec [12] the level is $M = \operatorname{lcm}(2^r, q^2)$, where $q \mid 2^m$ is the conductor of χ.) We may assume that $4 \mid M$. (We're only loosing some information by increasing the level.) Its Fourier expansion is

$$(\psi \otimes \chi)(z) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \chi(n) b_n W_{0, s_0 - \frac{1}{2}}(4\pi |n| y) e(nx)$$

with $b_n \in \mathbb{C}$ as above.

Definition 2.3.3. *For $s, w \in \mathbb{C}$ we define*

$$G_{\pm}(s, w) := \frac{1}{\Gamma(w \pm \frac{1}{4})} \int_0^{\infty} W_{\pm \frac{1}{4}, w - \frac{1}{2}}(2y) W_{0, s_0 - \frac{1}{2}}(2y) y^{s-1} \frac{dy}{y}.$$

We abbreviate $G_{\pm}(t) := G_{\pm}(\frac{1}{2} + it, \frac{1}{2} - it)$.

We recall some well-known properties of Gamma-functions and hypergeometric series. For the convenience of the reader and completeness we add proofs.

Lemma 2.3.4. *The Γ-function behaves asymptotically as follows:*

(i) $\Gamma(\sigma + it) = \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|} \left(\frac{|t|}{e}\right)^{it} \left(1 + O\left(\frac{1}{|t|}\right)\right)$ *as* $|t| \to \infty$.

(ii) $|\Gamma(\sigma + it)| \sim \sqrt{2\pi} |t|^{\sigma - \frac{1}{2}} e^{-\frac{\pi}{2}|t|}$ *as* $|t| \to \infty$.

Proof. Part (i) holds for $t > 0$ by Iwaniec [11], chapter B.1. For $t < 0$ it follows from the $t > 0$ case by using $\Gamma(\bar{z}) = \overline{\Gamma(z)}$. Part (ii) follows directly from part (i) by taking the absolute value on both sides. □

Definition 2.3.5. *The hypergeometric series ${}_{p+1}F_p$ is defined as*

$${}_{p+1}F_p \left({a_1, a_2, \ldots, a_{p+1} \atop b_1, b_2, \ldots, b_p} ; z \right) := \sum_{n=0}^{\infty} \frac{(a_1)_n, (a_2)_n, \ldots, (a_{p+1})_n}{(b_1)_n, (b_2)_n, \ldots, (b_p)_n} \frac{z^n}{n!},$$

where

$$(a)_0 := 1 \quad \text{and} \quad (a)_n := a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \text{ for } n \in \mathbb{Z}_{\geq 1}.$$

Lemma 2.3.6. *(Theorem 2.1.1 and Theorem 2.1.2 in Andrews, Askey, Roy [1]) The hypergeometric series ${}_{p+1}F_p\left({a_1,a_2,\dots,a_{p+1} \atop b_1,b_2,\dots,b_p}; z\right)$ converges absolutely for $|z| < 1$. Moreover, in the case $|z| = 1$ it converges absolutely if $\Re(\sum_{i=1}^{p} b_i - \sum_{j=1}^{p+1} a_j) > 0$.*

Lemma 2.3.7. *The hypergeometric series satisfies*

$$\lim_{|t|\to\infty} {}_3F_2\left({1-s_0, s_0, \frac{1}{2}-\frac{p}{4}\pm it \atop 1\pm 2it, 1}; 1\right) = {}_2F_1\left({1-s_0, s_0 \atop 1}; \frac{1}{2}\right)$$

for every $s_0 \in \mathbb{C}$ and $p \in \{\pm 1\}$.

Proof. By definition we have

$$\lim_{|t|\to\infty} {}_3F_2\left({1-s_0, s_0, \frac{1}{2}-\frac{p}{4}\pm it \atop 1\pm 2it, 1}; 1\right) = \lim_{|t|\to\infty} \sum_{n=0}^{\infty} \frac{(1-s_0)_n (s_0)_n (\frac{1}{2}-\frac{p}{4}\pm it)_n}{(1\pm 2it)_n (1)_n} \frac{1}{n!}.$$

We now show that we may use dominated convergence in order to interchange the limit and the sum. Indeed, for every $\epsilon \in (0, \frac{1}{4})$ we have

$$\begin{aligned}
\left|\frac{(1-s_0)_n (s_0)_n (\frac{1}{2}-\frac{p}{4}\pm it)_n}{(1\pm 2it)_n (1)_n} \frac{1}{n!}\right| &= \frac{1}{n!}\left|\frac{(1-s_0)_n (s_0)_n}{(1+\epsilon)_n}\right| \cdot \left|\frac{(1+\epsilon)_n (\frac{1}{2}-\frac{p}{4}\pm it)_n}{(1\pm 2it)_n (1)_n}\right| \\
&= \frac{1}{n!}\left|\frac{(1-s_0)_n (s_0)_n}{(1+\epsilon)_n}\right| \cdot \prod_{l=0}^{n-1}\left|\frac{(1+\epsilon+l)(\frac{1}{2}-\frac{p}{4}\pm it+l)}{(1\pm 2it+l)(1+l)}\right| \\
&\le \frac{1}{n!}\left|\frac{(1-s_0)_n (s_0)_n}{(1+\epsilon)_n}\right|.
\end{aligned}$$

In the last step we used that

$$\begin{aligned}
&\left|(1+\epsilon+l)\left(\tfrac{1}{2}-\tfrac{p}{4}\pm it+l\right)\right|^2 \\
&\quad= \left|l^2 + l\left(\tfrac{3}{2}+\epsilon-\tfrac{p}{4}\right) + (1+\epsilon)\left(\tfrac{1}{2}-\tfrac{p}{4}\right) \pm it(1+\epsilon+l)\right|^2 \\
&\quad= \Big(l^2 + l\underbrace{\left(\tfrac{3}{2}+\epsilon-\tfrac{p}{4}\right)}_{|\cdot|\le 2} + \underbrace{(1+\epsilon)\left(\tfrac{1}{2}-\tfrac{p}{4}\right)}_{|\cdot|\le\frac{5}{4}\cdot\frac{3}{4}\le 1}\Big)^2 + t^2\Big(\underbrace{1+\epsilon+l}_{|\cdot|\le\frac{5}{4}+l\le 2(l+1)}\Big)^2 \\
&\quad\le (l+1)^4 + 4t^2(l+1)^2 = |(l+1)^2 \pm 2ti(l+1)|^2 = |(1\pm 2it+l)(1+l)|^2
\end{aligned}$$

and hence,

$$\prod_{l=0}^{n-1}\left|\frac{1+\epsilon+l)(\frac{1}{2}-\frac{p}{4}\pm it+l)}{(1\pm 2it+l)(1+l)}\right| \le 1.$$

So, the summands are dominated by $\frac{1}{n!}\frac{(1-s_0)_n(s_0)_n}{(1+\epsilon)_n}$ and

$$\sum_{n=0}^{\infty}\frac{1}{n!}\frac{(1-s_0)_n(s_0)_n}{(1+\epsilon)_n} = {}_2F_1\left({1-s_0,s_0 \atop 1+\epsilon};1\right),$$

which converges absolutely by Lemma 2.3.6, since $\Re(1+\epsilon-(1-s_0)-s_0)=\epsilon>0$.

Using dominated convergence we can now interchange the limit and the sum. It remains to study the asymptotic behavior of each summand. Using Lemma 2.3.4, part (1) we have

$$\begin{aligned}
\frac{(1-s_0)_n(s_0)_n(\frac{1}{2}-\frac{p}{4}\pm it)_n}{(1\pm 2it)_n(1)_n}\frac{1}{n!} &= \frac{(1-s_0)_n(s_0)_n}{(1)_n n!}\frac{\Gamma(\frac{1}{2}-\frac{p}{4}\pm it+n)}{\Gamma(\frac{1}{2}-\frac{p}{4}\pm it)}\frac{\Gamma(1\pm 2it)}{\Gamma(1\pm 2it+n)}\\
&\sim \frac{(1-s_0)_n(s_0)_n}{(1)_n n!}\cdot\frac{\sqrt{2\pi}|t|^{n-\frac{p}{4}}e^{-\frac{\pi}{2}|t|}\left(\frac{|t|}{e}\right)^{\pm it}}{\sqrt{2\pi}|t|^{\frac{-p}{4}}e^{-\frac{\pi}{2}|t|}\left(\frac{|t|}{e}\right)^{\pm it}}\frac{\sqrt{2\pi}|2t|^{\frac{1}{2}}e^{-\pi|t|}\left(\frac{|2t|}{e}\right)^{\pm 2it}}{\sqrt{2\pi}|2t|^{n+\frac{1}{2}}e^{-\pi|t|}\left(\frac{|2t|}{e}\right)^{\pm 2it}}\\
&\sim \frac{(1-s_0)_n(s_0)_n}{(1)_n n!}\frac{1}{2^n} \quad \text{as } |t|\to\infty.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
\lim_{|t|\to\infty} {}_3F_2\left({1-s_0,s_0,\frac{1}{2}-\frac{p}{4}\pm it \atop 1\pm 2it,1};1\right) &= \sum_{n=0}^{\infty}\lim_{|t|\to\infty}\frac{(1-s_0)_n(s_0)_n(\frac{1}{2}-\frac{p}{4}\pm it)_n}{(1\pm 2it)_n(1)_n}\frac{1}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(1-s_0)_n(s_0)_n}{(1)_n}\frac{1}{n!}\left(\frac{1}{2}\right)^n\\
&= {}_2F_1\left({1-s_0,s_0 \atop 1};\frac{1}{2}\right).
\end{aligned}$$

Note that this is also absolutely convergent by Lemma 2.3.6. □

Theorem 2.3.8. *The functions $G_\pm(t)$ behave asymptotically as follows:*

$$G_\pm(t)\sim C\gamma(t)|t|^{-\frac{1}{2}}e^{-\pi|t|}\left((1\pm i)e^{\pi t}+(1\mp i)e^{-\pi t}\right) \quad \textit{as } |t|\to\infty,$$

where C is a constant independent of t, and $\gamma(t)$ satisfies $|\gamma(t)|=1$.

For $t \in \mathbb{R}$ and $p \in \{\pm 1\}$ we have

$$G_p(t) \overset{\text{Definition}}{=} \frac{1}{\Gamma(\frac{1}{2} - it + \frac{p}{4})} \int_0^\infty W_{0,s_0-\frac{1}{2}}(2y) W_{\frac{p}{4},-it}(2y) y^{-\frac{1}{2}+it} \frac{dy}{y}$$

$$\overset{x=2y}{=} \frac{2^{\frac{1}{2}-it}}{\Gamma(w + \frac{p}{4})} \int_0^\infty W_{0,s_0-\frac{1}{2}}(x) W_{\frac{p}{4},-it}(x) x^{-\frac{1}{2}+it} \frac{dx}{x}$$

$$= \frac{2^{\frac{1}{2}-it}}{\Gamma(\frac{1}{2} + \frac{p}{4} - it)} \Bigg[\frac{\Gamma(1-s_0)\Gamma(s_0)\Gamma(2it)}{\Gamma(\frac{1}{2} - \frac{p}{4} + it)\Gamma(1)} \cdot {}_3F_2\left({1-s_0, s_0, \frac{1}{2} - \frac{p}{4} - it \atop 1-2it, 1}; 1\right)$$
$$+ \frac{\Gamma(s_0 + 2it)\Gamma(1 - s_0 + 2it)\Gamma(-2it)}{\Gamma(\frac{1}{2} - \frac{p}{4} - it)\Gamma(1 + 2it)} \cdot {}_3F_2\left({s_0 + 2it, 1 - s_0 + 2it, \frac{1}{2} - \frac{p}{4} + it \atop 1+2it, 1+2it}; 1\right)\Bigg],$$

if $|\Re(s_0 - \frac{1}{2})| + |\Re(-it)| < \Re(-\frac{1}{2} + it) + 1$, i.e. if $\Re(s_0) \in (0, 1)$. The last equality is identity 7.611 7. from Gradshteyn and Ryzhik [10] with $\rho = -\frac{1}{2} + it$, $k = 0$, $\mu = s_0 - \frac{1}{2}$, $\lambda = \frac{p}{4}$ and $\nu = -it$. By formula (2.9) on page 1491 in Jakobson [14] with $a = \frac{1}{2} - \frac{p}{4} + it$, $b = s_0 + 2it$, $c = 1 - s_0 + 2it$ and $e = f = 1 + 2it$ we get further

$$= \frac{2^{\frac{1}{2}-it}}{\Gamma(\frac{1}{2} + \frac{p}{4} - it)} \Bigg[\frac{\Gamma(1-s_0)\Gamma(s_0)\Gamma(2it)}{\Gamma(\frac{1}{2} - \frac{p}{4} + it)} {}_3F_2\left({1-s_0, s_0, \frac{1}{2} - \frac{p}{4} - it \atop 1-2it, 1}; 1\right)$$
$$+ \frac{\Gamma(s_0 + 2it)\Gamma(1 - s_0 + 2it)\Gamma(-2it)\Gamma(\frac{1}{2} + \frac{p}{4} - it)}{\Gamma(\frac{1}{2} - \frac{p}{4} - it)\Gamma(\frac{1}{2} + \frac{p}{4} + it)} {}_3F_2\left({1-s_0, s_0, \frac{1}{2} - \frac{p}{4} + it \atop 1+2it, 1}; 1\right)\Bigg].$$

Using Lemma 2.3.7 this behaves as $|t| \to \infty$ as

$$\sim \frac{2^{\frac{1}{2}-it}}{\Gamma(\frac{1}{2} + \frac{p}{4} - it)} {}_2F_1\left({1-s_0, s_0 \atop 1}; \frac{1}{2}\right) \Bigg[\frac{\Gamma(1-s_0)\Gamma(s_0)\Gamma(2it)}{\Gamma(\frac{1}{2} - \frac{p}{4} + it)}$$
$$+ \frac{\Gamma(s_0 + 2it)\Gamma(1 - s_0 + 2it)\Gamma(-2it)\Gamma(\frac{1}{2} + \frac{p}{4} - it)}{\Gamma(\frac{1}{2} - \frac{p}{4} - it)\Gamma(\frac{1}{2} + \frac{p}{4} + it)}\Bigg]. \tag{2.3.1}$$

By Lemma 2.3.4, part (2) we have

$$\left| \frac{\Gamma(1-s_0)\Gamma(s_0)\Gamma(2it)}{\Gamma(\frac{1}{2} + \frac{p}{4} - it)\Gamma(\frac{1}{2} - \frac{p}{4} + it)} \right| \sim |\Gamma(1-s_0)\Gamma(s_0)| \cdot \frac{|2t|^{-\frac{1}{2}} e^{-\pi|t|}}{\sqrt{2\pi}|t|^{\frac{p}{4}} e^{-\frac{\pi}{2}|t|} |t|^{-\frac{p}{4}} e^{-\frac{\pi}{2}|t|}} \asymp |t|^{-\frac{1}{2}}$$

and

$$\left| \frac{\Gamma(s_0 + 2it)\Gamma(1 - s_0 + 2it)\Gamma(-2it)\Gamma(\frac{1}{2} + \frac{p}{4} - it)}{\Gamma(\frac{1}{2} + \frac{p}{4} - it)\Gamma(\frac{1}{2} - \frac{p}{4} - it)\Gamma(\frac{1}{2} + \frac{p}{4} + it)} \right|$$
$$\overset{s_0 = \sigma_0 + it_0}{\sim} \frac{\sqrt{2\pi}|2t + t_0|^{\sigma_0 - \frac{1}{2}} e^{-\frac{\pi}{2}|2t + t_0|} \cdot |2t - t_0|^{\frac{1}{2} - \sigma_0} e^{-\frac{\pi}{2}|2t - t_0|} \cdot |2t|^{-\frac{1}{2}} e^{-\pi|t|} \cdot |t|^{\frac{p}{4}} e^{-\frac{\pi}{2}|t|}}{|t|^{\frac{p}{4}} e^{-\frac{\pi}{2}|t|} \cdot |t|^{-\frac{p}{4}} e^{-\frac{\pi}{2}|t|} \cdot |t|^{\frac{p}{4}} e^{-\frac{\pi}{2}|t|}}$$
$$\asymp |t|^{-\frac{1}{2}} e^{-2\pi|t|} = o\left(|t|^{-\frac{1}{2}}\right).$$

We see that the second term of (2.3.1) can be neglected when $|t| \to \infty$. Putting everything together gives

$$\begin{aligned} G_p(t) &\sim \frac{2^{\frac{1}{2}} - it}{\Gamma(\frac{1}{2}+\frac{p}{4}-it)} {}_2F_1\left({1-s_0, s_0 \atop 1}; \tfrac{1}{2}\right) \frac{\Gamma(1-s_0)\Gamma(s_0)\Gamma(2it)}{\Gamma(\frac{1}{2}-\frac{p}{4}+it)} \\ &\sim 2^{\frac{1}{2}-it}\Gamma(1-s_0)\Gamma(s_0) \frac{{}_2F_1\left({1-s_0, s_0 \atop 1}; \frac{1}{2}\right)}{\Gamma(\frac{1}{2}+\frac{p}{4}-it)\Gamma(\frac{1}{2}-\frac{p}{4}+it)} \sqrt{2\pi}|2t|^{-\frac{1}{2}} e^{-\pi|t|} \left(\frac{2|t|}{e}\right)^{2it} \end{aligned}$$

Note that for $z \in \mathbb{C}\backslash\mathbb{Z}$ the Γ-function satisfies $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$. Using this with $z = \frac{1}{2} + \frac{p}{4} - it$ we get

$$\begin{aligned} \frac{1}{\Gamma(\frac{1}{2}+\frac{p}{4}-it)\Gamma(\frac{1}{2}-\frac{p}{4}+it)} &= \frac{\sin(\pi(\frac{1}{2}+\frac{p}{4}-it))}{\pi} = \frac{\cos(\pi(\frac{p}{4}-it))}{\pi} \\ &= \frac{1}{2\pi}\left(e^{i\pi\frac{p}{4}}e^{\pi t} + e^{-i\pi\frac{p}{4}}e^{-\pi t}\right) \\ &= \frac{1}{2\pi}\left(\frac{1+ip}{\sqrt{2}}e^{\pi t} + \frac{1-ip}{\sqrt{2}}e^{-\pi t}\right). \end{aligned}$$

Summarizing, we have

$$G_p(t) \sim C \underbrace{\left(\frac{\sqrt{2}|t|}{e}\right)^{2it}}_{=:\gamma(t)} |t|^{-\frac{1}{2}} e^{-\pi|t|} \left((1+ip)e^{\pi t} + (1-ip)e^{-\pi t}\right),$$

where $C := \frac{\Gamma(1-s_0)\Gamma(s_0)}{2\sqrt{\pi}} {}_2F_1\left({1-s_0, s_0 \atop 1}; \frac{1}{2}\right)$. This proves the Theorem.

Corollary 2.3.9. *We have*

$$G_+(t) \pm G_-(t) \asymp |t|^{-\frac{1}{2}} \quad \text{as } |t| \to \infty,$$

where $\asymp$ means asymptotically equivalent up to a constant multiple.

Proof. Using Theorem 2.3.8 we have

$$\begin{aligned} &|G_+(t) \pm G_-(t)| \\ &\quad \sim |C| \underbrace{|\gamma(t)|}_{=1} |t|^{-\frac{1}{2}} e^{-\pi|t|} \underbrace{\left|(1+i)e^{\pi t} + (1-i)e^{-\pi t} \pm (1-i)e^{\pi t} \pm (1+i)e^{-\pi t}\right|}_{\text{even function, i.e.} f(t)=f(-t), \text{ hence } f(t)=f(|t|)} \\ &\quad \sim |C||t|^{-\frac{1}{2}} e^{-\pi|t|} \left|(1+i)e^{\pi|t|} + (1-i)e^{-\pi|t|} \pm (1-i)e^{\pi|t|} \pm (1+i)e^{-\pi|t|}\right| \\ &\quad \sim |C||t|^{-\frac{1}{2}} e^{-\pi|t|} \left|(1+i)e^{\pi|t|} \pm (1-i)e^{\pi|t|}\right| \\ &\quad \sim |C||t|^{-\frac{1}{2}} |(1+i) \pm (1-i)| = 2|C||t|^{-\frac{1}{2}}. \end{aligned}$$

□

Definition 2.3.10. *Let ψ be a cuspidal Hecke newform of weight 0, trivial multiplier of $\Gamma_0(2^r)$, Laplace eigenvalue $s_0(1-s_0)$ and Fourier expansion as above. Let χ be a character modulo 2^m and let $M = 2^l$, $l \geq 2$ be the level of $\psi \otimes \chi$. Let χ' be a character modulo Ml. For $s, w \in \mathbb{C}$ we define the following Rankin-Selberg type integral:*

$$I(\psi, \chi, \chi', s, w) = \int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) E_{0,\chi'}(z, w, \tfrac{1}{2}) \overline{E_{\infty,\chi'}(z, \bar{s}, \tfrac{1}{2})} \, d\mu(z).$$

Theorem 2.3.11. *The Rankin-Selberg integral $I(\psi, \chi, \chi', s, w)$ can be expressed in terms of double Dirichlet series as follows:*

$$I(\psi, \chi, \chi', s, w) = \frac{\pi^w e^{-i\frac{\pi}{4}} i\chi'(-1)}{(2\pi)^{s-1} M^w \zeta^{(2)}(4w-1)\zeta^{(2)}(4s-1)} \Big(\chi(-1) b_{-1} Z_\psi(s, w, \chi, \chi') G_+(s, w) \\ + Z_\psi(s, w, \chi, \chi_4\chi') G_-(s, w) \Big).$$

Proof. The strategy is to plug in the definition for the Eisenstein series at ∞ and use the unfolding method. In order to unfold we use that $\psi \otimes \chi$ is of weight 0 i.e. $(\psi \otimes \chi)(\gamma \cdot z) = (\psi \otimes \chi)(z)$ for every $\gamma \in \Gamma_0(M)$ as well as the transformation property of the Eisenstein series, namely $E_{0,\chi'}(\gamma z, w, \frac{1}{2}) = \nu(\gamma) j_\gamma(z)^{\frac{1}{2}} E_{0,\chi'}(z, w, \frac{1}{2})$ for every $\gamma \in \Gamma_0(M)$ as in Proposition 2.2.2. We note that we implicitly first restrict to s, w such that all sums and integrals converge and then use meromorphic continuation to deduce the statement for all $s, w \in \mathbb{C}$. We have

$$\begin{aligned} I(\psi, \chi, \chi', s, w) &= \int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) E_{0,\chi'}(z, w, \tfrac{1}{2}) \overline{E_{\infty,\chi'}(z, \bar{s}, \tfrac{1}{2})} \, d\mu(z) \\ &= \int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) E_{0,\chi'}(z, w, \tfrac{1}{2}) \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(M)} \nu(\gamma) \overline{j_\gamma(z)^{-\frac{1}{2}}} \overline{\Im(\gamma z)^{\bar{s}}} \, d\mu(z) \\ &= \int_{\Gamma_0(M)\backslash\mathbb{H}} \sum_{\gamma \in \Gamma_\infty\backslash\Gamma_0(M)} (\psi \otimes \chi)(\gamma z) E_{0,\chi'}(\gamma z, w, \tfrac{1}{2}) \underbrace{j_\gamma(z)^{-\frac{1}{2}} \overline{j_\gamma(z)}^{-\frac{1}{2}}}_{=|j_\gamma(z)|^{-\frac{1}{2}}=1} \Im(\gamma z)^s \, d\mu(\gamma z) \\ &= \int_{\Gamma_\infty\backslash\mathbb{H}} (\psi \otimes \chi)(z) E_{0,\chi'}(z, w, \tfrac{1}{2}) \Im(z)^s \, d\mu(z). \end{aligned}$$

Plugging in the Fourier expansions for $\psi \otimes \chi$ and for $E_{0,\chi'}(z, w, \frac{1}{2})$ we further get

$$
\begin{aligned}
&= \int_0^\infty \int_0^1 \sum_{m\in\mathbb{Z}\setminus\{0\}} \chi(m) b_m W_{0,s_0-\frac{1}{2}}(4\pi|m|y) e(mx) \\
&\qquad \cdot \left(c_0(y) + \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n(y) e(nx) \right) y^{s-2}\, dx dy \\
&= \int_0^\infty \sum_{m\in\mathbb{Z}\setminus\{0\}} \chi(m) b_m W_{0,s_0-\frac{1}{2}}(4\pi|m|y) \\
&\qquad \cdot \Bigg(c_0(y) \underbrace{\int_0^1 e(mx)\, dx}_{=0} + \sum_{n\in\mathbb{Z}\setminus\{0\}} c_n(y) \underbrace{\int_0^1 e((m+n)x)\, dx}_{=\delta_{m+n=0}} \Bigg) y^{s-2}\, dy \\
&= \int_0^\infty \sum_{n\in\mathbb{Z}\setminus\{0\}} \chi(-n) b_{-n} W_{0,s_0-\frac{1}{2}}(4\pi|n|y) c_n(y) y^{s-2}\, dy.
\end{aligned}
$$

Note that since χ is a character modulo 2^m, we have $\chi(-n) = 0$ whenever n is even. Hence, the sum can be taken over odd n only. Recall from Proposition 2.2.2 that (for $n \neq 0$) the Fourier coefficient $c_n(y)$ equals

$$
\begin{aligned}
c_n(y) &= \sum_{\mathrm{id}\neq\gamma\in\Gamma_\infty\setminus\sigma_0^{-1}\Gamma_0(M)/\Gamma_\infty} \operatorname{sgn}(c)^{\frac{1}{2}} \frac{\overline{\nu_{0,\infty}(\gamma)}}{c^{2w}} e(n\tfrac{d}{c}) \pi^w e^{-i\frac{\pi}{4}} \frac{|n|^{w-1}}{\Gamma(w+\frac{n}{4|n|})} W_{\frac{n}{4|n|},w-\frac{1}{2}}(4\pi|n|y) \\
&= \frac{i\chi'(-1)}{M^w} \frac{L^*(2w-\frac{1}{2}, n, \chi')}{\zeta^{(2)}(4w-1)} \pi^w e^{-i\frac{\pi}{4}} \frac{|n|^{w-1}}{\Gamma(w+\frac{n}{4|n|})} W_{\frac{n}{4|n|},w-\frac{1}{2}}(4\pi|n|y),
\end{aligned}
$$

where we used Lemma 2.2.8.

So, we have

$$
\begin{aligned}
I(\psi,\chi,\chi',s,w) &= \frac{\pi^w e^{-i\frac{\pi}{4}} i\chi'(-1)}{M^w \zeta^{(2)}(4w-1)} \sum_{\substack{n\in\mathbb{Z}\\ \text{odd}}} \chi(-n) b_{-n} |n|^{w-1} L^*(2w-\tfrac{1}{2}, n, \chi') \\
&\qquad \cdot \underbrace{\frac{1}{\Gamma(w+\frac{n}{4|n|})} \int_0^\infty W_{0,s_0-\frac{1}{2}}(4\pi|n|y) W_{\frac{n}{4|n|},w-\frac{1}{2}}(4\pi|n|y) y^{s-2}\,dy}_{=(2\pi|n|)^{1-s} G_{\frac{n}{|n|}}(s,w)} \\
&= \frac{\pi^w e^{-i\frac{\pi}{4}} i\chi'(-1)}{M^w \zeta^{(2)}(4w-1)} \Bigg(\sum_{\substack{n\geq 1\\ \text{odd}}} \chi(-n) \underbrace{b_{-n}}_{=b_{-1} n^{-\frac{1}{2}} t_n} n^{w-1} L^*(2w-\tfrac{1}{2}, n, \chi') (2\pi n)^{1-s} G_+(s,w) \\
&\qquad + \sum_{\substack{m\geq 1\\ \text{odd}}} \chi(m) \underbrace{b_m}_{=m^{-\frac{1}{2}} t_m} m^{w-1} \underbrace{L^*(2w-\tfrac{1}{2}, -m, \chi')}_{=L^*(2w-\frac{1}{2}, m, \chi_4\chi')} (2\pi m)^{1-s} G_-(s,w) \Bigg) \\
&= \frac{\pi^w e^{-i\frac{\pi}{4}} i\chi'(-1)(2\pi)^{1-s}}{M^w \zeta^{(2)}(4w-1)\zeta^{(2)}(4s-1)} \Big(\chi(-1) b_{-1} Z_\psi(s,w,\chi,\chi') G_+(s,w) \\
&\qquad + Z_\psi(s,w,\chi,\chi_4\chi') G_-(s,w) \Big)
\end{aligned}
$$

This proves the theorem. □

2.4 Maass-Selberg relations

In this section we repeat the Maass-Selberg relations for Eisenstein series and use them to bound the Rankin-Selberg integral. Together with the asymptotic behaviour of a certain integral of Whittaker functions from the previous section, we can conclude the convexity bound for the double Dirichlet series.

Definition 2.4.1. *The truncation of an Eisenstein series $E_{\mathfrak{a}}(z,s,k)$ at height T is the function*

$$E^T_{\mathfrak{a}}(\cdot,s,k) : \mathbb{H} \to \mathbb{C}$$

defined as follows: On the fundamental domain

$$E^T_{\mathfrak{a}}(z,s,k) := \begin{cases} E_{\mathfrak{a}}(z,s,k) & \text{if } y < T \\ E_{\mathfrak{a}}(z,s,k) - c_0(y) & \text{if } y \geq T \end{cases}$$

and this can be extended to the whole upper half-plane by

$$E^T_{\mathfrak{a}}(\gamma\cdot z,s,k) = \nu(\gamma) j_\gamma(z)^k E^T_{\mathfrak{a}}(z,s,k).$$

In the following let $\mathfrak{a}_1, \mathfrak{a}_2, \dots, \mathfrak{a}_h$ be a complete set of open cusps of $\Gamma = \Gamma_0(M)$ and let $\varphi_{\mathfrak{a},\mathfrak{b}}$ denote the $(\mathfrak{a},\mathfrak{b})$th matrix-coefficient i.e. the coefficient of y^{1-s} of the 0-th Fourier coefficient of $E_{\mathfrak{a}}(z,s,k)$ at the cusp $\mathfrak{b}$. Note that by abuse of notation we identify cusps $\mathfrak{a}$ and $\mathfrak{b}$ with the numbers $1 \le i, j \le h$ satisfying $\mathfrak{a}_i = \mathfrak{a}$ and $\mathfrak{a}_j = \mathfrak{b}$, i.e. $\varphi_{\mathfrak{a},\mathfrak{b}}(w) = (\Phi(w))_{ij}$.

Recall that the scattering matrix $\Phi = (\varphi_{i,j})_{1\le i,j\le h}$ is a unitary matrix on the line $\Re(z) = \frac{1}{2}$. This implies that for $\Re(z) = \frac{1}{2}$ we have $1 = (\Phi(z)\Phi(z)^t)_{rr} = \sum_{j=1}^h |\varphi_{rj}(z)|^2$ and hence, $|\varphi_{rj}(z)| \le 1$ for every $1 \le r, j \le h$. Moreover, it is well-known that the coefficients of the scattering matrix satisfies $\left|\frac{\varphi'_{ij}(z)}{\varphi_{ij}(z)}\right| = O\left(\log|y|\right)$ for $z = \frac{1}{2} + iy$.

Theorem 2.4.2. *(Maass-Selberg relations, Lemma 11.2 in Roelcke [23]) For every weight k we have*

$$\int_{\mathcal{F}} E^T_{\mathfrak{a},\chi'}(z,s,k)\overline{E^T_{\mathfrak{b},\chi'}(z,s,k)}d\mu(z)$$
$$= \begin{cases} 2\delta_{\mathfrak{a},\mathfrak{b}}\log(T) - \sum_{j=1}^h \varphi_{\mathfrak{a},\mathfrak{a}_j}(s)\overline{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(s)} + \frac{1}{\bar{s}-s}\overline{\varphi_{\mathfrak{b},\mathfrak{a}}(s)}T^{\bar{s}-s} - \frac{1}{\bar{s}-s}\varphi_{\mathfrak{a},\mathfrak{b}}(s)T^{s-\bar{s}} & \\ \qquad\qquad \text{if } \Re(s) = \frac{1}{2}, s \ne \frac{1}{2}, \\ 2\delta_{\mathfrak{a},\mathfrak{b}}\log(T) + 2\overline{\varphi_{\mathfrak{b},\mathfrak{a}}}(\frac{1}{2})\log(T) - \sum_{j=1}^h \varphi_{\mathfrak{a},\mathfrak{a}_j}(\frac{1}{2})\overline{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(\frac{1}{2})} - \overline{\varphi'_{\mathfrak{b},\mathfrak{a}}(\frac{1}{2})} & \\ \qquad\qquad \text{if } s = \frac{1}{2}, \\ \frac{1}{\bar{s}+s-1}\delta_{\mathfrak{a},\mathfrak{b}}T^{\bar{s}+s-1} - \frac{1}{\bar{s}+s-1}\sum_{j=1}^h \varphi_{\mathfrak{a},\mathfrak{a}_j}(s)\overline{\varphi_{\mathfrak{b},\mathfrak{a}_j}(s)}T^{1-\bar{s}-s} + 2\overline{\varphi_{\mathfrak{b},\mathfrak{a}}(s)}\log(T) - \overline{\varphi'_{\mathfrak{b},\mathfrak{a}}(s)} & \\ \qquad\qquad \text{if } s \in \mathbb{R}\backslash\{\frac{1}{2}\}. \end{cases}$$

Lemma 2.4.3. *For $\Re(s) = \Re(w) = \frac{1}{2}$ the Rankin-Selberg integral satisfies the following bound:*

$$I(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}+iu) = O(\log|tu|) \quad \textit{as } |t|,|u| \to \infty.$$

Proof. First, note that any cusp form decays rapidly as the height goes to infinity. In particular, $(\psi\otimes\chi)(z) = O\left(\mathrm{height}(z)^{-A}\right)$ for every $A \in \mathbb{R}_{>2}$, where $\mathrm{height}(z) := \max_{\gamma\in\Gamma}(\Im(\gamma\cdot z))$. Let $\mathcal{F}$ denote the fundamental domain and $\mathcal{F}^T = \{z \in \mathcal{F} \mid \mathrm{height}(z) < T\}$. We then have

$$\begin{aligned} |I(\psi,\chi,\chi',s,w)| &= \left|\int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi\otimes\chi)(z)E_{0,\chi'}(z,w,\tfrac{1}{2})\overline{E_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})}d\mu(z)\right| \\ &\le \sum_{T\ge1}\left|\int_{\mathcal{F}^T} (\psi\otimes\chi)(z)E_{0,\chi'}(z,w,\tfrac{1}{2})\overline{E_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})}d\mu(z)\right| \\ &= O\left(\sum_{T\ge1} T^{-A}\left|\int_{\mathcal{F}^T} E^T_{0,\chi'}(z,w,\tfrac{1}{2})\overline{E^T_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})}d\mu(z)\right|\right). \end{aligned}$$

Using Cauchy-Schwarz inequality, we have further

$$= O\left(\sum_{T\geq 1} T^{-A}\left(\int_{\mathcal{F}^T}\left|E^T_{0,\chi'}(z,w,\tfrac{1}{2})\right|^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\mathcal{F}^T}\left|E^T_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})\right|^2 d\mu(z)\right)^{\frac{1}{2}}\right)$$
$$= O\left(\sum_{T\geq 1} T^{-A}\left(\int_{\mathcal{F}}\left|E^T_{0,\chi'}(z,w,\tfrac{1}{2})\right|^2 d\mu(z)\right)^{\frac{1}{2}}\left(\int_{\mathcal{F}}\left|E^T_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})\right|^2 d\mu(z)\right)^{\frac{1}{2}}\right).$$

By the Maass-Selberg relations (Theorem 2.4.2) we have

$$\begin{aligned}
I_{T,0} &:= \int_{\mathcal{F}}\left|E^T_{0,\chi'}(z,w,\tfrac{1}{2})\right| d\mu(z)\\
&= 2\log(T) - \sum_{j=1}^{h}\varphi_{0,\mathfrak{a}_j}(w)\overline{\varphi'_{0,\mathfrak{a}_j}(w)} + \frac{1}{\overline{w}-w}\overline{\varphi_{0,0}(w)}T^{\overline{w}-w} - \frac{1}{\overline{w}-w}\varphi_{0,0}(w)T^{w-\overline{w}}\\
&\leq 2\log(T) + \sum_{j=1}^{h}\underbrace{\left|\varphi_{0,\mathfrak{a}_j}(w)\right|^2}_{\leq 1}\underbrace{\left|\frac{\varphi'_{0,\mathfrak{a}_j}(w)}{\varphi_{0,\mathfrak{a}_j}(w)}\right|}_{=O(\log|u|)} + \frac{1}{2|u|}\underbrace{|\varphi_{0,0}(w)|}_{\leq 1} + \frac{1}{2|u|}\underbrace{|\varphi_{0,0}(w)|}_{\leq 1}\\
&= O(\log|u| + \log(T))
\end{aligned}$$

and, analogously,

$$I_{T,\infty} := \int_{\mathcal{F}}\left|E^T_{\infty,\chi'}(z,\bar{s},\tfrac{1}{2})\right| d\mu(z) = O(\log|t| + \log(T)).$$

Hence,

$$\begin{aligned}
(I_{T,0}I_{T,\infty})^{\frac{1}{2}} &= O\left(\sqrt{(\log|u| + \log(T))(\log|t| + \log(T))}\right)\\
&= O\left(\sqrt{\log(T)^2 + (\log|u| + \log|t|)\log(T) + \log|u|\log|t|}\right)\\
&= O\left(\log(T) + \sqrt{\log(|u||t|)\log(T)} + \max(\log|u|, \log|t|)\right)\\
&= O\left(\log(T) + \sqrt{\log(|u||t|)}\sqrt{\log(T)} + \log(|u||t|)\right).
\end{aligned}$$

Going back to our original integral we find

$$\begin{aligned}|I(\psi,\chi,\chi',s,w)| &= O\left(\sum_{T\geq 1} T^{-A}\left(I_{T,0}I_{T,\infty}\right)^{\frac{1}{2}}\right)\\ &= O\left(\sum_{T\geq 1} T^{-A}\left(\log(T)+\sqrt{\log|ut|}\sqrt{\log(T)}+\log|ut|\right)\right)\\ &= O\left(\sum_{T\geq 1}\frac{\log(T)}{T^A}+\sqrt{\log|ut|}\sum_{T\geq 1}\frac{\sqrt{\log(T)}}{T^A}+\log|ut|\sum_{T\geq 1}\frac{1}{T^A}\right)\\ &= O\left(\log|ut|\right) \quad \text{as } |t|,|u|\to\infty,\end{aligned}$$

where we used that the three sums converge by the integral criterion for $A > 2$. □

Definition 2.4.4. *We define*

$$\tilde{I}(\psi,\chi,\chi',s,w) := \chi(-1)b_{-1}Z_\psi(s,w,\chi,\chi')G_+(s,w) + Z_\psi(s,w,\chi,\chi_4\chi')G_-(s,w).$$

Corollary 2.4.5. *We have the bound*

$$\tilde{I}(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}+iu) = O(\log|t|\log|u|\max(\log|t|,\log|u|)) \quad \textit{as } |t|,|u|\to\infty.$$

Proof. By Lemma 2.4.3 we have $I(\psi,\chi,\chi',\frac{1}{2}+it,\frac{1}{2}+iu) = O(\log(|t||u|)) = O(\log|t| + \log|u|) = O(\max(\log|t|,\log|u|))$ as $|t|,|u|\to\infty$. Using Theorem 2.3.11 we get

$$\begin{aligned}&\tilde{I}(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}+iu)\\ &= \underbrace{2^{-\frac{1}{2}+it}M^{\frac{1}{2}+iu}\pi^{-\frac{1}{2}-iu}(-i)\chi'(-1)^{-1}e^{i\pi/4}}_{=O(1)}\underbrace{\zeta^{(2)}(1+4iu)}_{=O(\log|u|)}\underbrace{\zeta^{(2)}(1+4it)}_{=O(\log|t|)}\underbrace{I(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}+iu)}_{=O(\max(\log|t|,\log|u|))}\\ &= O\left(\log|t|\log|u|\max(\log|t|,\log|u|)\right) \quad \text{as } |t|,|u|\to\infty.\end{aligned}$$

Here, we used $\zeta(1+it) = O(\log|t|)$ (see for example Theorem 3.5 in [31]). □

Remark. The proof also shows that

$$\widetilde{I}(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}+iu) = O\left(\left|\zeta^{(2)}(1+4iu)\right|\left|\zeta^{(2)}(1+4it)\right|\max(\log|t|,\log|u|)\right).$$

Theorem 2.4.6. *The double Dirichlet series satisfy the convexity bound*

$$\begin{aligned}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') &= O\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right)\\ &= O\left(q(t)^{\frac{1}{4}+\epsilon}\right) \quad \textit{as } |t|\to\infty,\end{aligned}$$

where $q(t) := |t|^2$.

Proof. By definition we have

$$\chi(-1)b_{-1}\tilde{I}(\psi,\chi,\chi',s,w) \pm \tilde{I}(\psi,\chi,\chi_4\chi',s,w) = (Z(s,w,\chi,\chi') \pm \chi(-1)b_{-1}Z(s,w,\chi,\chi_4\chi')) \cdot (G_+(s,w) \pm G_-(s,w)) .$$

Using Corollary 2.3.9 and Corollary 2.4.5 we get

$$\begin{aligned} Z(\tfrac{1}{2}+it,&\tfrac{1}{2}-it,\chi,\chi') \pm \chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi') \\ &= \frac{\chi(-1)b_{-1}\tilde{I}(\psi,\chi,\chi',\frac{1}{2}+it,\frac{1}{2}-it) \pm \tilde{I}(\psi,\chi,\chi_4\chi',\frac{1}{2}+it,\frac{1}{2}-it)}{G_+(t)\pm G_-(t)} \\ &= O\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right) \\ &= O\left((\log|t|)^3|t|^{\frac{1}{2}}\right) = O\left(|t|^{\frac{1}{2}+\epsilon}\right) \quad \text{as } |t|\to\infty. \end{aligned}$$

It follows that

$$\begin{aligned} Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') &= \frac{1}{2}\Big(Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') + \chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi') \\ &\qquad + Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') - \chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi')\Big) \\ &= O\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right) \\ &= O\left(|t|^{\frac{1}{2}+\epsilon}\right) = O\left(q(t)^{\frac{1}{4}+\epsilon'}\right), \end{aligned}$$

□

Quantum Unique Ergodicity

3.1 Introduction

In this chapter we prove quantum unique ergodicity for half-integral weight Eisenstein series using an ergodic theoretic approach based on the works of Lindenstrauss [16], [15], Bourgain-Lindenstrauss [2] and many others. A very detailed treatment of this method in the case of normalized Laplace eigenfunctions on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ can be found in the unpublished lecture notes by Einsiedler-Ward [7].

The main tool from ergodic theory we will use is the following Theorem by Lindenstrauss [15]:

Theorem 3.1.1. *Let Γ be a congruence lattice over $\mathbb{Q}$, let $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ and let μ be a probability measure satisfying the following properties:*

[I] *μ is invariant under the geodesic flow,*

$[R]_p$ *μ is Hecke p-recurrent for a prime p, and*

[E] *the entropy of every ergodic component of μ is positive for the geodesic flow.*

Then, $\mu = m_X$ is the Haar measure on X.

The strategy is to lift a sequence of measures on $M = \Gamma\backslash\mathbb{H}$ to the space $X = \Gamma\backslash\mathrm{SL}_2(\mathbb{R})$ and to show that the three assumptions of Theorem 3.1.1 are satisfied for the weak*-limit of this lifted sequence of measures. By Theorem 3.1.1 it then follows that this weak*-limit, which is a measure on X, is the Haar measure on X. Projecting it back to the space M it follows that the weak*-limit of the original sequence of measures on M is the uniform measure.

Using this method, Lindenstrauss [15] and Soundararajan [30] have already shown the following arithmetic quantum unique ergodicity result:

Theorem 3.1.2. *Let Γ be a congruence lattice over $\mathbb{Q}$ and $M = \Gamma\backslash\mathbb{H}$. Every sequence (φ_i) of L^2-normalized Hecke Maass cusp forms with Laplace eigenvalues $\lambda_i \to \infty$ as $i \to \infty$ satisfies*

$$|\varphi_i|^2 \mathrm{dvol_M} \overset{\text{weak}^*}{\longrightarrow} \mathrm{dvol_M} \quad \textit{as } \mathrm{i} \to \infty,$$

i.e. the measures distribute uniformly.

There are two obstacles when applying the described method for weight $\frac{1}{2}$ Eisenstein series instead of Hecke Maass cusp forms. The first is that for half-integral automorphic forms we have to work on an appropriate cover of $\mathrm{SL}_2(\mathbb{R})$. The second issue is that the Eisenstein series is not in L^2, but the described method uses normalized, in particular finite measures. This second issue we will overcome by considering truncated Eisenstein series.

Definition 3.1.3. *Consider the finite-dimensional complex vector space consisting of all Eisenstein series of weight $\frac{1}{2}$ with fixed s, i.e.*

$$V_s := \left\langle E_{\mathfrak{a}}(z,s,\tfrac{1}{2}) \mid \mathfrak{a} \text{ is a cusp of } \Gamma_0(M) \right\rangle = \left\{ \sum_{\substack{\mathfrak{a} \text{ cusp} \\ \text{of } \Gamma_0(M)}} c(\mathfrak{a}) E_{\mathfrak{a}}(z,s,\tfrac{1}{2}) \mid c(\mathfrak{a}) \in \mathbb{C} \right\}.$$

As a direct consequence of Proposition 2.2.2 we get

Corollary 3.1.4. *Every $E(z,s) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(z,s,\frac{1}{2}) \in V_s$ satisfies the following properties*

(a) *$E(\gamma \cdot z, s) = \nu(\gamma) j_\gamma(z)^{\frac{1}{2}} E(z,s)$ for every $\gamma \in \Gamma_0(M)$.*

(b) *It has Fourier expansion $j_{\sigma_{\mathfrak{a}}}(z)^{-\frac{1}{2}} E(z,s) = \sum_{n\in\mathbb{Z}} \left(\sum_{\mathfrak{a}} c(\mathfrak{a}) c_{n,\mathfrak{a}}(y)\right) e(nx)$ with $c_{n,\mathfrak{a}}(y)$ as calculated in Proposition 2.2.2.*

(c) *$E(\cdot, s)$ admits a meromorphic continuation to $\mathbb{C}$.*

(d) *$\Delta_{\frac{1}{2}} E(z,s) = s(1-s)E(z,s)$.*

Proposition 3.1.5. *Let $\mathcal{T}_0 \subset \mathbb{R}$ be any sequence of real numbers tending off to infinity. For every $E(z,\frac{1}{2}+it) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(z,\frac{1}{2}+it,\frac{1}{2}) \in V_{\frac{1}{2}+it}$, where the coefficients $c(\mathfrak{a})$ are not all zero, there exists a subsequence $\mathcal{T} \subset \mathcal{T}_0$ such that*

$$\frac{1}{\log|t|} \int_{\mathcal{F}} |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) \longrightarrow c \quad \text{as } t \in \mathcal{T}, |t| \to \infty$$

for some constant $c \in \mathbb{R}_{>0}$. Here, $E^T(z,\frac{1}{2}+it) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E^T_{\mathfrak{a}}(z,\frac{1}{2}+it,\frac{1}{2})$ denotes the Eisenstein series truncated at height T as in Definition 2.4.1.

Proof. Using the Maass-Selberg relations in Theorem 2.4.2 we have

$$
\begin{aligned}
\int_{\mathcal{F}} |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) &= \int_{\mathcal{F}} \left| \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2}) \right|^2 d\mu(z) \\
&= \int_{\mathcal{F}} \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2}) \sum_{\mathfrak{b}} \overline{c(\mathfrak{b}) E_{\mathfrak{b}}^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2})} d\mu(z) \\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})} \int_{\mathcal{F}} E_{\mathfrak{a}}^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2}) \overline{E_{\mathfrak{b}}^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2})} d\mu(z) \\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})} \Bigg(2\delta_{\mathfrak{a},\mathfrak{b}} \log|t| - \sum_{j=1}^{h} \varphi_{\mathfrak{a},\mathfrak{a}_j}(\tfrac{1}{2}+it) \overline{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(\tfrac{1}{2}+it)} \\
&\qquad + \frac{1}{-2it} \overline{\varphi_{\mathfrak{b},\mathfrak{a}}(\tfrac{1}{2}+it)} |t|^{-2it} - \frac{1}{-2it} \varphi_{\mathfrak{a},\mathfrak{b}}(\tfrac{1}{2}+it) |t|^{2it} \Bigg) \qquad (3.1.1) \\
&\ll \sum_{\mathfrak{a},\mathfrak{b}} \left| c(\mathfrak{a})\overline{c(\mathfrak{b})} \right| \log|t| \ll \log|t|.
\end{aligned}
$$

Hence, there exists a constant $C \in \mathbb{R}_{\geq 0}$ such that

$$
\left| \frac{1}{\log|t|} \int_{\mathcal{F}} |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) \right| \leq C
$$

for every $t \in \mathcal{T}_0$. By the Bolzano-Weierstrass theorem, there exists a subsequence $\mathcal{T} \subset \mathcal{T}_0$ such that

$$
\frac{1}{\log|t|} \int_{\mathcal{F}} |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) \longrightarrow c \quad \text{for } t \in \mathcal{T}, |t| \to \infty
$$

for some $c \in \mathbb{R}_{\geq 0}$. Actually, by applying Stirling's approximation and the prime number theorem to (3.1.1), the terms in the bracket apart from $2\delta_{\mathfrak{a},\mathfrak{b}} \log|t|$ are of order $o(\log|t|)$. Hence, the constant c is actually non-zero. □

From now on, we will always implicitly take $t \in \mathcal{T}$ in order to avoid writing subscripts everywhere.

Definition 3.1.6. *We define measures μ_t on $M = \Gamma_0(2^l)\backslash\mathbb{H}$ by*

$$
\mu_t(f) := \frac{1}{c\log|t|} \int_{\mathcal{F}} f(z) |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) \quad \textit{for every } f \in C_c^\infty(M),
$$

where $E(z,\frac{1}{2}+it) \in V_{\frac{1}{2}+it}$ is a linear combination of Eisenstein series and $c \in \mathbb{R}_{>0}$ is the normalization constant described in Proposition 3.1.5.

In particular, any weak*-limit of μ_t is a probability measure.

Remark. Note that for every $f \in C_c^\infty(M)$ there exists some t_0 such that the support of f is contained in $M^{|t|}$ for every $|t| \geq t_0$. In particular, for $|t| \geq t_0$ we have

$$\begin{aligned}\mu_t(f) &= \frac{1}{c\log|t|}\int_{\mathcal{F}} f(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) = \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}} f(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z)\\ &= \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}} f(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) = \frac{1}{c\log|t|}\int_{\mathcal{F}} f(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z).\end{aligned}$$

Using the method described above, we can show the following:

Proposition 3.1.7. *Let (μ_t) be the sequence of measures on $M = \Gamma_0(2^l)\backslash\mathbb{H}$ with $l \geq 2$ as defined in Definition 3.1.6. Then, for every sequence $\mathcal{T}_0 \subset \mathbb{R}$ tending off to infinity, there exists a subsequence $\mathcal{T} \subset \mathcal{T}_0$ and some constant $0 \leq \lambda \leq 1$ such that*

$$\mu_t \overset{weak^*}{\longrightarrow} \lambda\,\mathrm{vol_M} \quad as \quad \mathrm{t} \in \mathcal{T}, |\mathrm{t}| \to \infty.$$

Proof. First, we show that for every sequence $\mathcal{T}_0 \subset \mathbb{R}$ tending off to infinity there exists a subsequence $\mathcal{T} \subset \mathcal{T}_0$ such that μ_t is weak*-convergent for $t \in \mathcal{T}$. By the Banach-Alaoglu theorem, the set

$$D := \{\varphi : C_c^\infty(M) \to \mathbb{R} \mid \sup_{\substack{f \in C_c^\infty(M)\\ ||f||_\infty \leq 1}} |\varphi(f)| \leq 1\}$$

is compact, i.e. every sequence has a convergent subsequence. Recall that

$$\begin{aligned}|\mu_t(f)| &= \frac{1}{c\log|t|}\int_{\mathcal{F}} f(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z)\\ &\leq ||f||_\infty \frac{1}{c\log|t|}\int_{\mathcal{F}} |E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) \longrightarrow ||f||_\infty\end{aligned}$$

for every $f \in C_c^\infty(M)$. In particular, for $|t|$ big enough, we have $|\mu_t(f)| \leq (1+\epsilon)||f||_\infty$ for every $\epsilon > 0$, hence $\frac{1}{2}\mu_t(f) \in D$. It follows that for every $\mathcal{T}_0 \subset \mathbb{R}$ there exists some $\mathcal{T} \subset \mathcal{T}_0$ such that $(\frac{1}{2}\mu_t)_{t\in\mathcal{T}}$, hence also $(\mu_t)_{t\in\mathcal{T}}$, is weak*-convergent. Call the weak*-limit of this sub subsequence $\mu_{\mathcal{T}}$. As described in the following sections we can construct a sequence of measures $(|\widetilde{\varphi}_t|^2\,\mathrm{dm}_{X^{|t|}})_{t\in\mathcal{T}}$ such that every weak*-limit point $\widetilde{\mu}$ of $|\widetilde{\varphi}_t|^2\,\mathrm{dm}_{X^{|t|}}$ is a measure on $\Gamma_0(2^l)\backslash\mathrm{SL}_2(\mathbb{R})$ that satisfies

- $\widetilde{\mu}$ is a lift of $\mu_{\mathcal{T}}$, i.e. $\int f d\widetilde{\mu} = \int f d\mu_{\mathcal{T}}$. [cf. Theorem 3.3.15, (a)]
- $\widetilde{\mu}$ is invariant under the geodesic flow. [cf. Theorem 3.3.15, (b)]

- $\widetilde{\mu}$ is Hecke p-recurrent for a prime p. [cf. Theorem 3.4.12]
- The entropy of every ergodic component of $\widetilde{\mu}$ is positive for the geodesic flow. [cf. Corollary 3.5.19]

Note that $\widetilde{\mu}$ is not necessarily a probability measure but has total mass at most 1. By Theorem 3.1.1, the measure $\widetilde{\mu}$ is a constant multiple of the Haar measure. Since it projects down to $\mu_{\mathcal{T}}$, it follows that $\mu_{\mathcal{T}}$ is a constant multiple of the uniform measure, i.e. there exists a constant $0 \leq \lambda \leq 1$ such that

$$\mu_t \overset{\text{weak*}}{\to} \mu_{\mathcal{T}} = \lambda \operatorname{vol}_M \quad \text{as} \quad t \in \mathcal{T}, |t| \to \infty$$

as claimed. $\square$

Theorem 3.1.8. *Let (μ_t) be the sequence of measures on $M = \Gamma_0(2^l)\backslash\mathbb{H}$ with $l \geq 2$ as defined in Definition 3.1.6. Then, there exists a constant $0 \leq \lambda \leq 1$ such that*

$$\mu_t \overset{weak^*}{\longrightarrow} \lambda \operatorname{vol_M} \quad as \quad |\mathrm{t}| \to \infty.$$

Proof. Assume by contradiction that μ_t does not converge in weak*-topology to a constant multiple $\lambda \operatorname{vol_M}$ of the uniform measure. Then, there exists some neighborhood U of $\lambda \operatorname{vol}_M$ such that for every $r \in \mathbb{R}$ there exists some $|t_r| \geq T$ with $\mu_{t_r} \notin U$. Consider the subsequence (μ_{t_r}). Since all of its elements lie outside of U, none of its subsubsequences can converge to $\lambda \operatorname{vol}_M$. But this gives a contradiction to Proposition 3.1.7. $\square$

3.2 The Metaplectic Group and Hecke Operators

In this section we repeat the constructions of the metaplectic group and of the Hecke operators. Standard references are e.g. Gelbart [9] and Shimura [28].

Definition 3.2.1. *The double cover of* $\mathrm{SL}_2(\mathbb{R})$ *is*

$$\{(g, \zeta) \mid g \in \mathrm{SL}_2(\mathbb{R}), \zeta \in \{\pm 1\}\},$$

with multiplication defined as

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = \Big(g_1 g_2, w_{\frac{1}{2}}(g_1, g_2)\zeta_1\zeta_2\Big).$$

Here,

$$w_{\frac{1}{2}}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) = \begin{cases} 1 & \textit{if } \arg(c_1 i + d_1) + \arg(c_2 i + a_2) \in (-\pi, \pi], \\ -1 & \textit{otherwise} \end{cases}$$

as in Proposition 2.1.3.

Remark. This defines a group, since

$$\begin{aligned}((g_1,\zeta_1)\cdot(g_2,\zeta_2))\cdot(g_3,\zeta_3) &= \Big(g_1g_2, w_{\frac{1}{2}}(g_1,g_2)\zeta_1\zeta_2\Big)\cdot(g_3,\zeta_3)\\ &= \Big(g_1g_2g_3, w_{\frac{1}{2}}(g_1g_2,g_3)w_{\frac{1}{2}}(g_1,g_2)\zeta_1\zeta_2\zeta_3\Big)\\ &= \Big(g_1g_2g_3, w_{\frac{1}{2}}(g_1,g_2g_3)w_{\frac{1}{2}}(g_2,g_3)\zeta_1\zeta_2\zeta_3\Big)\\ &= (g_1,\zeta_1)\cdot\Big(g_2g_3, w_{\frac{1}{2}}(g_2,g_3)\zeta_2\zeta_3\Big)\\ &= (g_1,\zeta_1)\cdot((g_2,\zeta_2)\cdot(g_3,\zeta_3)).\end{aligned}$$

The neutral element is $(\mathrm{id},1)$ and the inverse element is $(g,\zeta)^{-1}=(g^{-1},w_{\frac{1}{2}}(g,g^{-1})\zeta^{-1})$. For the latter we used that

$$w_{\frac{1}{2}}(g,g^{-1})=\begin{cases}-1 & \text{if } g=\left(\begin{smallmatrix} * & * \\ 0 & d\end{smallmatrix}\right) \text{ with } d>0\\ 1 & \text{else}\end{cases}=w_{\frac{1}{2}}(g^{-1},g).$$

For $\theta\in\{-\pi\le\theta<3\pi\}$ we denote by $[\theta]$ the projection onto $[-\pi,\pi)$.

Lemma 3.2.2. *The map*

$$\begin{aligned}\Phi' : [-\pi,3\pi) &\longrightarrow \{(g,\zeta)\mid g\in\mathrm{SL}_2(\mathbb{R}),\zeta\in\{\pm1\}\}\\ \theta &\mapsto (k_\theta,\zeta),\ \textit{where } \zeta=\zeta(\theta)=\begin{cases}1 & \textit{if } [\theta]=\theta\\ -1 & \textit{otherwise}\end{cases}\end{aligned}$$

is a homomorphism.

We denote $\widetilde{K}:=\Phi'([-\pi,3\pi))$.

Proof. Note that $\arg(-\sin(\theta)i+\cos(\theta))+\arg(-\sin(\theta')i+\cos(\theta'))=\arg(e^{-i\theta})+\arg(e^{-i\theta'})=-([\theta]+[\theta'])$, hence

$$\begin{aligned}w_{\frac{1}{2}}(k_\theta,k_{\theta'}) &= \begin{cases}1 & \text{if } -([\theta]+[\theta'])\in(-\pi,\pi]\\ -1 & \text{otherwise}\end{cases}=\begin{cases}1 & \text{if } [\theta]+[\theta']\in[-\pi,\pi)\\ -1 & \text{otherwise}\end{cases}\\ &= \begin{cases}1 & \text{if } [[\theta]+[\theta']]=[\theta]+[\theta'],\\ -1 & \text{otherwise.}\end{cases}\end{aligned}$$

It follows that

$$\begin{aligned}\Phi'(\theta)\Phi'(\theta') &= (k_\theta,\zeta(\theta))(k_{\theta'},\zeta(\theta'))=\Big(k_\theta k_{\theta'}, w_{\frac{1}{2}}(k_\theta,k_{\theta'})\zeta(\theta)\zeta(\theta')\Big)\\ &\overset{(*)}{=} (k_\theta k_{\theta'},\zeta(\theta+\theta'))=\Phi'(\theta+\theta'),\end{aligned}$$

where we used

$$w_{\frac{1}{2}}(k_\theta, k'_\theta)\zeta(\theta)\zeta(\theta') = \zeta(\theta + \theta'). \tag{3.2.1}$$

This identity can be checked for every case separately, i.e. distinguishing between $[\theta] = \theta$ and $[\theta] = \theta - 2\pi$, between $[\theta'] = \theta'$ and $[\theta'] = \theta' - 2\pi$, and between $[\theta + \theta'] = \theta + \theta'$ and $[\theta + \theta'] = \theta + \theta' - 2\pi$. This gives the following 8 cases:

- If $[\theta] = \theta$, $[\theta'] = \theta'$ and $[\theta + \theta'] = \theta + \theta'$, then we have $[[\theta] + [\theta']] = [\theta + \theta'] = \theta + \theta' = [\theta] + [\theta']$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = 1 \cdot 1 \cdot 1 = 1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta$, $[\theta'] = \theta'$ and $[\theta + \theta'] = \theta + \theta' - 2\pi$, then we have $[[\theta] + [\theta']] = [\theta + \theta'] = \theta + \theta' - 2\pi = [\theta] + [\theta'] - 2\pi$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = -1 \cdot 1 \cdot 1 = -1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta$, $[\theta'] = \theta' - 2\pi$ and $[\theta + \theta'] = \theta + \theta'$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 2\pi] = [\theta + \theta'] = \theta + \theta' = [\theta] + [\theta'] + 2\pi$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = (-1) \cdot 1 \cdot (-1) = 1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta$, $[\theta'] = \theta' - 2\pi$ and $[\theta + \theta'] = \theta + \theta' - 2\pi$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 2\pi] = [\theta + \theta'] = \theta + \theta' - 2\pi = [\theta] + [\theta']$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = 1 \cdot 1 \cdot (-1) = -1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta - 2\pi$, $[\theta'] = \theta'$ and $[\theta + \theta'] = \theta + \theta'$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 2\pi] = [\theta + \theta'] = \theta + \theta' = [\theta] + [\theta'] + 2\pi$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = (-1) \cdot (-1) \cdot 1 = 1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta - 2\pi$, $[\theta'] = \theta'$ and $[\theta + \theta'] = \theta + \theta' - 2\pi$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 2\pi] = [\theta + \theta'] = \theta + \theta' - 2\pi = [\theta] + [\theta']$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = 1 \cdot (-1) \cdot 1 = -1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta - 2\pi$, $[\theta'] = \theta' - 2\pi$ and $[\theta + \theta'] = \theta + \theta'$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 4\pi] = [\theta + \theta'] = \theta + \theta' = [\theta] + [\theta']$. Hence,
$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = 1 \cdot (-1) \cdot (-1) = 1 = \zeta(\theta + \theta').$$

- If $[\theta] = \theta - 2\pi$, $[\theta'] = \theta' - 2\pi$ and $[\theta + \theta'] = \theta + \theta' - 2\pi$, then we have $[[\theta] + [\theta']] = [\theta + \theta' - 4\pi] = [\theta + \theta'] = \theta + \theta' - 2\pi = [\theta] + [\theta'] + 2\pi$. Hence,

$$w_{\frac{1}{2}}(k_\theta, k_{\theta'})\zeta(\theta)\zeta(\theta') = (-1) \cdot (-1) \cdot (-1) = -1 = \zeta(\theta + \theta').$$

□

In order to speak about left-invariance under $\Gamma = \Gamma_0(M)$ for $M = 2^l$, $l \geq 2$, we have to embed $\Gamma_0(M)$ appropriately. For this purpose it is not sufficient to consider the double cover, but instead we need 2^{l-1} copies of the double cover. This leads to the following definition of the metaplectic group:

Definition 3.2.3. *The metaplectic group of* $\mathrm{SL}_2(\mathbb{R})$ *is*

$$\widetilde{\mathrm{SL}_2(\mathbb{R})} := \{(g, \zeta) \mid g \in \mathrm{SL}_2(\mathbb{R}), \zeta \in \mu_M\},$$

where $\mu_M = \mu_{2^l}$ *is the set of* 2^l*-th roots of unity. The multiplication is defined as*

$$(g_1, \zeta_1) \cdot (g_2, \zeta_2) = \left(g_1 g_2, w_{\frac{1}{2}}(g_1, g_2)\zeta_1\zeta_2\right).$$

Remark. Note that the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ says that every $g \in \mathrm{SL}_2(\mathbb{R})$ can be written as $g = n_x a_y k_\theta$ for some $n_x \in N$, $a_y \in A$ and $k_\theta \in K$. Here,

$$K = \mathrm{SO}_2(\mathbb{R}) = \left\{ k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \;\middle|\; \theta \in \mathbb{R}/2\pi\mathbb{Z} \right\}$$

and

$$N = \left\{ n_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\} \hookrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})} \quad , \quad n_x \mapsto (n_x, 1)$$
$$A = \left\{ a_y = \begin{pmatrix} y^{\frac{1}{2}} & 0 \\ 0 & y^{-\frac{1}{2}} \end{pmatrix} \;\middle|\; y \in \mathbb{R}_{>0} \right\} \hookrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})} \quad , \quad a_y \mapsto (a_y, 1)$$

are homomorphisms. Hence, every element $(g, \zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ can be written as

$$(g, \zeta) = (n_x a_y k_\theta, \zeta) = (n_x a_y, 1)(k_\theta, \zeta(\theta))(\mathrm{id}, \zeta') = (n_x a_y, \zeta')(k_\theta, \zeta(\theta))$$

for some $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$ and $\theta \in [-\pi, 3\pi)$. Here, $\zeta' = \zeta\zeta(\theta)$.

Definition 3.2.4. *We call a function* $\varphi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{C}$ *genuine if* $\varphi(g, \zeta) = \zeta^{-1}\varphi(g, 1)$ *for every* $(g, \zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Lemma 3.2.5. *Let $M = 2^l$ with $l \geq 2$ and let ν be the multiplier system on $\Gamma_0(M)$ defined in* (2.1.3)*. The function*

$$\Phi : \Gamma_0(M) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \mod M \right\} \longrightarrow \widetilde{\mathrm{SL}_2(\mathbb{R})}$$
$$\gamma \longmapsto (\gamma, \nu(\gamma))$$

is a homomorphism.

Proof. Since $\nu(\gamma) = \eta(d) \left(\frac{c}{d}\right) \epsilon_d^{-1}$ is a multiplier system for $\Gamma_0(M)$ the function is well-defined and we have

$$\Phi(\gamma\gamma') = (\gamma\gamma', \nu(\gamma\gamma')) = \Big(\gamma\gamma', w_{\frac{1}{2}}(\gamma, \gamma')\nu(\gamma)\nu(\gamma')\Big) = (\gamma, \nu(\gamma))(\gamma', \nu(\gamma')) = \Phi(\gamma)\Phi(\gamma')$$

for all $\gamma, \gamma' \in \Gamma_0(M)$. □

Definition 3.2.6. *We denote the image of this homomorphism by*

$$\Gamma_0(M)^* := \{(\gamma, \nu(\gamma)) \in \widetilde{\mathrm{SL}_2(\mathbb{R})} \mid \gamma \in \Gamma_0(M)\}.$$

Definition 3.2.7. *We define the weight $\frac{1}{2}$ slash operator on automorphic functions $\{f : \mathbb{H} \to \mathbb{C} \mid f(\gamma \cdot z) = \nu(\gamma) j_\gamma(z)^{\frac{1}{2}} f(z)$ for $\gamma \in \Gamma_0(M)\}$ by*

$$(f \mid [(g, \zeta)])(z) := f(g \cdot z)\zeta^{-1} j_g(z)^{-\frac{1}{2}}$$

for $(g, \zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$.

Proposition 3.2.8. *The weight $\frac{1}{2}$ slash operator satisfies*

(i) $f \mid [(g_1, \zeta_1)(g_2, \zeta_2)] = (f \mid [(g_1, \zeta_1)]) \mid [(g_2, \zeta_2)]$ *for all* $(g_1, \zeta_1), (g_2, \zeta_2) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$

(ii) $f \mid [(\gamma, \nu(\gamma))(g, \zeta)] = f \mid [(g, \zeta)]$ *for every $\gamma \in \Gamma_0(M)$, i.e. the slash operator is left $\Gamma_0(M)^*$-invariant.*

Proof. (i) We have

$$\begin{aligned}
((f \mid [(g_1, \zeta_1)]) \mid [(g_2, \zeta_2)])(z) &= (f \mid [(g_1, \zeta_1)])(g_2 \cdot z)\zeta_2^{-1} j_{g_2}(z)^{-\frac{1}{2}} \\
&= f(g_1 \cdot (g_2 \cdot z))\zeta_2^{-1} j_{g_2}(z)^{-\frac{1}{2}} \zeta_1^{-1} j_{g_1}(g_2 \cdot z)^{-\frac{1}{2}} \\
&= f((g_1 g_2) \cdot z)\zeta_1^{-1}\zeta_2^{-1} w_{\frac{1}{2}}(g_1, g_2)^{-1} j_{g_1 g_2}(z)^{-\frac{1}{2}} \\
&= \Big(f \mid [(g_1 g_2, w_{\frac{1}{2}}(g_1, g_2)\zeta_1\zeta_2)]\Big)(z) \\
&= (f \mid [(g_1, \zeta_1)(g_2, \zeta_2)])(z).
\end{aligned}$$

(ii) We have

$$\begin{aligned}(f \mid [(\gamma,\nu(\gamma))(g,\zeta)])\,(z) &= \left(f \mid [(\gamma g, w_{\frac{1}{2}}(\gamma,g)\nu(\gamma)\zeta)]\right)(z)\\ &= f(\gamma g\cdot z)w_{\frac{1}{2}}(\gamma,g)^{-1}\nu(\gamma)^{-1}\zeta^{-1}j_{\gamma g}(z)^{-\frac{1}{2}}\\ &= \nu(\gamma)j_\gamma(g\cdot z)^{\frac{1}{2}}f(g\cdot z)w_{\frac{1}{2}}(\gamma,g)^{-1}\nu(\gamma)^{-1}\zeta^{-1}j_{\gamma g}(z)^{-\frac{1}{2}}\\ &= f(g\cdot z)\zeta^{-1}j_g(z)^{-\frac{1}{2}} = (f \mid [(g,\zeta)])\,(z).\end{aligned}$$

□

For the definition of half-integral Hecke operators, we follow Shimura [28]. While Shimura is considering a cover of $\mathrm{GL}_2(\mathbb{R})$, we are however only using the metaplectic cover of $\mathrm{SL}_2(\mathbb{R})$ introduced above. This is sufficient for our purpose. Let $M = 2^l$, $l \geq 2$, and let p be an odd prime number. For a fixed integer $m \geq 0$ let $\alpha = \begin{pmatrix} p^{-\frac{m}{2}} & 0 \\ 0 & p^{\frac{m}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and let $\xi = \left(\begin{pmatrix} p^{-\frac{m}{2}} & 0 \\ 0 & p^{\frac{m}{2}} \end{pmatrix}, \zeta\right) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ be an element projecting to α. Consider the homomorphism

$$t : \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha \to \{\pm 1\} \quad , \quad t(\gamma) := \nu(\gamma)\nu(\alpha\gamma\alpha^{-1})^{-1}.$$

Remark. (i) Note that for any $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R})$ we have $\alpha g = \begin{pmatrix} p^{-\frac{m}{2}}a & p^{-\frac{m}{2}}b \\ p^{\frac{m}{2}}c & p^{\frac{m}{2}}d \end{pmatrix}$ and $g\alpha = \begin{pmatrix} p^{-\frac{m}{2}}a & p^{\frac{m}{2}}b \\ p^{-\frac{m}{2}}c & p^{\frac{m}{2}}d \end{pmatrix}$. Since $\arg(p^{\frac{m}{2}}ci + p^{\frac{m}{2}}d) = \arg(c_i + d)$ and $\arg(p^{-\frac{m}{2}}ci + p^{-\frac{m}{2}}a) = \arg(ci + a)$ it follows by Proposition 2.1.3 that $w_{\frac{1}{2}}(\alpha g, g') = w_{\frac{1}{2}}(g, g') = w_{\frac{1}{2}}(g', g\alpha)$ for every $g' \in \mathrm{SL}_2(\mathbb{R})$. Similarly, $w_{\frac{1}{2}}(\alpha^{-1}g, g') = w_{\frac{1}{2}}(g, g') = w_{\frac{1}{2}}(g', g\alpha^{-1})$. So,

$$\begin{aligned} t(\gamma_1\gamma_2) &= \nu(\gamma_1\gamma_2)\nu(\alpha\gamma_1\gamma_2\alpha^{-1})^{-1} = \nu(\gamma_1\gamma_2)\nu(\alpha\gamma_1\alpha^{-1}\alpha\gamma_2\alpha^{-1})^{-1}\\ &= w_{\frac{1}{2}}(\gamma_1,\gamma_2)\nu(\gamma_1)\nu(\gamma_2)\underbrace{w_{\frac{1}{2}}(\alpha\gamma_1\alpha^{-1},\alpha\gamma_2\alpha^{-1})^{-1}}_{=w_{\frac{1}{2}}(\gamma_1,\gamma_2)^{-1}}\nu(\alpha\gamma_1\alpha^{-1})^{-1}\nu(\alpha\gamma_2\alpha^{-1})^{-1}\\ &= t(\gamma_1)t(\gamma_2)\end{aligned}$$

and t is indeed a homomorphism.

(ii) With $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha$ we have

$$\alpha\gamma\alpha^{-1} = \begin{pmatrix} p^{-\frac{m}{2}} & 0 \\ 0 & p^{\frac{m}{2}} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p^{\frac{m}{2}} & 0 \\ 0 & p^{-\frac{m}{2}} \end{pmatrix} = \begin{pmatrix} a & p^{-m}b \\ p^m c & d \end{pmatrix}.$$

Hence,

$$t(\gamma) = \nu(\gamma)\nu(\alpha\gamma\alpha^{-1})^{-1} = \left(\frac{c}{d}\right)\epsilon_d^{-1}\left(\left(\frac{p^m c}{d}\right)\epsilon_d^{-1}\right)^{-1} = \left(\frac{p}{d}\right)^m = \begin{cases} 1 & \text{if } m \text{ even,} \\ \left(\frac{p}{d}\right) & \text{if } m \text{ odd.}\end{cases}$$

Lemma 3.2.9. *For every $\gamma \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha$ we have*

$$\Phi(\alpha\gamma\alpha^{-1}) = \xi\Phi(\gamma)\xi^{-1}(\mathrm{id}, t(\gamma)).$$

Proof. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha$, i.e. with $\alpha\gamma\alpha^{-1} \in \Gamma_0(M)$ we have

$$\begin{aligned}
\Phi(\alpha\gamma\alpha^{-1}) &= \Phi\left(\left(\begin{pmatrix} p^{-\frac{m}{2}} & 0 \\ 0 & p^{\frac{m}{2}} \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} p^{\frac{m}{2}} & 0 \\ 0 & p^{-\frac{m}{2}} \end{pmatrix}\right)\right) = \Phi\left(\left(\begin{pmatrix} a & p^{-m}b \\ p^m c & d \end{pmatrix}\right)\right) \\
&= \left(\alpha\gamma\alpha^{-1}, \left(\frac{p^m c}{d}\right)\epsilon_d^{-1}\right) = \left(\alpha\gamma\alpha^{-1}, \left(\frac{c}{d}\right)\epsilon_d^{-1}t(\gamma)\right) = \left(\alpha\gamma\alpha^{-1}, \nu(\gamma)\right)(\mathrm{id}, t(\gamma)) \\
&= (\alpha, \zeta)(\gamma, \nu(\gamma))\left(\alpha^{-1}, \zeta^{-1}\right)(\mathrm{id}, t(\gamma)) = \xi\Phi(\gamma)\xi^{-1}(\mathrm{id}, t(\gamma)).
\end{aligned}$$

Here, we used continuously that $w_{\frac{1}{2}}(\alpha, g) = w_{\frac{1}{2}}(g, \alpha) = 1$ for every $g \in \mathrm{SL}_2(\mathbb{R})$. □

Lemma 3.2.10.

$$\Gamma_0(M)^* \cap \xi^{-1}\Gamma_0(M)^*\xi = \Phi(\mathrm{Ker}(t)).$$

Proof. Using Proposition 2.1.3 we have $w_{\frac{1}{2}}(\alpha^{-1}, \gamma') = 1 = w_{\frac{1}{2}}(\alpha^{-1}\gamma', \alpha) = 1$ for every $\gamma' \in \Gamma_0(M)$. Hence, we have

$$\begin{aligned}
\Gamma_0(M)^* \cap \xi^{-1}\Gamma_0(M)^*\xi &= \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M)\} \cap \{\xi^{-1}(\gamma', \nu(\gamma'))\xi \mid \gamma' \in \Gamma_0(M)\} \\
&= \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M)\} \cap \{(\alpha^{-1}\gamma'\alpha, \nu(\gamma')) \mid \gamma' \in \Gamma_0(M)\} \\
&= \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M), \exists \gamma' \in \Gamma_0(M) \text{ s.t. } \gamma = \alpha^{-1}\gamma'\alpha \text{ and } \nu(\gamma) = \nu(\gamma')\} \\
&= \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha \text{ and } \nu(\gamma) = \nu(\alpha\gamma\alpha^{-1})\} \\
&= \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha \text{ and } t(\gamma) = 1\} \\
&= \{(\gamma, \nu(\gamma)) \mid \gamma \in \mathrm{Ker}(\gamma)\} = \Phi(\mathrm{Ker}(t)).
\end{aligned}$$

□

Proposition 3.2.11. *(i) If m is odd, then $f \mid [\Gamma_0(M)^*\xi\Gamma_0(M)^*] \equiv 0$ for every automorphic form f.*

(ii) If $m = 2n$ is even, then the projection map is an isomorphism

$$P : \Gamma_0(M)^*\xi\Gamma_0(M)^* \longrightarrow \Gamma_0(M)\alpha\Gamma_0(M).$$

In particular, $\{\xi_i\} \subset \widetilde{\mathrm{SL}_2(\mathbb{R})}$ satisfies $\Gamma_0(M)^*\xi\Gamma_0(M)^* = \coprod_i \Gamma_0(M)^*\xi_i$ if and only if it satisfies $\Gamma_0(M)\alpha\Gamma_0(M) = \coprod_i \Gamma_0(M)P(\xi_i)$.

Proof. (i) Since $\alpha^{-1}\Gamma_0(M)\alpha < \Gamma_0(M)$ is a normal subgroup, we can write

$$\Gamma_0(M) = \coprod_j \left(\Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha\right)\gamma_j$$

for a set of representatives $\{\gamma_j\} \subset \Gamma_0(M)$.

Moreover, since m is odd, we can write

$$\Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha = \mathrm{Ker}(t) \amalg \mathrm{Ker}(t)\gamma,$$

where $\gamma \in \Gamma_0(M) \cap \alpha^{-1}\Gamma_0(M)\alpha$ is a fixed element such that $t(\gamma) = -1$. Hence, we can write

$$\begin{aligned}\Gamma_0(M)^* = \Phi(\Gamma_0(M)) &= \Phi(\mathrm{Ker}(t))\left((\mathrm{id},1) \amalg \Phi(\gamma)\right)\coprod_j \Phi(\gamma_j)\\ &= \left(\Gamma_0(M)^* \cap \xi^{-1}\Gamma_0(M)^*\xi\right)\left((\mathrm{id},1) \amalg \Phi(\gamma)\right)\coprod_j \Phi(\gamma_j),\end{aligned}$$

where we used Lemma 3.2.10. It follows that

$$\begin{aligned}\Gamma_0(M)^*\xi\Gamma_0(M)^* &= \Gamma_0(M)^*\xi\left(\Gamma_0(M)^* \cap \xi^{-1}\Gamma_0(M)^*\xi\right)\left((\mathrm{id},1) \amalg \Phi(\gamma)\right)\coprod_j \Phi(\gamma_j)\\ &= \Gamma_0(M)^*\xi\left((\mathrm{id},1) \amalg \Phi(\gamma)\right)\coprod_j \Phi(\gamma_j)\end{aligned}$$

and

$$\begin{aligned}f \,\Big|\, [\Gamma_0(M)^*\xi\Gamma_0(M)^*] &= \sum_j \left(f \mid [\xi\Phi(\gamma_j)] + f \mid [\xi\Phi(\gamma)\Phi(\gamma_j)]\right)\\ &= \sum_j \left((f \mid [\xi]) \mid [\Phi(\gamma_j)] + (f \mid [\xi\Phi(\gamma)]) \mid [\Phi(\gamma_j)]\right).\end{aligned}$$

By Lemma 3.2.9 we have $\xi\Phi(\gamma) = \Phi(\alpha\gamma\alpha^{-1})\xi(\mathrm{id}, t(\gamma)^{-1}) = \Phi(\alpha\gamma\alpha^{-1})\xi(\mathrm{id}, -1)$. Note that

$$\begin{aligned}\left(f \mid [\xi\Phi(\gamma)]\right)(z) = \left(f \mid [\xi(\mathrm{id},-1)]\right)(z) = \left(f \mid [(\alpha,-\zeta)]\right)(z) = f(\alpha\cdot z) - \zeta^{-1}j_\alpha(z)^{-\frac{1}{2}}\\ = -\left(f \mid [\xi]\right)(z).\end{aligned}$$

It follows that $f \,\Big|\, [\Gamma_0(M)^*\xi\Gamma_0(M)^*] = 0$.

(ii) If $m = 2n$ is even, then $t \equiv 1$ and

$$\Gamma_0(M)^*\xi\Gamma_0(M)^* = \Gamma_0(M)^*\xi\coprod_j \Phi(\gamma_j)$$

and

$$\Gamma_0(M)\alpha\Gamma_0(M) = \coprod_j \Gamma_0(M)\alpha\gamma_j$$

similarly as in the proof of (i). Since the projection gives a bijection $\Gamma_0(M)^*\xi\Phi(\gamma_j) \to \Gamma_0(M)\alpha\gamma_j$ for every j, it is also an isomorphism $\Gamma_0(M)^*\xi\Gamma_0(M)^* \to \Gamma_0(M)\alpha\Gamma_0(M)$. Hence, $\{\xi_i\}$ satisfies $\Gamma_0(M)^*\xi\Gamma_0(M)^* = \coprod_i \Gamma_0(M)^*\xi_i$ if and only if it satisfies $\Gamma_0(M)\alpha\Gamma_0(M) = \coprod_i \Gamma_0(M)P(\xi_i)$. □

From the second statement in the Proposition we immediately get

Corollary 3.2.12. *If $m = 2n$ is even, then the map*

$$\Gamma_0(M)\alpha\Gamma_0(M) \to \Gamma_0(M)^*\xi\Gamma_0(M)^* \quad , \quad \gamma_1\alpha\gamma_2 \mapsto \Phi(\gamma_1)\xi\Phi(\gamma_2)$$

is bijective.

From now on we consider only $m = 2n$ even, as for odd m the Hecke operator will be identically vanishing by part (i) of Proposition 3.2.11.

Definition 3.2.13. *Let $\alpha = \begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and let $\xi = \left(\left(\begin{smallmatrix} p^{-n} & 0 \\ 0 & p^n \end{smallmatrix}\right), 1 \right) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ be an element projecting to α. Let $\{\xi_i\}$ be a complete set of representatives such that $\Gamma_0(M)^*\xi\Gamma_0(M)^* = \coprod_i \Gamma_0(M)^*\xi_i$. For $n \geq 0$ we define the p^{2n}-th Hecke operator acting on automorphic functions $f : \mathbb{H} \to \mathbb{C}$ as*

$$T_{p^{2n}}(f) = f \mid [\Gamma_0(M)^*\xi\Gamma_0(M)^*] = \sum_i f \mid [\xi_i].$$

Remark. If we would take $\xi = (\alpha, \zeta)$ instead, the definition of the Hecke operator would change by a factor of ζ^{-1} only. Thus, we can choose a normalization corresponding to $\zeta = 1$.

3.3 Lift and Invariance

In this section we construct the microlocal lift $\tilde{\varphi}_t$ of a linear combination of weight $\frac{1}{2}$ Eisenstein series at $s = \frac{1}{2} + it$. We show that $\tilde{\varphi}_t$ is indeed a lift and that the corresponding limiting measure is invariant under the geodesic flow.

We mainly follow the proof for the case where $\varphi : \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ is an L^2-normalized eigenfunction of $\Delta = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ with eigenvalue $-(\frac{1}{4} + t^2)$ which is worked out in details e.g. in [7]. (Note that this definition of the Laplacian differs by a minus sign from the one we used in chapter 1. However, we will use this convention for this second chapter as the Casimir operator Ω_c defined below looks nicer for this choice.)

Note that the Lie algebra of the metaplectic group equals

$$\mathfrak{sl}_2(\mathbb{R}) = \{m \in \mathrm{Mat}_2(\mathbb{R}) \mid \mathrm{Tr}(m) = 0\}\,.$$

The action of a Lie algebra element $m \in \mathfrak{sl}_2(\mathbb{R})$ on a function $\varphi : \widetilde{\mathrm{SL}_(\mathbb{R})} \to \mathbb{C}$ is given by

$$(m * \varphi)(g,\zeta) := \frac{\partial}{\partial t}\varphi\left((g,\zeta)(\exp(tm),1)\right)\Big|_{t=0}.$$

We consider the following basis of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$:

$$\mathcal{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad \mathcal{U}^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \quad \mathcal{U}^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

with

$$[\mathcal{H},\mathcal{U}^+] = 2\mathcal{U}^+ \quad , \quad [\mathcal{H},\mathcal{U}^-] = -2\mathcal{U}^- \quad , \quad [\mathcal{U}^+,\mathcal{U}^-] = \mathcal{H}.$$

We will throughout this section use the following elements in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) = \mathfrak{sl}_2(\mathbb{R}) + i\mathfrak{sl}_2(\mathbb{R})$:

$$\begin{aligned}
\mathcal{W} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathcal{U}^+ - \mathcal{U}^- \\
\mathcal{E}^+ &= \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} = \tfrac{1}{2}\mathcal{H} + \tfrac{i}{2}\left(\mathcal{U}^+ + \mathcal{U}^-\right) \\
\mathcal{E}^- &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix} = \tfrac{1}{2}\mathcal{H} - \tfrac{i}{2}\left(\mathcal{U}^+ + \mathcal{U}^-\right) \\
\Omega_c &= \tfrac{1}{4}\mathcal{H}\circ\mathcal{H} + \tfrac{1}{2}\left(\mathcal{U}^+\circ\mathcal{U}^- + \mathcal{U}^-\circ\mathcal{U}^+\right) \\
&= \mathcal{E}^-\circ\mathcal{E}^+ - \tfrac{1}{4}\mathcal{W}\circ\mathcal{W} - \tfrac{i}{2}\mathcal{W} = \mathcal{E}^+\circ\mathcal{E}^- - \tfrac{1}{4}\mathcal{W}\circ\mathcal{W} + \tfrac{i}{2}\mathcal{W}
\end{aligned}$$

For any space of functions $Y \subset \{\varphi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{C}\}$ we denote by

$$Y \cap \mathcal{A}_k := \{\varphi \in Y \mid \varphi((g,\zeta)(k_\theta,\zeta(\theta))) = e^{ik\theta}\varphi(g,\zeta) \text{ for all } \theta \in [-\pi,3\pi) \text{ and } (g,\zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}\}.$$

the subspace of $\widetilde{K}$-eigenfunctions of weight $k \in \frac{1}{2}\mathbb{Z}$. If the function space Y is clear we write for short $\mathcal{A}_k$ instead of $Y \cap \mathcal{A}_k$. Note that with this abbreviation $\mathcal{A}_k \cdot \mathcal{A}_m \subset \mathcal{A}_{k+m}$. We say that a function φ is $\widetilde{K}$-finite if there exists some $L \in \mathbb{N}$ such that $\varphi \in \bigoplus_{l=-L}^{L} \mathcal{A}_l$. If we have $\varphi \in \mathcal{A}_k$ as an assumption in some statement without specifying the space of functions, we always mean a nice enough function so that differential operators are defined.

Remark. Note that if we would consider the analog definition for a space of functions $Y \subset \{f : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}\}$, namely $Y \cap \mathcal{A}'_n = \{f \in Y \mid f(gk_\theta) = e^{in\theta} \text{ for all } k_\theta \in K\}$, then for $n \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$ this space would consist of the zero function only. Indeed, for e.g. $f \in Y \cap \mathcal{A}'_n$ with $n \in \frac{1}{2}\mathbb{Z} \smallsetminus \mathbb{Z}$ we would have

$$f(g) = e^{-in\theta} f(gk_\theta) = e^{-in\theta} f(gk_{\theta+2\pi}) = e^{-in\theta} e^{in(\theta+2\pi)} f(g) = e^{2\pi i n} f(g) = -f(g),$$

for every $g \in \mathrm{SL}_2(\mathbb{R})$. Hence, $f \equiv 0$. Since we are working with weight $\frac{1}{2}$ Eisenstein series, we really have to consider the metaplectic group instead.

Definition 3.3.1. *For $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ We define*

$$\varphi(\cdot, s, k) : \widetilde{\mathrm{SL}_2(\mathbb{R})} \longrightarrow \mathbb{C}$$

by

$$\varphi((g,\zeta), s, k) := E(g \cdot i, s, k)\zeta^{-1} j_g(i)^{-k},$$

where $E(g \cdot i, s, k) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(g \cdot i, s, k) \in V_s$ is a linear combination of weight k Eisenstein series. For weight $\frac{1}{2}$ we abbreviate $\varphi((g,\zeta), s) = \varphi((g,\zeta), s, \frac{1}{2})$.

Lemma 3.3.2. *The function $\varphi((g,\zeta), s, k)$ satisfies the following properties:*

(i) It is left Γ-invariant, i.e. $\varphi((\gamma, \nu(\gamma))(g,\zeta), s, k) = \varphi((g,\zeta), s, k)$ for every $\gamma \in \Gamma = \Gamma_0(M)$.

(ii) It is genuine, i.e. $\varphi((g,\zeta), s, k) = \zeta^{-1}\varphi((g,1), s, k)$ for every $(g,\zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$.

(iii) $\varphi((g,\zeta)(k_\theta, \zeta(\theta)), s, k) = e^{ii\theta}\varphi((g,\zeta), s, k)$ for every $\theta \in [-\pi, 3\pi)$.

Proof.

(i) Using the property of the Eisenstein series, we have

$$\begin{aligned}\varphi((\gamma, \nu(\gamma))(g,\zeta), s, k) &= \varphi((\gamma g, w_{\frac{1}{2}}(\gamma, g)\nu(\gamma)\zeta), s, k) \\ &= E(\gamma g \cdot i, s, k) w_k(\gamma, g)^{-1}\nu(\gamma)^{-1}\zeta^{-1} j_{\gamma g}(i)^k \\ &= E(g \cdot i, s, k)\nu(\gamma) j_\gamma(g \cdot i)^k w_k(\gamma, g)^{-1}\nu(\gamma)^{-1}\zeta^{-1} j_{\gamma g}(i)^{-k} \\ &= E(g \cdot i, s, k)\zeta^{-1} j_g(i)^{-k} = \varphi((g,\zeta), s, k)\end{aligned}$$

(ii) We have

$$\varphi((g,\zeta), s, k) = E(g \cdot i, s, k)\zeta^{-1} j_g(i)^{-k} = \zeta^{-1}\varphi((g,1), s, k).$$

(iii) We have

$$\begin{aligned}
\varphi((g,\zeta)(k_\theta,\zeta(\theta)),s,k) &= \varphi((gk_\theta, w_k(g,k_\theta)\zeta\zeta(\theta)),s,k) \\
&= E(gk_\theta\cdot i,s,k)\underbrace{w_k(g,k_\theta)^{-1}}_{=w_k(g,k_\theta)}\zeta^{-1}\underbrace{\zeta(\theta)^{-1}}_{=\zeta(\theta)}j_{gk_\theta}(i)^{-k} \\
&= E(g\cdot i,s)\zeta^{-1}\zeta(\theta)j_g(\underbrace{k_\theta\cdot i}_{=i})^{-\frac{1}{2}}j_{k_\theta}(i)^{-\frac{1}{2}} \\
&= \zeta(\theta)\varphi((g,\zeta),s,k)j_{k_\theta}(i)^{-k} = \zeta(\theta)e^{-ik\arg(-\sin(\theta)i+\cos(\theta))}\varphi((g,\zeta),s,k) \\
&= \zeta(\theta)e^{-ik\arg(e^{-i\theta})}\varphi((g,\zeta),s,k) = \zeta(\theta)e^{ik[\theta]}\varphi((g,\zeta),s,k) \\
&= e^{ik\theta}\varphi((g,\zeta),s,k).
\end{aligned}$$

In the last step we used that for $\theta \in [-\pi,\pi)$ we have

$$\zeta(\theta)e^{ik[\theta]} = e^{ik\theta}$$

and for $\theta \in [\pi, 3\pi)$ we have

$$\zeta(\theta)e^{ik[\theta]} = (-1)e^{ik(\theta-2\pi)} = -e^{ik\theta}e^{-2\pi ik} = -e^{ik\theta}e^{-\pi i-2\pi in} = e^{ik\theta},$$

since $k = \frac{1}{2} + n \in \frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$. □

Remark. Note that the function $|\varphi((g,\zeta),s,k)|^2 = |E(g\cdot i,s,k)|^2$ is independent of $\zeta \in \mu_M$, since $\varphi((g,\zeta),s,k)$ is genuine. Hence, this function can also be considered as a function on $\Gamma\setminus\mathrm{SL}_2(\mathbb{R})$.

The idea of the construction of the microlocal lift is to apply appropriate differential operators to a function in $\mathcal{A}_k$ in order to get functions of weight $\dots, k-4, k-2, k, k+2, k+4, \dots$ and then consider a sum of these functions of different weights. We first consider how the standard differential operators act on a function in $\mathcal{A}_k$:

Proposition 3.3.3. *Let $\varphi : \widetilde{\mathrm{SL}_2(\mathbb{R})} \to \mathbb{C}$ be a function in $\mathcal{A}_k$ ($k \in \frac{1}{2}\mathbb{Z}$), i.e. for every $(k_\theta,\zeta(\theta)) \in \widetilde{K}$ it satisfies $\varphi((g,\zeta)(k_\theta,\zeta(\theta))) = e^{ik\theta}\varphi(g,\zeta)$. Use the Iwasawa decomposition to write every $(g,\zeta) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ in the form $(g,\zeta) = (n_xa_y,\zeta')(k_\theta,\zeta(\theta))$ for some $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$ and $\theta \in [-\pi, 3\pi)$. Then, we have*

(i) $\Omega_c * \varphi \in \mathcal{A}_k$

(ii) $\mathcal{W} * \varphi = ik\varphi$ *(Moreover, $\varphi \in \mathcal{A}_k \Leftrightarrow \mathcal{W} * \varphi = ik\varphi$.)*

(iii) $\mathcal{E}^+ * \varphi \in \mathcal{A}_{k+2}$ *and* $\mathcal{E}^- * \varphi \in \mathcal{A}_{k-2}$

(iv) $(\mathcal{H} * \varphi)(n_x a_y, \zeta) = 2y\frac{\partial}{\partial y}\varphi(n_x a_y, \zeta)$

(v) $(\mathcal{U}^+ * \varphi)(n_x a_y, \zeta) = y\frac{\partial}{\partial x}\varphi(n_x a_y, \zeta)$

(vi) $(\Omega_c * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) = e^{ik\theta}\left(y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}\right)\varphi(n_x a_y, \zeta)$

(vii) $(\mathcal{E}^+ * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) = e^{i(k+2)\theta}\left(y\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} + \frac{k}{2}\right)\varphi(n_x a_y, \zeta)$

(viii) $(\mathcal{E}^- * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) = e^{i(k-2)\theta}\left(y\frac{\partial}{\partial y} - iy\frac{\partial}{\partial x} - \frac{k}{2}\right)\varphi(n_x a_y, \zeta)$

Proof. (i) Since $\varphi \in \mathcal{A}_k$ we have for every $\theta \in [-\pi, 3\pi)$ that

$$\begin{aligned}
&(\Omega_c * \varphi)((g,\zeta)(k_\theta,\zeta(\theta))) = \frac{\partial}{\partial t}\varphi((g,\zeta)(k_\theta,\zeta(\theta))(\exp(t\Omega_c),1))\Big|_{t=0} \\
&= \frac{\partial}{\partial t}e^{ik\theta}\varphi((g,\zeta)(k_\theta,\zeta(\theta))(\exp(t\Omega_c),1)(k_{-\theta},\zeta(-\theta)))\Big|_{t=0} \\
&= e^{ik\theta}\frac{\partial}{\partial t}\varphi\Big((g,\zeta)(k_\theta\exp(t\Omega_c)k_\theta^{-1}, w_{\frac{1}{2}}(k_\theta\exp(t\Omega_c),k_\theta^{-1})w_{\frac{1}{2}}(k_\theta,\exp(t\Omega_c))\zeta(\theta)\zeta(-\theta))\Big)\Big|_{t=0} \\
&= e^{ik\theta}\frac{\partial}{\partial t}\varphi\Big((g,\zeta)(k_\theta\exp(t\Omega_c)k_\theta^{-1}, w_{\frac{1}{2}}(k_\theta\exp(t\Omega_c),k_\theta^{-1})w_{\frac{1}{2}}(k_\theta,\exp(t\Omega_c))w_{\frac{1}{2}}(k_\theta,k_\theta^{-1}))\Big)\Big|_{t=0},
\end{aligned}$$

where we used identity (3.2.1). Note that the Casimir operator commutes with every Lie algebra element. Hence,

$$k_\theta\exp(t\Omega_c)k_\theta^{-1} = \exp(tk_\theta\Omega_c k_\theta^{-1}) = \exp(t\Omega_c).$$

In particular, k_θ and $\exp(t\Omega_c)$ commute with each other. Using parts (i), (iii) and (iv) of Proposition 2.1.2 we have

$$\begin{aligned}
w_{\frac{1}{2}}(k_\theta\exp(t\Omega_c),k_\theta^{-1}) &= w_{\frac{1}{2}}(\exp(t\Omega_c)k_\theta,k_\theta^{-1}) \\
&= w_{\frac{1}{2}}(\exp(t\Omega_c),k_\theta))\underbrace{w_{\frac{1}{2}}(\exp(t\Omega_c),\mathrm{id})}_{=1}w_{\frac{1}{2}}(k_\theta,k_\theta^{-1}) \\
&= w_{\frac{1}{2}}(k_\theta,\exp(t\Omega_c)))w_{\frac{1}{2}}(k_\theta,k_\theta^{-1}).
\end{aligned}$$

This gives further

$$(\Omega_c * \varphi)((g,\zeta)(k_\theta,\zeta(\theta))) = e^{ik\theta}\frac{\partial}{\partial t}\varphi\left((g,\zeta)(\exp(t\Omega_c),1)\right)\Big|_{t=0} = e^{ik\theta}\left(\Omega_c * \varphi\right)(g,\zeta).$$

(ii) Note that $\exp(t\mathcal{W}) = \exp\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{smallmatrix}\right) = k_t$. For $\varphi \in \mathcal{A}_k$ we have

$$\begin{aligned}(\mathcal{W} * \varphi)(n_x a_y k_\theta, \zeta(\theta)) &= \frac{\partial}{\partial t}\varphi((n_x a_y k_\theta, \zeta(\theta))(\exp(t\mathcal{W}), 1))\Big|_{t=0} \\ &= \frac{\partial}{\partial t}\varphi((n_x a_y k_\theta, \zeta(\theta))(k_t, \zeta(t)))\Big|_{t=0} = \frac{\partial}{\partial t}e^{ikt}\varphi(n_x a_y k_\theta, \zeta(\theta))\Big|_{t=0} \\ &= ike^{ikt}\varphi(n_x a_y k_\theta, \zeta(\theta))\big|_{t=0} = ik\varphi(n_x a_y k_\theta, \zeta(\theta)).\end{aligned}$$

This shows that $\mathcal{W} * \varphi = ik\varphi$. Conversely, assume that $\mathcal{W} * \varphi = ik\varphi$ and note that $\zeta(t) = 0$ for t close to 0. Hence, $ik\varphi(g, \zeta) = (\mathcal{W} * \varphi)(g, \zeta) = \frac{\partial}{\partial t}\varphi((g, \zeta)(k_t, 1))\big|_{t=0} = \frac{\partial}{\partial t}\varphi((g, \zeta)(k_t, \zeta(t)))\big|_{t=0}$, and

$$\begin{aligned}&\frac{\partial}{\partial t}\left(e^{-ikt}\varphi((g, \zeta)(k_t, \zeta(t)))\right)\Big|_{t=0} \\ &= -ike^{-ik\theta}\varphi((g, \zeta)(k_t, \zeta(t))) + e^{-ikt}\frac{\partial}{\partial t}\varphi((g, \zeta)(k_t, \zeta(t)))\Big|_{t=0} \\ &= -ik\varphi(g, \zeta) + \frac{\partial}{\partial t}\varphi((g, \zeta)(k_t, \zeta(t)))\Big|_{t=0} = 0.\end{aligned}$$

So, $e^{-ikt}\varphi((g, \zeta)(k_t, \zeta(t)))$ is constant in t, i.e. $e^{-ikt}\varphi((g, \zeta)(k_t, \zeta(t))) = \varphi((g, \zeta)(k_0, 1)) = \varphi(g, \zeta)$. Hence, $\varphi((g, \zeta)(k_t, \zeta(t))) = e^{ikt}\varphi(g, \zeta)$, i.e. $\varphi \in \mathcal{A}_k$.

(iii) Note that

$$[\mathcal{W}, \mathcal{E}^+] = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}\right] = \begin{pmatrix} i & -1 \\ -1 & -i \end{pmatrix} = 2i\mathcal{E}^+$$

and

$$[\mathcal{W}, \mathcal{E}^-] = \left[\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}\right] = \begin{pmatrix} -i & -1 \\ -1 & i \end{pmatrix} = -2i\mathcal{E}^-.$$

For $\varphi \in \mathcal{A}_k$ we have

$$\mathcal{W} * (\mathcal{E}^\pm * \varphi) = \mathcal{E}^\pm * (\mathcal{W} * \varphi) + [\mathcal{W}, \mathcal{E}^\pm] * \varphi = ik\mathcal{E}^\pm * \varphi \pm 2i\mathcal{E}^\pm * \varphi = (k \pm 2)i\mathcal{E}^\pm * \varphi.$$

By (ii) it follows that $\mathcal{E}^\pm * \varphi \in \mathcal{A}_{k\pm2}$.

(iv) Since $\exp(t\mathcal{H}) = \exp\left(\begin{smallmatrix} t & 0 \\ 0 & -t \end{smallmatrix}\right) = \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t} \end{smallmatrix}\right) = a_{e^{2t}}$ we have

$$\begin{aligned}(\mathcal{H} * \varphi)(n_x a_y, \zeta) &= \frac{\partial}{\partial t}\varphi((n_x a_y, \zeta)(a_{e^{2t}}, 1))\Big|_{t=0} = \frac{\partial}{\partial t}\varphi(n_x a_{ye^{2t}}, \zeta)\Big|_{t=0} \\ &\overset{s=ye^{2t}}{=} 2y\,\frac{\partial}{\partial s}\varphi(n_x a_s, \zeta)\Big|_{s=y} = 2y\frac{\partial}{\partial y}\varphi(n_x a_y, \zeta).\end{aligned}$$

(v) Since $\exp(t\mathcal{U}^+) = \exp\left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) = n_t$ we have

$$(\mathcal{U}^+ * \varphi)(n_x a_y, \zeta) = \frac{\partial}{\partial t}\varphi((n_x a_y, \zeta)(\exp(t\mathcal{U}^+), 1)|_{t=0} = \frac{\partial}{\partial t}\varphi(n_x \underbrace{a_y n_t}_{=n_{ty}a_y}, \zeta)\Big|_{t=0}$$
$$= \frac{\partial}{\partial t}\varphi(n_{x+ty}a_y, \zeta)\Big|_{t=0} = y\frac{\partial}{\partial s}\varphi(n_s a_y, \zeta)|_{s=x} = y\frac{\partial}{\partial x}\varphi(n_x a_y, \zeta).$$

(vi) Recall that $[\mathcal{U}^+, \mathcal{U}^-] = \mathcal{H}$, so $\mathcal{U}^- * (\mathcal{U}^+ * \varphi) = \mathcal{U}^+ * (\mathcal{U}^- * \varphi) - \mathcal{H} * \varphi$. Also note that $\mathcal{W} = \mathcal{U}^+ - \mathcal{U}^-$. Since $\Omega_c * \varphi \in \mathcal{A}_k$ by (i) we have

$$\begin{aligned}
&(\Omega_c * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) = e^{ik\theta}(\Omega_c * \varphi)(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(\tfrac{1}{4}\mathcal{H} * (\mathcal{H} * \varphi) + \tfrac{1}{2}\mathcal{U}^+ * (\mathcal{U}^- * \varphi) + \frac{1}{2}\mathcal{U}^- * (\mathcal{U}^+ * \varphi)\right)(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(\tfrac{1}{4}\mathcal{H} * (\mathcal{H} * \varphi) + \mathcal{U}^+ * (\mathcal{U}^- * \varphi) - \tfrac{1}{2}\mathcal{H} * \varphi\right)(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(\tfrac{1}{4}\mathcal{H} * (\mathcal{H} * \varphi) + \mathcal{U}^+ * (\mathcal{U}^+ * \varphi) - \mathcal{U}^+ * (\mathcal{W} * \varphi) - \tfrac{1}{2}\mathcal{H} * \varphi\right)(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(\frac{1}{4}2y\frac{\partial}{\partial y}\left(2y\frac{\partial}{\partial y}\right) + y\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial x}ik - y\frac{\partial}{\partial y}\right)\varphi(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(y\frac{\partial}{\partial y} + y^2\frac{\partial^2}{\partial y^2} + y^2\frac{\partial^2}{\partial x^2} - iky\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)\varphi(n_x a_y, \zeta)\\
&= e^{ik\theta}\left(y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}\right)\varphi(n_x a_y, \zeta)
\end{aligned}$$

for every $x \in \mathbb{R}$, $y \in \mathbb{R}_{>0}$, $\theta \in [-\pi, 3\pi)$ and $\zeta \in \mu_M$.

(vii) Recall that $\mathcal{E}^+ = \frac{1}{2}\mathcal{H} + \frac{i}{2}(\mathcal{U}^+ + \mathcal{U}^-) = \frac{1}{2}\mathcal{H} + i\mathcal{U}^+ - \frac{i}{2}\mathcal{W}$. Since $\mathcal{E}^+ * \varphi \in \mathcal{A}_{k+2}$ by (iii), we have

$$\begin{aligned}
(\mathcal{E}^+ * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) &= e^{i(k+2)\theta}(\mathcal{E}^+ * \varphi)(n_x a_y, \zeta)\\
&= e^{i(k+2)\theta}\left(\tfrac{1}{2}\mathcal{H} * \varphi + i\mathcal{U}^+ * \varphi - \tfrac{i}{2}\mathcal{W} * \varphi\right)(n_x a_y, \zeta)\\
&= e^{i(k+2)\theta}\left(y\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} + \frac{k}{2}\right)\varphi(n_x a_y, \zeta)
\end{aligned}$$

(viii) Recall that $\mathcal{E}^- = \frac{1}{2}\mathcal{H} - \frac{i}{2}(\mathcal{U}^+ + \mathcal{U}^-) = \frac{1}{2}\mathcal{H} - i\mathcal{U}^+ + \frac{i}{2}\mathcal{W}$. Since $\mathcal{E}^+ * \varphi \in \mathcal{A}_{k-2}$ by (iii), we have

$$\begin{aligned}(\mathcal{E}^- * \varphi)((n_x a_y, \zeta)(k_\theta, \zeta(\theta))) &= e^{i(k-2)\theta}(\mathcal{E}^+ * \varphi)(n_x a_y, \zeta)\\ &= e^{i(k-2)\theta}\left(\tfrac{1}{2}\mathcal{H} * \varphi - i\mathcal{U}^+ * \varphi + \tfrac{i}{2}\mathcal{W} * \varphi\right)(n_x a_y, \zeta)\\ &= e^{i(k-2)\theta}\left(y\frac{\partial}{\partial y} - iy\frac{\partial}{\partial x} - \frac{k}{2}\right)\varphi(n_x a_y, \zeta)\end{aligned}$$

□

Theorem 3.3.4. *The Casimir operator and the raising and lowering operators act on the Eisenstein series $\varphi(\cdot, s, k)$ defined in Defionition 3.3.1 as follows:*

(i) $(\Omega_c * \varphi(\cdot, s, k))(g, \zeta) = s(s-1)\varphi((g, \zeta), s, k)$,

(ii) $(\mathcal{E}^+ * \varphi(\cdot, s, k))(g, \zeta) = \left(s + \frac{k}{2}\right)\varphi((g, \zeta), s, k+2)$,

(iii) $(\mathcal{E}^- * \varphi(\cdot, s, k))(g, \zeta) = \left(s - \frac{k}{2}\right)\varphi((g, \zeta), s, k-2)$.

Proof. Note that (for $\Re(s) > 1$) we have

$$\begin{aligned}\varphi((g, \zeta), s, k) &= E(g \cdot i, s, k)\zeta^{-1} j_g(i)^{-k}\\ &= \zeta^{-1}\sum_{\mathfrak{a}} c(\mathfrak{a}) \sum_{\Gamma_\mathfrak{a}\backslash\Gamma} \overline{\nu(\gamma) w_k(\sigma_\mathfrak{a}^{-1}, \gamma)} j_{\sigma_\mathfrak{a}^{-1}\gamma}(z)^{-k}\Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s j_g(i)^{-k}\\ &= \zeta^{-1}\sum_{\mathfrak{a}} c(\mathfrak{a}) \sum_{\Gamma_\mathfrak{a}\backslash\Gamma} \overline{\nu(\gamma) w_k(\sigma_\mathfrak{a}^{-1}, \gamma)} w_k(\sigma_\mathfrak{a}^{-1}\gamma, g) j_{\sigma_\mathfrak{a}^{-1}\gamma g}(i)^{-k}\Im(\sigma_\mathfrak{a}^{-1}\gamma z)^s\\ &= \sum_{\mathfrak{a}} c(\mathfrak{a}) \sum_{\Gamma_\mathfrak{a}\backslash\Gamma} \overline{\nu(\gamma) w_{k+2}(\sigma_\mathfrak{a}^{-1}, \gamma)} w_{k+2}(\sigma_\mathfrak{a}^{-1}\gamma, g) F_k(\sigma_\mathfrak{a}^{-1}\gamma g, \zeta),\end{aligned}$$

where

$$F_k(g, \zeta) = \zeta^{-1} j_g(i)^{-k}\Im(g \cdot i)^s.$$

First note that with $k \in \frac{1}{2}\mathbb{Z}\backslash\mathbb{Z}$ and $(g, \zeta) = (n_x a_y, \zeta')(k_\theta, \zeta(\theta))$ we have

$$\begin{aligned}j_g(i)^{-k} &= j_{n_x a_y k_\theta}(i)^{-k} = e^{-ik\arg(-y^{-\frac{1}{2}}\sin(\theta)i + y^{-\frac{1}{2}}\cos\theta)} = e^{-ik\arg(-\sin(\theta)i + \cos\theta)} = e^{-\frac{1}{2}i\arg(e^{-i\theta})}\\ &= e^{ik[\theta]} = \zeta(\theta)e^{ik\theta},\end{aligned}$$

hence

$$F_k((n_x a_y, \zeta')(k_\theta, \zeta(\theta))) = \zeta'^{-1}\zeta(\theta)^{-1}\zeta(\theta)e^{ik\theta}y^s = \zeta'^{-1}e^{ik\theta}y^s.$$

In particular, $F_k \in \mathcal{A}_k$. Since $w_{\frac{1}{2}}(\sigma_\mathfrak{a}^{-1}\gamma, g)^{-1}$ is constant (± 1) for generic z, it suffices to understand the action of the differential operators on the function F_k.

(i) Since $F_k \in \mathcal{A}_k$, by part (vi) of Proposition 3.3.3 have

$$\begin{aligned}(\Omega_c * F_k)((n_x a_y, \zeta')(k_\theta, \zeta(\theta))) &= e^{ik\theta}\left(\left(y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right) - iky\frac{\partial}{\partial x}\right) * F_k\right)(n_x a_y, \zeta') \\ &= e^{ik\theta} s(s-1)\zeta'^{-1} y^s = s(s-1)F_k((n_x a_y, \zeta')(k_\theta, \zeta(\theta)))\end{aligned}$$

Since left translation commutes with differential operators it follows that also

$$(\Omega_c * F_k)(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta) = s(s-1)F_k(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta),$$

hence

$$\Omega_c * \varphi(\cdot, s, k) = s(s-1)\varphi(\cdot, s, k).$$

(ii) By part (vii) of Proposition 3.3.3 have

$$\begin{aligned}(\mathcal{E}^+ * F_k)((n_x a_y, \zeta')(k_\theta, \zeta(\theta))) &= e^{i(k+2)\theta}\left(\left(y\frac{\partial}{\partial y} + iy\frac{\partial}{\partial x} + \frac{k}{2}\right) * F_k\right)(n_x a_y, \zeta') \\ &= e^{i(k+2)\theta}\left(s+\tfrac{k}{2}\right)\zeta'^{-1} y^s = \left(s+\tfrac{k}{2}\right)F_{k+2}((n_x a_y, \zeta')(k_\theta, \zeta(\theta)))\end{aligned}$$

Since left translation commutes with differential operators it follows that also

$$(\mathcal{E}^+ * F_k)(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta) = \left(s+\tfrac{k}{2}\right)F_{k+2}(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta),$$

hence

$$\mathcal{E}^+ * \varphi(\cdot, s, k) = \left(s+\tfrac{k}{2}\right)\varphi(\cdot, s, k+2).$$

(iii) By part (viii) of Proposition 3.3.3 have

$$\begin{aligned}(\mathcal{E}^- * F_k)((n_x a_y, \zeta')(k_\theta, \zeta(\theta))) &= e^{i(k-2)\theta}\left(\left(y\frac{\partial}{\partial y} - iy\frac{\partial}{\partial x} - \frac{k}{2}\right) * F_k\right)(n_x a_y, \zeta') \\ &= e^{i(k-2)\theta}\left(s-\tfrac{k}{2}\right)\zeta'^{-1} y^s = \left(s-\tfrac{k}{2}\right)F_{k-2}((n_x a_y, \zeta')(k_\theta, \zeta(\theta)))\end{aligned}$$

Since left translation commutes with differential operators it follows that also

$$(\mathcal{E}^- * F_k)(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta) = \left(s-\tfrac{k}{2}\right)F_{k-2}(\sigma_{\mathfrak{a}}^{-1}\gamma g, \zeta),$$

hence

$$\mathcal{E}^- * \varphi(\cdot, s, k) = \left(s-\tfrac{k}{2}\right)\varphi(\cdot, s, k-2).$$

□

We denote the metaplectic group by $\widetilde{G} = \widetilde{\mathrm{SL}_2(\mathbb{R})}$ and by

$$\Gamma^* = \{(\gamma, \nu(\gamma)) \mid \gamma \in \Gamma_0(M)\} \subset \widetilde{\mathrm{SL}_2(\mathbb{R})}$$

and

$$\widetilde{K} = \{(k_\theta, \zeta(\theta)) \mid \theta \in \mathbb{R}/4\pi\mathbb{Z}\} \subset \widetilde{\mathrm{SL}_2(\mathbb{R})}$$

the section of Γ and the lift of K to the metaplectic group, respectively. We denote

$$\widetilde{X} = \Gamma^* \backslash \widetilde{G} \quad \text{and} \quad \widetilde{M} = \Gamma^* \backslash \widetilde{G}/\widetilde{K}.$$

We will consider the truncated space

$$\widetilde{X}^T := \{(g, \zeta) \mid g \in X^T, \zeta \in \mu_M\} \subset \widetilde{\mathrm{SL}_2(\mathbb{R})},$$

where

$$X^T := \{g \in X \mid \mathrm{height}_X(g) < T\} = \{g \in X \mid \max_{\gamma \in \mathrm{SL}_2(\mathbb{Z})} \Im(\gamma g \cdot i) < T\}.$$

We also define the space

$$\widetilde{\mathcal{C}}^T := \{f \in C^\infty_{\mathrm{bd}}(\widetilde{X}^T) \mid Df \in C^\infty_{\mathrm{bd}}(\widetilde{X}^T) \text{ for all } D \in \mathcal{E}(\mathfrak{sl}_2(\mathbb{R}))\},$$

where $C^\infty_{\mathrm{bd}}(\widetilde{X}^T) = C^\infty(\widetilde{X}^T) \cap L^\infty(\widetilde{X}^T)$.

Definition 3.3.5. *We define $\varphi_{\frac{1}{2}} : \widetilde{X} \to \mathbb{C}$ by*

$$\begin{aligned}
\varphi_{\frac{1}{2}}((g,\zeta)) := \varphi_{t,\frac{1}{2}}((g,\zeta)) &:= \frac{1}{\sqrt{c \log |t|}} \varphi((g,\zeta), \tfrac{1}{2} + it, \tfrac{1}{2}) \\
&= \frac{1}{\sqrt{c \log |t|}} E(g \cdot i, \tfrac{1}{2} + it) \zeta^{-1} j_g(i)^{-\frac{1}{2}} \\
&= \frac{1}{\sqrt{c \log |t|}} \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(g \cdot i, \tfrac{1}{2} + it, \tfrac{1}{2}) \zeta^{-1} j_g(i)^{-\frac{1}{2}},
\end{aligned}$$

where $c \in \mathbb{R}_{>0}$ is the normalization constant as in Definition 3.1.6.

Note that $|\varphi_{t,\frac{1}{2}}| : \mathrm{SL}_2(\mathbb{R}) \to \mathbb{C}$ and moreover, the measures μ_t and $|\varphi_{t,\frac{1}{2}}|^2 \, \mathrm{dm}_{X^{|t|}}$ coincide on $C_c^\infty(M^{|t|})$, where $M^{|t|} = \{z \in M \mid \mathrm{height}(z) < |t|\}$. In particular, for every $f \in C_c^\infty(M)$ there exists some t_0 such that $\mu_t(f) = \int_{\widetilde{X}^{|t|}} f(g \cdot i) |\varphi_{t,\frac{1}{2}}((g,1))|^2 \, \mathrm{dm}_{\widetilde{X}^{|t|}}$ for $|t| \geq t_0$.

REMARK. By Lemma 3.3.2 this is a well-defined genuine function. Moreover, $\varphi_{\frac{1}{2}} \in \mathcal{A}_{\frac{1}{2}}$ and we know explicitely how the Casimir operator and raising and lowering operators act on it.

Lemma 3.3.6. *Let $f \in \mathcal{A}_{\frac{1}{2}+2n}$ for some $n \in \mathbb{Z}$ with $\Omega_c * f = \lambda f$ for $\lambda = -\left(\frac{1}{4}+t^2\right)$. Then,*

(i) $(\mathcal{E}^+ \circ \mathcal{E}^-) * f = \left(\lambda - n^2 + \frac{1}{2}n + \frac{3}{16}\right) f$

(ii) $(\mathcal{E}^- \circ \mathcal{E}^+) * f = \left(\lambda - n^2 - \frac{3}{2}n - \frac{5}{16}\right) f$

Proof. (i) Recall that $\mathcal{E}^+ \circ \mathcal{E}^- = \Omega_c + \frac{1}{4}\mathcal{W} \circ \mathcal{W} - \frac{i}{2}\mathcal{W}$. Hence, we have

$$\begin{aligned}\left(\mathcal{E}^+ \circ \mathcal{E}^-\right) * f &= \Omega_c * f + \tfrac{1}{4}\mathcal{W} * \underbrace{(\mathcal{W} * f)}_{=(\frac{1}{2}+2n)if} - \tfrac{i}{2}\mathcal{W} * f \\ &= \lambda f + \tfrac{1}{4}\left((\tfrac{1}{2}+2n)i\right)^2 f - \tfrac{i}{2}\left(\tfrac{1}{2}+2n\right) if \\ &= \left(\lambda - \tfrac{1}{4}\left(\tfrac{1}{4}+2n+4n^2\right) + \tfrac{1}{4} + n\right) f \\ &= \left(\lambda - n^2 + \tfrac{1}{2}n + \tfrac{3}{16}\right) f.\end{aligned}$$

(ii) Similarly, we have $\mathcal{E}^- \circ \mathcal{E}^+ = \Omega_c + \frac{1}{4}\mathcal{W} \circ \mathcal{W} + \frac{i}{2}\mathcal{W}$ and hence,

$$\begin{aligned}\left(\mathcal{E}^- \circ \mathcal{E}^+\right) * f &= \Omega_c * f + \tfrac{1}{4}\mathcal{W} * (\mathcal{W} * f) + \tfrac{i}{2}\mathcal{W} * f \\ &= \lambda f + \tfrac{1}{4}\left((\tfrac{1}{2}+2n)i\right)^2 f + \tfrac{i}{2}\left(\tfrac{1}{2}+2n\right) if \\ &= \left(\lambda - \tfrac{1}{4}\left(\tfrac{1}{4}+2n+4n^2\right) - \tfrac{1}{4} - n\right) f \\ &= \left(\lambda - n^2 - \tfrac{3}{2}n - \tfrac{5}{16}\right) f.\end{aligned}$$

□

In the following we denote by $\langle\cdot,\cdot\rangle$ always the L^2-inner product on the truncated space $\widetilde{X}^{|t|}$, i.e.

$$\langle f, g\rangle := \int_{\widetilde{X}^{|t|}} f\bar{g}\,\mathrm{dm}_{\widetilde{X}^{|t|}},$$

whenever $f\bar{g} \in L^2(\widetilde{X}^{|t|})$.

Definition 3.3.7. *We define*

$$\begin{cases}\varphi_{\frac{1}{2}+2n+2} = \frac{1}{\frac{3}{4}+n+it}\ \mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} & \text{for } n \in \mathbb{Z}_{\geq 0}, \\ \varphi_{\frac{1}{2}+2n-2} = \frac{1}{\frac{1}{4}-n+it}\ \mathcal{E}^- * \varphi_{\frac{1}{2}+2n} & \text{for } n \in \mathbb{Z}_{\leq 0}.\end{cases} \tag{3.3.1}$$

We define the microlocal lift to be

$$\widetilde{\varphi} := \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}$$

for $N = N(\lambda) = N(t)$ to be chosen big enough.

Proposition 3.3.8. *The functions $\varphi_{\frac{1}{2}+2n} := \varphi_{t,\frac{1}{2}+2n}$ for $n \in \mathbb{Z}$ constructed from $\varphi_{\frac{1}{2}} := \varphi_{t,\frac{1}{2}}$ by the formulas (*) and the microlocal lift $\widetilde{\varphi} := \widetilde{\varphi}_t$ satisfy the following properties:*

(i) $\Omega_c * \varphi_{\frac{1}{2}+2n} = \lambda \varphi_{\frac{1}{2}+2n}$ *for every $n \in \mathbb{Z}$*

(ii) $\Omega_c * \widetilde{\varphi} = \lambda \widetilde{\varphi}$

(iii) The formulas (3.3.1) *hold for all $n \in \mathbb{Z}$.*

Proof. (i) Since the Casimir operator Ω_c commutes with the action of $\mathfrak{sl}_2(\mathbb{C})$, it particularly commutes with the raising and lowering operators. Hence, $\Omega_c * \varphi_{\frac{1}{2}} = \lambda \varphi_{\frac{1}{2}}$ implies that also $\Omega_c * \varphi_{\frac{1}{2}+2n} = \lambda \varphi_{\frac{1}{2}+2n}$ for every $n \in \mathbb{Z}$ using the formulas (*).

(ii) By definition of the microlocal lift, we have

$$\Omega_c * \widetilde{\varphi} = \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \underbrace{\Omega_c * \varphi_{\frac{1}{2}+2n}}_{\overset{(i)}{=} \lambda \varphi_{\frac{1}{2}+2n}} = \lambda \frac{1}{\sqrt{2N+1}} \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n} = \lambda \widetilde{\varphi}.$$

(iii) For every $n \in \mathbb{Z}_{\leq 0}$ we have

$$\begin{aligned}
\mathcal{E}^+ * \varphi_{\frac{1}{2}+2n-2} &\overset{(3.3.1)}{=} \frac{1}{\frac{1}{4} - n + it} \left(\mathcal{E}^+ \circ \mathcal{E}^-\right) * \varphi_{\frac{1}{2}+2n} \\
&\overset{\text{Lemma 3.3.6},(i)}{=} \frac{1}{\frac{1}{4} - n + it} \left(\lambda - n^2 + \tfrac{1}{2}n + \tfrac{3}{16}\right) \varphi_{\frac{1}{2}+2n} \\
&= -\frac{t^2 + n^2 - \frac{1}{2}n + \frac{1}{16}}{\frac{1}{4} - n + it} \varphi_{\frac{1}{2}+2n} = -\frac{\left|\frac{1}{4} - n + it\right|^2}{\frac{1}{4} - n + it} \varphi_{\frac{1}{2}+2n} \\
&= -\left(\overline{\tfrac{1}{4} - n + it}\right) \varphi_{\frac{1}{2}+2n} = \left(-\tfrac{1}{4} + n + it\right) \varphi_{\frac{1}{2}+2n} \\
&= \left(\tfrac{3}{4} + (n-1) + it\right) \varphi_{\frac{1}{2}+2(n-1)+2}.
\end{aligned}$$

So, we have

$$\varphi_{\frac{1}{2}+2(n-1)+2} = \frac{1}{\frac{3}{4} + (n-1) + it} \mathcal{E}^+ * \varphi_{\frac{1}{2}+2(n-1)} \quad \text{for all } n \in \mathbb{Z}_{\leq 0}$$

which means

$$\varphi_{\frac{1}{2}+2n+2} = \frac{1}{\frac{3}{4}+n+it}\mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} \quad \text{for all } n \in \mathbb{Z}_{\leq -1}.$$

Similarly, we have for every $n \in \mathbb{Z}_{\geq 0}$ that

$$\begin{aligned}
\mathcal{E}^- * \varphi_{\frac{1}{2}+2n+2} &\overset{(3.3.1)}{=} \frac{1}{\frac{3}{4}+n+it}\left(\mathcal{E}^- \circ \mathcal{E}^+\right) * \varphi_{\frac{1}{2}+2n} \\
&\overset{\text{Lemma 3.3.6},(ii)}{=} \frac{1}{\frac{3}{4}+n+it}\left(\lambda - n^2 - \tfrac{3}{2}n - \tfrac{5}{16}\right)\varphi_{\frac{1}{2}+2n} \\
&= -\frac{t^2+n^2+\frac{3}{2}n+\frac{9}{16}}{\frac{3}{4}+n+it}\varphi_{\frac{1}{2}+2n} = -\frac{\left|\frac{3}{4}+n+it\right|^2}{\frac{3}{4}+n+it}\varphi_{\frac{1}{2}+2n} \\
&= -\left(\overline{\tfrac{3}{4}+n+it}\right)\varphi_{\frac{1}{2}+2n} = \left(-\tfrac{3}{4}-n+it\right)\varphi_{\frac{1}{2}+2n} \\
&= \left(\tfrac{1}{4}-(n+1)+it\right)\varphi_{\frac{1}{2}+2(n+1)-2}.
\end{aligned}$$

So, we have

$$\varphi_{\frac{1}{2}+2(n+1)-2} = \frac{1}{\frac{1}{4}-(n+1)+it}\mathcal{E}^- * \varphi_{\frac{1}{2}+2(n+1)} \quad \text{for all } n \in \mathbb{Z}_{\geq 0}$$

which means

$$\varphi_{\frac{1}{2}+2n-2} = \frac{1}{\frac{1}{4}-n+it}\mathcal{E}^- * \varphi_{\frac{1}{2}+2n} \quad \text{for all } n \in \mathbb{Z}_{\geq 1}.$$

Together with (3.3.1) This gives

$$\begin{cases} \varphi_{\frac{1}{2}+2n+2} = \frac{1}{\frac{3}{4}+n+it}\ \mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} \\ \varphi_{\frac{1}{2}+2n-2} = \frac{1}{\frac{1}{4}-n+it}\ \mathcal{E}^- * \varphi_{\frac{1}{2}+2n} \end{cases} \quad \text{for all } n \in \mathbb{Z}. \tag{3.3.2}$$

□

Even though, $\varphi_{\frac{1}{2}}$ does not lie in $L^2(\widetilde{X})$ we have

Proposition 3.3.9. *The L^2-norms can be bounded as follows:*

(i) $||\varphi_{\frac{1}{2}+2n}||_{L^2(\widetilde{X}^{|t|})} = O(1)$ *for every* $n \in \mathbb{Z}$.

(ii) $||\widetilde{\varphi}||_{L^2(\widetilde{X}^{|t|})} = O(1)$

Here, the constants are independent of t. They however might depend on the level M.

Proof. (i) By Theorem 3.3.4 we know that

$$\left(\mathcal{E}^{+}\right)^{n} * \varphi_{\frac{1}{2}} = \left(s+\tfrac{1}{4}\right)\left(s+\tfrac{5}{4}\right)\left(s+\tfrac{9}{4}\right)\cdots\left(s+\tfrac{1+4(n-1)}{4}\right)\varphi(\cdot,s,\tfrac{1}{2}+2n)$$

and

$$\left(\mathcal{E}^{-}\right)^{n} * \varphi_{\frac{1}{2}} = \left(s-\tfrac{1}{4}\right)\left(s-\tfrac{5}{4}\right)\left(s-\tfrac{9}{4}\right)\cdots\left(s-\tfrac{1+4(j-1)}{4}\right)\varphi(\cdot,s,\tfrac{1}{2}-2j)$$

for any $n \in \mathbb{Z}_{\geq 0}$. With $s = \frac{1}{2}+it$ and using (3.3.2) this gives

$$||\varphi_{\frac{1}{2}+2n}||_{L^2(\widetilde{X}^{|t|})} = ||(\mathcal{E}^{+})^{n} * \varphi_{\frac{1}{2}}||_{L^2(\widetilde{X}^{|t|})} \ll ||\varphi(\cdot,s,\tfrac{1}{2}+2n)||_{L^2(\widetilde{X}^{|t|})}.$$

and

$$||\varphi_{\frac{1}{2}-2n}||_{L^2(\widetilde{X}^{|t|})} = ||(\mathcal{E}^{-})^{n} * \varphi_{\frac{1}{2}}||_{L^2(\widetilde{X}^{|t|})} \ll ||\varphi(\cdot,s,\tfrac{1}{2}-2n)||_{L^2(\widetilde{X}^{|t|})}.$$

Let $\mathfrak{a}_1,\ldots,\mathfrak{a}_h$ denote the open cusps of $\Gamma_0(M)$. Using Maass-Selberg (Theorem 2.4.2) we have

$$\begin{aligned}
&\int_{\widetilde{X}^{|t|}} |\varphi((g,\zeta,\tfrac{1}{2}\pm 2n))|^2 \,\mathrm{dm}_{\widetilde{X}^{|t|}} = \frac{1}{c\log|t|}\int_{\widetilde{X}^{|t|}} \left|E(g\cdot i,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\zeta^{-1}j_g(i)^{-\frac{1}{2}}\right|^2 \mathrm{dm}_{\widetilde{X}^{|t|}}\\
&= \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}} \left|E(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\right|^2 d\mu(z) = \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}} \left|E^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\right|^2 d\mu(z)\\
&\leq \frac{1}{c\log|t|}\int_{\mathcal{F}} \left|E^{|t|}(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\right|^2 d\mu(z)\\
&= \frac{1}{c\log|t|}\int_{\mathcal{F}} \left|\sum_{\mathfrak{a}} c(\mathfrak{a})E_{\mathfrak{a}}(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\right|^2 d\mu(z)\\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})}\frac{1}{c\log|t|}\int_{\mathcal{F}} E_{\mathfrak{a}}(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)\overline{E_{\mathfrak{b}}(z,\tfrac{1}{2}+it,\tfrac{1}{2}\pm 2n)}d\mu(z)\\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})}\frac{1}{c\log|t|}\Bigg(2\delta_{\mathfrak{a},\mathfrak{b}}\log|t| - \sum_{j=1}^{h} \underbrace{\varphi_{\mathfrak{a},\mathfrak{a}_j}(\tfrac{1}{2}+it)}_{=O(1)}\underbrace{\overline{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(\tfrac{1}{2}+it)}}_{=O(\log|t|)}\\
&\qquad + \frac{1}{-2it}\underbrace{\overline{\varphi_{\mathfrak{b},\mathfrak{a}}(\tfrac{1}{2}+it)}}_{=O(1)}|t|^{-2it} - \frac{1}{-2it}\underbrace{\varphi_{\mathfrak{a},\mathfrak{b}}(\tfrac{1}{2}+it)}_{=O(1)}|t|^{2it}\Bigg)\\
&= O(1).
\end{aligned}$$

Hence, $||\varphi_{\frac{1}{2}+2n}||_{L^2(\widetilde{X}^{|t|})} = O(1)$ for every $n \in \mathbb{Z}$.

(ii) Since $\mathcal{A}_{\frac{1}{2}+2n} \perp \mathcal{A}_{\frac{1}{2}+2m}$ for $n \neq m$, we have

$$\begin{aligned}||\widetilde{\varphi}||_{L^2(\widetilde{X}^{|t|})} &= \langle \widetilde{\varphi}, \widetilde{\varphi} \rangle = \frac{1}{2N+1} \sum_{n,m=-N}^{N} \langle \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}+2m} \rangle \\ &= \frac{1}{2N+1} \sum_{n=-N}^{N} \langle \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}+2n} \rangle = \frac{1}{2N+1} \sum_{n=-N}^{N} \underbrace{||\varphi_{\frac{1}{2}+2n}||_{L^2(\widetilde{X}^{|t|})}}_{=O(1)} \\ &= O(1).\end{aligned}$$

□

Proposition 3.3.10. *Let $f \in C_c^\infty(\widetilde{X}^{|t|})$ be a K-finite function. Then,*

$$\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \rangle_{L^2(\widetilde{X}^{|t|})} = \langle f\varphi_{\frac{1}{2}+2m-2n}, \varphi_{\frac{1}{2}} \rangle_{L^2(\widetilde{X}^{|t|})} + O_f\left(\frac{N}{|t|}\right)$$

for all $m, n \in \mathbb{Z}$ with $|n| \leq N$.

Proof. We prove the statement by induction on $n \geq 0$ and on $n \leq 0$, separately. For $n = 0$ the statement holds trivially for every $m \in \mathbb{Z}$. Assume that the statement holds for $n \in \mathbb{Z}_{\geq 0}$ and every $m \in \mathbb{Z}$. We want to show that it also holds for $n+1$ and every $m \in \mathbb{Z}$. Let $m \in \mathbb{Z}$ be arbitrary. Using equation (3.3.2) we get

$$\begin{aligned}\left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n+2} \right\rangle &= \left\langle f\varphi_{\frac{1}{2}+2m}, \frac{1}{\frac{3}{4}+n+it}\mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} \right\rangle = \frac{1}{\frac{3}{4}+n-it} \left\langle f\varphi_{\frac{1}{2}+2m}, \mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} \right\rangle \\ &= \frac{-1}{\frac{3}{4}+n-it} \left\langle \mathcal{E}^- * (f\varphi_{\frac{1}{2}+2m}), \varphi_{\frac{1}{2}+2n} \right\rangle.\end{aligned}$$

Here, we can use the adjoint $-\mathcal{E}^-$ of $\mathcal{E}^+$, respectively integration by parts without any boundary terms, since we are integrating against some compactly supported function. By the product rule, we have $\mathcal{E}^- * (f\varphi_{\frac{1}{2}+2m}) = (\mathcal{E}^- * f)\varphi_{\frac{1}{2}+2m} + f(\mathcal{E}^- * \varphi_{\frac{1}{2}+2m})$. Since $\mathcal{E}^- * f \in C_c^\infty(\widetilde{X}^{|t|})$ as well and using the L^2-bounds from Proposition 3.3.9, we get further

$$\begin{aligned}&= \frac{-1}{\frac{3}{4}+n-it} \underbrace{\left\langle (\mathcal{E}^- * f)\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \right\rangle}_{=O_f(1) \text{ by Cauchy-Schwarz}} + \frac{-1}{\frac{3}{4}+n-it} \left\langle f(\mathcal{E}^- * \varphi_{\frac{1}{2}+2m}), \varphi_{\frac{1}{2}+2n} \right\rangle \\ &\overset{(3.3.2)}{=} O_f\left(\frac{1}{|t|}\right) - \frac{1}{\frac{3}{4}+n-it} \left\langle f\left(\tfrac{1}{4}-m+it\right) \varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n} \right\rangle \\ &= \frac{m-\frac{1}{4}-it}{n+\frac{3}{4}-it} \left\langle f\varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n} \right\rangle + O_f\left(\frac{1}{|t|}\right).\end{aligned}$$

We can write $\frac{m-\frac{1}{4}-it}{n+\frac{3}{4}-it} = \frac{n+\frac{3}{4}-it+m-n-1}{n+\frac{3}{4}-it} = 1 + \frac{m-n-1}{n+\frac{3}{4}-it}$. Hence,

$$\begin{aligned}
&= \left\langle f\varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n}\right\rangle + \frac{m-n-1}{n+\frac{3}{4}-it}\underbrace{\left\langle f\varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n}\right\rangle}_{=0 \text{ unless } m-n-1\in[-L,L]} + O_f\left(\frac{1}{|t|}\right)\\
&= \left\langle f\varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n}\right\rangle + O_f\left(\frac{L}{|t|}\right) + O_f\left(\frac{1}{|t|}\right)\\
&= \left\langle f\varphi_{\frac{1}{2}+2m-2}, \varphi_{\frac{1}{2}+2n}\right\rangle + O_f\left(\frac{1}{|t|}\right).
\end{aligned}$$

By induction hypothesis, this is

$$= \left\langle f\varphi_{\frac{1}{2}+2m-2-2n}, \varphi_{\frac{1}{2}}\right\rangle + O_f\left(\frac{N}{|t|}\right) + O_f\left(\frac{1}{|t|}\right) = \left\langle f\varphi_{\frac{1}{2}+2m-2(n+1)}, \varphi_{\frac{1}{2}}\right\rangle + O_f\left(\frac{N}{|t|}\right)$$

which proves the statement for $n+1$. Hence, we get the desired statement for all $n \in \mathbb{Z}_{\geq 0}$. The argumentation to get it for negative n is analogous: Assuming the statement for some $n \in \mathbb{Z}_{\leq 0}$ the statement also holds for $n-1$ as follows:

$$\begin{aligned}
&\left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n-2}\right\rangle \overset{(3.3.2)}{=} \left\langle f\varphi_{\frac{1}{2}+2m}, \frac{1}{\frac{1}{4}-n+it}\mathcal{E}^- * \varphi_{\frac{1}{2}+2n}\right\rangle\\
&= \frac{-1}{\frac{1}{4}-n-it}\left\langle \mathcal{E}^+ * (f\varphi_{\frac{1}{2}+2m}), \varphi_{\frac{1}{2}+2n}\right\rangle\\
&= \frac{-1}{\frac{1}{4}-n-it}\left\langle (\mathcal{E}^+ * f)\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n}\right\rangle + \frac{-1}{\frac{1}{4}-n-it}\left\langle f(\mathcal{E}^+ * \varphi_{\frac{1}{2}+2m}), \varphi_{\frac{1}{2}+2n}\right\rangle\\
&= O_f\left(\frac{1}{|t|}\right) + \frac{-\frac{3}{4}-m-it}{\frac{1}{4}-n-it}\left\langle f\varphi_{\frac{1}{2}+2m+2}, \varphi_{\frac{1}{2}+2n}\right\rangle\\
&= O_f\left(\frac{1}{|t|}\right) + \left\langle f\varphi_{\frac{1}{2}+2m+2}, \varphi_{\frac{1}{2}+2n}\right\rangle + \frac{n-m-1}{\frac{1}{4}-n-it}\left\langle f\varphi_{\frac{1}{2}+2m+2}, \varphi_{\frac{1}{2}+2n}\right\rangle\\
&= \left\langle f\varphi_{\frac{1}{2}+2m+2}, \varphi_{\frac{1}{2}+2n}\right\rangle + O_f\left(\frac{1}{|t|}\right)
\end{aligned}$$

and by induction hypothesis we get further

$$= \left\langle f\varphi_{\frac{1}{2}+2m+2-2n}, \varphi_{\frac{1}{2}}\right\rangle + O_f\left(\frac{N}{|t|}\right) = \left\langle f\varphi_{\frac{1}{2}+2m-2(n-1)}, \varphi_{\frac{1}{2}}\right\rangle + O_f\left(\frac{N}{|t|}\right).$$

Hence, the statement also holds for all $n \in \mathbb{Z}_{\leq 0}$. □

Theorem 3.3.11. *The microlocal lift $\widetilde{\varphi}$ is almost a lift of $\varphi_{\frac{1}{2}}$ in the following sense:*

a) For every $f \in C_c^\infty(M^{|t|})$ we have

$$\int f|\widetilde{\varphi}|^2 \,\mathrm{dm}_{X^{|t|}} = \int f|\varphi_{\frac{1}{2}}|^2 \,\mathrm{dvol}_{M^{|t|}} + O_f\left(\tfrac{N}{|t|}\right).$$

b) More generally, if $f \in C_c^\infty(X^{|t|})$ is K-finite, i.e. $f \in \sum_{l=-L}^{L} \mathcal{A}_{2l}$, then

$$\int f|\widetilde{\varphi}|^2 \,\mathrm{dm}_{X^{|t|}} = \left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} + O_f\left(\max\left\{\tfrac{1}{N}, \tfrac{N}{|t|}\right\}\right).$$

Here, we consider f as a function on $\widetilde{X}^{|t|}$ by setting $f((g,\zeta)) := f(g)$. Moreover, the N in the definition of the microlocal lift is taken to be bigger than L.

Remark. Recall that even though $\varphi_{\frac{1}{2}}$ is a function on $\widetilde{X}$, its absolute value $|\varphi_{\frac{1}{2}}|$ can be considered as a function on X, as $\varphi_{\frac{1}{2}}$ is genuine. By proposition 3.3.3, parts (vii) and (viii), also $\mathcal{E}^{\pm} * \varphi_{\frac{1}{2}}$ are genuine functions. Hence, $\varphi_{\frac{1}{2}+2n}$ for $n \in \mathbb{Z}$ are genuine, and so is the microlocal lift $\widetilde{\varphi}$. Thus, $|\widetilde{\varphi}|$ can be considered as a function on X. In particular, the integrals appearing in Theorem 3.3.11 are well-defined.

Proof. We distinguish the two cases $\frac{N}{|t|} \geq 1$ and $\frac{N}{|t|} < 1$. First, we consider the case $\frac{N}{|t|} \geq 1$.

(a) We have

$$\int f|\tilde{\varphi}|^2 \,\mathrm{dm}_{X^{|t|}} \leq ||f||_{L^\infty(X^{|t|})} ||\widetilde{\varphi}||_{L^2(X^{|t|})} = O_f(1).$$

and also

$$\int f|\varphi_{\frac{1}{2}}|^2 \,\mathrm{dvol}_{M^{|t|}} \leq ||f||_{L^\infty(X^{|t|})} ||\varphi_{\frac{1}{2}}||_{L^2(X^{|t|})} = O_f(1)$$

by proposition 3.3.9. Since we are in the case $\frac{N}{|t|} \geq 1$, it follows that

$$\int f|\tilde{\varphi}|^2 \,\mathrm{dm}_{X^{|t|}} = \int f|\varphi_{\frac{1}{2}}|^2 \,\mathrm{dvol}_{M^{|t|}} + O_f(1) = \int f|\varphi_{\frac{1}{2}}|^2 \,\mathrm{dvol}_{M^{|t|}} + O_f\left(\frac{N}{|t|}\right).$$

(b) Let f be a K-finite function, i.e. $f \in \sum_{l=-L}^{L} \mathcal{A}_{2l}$. Hence, $f\varphi_{\frac{1}{2}+2n} \in \sum_{l=-L}^{L} \mathcal{A}_{\frac{1}{2}+2(n+l)}$. By orthogonality, $\langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \rangle_{L^2(\widetilde{X}^{|t|})} = 0$ if $n \notin [-L, L]$. Hence,

$$\begin{aligned}
\left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} &= \sum_{n=-N}^{N} \langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \rangle_{L^2(\widetilde{X}^{|t|})} \\
&= \sum_{n \in [-N,N] \cap [-L,L]} \underbrace{\langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \rangle_{L^2(\widetilde{X}^{|t|})}}_{=O_f(1) \text{ by Cauchy-Schwarz}} \\
&\ll (2L+1) O_f(1) = O_f(1).
\end{aligned}$$

Here we used that L depends on f only. With the estimate $\int f|\tilde{\varphi}|^2 = O_f(1)$ as in (a), we get

$$\begin{aligned}\int f|\widetilde{\varphi}|^2 \, \mathrm{dm}_{X^{|t|}} = O_f(1) &= \left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} + O_f(1) \\ &= \left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} + O_f\left(\frac{N}{|t|}\right) \\ &= \left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} + O_f\left(\max\left\{\tfrac{1}{N}, \tfrac{N}{|t|}\right\}\right).\end{aligned}$$

This ends the prove in the case $\frac{N}{|t|} \geq 1$. Let's now assume that $\frac{N}{|t|} < 1$.

(a) For $f \in \mathcal{A}_0$ we have $\left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \right\rangle = 0$ unless $m = n$. Hence,

$$\begin{aligned}\int f|\widetilde{\varphi}|^2 \, \mathrm{dm}_{X^{|t|}} = \langle f\widetilde{\varphi}, \widetilde{\varphi}\rangle_{L^2(\widetilde{X}^{|t|})} &= \frac{1}{2N+1} \sum_{m,n=-N}^{N} \left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \right\rangle \\ &= \frac{1}{2N+1} \sum_{n=-N}^{N} \left\langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}+2n} \right\rangle \\ &\overset{(*)}{=} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\left\langle f\varphi_{\frac{1}{2}}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right)\right) \\ &= \left\langle f\varphi_{\frac{1}{2}}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right) = \int f|\varphi_{\frac{1}{2}}|^2 \, \mathrm{dvol}_{M^{|t|}} + O_f\left(\frac{N}{|t|}\right).\end{aligned}$$

In equation (*) we used Proposition 3.3.10 and $\langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \rangle = 0$ if $|n| > L$.

(b) For $f \in \sum_{l=-L}^{L} \mathcal{A}_{2l}$ we have

$$\begin{aligned}
\int f|\widetilde{\varphi}|^2 \,\mathrm{dm}_{X^{|t|}} &= \frac{1}{2N+1} \sum_{m,n=-N}^{N} \underbrace{\left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \right\rangle_{L^2(\widetilde{X}^{|t|})}}_{=0 \text{ unless } m-n\in[-L,L]} \\
&= \frac{1}{2N+1} \sum_{\substack{m,n=-N \\ m-n\in[-L,L]}}^{N} \left\langle f\varphi_{\frac{1}{2}+2m}, \varphi_{\frac{1}{2}+2n} \right\rangle \\
&= \frac{1}{2N+1} \sum_{\substack{m,n=-N \\ m-n\in[-L,L]}}^{N} \left(\left\langle f\varphi_{\frac{1}{2}+2(m-n)}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right)\right),
\end{aligned}$$

where we used again Proposition 3.3.10. Note that the last summand depends on $m-n$ only. The number of ways to write $l \in [-L, L]$ as $l = m - n$ for $m, n \in \{-N, \ldots, N\}$ equals the number of choices for $n \in \{-N, \ldots, N\}$ such that $n \in \{-N - l, \ldots, N - l\}$ i.e. equals $|\{-N, \ldots, N\} \cap \{-N - l, \ldots, N - l\}| = 2N + 1 - |l|$. Thus, we get further

$$\begin{aligned}
&= \frac{1}{2N+1} \sum_{l=-L}^{L} (2N+1-|l|) \left(\left\langle f\varphi_{\frac{1}{2}+2l}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right)\right) \\
&= \sum_{l=-L}^{L} \left(1 - \frac{|l|}{2N+1}\right) \left(\left\langle f\varphi_{\frac{1}{2}+2l}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right)\right) \\
&= \sum_{l=-L}^{L} \left(1 + O_f\left(\frac{1}{N}\right)\right) \left(\left\langle f\varphi_{\frac{1}{2}+2l}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{N}{|t|}\right)\right) \\
&= \sum_{l=-L}^{L} \left\langle f\varphi_{\frac{1}{2}+2l}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{1}{N}\right) \underbrace{\sum_{l=-L}^{L} \left\langle f\varphi_{\frac{1}{2}+2l}, \varphi_{\frac{1}{2}} \right\rangle}_{=O_f(1)} + O_f\left(\frac{N}{|t|}\right) \\
&\overset{(*)}{=} \sum_{n=-N}^{N} \left\langle f\varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\frac{1}{N}\right) + O_f\left(\frac{N}{|t|}\right) \\
&= \left\langle f \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle + O_f\left(\max\left\{\frac{1}{N} + \frac{N}{|t|}\right\}\right).
\end{aligned}$$

Equation $(*)$ uses the assumption $N \geq L$.

□

Theorem 3.3.12. *If $f \in C_c^\infty(X^{|t|})$ is a K-finite function, say $f \in \sum_{l=-L}^{L} \mathcal{A}_{2l}$, and $N > L+1$, then*

$$\left\langle ((t\mathcal{H}+\mathcal{V}) * f) \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} = 0$$

for some degree-two differential operator $\mathcal{V}$ independent of t. In particular,

$$\left\langle (\mathcal{H} * f) \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle_{L^2(\widetilde{X}^{|t|})} = O_f\left(\frac{1}{|t|}\right).$$

Proof. We denote $\psi := \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}$. We have

$$\begin{aligned}
\mathcal{E}^+ * \psi &= \sum_{n=-N}^{N} \mathcal{E}^+ * \varphi_{\frac{1}{2}+2n} = \sum_{n=-N}^{N} \left(\tfrac{3}{4}+n+it\right) \varphi_{\frac{1}{2}+2n+2} \\
&= \sum_{n=-N}^{N} \left(-\tfrac{1}{2}+it\right) \varphi_{\frac{1}{2}+2n+2} + \sum_{n=-N}^{N} \tfrac{1}{2}\left(\tfrac{1}{2}+2n+2\right) \varphi_{\frac{1}{2}+2n+2} \\
&= \left(-\tfrac{1}{2}+it\right) \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n+2} - \tfrac{i}{2}\mathcal{W} * \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n+2} \\
&= \left(-\tfrac{1}{2}+it-\tfrac{i}{2}\mathcal{W}\right) * \sum_{m=-N+1}^{N+1} \varphi_{\frac{1}{2}+2m} = \left(-\tfrac{1}{2}+it-\tfrac{i}{2}\mathcal{W}\right) * \left(\psi + \varphi_{\frac{1}{2}+2N+2} - \varphi_{\frac{1}{2}-2N}\right)
\end{aligned}$$

and analogously

$$\begin{aligned}
\mathcal{E}^- * \psi &= \sum_{n=-N}^{N} \left(\tfrac{1}{4}-n+it\right) \varphi_{\frac{1}{2}+2n-2} = \sum_{n=-N}^{N} \left(-\tfrac{1}{2}+it-\tfrac{1}{2}\left(\tfrac{1}{2}+2n-2\right)\right) \varphi_{\frac{1}{2}+2n-2} \\
&= \sum_{n=-N}^{N} \left(-\tfrac{1}{2}+it+\tfrac{i}{2}\mathcal{W}\right) * \varphi_{\frac{1}{2}+2n-2} = \left(-\tfrac{1}{2}+it+\tfrac{i}{2}\mathcal{W}\right) * \sum_{m=-N-1}^{N-1} \varphi_{\frac{1}{2}+2m} \\
&= \left(-\tfrac{1}{2}+it+\tfrac{i}{2}\mathcal{W}\right) * \left(\psi + \varphi_{\frac{1}{2}-2N-2} - \varphi_{\frac{1}{2}+2N}\right).
\end{aligned}$$

By Proposition 3.3.8 we have $\Omega_c * \psi = \lambda\psi$. Hence, for every $f \in C_c^\infty\left(X^{|t|}\right)$ and $\lambda = \frac{1}{4}+t^2 \in \mathbb{R}$ we have

$$\left\langle f\psi, \Omega_c * \varphi_{\frac{1}{2}} \right\rangle = \left\langle f\psi, \lambda\varphi_{\frac{1}{2}} \right\rangle = \lambda \left\langle f\psi, \varphi_{\frac{1}{2}} \right\rangle = \left\langle f(\lambda\psi), \varphi_{\frac{1}{2}} \right\rangle = \left\langle f \cdot (\Omega_c * \psi), \varphi_{\frac{1}{2}} \right\rangle. \quad (3.3.3)$$

On the other hand, we can write

$$\begin{aligned}\Omega_c * \varphi_{\frac{1}{2}} &= \left(\mathcal{E}^- \circ \mathcal{E}^+ - \tfrac{1}{4}\mathcal{W}\circ\mathcal{W} - \tfrac{i}{2}\mathcal{W}\right) * \varphi_{\frac{1}{2}} = \left(\mathcal{E}^- \circ \mathcal{E}^+ - \tfrac{1}{4}\left(\tfrac{i}{2}\right)^2 - \tfrac{i}{2}\cdot\tfrac{i}{2}\right) * \varphi_{\frac{1}{2}}\\ &= \left(\mathcal{E}^- \circ \mathcal{E}^+ + \tfrac{5}{16}\right) * \varphi_{\frac{1}{2}}.\end{aligned}$$

Thus, we also have

$$\begin{aligned}\left\langle f\psi, \Omega_c * \varphi_{\frac{1}{2}}\right\rangle &= \left\langle f\psi, \left(\mathcal{E}^- \circ \mathcal{E}^+\right) * \varphi_{\frac{1}{2}}\right\rangle + \tfrac{5}{16}\left\langle f\varphi, \varphi_{\frac{1}{2}}\right\rangle\\
&= \left\langle \left(\mathcal{E}^- \circ \mathcal{E}^+\right) * (f\psi), \varphi_{\frac{1}{2}}\right\rangle + \tfrac{5}{16}\left\langle f\varphi, \varphi_{\frac{1}{2}}\right\rangle\\
&= \left\langle \left(\mathcal{E}^- * \left(\mathcal{E}^+ * f\right)\cdot\psi + f\cdot\left(\mathcal{E}^+ * \psi\right)\right) + \tfrac{5}{16}f\psi, \varphi_{\frac{1}{2}}\right\rangle\\
&= \Big\langle \left(\mathcal{E}^- \circ \mathcal{E}^+\right) * f\cdot\psi + \left(\mathcal{E}^+ * f\right)\cdot\left(\mathcal{E}^- * \psi\right) + \left(\mathcal{E}^- * f\right)\cdot\left(\mathcal{E}^+ * \psi\right)\\
&\qquad + f\cdot\left(\mathcal{E}^- \circ \mathcal{E}^+\right) * \psi + \tfrac{5}{16}f\psi, \varphi_{\frac{1}{2}}\Big\rangle\\
&= \Big\langle \left(\left(\mathcal{E}^- \circ \mathcal{E}^+\right) * f\right)\cdot\psi + \left(\mathcal{E}^+ * f\right)\cdot\left(-\tfrac{1}{2} + it + \tfrac{i}{2}\mathcal{W}\right) * \left(\psi + \varphi_{\frac{1}{2}-2N-2} - \varphi_{\frac{1}{2}+2N}\right)\\
&\qquad + \left(\mathcal{E}^- * f\right)\cdot\left(-\tfrac{1}{2} + it - \tfrac{i}{2}\mathcal{W}\right) * \left(\psi + \varphi_{\frac{1}{2}+2N+2} - \varphi_{\frac{1}{2}-2N}\right)\\
&\qquad + f\cdot\left(\Omega_c + \tfrac{1}{4}\mathcal{W}\circ\mathcal{W} + \tfrac{i}{2}\mathcal{W}\right) * \psi + \tfrac{5}{16}f\psi, \varphi_{\frac{1}{2}}\Big\rangle.\end{aligned}$$

Note that $\mathcal{E}^+ * f \in \sum_{m=-L+1}^{L+1}\mathcal{A}_{2m}$ and $\mathcal{E}^- * f \in \sum_{m=-L-1}^{L-1}\mathcal{A}_{2m}$. By orthogonality, the terms with $\varphi_{\frac{1}{2}-2N-2}$, $\varphi_{\frac{1}{2}+2N}$, $\varphi_{\frac{1}{2}+2N+2}$ and $\varphi_{\frac{1}{2}-2N}$ vanish if $|N| > L+1$. Comparing with equation (3.3.3) we get

$$\begin{aligned}0 = \Big\langle &\left(\left(\mathcal{E}^- \circ \mathcal{E}^+\right) * f\right)\cdot\psi + \left(\mathcal{E}^+ * f\right)\cdot\left(-\tfrac{1}{2} + it + \tfrac{i}{2}\mathcal{W}\right) * \psi\\
&+ \left(\mathcal{E}^- * f\right)\cdot\left(-\tfrac{1}{2} + it - \tfrac{i}{2}\mathcal{W}\right) * \psi + f\cdot\left(\tfrac{1}{4}\mathcal{W}\circ\mathcal{W} + \tfrac{i}{2}\mathcal{W}\right) * \psi + \tfrac{5}{16}f\psi, \varphi_{\frac{1}{2}}\Big\rangle.\end{aligned}$$

Note that for f_1, f_2 we have

$$\begin{aligned}\left\langle f_1\cdot\left(\mathcal{W} * f_2\right), \varphi_{\frac{1}{2}}\right\rangle_{L^2(X^{|t|})} &= \left\langle \mathcal{W} * \left(f_1 f_2\right) - \left(\mathcal{W} * f_1\right)\cdot f_2, \varphi_{\frac{1}{2}}\right\rangle\\
&= \left\langle f_1 f_2, -\mathcal{W} * \varphi_{\frac{1}{2}}\right\rangle - \left\langle \left(\mathcal{W} * f_1\right)\cdot f_2, \varphi_{\frac{1}{2}}\right\rangle\\
&= \tfrac{i}{2}\left\langle f_1 f_2, \varphi_{\frac{1}{2}}\right\rangle - \left\langle \left(\mathcal{W} * f_1\right)\cdot f_2, \varphi_{\frac{1}{2}}\right\rangle\\
&= \left\langle \left(\left(\tfrac{i}{2} - \mathcal{W}\right) * f_1\right)\cdot f_2, \varphi_{\frac{1}{2}}\right\rangle.\end{aligned}$$

In particular,

$$\begin{aligned}\left\langle \left(\mathcal{E}^{\pm} * f\right) \cdot \left(\mathcal{W} * \psi\right), \varphi_{\frac{1}{2}} \right\rangle &= \left\langle \left(\left(\tfrac{i}{2} - \mathcal{W}\right) * \left(\mathcal{E}^{\pm} * f\right)\right) \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle \\ &= \left\langle \left(\left(\tfrac{i}{2}\mathcal{E}^{\pm} - \mathcal{W} \circ \mathcal{E}^{\pm}\right) * f\right) \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle,\end{aligned}$$

and

$$\begin{aligned}\left\langle f \cdot (\mathcal{W} \circ \mathcal{W}) * \psi, \varphi_{\frac{1}{2}} \right\rangle &= \left\langle \left(\left(\tfrac{i}{2} - \mathcal{W}\right) * f\right) \cdot \left(\mathcal{W} * \psi\right), \varphi_{\frac{1}{2}} \right\rangle \\ &= \left\langle \left(\left(\tfrac{i}{2} - \mathcal{W}\right) \circ \left(\tfrac{i}{2} - \mathcal{W}\right)\right) * f \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle \\ &= \left\langle \left(-\tfrac{1}{4} - i\mathcal{W} + \mathcal{W} \circ \mathcal{W}\right) * f \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle.\end{aligned}$$

Combining this, we can bring all differential operators to the left of f, namely

$$\begin{aligned}0 = \Big\langle & \left(\left(\mathcal{E}^{-} \circ \mathcal{E}^{+}\right) * f\right) \cdot \psi + \left(\left(-\tfrac{1}{2} + it\right)\left(\mathcal{E}^{+} + \mathcal{E}^{-}\right) * f\right) \cdot \psi + \left(-\tfrac{1}{4} - \tfrac{i}{2}\mathcal{W} \circ \mathcal{E}^{+}\right) * f \cdot \psi \\ & + \left(\tfrac{1}{4} + \tfrac{i}{2}\mathcal{W} \circ \mathcal{E}^{-}\right) * f \cdot \psi + \left(-\tfrac{1}{16} - \tfrac{i}{4}\mathcal{W} + \tfrac{1}{4}\mathcal{W} \circ \mathcal{W}\right) * f \cdot \psi + \left(-\tfrac{1}{4} - \tfrac{i}{2}\mathcal{W}\right) * f \cdot \psi \\ & + \tfrac{5}{16} f\psi, \varphi_{\frac{1}{2}} \Big\rangle \\ = & \left\langle \left(\mathcal{E}^{-} \circ \mathcal{E}^{+} - \tfrac{1}{2}\mathcal{H} + it\mathcal{H} - \tfrac{i}{2}\mathcal{W} \circ \mathcal{E}^{+} + \tfrac{i}{2}\mathcal{W} \circ \mathcal{E}^{-} + \tfrac{1}{16}\mathcal{W} \circ \mathcal{W} - \tfrac{3i}{4}\mathcal{W}\right) * f \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle \\ = & \left\langle \left(it\mathcal{H} + i\mathcal{V}\right) * f \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle,\end{aligned}$$

where

$$\mathcal{V} = -i\mathcal{E}^{-} \circ \mathcal{E}^{+} + \tfrac{i}{2}\mathcal{H} - \tfrac{1}{2}\mathcal{W} \circ \mathcal{E}^{+} + \tfrac{1}{2}\mathcal{W} \circ \mathcal{E}^{-} - \tfrac{i}{16}\mathcal{W} \circ \mathcal{W} - \tfrac{3}{4}\mathcal{W}$$

is a second order differential operator. It follows that

$$\left\langle (t\mathcal{H} + \mathcal{V}) * f \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle = 0.$$

Moreover, since $\mathcal{V} * f \in \sum_{l=-L-2}^{L+2} \mathcal{A}_{2l}$ we have by orthogonality

$$\begin{aligned}\left\langle (\mathcal{H} * f) \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle = -\frac{1}{t}\left\langle (\mathcal{V} * f) \cdot \psi, \varphi_{\frac{1}{2}} \right\rangle &= -\frac{1}{t}\left\langle (\mathcal{V} * f) \cdot \sum_{n=-N}^{N} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle \\ &= -\frac{1}{t}\left\langle (\mathcal{V} * f) \cdot \sum_{n=-L-2}^{L+2} \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle \\ &\leq \frac{2L+4}{|t|} \cdot \max_{n \in \{-L-2, \ldots, L+2\}} \left| \left\langle (\mathcal{V} * f) \cdot \varphi_{\frac{1}{2}+2n}, \varphi_{\frac{1}{2}} \right\rangle \right| = O_f\left(\frac{1}{|t|}\right).\end{aligned}$$

□

Definition 3.3.13. *For $f \in C_c^\infty(X)$ consider the K-finite approximations*

$$f^{(L)} := \sum_{l=-L}^{L} \pi_l(f), \quad \text{where} \quad \pi_l(f)(x) := \int_K f(xk_\theta)e^{-il\theta}\,\mathrm{dm}_K(k_\theta) \in \mathcal{A}_l.$$

Lemma 3.3.14. *Let $f \in C_c^\infty(X)$. Then,*

(a) $f^{(L)} \longrightarrow f$ uniformly as $L \to \infty$.

(b) $\mathcal{H} * f^{(L)} \longrightarrow \mathcal{H} * f$ uniformly as $L \to \infty$.

Proof. It is sufficient to prove the statements for $f \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. Indeed, take a collection $\{U_j\} \subset \mathrm{SL}_2(\mathbb{R})$ of open sets such that the projection $\mathrm{pr} : \mathrm{SL}_2(\mathbb{R}) \to \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R})$ is injective on every U_j and such that $\mathrm{supp}(f) \subset \cup_j \mathrm{pr}(U_j)$. Consider a smooth partition of unity on $\mathrm{supp}(f) \subset \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R})$ relative to $\{\mathrm{pr}(U_j)\}$, i.e. consider smooth functions $\rho_j : \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R}) \to [0,1]$ such that $\mathrm{supp}(\rho_j) \subset \mathrm{pr}(U_j)$ for every j and $\sum_j \rho_j(x) = 1$ for every $x \in \mathrm{supp}(f)$. Now, if the claimed convergence result holds for functions in $C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$, then it holds for functions in $C_c^\infty(\cup_j \mathrm{pr}(U_j))$, hence by the partition of unity also for functions in $C_c^\infty(\Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R}))$. So, let $f \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$.

(a) Using integration by parts twice, we have

$$\begin{aligned}
\pi_l(f)(x) &= \int_K f(xk_\theta)e^{-il\theta}\,\mathrm{dm}_K(k_\theta) \\
&= \underbrace{\left[f(xk_\theta)\tfrac{i}{l}e^{-il\theta}\right]_{\theta=0}^{2\pi}}_{=0} - \int_K \frac{\partial}{\partial k_\theta} f(xk_\theta)\tfrac{i}{l}e^{-il\theta}\,\mathrm{dm}_K(k_\theta) = -\frac{i}{l}\int_K \frac{\partial}{\partial k_\theta} f(xk_\theta)e^{-il\theta}\,\mathrm{dm}_K(k_\theta) \\
&= -\frac{i}{l}\left(\underbrace{\left[\frac{\partial}{\partial k_\theta} f(xk_\theta)\tfrac{i}{l}e^{-il\theta}\right]_{\theta=0}^{2\pi}}_{=0} - \int_K \frac{\partial^2}{\partial k_\theta^2} f(xk_\theta)\tfrac{i}{l}e^{-il\theta}\,\mathrm{dm}_K(k_\theta)\right) \\
&= -\frac{1}{l^2}\int_K \frac{\partial^2}{\partial k_\theta^2} f(xk_\theta)\tfrac{i}{l}e^{-il\theta}\,\mathrm{dm}_K(k_\theta) \ll \frac{1}{l^2}\max_{k_\theta\in K}\left|\frac{\partial^2}{\partial k_\theta^2} f(xk_\theta)\right|\mu(K) \ll \frac{||f''||_\infty}{l^2} \ll_f \frac{1}{l^2}.
\end{aligned}$$

By the Weierstrass M-Test, the series

$$\sum_{l=-L}^{L} \pi_l(f)(x) \ll_f 1 + 2\sum_{l=1}^{L}\frac{1}{l^2}$$

converges absolutely and uniformly as $L \to \infty$. Note that

$$f \in L^2(\mathrm{SL}_2(\mathbb{R})) \cong \bigoplus_{l\in\mathbb{Z}} L^2(\mathrm{SL}_2(\mathbb{R})) \cap \mathcal{A}_l,$$

so the uniform limit of the K-finite approximation $f^{(L)} = \sum_{l=-L}^{L} \pi_l(f)$ has to be the function f.

(b) For $f \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$ we calculate

$$\begin{aligned}(\mathcal{H} * \pi_l(f))(x) &= \frac{\partial}{\partial t}\pi_l(f)(x\exp(t\mathcal{H}))\Big|_{t=0} = \frac{\partial}{\partial t}\int_K f(x\exp(t\mathcal{H})k_\theta)e^{-il\theta}\,\mathrm{dm}_K(k_\theta)\Big|_{t=0} \\ &= \int_K \frac{\partial}{\partial t} f(xk_\theta\exp(t\underbrace{k_\theta^{-1}\mathcal{H}k_\theta}_{=\mathrm{Ad}_{k_\theta^{-1}}(\mathcal{H})}))\Big|_{t=0} e^{-il\theta}\,\mathrm{dm}_K(k_\theta) \\ &= \int_K \left(\mathrm{Ad}_{k_\theta}^{-1}(\mathcal{H}) * f\right)(xk_\theta)e^{-il\theta}\,\mathrm{dm}_K(k_\theta).\end{aligned}$$

Since $\mathrm{Ad}_{k_\theta^{-1}} * f \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$, we can apply integration by parts twice to get $||\mathcal{H} * \pi_l(f)|| \ll_f \frac{1}{l^2}$. So, by Weierstrass M-test, $\mathcal{H} * f^{(L)}$ converges absolutely and uniformly to some $h \in C_c^\infty(\mathrm{SL}_2(\mathbb{R}))$. It remains to show that $h = \mathcal{H}*f$. Denote $a_t := \left(\begin{smallmatrix} e^t & 0 \\ 0 & e^{-t}\end{smallmatrix}\right) \in A$ for $t \in \mathbb{R}$. Let $g \in \mathrm{SL}_2(\mathbb{R})$ such that $g \notin \mathrm{supp}(f)K$. Integration along the geodesic flow gives

$$\begin{aligned}\int_0^t (\mathcal{H} * f^{(L)})(ga_s)\,\mathrm{ds} &= \int_0^t \frac{\partial}{\partial r} f^{(L)}(ga_s\exp(r\mathcal{H}))\Big|_{r=0}\mathrm{ds} = \frac{\partial}{\partial r}\int_0^t f^{(L)}(ga_{s+r})\,\mathrm{ds}\Big|_{r=0} \\ &= \frac{\partial}{\partial r}\int_r^{t+r} f^{(L)}(ga_u)\mathrm{d}u\Big|_{r=0} = f^{(L)}(ga_{t+r}) - f^{(L)}(ga_r)\big|_{r=0} \\ &= f^{(L)}(ga_t) - f^{(L)}(g) = f^{(L)}(ga_t).\end{aligned}$$

In the last equality we used that $g \notin \mathrm{supp}(f)K$, hence $\pi_l(f)(g) = \int_K f(gk_\theta)e^{-il\theta}\,\mathrm{dm}_K = 0$ for every l. In particular, we have

$$f(ga_t) \overset{\text{part }(a)}{=} \lim_{L\to\infty} f^{(L)}(ga_t) = \lim_{L\to\infty}\int_0^t (\mathcal{H} * f^{(L)})(ga_s)\,\mathrm{ds} = \int_0^t h(ga_s)\,\mathrm{ds}. \tag{3.3.4}$$

Now,

$$\begin{aligned}(\mathcal{H} * f)(ga_t) &= \frac{\partial}{\partial r}f(ga_t\exp(r\mathcal{H}))\Big|_{r=0} = \frac{\partial}{\partial r}f(ga_{t+r})\Big|_{r=0} = \frac{\partial}{\partial u}f(ga_u)\Big|_{u=t} \\ &= \frac{\partial}{\partial t}f(ga_t) \overset{(3.3.4)}{=} \frac{\partial}{\partial t}\int_0^t h(ga_s)\,\mathrm{ds} = h(ga_t).\end{aligned}$$

So, we've shown that for every $g \notin \mathrm{supp}(f)K$ and every $t \in \mathbb{R}$ we have

$$(\mathcal{H} * f)(ga_t) = h(ga_t).$$

Now, let $g \in \mathrm{SL}_2(\mathbb{R})$ be arbitrary. Since f is compactly supported, there exists some $t \in \mathbb{R}$ such that $ga_t^{-1} \notin \mathrm{supp}(f)K$. Hence,

$$(\mathcal{H} * f)(g) = (\mathcal{H} * f)(ga_t^{-1}a_t) = h(ga_t^{-1}a_t) = h(g).$$

This shows that $\mathcal{H} * f^{(L)} \longrightarrow h = \mathcal{H} * f$ uniformly as $L \to \infty$. □

Theorem 3.3.15. *Let $N = N(t)$ be such that $\frac{N}{|t|} \to 0$ and $\frac{1}{N} \to 0$ as $|t| \to \infty$. Let $\{\mu_t\}$ be the sequence of measures defined in Definition 3.1.6. Then, any weak*-limit $\tilde{\mu}$ of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$*

(a) is a lift of the weak-limit μ of μ_t,*

(b) is invariant under the geodesic flow, i.e. $\int_X f(xa_T)d\tilde{\mu} = \int_X f(x)d\tilde{\mu}$ for every $T \in \mathbb{R}$ and every $f \in C_c^\infty(X)$.

Proof. (a) For every $f \in C_c^\infty(M)$ there exists some t_0 such that $f \in C_c^\infty(M^{|t|})$ for $|t| > t_0$. By Theorem 3.3.11, part (a) we have

$$\int f|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}} = \int f|\varphi_{t,\frac{1}{2}}|^2 \,\mathrm{dvol}_{M^{|t|}} + O_f\left(\frac{N}{|t|}\right).$$

Taking the limits as $|t| \to \infty$ this gives

$$\int f d\widetilde{\mu} = \int f d\mu,$$

i.e. $\widetilde{\mu}$ is a lift of μ as claimed.

(b) Denote $\psi_t = \sum_{n=-N(t)}^{N(t)} \varphi_{t,\frac{1}{2}+2n}$. In order to prove invariance, let first $f \in C_c^\infty(X)$ be K-finite. Note that then also $\mathcal{H} * f \in C_c^\infty(X)$ is K-finite. Using part (b) of Theorem 3.3.11 and Theorem 3.3.12 we have (for $|t|$ big enough)

$$\begin{aligned}\int_X (\mathcal{H} * f)|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_X &\overset{\text{Thm 3.3.11}}{=} \left\langle (\mathcal{H} * f)\psi_t, \varphi_{t,\frac{1}{2}} \right\rangle + O_f\left(\max\left\{\frac{1}{N}, \frac{N}{|t|}\right\}\right) \\ &\overset{\text{Thm 3.3.12}}{=} O_f\left(\frac{1}{|t|}\right) + O_f\left(\max\left\{\frac{1}{N}, \frac{N}{|t|}\right\}\right).\end{aligned}$$

By assumption on $N = N(t)$ this expression converges to 0 as $|t| \to \infty$. On the other hand, by weak*-convergence we have $\int_X (\mathcal{H} * f)|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_X \to \int_X (\mathcal{H} * f) d\widetilde{\mu}$ as $|t| \to \infty$. Hence, this gives

$$\int_X (\mathcal{H} * f) d\widetilde{\mu} = 0 \quad \text{for every } K\text{-finite } f \in C_c^\infty(X). \tag{3.3.5}$$

Now, let $f \in C_c^\infty(X)$ be arbitrary. By Lemma 3.3.14 we can write

$$\int_X (\mathcal{H} * f) d\widetilde{\mu} = \int_X \lim_{L\to\infty} (\mathcal{H} * f^{(L)}) d\widetilde{\mu} = \lim_{L\to\infty} \int_X (\mathcal{H} * f^{(L)}) d\widetilde{\mu} = 0. \tag{3.3.6}$$

The expression vanishes, because $f^{(L)} \in C_c^\infty(X)$ is K-finite, so we can apply (3.3.5). For $s \in \mathbb{R}$ denote $a_s = \left(\begin{smallmatrix} e^s & 0 \\ 0 & e^{-s} \end{smallmatrix}\right) \in A$. Note that

$$\int_0^T (\mathcal{H} * f)(xa_s)ds = \int_0^T \frac{\partial}{\partial r} f(xa_s \exp(r\mathcal{H}))\Big|_{r=0} ds = \frac{\partial}{\partial r}\left(\int_0^T f(xa_{s+r})ds\right)\Big|_{r=0}$$
$$= \frac{\partial}{\partial r}\left(\int_r^{r+T} f(xa_u)du\right)\Big|_{r=0} = f(xa_T) - f(x) \tag{3.3.7}$$

and

$$(\mathcal{H} * (R_{a_s} f))(x) = \frac{\partial}{\partial r}(R_{a_s} f)(x\exp(r\mathcal{H}))\Big|_{r=0} = \frac{\partial}{\partial r} f(x\exp(r\mathcal{H}a_s))\Big|_{r=0}$$
$$= \frac{\partial}{\partial r} f(xa_s \exp(r\mathcal{H}))\Big|_{r=0} = (\mathcal{H} * f)(xa_s), \tag{3.3.8}$$

where R_{a_s} denotes right translation by a_s. Hence, for every $T \in \mathbb{R}$ we have

$$\int_x f(xa_T)d\widetilde{\mu} - \int_X f(x)d\widetilde{\mu} = \int_X f(xa_T) - f(x)d\widetilde{\mu} \overset{(3.3.7)}{=} \int_X \int_0^T (\mathcal{H} * f)(xa_s)dsd\widetilde{\mu}$$
$$= \int_0^T \int_X (\mathcal{H} * f)(xa_s)d\widetilde{\mu}ds \overset{(3.3.8)}{=} \int_0^T \int_X (\mathcal{H} * (R_{a_s} f))(x)d\widetilde{\mu}ds.$$

Since $f \in C_c^\infty(X)$, also $R_{a_s} f \in C_c^\infty(X)$, so

$$\int_X (\mathcal{H} * (R_{a_s} f))d\widetilde{\mu} = 0$$

by equation (3.3.6). Hence,

$$\int_X f(xa_T)d\widetilde{\mu} = \int_X f(x)d\widetilde{\mu}$$

for every $f \in C_c^\infty(X)$ and every $T \in \mathbb{R}$.

□

3.4 Hecke Recurrence

In this section we show that any weak*-limit of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$ is Hecke p-recurrent for some (actually every) prime $p \neq 2$. In order to understand the action of the Hecke operator, we first need to explicitely calculate the representatives appearing in the definition of the Hecke operator:

Let $M = 2^l$ with $l \geq 2$, and let $p \neq 2$ be an odd prime. For $n \geq 1$ write $\alpha = \begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$.

Theorem 3.4.1. *We can decompose $\Gamma_0(M)\alpha\Gamma_0(M)$ into cosets as follows:*

(a) A complete set of representatives $\{\alpha_i\}$ such that

$$\Gamma_0(M)\alpha\Gamma_0(M) = \coprod_i \Gamma_0(M)\alpha_i$$

is given by the following $p^{2n} + p^{2n-1}$ elements:

$$\begin{array}{lll}
\begin{pmatrix} p^{-n} & p^{-n}j \\ 0 & p^{n} \end{pmatrix} & , \quad 0 \le j < p^{2n} & (p^{2n} \text{ elements}) \\
\begin{pmatrix} p^{1-n} & p^{-n}j \\ 0 & p^{n-1} \end{pmatrix} & , \quad 0 < j < p^{2n-1}, p \nmid j & (p^{2n-1} - p^{2n-2} \text{ elements}) \\
\begin{pmatrix} p^{2-n} & p^{-n}j \\ 0 & p^{n-2} \end{pmatrix} & , \quad 0 < j < p^{2n-2}, p \nmid j & (p^{2n-2} - p^{2n-3} \text{ elements}) \\
\vdots & & \\
\begin{pmatrix} p^{n-2} & p^{-n}j \\ 0 & p^{2-n} \end{pmatrix} & , \quad 0 < j < p^{2}, p \nmid j & (p^{2} - p \text{ elements}) \\
\begin{pmatrix} p^{n-1} & p^{-n}j \\ 0 & p^{1-n} \end{pmatrix} & , \quad 0 < j < p & (p - 1 \text{ elements}) \\
\begin{pmatrix} p^{n} & 0 \\ 0 & p^{-n} \end{pmatrix} & & (1 \text{ element}).
\end{array}$$

(b) A complete set of representatives $\{\beta_i\}$ such that

$$\Gamma_0(M)\alpha\Gamma_0(M) = \coprod_i \beta_i\Gamma_0(M)$$

is given by the following $p^{2n} + p^{2n-1}$ elements:

$$\beta_{l,j} = \begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l} \end{pmatrix},$$

where $0 \le l \le 2n$, $0 \le j < p^l$ and $p \nmid j$ whenever $l \ne 0, 2n$.

Proof. (a) An arbitrary element in $\alpha\Gamma_0(M)$ is of the form

$$\begin{pmatrix} p^{-n} & 0 \\ 0 & p^{n} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p^{-n}a & p^{-n}b \\ p^{n}c & p^{n}d \end{pmatrix}$$

with $ad-bc=1$ and $c\equiv 0 \mod M$. Let $l\in\{0,1,\dots,2n\}$ be maximal with $p^l \mid a$. Then, $\left(\frac{a}{p^l},p^{2n-l}c\right)=1$, i.e. there exist $x,y\in\mathbb{Z}$ such that $\frac{a}{p^l}x+p^{2n-l}cy=1$. Hence, $\begin{pmatrix} x+p^{2n}cdy & y-ady \\ -p^{2n-l}c & \frac{a}{p^l}\end{pmatrix}\in\Gamma_0(M)$ and

$$\begin{pmatrix} x+p^{2n}cdy & y-ady \\ -p^{2n-l}c & \frac{a}{p^l}\end{pmatrix}\begin{pmatrix} p^{-n}a & p^{-n}b \\ p^nc & p^nd\end{pmatrix}=\begin{pmatrix} p^{l-n} & p^{-n}bx \\ 0 & p^{n-l}\end{pmatrix}.$$

Note that if $1\le l\le 2n-1$, then $ad-bc=1$ implies that $p\nmid b$, and $\frac{a}{p^l}x+p^{2n-l}cy=1$ implies that $p\nmid x$. Hence, $p\nmid bx$. Thus, a representative is

$$\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l}\end{pmatrix},$$

where $0\le j\le p^{2n-l}$ and $p\nmid j$ if $1\le l\le 2n-1$. With $l=0,1,\dots,2n$ this gives the claimed representatives. Note that for any integer $m\in\mathbb{Z}$ we have $\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)\in\Gamma_0(M)$ and

$$\begin{pmatrix} 1 & m \\ 0 & 1\end{pmatrix}\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l}\end{pmatrix}=\begin{pmatrix} p^{l-n} & p^{-n}j+mp^{n-l} \\ 0 & p^{n-l}\end{pmatrix}=\begin{pmatrix} p^{l-n} & p^{-n}\left(j+mp^{2n-l}\right) \\ 0 & p^{n-l}\end{pmatrix}.$$

Hence, the integer j is running modulo p^{2n-1}, e.g. $0\le j\le p^{2n-l}$. This gives the claimed representatives.

(b) This is shown completely analogous to part (a): An arbitrary element in $\Gamma_0(M)\alpha$ is of the form

$$\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} p^{-n} & 0 \\ 0 & p^n\end{pmatrix}=\begin{pmatrix} p^{-n}a & p^nb \\ p^{-n}c & p^nd\end{pmatrix}$$

with $ad-bc=1$ and $c\equiv 0 \mod M$. Let $l\in\{0,1,\dots,2n\}$ be minimal with $p^{2n-l}\mid c$. Then, $\left(\frac{c}{p^{2n-l}},p^ld\right)=1$, i.e. there exists $x,y\in\mathbb{Z}$ sucht that $\frac{c}{p^{2n-l}}x+p^ldy=1$. Hence, $\begin{pmatrix} p^ld & x-p^{2n}bdy \\ -\frac{c}{p^{2n-l}} & y+bcy\end{pmatrix}\in\Gamma_0(M)$ and

$$\begin{pmatrix} p^{-n}a & p^nb \\ p^{-n}c & p^nd\end{pmatrix}\begin{pmatrix} p^ld & x-p^{2n}bdy \\ -\frac{c}{p^{2n-l}} & y+bcy\end{pmatrix}=\begin{pmatrix} p^{l-n} & p^{-n}ax \\ 0 & p^{n-l}\end{pmatrix}.$$

Note that if $1\le l\le 2n-1$, then $ad-bc=1$ implies $p\nmid a$ and $\frac{c}{p^{2n-l}}x+p^ldy=1$ implies $p\nmid x$. Hence, $p\nmid ax$. Thus, a representative is

$$\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l}\end{pmatrix},$$

where $p \nmid j$ if $1 \le l \le 2n-1$. Since for any integer $m \in \mathbb{Z}$, the matrix $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(M)$ and

$$\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l} \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p^{l-n} & p^{l-n}m + p^{-n}j \\ 0 & p^{n-l} \end{pmatrix} = \begin{pmatrix} p^{l-n} & p^{-n}\left(j + mp^{l}\right) \\ 0 & p^{n-l} \end{pmatrix}$$

the integer j is running modulo p^l. This gives the claimed representatives.

□

Remark. Another choice of representatives would be

$$\alpha_{lj,0} = \begin{pmatrix} p^{n-l} & 0 \\ p^{-n}Mj & p^{l-n} \end{pmatrix} \quad , \quad \text{with } 0 \le l \le 2n, 0 \le j < p^{2n-l} \text{ and } p \nmid j \text{ whenever } l \neq 0, 2n$$

and

$$\beta_{lj,0} = \begin{pmatrix} p^{n-l} & 0 \\ p^{-n}Mj & p^{l-n} \end{pmatrix} \quad , \quad \text{with } 0 \le l \le 2n, 0 \le j < p^{l} \text{ and } p \nmid j \text{ whenever } l \neq 0, 2n$$

Corollary 3.4.2. *For $\xi = (\alpha, 1) \in \widetilde{\mathrm{SL}_2(\mathbb{R})}$ we have*

$$\Gamma_0(M)^* \xi \Gamma_0(M)^* = \coprod_{\substack{l=0 \\ j}}^{2n} \Gamma_0(M)^* \xi_{l,j} = \coprod_{\substack{l=0 \\ j}}^{2n} \xi'_{l,j} \Gamma_0(M)^*$$

with a choice of representatives $\{\xi_{l,j} \in \widetilde{\mathrm{SL}_2(\mathbb{R})}\}$ given as

$$\xi_{l,j} = \left(\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l} \end{pmatrix}, \left(\frac{-j}{p}\right)^{l} \epsilon_{p^l}^{-1} \right),$$

where $0 \le l \le 2n$ and $0 \le j < p^{2n-l}$ with $p \nmid j$ whenever $1 \le l \le 2n - l$, and $\{\xi'_{l,j} \in \widetilde{\mathrm{SL}_2(\mathbb{R})}\}$ given as

$$\xi'_{l,j} = \left(\begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l} \end{pmatrix}, \left(\frac{-j}{p}\right)^{l} \epsilon_{p^l}^{-1} \right),$$

where $0 \le l \le 2n$ and $0 \le j < p^{l}$ with $p \nmid j$ whenever $1 \le l \le 2n - l$.

Proof. Note that any representative $\alpha_{l,j}$ as in Theorem 3.4.1 (a) can be written as

$$\alpha_{l,j} = \begin{pmatrix} p^{l-n} & p^{-n}j \\ 0 & p^{n-l} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ p^{2n-l}Ms & 1 \end{pmatrix} \begin{pmatrix} p^{-n} & 0 \\ 0 & p^{n} \end{pmatrix} \begin{pmatrix} p^{l} & j \\ -Ms & r \end{pmatrix}$$

for some $r, s \in \mathbb{Z}$ with $p^l r + Msj = 1$. Using the isomorphism

$$\Gamma_0(M)\alpha\Gamma_0(M) \to \Gamma_0(M)^*\xi\Gamma_0(M)^* \quad , \quad \gamma_1\alpha\gamma_2 \mapsto \Phi(\gamma_1)\xi\Phi(\gamma_2)$$

as in Proposition 3.2.11 we get the corresponding representatives of $\Gamma_0(M)^*\xi\Gamma_0(M)^*$ as

$$\begin{aligned}
\xi_{l,j} &= \Phi\begin{pmatrix} 1 & 0 \\ p^{2n-l}Ms & 1 \end{pmatrix} \xi \Phi\begin{pmatrix} p^l & j \\ -Ms & r \end{pmatrix} \\
&= \left(\begin{pmatrix} 1 & 0 \\ p^{2n-l}Ms & 1 \end{pmatrix}, 1\right)\left(\begin{pmatrix} p^{-n} & 0 \\ 0 & p^n \end{pmatrix}, 1\right)\left(\begin{pmatrix} p^l & j \\ -Ms & r \end{pmatrix}, \left(\frac{-Ms}{r}\right)\epsilon_r^{-1}\right) \\
&= \left(\alpha_{l,j}, w_{\frac{1}{2}}\left(\begin{pmatrix} p^{-n} & 0 \\ p^{n-l}Ms & p^n \end{pmatrix}, \begin{pmatrix} p^l & j \\ -Ms & r \end{pmatrix}\right)\left(\frac{-Ms}{r}\right)\epsilon_r^{-1}\right) \\
&= \left(\alpha_{l,j}, \left(\frac{-Ms}{r}\right)\epsilon_r^{-1}\right).
\end{aligned}$$

Here, we used that $\arg(p^{n-l}Msi + p^n) + \arg(-Msi + p^l) = 0 \in (-\pi, \pi]$. Since $p^l r = 1 - Msj \equiv 1 \mod 4$, we have $r \equiv p^l \mod 4$, hence $\epsilon_r^{-1} = \epsilon_{p^l}^{-1}$. Moreover, since $Msj = 1 - p^l r \equiv 1 \mod r$, we get

$$\left(\frac{-Ms}{r}\right) = \left(\frac{Msj}{r}\right)\left(\frac{-j}{r}\right) = \left(\frac{-j}{r}\right) = \left(\frac{-j}{p^l}\right)\left(\frac{-j}{p^l r}\right).$$

Writing $j = 2^a j'$ with j' odd and using quadratic reciprocity we get further

$$\left(\frac{-j}{p^l r}\right) = \left(\frac{-2^a j'}{p^l r}\right) = \left(\frac{2}{p^l r}\right)^a \left(\frac{-j'}{p^l r}\right) = \left(\frac{2}{p^l r}\right)^a \left(\frac{p^l r}{-j'}\right)(-1)^{\frac{p^l r-1}{2}\frac{-j'-1}{2}} = \left(\frac{2}{p^l r}\right)^a \left(\frac{p^l r}{-j'}\right),$$

since $p^l r \equiv 1 \mod 4$. Moreover, if $a \geq 1$ (i.e. j is even), then $p^l r = 1 - Msj \equiv 1 \mod 8$. Thus, $\left(\frac{2}{p^l r}\right)^a = 1$ for every $a \geq 0$. Hence,

$$\left(\frac{-j}{p^l r}\right) = \left(\frac{p^l r}{-j'}\right) = \left(\frac{1 - Msj}{-j'}\right) = 1.$$

Overall, the representative becomes

$$\xi_{l,j} = \left(\alpha_{l,j}, \left(\frac{-j}{p^l}\right)\epsilon_{p^l}^{-1}\right) = \begin{cases} (\alpha_{l,j}, 1) & \text{if } l \text{ is even,} \\ \left(\alpha_{l,j}, \left(\frac{-j}{p}\right)\epsilon_p^{-1}\right) & \text{if } l \text{ is odd.} \end{cases}$$

The identical argumentation applies to $\beta_{l,j}$ giving $\xi'_{l,j}$. □

Remark. Another choice of representatives would be $\{\xi_{l,j,0} \in \widetilde{\mathrm{SL}_2(\mathbb{R})}\}$ given as

$$\xi_{l,j,0} = \left(\begin{pmatrix} p^{n-l} & 0 \\ p^{-n}Mj & p^{l-n} \end{pmatrix}, \left(\frac{Mj}{p}\right)^l \epsilon_{p^l}^{-1} \right),$$

where $0 \le l \le 2n$ and $0 \le j < p^{2n-l}$ with $p \nmid j$ whenever $1 \le l \le 2n-l$, and $\{\xi'_{l,j,0} \in \widetilde{\mathrm{SL}_2(\mathbb{R})}\}$ given as

$$\xi'_{l,j,0} = \left(\begin{pmatrix} p^{n-l} & 0 \\ p^{-n}Mj & p^{l-n} \end{pmatrix}, \left(\frac{Mj}{p}\right)^l \epsilon_{p^l}^{-1} \right),$$

where $0 \le l \le 2n$ and $0 \le j < p^l$ with $p \nmid j$ whenever $1 \le l \le 2n-l$.

Theorem 3.4.3. *The p^{2n}-th Hecke eigenvalue of the Eisenstein series $E_{\mathfrak{a}}(z, s, \frac{1}{2})$ of weight $\frac{1}{2}$ on $\Gamma_0(M)$ is*

$$\lambda_{p^{2n}} = p^{2ns} + p^{2n(1-s)} + (p-1)\frac{p^{2(n-1)(1-s)} - p^{2(n-1)s}}{p^{1-2s} - p^{2s-1}}.$$

For $n = 0$, the Hecke operator T_{p^0} is the identity and has eigenvalue 1. Moreover, these $\lambda_{p^{2n}}$ are also the Hecke eigenvalues for linear combinations $E(z,s) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(z, s, \frac{1}{2}) \in V_s$.

Proof. We will prove in detail the case of cusp $\mathfrak{a} = \infty$ and explain why the analog argument works for the cusp $\mathfrak{a} = 0$. For general cusp, the same method is expected to work as well, but we omit the technical details in the case of other cusps here, since we only need the statement for the cusps ∞ and 0 for the conclusion in chapter 4.

We let $n \ge 1$ and write for short $E_{\mathfrak{a}}(z) := E_{\mathfrak{a}}(z, s, \frac{1}{2})$. Then, with $\xi_i = (\alpha_i, \zeta(\alpha_i))$ running over the set of representatives as in Corollary 3.4.2 we have

$$\begin{aligned}
(T_{p^{2n}} E_\infty)(z) &= \sum_i (E_\infty \mid [\xi_i])(z) = \sum_i E_\infty(\alpha_i \cdot z) \zeta(\alpha_i)^{-1} j_{\alpha_i}(z)^{-\frac{1}{2}} \\
&= \sum_i \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(M)} \overline{\nu(\gamma)} j_\gamma(\alpha_i \cdot z)^{-\frac{1}{2}} \Im(\gamma\alpha_i \cdot z)^s \zeta(\alpha_i)^{-1} j_{\alpha_i}(z)^{-\frac{1}{2}} \\
&= \sum_{\substack{i \\ \gamma \in \Gamma_\infty \backslash \Gamma_0(M)}} \zeta(\gamma)^{-1} \zeta(\alpha_i)^{-1} w_{\frac{1}{2}}(\gamma, \alpha_i)^{-1} j_{\gamma\alpha_i}(z)^{-\frac{1}{2}} \Im(\gamma\alpha_i \cdot z)^s \\
&= \sum_{\substack{i \\ \gamma \in \Gamma_\infty \backslash \Gamma_0(M)}} \zeta(\gamma\alpha_i)^{-1} j_{\gamma\alpha_i}(z)^{-\frac{1}{2}} \Im(\gamma\alpha_i \cdot z)^s \\
&= \sum_{\gamma' \in \Gamma_\infty \backslash \Gamma_0(M)\alpha\Gamma_0(M)} \zeta(\gamma')^{-1} j_{\gamma'}(z)^{-\frac{1}{2}} \Im(\gamma' \cdot z)^s,
\end{aligned}$$

where in the last equality we used the isomorphism

$$\Gamma_\infty\backslash\Gamma_0(M)\times\{i \mid \{\alpha_i\} \text{ as in Theorem 3.4.1}(a)\}\to\Gamma_\infty\backslash\Gamma_0(M)\alpha\Gamma_0(M) \quad , \quad (\gamma,i)\mapsto\gamma\alpha_i.$$

Surjectivity holds by construction, respectively by Theorem 3.4.1 (a). Injectivity can be shown as follows: If $\gamma_1,\gamma_2\in\Gamma_0(M)$ and i,j such that $\Gamma_\infty\gamma_1\alpha_i=\Gamma_\infty\gamma_2\alpha_j$, then $\Gamma_0(M)\alpha_i\cap\Gamma_0(M)\alpha_j\neq\emptyset$. But this implies that $i=j$. It follows that $\Gamma_\infty\gamma_1=\Gamma_\infty\gamma_2$, which finishes the proof of injectivity.

Now, by constructions of the $\{\beta_i\}$ as in Theorem 3.4.1 (b) we have

$$\begin{aligned}\Gamma_\infty\backslash\Gamma_0(M)\alpha\Gamma_0(M)&=\Gamma_\infty\backslash\coprod_i\beta_i\Gamma_0(M)=\Gamma_\infty\backslash\coprod_{l=0}^{2n}\coprod_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0,2n}}^{p^l-1}\beta_{l,j}\Gamma_0(M)\\&=\coprod_{l=0}^{2n}\coprod_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0,2n}}^{p^{\min(l,2n-l)}-1}\Gamma_\infty\backslash\beta_{l,j}\Gamma_0(M)\end{aligned}$$

The last equality holds because, similarly as in the proof of Theorem 3.4.1 (a), $\beta_{l,j}$ and $\beta_{l,j+p^{2n-l}m}$ describe the same elements in the quotient for every $m\in\mathbb{Z}$. We have further for every l and every j a bijection

$$\begin{aligned}\varphi_{l,j}:\Gamma_\infty\backslash\beta_{l,j}\left(\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_\infty\beta_{l,j})\backslash\Gamma_0(M)\right)&\to\Gamma_\infty\backslash\beta_{l,j}\Gamma_0(M)\\ \beta_{l,j}\,\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_\infty\beta_{l,j})\gamma&\mapsto\beta_{l,j}\gamma.\end{aligned}$$

Note that it is indeed injective: If $\gamma_1,\gamma_2\in\Gamma_0(M)$ such that $\Gamma_\infty\beta_{l,j}\gamma_1=\Gamma_\infty\beta_{l,j}\gamma_2$, then $\Gamma_\infty\beta_{l,j}\gamma_1\gamma_2^{-1}=\Gamma_\infty\beta_{l,j}$, i.e. $\gamma_1\gamma_2^{-1}\in\mathrm{Stab}_{\Gamma_0(N)}(\Gamma_\infty\beta_{l,j})$. Hence, $\mathrm{Stab}_{\Gamma_0(N)}(\Gamma_\infty\beta_{l,j})\gamma_2=\mathrm{Stab}_{\Gamma_0(N)}(\Gamma_\infty\beta_{l,j})\gamma_1\gamma_2^{-1}\gamma_2=\mathrm{Stab}_{\Gamma_0(N)}(\Gamma_\infty\beta_{l,j})\gamma_1$ and the map is indeed injective.

The stabilizers are

$$\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_\infty\beta_{l,j})=\{\gamma\in\Gamma_0(M)\mid\Gamma_\infty\beta_{l,j}\gamma=\Gamma_\infty\beta_{l,j}\}.$$

Note that, since elements in Γ_∞ as well as $\beta_{l,j}$ are upper triangular matrices, every element in the stabilizer has to be upper triangular, i.e. in Γ_∞ as well. If $\gamma=\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)$ is an element in the stabilizer, then it has to satisfy

$$\begin{aligned}\Gamma_\infty\begin{pmatrix}p^{l-n}&p^{-n}j\\0&p^{n-l}\end{pmatrix}&\overset{!}{=}\Gamma_\infty\begin{pmatrix}p^{l-n}&p^{-n}j\\0&p^{n-l}\end{pmatrix}\begin{pmatrix}1&m\\0&1\end{pmatrix}=\Gamma_\infty\begin{pmatrix}p^{l-n}&p^{-n}j+p^{l-n}m\\0&p^{n-l}\end{pmatrix}\\&=\Gamma_\infty\begin{pmatrix}1&p^{2l-2n}m\\0&1\end{pmatrix}\begin{pmatrix}p^{l-n}&p^{-n}j\\0&p^{n-l}\end{pmatrix}.\end{aligned}$$

For this condition to hold we need $p^{2l-2n}m \in \mathbb{Z}$. Thus, the stabilizer is

$$\begin{aligned}\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_\infty\beta_{l,j}) &= \left\{ \pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} \in \Gamma_\infty \;\middle|\; p^{2l-2n}m \in \mathbb{Z}\right\}\\ &= \begin{cases}\Gamma_\infty & \text{if } n \le l \le 2n,\\ \left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \mid m \in p^{2n-2l}\mathbb{Z}\right\} & \text{if } 0 \le l \le n-1.\end{cases}\end{aligned}$$

Combining the described bijections, we get

$$\begin{aligned}\Gamma_\infty\backslash\Gamma_0(M)\alpha\Gamma_0(M) &= \coprod_{l=0}^{2n}\coprod_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0,2n}}^{p^{\min(l,2n-l)}-1} \Gamma_\infty\backslash\beta_{l,j}\left(\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_\infty\beta_{l,j})\backslash\Gamma_0(M)\right)\\ &= \coprod_{l=0}^{n-1}\coprod_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0,2n}}^{p^l-1} \Gamma_\infty\backslash\alpha_{l,j}\left(\left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \mid m \in p^{2n-2l}\mathbb{Z}\right\}\backslash\Gamma_0(M)\right)\\ &\qquad \sqcup \coprod_{l=n}^{2n}\coprod_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0,2n}}^{p^{2n-l}-1} \Gamma_\infty\backslash\alpha_{l,j}\left(\Gamma_\infty\backslash\Gamma_0(M)\right),\end{aligned}$$

and hence,

$$\begin{aligned}(T_{p^{2n}}E_\infty)(z) &= \sum_{\gamma'\in\Gamma_\infty\backslash\Gamma_0(M)\alpha\Gamma_0(M)} \zeta(\gamma')^{-1} j_{\gamma'}(z)^{-\frac{1}{2}}\Im(\gamma'\cdot z)^s\\ &= \sum_{l=0}^{n-1}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0}}^{p^l-1}\sum_{\gamma\in\left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)\middle| m\in p^{2n-2l}\mathbb{Z}\right\}\backslash\Gamma_0(M)} \zeta(\alpha_{l,j}\gamma)^{-1} j_{\alpha_{l,j}\gamma}(z)^{-\frac{1}{2}}\Im(\alpha_{l,j}\gamma\cdot z)^s\\ &\quad + \sum_{l=n}^{2n}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 2n}}^{p^{2n-l}-1}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)} \zeta(\alpha_{l,j}\gamma)^{-1} j_{\alpha_{l,j}\gamma}(z)^{-\frac{1}{2}}\Im(\alpha_{l,j}\gamma\cdot z)^s.\end{aligned}$$

Note that $\zeta(\alpha_{l,j}\gamma)^{-1} = \zeta(\alpha_{l,j})^{-1}\zeta(\gamma)^{-1}w_{\frac{1}{2}}(\alpha_{l,j},\gamma)^{-1} = \left(\frac{-j}{p^l}\right)\epsilon_{p^l}\overline{\nu(\gamma)}$. Moreover,

$$\alpha_{l,j}\gamma = \begin{pmatrix}p^{l-n} & p^{-n}j\\ 0 & p^{n-l}\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}p^{l-n}a + p^{-n}cj & p^{l-n}b + p^{-n}dy\\ p^{n-l}c & p^{n-l}d\end{pmatrix},$$

hence

$$j_{\alpha_{l,j}\gamma}(z) = \frac{p^{n-l}cz + p^{n-l}d}{|p^{n-l}cz + p^{n-l}d|} = \frac{cz+d}{|cz+d|} = j_\gamma(z).$$

Also,

$$\Im(\alpha_{l,j}\cdot(\gamma\cdot z)) = \Im\left(\frac{p^{l-n}\gamma\cdot z + p^{-n}j}{p^{n-l}}\right) = p^{2l-2n}\Im(\gamma\cdot z).$$

Hence, we get further

$$\begin{aligned}
(T_{p^{2n}}E_\infty)(z) &= \sum_{l=0}^{n-1}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0}}^{p^l-1}\ \sum_{\gamma\in\left\{\pm\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\middle| m\in p^{2n-2l}\mathbb{Z}\right\}\backslash\Gamma_0(M)}\left(\frac{-j}{p^l}\right)\epsilon_{p^l}\overline{\nu(\gamma)}j_\gamma(z)^{-\frac{1}{2}}p^{(2l-2n)s}\Im(\gamma\cdot z)^s\\
&\quad+\sum_{l=n}^{2n}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 2n}}^{p^{2n-l}-1}\ \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)}\left(\frac{-j}{p^l}\right)\epsilon_{p^l}\overline{\nu(\gamma)}j_\gamma(z)^{-\frac{1}{2}}p^{(2l-2n)s}\Im(\gamma\cdot z)^s\\
&=\sum_{l=0}^{n-1}p^{(2l-2n)s}\epsilon_{p^l}\sum_{\gamma\in\left\{\pm\left(\begin{smallmatrix}1&m\\0&1\end{smallmatrix}\right)\middle| m\in p^{2n-2l}\mathbb{Z}\right\}\backslash\Gamma_0(M)}\ \sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0}}^{p^l-1}\left(\frac{-j}{p^l}\right)\overline{\nu(\gamma)}j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s\\
&\quad+\sum_{l=n}^{2n}p^{(2l-2n)s}\epsilon_{p^l}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)}\ \sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 2n}}^{p^{2n-l}-1}\left(\frac{-j}{p^l}\right)\overline{\nu(\gamma)}j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s.
\end{aligned}$$

We note that

$$\epsilon_{p^l}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 0}}^{p^l-1}\left(\frac{-j}{p^l}\right) = \begin{cases}0 & \text{if } l \text{ is odd}\\ 1 & \text{if } l=0\\ p^l-p^{l-1} & \text{if } l\neq 0 \text{ is even}\end{cases}$$

and analogous

$$\epsilon_{p^l}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 2n}}^{p^{2n-l}-1}\left(\frac{-j}{p^l}\right) = \epsilon_{p^l}\sum_{\substack{j=0\\ p\nmid j \text{ if } l\neq 2n}}^{p^{2n-l}-1}\left(\frac{-j}{p^{2n-l}}\right) = \begin{cases}0 & \text{if } l \text{ is odd,}\\ 1 & \text{if } l=2n,\\ p^{2n-l}-p^{2n-l-1} & \text{if } l\neq 2n \text{ is even.}\end{cases}$$

Hence, all summands with l odd vanish and we get

$$\begin{aligned}(T_{p^{2n}}E_\infty)(z) = p^{-2ns} &\sum_{\gamma\in\left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)\middle| m\in p^{2n}\mathbb{Z}\right\}\backslash\Gamma_0(M)} \overline{\nu(\gamma)} j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s \\ &+ \sum_{\substack{l=2\\ \text{even}}}^{n-1} p^{(2l-2n)s}\left(p^l - p^{l-1}\right) \sum_{\gamma\in\left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right)\middle| m\in p^{2n-2l}\mathbb{Z}\right\}\backslash\Gamma_0(M)} \overline{\nu(\gamma)} j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s \\ &+ \sum_{\substack{l=n\\ \text{even}}}^{2n-2} p^{(2l-2n)s}\left(p^{2n-l} - p^{2n-l-1}\right) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)} \overline{\nu(\gamma)} j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s \\ &+ p^{2ns} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)} \overline{\nu(\gamma)} j_\gamma(z)^{-\frac{1}{2}}\Im(\gamma\cdot z)^s.\end{aligned}$$

Note that $H_l := \left\{\pm\left(\begin{smallmatrix}1 & m\\ 0 & 1\end{smallmatrix}\right) \,\middle|\, m\in p^{2n-2l}\mathbb{Z}\right\} < \Gamma_\infty$ is a normal subgroup of index p^{2n-2l}. Using $H_l\backslash\Gamma_0(M) = (H_l\backslash\Gamma_\infty)\times(\Gamma_\infty\backslash\Gamma_0(M))$ and the invariance of all involved functions by multiplication of the argument on the left with elements in Γ_∞ we get further

$$\begin{aligned}(T_{p^{2n}}E_\infty)(z) = p^{-2ns}p^{2n}E_\infty(z) &+ \sum_{\substack{l=2\\ \text{even}}}^{n-1} p^{(2l-2n)s}\left(p^l - p^{l-1}\right)p^{2n-2l}E_\infty(z) \\ &+ \sum_{\substack{l=n\\ \text{even}}}^{2n-2} p^{(2l-2n)s}\left(p^{2n-l} - p^{2n-l-1}\right)E_\infty(z) + p^{2ns}E_\infty(z).\end{aligned}$$

Hence, for $n\geq 1$, the eigenvalue is

$$\begin{aligned}\lambda_{p^{2n}} &= p^{2ns} + p^{2n(1-s)} + (p-1)\sum_{\substack{l=2\\ \text{even}}}^{2n-2} p^{(2l-2n)s}p^{2n-l-1} \\ &= p^{2ns} + p^{2n(1-s)} + (p-1)p^{2n(1-s)-1}\sum_{r=1}^{n-1}\left(p^{4s-2}\right)^r \\ &= p^{2ns} + p^{2n(1-s)} + (p-1)p^{2n(1-s)-1}\left(\frac{1-p^{(4s-2)n}}{1-p^{4s-2}} - 1\right) \\ &= p^{2ns} + p^{2n(1-s)} + (p-1)p^{2n(1-s)-1}\frac{p^{4s-2} - p^{(4s-2)n}}{1-p^{4s-2}} \\ &= p^{2ns} + p^{2n(1-s)} + (p-1)\frac{p^{2(n-1)(1-s)} - p^{2(n-1)s}}{p^{1-2s} - p^{2s-1}}.\end{aligned}$$

For cusp $\mathfrak{a} = 0$ the argument is completely analogous. The main difference is that from the definition of the Eisenstein series there are additional σ_0 appearing and that we work with the representatives as in the Remark after Corollary 3.4.2, which are lower triangular matrices as are the elements of Γ_0. Hence, also elements in the stabilizer are lower triangular, more precisely,

$$\begin{aligned}\mathrm{Stab}_{\Gamma_0(M)}(\Gamma_0\beta_{l,j}) &= \left\{\pm\begin{pmatrix}1 & 0\\ Mm & 1\end{pmatrix}\in\Gamma_0 \,\middle|\, p^{2l-2n}m\in\mathbb{Z}\right\}\\ &= \begin{cases}\Gamma_0 & \text{if } n\le l\le 2n,\\ \left\{\pm\left(\begin{smallmatrix}1 & 0\\ Mm & 1\end{smallmatrix}\right) \mid m\in p^{2n-2l}\mathbb{Z}\right\} & \text{if } 0\le l\le n-1.\end{cases}\end{aligned}$$

For general cusp $\mathfrak{a}$ the argumentation with triangular matrices does not apply anymore and it gets more technical to calculate the stabilizer, which we omit here. □

Corollary 3.4.4. *The p^{2n}-th Hecke eigenvalue of any linear combination of weight $\frac{1}{2}$ Eisenstein series $E(z,\frac{1}{2}+it)=\sum_{\mathfrak{a}} c(\mathfrak{a})E_{\mathfrak{a}}(z,\frac{1}{2}+it,\frac{1}{2})\in V_{\frac{1}{2}+it}$ for $n\ge 1$ is*

$$\lambda_{p^{2n}} = 2p^n\cos(2nt\log(p)) + \left(p^n - p^{n-1}\right)\frac{\sin(2(n-1)t\log(p))}{\sin(2t\log(p))}.$$

Proof. By Theorem 3.4.3 the p^{2n}-th Hecke eigenvalue is

$$\begin{aligned}\lambda_{p^{2n}} &= p^{2ns} + p^{2n(1-s)} + (p-1)\frac{p^{2(n-1)(1-s)} - p^{2(n-1)s}}{p^{1-2s} - p^{2s-1}}\\ &= p^{n+2nit} + p^{n-2nit} + (p-1)\frac{p^{(n-1)(1-2it)} - p^{(n-1)(1+2it)}}{p^{-2it} - p^{2it}}\\ &= p^n\left(p^{2nit} + p^{-2nit}\right) + \left(p^n - p^{n-1}\right)\frac{p^{2(n-1)it} - p^{-2(n-1)it}}{p^{2it} - p^{-2it}}\\ &= 2p^n\cos(2nt\log(p)) + \left(p^n - p^{n-1}\right)\frac{\sin(2(n-1)t\log(p))}{\sin(2t\log(p))}.\end{aligned}$$

□

Proposition 3.4.5. *The Hecke eigenvalues of $E(z,\frac{1}{2}+it)\in V_{\frac{1}{2}+it}$ satisfy the following recurrence relation:*

$$\lambda_{p^{2(n+1)}} = \lambda_{p^2}\lambda_{p^{2n}} - p^2\lambda_{p^{2(n-1)}} \quad \textit{for } n\ge 2.$$

In particular,

$$T_{p^{2(n+1)}}E(z,\tfrac{1}{2}+it) = \left(T_{p^2}\circ T_{p^{2n}} - p^2 T_{p^{2(n-1)}}\right)E(z,\tfrac{1}{2}+it) \quad \textit{for } n\ge 2.$$

Proof. Recall the expressions for the Hecke eigenvalues from Corollary 3.4.4. With the abbreviation $\theta = 2t\log(p)$ and using the addition theorems of sine and cosine several times, we have

$$\begin{aligned}
\lambda_{p^2}\lambda_{p^{2n}} - p^2\lambda_{p^{2(n-1)}} &= 2p\cos(\theta)\left(2p^n\cos(n\theta) + \left(p^n - p^{n-1}\right)\frac{\sin((n-1)\theta)}{\sin(\theta)}\right) \\
&\quad - p^2\left(2p^{n-1}\cos((n-1)\theta) + \left(p^{n-1} - p^{n-2}\right)\frac{\sin((n-2)\theta)}{\sin(\theta)}\right) \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + 2\left(p^{n+1} - p^n\right)\frac{\cos(\theta)\sin((n-1)\theta)}{\sin(\theta)} \\
&\quad - 2p^{n+1}\cos((n-1)\theta) - \left(p^{n+1} - p^n\right)\frac{\sin((n-1)\theta - \theta)}{\sin(\theta)} \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + 2\left(p^{n+1} - p^n\right)\frac{\cos(\theta)\sin((n-1)\theta)}{\sin(\theta)} \\
&\quad - 2p^{n+1}\cos((n-1)\theta) - \left(p^{n+1} - p^n\right)\left(\frac{\sin((n-1)\theta)\cos(\theta)}{\sin(\theta)} - \cos((n-1)\theta)\right) \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + \left(p^{n+1} - p^n\right)\frac{\cos(\theta)\sin((n-1)\theta)}{\sin(\theta)} - \left(p^{n+1} + p^n\right)\cos((n-1)\theta) \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + \left(p^{n+1} - p^n\right)\frac{\sin((n-1)\theta + \theta) - \cos((n-1)\theta)\sin(\theta)}{\sin(\theta)} \\
&\quad - \left(p^{n+1} + p^n\right)\cos((n-1)\theta) \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + \left(p^{n+1} - p^n\right)\frac{\sin(n\theta)}{\sin(\theta)} - 2p^{n+1}\cos((n-1)\theta) \\
&= 4p^{n+1}\cos(\theta)\cos(n\theta) + \left(p^{n+1} - p^n\right)\frac{\sin(n\theta)}{\sin(\theta)} - 2p^{n+1}\left(\cos(\theta)\cos(n\theta) + \sin(\theta)\sin(n\theta)\right) \\
&= 2p^{n+1}\left(\cos(\theta)\cos(n\theta) - \sin(\theta)\sin(n\theta)\right) + \left(p^{n+1} - p^n\right)\frac{\sin(n\theta)}{\sin(\theta)} \\
&= 2p^{n+1}\cos((n+1)\theta) + \left(p^{n+1} - p^n\right)\frac{\sin(n\theta)}{\sin(\theta)} = \lambda_{p^{2(n+1)}}.
\end{aligned}$$

□

Recall the function

$$\varphi_{t,\frac{1}{2}} : \widetilde{X} \to \mathbb{C} \quad \text{given by} \quad \varphi_{t,\frac{1}{2}}(g,\zeta) = \frac{1}{\sqrt{c\log|t|}}E(g\cdot i, \tfrac{1}{2} + it)\zeta^{-1}j_g(i)^{-\frac{1}{2}}$$

for $E(g\cdot i, \frac{1}{2}+it) \in V_{\frac{1}{2}+it}$. Note that the Hecke operators act on functions $f : \widetilde{X} \to \mathbb{C}$ by $T_{p^{2n}}(f)(g,\zeta) = \sum_i f(\xi\cdot(g,\zeta)) = \sum_i f(\alpha_i g, w_{\frac{1}{2}}(\alpha_i, g)\zeta(\alpha_i)\zeta)$, where the sum is over a set of representatives $\{\xi_i\} = \{(\alpha_i, \zeta(\alpha_i))\}$ as in Corollary 3.4.2. Also note the following property of Hecke operators:

Proposition 3.4.6. *Every differential operator $m \in \mathfrak{sl}_2(\mathbb{C})$ commutes with the Hecke operators $T_{p^{2n}}$.*

Proof. For every function $f : \widetilde{X} \to \mathbb{C}$ we have

$$\begin{aligned}(m * T_{p^{2n}}(f))(g,\zeta) &= \frac{\partial}{\partial r} T_{p^{2n}}(f)((g,\zeta)(\exp(rm),1))\Big|_{r=0} \\ &= \frac{\partial}{\partial r}\sum_i f((\alpha_i,\zeta(\alpha_i))(g,\zeta)(\exp(rm),1))\Big|_{r=0} \\ &= \sum_i \frac{\partial}{\partial r} f((\alpha_i,\zeta(\alpha_i))(g,\zeta)(\exp(rm),1))\Big|_{r=0} \\ &= \sum_i (m*f)((\alpha_i,\zeta(\alpha_i))(g,\zeta)) \\ &= T_{p^{2n}}(m*f)(g,\zeta).\end{aligned}$$

□

Corollary 3.4.7. *The Hecke eigenvalues of the microlocal lift $\widetilde{\varphi}_t = \frac{1}{\sqrt{2N+1}}\sum_{n=-N}^{N}\varphi_{t,\frac{1}{2}+2n}$, where $\varphi_{t,\frac{1}{2}+2n}$ are constructed from $\varphi_{t,\frac{1}{2}}$ by raising and lowering operator as in the previous section, are the same as for $E(z,\frac{1}{2}+it)$, i.e.*

$$\lambda_{p^{2n}} = 2p^n\cos(2nt\log(p)) + (p^n - p^{n-1})\frac{\sin(2(n-1)t\log(p))}{\sin(2t\log(p))}.$$

Proof. We can show by induction that $T_{p^{2n}}\varphi_{t,\frac{1}{2}+2l} = \lambda_{p^{2n}}\varphi_{t,\frac{1}{2}+2l}$. Indeed, for $l = 0$ we have

$$\begin{aligned}\left(T_{p^{2n}}\varphi_{t,\frac{1}{2}}\right)(g,\zeta) &= \sum_i \varphi_{t,\frac{1}{2}}(\alpha_i g, \zeta(\alpha_i)\zeta) \\ &= \sum_i \frac{1}{\sqrt{c\log|t|}} E(\alpha_i g\cdot i, \tfrac{1}{2}+it)\zeta(\alpha_i)^{-1}\zeta^{-1} j_{\alpha_i g}(i)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{c\log|t|}}\zeta^{-1} j_g(i)^{-\frac{1}{2}} \sum_i E(\alpha_i g\cdot i, \tfrac{1}{2}+it)\zeta(\alpha_i)^{-1} j_{\alpha_i}(g\cdot i)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{c\log|t|}}\zeta^{-1} j_g(i)^{-\frac{1}{2}} \underbrace{\left(T_{p^{2n}}E(\cdot,\tfrac{1}{2}+it)\right)(g\cdot i)}_{=\lambda_{p^{2n}}E(g\cdot i,\frac{1}{2}+it)\text{ by Theorem 3.4.3}} \\ &= \lambda_{p^{2n}}\frac{1}{\sqrt{c\log|t|}} E(g\cdot i,\tfrac{1}{2}+it)\zeta^{-1} j_g(i)^{-\frac{1}{2}} = \lambda_{p^{2n}}\varphi_{t,\frac{1}{2}}(g,\zeta).\end{aligned}$$

In the third equality we used that $j_{\alpha_i g}(i)^{-\frac{1}{2}} = j_{\alpha_i}(g\cdot i)^{-\frac{1}{2}} j_g(i)^{-\frac{1}{2}}$. This indeed holds, since the α_i are upper triangular matrices, hence $w_{\frac{1}{2}}(\alpha_i, g) = 1$. Assume that we know the statement for some l. Since the Hecke operator commutes with any differential operator, in particular with the raising and lowering operators, we can use equation (3.3.1) to get

$$\begin{aligned} T_{p^{2n}}(\varphi_{t,\frac{1}{2}+2(l+1)}) &\overset{(3.3.1)}{=} T_{p^{2n}}\left(\frac{1}{\frac{3}{4}+l+it}\mathcal{E}^+ * \varphi_{t,\frac{1}{2}+2l}\right) = \frac{1}{\frac{3}{4}+l+it}\mathcal{E}^+ * \left(T_{p^{2n}}\varphi_{t,\frac{1}{2}+2l}\right) \\ &= \lambda_{p^{2n}}\frac{1}{\frac{3}{4}+l+it}\mathcal{E}^+ * \varphi_{t,\frac{1}{2}+2l} = \lambda_{p^{2n}}\varphi_{t,\frac{1}{2}+2(l+1)} \end{aligned}$$

and

$$\begin{aligned} T_{p^{2n}}(\varphi_{t,\frac{1}{2}+2(l-1)}) &\overset{(3.3.1)}{=} T_{p^{2n}}\left(\frac{1}{\frac{1}{4}-l+it}\mathcal{E}^- * \varphi_{t,\frac{1}{2}+2l}\right) = \frac{1}{\frac{1}{4}-l+it}\mathcal{E}^- * \left(T_{p^{2n}}\varphi_{t,\frac{1}{2}+2l}\right) \\ &= \lambda_{p^{2n}}\frac{1}{\frac{1}{4}-l+it}\mathcal{E}^- * \varphi_{t,\frac{1}{2}+2l} = \lambda_{p^{2n}}\varphi_{t,\frac{1}{2}+2(l-1)}. \end{aligned}$$

Hence, also the microlocal lift satisfies

$$T_{p^{2n}}\widetilde{\varphi}_t = \frac{1}{\sqrt{2N+1}}\sum_{l=-N}^{N} T_{p^{2n}}\varphi_{t,\frac{1}{2}+2l} = \lambda_{p^{2n}}\frac{1}{\sqrt{2N+1}}\sum_{l=-N}^{N}\varphi_{t,\frac{1}{2}+2l} = \lambda_{p^{2n}}\widetilde{\varphi}_t.$$

□

Lemma 3.4.8. *The Hecke eigenvalues of $\widetilde{\varphi}_t$ satisfy*

$$\sum_{\substack{n=0\\ even}}^{2m} p^{-\frac{n}{2}}\lambda_{p^{2n}} = p^m\frac{\sin((2m+1)\theta)}{\sin(\theta)},$$

where $\theta = 2t\log(p)$.

Proof. With the abbreviation $\theta = 2t\log(p)$ we have

$$\lambda_{p^{2n}} = 2p^n\cos(n\theta) + \left(p^n - p^{n-1}\right)\frac{\sin((n-1)\theta)}{\sin(\theta)}$$

for $n \geq 1$. We prove the Lemma by induction on $m \geq 0$. For $m = 0$, the equation $1 = 1$ is trivially true. Let $m \geq 1$ and assume the equation holds for $m-1$. Then, we have by induction hypothesis

$$\begin{aligned}\sum_{\substack{n=0\\\text{even}}}^{2m} p^{-\frac{n}{2}}\lambda_{p^{2n}} &= \sum_{\substack{n=0\\\text{even}}}^{2(m-1)} p^{-\frac{n}{2}}\lambda_{p^{2n}} + p^{-m}\lambda_{p^{4m}} \\ &= p^{m-1}\frac{\sin((2m-1)\theta)}{\sin(\theta)} + 2p^m\cos(2m\theta) + \left(p^m - p^{m-1}\right)\frac{\sin((2m-1)\theta)}{\sin(\theta)} \\ &= 2p^m\cos(2m\theta) + p^m\frac{\sin((2m-1)\theta)}{\sin(\theta)}.\end{aligned}$$

Using that

$$\sin(2m\theta-\theta) = \sin(2m\theta)\cos(\theta) - \cos(2m\theta)\sin(\theta) = \sin(2m\theta+\theta) - 2\cos(2m\theta)\sin(\theta)$$

we get

$$\begin{aligned}\sum_{\substack{n=0\\\text{even}}}^{2m} p^{-\frac{n}{2}}\lambda_{p^{2n}} &= p^m\frac{2\cos(2m\theta)\sin(\theta) + \sin((2m-1)\theta)}{\sin(\theta)} \\ &= p^m\frac{\sin((2m+1)\theta)}{\sin(\theta)}.\end{aligned}$$

□

Definition 3.4.9. *A finite measure $\widetilde{\mu}$ on $X = \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ is called Hecke-p-recurrent if for every Borel set $B \subset X$ and $\widetilde{\mu}$-almost every $x \in B$ we have $x \in T_{p^{2n}}(\mathbb{1}_B)$ for infinitely many n, or equivalently, for every $k \in \mathbb{Z}_{\geq 1}$ there exists some $m_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{n=0}^{m_k} T_{p^{2n}}(\mathbb{1}_B)(x) \geq k$.*

The proof of the Hecke recurrence uses the following key Proposition (cf Proposition 3.22 in [7] and Lemma 38 in [19]).

Proposition 3.4.10. *There exists an absolute constant $c_0 > 0$ such that*

$$\sum_{n=0}^{m} T_{p^{2n}}|\widetilde{\varphi}_t|^2 \geq c_0 m|\widetilde{\varphi}_t|^2 \quad \textit{for every } m \geq 0.$$

Proof. We have

$$\begin{aligned}\sum_{n=0}^{m} T_{p^{2n}}|\widetilde{\varphi}_t(g,\zeta)|^2 &= \sum_{n=0}^{m}\sum_i |\widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)|^2 \geq \sum_{\substack{n=0\\\text{even}}}^{m}\sum_i |\widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)|^2 \\ &\geq \max_{k\leq\frac{m}{2}} \sum_{\substack{n=0\\\text{even}}}^{2k}\sum_i \left|p^{-\frac{n}{2}}\widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)\right|^2.\end{aligned}$$

Here, we used that $p^{-\frac{n}{2}} \leq 1$. By Cauchy-Schwarz inequality we have

$$\sum_{\substack{n=0\\ \text{even}}}^{2k} \sum_i \left| p^{-\frac{n}{2}} \widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta) \right|^2 \geq \left| \sum_{\substack{n=0\\ \text{even}}}^{2k} \sum_i p^{-\frac{n}{2}} \widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta) \right|^2 \cdot \frac{1}{\sum_{\substack{n=0\\ \text{even}}}^{2k} \sum_i 1}.$$

We know from Theorem 3.4.1 (a) that i is ranging over $p^{2n} + p^{2n-1}$ elements. Hence,

$$\sum_{\substack{n=0\\ \text{even}}}^{2k} \sum_i 1 = \sum_{n=0}^{k} \sum_i 1 = \sum_{n=0}^{k} \left(p^{2n} + p^{2n-1}\right) = \left(1 + \frac{1}{p}\right) \sum_{n=0}^{k} p^{2n} = \frac{p+1}{p} \cdot \frac{p^{2k+2} - 1}{p^2 - 1}$$
$$= \frac{p^{2k+3} + p^{2k+2} - p - 1}{p^3 - p} \leq \frac{2p^{2k+3}}{\frac{1}{2}p^3} = 4p^{2k}$$

Thus, we get

$$\begin{aligned}
\sum_{n=0}^{m} T_{p^{2n}} |\widetilde{\varphi}_t(g, \zeta)|^2 &\geq \max_{k \leq \frac{m}{2}} \frac{1}{4p^{2k}} \left| \sum_{\substack{n=0\\ \text{even}}}^{2k} p^{-\frac{n}{2}} \sum_i \widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta) \right|^2 \\
&= \max_{k \leq \frac{m}{2}} \frac{1}{4p^{2k}} \left| \sum_{\substack{n=0\\ \text{even}}}^{2k} p^{-\frac{n}{2}} T_{p^{2n}} \widetilde{\varphi}_t(g, \zeta) \right|^2 = \max_{k \leq \frac{m}{2}} \frac{1}{4p^{2k}} \left| \sum_{\substack{n=0\\ \text{even}}}^{2k} p^{-\frac{n}{2}} \lambda_{p^{2n}} \widetilde{\varphi}_t(g, \zeta) \right|^2 \\
&= \max_{k \leq \frac{m}{2}} \frac{1}{4p^{2k}} \left| \sum_{\substack{n=0\\ \text{even}}}^{2k} p^{-\frac{n}{2}} \lambda_{p^{2n}} \right|^2 |\widetilde{\varphi}_t(g, \zeta)|^2 \\
&= \max_{k \leq \frac{m}{2}} \frac{1}{4p^{2k}} \left| p^k \frac{\sin((2k+1)\theta)}{\sin(\theta)} \right|^2 |\widetilde{\varphi}_t(g, \zeta)|^2 \\
&= \frac{1}{4} \max_{k \leq \frac{m}{2}} \left| \frac{\sin((2k+1)\theta)}{\sin(\theta)} \right|^2 |\widetilde{\varphi}_t(g, \zeta)|^2,
\end{aligned}$$

where we used again the abbreviation $\theta = 2t\log(p)$. On the other hand, we can apply

Cauchy-Schwarz to the inner summation only:

$$\begin{aligned}
T_{p^{2n}}|\widetilde{\varphi}_t(g,\zeta)|^2 &= \sum_i |\widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)|^2 \geq \frac{1}{p^{2n}+p^{2n-1}} \left|\sum_i \widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)\right|^2 \\
&= \frac{1}{\underbrace{(p+1)}_{\leq 2p} p^{2n-1}} \left|T_{p^{2n}}\widetilde{\varphi}_t(g,\zeta)\right|^2 \geq \frac{1}{2p^{2n}} \left|\lambda_{p^{2n}}\widetilde{\varphi}_t(g,\zeta)\right|^2 \\
&= \frac{1}{2p^n} \left|p^{-\frac{n}{2}}\lambda_{p^{2n}}\right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 .
\end{aligned}$$

Bounding again by summing over the even terms only and apply Lemma 3.4.8, we have

$$\begin{aligned}
\sum_{n=0}^{m} T_{p^{2n}} \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 &\geq \sum_{\substack{n=0\\ \text{even}}}^{m} \frac{1}{2p^n} \left|p^{-\frac{n}{2}}\lambda_{p^{2n}}\right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 \\
&= \sum_{\substack{n=0\\ \text{even}}}^{m} \frac{1}{2p^n} \left| \sum_{\substack{k=0\\ \text{even}}}^{n} p^{-\frac{k}{2}}\lambda_{p^{2k}} - \sum_{\substack{k=0\\ \text{even}}}^{n-2} p^{-\frac{k}{2}}\lambda_{p^{2k}} \right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 \\
&= \sum_{\substack{n=0\\ \text{even}}}^{m} \frac{1}{2p^n} \left| p^{\frac{n}{2}} \frac{\sin((n+1)\theta)}{\sin(\theta)} - p^{\frac{n}{2}-1} \frac{\sin((n-1)\theta)}{\sin(\theta)} \right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 \\
&= \sum_{\substack{n=0\\ \text{even}}}^{m} \frac{1}{2} \left| \frac{\sin((n+1)\theta)}{\sin(\theta)} - \frac{\sin((n-1)\theta)}{p\sin(\theta)} \right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 \\
&= \frac{1}{2} \sum_{n=0}^{\frac{m}{2}} \left| \frac{\sin((2n+1)\theta)}{\sin(\theta)} - \frac{\sin((2n-1)\theta)}{p\sin(\theta)} \right|^2 \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 .
\end{aligned}$$

We've shown that

$$\sum_{n=0}^{m} T_{p^{2n}} \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 \gg c_{p,\theta}(m) \left|\widetilde{\varphi}_t(g,\zeta)\right|^2 ,$$

where

$$c_{p,\theta}(m) := \max_{k\leq\frac{m}{2}} \left| \frac{\sin((2k+1)\theta)}{\sin(\theta)} \right|^2 + \sum_{n=0}^{\frac{m}{2}} \left| \frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)} \right|^2 .$$

Hence, it remains to show that $c_{p,\theta}(m) \gg m$ uniformly in θ (and also in the prime p). Suppose by contradiction that this is not true. Then, there exists a sequence of integers

$j \to \infty$ and tuples (p_j, θ_j, m_j) such that $m_j \to \infty$ and $c_{p_j,\theta_j}(m_j) = o(m_j)$ as $j \to \infty$. For better readability we are not writing the j throughout the argument. Since we want to get a contradiction, we may pass to subsequences in order to consider the cases $\sin(2\theta) \ggg \frac{1}{m}$ and $\sin(2\theta) \lll \frac{1}{m}$ separately. Here, the notation $f \lll g$ means $f = o(g)$. (This is analogous to the case distinction used in [19].) We first consider the case $\sin(2\theta) \ggg \frac{1}{m}$. In that case $c_{p,\theta}(m)$ is bounded from below as follows:

$$\begin{aligned} c_{p,\theta}(m) &\geq \sum_{n=0}^{\frac{m}{2}} \left| \frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)} \right|^2 \\ &= \frac{1}{4|\sin(\theta)|^2} \sum_{n=0}^{\frac{m}{2}} \left| e^{(2n+1)i\theta)} - e^{-(2n+1)i\theta} - \frac{1}{p} e^{(2n-1)i\theta)} + \frac{1}{p} e^{-(2n-1)i\theta)} \right|^2 \\ &\geq \frac{1}{4} \sum_{n=0}^{\frac{m}{2}} \left| \left(e^{i\theta} - \frac{1}{p} e^{-i\theta} \right) e^{2ni\theta} - \left(e^{-i\theta} - \frac{1}{p} e^{i\theta} \right) e^{-2ni\theta} \right|^2, \end{aligned}$$

where we used the trivial inequality $|\sin(\theta)| \leq 1$. We denote $\alpha := e^{i\theta} - \frac{1}{p}e^{-i\theta}$ and $\beta := e^{-i\theta} - \frac{1}{p}e^{i\theta}$. Note that $\frac{1}{2} \leq |\alpha|, |\beta| \leq \frac{3}{2}$. We get further

$$\begin{aligned} c_{p,\theta}(m) &\geq \frac{1}{4} \sum_{n=0}^{\frac{m}{2}} \left(\alpha e^{2ni\theta} - \beta e^{-2ni\theta} \right) \left(\overline{\alpha} e^{-2ni\theta} - \overline{\beta} e^{2ni\theta} \right) \\ &= \frac{1}{4} \left(\lfloor \tfrac{m}{2} \rfloor + 1 \right) \left(|\alpha|^2 + |\beta|^2 \right) - \frac{1}{4} \left(\alpha\overline{\beta} \sum_{n=0}^{\frac{m}{2}} e^{4ni\theta} + \overline{\alpha}\beta \sum_{n=0}^{\frac{m}{2}} e^{-4ni\theta} \right). \end{aligned}$$

Summing the geometric series and use $\frac{1}{\sin(2\theta)} \lll m$ as well as $|\alpha\overline{\beta}| \leq \frac{9}{4} \ll 1$ and $|\overline{\alpha}\beta| \ll 1$ we have

$$\begin{aligned} \alpha\overline{\beta} \sum_{n=0}^{\frac{m}{2}} e^{4ni\theta} + \overline{\alpha}\beta \sum_{n=0}^{\frac{m}{2}} e^{-4ni\theta} &= \alpha\overline{\beta} \frac{1 - e^{4(\lfloor \frac{m}{2} \rfloor + 1)i\theta}}{1 - e^{4i\theta}} + \overline{\alpha}\beta \frac{1 - e^{-4(\lfloor \frac{m}{2} \rfloor + 1)i\theta}}{1 - e^{-4i\theta}} \\ &= 2i\alpha\overline{\beta} \frac{e^{(4\lfloor \frac{m}{2} \rfloor + 2)i\theta} - e^{-2i\theta}}{\sin(2\theta)} + 2i\overline{\alpha}\beta \frac{e^{2i\theta} - e^{-(4\lfloor \frac{m}{2} \rfloor + 2)i\theta}}{\sin(2\theta)} \\ &\lll m. \end{aligned}$$

On the other hand, $|\alpha|^2 + |\beta|^2 \geq \frac{1}{2}$, hence

$$\frac{1}{4} \left(\lfloor \tfrac{m}{2} \rfloor + 1 \right) \left(|\alpha|^2 + |\beta|^2 \right) \gg m.$$

Thus, $c_{p,\theta}(m) \gg m$, which is a contradiction.

For the second case, $\sin(2\theta) \ll \frac{1}{m}$, we do another case distinction. Note that $\sin(2\theta) \ll \frac{1}{m}$ implies that $2\theta - r\pi \ll \frac{1}{m}$, i.e. $\theta - r\frac{\pi}{2} \ll \frac{1}{m}$ for some $r \in \mathbb{Z}$. We distinguish between r odd and r even. If r is odd, then

$$\cos(r\tfrac{\pi}{2}) = 0 \quad \text{and} \quad \sin(r\tfrac{\pi}{2}) = \chi_4(r) = \begin{cases} 1 & \text{if } r \equiv 1 \mod 4, \\ -1 & \text{if } r \equiv 3 \mod 4. \end{cases}$$

In this case we use again

$$c_{p,\theta}(m) \geq \sum_{n=0}^{\frac{m}{2}} \left| \frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)} \right|^2 .$$

Since θ is close to an odd multiple of $\frac{\pi}{2}$, the denominator $\sin(\theta)$ of the fraction is close to 1. We want to bound the numerators from below. By assumption $|\theta - r\frac{\pi}{2}| \leq \frac{C}{m}$ for some constant C. Set $\epsilon := \frac{1}{2C}$. Using Fourier expansions around $\theta_0 = r\frac{\pi}{2}$ the numerator is

$$\begin{aligned}\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta) = &\sum_{l=0}^{\infty} \frac{(-1)^l \chi_4((2n+1)r)}{(2l)!}(2n+1)^{2l}(\theta - r\tfrac{\pi}{2})^{2l} \\ &- \frac{1}{p}\sum_{l=0}^{\infty} \frac{(-1)^l \chi_4((2n-1)r)}{(2l)!}(2n-1)^{2l}(\theta - r\tfrac{\pi}{2})^{2l}.\end{aligned}$$

Since $\chi_4((2n-1)r) = \chi_4(2n-1)\chi_4(r) = -\chi_4(2n+1)\chi_4(r) = -\chi_4((2n+1)r)$ we get further

$$\begin{aligned}&= \chi_4((2n+1)r)\sum_{l=0}^{\infty} \frac{(-1)^l}{(2l)!}(\theta - r\tfrac{\pi}{2})^{2l}\left((2n+1)^{2l} + \frac{1}{p}(2n-1)^{2l}\right) \\ &= \chi_4((2n+1)r)\left(1 + \frac{1}{p} + \sum_{l=1}^{\infty} \frac{(-1)^l}{(2l)!}(\theta - r\tfrac{\pi}{2})^{2l}\left((2n+1)^{2l} + \frac{1}{p}(2n-1)^{2l}\right)\right)\end{aligned}$$

For $n \leq \epsilon m = \frac{m}{2C}$ we have

$$\begin{aligned}(2n+1)^{2l} + \frac{1}{p}(2n-1)^{2l} &= \sum_{s=0}^{2l} \binom{2l}{s}(2n)^s + \frac{1}{p}\sum_{s=0}^{2l}\binom{2l}{s}(2n)^s(-1)^{2l-s} \\ &= \sum_{s=0}^{2l}\binom{2l}{s}(2n)^s\left(1 + \frac{(-1)^s}{p}\right) \leq \sum_{s=0}^{2l}\binom{2l}{s}\left(\frac{m}{C}\right)^s\left(1 + \frac{1}{p}\right) \\ &= \left(1 + \frac{1}{p}\right)\left(\frac{m}{C} + 1\right)^{2l}.\end{aligned}$$

Hence, for $n \leq \epsilon m$ we have

$$\left|\sum_{l=1}^{\infty} \frac{(-1)^l}{(2l)!}(\theta - r\tfrac{\pi}{2})^{2l}\left((2n+1)^{2l} + \frac{1}{p}(2n-1)^{2l}\right)\right| \leq \sum_{l=1}^{\infty} \frac{1}{(2l)!}\left(\frac{C}{m}\right)^{2l}\left(1+\frac{1}{p}\right)\left(\frac{m}{C}+1\right)^{2l}$$
$$= \left(1+\frac{1}{p}\right)\sum_{l=1}^{\infty}\frac{1}{(2l)!}\left(1+\frac{C}{m}\right)^{2l} = \left(1+\frac{1}{p}\right)\left(\cosh\left(1+\frac{C}{m}\right)-1\right)$$
$$\leq \frac{3}{2}\left(\cosh\left(1+\frac{C}{m}\right)-1\right).$$

For m big enough this expression is $\leq \frac{3}{2}\cdot\frac{3}{5} = \frac{9}{10} < 1$. So, for $n \leq \epsilon m$ the numerator is

$$\left|\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)\right| = 1 + \frac{1}{p} + \underbrace{(\dots)}_{|\cdot|\leq 1}.$$

Hence, for $n \leq \epsilon m$ we've shown that

$$\frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)} \gg 1,$$

thus

$$c_{p,\theta}(m) \geq \sum_{n=0}^{\frac{m}{2}}\left|\frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)}\right|^2$$
$$\geq \sum_{n=0}^{\epsilon m}\left|\frac{\sin((2n+1)\theta) - \frac{1}{p}\sin((2n-1)\theta)}{\sin(\theta)}\right|^2 \gg \sum_{n=0}^{\epsilon m} 1 \gg m,$$

which gives the desired contradiction.

The remaining case is $\sin(2\theta) \ll \frac{1}{m}$ with $\theta - r\frac{\pi}{2} \ll \frac{1}{m}$ for r even. Note that this implies $\sin(\theta) \ll \frac{1}{m}$. In this case we are using the lower bound

$$c_{p,\theta}(m) \geq \max_{k\leq\frac{m}{2}}\left|\frac{\sin((2k+1)\theta)}{\sin(\theta)}\right|^2.$$

By assumption there exists a constant C such that $|\sin(\theta_j)| \leq \frac{C}{10m_j}$, hence $\frac{m_j}{C}|\sin(\theta_j)| \leq \frac{1}{10}$ for all $j \geq j_0$. Since $m_j \to \infty$ there exists a positive integer k with $k|\sin(\theta_j)| \leq \frac{1}{10}$. Let $1 \leq k_j \leq \frac{m_j}{2}$ be the maximal positive integer with $k_j|\sin(\theta_j)| \leq \frac{1}{10}$ for every $j \geq j_0$. Note that $k_j \geq \lfloor\frac{m_j}{C}\rfloor \gg m_j$, hence $k_j \gg m_j$. Since $|\sin(\theta_j)| \leq \frac{1}{10k_j} \leq \frac{1}{10}$ is small, θ_j is close to a multiple of π. Hence, we have $\lim_{j\to\infty}\left|\frac{\sin((2k+1)\theta)}{\sin(\theta)}\right| = 2k+1$. As a lower bound we get $\frac{\sin((2k+1)\theta)}{\sin(\theta)} \gg k$ for this choice of k. Overall, we get in this second case the estimate

$$c_{p,\theta}(m) \geq \max_{k\leq \frac{m}{2}} \left|\frac{\sin((2k+1)\theta)}{\sin(\theta)}\right|^2 \gg k^2 \gg m^2 \geq m$$

which is again a contradiction to the assumption. This proves that $c_{p,\theta}(m) \gg m$, hence the Proposition holds. □

Corollary 3.4.11. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$, where $\widetilde{\varphi}_t$ is the microlocal lift. Then, there exists some constant $c_0 > 0$ such that for every non-negative measurable function f on $X = \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ we have the inequality*

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(f) d\widetilde{\mu} \geq c_0 m \int_X f d\widetilde{\mu} \quad \textit{for every } m \geq 0.$$

Proof. By Proposition 3.4.10 there exists some constant $c_0 \geq 0$ such that $\sum_{n=0}^{m} T_{p^{2n}}|\widetilde{\varphi}_t|^2 \geq c_0 m|\widetilde{\varphi}_t|^2$ for every $m \geq 0$. We first restrict to $f \in C_c(X)$ non-negative. Since f is compactly supported and so are the $T_{p^{2n}}f$, the following integral converges for every $|t|$ big enough and we can also write it as an inner product. Since the Hecke operators are self-adjoint and $f \geq 0$, we have

$$\begin{aligned}\int_X \sum_{n=0}^{m} T_{p^{2n}}(f)|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_X &= \left\langle \sum_{n=0}^{m} T_{p^{2n}}(f), |\widetilde{\varphi}_t|^2 \right\rangle_{L^2(X)} = \left\langle f, \sum_{n=0}^{m} T_{p^{2n}}|\widetilde{\varphi}_t|^2 \right\rangle_{L^2(X)} \\ &\geq \left\langle f, c_0 m|\widetilde{\varphi}_t|^2\right\rangle_{L^2(X)} = c_0 m \int_X f|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_X.\end{aligned}$$

Taking the weak*-limit, we get

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(f) d\widetilde{\mu} \geq c_0 m \int_X f d\widetilde{\mu} \quad \text{for every } m \geq 0$$

in the case of $f \in C_c(X)$ non-negative. We extend the statement to any measurable function $f \geq 0$ as follows:

- For $\Omega \subset C$ compact or $\Omega = \cup_n \Omega_n$ a countable union of compact sets, there exists a sequence of non-negative functions $(f_k) \subset C_c(X)$ such that $\lim_{k\to\infty} f_k = \mathbb{1}_\Omega$. By dominated convergence theorem and using continuity of the Hecke operators, we have

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_\Omega) d\widetilde{\mu} = \lim_{k\to\infty} \int_X \sum_{n=0}^{m} T_{p^{2n}}(f_k) d\widetilde{\mu} \geq c_0 m \lim_{k\to\infty} \int_X f_k d\widetilde{\mu} = c_0 m \int_X \mathbb{1}_\Omega d\widetilde{\mu}.$$

- For $B \subset X$ a Borel set there exists a countable union $\Omega = \cup_n \Omega_n$ of compact sets approximating B in the sense that $\Omega \subset B$ and $\widetilde{\mu}(B \backslash \Omega) = 0$. Hence,

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_B) d\widetilde{\mu} \geq \int_X \sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_\Omega) d\widetilde{\mu} \geq c_0 m \int_X \mathbb{1}_\Omega d\widetilde{\mu} = c_0 m \widetilde{\mu}(\Omega) = c_0 m \widetilde{\mu}(B)$$
$$= c_0 m \int_X \mathbb{1}_B d\widetilde{\mu}.$$

- Every non-negative measurable function f can be approximated by an increasing sequence of simple functions. Since the inequality holds for characteristic functions on measurable sets, it also holds for linear combinations of those, i.e. for simple functions. By the monotone convergence theorem it also holds for every non-negative measurable function f as desired.

□

Theorem 3.4.12. *Any weak*-limit $\widetilde{\mu}$ of $|\widetilde{\varphi}_t|^2 \, \mathrm{dm}_{X^{|t|}}$ is Hecke-p-recurrent.*

Proof. Assume by contradiction that $\widetilde{\mu}$ is not Hecke-p-recurrent. Then, there exists some Borel set $B \subset X$ of positive measure $\widetilde{\mu}(B) > 0$ such that for every $x \in B$ there exists some $C \geq 1$ such that

$$\sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_B)(x) \leq C \quad \text{for every } m \geq 0.$$

Hence,

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_B) d\widetilde{\mu} \leq C \widetilde{\mu}(X) = C \quad \text{for every } m \geq 0.$$

Applying Corollary 3.4.11 to $f = \mathbb{1}_B$ gives the existence of a constant $c_0 > 0$ such that

$$\int_X \sum_{n=0}^{m} T_{p^{2n}}(\mathbb{1}_B) d\widetilde{\mu} \geq c_0 m \int_X \mathbb{1}_B d\widetilde{\mu} = c_0 m \widetilde{\mu}(B) \quad \text{for every } m \geq 0.$$

Together we have

$$c_0 m \widetilde{\mu}(B) \leq C$$

for every $m \geq 0$. This inequality cannot hold for all $m \geq 0$ unless $\widetilde{\mu}(B) = 0$. So, this gives the desired contradiction. □

3.5 Positive Entropy

Let $M = 2^l$, $l \geq 2$, and $p \neq 2$ be an odd prime as usual. In order to show that every ergodic component has positive entropy, we follow the standard argument, see e.g. [7].

Lemma 3.5.1. *The eigenvalues $\lambda_{p^{2n}}$ of the Hecke operator $T_{p^{2n}}$ applied to the microlocal lift $\widetilde{\varphi}_t$ satisfy either*

$$|\lambda_{p^2}| \gg p$$

or

$$|\lambda_{p^4}| \gg p^2.$$

Proof. By Corollary 3.4.4, $\lambda_{p^2} = 2p\cos(\theta)$ and

$$\lambda_{p^4} = 2p^2\cos(2\theta) + p^2 - p = 4p^2\cos(\theta)^2 - p^2 - p = (\lambda_{p^2})^2 - p^2 - p.$$

Hence, λ_{p^4} and $(\lambda_{p^2})^2$ cannot both be small compared to p^2, i.e. either $|\lambda_{p^4}| \gg p^2$ or $|\lambda_{p^2}| \gg p$. □

Lemma 3.5.2. *The microlocal lift satisfies*

$$T_{p^2}|\widetilde{\varphi}_t|^2 + T_{p^4}|\widetilde{\varphi}_t|^2 \gg |\widetilde{\varphi}_t|^2.$$

Proof. If $|\lambda_{p^2}| \gg p$, then by Cauchy-Schwarz inequality we have

$$\begin{aligned} p^2|\widetilde{\varphi}_t(g,\zeta)|^2 &\ll |\lambda_{p^2}\widetilde{\varphi}_t(g,\zeta)|^2 = |T_{p^2}\widetilde{\varphi}_t(g,\zeta)|^2 = \left|\sum_i \widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)\right|^2 \\ &\leq \sum_i |\widetilde{\varphi}_t(\alpha_i g, \zeta(\alpha_i)\zeta)|^2 \cdot \sum_i 1 = T_{p^2}|\widetilde{\varphi}_t(g,\zeta)|^2 \cdot \underbrace{(p^2+p)}_{\leq 2p^2} \ll p^2 T_{p^2}|\widetilde{\varphi}_t(g,\zeta))|^2. \end{aligned}$$

Here, the sum is over the set $\{i\}$ of p^2+p elements such that $\Gamma_0(M)\begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix}\Gamma_0(M) = \coprod_i \Gamma_0(M)\alpha_i$ as in Theorem 3.4.1. Now, if $|\lambda_{p^2}| \not\gg p$, then $|\lambda_{p^4}| \gg p^2$ by Lemma 3.5.1. The analog argument gives in this case

$$\begin{aligned} p^4|\widetilde{\varphi}_t(g,\zeta)|^2 &\ll |\lambda_{p^4}\widetilde{\varphi}_t(g,\zeta)|^2 = |T_{p^4}\widetilde{\varphi}_t(g,\zeta)|^2 = \left|\sum_j \widetilde{\varphi}_t(\alpha_j g, \zeta(\alpha_j)\zeta)\right|^2 \\ &\leq \sum_j |\widetilde{\varphi}_t(\alpha_j g, \zeta(\alpha_j)\zeta)|^2 \cdot \sum_j 1 = T_{p^4}|\widetilde{\varphi}_t(g,\zeta)|^2 \cdot \underbrace{(p^4+p^3)}_{\leq 2p^4} \ll p^4 T_{p^4}|\widetilde{\varphi}_t(g,\zeta)|^2. \end{aligned}$$

Here, the sum is over the set $\{j\}$ of p^4+p^3 elements such that $\Gamma_0(M)\begin{pmatrix} p^{-2} & 0 \\ 0 & p^2 \end{pmatrix}\Gamma_0(M) = \coprod_j \Gamma_0(M)\alpha_j$ as in Theorem 3.4.1. Hence, we have in the first case $T_{p^2}|\widetilde{\varphi}_t|^2 \gg |\widetilde{\varphi}_t|^2$ and in the second case $T_{p^4}|\widetilde{\varphi}_t|^2 \gg |\widetilde{\varphi}_t|^2$. Together this gives

$$T_{p^2}|\widetilde{\varphi}_t|^2 + T_{p^4}|\widetilde{\varphi}_t|^2 \gg |\widetilde{\varphi}_t|^2.$$

□

Lemma 3.5.3. *Let $\widetilde{\mu}$ denote any weak*-limit of the measures $|\widetilde{\varphi}_t|^2\,\mathrm{dm}_{X^{|t|}}$. Then, for every non-negative measurable function $f : X \to \mathbb{C}$ and every prime $p \neq 2$ we have*

$$\int (T_{p^2}(f) + T_{p^4}(f))\, d\widetilde{\mu} \gg \int f d\widetilde{\mu}.$$

Moreover, for any $P \in \mathbb{R}_{>0}$ we have

$$\int \sum_{\substack{p\le P\\ prime}} (T_{p^2}(f) + T_{p^4}(f))\, d\widetilde{\mu} \gg \sqrt{P} \int f d\widetilde{\mu}.$$

Proof. First, fix a compactly supported function $f \in C_c^\infty(X)$. By Lemma 3.5.2 we have for $|t|$ big enough

$$\begin{aligned}\int (T_{p^2}(f) + T_{p^4}(f))\, |\widetilde{\varphi}_t|^2\,\mathrm{dm}_X &= \left\langle T_{p^2}(f) + T_{p^4}(f), |\widetilde{\varphi}_t|^2 \right\rangle \\ &= \left\langle f, T_{p^2}(|\widetilde{\varphi}_t|^2) + T_{p^4}(|\widetilde{\varphi}_t|^2) \right\rangle \\ &\gg \left\langle f, |\widetilde{\varphi}_t|^2 \right\rangle \\ &= \int f|\widetilde{\varphi}_t|^2\,\mathrm{dm}_X,\end{aligned}$$

where we used that the Hecke operators are self-adjoint. Taking the limit as $|t| \to \infty$ gives

$$\int (T_{p^2}(f) + T_{p^4}(f))\, d\widetilde{\mu} \gg \int f d\widetilde{\mu}.$$

So, the claimed statement holds for every $f \in C_c^\infty(X)$. This can be extended to any non-negative measurable function f by dominated and monotone convergence. By the Prime Number Theorem, the prime counting function satisfies the asymptotic lower bound $\pi(x) \gg \sqrt{x}$. Hence,

$$\int \sum_{p\le P} (T_{p^2}(f) + T_{p^4}(f))\, d\widetilde{\mu} \gg \sqrt{P} \int f d\widetilde{\mu} \qquad \text{for any } P \in \mathbb{R}_{>0}.$$

□

Definition 3.5.4. *For $x \in \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ and $\delta > 0$ sufficiently small we define the Bowen m-ball as*

$$B_m := x \bigcap_{i=-m}^{m} a^i B_\delta\, a^{-i},$$

where B_δ is a ball of radius δ around the identity matrix, and

$$a = \begin{pmatrix} e^{\frac{1}{2}} & 0 \\ 0 & e^{-\frac{1}{2}} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R}).$$

We consider the time-one map

$$\begin{aligned} T : \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R}) &\to \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R}) \\ x &\mapsto T(x) := xa. \end{aligned}$$

Lemma 3.5.5. *For $\delta > 0$ small enough, we have*

$$\bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i} \subset \exp\left([-\kappa, \kappa]\mathcal{H} + [-\kappa e^{-m}, \kappa e^{-m}]\mathcal{U}^+ + [-\kappa e^{-m}, \kappa e^{-m}]\mathcal{U}^-\right).$$

In particular, every element in $\bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$ can be written as $\exp(t\mathcal{H})\exp(v)$ for some $t \in \mathbb{R}$ with $t \ll \delta$ and some $v \in \mathfrak{sl}_2(\mathbb{R})$ with $||v|| \ll \delta e^{-m}$.

Proof. We may choose δ small enough so that the exponential map maps a neighborhood of $0 \in \mathfrak{sl}_2(\mathbb{R})$ injectively onto $B_{2\delta}$, i.e. $B_{2\delta} = \exp(U)$ for some neighborhood U of $0 \in \mathfrak{sl}_2(\mathbb{R})$. Then,

$$\begin{aligned} a^i B_{2\delta} a^{-i} = a^i \exp(U) a^{-i} = \exp(a^i U a^{-i}) &= \{\exp\left(a^i(\alpha\mathcal{H} + \beta\mathcal{U}^+ + \gamma\mathcal{U}^-)a^{-i}\right) \mid \alpha, \beta, \gamma \ll \delta\} \\ &= \{\exp\left(\alpha a^i \mathcal{H} a^{-i} + \beta a^i \mathcal{U}^+ a^{-i} + \gamma a^i \mathcal{U}^- a^{-i}\right) \mid \alpha, \beta, \gamma \ll \delta\} \\ &= \{\exp\left(\alpha\mathcal{H} + \beta e^i \mathcal{U}^+ + \gamma e^{-i}\mathcal{U}^-\right) \mid \alpha, \beta, \gamma \ll \delta\}. \end{aligned}$$

Hence,

$$\bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i} \subset \exp\left([-\kappa, \kappa]\mathcal{H} + [-\kappa e^{-m}, \kappa e^{-m}]\mathcal{U}^+ + [-\kappa e^{-m}, \kappa e^{-m}]\mathcal{U}^-\right)$$

for some $\kappa \ll \delta$, which implies the existence of t and v as claimed. □

Let $\Omega \subset X = \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ be a compact set such that every A-orbit intersects Ω non-trivially and such that for every $x \in X$ there exists some $n \in \mathbb{Z}$ with $T^n x \in \Omega$. Such a compact set Ω exists e.g. by the discussion in section 4.3.2 in [7].

Lemma 3.5.6. *Let $x \in \Omega \subset \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ and write $x = \Gamma_0(M)g$ for some $g \in \mathrm{SL}_2(\mathbb{R})$. Let B_m be the Bowen m-ball associated to x.*

(a) Suppose that $y \in \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ such that

$$T_q(\mathbb{1}_{B_m})(y) > 1,$$

where $q \in \{p^2, p^4\}$ for some prime number p. Then, there exist $b_1, b_2 \in \bigcap_{i=-m}^m a^i B_\delta a^{-i}$ and $\gamma_1, \gamma_2 \in \mathrm{Mat}_2(\mathbb{Z})$ not divisible by p with $\det(\gamma_1) = \det(\gamma_2) = q$, $\gamma_2^{-1}\gamma_1 \notin \mathbb{R}\cdot \mathrm{id}$ and $\Gamma_0(M)\gamma_1 \neq \Gamma_0(M)\gamma_2$, such that

$$\sqrt{q}y = \Gamma_0(M)\gamma_1 g b_1 = \Gamma_0(M)\gamma_2 g b_2$$

and

$$\gamma_1 g b_1 = \gamma_2 g b_2.$$

(b) Suppose that $y \in \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ such that

$$T_{q_1}(\mathbb{1}_{B_m})(y) + T_{q_2}(\mathbb{1}_{B_m})(y) > 1,$$

where $q_i \in \{p_i^2, p_i^4\}$ with $q_1 \neq q_2$ for primes p_1, p_2. Then, there exist elements $b_1, b_2 \in \bigcap_{i=-m}^m a^i B_\delta a^{-i}$ and $\gamma_1, \gamma_2 \in \mathrm{Mat}_2(\mathbb{Z})$ not divisible by p_1, respectively p_2, with $\det(\gamma_1) = q_1$, $\det(\gamma_2) = q_2$, $\gamma_2^{-1}\gamma_1 \notin \mathbb{R}\cdot \mathrm{id}$ and $\Gamma_0(M)\gamma_1 \neq \Gamma_0(M)\gamma_2$, such that

$$y = \frac{1}{\sqrt{q_1}}\Gamma_0(M)\gamma_1 g b_1 = \frac{1}{\sqrt{q_2}}\Gamma_0(M)\gamma_2 g b_2$$

and

$$\gamma_1 g b_1 = \sqrt{\frac{q_1}{q_2}}\gamma_2 g b_2.$$

Proof. (a) First note that $\mathbb{1}_{B_m}$ is left $\Gamma_0(M)$-invariant, and hence the Hecke operator is defined on this characteristic function:

$$\mathbb{1}_{B_m}(y) = \mathbb{1}_{\Gamma_0(M)g\bigcap_{i=-m}^m a^i B_\delta a^{-i}}(y) = \mathbb{1}_{\Gamma_0(M)g\bigcap_{i=-m}^m a^i B_\delta a^{-i}}(\gamma\cdot y) = \mathbb{1}_{B_m}(\gamma\cdot y)$$

for every $\gamma \in \Gamma_0(M)$. By definition of the Hecke operator we have

$$1 < T_q(\mathbb{1}_{B_m})(y) = \sum_i \mathbb{1}_{B_m}(\alpha_i \cdot y) = \sum_i \mathbb{1}_{\Gamma_0(M)g\bigcap_{i=-m}^m a^i B_\delta a^{-i}}(\alpha_i\cdot y),$$

where the sum is over the representatives $\{\alpha_i\}$ as in Theorem 3.4.1, part (a), with $p^{2n} = q$. Hence, there exist two such representatives α_1, α_2 such that

$$\alpha_1 \cdot y, \alpha_2 \cdot y \in \Gamma_0(M) g \bigcap_{i=-m}^{m} a^i B_\delta a^{-i}.$$

So, there exist $\eta_1, \eta_2 \in \Gamma_0(M)$ and $b_1, b_2 \in \bigcap_{i=-m}^{m} a^i B_\delta a^{-i}$ such that

$$\alpha_1 y = \eta_1 g b_1 \quad \text{and} \quad \alpha_2 y = \eta_2 g b_2.$$

In particular,

$$y = \alpha_1^{-1} \eta_1 g b_1 = \alpha_2^{-1} \eta_2 g b_2.$$

Set $\gamma_1 := \sqrt{q}\alpha_1^{-1}\eta_1$ and $\gamma_2 := \sqrt{q}\alpha_2^{-1}\eta_2$. Note that $\gamma_1, \gamma_2 \in \mathrm{Mat}_2(\mathbb{Z})$ are not divisible by p and $\det(\gamma_1) = \det(\gamma_2) = q$. Moreover,

$$y = \alpha_1^{-1}\eta_1 g b_1 = \frac{1}{\sqrt{q}} \gamma_1 g b_1 \quad \text{and} \quad y = \alpha_2^{-1}\eta_2 g b_2 = \frac{1}{\sqrt{q}} \gamma_2 g b_2,$$

thus $\gamma_1 g b_1 = \gamma_2 g b_2$. It remains to show that $\gamma_2^{-1}\gamma_1 \notin \mathbb{R} \cdot \mathrm{id}$ and $\Gamma_0(M)\gamma_1 \neq \Gamma_0(M)\gamma_2$.

Suppose by contradiction first that $\gamma_2^{-1}\gamma_1 \in \mathbb{R} \cdot \mathrm{id}$, i.e. $\eta_2^{-1}\alpha_2\alpha_1^{-1}\eta_1 = c \cdot \mathrm{id}$ for some $c \in \mathbb{R}$. Taking determinants on both sides, it follows that $c = \pm 1$. Hence, $\eta_2^{-1}\alpha_2 = \pm\eta_1^{-1}\alpha_1$, but $\alpha_2 = \pm\eta_2\eta_1^{-1}\alpha_1 \in \Gamma_0(M)\alpha_1$ gives a contradiction.

Suppose now, again by contradiction, that $\Gamma_0(M)\gamma_1 = \Gamma_0(M)\gamma_2$. Then, $\gamma_1 \in \Gamma_0(M)\gamma_2$, i.e. $\gamma_1^{-1} \in \gamma_2^{-1}\Gamma_0(M)$. It follows that $\eta_1^{-1}\alpha_1 \in \eta_2^{-1}\alpha_2\Gamma_0(M)$, thus $\alpha_1 \in \Gamma_0(M)\alpha_2\Gamma_0(M)$. So, there exist $\gamma, \gamma' \in \Gamma_0(M)$ such that $\alpha_1 = \gamma\alpha_2\gamma'$. Hence,

$$\eta_2^{-1}\alpha_2 y = g b_2 \quad \Rightarrow \quad \mathrm{id} = \alpha_2^{-1}\eta_2 g b_2 y^{-1} \in \alpha_2^{-1}\eta_2 g \bigcap_{i=-m}^{m} a^i B_\delta a^{-i} y^{-1}$$

and

$$\eta_1^{-1}\gamma\alpha_2\gamma' y = g b_1 \quad \Rightarrow \quad \gamma' = \alpha_2^{-1}\gamma^{-1}\eta_1 g b_1 y^{-1} \in \alpha_2^{-1}\gamma^{-1}\eta_1 g \bigcap_{i=-m}^{m} a^i B_\delta a^{-i} y^{-1}.$$

Note that g lies in a fixed compact set, hence so does y. Since δ is small and $\Gamma_0(M)$ is discrete, it follows that $\gamma' = \mathrm{id}$. Hence, $\alpha_1 = \gamma\alpha_2 \in \Gamma_0(M)\alpha_2$. But since α_1 and α_2 are representing distinct right cosets, this gives a contradiction.

(b) Similar as in part (a), the condition

$$1 < T_{q_1}(\mathbb{1}_{B_m})(y) + T_{q_2}(\mathbb{1}_{B_m})(y) = \sum_i \mathbb{1}_{B_m}(\alpha_i \cdot y) + \sum_j \mathbb{1}_{B_m}(\alpha_j \cdot y)$$

implies that there exist α_1, α_2 as in Theorem 3.4.1, part (a) (but for different p^n, corresponding to q_1 and q_2, respectively), $\eta_1, \eta_2 \in \Gamma_0(M)$ and $b_1, b_2 \in \bigcap_{i=-m}^m a^i B_\delta a^{-i}$ such that

$$\alpha_1 \cdot y = \eta_1 g b_1$$
$$\alpha_2 \cdot y = \eta_2 g b_2.$$

In particular, $y = \alpha_1^{-1}\eta_1 g b_1 = \alpha_2^{-1}\eta_2 g b_2$. Set $\gamma_1 := \sqrt{q_1}\alpha_1^{-1}\eta_1, \gamma_2 := \sqrt{q_2}\alpha_2^{-1}\eta_2 \in \mathrm{Mat}_2(\mathbb{Z})$. Note that γ_i is not divisible by p_i and $\det(\gamma_i) = q_i$ for $i = 1, 2$. Moreover,

$$y = \alpha_1^{-1}\eta_1 g b_1 = \frac{1}{\sqrt{q_1}}\gamma_1 g b_1$$
$$y = \alpha_2^{-1}\eta_2 g b_2 = \frac{1}{\sqrt{q_2}}\gamma_2 g b_2.$$

Hence,

$$\gamma_1 g b_1 = \sqrt{\frac{q_1}{q_2}}\gamma_2 g b_2$$

and

$$y = \frac{1}{\sqrt{q_1}}\Gamma_0(M)\gamma_1 g b_1 = \frac{1}{\sqrt{q_2}}\Gamma_0(M)\gamma_2 g b_2.$$

It remains to show that $\gamma_2^{-1}\gamma_1 \notin \mathbb{R}\cdot \mathrm{id}$ and $\Gamma_0(M)\gamma_1 \neq \Gamma_0(M)\gamma_2$. Suppose by contradiction first that $\gamma_2^{-1}\gamma_1 = c \cdot \mathrm{id}$ for some $c \in \mathbb{R}$. Taking determinants on both sides gives $c = \pm\sqrt{\frac{q_1}{q_2}}$. Hence, $\sqrt{q_2}\gamma_1 = \pm\sqrt{q_1}\gamma_2$. Since γ_i is not divisible by p_i this implies that $q_1 = q_2$ which is a contradiction. Now assume, also by contradiction, that $\Gamma_0(M)\gamma_1 = \Gamma_0(M)\gamma_2$, i.e. there exists some $\eta \in \Gamma_0(M)$ such that $\gamma_1 = \eta\gamma_2$. Taking determinants on both sides gives $q_1 = q_2$ which is a contradiction.

□

Lemma 3.5.7. *For $i = 1, 2$ let $q_i \in \{p_i^2, p_i^4\}$, where $p_1, p_2 \leq P(m)$ be two not necessarily distinct primes. Let $x = \Gamma_0(M)g \in \Omega$. Let $\gamma_1, \gamma_2 \in \mathrm{Mat}_2(\mathbb{Z})$, $\gamma_2^{-1}\gamma_1 \notin \mathbb{R}\cdot\mathrm{id}$, with $\det(\gamma_i) = q_i$ and $b_1, b_2 \in \bigcap_{i=-m}^m a^i B_\delta a^{-i}$ be such that $\gamma_1 g b_1 = \sqrt{\frac{q_1}{q_2}}\gamma_2 g b_2$. Then, there exists $\eta_m \in \mathrm{Mat}_2(\mathbb{Z}) \setminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ with $\det(\eta_m) \leq P(m)^8$ such that*

$$\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}.$$

Proof. Let $\eta_m := q_2\gamma_2^{-1}\gamma_1$. Since $q_i \in \{p_i^2, p_i^4\}$ and $p_i \leq P(m)$ we have

$$\det(\eta_m) = q_2^2 q_2^{-1} q_1 = q_1 q_2 \leq P(m)^8.$$

Moreover,

$$g^{-1}\eta_m g = g^{-1}q_2\gamma_2^{-1}\gamma_1 g = g^{-1}q_2\gamma_2^{-1}\sqrt{\frac{q_1}{q_2}}\gamma_2 g b_2 b_1^{-1} = \sqrt{q_1 q_2}\, b_2 b_1^{-1},$$

so $g^{-1}\eta_m g$ is a multiple of $b_2 b_1^{-1} \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$. Hence,

$$\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}.$$

It remains to show that $\eta_m \notin \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$. Assume by contradiction that $\eta_m = cA$ for some $c \in \mathbb{Q}$ and $A \in \mathrm{GL}_2(\mathbb{Z})$. Taking determinants on both sides gives $q_1 q_2 = \pm c^2$, hence $c = \sqrt{q_1 q_2}$ and $\det(A) = 1$. But, since

$$A = \frac{1}{\sqrt{q_1 q_2}}\eta_m = \frac{1}{\sqrt{\det(\eta_m)}}\eta_m \in g \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$$

with δ small, g in a fixed compact set, and $\mathrm{GL}_2(\mathbb{Z})$ is discrete, it follows that $A = \mathrm{id}$. Hence, $\gamma_2^{-1}\gamma_1 = q_2^{-1}\eta_m = q_2^{-1}c\,\mathrm{id} \in \mathbb{R}\cdot\mathrm{id}$ gives a contradiction. □

Lemma 3.5.8. *Let $g \in \mathrm{SL}_2(\mathbb{R})$ be in some fixed compact set. For $m \in \mathbb{Z}_{\geq 1}$, let $\eta_m \in \mathrm{Mat}_2(\mathbb{Z}) \smallsetminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ be such that $\det(\eta_m) \ll e^{\frac{(1-\epsilon)m}{2}}$ for some $\epsilon > 0$. Moreover, assume that $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$. Then, there exists some $m_0 \in \mathbb{Z}_{\geq 1}$ such that η_m and η_{m+1} commute for every $m \geq m_0$.*

Proof. We want to show that $[\eta_m, \eta_{m+1}] = \eta_m^{-1}\eta_{m+1}^{-1}\eta_m\eta_{m+1} = \mathrm{id}$. Note that the commutator satisfies $[c_1 g_1, c_2 g_2] = c_1^{-1}g_1^{-1}c_2^{-1}g_2^{-1}c_1 g_1 c_2 g_2 = g_1^{-1}g_2^{-1}g_1 g_2 = [g_1, g_2]$ for all $c_1, c_2 \in \mathbb{R}$ and $g_1, g_2 \in \mathrm{GL}_2(\mathbb{R})$. Hence, we have

$$\begin{aligned} g^{-1}[\eta_m, \eta_{m+1}]g = [g^{-1}\eta_m g, g^{-1}\eta_{m+1} g] &= \Bigg[\underbrace{\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g}_{\in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}}, \underbrace{\frac{1}{\sqrt{\det(\eta_{m+1})}} g^{-1}\eta_{m+1} g}_{\in \bigcap_{i=-(m+1)}^{m+1} a^i B_{2\delta} a^{-i}}\Bigg] \\ &= [\exp(t_n\mathcal{H})\exp(v_n), \exp(t_{n+1}\mathcal{H})\exp(v_{n+1})] \end{aligned}$$

for some $t_n, t_{n+1} \ll \delta$ and $||v_n||, ||v_{n+1}|| \ll \delta e^{-m}$ as in Lemma 3.5.5. This can be written further as

$$\begin{aligned} &= \exp(-v_n)\exp(-t_n\mathcal{H})\exp(-v_{n+1})\exp(-t_{n+1}\mathcal{H})\exp(t_n\mathcal{H})\exp(v_n)\exp(t_{n+1}\mathcal{H})\exp(v_{n+1}) \\ &= \exp(-v_n)\cdot\exp(-t_n\mathcal{H})\exp(-v_{n+1})\exp(t_n\mathcal{H})\cdot\exp(-t_{n+1}\mathcal{H})\exp(v_n)\exp(t_{n+1}\mathcal{H})\cdot\exp(v_{n+1}), \end{aligned}$$

where all four factors are of distance $\ll e^{-m}$ from the identity. Since g is in some fixed compact set, also $[\eta_m, \eta_{m+1}]$ lies at distance $\ll e^{-m}$ from the identity. Note that $[\eta_m, \eta_{m+1}] \in \mathrm{Mat}_2(\mathbb{Q})$ and the minimal $c \in \mathbb{Z}_{\geq 1}$ such that $c[\eta_m, \eta_{m+1}] \in \mathrm{Mat}_2(\mathbb{Z})$ is

$$|c| \leq \det(\eta_m)\det(\eta_{m+1}) \ll e^{(1-\epsilon)m}.$$

Assume by contradicition that $[\eta_m, \eta_{m+1}] \neq \mathrm{id}$. Then,

$$||[\eta_m, \eta_{m+1}] - \mathrm{id}|| = \frac{1}{|c|}||c[\eta_m, \eta_{m+1}] - c \cdot \mathrm{id}|| \geq \frac{1}{|c|} \gg e^{(\epsilon-1)m},$$

where we used that every non-zero integral matrix has norm ≥ 1. Above we argued that the distance from the identity is $\ll e^{-m}$, hence we have

$$e^{(\epsilon-1)m} \ll e^{-m}, \quad \text{i.e.} \quad e^{\epsilon m} \leq C,$$

for some constant C. But this can be true only for finitely many m. Thus, we find some m_0 such that for every $m \geq m_0$ this cannot be true, hence gives the desired contradiction and $[\eta_m, \eta_{m+1}] = \mathrm{id}$ for $m \geq m_0$ as claimed. □

Lemma 3.5.9. *Let $g \in \mathrm{SL}_2(\mathbb{R})$ be in some fixed compact set. For $m \in \mathbb{Z}_{\geq 1}$, let $\eta_m \in \mathrm{Mat}_2(\mathbb{Z}) \smallsetminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ be such that $\det(\eta_m) \ll e^{\frac{(1-\epsilon)m}{2}}$ for some $\epsilon > 0$. Moreover, assume that $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$. Then, there exists some $m_0 \in \mathbb{Z}_{\geq 1}$ such that for every $m \geq m_0$ the matrix $\frac{\eta_m}{\det(\eta_m)}$ is a hyperbolic element, i.e. conjugated to an element in $A = \left\{ \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} \middle| \ y \in \mathbb{R}_{>0} \right\}$.*

Proof. Since every element in $\mathrm{SL}_2(\mathbb{R})$ is conjugated to an element in either A, N or K, for every m there is some $h_m \in \mathrm{SL}_2(\mathbb{R})$ such that $h_m^{-1}\frac{\eta_m}{\sqrt{\det(\eta_m)}}h_m$ lies either in A, N or K. We write $h_m^{-1}\frac{\eta_m}{\sqrt{\det(\eta_m)}}h_m = g_m \in G_m$, where $g_m \in \{a_y, n_x, k_\theta\}$ and $G_m \in \{A, N, K\}$. Since the centralizer of a non-trivial element in G_m is equal to G_m and $\eta_m \notin \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$, for any choice of $G_m \in \{A, N, K\}$, we have

$$h_m^{-1}C(\eta_m)h_m = C\left(h_m^{-1}\frac{\eta_m}{\sqrt{\det(\eta_m)}}h_m\right) = C(g_m) = G_m.$$

First note that by Lemma 3.5.8, there exists some m_0 such that η_m and η_{m+1} commute for all $m \geq m_0$. Hence, for all $m \geq m_0$ we have $\eta_{m+1} \in C(\eta_m)$, hence $\frac{1}{\sqrt{\det(\eta_{m+1})}}h_m^{-1}\eta_{m+1}h_m \in G_m$. So, $\frac{1}{\sqrt{\det(\eta_{m+1})}}\eta_{m+1}$ is conjugated to some element in G_m but it was also conjugated to some element in G_{m+1}. It follows that $G_{m+1} = G_m$ for every $m \geq m_0$. This means that either $\frac{1}{\sqrt{\det(\eta_m)}}\eta_m \in A$ for every $m \geq m_0$ or $\frac{1}{\sqrt{\det(\eta_m)}}\eta_m \in N$ for every $m \geq m_0$, or $\frac{1}{\sqrt{\det(\eta_m)}}\eta_m \in K$ for every $m \geq m_0$. Assume by contradiction that the second or third possibility happens, i.e. that the trace satisfies $\left|\operatorname{Tr}\left(\frac{1}{\sqrt{\det(\eta_m)}}t^{-1}\eta_m g\right)\right| \leq 2$. On the other hand, by Lemma 3.5.5 we may write

$$\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g = \exp(t_m\mathcal{H})\exp(v_m)$$

for some $t_m \in \mathbb{R}$ with $t_m \ll \delta$ and $v \in {}_2(\mathbb{R})$ with $\|v\| \ll \delta e^{-m}$. Since v is close to 0, we may write $\exp(v_m) = \mathrm{id} + w_m$ with w_m close to 0, i.e. $|w_{ij}| \ll e^{-m}$. Since $t_m \ll \delta$, this gives

$$\begin{aligned}\operatorname{Tr}\left(\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g\right) &= \operatorname{Tr}\left(\exp(t_m\mathcal{H})(\mathrm{id}+w_m)\right) = \operatorname{Tr}\left(\begin{pmatrix} e^{t_m} & 0 \\ 0 & e^{-t_m}\end{pmatrix}\begin{pmatrix} 1+w_{11} & w_{12} \\ w_{21} & 1+w_{22}\end{pmatrix}\right) \\ &= e^{t_m} + w_{11}e^{t_m} + e^{-t_m} + w_{22}e^{-t_m} = e^{t_m} + e^{-t_m} + O(e^{-m}) \\ &= 2\cosh(t_m) + O(e^{-m}).\end{aligned}$$

Since the trace is bounded by 2, we get the condition

$$\cosh(t_m) - 1 = O(e^{-m}) \quad \text{i.e.} \quad \frac{t_m^2}{2} + \frac{t_m^4}{4!} + \ldots = O(e^{-m}).$$

Hence, $t_m \ll e^{-\frac{m}{2}}$ and $\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g$ has distance $\ll e^{-\frac{m}{2}}$ from the identity. Recall that $\det(\eta_m) \ll e^{\frac{(1-\epsilon)m}{2}}$, hence the distance to the identity can be bounded from below by

$$\left\|\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g - \mathrm{id}\right\| = \frac{1}{\sqrt{\det(\eta_m)}}\left\|g^{-1}\eta_m g - \sqrt{\det(\eta_m)}\cdot\mathrm{id}\right\| \geq \frac{1}{\sqrt{\det(\eta_m)}} \gg e^{\frac{(\epsilon-1)m}{4}}.$$

Together, we have

$$e^{\frac{(\epsilon-1)m}{4}} \ll e^{-\frac{m}{2}}, \quad \text{i.e.} \quad e^{\frac{(1+\epsilon)m}{4}} \ll 1 \qquad \text{for all } m \geq m_0.$$

This gives a contradiction as the inequality cannot hold for infinitely many m. □

Lemma 3.5.10. *Let $g \in \mathrm{SL}_2(\mathbb{R})$ be in some fixed compact set. For $m \in \mathbb{Z}_{\geq 1}$, let $\eta_m \in \mathrm{Mat}_2(\mathbb{Z}) \smallsetminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ be such that $\det(\eta_m) \ll e^{\frac{(1-\epsilon)m}{2}}$ for some $\epsilon > 0$. Moreover, assume that $\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g \in \bigcap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$. Then, there exists some $m_0 \in \mathbb{Z}_{\geq 1}$ such that*

$$\frac{1}{\sqrt{\det(\eta_m)}}g^{-1}\eta_m g \in A \quad \textit{for every } m \geq m_0.$$

Proof. By Lemma 3.5.5 we can write $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g = \exp(t_m\mathcal{H} + w_m)$ with $t_m \ll \delta$ and $\|w_m\| \ll e^{-m}$. Moreover, we may assume $|t_m| > e^{-\frac{m}{2}}$ as otherwise we get a contradiction as in the proof of Lemma 3.5.9. Let $s_m := \|t_m\mathcal{H} + w_m\|$ and $\widetilde{w_m} := \frac{t_m\mathcal{H}+w_m}{s_m}$. In particular, $\|\widetilde{w_m}\| = 1$ and $s_m \gg e^{-\frac{m}{2}}$. It follows that

$$\left\|\widetilde{w_m} - \frac{t_m}{s_m}\mathcal{H}\right\| = \left\|\frac{w_m}{s_m}\right\| \ll \frac{e^{-m}}{e^{-\frac{m}{2}}} = e^{-\frac{m}{2}}. \tag{3.5.1}$$

Note that $\widetilde{w_m} \neq 0$ as $\eta_m \notin \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$. In particular, $\widetilde{w_m}$ is non-central, hence its centralizer is 1-dimensional: $C(\widetilde{w_m}) = \mathbb{R}\cdot\widetilde{w_m}$. Recall that η_m and η_{m+1} commute for $m \geq m_0'$ by Lemma 3.5.8. Hence,

$$\mathrm{id} = [\exp(s_m\widetilde{w_m}), \exp(s_{m+1}\widetilde{w_{m+1}})] = \exp([s_m\widetilde{w_m}), s_{m+1}\widetilde{w_{m+1}}])$$

and

$$\begin{aligned} 0 &= [s_m\widetilde{w_m}, s_{m+1}\widetilde{w_{m+1}}] = s_m\widetilde{w_m}s_{m+1}\widetilde{w_{m+1}} - s_{m+1}\widetilde{w_{m+1}}s_m\widetilde{w_m} \\ &= s_m s_{m+1}(\widetilde{w_m}\widetilde{w_{m+1}} - \widetilde{w_{m+1}}\widetilde{w_m}). \end{aligned}$$

So, $\widetilde{w_{m+1}} \in C(\widetilde{w_m}) = \mathbb{R}\cdot\widetilde{w_m}$. Since both, $\|\widetilde{w_m}\| = \left\|\widetilde{w_{m+1}}\right\| = 1$, it follows that $\widetilde{w_{m+1}} = \pm\widetilde{w_m}$ for all $m \geq m_0'$. Using equation (3.5.1) there exists some m_0 such that $\widetilde{w_m} \in \mathbb{R}\mathcal{H}$ for all $m \geq m_0$. In particular, $w_m = 0$ and $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g = \exp(t_m\mathcal{H}) \in A$ for every $m \geq m_0$. □

Lemma 3.5.11. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2\,\mathrm{dm}_{X^{|t|}}$ and let $x = \Gamma_0(M)g \in \Omega \subset \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R})$. Let $\eta \in \mathrm{GL}_2(\mathbb{Z}) \smallsetminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ be such that $\frac{1}{\sqrt{\det(\eta)}} g^{-1}\eta g \in A$. Assume that η is diagonalizable over $\mathbb{Q}$. Then, x belongs to some set of measure zero.*

Proof. Since η is diagonalizable over $\mathbb{Q}$, there exists some $g_0 \in \mathrm{GL}_2(\mathbb{Q})$ with $\frac{1}{\sqrt{\det(\eta)}} g_0^{-1}\eta g_0 \in A$. Hence, $g \in \frac{1}{\sqrt{\det(g_0)}} g_0 A \cup \frac{1}{\sqrt{\det(g_0)}} g_0 w A$, where $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ is the Weyl element. Thus,

$$x \in \Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}} g_0 A \cup \Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}} g_0 w A.$$

Note that the orbits $\Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}} g_0 A$ and $\Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}} g_0 w A$ are both divergent. Indeed, with $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ we have

$$\frac{1}{\sqrt{\det(g_0)}} g_0 \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot i = \frac{ae^t i + be^{-t}}{ce^t i + de^{-t}} \overset{t\to\infty}{\longrightarrow} \frac{a}{c}$$

and

$$\frac{1}{\sqrt{\det(g_0)}} g_0 w \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot i = \frac{-be^t i + ae^{-t}}{-de^t i + ce^{-t}} \xrightarrow{t\to\infty} \frac{b}{d}$$

which are in the limit both cusps in $X = \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$. So, for every n, the set $E_n := \{\Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}} g_0 \begin{pmatrix} e^y & 0 \\ 0 & e^-y \end{pmatrix} \mid |y| \leq n\} \subset \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ is measurable and for every $x_0 \in E_n$ there exists some N_0 such that $T^m x = xa^m \notin E_n$ for every $m \geq N_0$. Since $\widetilde{\mu}$ is A-invariant by Theorem 3.3.15, it follows by Poincaré recurrence (see e.g. Theorem 2.11 in [8]) that $\widetilde{\mu}(E_n) = 0$. Letting n go to infinity and applying the same argument to the other orbit, it follows that both orbits are sets of measure zero. □

Lemma 3.5.12. *Let $x = \Gamma_0(M)g \in X = \Gamma_0(M)\backslash \mathrm{SL}_2(\mathbb{R})$ and assume that the orbit xA is periodic (i.e. compact). Then, there exists some $a \in A \setminus \{\mathrm{id}\}$ and some $\gamma \in \Gamma_0(M)$ such that $ga = \gamma g$. Moreover, the algebra $\mathbb{F} = \mathbb{Q}[\gamma] \subset \mathrm{Mat}_2(\mathbb{Q})$ is a real quadratic field extension of $\mathbb{Q}$ and $\gamma \in \mathcal{O}_{\mathbb{F}}^\times$ is an algebraic unit.*

Proof. Since xA is periodic, there exists some $a \in A \setminus \{\mathrm{id}\}$ such that $xa = x$, i.e. such that $\Gamma_0(M)ga = \Gamma_0(M)g$, i.e. such that $ga \in \Gamma_0(M)g$. Hence, there exists some $a \in A \setminus \{\mathrm{id}\}$ and some $\gamma \in \Gamma_0(M)$ such that $ga = \gamma g$. Since $\gamma = gag^{-1}$, the characteristic polynomial of γ is

$$\begin{aligned} p_\lambda(t) &= \det(\gamma - \lambda \cdot \mathrm{id}) = \det(gag^{-1} - \lambda \cdot \mathrm{id}) = \det(g(a - \lambda \cdot \mathrm{id})g^{-1}) = \det(a - \lambda \cdot \mathrm{id}) \\ &= (e^t - \lambda)(e^{-t} - \lambda) = \lambda^2 - (e^t + e^{-t})\lambda + 1. \end{aligned}$$

Since $t \neq 0$ we have $e^{\pm t} \notin \mathbb{Q}$ and the characteristic polynomial is irreducible over $\mathbb{Q}$. Thus, $\mathbb{F} = \mathbb{Q}[\gamma] \cong \mathbb{Q}[X]/(X^2 - (e^t + e^{-t})X + 1)$ is indeed a field. It has two real embeddings, namely $\mathbb{Q}[\gamma] \cong \mathbb{Q}[e^t] \longrightarrow \mathbb{R}$ given by $\gamma \mapsto e^t \mapsto e^{\pm t}$. So, $\mathbb{F} = \mathbb{Q}[\gamma]$ is indeed a real quadratic field extension of $\mathbb{Q}$. The norm is $N_{\mathbb{F}/\mathbb{Q}}(\gamma) = \det(\gamma) = 1$, hence $\gamma \in \mathcal{O}_{\mathbb{F}}^\times$ is indeed and algebraic unit. □

Lemma 3.5.13. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$. Then, every periodic (i.e. compact) A-orbit has measure zero.*

Proof. Assume by contradiction that there is some $x = \Gamma_0(M)g \in X$ such that xA is periodic and $\widetilde{\mu}(xA) > 0$. By Lemma 3.5.12 there exists some $\gamma \in \Gamma_0(M)$ such that $g^{-1}\gamma g = a \in A \setminus \{\mathrm{id}\}$ and such that $\mathbb{F} = \mathbb{Q}[\gamma]$ is a real quadratic field extension of $\mathbb{Q}$ with $\gamma \in \mathcal{O}_{\mathbb{F}}^\times$. Let p be a prime that is inert for $\mathbb{F}/\mathbb{Q}$. Recall that $\widetilde{\mu}$ is Hecke-p-recurrent by Theorem 3.4.12. Hence, for $\widetilde{\mu}$-a.e. $y \in xA$, say $y = \Gamma_0(M)xa$ with $a \in A$, there exists a sequence $(n_k)_{k\geq 1}$ of disjoint natural numbers such that $T_{p^{2n_k}}(\mathbb{1}_{xA})(y) = 1$ for every $k \geq 1$. This implies that for every $k \geq 1$ there exists some $\alpha_k \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ as in Theorem 3.4.1 (a) with $n = n_k$ such that $\mathbb{1}_{xA}(\alpha_k ga) = 1$, i.e. $\alpha_k ga \in \Gamma_0(M)gA$. It follows that for every $k \geq 1$ there exists some $\gamma'_k \in \Gamma_0(M)$ and some $a'_k \in A$ such that

$$\alpha_k ga = \gamma'_k g a'_k.$$

Hence, for every $k \geq 1$, there exists some $\widetilde{a_k} = a'_k a^{-1} \in A$ and some upper triangular matrix $\widetilde{\gamma_k} = \gamma'^{-1}_k \alpha_k \in \mathrm{SL}_2(\mathbb{Z}[\frac{1}{p}])$ such that

$$\widetilde{\gamma_k} g = g\widetilde{a_k}.$$

Hence, also for every $k \geq 1$ there exists some $a_k = p^{n_k}\widetilde{a_k} \in p^{n_k}A$ and some upper triangular matrix $\gamma_k = p^{n_k}\widetilde{\gamma_k} \in \mathrm{Mat}_2(\mathbb{Z})$ with $\det(\gamma_k) = p^{2n_k}$ such that

$$\gamma_k g = g a_k.$$

Since γ and γ_k are diagonalized by the same matrix g, it follows that $\gamma_k \in C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma)$ for every $k \geq 1$. Note that the centralizer of γ in $\mathrm{Mat}_2(\mathbb{R})$ is 2-dimensional. Indeed,

$$\begin{aligned} C_{\mathrm{Mat}_2(\mathbb{R})}(\gamma) &= \{\eta \in \mathrm{Mat}_2(\mathbb{R}) \mid \eta\gamma = \gamma\eta\} = \{\eta \in \mathrm{Mat}_2(\mathbb{R}) \mid g^{-1}\eta g g^{-1}\gamma g = g^{-1}\gamma g g^{-1}\eta g\} \\ &= \{\eta' \in \mathrm{Mat}_2(\mathbb{R}) \mid \eta' a = a\eta'\}, \end{aligned}$$

where $a \in A \setminus \{\mathrm{id}\}$ as in Lemma 3.5.12. With $\eta' = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $a = \begin{pmatrix} y & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$ with $y \neq 1$ we have

$$\eta' a = a\eta' \quad \Leftrightarrow \quad \begin{pmatrix} ay & \frac{b}{y} \\ cy & \frac{d}{y} \end{pmatrix} = \begin{pmatrix} ay & by \\ \frac{c}{y} & \frac{d}{y} \end{pmatrix} \quad \Leftrightarrow \quad b = c = 0,\ a, d \in \mathbb{R},$$

hence

$$\dim_{\mathbb{R}}(C_{\mathrm{Mat}_2(\mathbb{R})}(\gamma)) = 2.$$

Note that the sequences $0 \to C_{\mathrm{Mat}_2(\mathbb{R})}(\gamma) \hookrightarrow \mathrm{Mat}_2(\mathbb{R}) \xrightarrow{f_2} \mathrm{Mat}_2(\mathbb{R})$ with $f_2(\eta) = [\eta, \gamma]$ and $0 \to C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma) \hookrightarrow \mathrm{Mat}_2(\mathbb{Q}) \xrightarrow{f_1} \mathrm{Mat}_2(\mathbb{Q})$ with $f_1(\eta) = [\eta, \gamma]$, hence ($\mathbb{R}$ is a flat $\mathbb{Q}$-module) also $0 \to C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma) \otimes \mathbb{R} \to \mathrm{Mat}_2(\mathbb{Q}) \otimes \mathbb{R} \xrightarrow{f_1 \otimes 1} \mathrm{Mat}_2(\mathbb{Q}) \otimes \mathbb{R}$ are all exact. Hence, we get the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma) \otimes_{\mathbb{Q}} \mathbb{R} & \xhookrightarrow{i} & \mathrm{Mat}_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} & \xrightarrow{f_1} & \mathrm{Mat}_2(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{R} \\ & & & & \downarrow \cong & & \downarrow \cong \\ 0 & \hookrightarrow & C_{\mathrm{Mat}_2(\mathbb{R})}(\gamma) & \xhookrightarrow{j} & \mathrm{Mat}_2(\mathbb{R}) & \xrightarrow{f_2} & \mathrm{Mat}_2(\mathbb{R}) \end{array}$$

It follows that

$$\begin{aligned}\dim_{\mathbb{Q}}(C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma)) &= \dim_{\mathbb{R}}(C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma)\otimes_{\mathbb{Q}}\mathbb{R}) = \dim_{\mathbb{R}}(\mathrm{Image}(i)) = \dim_{\mathbb{R}}(\mathrm{Kern}(f_1))\\ &= \dim_{\mathbb{R}}(\mathrm{Kern}(f_2)) = \dim_{\mathbb{R}}(\mathrm{Image}(j)) = \dim_{\mathbb{R}}(C_{\mathrm{Mat}_2(\mathbb{R})}(\gamma) = 2.\end{aligned}$$

So, $\mathbb{F} = \mathbb{Q}[\gamma]$ and $C_{\mathrm{Mat}_2(\mathbb{Q})}(\gamma)$ are two-dimensional vector spaces over $\mathbb{Q}$ that both contain the identity and $\gamma \neq \mathrm{id}$. Hence, they must be equal, so $\gamma_k \in \mathbb{F}$ for every $k \geq 1$. For every $k \geq 1$ we consider the $\mathbb{Z}$-algebra $\mathbb{Z}[\gamma_k]$. Since γ_k has integer entries, it follows that its characteristic polynomial $x^2 - \mathrm{Tr}(\gamma_k)x + \det(\gamma_k)$ is a monic polynomial with integer coefficients. By the Cayley-Hamilton Theorem, γ_k is a root of this polynomial. Hence, γ_k is integral, i.e. $\gamma_k \in \mathcal{O}_{\mathbb{F}}$. Since

$$N_{\mathbb{F}|\mathbb{Q}}(\gamma_k) = \det(\gamma_k) = p^{2n_k}$$

we have $\gamma_k \in \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$. Note that by Dirichlet's S-unit Theorem (here S consists of only one prime ideal, namely (p), as p was chosen to be inert) we get the rank

$$\begin{aligned}\mathrm{rank}\left(\left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}\right) &= |\{\text{real embeddings}\}| + |\{\text{pairs of complex embeddings}\}| - 1 + |S|\\ &= 2 + 0 - 1 + 1 = 2.\end{aligned}$$

Also note that $\left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ is abelian. We now consider the subgroup $\mathcal{O}_{\mathbb{F}}^{\times}p^{\mathbb{Z}} \leq \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$. By Dirichlet's unit Theorem we have $\mathrm{rank}\left(\mathcal{O}_{\mathbb{F}}^{\times}\right) = 2+0-1 = 1$, hence $\mathcal{O}_{\mathbb{F}}^{\times}p^{\mathbb{Z}}$ has rank 2 as well. Thus, $\mathcal{O}_{\mathbb{F}}^{\times}p^{\mathbb{Z}} \subset \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ is a finite index subgroup. Note that $\mathcal{O} := \mathbb{F} \cap \mathrm{Mat}_2(\mathbb{Z})$ is an order in $\mathbb{F}$, hence a finite index subgroup of the maximal order $\mathcal{O}_{\mathbb{F}}$. It follow that also $\mathcal{O}^{\times} \subset \mathcal{O}_{\mathbb{F}}^{\times}$ is of finite index. To summarize we have $\mathcal{O}^{\times} \subset \mathcal{O}_{\mathbb{F}}^{\times}$ of finite index and $\mathcal{O}_{\mathbb{F}}^{\times}p^{\mathbb{Z}} \subset \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ of finite index. Hence, also $\mathcal{O}^{\times}p^{\mathbb{Z}} \subset \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ is of finite index, i.e. there are only a finitely many disjoint cosets. Hence, there exists some $m \in \mathbb{Z}_{\geq 1}$ and $\eta_1, \ldots, \eta_m \in \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ such that

$$\bigcup_{j=1}^{m} \mathcal{O}^{\times}p^{\mathbb{Z}}\eta_j = \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}.$$

Recall that $\gamma_k \in \left(\mathcal{O}_{\mathbb{F}}[\tfrac{1}{p}]\right)^{\times}$ for every $k \geq 1$. It follows that there exists $k \neq k'$ such that $\gamma_k, \gamma_{k'}$ lie in the same coset, i.e. $\mathcal{O}^{\times}p^{\mathbb{Z}}\gamma_k = \mathcal{O}^{\times}p^{\mathbb{Z}}\gamma_{k'}$. Since $\gamma \in \Gamma_0(M)$ we have $\mathcal{O}^{\times} \subset \Gamma_0(M)$.

By construction we have $\gamma_k = p^{n_k}\widetilde{\gamma}_k \in \Gamma_0(M)p^{n_k}\alpha_k$ and $\gamma_{k'} \in \Gamma_0(M)p^{n_{k'}}\alpha_{k'}$ with $\alpha_k, \alpha_{k'}$ as in Theorem 3.4.1. hence we find $k \neq k'$ such that

$$\Gamma_0(M)p^m\alpha_k = \Gamma_0(M)p^{m'}\alpha_{k'}.$$

Since $\alpha_k, \alpha_{k'}$ both have determinant 1 it follows that $m = n$, hence

$$\Gamma_0(M)\alpha_k = \Gamma_0(M)\alpha_{k'},$$

which gives a contradiction to $k \neq k'$. □

Lemma 3.5.14. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$ and let $x = \Gamma_0(M)g \in \Omega \subset \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R})$. Let $\eta \in \mathrm{GL}_2(\mathbb{Z}) \setminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ be such that $\frac{1}{\sqrt{\det(\eta)}}g^{-1}\eta g \in A$. Assume that η is diagonalizable over $\mathbb{R}$ but not over $\mathbb{Q}$. Then, x belongs to some set of measure zero.*

Proof. Since η is diagonalizable over $\mathbb{R}$, there exists $g_0 \in \mathrm{SL}_2(\mathbb{R})$ such that $\frac{1}{\sqrt{\det(\eta)}}g_0^{-1}\eta g \in A$. Hence,

$$x \in \Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}}g_0A \cup \Gamma_0(M)\frac{1}{\sqrt{\det(g_0)}}g_0wA.$$

Note that in the limits $t \to \pm\infty$,

$$\frac{1}{\sqrt{\det(g_0)}}g_0\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot i \quad \text{and} \quad \frac{1}{\sqrt{\det(g_0)}}g_0w\begin{pmatrix} e^t & 0 \\ 0 & e^{-t}\end{pmatrix}\cdot i$$

go both to $\frac{a}{c}$ and $\frac{b}{d}$. Since η is not diagonalizable over $\mathbb{Q}$, both $\frac{a}{c}$ and $\frac{b}{d}$ are irrational. Indeed, if $\frac{a}{c}, \frac{b}{d} \in \mathbb{Q}$, then η is also diagonalizable by $\begin{pmatrix} \frac{a}{c} & \frac{b}{d} \\ 1 & 1\end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$, hence diagonalizable over $\mathbb{Q}$ which is a contradiction. Hence, at least one eigenvector, say $v_1 = \begin{pmatrix} a \\ c\end{pmatrix}$ is such that $\frac{a}{c} \notin \mathbb{Q}$. But then, the corresponding eigenvalue satisfies $\lambda_1 \in \mathbb{R} \setminus \mathbb{Q}$. Since $\lambda_1\lambda_2 = \det(\eta) \in \mathbb{Q}$, also $\lambda_2 \notin \mathbb{Q}$ and $v_2 = \begin{pmatrix} b \\ d\end{pmatrix}$ is such that $\frac{b}{d} \notin \mathbb{Q}$. Hence, both orbits are compact (i.e. periodic). Hence, by Lemma 3.5.13 they have measure zero. □

Theorem 3.5.15. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2 \,\mathrm{dm}_{X^{|t|}}$. There exists a constant $h > 0$ such that for almost every $x \in \Omega$ there are infinitely many $m \geq 1$ with*

$$\widetilde{\mu}\left(x\bigcap_{i=-m}^{m} a^iB_\delta a^{-i}\right) \leq e^{-mh}.$$

Proof. Let $x = \Gamma_0(M)g \in \Omega$ with $g \in \mathrm{SL}_2(\mathbb{R})$. Fix some $h < \frac{1}{32}$, say $h = \frac{1-\epsilon}{32}$, and assume that there exists some $m_0 \in \mathbb{N}$ with $\widetilde{\mu}(B_m) \geq e^{-mh}$ for all $m \geq m_0$. We want to show that there exists some null-set that contains all such x. By Lemma 3.5.3 there exists some constant $c > 0$ such that

$$\int_X \sum_{p \leq P} \left(T_{p^2}(f) + T_{p^4}(f)\right) d\widetilde{\mu} \geq c\sqrt{P} \int_X f d\widetilde{\mu}$$

for every non-negative measurable function f. Let $P(m) := \frac{e^{2hm}}{c^2}$. Applying the inequality to $f = \mathbb{1}_{B_m}$ gives

$$\int_X \sum_{p \leq P(m)} \left(T_{p^2}(\mathbb{1}_{B_m}) + T_{p^4}(\mathbb{1}_{B_m})\right) d\widetilde{\mu} \geq e^{hm}\widetilde{\mu}(B_m) \geq 1.$$

for all $m \geq m_0$. It follows that for every $m \geq m_0$ there exists some $y_m \in X$ such that

$$\sum_{p \leq P(m)} \left(T_{p^2}(\mathbb{1}_{B_m}) + T_{p^4}(\mathbb{1}_{B_m})\right)(y_m) d\widetilde{\mu} > 1.$$

Thus, either

(i) There is some prime $p \leq P(m)$ and $q \in \{p^2, p^4\}$ such that $T_q(\mathbb{1}_{B_m})(y_m) > 1$, or

(ii) There are primes $p_1, p_2 \leq P(m)$ and $q_1 \in \{p_1^2, p_1^4\}$, $q_2 \in \{p_2^2, p_2^4\}$ such that $T_{q_1}(\mathbb{1}_{B_m})(y_m) + T_{q_2}(\mathbb{1}_{B_m})(y_m) > 1$.

By Lemma 3.5.6 and Lemma 3.5.7 for every $m \in \mathbb{Z}_{\geq 1}$ there exists $\eta_m \in \mathrm{Mat}_2(\mathbb{Z}) \setminus \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ with $\det(\eta_m) \leq P(m)^8 \ll e^{16hm} = e^{\frac{(1-\epsilon)m}{2}}$ and such that $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in \cap_{i=-m}^{m} a^i B_{2\delta} a^{-i}$. By Lemma 3.5.10 there exists some m_0 such that $\frac{1}{\sqrt{\det(\eta_m)}} g^{-1}\eta_m g \in A$ for all $m \geq m_0$. So, either η_m is diagonalizable over $\mathbb{Q}$, or over $\mathbb{R}$ but not over $\mathbb{Q}$. For every $\eta \in \mathrm{GL}_2(\mathbb{Q}) \backslash \mathbb{Q}\,\mathrm{GL}_2(\mathbb{Z})$ we consider a set $B_\eta \subset X$ of measure zero as in the proofs of Lemma 3.5.11 and 3.5.14. By these Lemmas x belongs to the union $B = \cup_\eta B_\eta$. But since $\mathrm{GL}_2(\mathbb{Q})$ is countable, this is a countable union of null sets, hence again a set of measure zero. So, every $x \in \Omega$ that satisfies $\mu(B_m) \geq e^{-mh}$ for all $m \geq m_0$ belongs to the null set B which proves the theorem. □

Definition 3.5.16. *For a countable partition $\xi = \{A_1, A_2, A_3, \ldots\}$ of a space X and a transformation T on X, we define the refined partition $\bigvee_{i=0}^{n-1} T^{-i}\xi$ as*

$$\bigvee_{i=0}^{n-1} T^{-i}\xi = \{A_{i_0} \cap T^{-1}A_{i_1} \cap T^{-2}A_{i_2} \cap \ldots \cap T^{1-n}A_{i_{n-1}} \mid i_0, i_1, i_2, \ldots i_{n-1} \geq 1\}.$$

The dynamical entropy of the partition ξ with respect to the transformation T is

$$h_\mu(T,\xi) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{i=0}^{n-1} T^{-i}\xi\right),$$

where $H_\mu(\xi) := -\sum_{A_i\in\xi} \mu(A_i)\log\mu(A_i)$ is the static entropy. The entropy of the transformation T is

$$h_\mu(T) := \sup_\xi h_\mu(T,\xi).$$

Theorem 3.5.17. *(Shannon-McMillan-Breiman [27], [18], [4], [3]) Let T be a measure-preserving transformation of the probability space (X,μ) and let ξ be a countable partition of X with finite entropy. Then, for almost every $x\in X$ we have*

$$-\frac{1}{2m+1}\log\mu\left([x]_{\bigvee_{i=-m}^{m} T^{-i}\xi}\right) \longrightarrow h_{\mu_x}(T,\xi) \quad \text{as } m\to\infty.$$

Here, $h_{\mu_x}(T,\xi)$ is the dynamical entropy of the ergodic component μ_x of the measure μ.

Lemma 3.5.18. *(see [6] for a proof) Let Ω be the compact subset of $X = \Gamma_0(M)\backslash\mathrm{SL}_2(\mathbb{R})$ as before. There exists a countable partition ξ of X with $X\backslash\Omega\in\xi$ and such that whenever $x\in\Omega$ with $x\in A\in\bigvee_{i=-m}^{m} T^{-i}\xi$, then*

$$A \subset x\bigcap_{i=-m}^{m} a^i B_\delta a^{-i}$$

for some small enough δ. Moreover, ξ has finite entropy for any T-invariant probability measure.

Corollary 3.5.19. *Let $\widetilde{\mu}$ be any weak*-limit of $|\widetilde{\varphi}_t|^2\,\mathrm{dm}_{X^{|t|}}$. Then, the entropy of every ergodic component of $\widetilde{\mu}$ is positive for the geodesic flow, i.e. for the time-one map T.*

Proof. Consider first the case $x\in\Omega$. By Theorem 3.5.15 there exists a constant $h>0$ such that for $\widetilde{\mu}$-a.e. $x\in\Omega$ we have

$$\widetilde{\mu}\left(x\bigcap_{i=-m}^{m} a^i B_\delta a^{-i}\right) \le e^{-mh}$$

for infinitely many $m\ge 1$. On the other hand, by Lemma 3.5.18, there is a countable partition ξ of finite entropy such that

$$[x]_{\bigvee_{i=-m}^{m} T^{-i}\xi} \subset x\bigcap_{i=-m}^{m} a^i B_\delta a^{-i}.$$

Hence, for $\widetilde{\mu}$-a.e. $x \in \Omega$ we have

$$\widetilde{\mu}\left([x]_{\bigvee_{i=-m}^{m} T^{-i}\xi}\right) \leq \widetilde{\mu}\left(x \bigcap_{i=-m}^{m} a^i B_\delta a^{-i}\right) \leq e^{-mh}$$

for infinitely many $m \geq 1$. Thus,

$$-\frac{1}{2m+1} \log \widetilde{\mu}\left([x]_{\bigvee_{i=-m}^{m} T^{-i}\xi}\right) \geq \frac{mh}{2m+1} \longrightarrow \frac{h}{2} > 0 \quad \text{as } m \to \infty.$$

By the Shannon-McMillan-Breiman Theorem it follows that

$$h_{\widetilde{\mu}_x}(T, \xi) \geq \frac{h}{2} > 0$$

for $\widetilde{\mu}$-a.e. $x \in \Omega$. Now, let $x \in X \backslash \Omega$. By construction of Ω there exists some $n \in \mathbb{Z}$ such that $T^n x \in \Omega$. Applying twice the Shannon-McMillan-Breiman Theorem, we get

$$\begin{aligned} h_{\widetilde{\mu}_x}(T, T^n\xi) &= \lim_{m\to\infty} -\frac{1}{2m+1} \log \widetilde{\mu}\left([x]_{\bigvee_{i=-m}^{m} T^{-i}T^n\xi}\right) \\ &= \lim_{m\to\infty} -\frac{1}{2m+1} \log \widetilde{\mu}\left([T^n x]_{\bigvee_{i=-m}^{m} T^{-i}\xi}\right) = h_{\widetilde{\mu}_{T^n x}}(T, \xi). \end{aligned}$$

Since $T^n x \in \Omega$ it follows that $h_{\widetilde{\mu}_{T^n x}}(T, \xi) \geq \frac{h}{2} > 0$, hence also

$$h_{\widetilde{\mu}_x}(T, T^n\xi) = \frac{h}{2} > 0.$$

Hence, the entropy of $\widetilde{\mu}$-almost every ergodic component is

$$h_{\widetilde{\mu}_x}(T) = \sup_{\xi} h_{\widetilde{\mu}_x}(T, \xi) \geq \frac{h}{2} > 0.$$

This finishes the proof of positive entropy. □

Improving on the convexity bound

In this chapter we are using the quantum unique ergodicity result from chapter 3 in order to get better bounds for the Rankin-Selberg integrals and, as a consequence, also for the Double Dirichlet series.

4.1 Bounds for Rankin-Selberg Integrals

First, we recall the trivial bound and prove it for completeness.

Theorem 4.1.1. *For every Maass cusp form ψ and every linear combination of weight $\frac{1}{2}$ Eisenstein series $E(z, \frac{1}{2}+it) = \sum_{\mathfrak{a}} c(\mathfrak{a}) E_{\mathfrak{a}}(z, \frac{1}{2}+it, \frac{1}{2}) \in V_{\frac{1}{2}+it}$ the following trivial bound holds:*

$$\int_{\Gamma_0(M)\backslash\mathbb{H}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 |d\mu(z) = O_\psi(\log|t|).$$

Proof. Since the cusp form ψ is rapidly decaying at the cusps, for every $A \in \mathbb{R}_{>2}$ we have $\psi(z) = O(\text{height}(z)^{-A})$, where the height is defined as $\text{height}(z) := \max_{\gamma\in\Gamma_0(M)} \Im(\gamma \cdot z)$. We divide the fundamental domain into subsets with height between X and $2X$, i.e.

$$\begin{aligned}\mathcal{F} &= \{z \in \mathcal{F} \mid \text{height}(z) < 1\} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4 \cup \ldots \\ &= \{z \in \mathcal{F} \mid \text{height}(z) < 1\} \cup \bigcup_{n\geq 0} \mathcal{F}_{2^n}\end{aligned}$$

where $\mathcal{F}_X := \{z \in \mathcal{F} \mid X \leq \text{height}(z) < 2X\}$. Note that $\mathcal{F}_X \subset \mathcal{F}^{2X}$, i.e. $\mathcal{F}_X$ is contained in the fundamental domain truncated at height $2X$. With this notation, we have

$$
\begin{aligned}
&\int_{\mathcal{F}} \psi(z)|E(z,\tfrac12+it)|^2 d\mu(z)\\
&= \int_{\{z\in\mathcal{F}|\text{height}(z)<1\}} \underbrace{\psi(z)}_{=O_\psi(1)} |E(z,\tfrac12+it)|^2 d\mu(z) + \sum_{n\geq 0}\int_{\mathcal{F}_{2^n}} \underbrace{\psi(z)}_{=O(2^{-An})} |E(z,\tfrac12+it)|^2 d\mu(z)\\
&\ll_\psi \int_{\{z\in\mathcal{F}|\text{height}(z)<1\}} |E^1(z,\tfrac12+it)|^2 d\mu(z) + \sum_{n\geq 0}\frac{1}{2^{An}}\int_{\mathcal{F}_{2^n}} |E^{2^{n+1}}(z,\tfrac12+it)|^2 d\mu(z)\\
&\ll \int_{\mathcal{F}} |E^1(z,\tfrac12+it)|^2 d\mu(z) + \sum_{n\geq 0}\frac{1}{2^{An}}\int_{\mathcal{F}} |E^{2^{n+1}}(z,\tfrac12+it)|^2 d\mu(z)\\
&\ll \log|t| + \sum_{n\geq 0}\frac{1}{2^{An}}\left(\log(2^{n+1})+\log|t|\right) = \log|t| + \log(2)\sum_{n\geq 0}\frac{n+1}{2^{An}} + \log|t|\sum_{n\geq 0}\frac{1}{2^{An}}\\
&\ll \log|t|.
\end{aligned}
$$

Here we used the Maass-Selberg relations from Theorem 2.4.2, namely

$$
\begin{aligned}
\int_{\mathcal{F}} |E^T(z,\tfrac12+it)|^2 d\mu(z) &= \int_{\mathcal{F}} \left|\sum_{\mathfrak{a}} c(\mathfrak{a}) E^T_{\mathfrak{a}}(z,\tfrac12+it,\tfrac12)\right|^2 d\mu(z)\\
&= \int_{\mathcal{F}} \sum_{\mathfrak{a}} c(\mathfrak{a}) E^T_{\mathfrak{a}}(z,\tfrac12+it,\tfrac12) \sum_{\mathfrak{b}} \overline{c(\mathfrak{b}) E^T_{\mathfrak{b}}(z,\tfrac12+it,\tfrac12)} d\mu(z)\\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})} \int_{\mathcal{F}} E^T_{\mathfrak{a}}(z,\tfrac12+it,\tfrac12)\overline{E^T_{\mathfrak{b}}(z,\tfrac12+it,\tfrac12)} d\mu(z)\\
&= \sum_{\mathfrak{a},\mathfrak{b}} c(\mathfrak{a})\overline{c(\mathfrak{b})}\Bigg(2\delta_{\mathfrak{a},\mathfrak{b}}\log(T) - \sum_{j=1}^{h} \underbrace{\varphi_{\mathfrak{a},\mathfrak{a}_j}(\tfrac12+it)}_{|\cdot|=1} \underbrace{\overline{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(\tfrac12+it)}}_{|\cdot|=\left|\frac{\varphi'_{\mathfrak{b},\mathfrak{a}_j}(\frac12+it)}{\varphi_{\mathfrak{b},\mathfrak{a}_j}(\frac12+it)}\right|\ll\log|t|}\\
&\qquad + \frac{1}{-2it}\underbrace{\overline{\varphi_{\mathfrak{b},\mathfrak{a}}(\tfrac12+it)}}_{|\cdot|=1}\underbrace{T^{-2it}}_{|\cdot|=1} - \frac{1}{-2it}\underbrace{\varphi_{\mathfrak{a},\mathfrak{b}}(\tfrac12+it)}_{|\cdot|=1}\underbrace{T^{2it}}_{|\cdot|=1}\Bigg)\\
&\ll \log(T)+\log|t|.
\end{aligned}
$$

□

We want to improve on this trivial bound and get a $o(\log|t|)$ instead. The following Lemma gives a bound for the integral outside of compact sets:

Lemma 4.1.2. *For every Maass cusp form ψ and every $\epsilon > 0$ there exists some $H = H(\psi, \epsilon)$ such that*

$$\int_{\{z\in\mathcal{F}|\text{height}(z)\geq H\}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 d\mu(z) = O_\psi(\epsilon \log|t|) \text{ as } |t| \to \infty.$$

Proof. Let $\mathcal{F}_X := \{z \in \mathcal{F} \mid X \leq \text{height}(z) < 2X\}$ and

$$\begin{aligned}\mathcal{F} &= \{z \in \mathcal{F} \mid \text{height}(z) < 1\} \cup \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4 \cup \dots \\ &= \{z \in \mathcal{F} \mid \text{height}(z) < 1\} \cup \bigcup_{n\geq 0} \mathcal{F}_{2^n}.\end{aligned}$$

For $A \in \mathbb{R}_{>2}$ we have $\psi(z) = O(\text{height}(z)^{-A})$, hence

$$\begin{aligned}\int_{\{z\in\mathcal{F}|\text{height}(z)\geq H\}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 d\mu(z) &= \sum_{n\geq 0} \int_{\mathcal{F}_{2^n H}} \underbrace{\psi(z)}_{=O(2^{-An}H^{-A})} \underbrace{|E(z, \tfrac{1}{2}+it)|^2}_{=|E^{2^{n+1}H}(z,\frac{1}{2}+it)|^2} d\mu(z) \\ &\ll \sum_{n\geq 0} \frac{1}{2^{An}H^A} \int_{\mathcal{F}} |E^{2^{n+1}H}(z, \tfrac{1}{2}+it)|^2 d\mu(z) \\ &\ll \frac{1}{H^A} \sum_{n\geq 0} \frac{1}{2^{An}} \left(\log(2^{n+1}H) + \log|t|\right) \\ &\ll \frac{1}{H^A} \sum_{n\geq 0} \frac{(n+1)\log(2) + \log H + \log|t|}{2^{An}} \\ &\ll \frac{1}{H^A} (\log H + \log|t|) \ll \frac{1}{H^A} \log|t|.\end{aligned}$$

So, indeed for $H \geq \epsilon^{-\frac{1}{A}}$ we have $\ll \epsilon \log|t|$. ∎

Remark. In particular, this Lemma tells us that given ψ and ϵ, then for big enough $|t|$ we have

$$\int_{\{z\in\mathcal{F}|\text{height}(z)\geq |t|\}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 d\mu(z) \ll \epsilon \log|t|.$$

Corollary 4.1.3. *For every $E(z, \frac{1}{2}+it) \in V_{\frac{1}{2}+it}$ and every Maass cusp form ψ we have*

$$\frac{1}{c\log|t|} \int_{\mathcal{F}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 d\mu(z) \to \int_{\mathcal{F}} \psi(z) d\mu(z) \text{ as } |t| \to \infty.$$

Proof. By Lemma 4.1.2, for every $\epsilon > 0$ we know that for $|t| \geq H = \epsilon^{-\frac{1}{A}}$ we have

$$\frac{1}{\log|t|} \int_{\{z\in\mathcal{F}|\text{height}(z)\geq |t|\}} \psi(z)|E(z, \tfrac{1}{2}+it)|^2 d\mu(z) = O(\epsilon).$$

In particular,

$$\frac{1}{\log|t|}\int_{\mathcal{F}}\psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z)=\frac{1}{\log|t|}\int_{\mathcal{F}^{|t|}}\psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z)+O(\epsilon). \quad (4.1.1)$$

Similarly, we also have

$$\begin{aligned}\int_{\{z\in\mathcal{F}|\mathrm{height}(z)\geq|t|\}}\underbrace{\psi(z)}_{=O(\mathrm{height}(z)^{-A})}d\mu(z) &\ll \frac{1}{|t|^A}\int_{\{z\in\mathcal{F}|\mathrm{height}(z)\geq|t|\}}1d\mu(z)\\ &\leq \frac{1}{|t|^A}\int_{\mathcal{F}}1d\mu(z)=\frac{1}{|t|^A}\leq\epsilon,\end{aligned}$$

thus

$$\int_{\mathcal{F}}\psi(z)d\mu(z)=\int_{\mathcal{F}^{|t|}}\psi(z)d\mu(z)+O(\epsilon). \quad (4.1.2)$$

Now, let $(f_n)\subset C_c(M)$ be a non-decreasing sequence of continuous functions on M with compact support such that $f_n \overset{n\to\infty}{\longrightarrow} \begin{cases}\psi(z) & \text{if } z\in\mathcal{F}^{|t|},\\ 0 & \text{otherwise.}\end{cases}$ So, we get

$$\begin{aligned}\frac{1}{c\log|t|}\int_{\mathcal{F}}\psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) &\overset{(4.1.1)}{=} \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}}\psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z)+O(\epsilon)\\ &= \frac{1}{c\log|t|}\int_{\mathcal{F}^{|t|}}\psi(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z)+O(\epsilon)\\ &= \frac{1}{c\log|t|}\int_{\mathcal{F}}\lim_{n\to\infty}f_n(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z)+O(\epsilon)\\ &= \lim_{n\to\infty}\frac{1}{c\log|t|}\int_{\mathcal{F}}f_n(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z)+O(\epsilon),\end{aligned}$$

where we could interchange integral and limit by monotone convergence theorem. On the other hand, we analogously have

$$\begin{aligned}\int_{\mathcal{F}}\psi(z)d\mu(z) &\overset{(4.1.2)}{=} \int_{\mathcal{F}^{|t|}}\psi(z)d\mu(z)+O(\epsilon)=\int_{\mathcal{F}}\lim_{n\to\infty}f_n(z)d\mu(z)+O(\epsilon)\\ &= \lim_{n\to\infty}\int_{\mathcal{F}}f_n(z)d\mu(z)+O(\epsilon).\end{aligned}$$

By our QUE result from chapter 2 we have the weak*-convergence

$$\mu_t \overset{weak^*}{\longrightarrow} \mathrm{dvol_M}$$

as $|t| \to \infty$, which means that

$$\frac{1}{c\log|t|}\int_{\mathcal{F}} f(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) = \int_{\mathcal{F}} f d\mu_t = \mu_t(f) \longrightarrow \mathrm{dvol_M(f)} = \int_{\mathcal{F}} \mathrm{f(z)d}\mu\mathrm{(z)}$$

for every $f \in C_c(M)$. In particular, this holds for the f_n defined above. Hence,

$$\frac{1}{c\log|t|}\int_{\mathcal{F}} \psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) = \lim_{n\to\infty}\frac{1}{c\log|t|}\int_{\mathcal{F}} f_n(z)|E^{|t|}(z,\tfrac{1}{2}+it)|^2 d\mu(z) + O(\epsilon)$$
$$\overset{|t|\to\infty}{\longrightarrow} \lim_{n\to\infty}\int_{\mathcal{F}} f_n(z)d\mu(z) + O(\epsilon) = \int_{\mathcal{F}} \psi(z)d\mu(z) + O(\epsilon),$$

This holds for every $\epsilon > 0$. So, we let $\epsilon \to 0$ to get

$$\frac{1}{c\log|t|}\int_{\mathcal{F}} \psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) \to \int_{\mathcal{F}} \psi(z)d\mu(z) \quad \text{as } |t| \to \infty.$$

□

Corollary 4.1.4. *For every $E(z,\frac{1}{2}+it) \in V_{\frac{1}{2}+it}$ and every Maass cusp form ψ we have the upper bound*

$$\int_{\mathcal{F}} \psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) = o(\log|t|).$$

Proof. Note that by the spectral decomposition, Maass cusp forms are orthogonal to the constant functions, i.e.

$$\int_{\mathcal{F}} \psi(z)d\mu(z) = 0.$$

So, Corollary 4.1.3 implies

$$\frac{1}{c\log|t|}\int_{\mathcal{F}} \psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) \to 0 \quad \text{as } |t| \to \infty.$$

Hence,

$$\int_{\mathcal{F}} \psi(z)|E(z,\tfrac{1}{2}+it)|^2 d\mu(z) = o(\log|t|).$$

□

Remark. Note that the bound also holds for ψ replaced by $\psi \otimes \chi$ for any character χ as this is again a Maass cusp form.

4.2 Bounds for Double Dirichlet Series

Using the bounds for the Rankin-Selberg integrals from the previous section, we are deducing corresponding bounds for the double Dirichlet series.

Proposition 4.2.1. *Let ψ be a cuspidal Hecke newform. Let χ be a Dirichlet character modulo 2^m and η a Dirichlet character modulo $M = 2^l$. Then, for $s = \frac{1}{2} + it$ and $w = \frac{1}{2} - it$ the following bounds hold:*

$$\int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) \left|E_{0,\eta}(z, w, \tfrac{1}{2})\right|^2 d\mu(z) = o(\log |t|)$$

$$\int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) \left|E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 d\mu(z) = o(\log |t|)$$

$$\int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) \left|E_{0,\eta}(z, w, \tfrac{1}{2}) + iE_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 d\mu(z) = o(\log |t|)$$

$$\int_{\Gamma_0(M)\backslash\mathbb{H}} (\psi \otimes \chi)(z) \left|E_{0,\eta}(z, w, \tfrac{1}{2}) + E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 d\mu(z) = o(\log |t|).$$

Proof. Since $\psi \otimes \chi$ is a Maass cusp form, these bounds hold by Corollary 4.1.4. □

Theorem 4.2.2. *Let ψ be a cuspidal Hecke newform. Let χ be a Dirichlet character modulo 2^m and η a Dirichlet character modulo $M = 2^l$. Then, we have the bound*

$$I(\psi, \chi, \eta, \tfrac{1}{2} + it, \tfrac{1}{2} - it) = o(\log |t|).$$

Proof. Note that

$$\begin{aligned}&\left|E_{0,\eta}(z, w, \tfrac{1}{2}) + E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 \\ &= \left|E_{0,\eta}(z, w, \tfrac{1}{2})\right|^2 + \left|E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 + E_{0,\eta}(z, w, \tfrac{1}{2})\overline{E_{\infty,\eta}(z, w, \tfrac{1}{2})} + \overline{E_{0,\eta}(z, w, \tfrac{1}{2})}E_{\infty,\eta}(z, w, \tfrac{1}{2})\end{aligned}$$

and

$$\begin{aligned}&\left|E_{0,\eta}(z, w, \tfrac{1}{2}) + iE_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 \\ &= \left|E_{0,\eta}(z, w, \tfrac{1}{2})\right|^2 + \left|E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 - iE_{0,\eta}(z, w, \tfrac{1}{2})\overline{E_{\infty,\eta}(z, w, \tfrac{1}{2})} + i\overline{E_{0,\eta}(z, w, \tfrac{1}{2})}E_{\infty,\eta}(z, w, \tfrac{1}{2}).\end{aligned}$$

Rearranging the equations while multiplying the second with i, we have

$$\begin{aligned}&E_{0,\eta}(z, w, \tfrac{1}{2})\overline{E_{\infty,\eta}(z, w, \tfrac{1}{2})} + \overline{E_{0,\eta}(z, w, \tfrac{1}{2})}E_{\infty,\eta}(z, w, \tfrac{1}{2}) \\ &\qquad = \left|E_{0,\eta}(z, w, \tfrac{1}{2}) + E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2 - \left|E_{0,\eta}(z, w, \tfrac{1}{2})\right|^2 - \left|E_{\infty,\eta}(z, w, \tfrac{1}{2})\right|^2\end{aligned}$$

and

$$\begin{aligned}&E_{0,\eta}(z,w,\tfrac{1}{2})\overline{E_{\infty,\eta}(z,w,\tfrac{1}{2})}-\overline{E_{0,\eta}(z,w,\tfrac{1}{2})}E_{\infty,\eta}(z,w,\tfrac{1}{2})\\&\quad=i\left|E_{0,\eta}(z,w,\tfrac{1}{2})+iE_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2-i\left|E_{0,\eta}(z,w,\tfrac{1}{2})\right|^2-i\left|E_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2.\end{aligned}$$

Adding both equations gives

$$\begin{aligned}E_{0,\eta}(z,w,\tfrac{1}{2})\overline{E_{\infty,\eta}(z,w,\tfrac{1}{2})}&=\tfrac{1}{2}\left|E_{0,\eta}(z,w,\tfrac{1}{2})+E_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2\\&\quad+\tfrac{1}{2}i\left|E_{0,\eta}(z,w,\tfrac{1}{2})+iE_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2\\&\quad-\tfrac{1}{2}(1+i)\left|E_{0,\eta}(z,w,\tfrac{1}{2})\right|^2-\tfrac{1}{2}(1+i)\left|E_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2.\end{aligned}$$

Hence, with $s=\frac{1}{2}+it$ and $w=\frac{1}{2}-it$, we have

$$\begin{aligned}I(\psi,\chi,\eta,s,w)&=\int_{\Gamma_0(M)\backslash\mathbb{H}}(\psi\otimes\chi)(z)E_{0,\eta}(z,w,\tfrac{1}{2})\overline{E_{\infty,\eta}(z,w,\tfrac{1}{2})}d\mu(z)\\&=\tfrac{1}{2}\int_{\Gamma_0(M)\backslash\mathbb{H}}(\psi\otimes\chi)(z)\left|E_{0,\eta}(z,w,\tfrac{1}{2})+E_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2d\mu(z)\\&\quad+\tfrac{1}{2}i\int_{\Gamma_0(M)\backslash\mathbb{H}}(\psi\otimes\chi)(z)\left|E_{0,\eta}(z,w,\tfrac{1}{2})+iE_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2d\mu(z)\\&\quad-\tfrac{1}{2}(1+i)\int_{\Gamma_0(M)\backslash\mathbb{H}}(\psi\otimes\chi)(z)\left|E_{0,\eta}(z,w,\tfrac{1}{2})\right|^2d\mu(z)\\&\quad-\tfrac{1}{2}(1+i)\int_{\Gamma_0(M)\backslash\mathbb{H}}(\psi\otimes\chi)(z)\left|E_{\infty,\eta}(z,w,\tfrac{1}{2})\right|^2d\mu(z)\\&=o(\log|t|),\end{aligned}$$

since all integrals appearing are $o(\log|t|)$ by Proposition 4.2.1. □

Corollary 4.2.3. *Let ψ be a Hecke Maass cusp new form of conductor some power of 2 and let χ and χ' be Dirichlet characters some power of 2. Then, the following bound holds for the Double Dirichlet series:*

$$Z_\psi(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi')=o\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right),$$

where $q(t)=|t|^2$.

Proof. From the bound on $I(\psi,\chi,\chi',s,w)$ it follows that

$$\begin{aligned}\tilde{I}(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}-it)&=O\left(\left|\zeta^{(2)}(1-4it)\zeta^{(2)}(1+4it)I(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}-it)\right|\right)\\&=o\left(|\zeta^{(2)}(1-4it)||\zeta^{(2)}(1+4it)|\log|t|\right).\end{aligned}$$

We use the same argumentation as in chapter 1, but now with the stronger bound on the Rankin-Selberg integral, to get

$$\begin{aligned}
&Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') \pm \chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi') \\
&\qquad = \frac{\tilde{I}(\psi,\chi,\overline{\chi'},\tfrac{1}{2}+it,\tfrac{1}{2}-it) \pm \chi(-1)b_{-1}\tilde{I}(\psi,\chi,\chi_4\overline{\chi'},\tfrac{1}{2}+it,\tfrac{1}{2}-it)}{G_+(t)\pm G_-(t)} \\
&\qquad = o\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') &= \frac{1}{2}\Big(Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi')+\chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi') \\
&\qquad + Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi')-\chi(-1)b_{-1}Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi_4\chi')\Big) \\
&= o\left(|t|^{\frac{1}{2}}\log|t|\,|\zeta^{(2)}(1+4it)||\zeta^{(2)}(1-4it)|\right).
\end{aligned}$$

□

Remark. If one could show a stronger QUE result which would lead to the bounds $O(|t|^{-\delta})$ in Proposition 4.2.1, then even subconvexity would follow for the Double Dirichlet series. Indeed, in that case we would get an $O(|t|^{-\delta})$-bound for the Rankin-Selberg integral in Theorem 4.2.2. Consequently,

$$\tilde{I}(\psi,\chi,\chi',\tfrac{1}{2}+it,\tfrac{1}{2}-it) = O(|t|^{-\delta'})$$

and

$$Z(\tfrac{1}{2}+it,\tfrac{1}{2}-it,\chi,\chi') = O\left(|t|^{\frac{1}{2}-\delta'}\right) = O\left(q(t)^{\frac{1}{4}-\hat{\delta}}\right).$$

However, using the ergodic theoretic method we presented in chapter 2, this is out of reach.

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