hp FEM for Reaction-Diffusion Equations. II: Regularity Theory

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Abstract

A singularly perturbed reaction-diffusion equation in two dimensions is considered. We assume analyticity of the input data, i.e., the boundary of the domain is an analytic curve, the boundary data are analytic, and the right hand side is analytic. We give asymptotic expansions of the solution and new error bounds that are uniform in the perturbation parameter as well as in the expansion order. Additionally, we provide growth estimates for higher derivatives of the solution where the dependence on the perturbation parameter appears explicitly. These error bounds and growth estimates are used in the first part of this work to construct hp versions of the finite element method which feature *robust exponential convergence*, i.e., the rate of convergence is exponential and independent of the perturbation parameter ε.

Keywords: boundary layer, singularly perturbed problem, asymptotic expansions, error bounds

1 Introduction

Numerous partial differential equation models contain large or small parameters. We mention only the Navier–Stokes equations at small viscosity, the plate- and shell equations at small thickness, nearly incompressible solids and so on. The presence of small parameters often implies that the problem is singularly perturbed and much attention has been devoted in the past decades to the asymptotic analysis of the solution; we mention here only [1], [2]. Typically, the solutions admit decompositions into a smooth part, so-called boundary layers, and, in nonsmooth domains, corner layers. While the asymptotic structure of the solution is known for many problems (see, e.g., [3], [4], [5], [6]), the asymptotic expansions are often too complex to allow for the quantitative solution of specific problems, and one has to resort to numerical solutions of the boundary value problem (BVP) of interest. Here the singular perturbation character of the problem and the boundary layer components of the solution cause stability (locking) and approximability problems. The key to the convergence of a stable numerical method for these BVPs is the regularity of the solution, particularly, bounds on higher derivatives.
To analyze the parameter dependence of solution derivatives of arbitrary order for a class of elliptic, singularly perturbed BVPs is the purpose of the present paper. The main results are new growth estimates for higher order derivatives that are explicit in the small parameter ε and new, sharp error bounds of the asymptotic expansions of the solutions. These bounds are used in the first part of this work to analyze an *hp* Finite Element Method (*hp*-FEM) with *robust exponential convergence* for this problem class [7]. The techniques we employ, namely, Morrey's regularity theory and asymptotic expansions, are applicable to general elliptic systems and results analogous to the ones obtained here likely hold true for many other, singularly perturbed elliptic problems; this will be explored in future work.

1.1 The Model Problem

We consider the following model problem

$$\begin{aligned} L_\varepsilon u_\varepsilon \equiv -\varepsilon^2 \Delta u_\varepsilon + u_\varepsilon &= f \qquad \text{on } \Omega \subset \mathbb{R}^2, \\ u_\varepsilon &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

where $\partial\Omega$ is a closed, non-selfintersecting, analytic curve, f is analytic on $\overline{\Omega}$, g is analytic on $\partial\Omega$, and $\varepsilon \in (0,1]$ is a small parameter.
As usual, we denote by $L^2(\Omega)$ the square integrable functions on Ω and by $H^1(\Omega)$ those functions of $L^2(\Omega)$ whose (distributional) derivative is also in $L^2(\Omega)$. The trace operator maps $H^1(\Omega)$ onto the space $H^{1/2}(\partial\Omega)$ by restricting the elements of $H^1(\Omega)$ to the boundary $\partial\Omega$. $H^1_0(\Omega)$ denotes the kernel of the trace operator, that is, it is given by those functions in $H^1(\Omega)$ whose trace on $\partial\Omega$ is zero.
The weak formulation of (1.1) is to find $u_\varepsilon \in H^1(\Omega)$ such that $u_\varepsilon|_{\partial\Omega} = g$ and

$$B_\varepsilon(u_\varepsilon, v) := \varepsilon^2 \int_\Omega \nabla u_\varepsilon \cdot \nabla v \, dxdy + \int_\Omega u_\varepsilon v \, dxdy = F(v) := \int_\Omega f v \, dxdy \qquad \forall v \in H^1_0(\Omega). \tag{1.2}$$

Associated with this problem is the notion of an "energy"

$$\|u\|^2_{\varepsilon,\Omega} := B_\varepsilon(u,u) = \varepsilon^2 \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}$$

and an energy norm, being the square root of the energy. We have the a-priori estimate

$$\|u_\varepsilon\|_{\varepsilon,\Omega} \le \|f\|_{L^2(\Omega)} + C\|g\|_{H^{1/2}(\partial\Omega)} \tag{1.3}$$

for some $C > 0$ independent of ε.

The purpose of this paper to analyze the growth of the derivatives of the exact solution u_ε of (1.1). As the input data is analytic, standard elliptic regularity theory implies that the exact solution u_ε is analytic on $\overline{\Omega}$, i.e., it satisfies estimates of the form

$$\|D^\alpha u_\varepsilon\|_{L^\infty(\Omega)} \le |\alpha|! C_\varepsilon K_\varepsilon^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2.$$

However, the constants C_ε and K_ε depend on ε and our aim here is to control explicitly the dependence on ε of the derivatives of u_ε. Using the techniques of Morrey [8], we show in Section 3 (Theorem 3.1) the following estimate:

$$\|D^\alpha u_\varepsilon\|_{L^2(\Omega)} \le C K^{|\alpha|} \max\left(|\alpha|, \varepsilon^{-1}\right)^{|\alpha|} \qquad \forall \alpha \in \mathbb{N}_0^2$$

with C, $K > 0$ independent of ε. Note that for $|\alpha| \ge \varepsilon^{-1}$, this yields an estimate independent of ε; roughly speaking, this means that derivatives of order higher than ε^{-1} "don't see" the boundary layers introduced by the singular perturbation. This estimate is also sufficient to prove that polynomials of degree p can approximate the solution u_ε at a robust exponential rate provided that the polynomial degree p is at least $O(\varepsilon^{-1})$.

For a description of the behavior of the derivatives $D^\alpha u_\varepsilon$ of order $|\alpha| < \varepsilon^{-1}$, a more careful analysis is necessary. It is well-known that the solutions of (1.1) exhibit boundary layers, that is, in a neighborhood of the boundary $\partial\Omega$, the behavior of the solution normal to the boundary differs substantially from the behavior in the tangential direction. The description of the boundary layers is done in terms of asymptotic expansions. The main purpose of our analysis in Section 2.3 is to provide new error bounds for the remainder depending explicitly on the perturbation parameter ε and the expansion order (Theorem 2.7).

1.2 Notation

We introduce *boundary fitted coordinates* to define later on the asymptotic expansions of the exact solution. Let $(X(\theta), Y(\theta))$, $\theta \in [0, L)$ be an analytic, L-periodic parametrization by arclength of the boundary $\partial\Omega$ such that the normal vector $(-Y'(\theta), X'(\theta))$ always points into the domain Ω. Introduce the notation $\kappa(\theta)$ for the curvature of the boundary curve and denote by $\mathbb{T}_L$ the one dimensional torus of length L, i.e., $\mathbb{R}/_{[0,L)}$ endowed with the usual topology. The functions X, Y, and hence also κ are analytic on $\mathbb{T}_L$ by the analyticity of $\partial\Omega$. For the remainder of this paper, let $\rho_0 > 0$ be fixed such that

$$0 < \rho_0 < \frac{1}{\|\kappa\|_{L^\infty([0,L))}}. \tag{1.4}$$

Then the mapping

$$\begin{array}{rcl} \psi : [0, \rho_0] \times \mathbb{T}_L & \to & \overline{\Omega} \\ (\rho, \theta) & \mapsto & (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta)) \end{array} \tag{1.5}$$

is real analytic on $[0,\rho_0]\times\mathbb{T}_L$. The function ψ maps the rectangle $(0,\rho_0)\times[0,L)$ onto a tubular neighborhood Ω_0 of $\partial\Omega$. Furthermore by the choice of ρ_0, the inverse $\psi^{-1}:\overline{\Omega}_0\to[0,\rho_0]\times\mathbb{T}_L$ exists and is also real analytic on the closed set $\overline{\Omega}_0$.

For technical reasons we will be able to define the boundary layer expansion, that is, the inner expansion, only in a neighborhood of the boundary $\partial\Omega$. Therefore, we introduce a cut-off function χ supported by a neighborhood of $\partial\Omega$. For ease of notation, let us define χ in the neighborhood of $\partial\Omega$ in boundary fitted coordinates (ρ,θ). Fix

$$0<\rho_1<\rho_0, \tag{1.6}$$

and let χ be a smooth cut-off function, defined on $[0,\infty)\times\mathbb{T}_L$, satisfying

$$\chi=\begin{cases}1 & \text{for } 0\le\rho\le\rho_1\\ 0 & \text{for } \rho\ge(\rho_1+\rho_0)/2.\end{cases} \tag{1.7}$$

The boundary layer functions u_M^{BL} to be defined and analyzed in Section 2 decay exponentially away from the boundary. In order to describe this exponential decay, we introduce exponentially weighted spaces.

Definition 1.1 *Let $\alpha\in\mathbb{R}$. Define the spaces H^0_α, H^1_α as the completion of the smooth function on $[0,\infty)$ which have bounded support under the norms $\|\cdot\|_{0,\alpha}$ and $\|\cdot\|_{1,\alpha}$. These norms are given by*

$$\begin{aligned}\|f\|_{0,\alpha} &:= \left\{\int_0^\infty e^{2\alpha x}|f(x)|^2\,dx\right\}^{1/2},\\ \|f\|_{1,\alpha} &:= \left\{\int_0^\infty e^{2\alpha x}\left(|f(x)|^2+|f'(x)|^2\right)\,dx\right\}^{1/2}.\end{aligned}$$

Similarly, we define for functions $f:[0,\infty)\times\mathbb{T}_L$ the norms $\|\cdot\|_{0,\alpha,\infty}$ and $\|\cdot\|_{1,\alpha,\infty}$ via

$$\begin{aligned}\|f\|_{0,\alpha,\infty} &:= \left\{\sup_{y\in\mathbb{T}_L}\int_0^\infty e^{2\alpha x}|f(x,y)|^2\,dx\right\}^{1/2},\\ \|f\|_{1,\alpha,\infty} &:= \left\{\sup_{y\in\mathbb{T}_L}\int_0^\infty e^{2\alpha x}\left(|f(x,y)|^2+|\partial_x f(x,y)|^2\right)\,dx\right\}^{1/2}.\end{aligned}$$

For functions f of two variables, we introduce the short hand notation

$$|\nabla^p f|^2 := \sum_{|\alpha|=p}\frac{|\alpha|!}{\alpha!}|D^\alpha f|^2=\sum_{\beta_1,\dots,\beta_p=1}^{2}|D^{\beta_1\cdots\beta_p}f|^2$$

to control all derivatives of order p simultaneously.

Finally, as the right hand side f of (1.1) is assumed to be analytic on $\overline{\Omega}$ there is a complex neighborhood $\tilde{\Omega}\subset\mathbb{C}\times\mathbb{C}$ of $\overline{\Omega}$ and a holomorphic extension of f (for convenience again denoted by f) to $\tilde{\Omega}$ which satisfies

$$\|\nabla^p f\|_{L^\infty(\tilde{\Omega})}\le C_f p!\gamma_f^p \qquad \forall p\in\mathbb{N}_0 \tag{1.8}$$

for some C_f, $\gamma_f>0$. As f is holomorphic on $\tilde{\Omega}$, there is a constant $\gamma_{\Delta f}\ge\gamma_f$ such that

$$\|\Delta^{(i)}f\|_{L^\infty(\Omega)}\le C_f(2i)!\gamma_{\Delta f}^{2i} \qquad \forall i\in\mathbb{N}_0 \tag{1.9}$$

where $\Delta^{(i)}$ denotes the iterated Laplace operator, i.e., $\Delta^{(0)}=Id$, $\Delta^{(1)}=\Delta$, $\Delta^{(2)}=\Delta\Delta$, etc.

2 Analysis of the Asymptotic Expansion

In this section we present classical asymptotic expansions for the solution of (1.1). Our main result is a new error bound for the remainder, Theorem 2.7.
The asymptotic expansions (defined more precisely in the next subsection) allow us to decompose the solution u_ε as

$$u_\varepsilon = w_M + \chi u_M^{BL} + r_M$$

where $M \in \mathbb{N}_0$ indicates the expansion order, w_M is the truncated outer expansion, u_M^{BL} is the truncated inner expansion, χ is the cut-off function defined in (1.7), and r_M is a remainder. That the boundary layer functions u_M^{BL} decay indeed exponentially away from the boundary $\partial\Omega$ is proved in Section 2.2. The error bounds for the asymptotic expansion, i.e., bounds on the remainder r_M, can be found in Section 2.3.

2.1 Inner and Outer Expansion

For every $M \in \mathbb{N}_0$ the *outer expansion* of order $2M$ is given by

$$w_M := \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} f. \tag{2.1}$$

The function $u_\varepsilon - w_M$ then satisfies

$$L_\varepsilon(u_\varepsilon - w_M) = f - L_\varepsilon w_M = \varepsilon^{2M+2}\Delta^{(M+1)} f. \tag{2.2}$$

So, asymptotically as ε tends to zero, the functions w_M satisfy the differential equation in Ω. However, the functions w_M do not satisfy the given boundary conditions g. We therefore introduce a correction u^{BL} of w_M, which will lead to the inner expansion. The correction u^{BL} is defined as the solution of

$$\begin{aligned} L_\varepsilon u^{BL} &= 0 && \text{in } \Omega, \\ u^{BL} &= g - \sum_{i=0}^{M} \varepsilon^{2i} \left[\Delta^{(i)} f\right]_{\partial\Omega} && \text{on } \partial\Omega. \end{aligned} \tag{2.3}$$

The *inner expansion* is now an asymptotic expansion for this correction function u^{BL}. In order to define this expansion, we need to rewrite the differential operator L_ε in the boundary fitted coordinates (ρ, θ). If we introduce the curvature $\kappa(\theta)$ of $\partial\Omega$ and the function

$$\sigma(\rho, \theta) = \frac{1}{1 - \kappa(\theta)\rho}$$

we have (see, for example, [3])

$$\Delta u(\rho, \theta) = \partial_\rho^2 u - \kappa(\theta)\sigma(\rho,\theta)\partial_\rho u + \sigma^2(\rho,\theta)\partial_\theta^2 u + \rho\kappa'(\theta)\sigma^3(\rho,\theta)\partial_\theta u.$$

Expanding the function σ in a converging geometric series gives

$$\sigma(\rho, \theta) = \sum_{i=0}^{\infty} [\kappa(\theta)\rho]^i = \sum_{i=0}^{\infty} \varepsilon^i [\kappa(\theta)\widehat{\rho}]^i$$

where we introduced the *stretched variable* notation $\widehat{\rho} = \rho/\varepsilon$. Note that we chose $\rho_0 < \|\kappa\|_{L^\infty([0,L))}$ in (1.4) so that the power series expansion converges uniformly in $(\rho,\theta) \in [0,\rho_0] \times [0,L]$.
Recall that Ω_0 is the tubular neighborhood $\partial\Omega$ which is the image of the rectangle $(0,\rho_0)\times[0,L)$ under the map ψ. In this tubular neighborhood Ω_0 the differential equation (2.3) takes the form

$$-\varepsilon^2\left\{\partial_\rho^2 u^{BL} + \sum_{i=0}^{\infty}\rho^i\left(a_1^i\partial_\rho u^{BL} + a_2^i\partial_\theta^2 u^{BL} + a_3^i\partial_\theta u^{BL}\right)\right\} + u^{BL} = 0 \qquad \text{in } \Omega_0 \tag{2.4}$$

where we introduced the abbreviations

$$a_1^i = -[\kappa(\theta)]^{i+1}, \quad a_2^i = (i+1)[\kappa(\theta)]^i, \quad a_3^i = \frac{i(i+1)}{2}[\kappa(\theta)]^{i-1}\kappa'(\theta). \tag{2.5}$$

For technical convenience let us also formulate (2.4) in terms of the stretched variable $\widehat{\rho}$:

$$-\partial_{\widehat{\rho}}^2 u^{BL} - \sum_{i=0}^{\infty}(\varepsilon\widehat{\rho})^i\left(\varepsilon a_1^i\partial_{\widehat{\rho}} u^{BL} + \varepsilon^2 a_2^i\partial_\theta^2 u^{BL} + \varepsilon^2 a_3^i\partial_\theta u^{BL}\right) + u^{BL} = 0. \tag{2.6}$$

Now, in order to define the inner expansion, we make the formal ansatz $u^{BL} = \sum_{i=0}^\infty \varepsilon^i \widehat{U}_i(\widehat{\rho},\theta)$ where the functions $\widehat{U}_i$ are to be determined. Inserting this ansatz in (2.4) and equating like powers of ε we obtain a recurrence relation for the functions $\widehat{U}_i$:

$$\begin{aligned}
-\partial_{\widehat{\rho}}^2\widehat{U}_i + \widehat{U}_i &= \widehat{F}_i, \qquad i = 0,1,\dots\,, \\
\widehat{F}_i &= \widehat{F}_i^1 + \widehat{F}_i^2 + \widehat{F}_i^3, \\
\widehat{F}_i^1 &= \sum_{j=0}^{i-1}\widehat{\rho}^j a_1^j \partial_{\widehat{\rho}}\widehat{U}_{i-1-j}, \\
\widehat{F}_i^2 &= \sum_{j=0}^{i-2}\widehat{\rho}^j a_2^j \partial_\theta^2\widehat{U}_{i-2-j}, \\
\widehat{F}_i^3 &= \sum_{j=0}^{i-2}\widehat{\rho}^j a_3^j \partial_\theta\widehat{U}_{i-2-j}
\end{aligned}$$

where we used the tacit convention that empty sums take the value zero. As we expect the boundary layer function u^{BL} to decay away from the boundary $\partial\Omega$ and as we want to satisfy the boundary conditions, we supplement these ODEs for the $\widehat{U}_i$ with the boundary conditions

$$\begin{aligned}
\widehat{U}_i &\to 0 \qquad \text{as } \widehat{\rho}\to\infty, \\
[\widehat{U}_i]_{\partial\Omega} &= G_i := \begin{cases} g - [f]_{\partial\Omega} & \text{if } i = 0 \\ -[\Delta^{(i/2)} f]_{\partial\Omega} & \text{if } 0 < i \le 2M \text{ is even} \\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

The *inner expansion* of order $2M+1$ is defined as the function

$$u_M^{BL}(\rho,\theta) := \sum_{i=0}^{2M+1}\varepsilon^i\widehat{U}_i(\widehat{\rho},\theta) = \sum_{i=0}^{2M+1}\varepsilon^i\widehat{U}_i(\rho/\varepsilon,\theta), \tag{2.7}$$

and it satisfies the boundary conditions

$$[u_M^{BL}]_{\partial\Omega} = g - \sum_{i=0}^{M} \varepsilon^{2i}[\Delta^{(i)} f]_{\partial\Omega}.$$

Remark 2.1: We defined u_M^{BL} as the inner expansion of order $2M+1$ so that the first neglected term of the formal asymptotic expansion $\sum_{i=0}^{\infty}\varepsilon^i \widehat{U}_i$ is of order ε^{2M+2}. This is precisely the same power of ε as the first neglected term of the outer expansion $\sum_{i=0}^{\infty}\varepsilon^{2i}\Delta^{(i)} f$ truncated after the ε^{2M} term.

2.2 Properties of the Boundary Layer functions

By Lemma B.4 we see that the functions a_l^i, $l=1,2,3$, $i \in \mathbb{N}_0$ of (2.5) satisfy

$$\|D^p a_l^i\|_{L^\infty([0,L))} \le C_A A^p \tilde{A}^i \qquad \forall p, i \in \mathbb{N}_0, l = 1,2,3 \tag{2.8}$$

for some appropriate C_A, A, and $\tilde{A}$. In fact, Lemma B.4 allows us to choose $\tilde{A} > \|\kappa\|_{L^\infty([0,L))}$ arbitrarily close to $\|\kappa\|_{L^\infty([0,L))}$, so that we may assume that

$$\rho_0 \tilde{A} =: q < 1. \tag{2.9}$$

Similarly, we see that independently of M there are C_G, G, and $\tilde{G}$ such that

$$\|D^p G_i\|_{L^\infty([0,L))} \le C_G (i+p)^{i+p} \tilde{G}^p G^i. \tag{2.10}$$

We are now in position to formulate the following two propositions which clarify the properties of the functions $\widehat{U}_i$. The proofs are deferred to Appendix A.

Proposition 2.2 *For each $\alpha \in [0,1)$ the functions $\widehat{U}_i$ defined above satisfy*

$$\|\partial_\theta^m \widehat{U}_i\|_{1,\alpha,\infty} \le C_U \frac{K_1^m K_2^i}{(1-\alpha)^{i+1}} (i+m)^{i+m} \qquad \forall i, m \in \mathbb{N}_0 \tag{2.11}$$

for C_U, K_1, K_2 chosen such that

$$C_U := 6C_G + 1, \tag{2.12}$$

$$K_1 := 2\max(\tilde{G}, A), \tag{2.13}$$

$$K_2 \ge 2\max(2\tilde{A}, G+1) \qquad \textit{such that } (K_2^{-1} + K_2^{-2}K_1^2 + K_2^{-2}K_1) 12 C_A \le \frac{1}{2}. \tag{2.14}$$

Proposition 2.3 *Let $\alpha \in [0,1)$. Under the same hypotheses as Proposition 2.2, the functions $\widehat{U}_i$ satisfy*

$$\|\partial_{\hat{\rho}}^n \partial_\theta^m \widehat{U}_i\|_{0,\alpha,\infty} \le C_U \frac{K_1^m K_2^i K_3^n}{(1-\alpha)^{i+1}} \{i+m+\max(n-2,0)\}^{i+m} \qquad \forall i,m,n \in \mathbb{N}_0 \tag{2.15}$$

where C_U, K_1, K_2 are as defined in Proposition 2.2 and $K_3 > e$ satisfies

$$C_A \frac{K_2^{-1}K_3^{-1} + K_1^2 K_2^{-2} K_3^{-2} + K_1 K_2^{-2} K_3^{-2}}{(1-A/K_1)(1-2\tilde{A}/K_2)(1-e/K_3)} + K_3^{-2} \le 1. \tag{2.16}$$

Proposition 2.3 yields the following corollary.

Corollary 2.4 *Let K_1, K_2, K_3 and C_U be as in Proposition 2.3. The functions $\widehat{U}_i$ can be extended to functions holomorphic on*

$$\mathbb{C} \times \{z \,|\, |\mathrm{Im}\, z| < K_1^{-1} e^{-2}\}.$$

On setting $K_4 := K_3 e^2$, $K_5 := K_1 e^2$, the functions $\widehat{U}_i$ satisfy for all $\widehat{\rho} \geq 0$, $i \in \mathbb{N}_0$, $z \in \mathbb{C}$, $\zeta \in \mathbb{C}$ with $|\zeta| < K_5^{-1}$ the estimate

$$\left|\widehat{U}_i(\widehat{\rho} + z, \theta + \zeta)\right| \leq C(\alpha) e^{-\alpha \hat{\rho}} e^{K_4 |z|} \left(\frac{eK_2}{1-\alpha}\right)^i i^i \left(1 - K_5|\zeta|\right)^{-1}$$

where the constant $C(\alpha) > 0$ depends only on $\alpha \in (0,1)$, C_U, and K_3.

Proof: Proposition 2.3 allows us to control the growth of the derivatives of the functions $\widehat{U}_i$, and we are therefore able to get bounds on the power series expansion of $\widehat{U}_i$. By Lemma B.3, we have for $\widehat{\rho} \geq 0$ the estimate

$$|\partial_{\hat{\rho}}^n \, \partial_\theta^m \, \widehat{U}_i(\widehat{\rho}, \theta)| \leq \frac{1}{\sqrt{2\alpha}} e^{-\alpha \hat{\rho}} \|\partial_{\hat{\rho}}^{n+1} \, \partial_\theta^m \, \widehat{U}_i\|_{0,\alpha,\infty} \leq \frac{1}{\sqrt{2\alpha}} e^{-\alpha \hat{\rho}} C_U \frac{K_1^m K_2^i K_3^{n+1}}{(1-\alpha)^{i+1}} (i+m+n)^{i+m}.$$

Therefore, for $\widehat{\rho} \geq 0$ and $z \in \mathbb{C}$, $\zeta \in \mathbb{C}$ with $|\zeta| < K_5^{-1}$ we have

$$\sum_{n,m=0}^{\infty} \frac{\left|\partial_{\hat{\rho}}^n \, \partial_\theta^m \, \widehat{U}_i(\widehat{\rho}, \theta)\right|}{n!m!} |z|^n |\zeta|^m \leq$$
$$\frac{1}{\sqrt{2\alpha}(1-\alpha)} C_U K_3 \left(\frac{K_2}{(1-\alpha)}\right)^i \sum_{n,m=0}^{\infty} \frac{K_1^m K_3^n (i+m+n)^{i+m}}{n!m!} |z|^n |\zeta|^m.$$

From the estimate $(a+b+c)^{a+b} \leq a^a b^b e^{a+b+2c}$, valid for non-negative a, b, and c, we obtain

$$(i+m+n)^{i+m} \leq i^i m^m e^{i+m+2n}.$$

With the aid of Stirling's formula, $m! \geq C m^m e^{-m}$, we arrive at

$$\begin{aligned}
\sum_{n,m=0}^{\infty} \frac{\left|\partial_{\hat{\rho}}^n \, \partial_\theta^m \, \widehat{U}_i(\widehat{\rho}, \theta)\right|}{n!m!} |z|^n |\zeta|^m &\leq C(\alpha) e^{-\alpha \hat{\rho}} \left(\frac{eK_2}{(1-\alpha)}\right)^i i^i \sum_{n,m=0}^{\infty} K_1^m K_3^n \frac{m^m e^{m+2n}}{n! m^m e^{-m}} |z|^n |\zeta|^m \\
&\leq C(\alpha) e^{-\alpha \hat{\rho}} \left(\frac{eK_2}{(1-\alpha)}\right)^i i^i \sum_{n,m=0}^{\infty} (e^2 K_1)^m (e^2 K_3)^n \frac{1}{n!} |z|^n |\zeta|^m \\
&= C(\alpha) e^{-\alpha \hat{\rho}} \left(\frac{eK_2}{(1-\alpha)}\right)^i i^i e^{e^2 K_3 |z|} \left(1 - K_1 e^2 |\zeta|\right)^{-1}.
\end{aligned}$$

This estimate shows that the functions $\widehat{U}_i$ are indeed entire in the first variable and holomorphic in the second variable provided that $|\zeta| < K_1^{-1} e^{-2}$. □

Lemma 2.5 *Let K_2, K_4, and K_5 be given by Corollary 2.4. For every $\alpha \in (0,1)$ there is $C_\alpha > 0$ depending only on α, the function f, the boundary data g, and the function κ (i.e., the geometry of the domain Ω) such that*

$$\begin{aligned} \left|\partial_\rho^p \partial_\theta^m u_M^{BL}(\rho,\theta)\right| &\leq C_\alpha S_M m!(2K_5)^m e^{K_4 p}\varepsilon^{-p}e^{-\alpha\rho/\varepsilon} \qquad p,m \in \mathbb{N}_0, \quad \rho \geq 0 \\ \sup_{\theta\in[0,L)} \|\partial_\rho^p \partial_\theta^m u_M^{BL}(\cdot,\theta)\|_{L^2(\rho,\infty)} &\leq C_\alpha S_M m!(2K_5)^m e^{K_4 p}\varepsilon^{1/2-p}e^{-\alpha\rho/\varepsilon} \qquad p,m \in \mathbb{N}_0, \quad \rho \geq 0 \end{aligned}$$

where S_M is given by

$$S_M = \sum_{i=0}^{2M+1}\left(\frac{\varepsilon e K_2(2M+1)}{1-\alpha}\right)^i.$$

Remark 2.6: Under the assumption $\varepsilon e K_2(2M+1)/(1-\alpha) \leq q_0 < 1$, we get the simplified bound

$$S_M \leq C(q_0)$$

where $C(q_0)$ is independent of M and ε. As we shall see shortly, under a similar assumption ($2M\varepsilon$ sufficiently small), the remainder r_M is small in ε. In the complementary case, i.e., $2\varepsilon M >> 1$, the asymptotic expansion loses its meaning; this is the reason why we prove estimates for high order derivatives of the solution u_ε of (1.1) separately in the next section without reference to asymptotic expansions.

Proof of Lemma 2.5: By Cauchy's integral theorem for derivatives we have for $R > 0$

$$\partial_\rho^p \partial_\theta^m \widehat{U}_i(\rho/\varepsilon,\theta) = -\varepsilon^{-p}\frac{p!m!}{4\pi^2}\int_{|z|=R}\int_{|\zeta|=1/(2K_5)} \frac{\widehat{U}_i(\widehat{\rho}+z,\theta+\zeta)}{(-z)^{p+1}(-\zeta)^{m+1}}\,dzd\zeta$$

On using the parametrization $z = R\cos t + iR\sin t$, $t \in [0,2\pi)$, Corollary 2.4 implies

$$\begin{aligned} \left|\partial_\rho^p \partial_\theta^m \widehat{U}_i(\widehat{\rho},\theta)\right| &\leq C_\alpha \varepsilon^{-p} m! p! R^{-p}(2K_5)^m e^{-\alpha\widehat{\rho}} e^{K_4 R}\left(\frac{eK_2}{1-\alpha}\right)^i i^i \\ &\leq C_\alpha \varepsilon^{-p} m!(2K_5)^m e^{-\alpha\widehat{\rho}}\left(\frac{eK_2}{1-\alpha}\right)^i i^i e^{K_4 p}\left(\frac{eK_2}{1-\alpha}\right)^i i^i. \end{aligned}$$

where we chose $R = p+1$ and used Stirling formula $p! \leq Cp^p e^{-p}\sqrt{2\pi(p+1)}$ in the last estimate. Therefore, we can conclude

$$\begin{aligned} |\partial_\rho^p \partial_\theta^m u_M^{BL}(\rho,\theta)| &\leq \sum_{i=0}^{2M+1} \varepsilon^{-p}\varepsilon^i |\partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{U}_i(\widehat{\rho},\theta)| \leq C(\alpha)\varepsilon^{-p}e^{-\alpha\widehat{\rho}} m!(2K_5)^m e^{K_4 p}\sum_{i=0}^{2M+1}\left(\frac{\varepsilon e K_2 i}{1-\alpha}\right)^i \\ &\leq C(\alpha)\varepsilon^{-p}e^{-\alpha\widehat{\rho}} m!(2K_5)^m e^{K_4 p}\sum_{i=0}^{2M+1}\left(\frac{\varepsilon e K_2(2M+1)}{1-\alpha}\right)^i \end{aligned}$$

which proves the first estimate. The second estimate follows immediately from the first one. $\square$

2.3 Controlling the Remainder

Let w_M and u_M^{BL} be the truncated outer and inner expansions defined in (2.1), (2.7), and let χ be the cut-off function defined in (1.7). Then then remainder r_M is defined by

$$u_\varepsilon = w_M + \chi u_M^{BL} + r_M.$$

(We should note that the boundary layer function u_M^{BL} and the cut-off function are defined in boundary fitted coordinates whereas w_M and u_ε are defined in the usual x, y coordinates so that, strictly speaking, the term χu_M^{BL} has to be understood as $(\chi u_M^{BL}) \circ \psi^{-1}$ on the tubular neighborhood Ω_0 where ψ is the boundary fitted coordinate transformation defined in (1.5) and χu_M^{BL} is understood to vanish outside Ω_0). The following theorem gives a bound in energy norm for the remainder r_M which depends explicitly on the perturbation parameter ε and the expansion order M.

Theorem 2.7 *For every $M \in \mathbb{N}_0$ the remainder r_M satisfies $r_M = 0$ on $\partial\Omega$, and there are constants C, $K > 0$ depending only on the right hand side f, the boundary data g, the function κ (i.e., the geometry of the domain), and the cut-off function χ such that*

$$\|r_M\|_{\varepsilon,\Omega} \leq C\left(\varepsilon K(2M+2)\right)^{2M+2}.$$

Proof: For any $M \in \mathbb{N}_0$ we the remainder r_M is defined as $r_M = u_\varepsilon - w_M - \chi u_M^{BL}$ where w_M is defined by (2.1), u_M^{BL} is defined by (2.7) and χ is the cut-off function of (1.7). Hence, by construction of u_M^{BL}, $r_M = 0$ on $\partial\Omega$. Furthermore, the remainder r_M solves the following elliptic equation:

$$\begin{aligned} L_\varepsilon r_M &= L_\varepsilon\left(u - w_M - \chi u_M^{BL}\right) = \varepsilon^{2M+2}\Delta^{(M+1)} f - L_\varepsilon \chi u_M^{BL} \\ &= \varepsilon^{2M+2}\Delta^{(M+1)} f + \varepsilon^2 \Delta\chi u_M^{BL} + 2\varepsilon\nabla\chi\cdot\nabla u_M^{BL} - \chi L_\varepsilon u_M^{BL}. \end{aligned}$$

Let us now estimate the L^2 norm of the right hand side. By the assumptions on f, cf. (1.9), we have

$$\|\varepsilon^{2M+2}\Delta^{(M+1)} f\|_{L^\infty(\Omega)} \leq C_f\left(\varepsilon\gamma_{\Delta f}(2M+2)\right)^{2M+2}.$$

Let us fix $\alpha \in (0,1)$ for the remainder of this proof. As $\chi \equiv 1$ for $0 < \rho < \rho_1$ and $\chi \equiv 0$ for $\rho > (\rho_1 + \rho_0)/2$, we obtain with the aid of Lemma 2.5

$$\begin{aligned} \varepsilon^2\|\Delta\chi u_M^{BL}\|_{L^2(\Omega)} &\leq C_\alpha \varepsilon^{5/2} S_M e^{-\alpha\rho_1/\varepsilon}, \\ \varepsilon\|\nabla\chi\cdot\nabla u_M^{BL}\|_{L^2(\Omega)} &\leq C_\alpha \varepsilon^{1/2} S_M e^{-\alpha\rho_1/\varepsilon} \end{aligned}$$

with S_M defined in Lemma 2.5. Finally, by Lemma A.6, we have

$$\|\chi L_\varepsilon u_M^{BL}\|_{L^2(\Omega)} \leq C\varepsilon^{1/2}\left(K\varepsilon(2M+2)\right)^{2M+2}$$

for some $K > 0$ independent of ε and M. Using the energy estimate $\varepsilon^2\|\nabla r_M\|^2_{L^2(\Omega)} + \|r_M\|^2_{L^2(\Omega)} \leq \|L_\varepsilon r_M\|^2_{L^2(\Omega)}$ (cf. (1.3)), we obtain the following estimate for r_M:

$$\|r_M\|_{\varepsilon,\Omega} \leq C\left\{\varepsilon^{2M+2}\|\Delta^{(M+1)} f\|_{L^\infty(\Omega)} + \varepsilon^{1/2} S_M e^{-\alpha\rho_1/\varepsilon} + \varepsilon^{1/2}\left(K\varepsilon(2M+2)\right)^{2M+2}\right\} \tag{2.17}$$

where $C > 0$ is independent of ε and M. As $\alpha \in (0,1)$ is fixed, we can bound

$$S_M \leq (2M+2) \max \left(1, (K'\varepsilon(2M+1))^{2M+1}\right)$$

for some appropriately large K' independent of M and ε. Using the bounds

$$\begin{aligned} \varepsilon^{-(2M+2)} e^{-\alpha\rho_1/\varepsilon} &\leq \left(\frac{2M+2}{\alpha\rho_1}\right)^{2M+2} e^{-(2M+2)} =: (K''(2M+2))^{2M+2}, \\ e^{-\alpha\rho_1/\varepsilon} &\leq \varepsilon(\alpha\rho_1 e)^{-1} \end{aligned}$$

we infer

$$S_M e^{-\alpha\rho_1/\varepsilon} \leq (2M+2) \max \left((K''(2M+2)\varepsilon)^{2M+2}, \varepsilon(\alpha\rho_1 e)^{-1}(K'\varepsilon(2M+1))^{2M+1}\right)$$

which allows us to complete the proof of Theorem 2.7. □

Remark 2.8: The proof of Theorem 2.7 shows that a slightly stronger statement holds. We have actually proved the existence of constants K, C independent of ε and M such that

$$\|r_M\|_{\varepsilon,\Omega} \leq C\left\{\varepsilon^{2M+2}\|\Delta^{(M+1)} f\|_{L^\infty(\Omega)} + \varepsilon^{1/2}\left(K\varepsilon(2M+2)\right)^{2M+2}\right\}.$$

Hence, if the right hand side f satisfies $\Delta^{(M+1)} f = 0$, e.g., if f is a polynomial of degree $2M+1$, then the ε-dependence of the estimate is actually improved by a factor $\varepsilon^{1/2}$.

Remark 2.9: In the proof of Theorem 2.7, with the exception of $\Delta^{(M+1)} f$, all the terms could be bounded in exponentially weighted spaces. This means that, if $\Delta^{(M+1)} f = 0$ then we have estimates of the form

$$\|e^{\beta d(x)/\varepsilon} L_\varepsilon r_M\|_{L^2(\Omega)} \leq C(\Omega,\beta)\varepsilon^{1/2}\left(K\varepsilon(2M+2)\right)^{2M+2}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and $\beta > 0$ appropriately. From this, one can infer estimates on r_M in exponentially weighted energy norms as the bilinear form B_ε in (1.2) can be seen to satisfy an inf-sup condition on pairs of exponentially weighted spaces (cf. Proposition A.1 for a one dimensional analog).

Remark 2.10: The proof of Theorem 2.7 shows that we have

$$\|L_\varepsilon r_M\|_{L^\infty(\Omega)} \leq C\left(K\varepsilon(2M+2)\right)^{2M+2}.$$

As $r_M = 0$ on $\partial\Omega$, the classical maximum principle gives us the pointwise bound

$$\|r_M\|_{L^\infty(\Omega)} \leq C\left(K\varepsilon(2M+2)\right)^{2M+2}.$$

As the boundary $\partial\Omega$ is smooth, we can actually use the shift theorem for $-\Delta$ in order to control higher derivatives of r_M.

Corollary 2.11 *Assume the same hypotheses as in Theorem 2.7. Then for each $k \in \mathbb{N}_0$ there are constants C_k, $K > 0$ depending only on k, f, g, χ, and κ (i.e., the geometry of Ω) such that*

$$\|r_M\|_{H^k(\Omega)} \leq C_k \varepsilon^{-k} \left(\varepsilon K(2M+2)\right)^{2M+2}, \qquad k \in \mathbb{N}_0.$$

Proof: The proof is an application of the classical shift theorem and an induction argument on k. We note that the corollary holds true for $k = 0$ and $k = 1$ by Theorem 2.7. Furthermore, r_M solves

$$\begin{aligned} -\Delta r_M &= \varepsilon^{-2} L_\varepsilon r_M - \varepsilon^{-2} r_M \qquad \text{in } \Omega, \\ r_M &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{2.18}$$

If we proceed as in the proof of Theorem 2.7 but use Lemma A.7 instead of Lemma A.6, we can estimate

$$\|L_\varepsilon u_M^{BL}\|_{H^{k-2}(\Omega)} \leq C_k \varepsilon^{2-k} \left(\varepsilon K(2M+2)\right)^{2M+2}, \qquad k \geq 2.$$

Hence the shift theorem allows us to conclude

$$\|r_M\|_{H^k(\Omega)} \leq C_k \left(\varepsilon^{-k} \left(\varepsilon K(2M+2)\right)^{2M+2} + \varepsilon^{-2} \|r_M\|_{H^{k-2}(\Omega)}\right)$$

for $k \geq 2$. The obvious induction argument concludes the proof. □

3 Growth Estimates for the Derivatives

Theorem 3.1 *Let u_ε be the solution of (1.1). Then there are C and $K > 0$ depending only on f, g, and the geometry of Ω (in particular, C, K are independent of ε) such that*

$$\|\nabla^p u_\varepsilon\|_{L^2(\Omega)} \leq C K^p \max\left(p, \varepsilon^{-1}\right)^p \left(1 + \|u_\varepsilon\|_{\varepsilon,\Omega}\right) \qquad \forall p \in \mathbb{N}_0. \tag{3.1}$$

Remark 3.2: The proof of Theorem 3.1 actually shows that a similar statement holds for the Helmholtz equation: If u_ε solves

$$\begin{aligned} \varepsilon^2 \Delta u_\varepsilon + u_\varepsilon &= f \qquad \text{on } \Omega \subset \mathbb{R}^2, \\ u_\varepsilon &= g \qquad \text{on } \partial\Omega \end{aligned} \tag{3.2}$$

then Theorem 3.1 still holds true.

Proof of Theorem 3.1: Let $B_R \subset \Omega$ be a ball of radius R and denote $B_{R/2}$ the ball of radius $R/2$ with the same center. Proposition 3.3 below yields

$$\|\nabla^p u_\varepsilon\|_{L^2(B_{R/2})} \leq C K^p \max\left(p, \varepsilon^{-1}\right)^p \left(1 + \|u_\varepsilon\|_{\varepsilon,R}\right) \qquad \forall p \in \mathbb{N}_0$$

where C, $K > 0$ are independent of ε and p. Let us now consider estimates at the boundary. First, we see that we may consider the case of homogeneous Dirichlet data: As the boundary data g is analytic, it can be extended analytically into Ω, e.g., by taking as the extended function G defined by

$$\begin{aligned} -\Delta G &= 0 \qquad \text{on } \Omega, \\ G &= g \qquad \text{on } \partial\Omega. \end{aligned}$$

As $\partial\Omega$ and g are assumed to be analytic, standard elliptic theory gives that G is analytic on $\overline{\Omega}$. Note that G is independent of ε. The auxiliary function $\tilde{u} = u - G$ solves

$$\begin{aligned} -\varepsilon^2\Delta\tilde{u} + \tilde{u} &= \tilde{f} := f + \varepsilon^2\Delta G - G = f - G \qquad \text{on } \Omega, \\ \tilde{u} &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

and by the triangle inequality we can bound

$$\|\nabla^p u\|_{L^2(B_R\cap\Omega)} \le \|\nabla^p\tilde{u}\|_{L^2(B_R\cap\Omega)} + \|\nabla^p G\|_{L^2(B_R\cap\Omega)} \qquad \forall p \in \mathbb{N}_0$$

for balls B_R. It suffices therefore to get the desired bounds for $\tilde{u}$.
In order to apply Proposition 3.12, we introduce a mapping to flatten the boundary locally: For $R > 0$ and a point $x_0 \in \partial\Omega$, we introduce the conformal map ζ which maps $\Omega\cap B_{2R}(x_0)$ conformally onto

$$G_{2R} = \{(x,y)\,|\,x^2+y^2 < 4R^2, \quad y > 0\}.$$

The transformed functions $\hat{u} = \tilde{u}\circ\zeta^{-1}$, $\hat{f} = \tilde{f}\circ\zeta^{-1}$ then solve

$$\begin{aligned} -\varepsilon^2\Delta\hat{u} + |(\zeta^{-1})'|^2\hat{u} &= \hat{f}|(\zeta^{-1})'|^2 \qquad \text{on } G_{2R}, \\ \hat{u}(x,0) &= 0, \qquad -2R < x < 2R. \end{aligned}$$

Furthermore, by the analyticity of $\partial\Omega$, the function $|(\zeta^{-1})'|^2$ is (real) analytic on $\overline{G}_R$ and hence Proposition 3.12 is applicable (note that $\tilde{f}$ and hence $\hat{f}|(\zeta^{-1})'|^2$ are independent of ε), and we get the desired estimate for $\hat{u}$, i.e.,

$$\|\nabla^p\hat{u}\|_{L^2(G_{R/2})} \le CK^p\max(p,\varepsilon^{-1})^p\,(1 + \|\hat{u}\|_{\varepsilon,G_R}) \qquad \forall p \in \mathbb{N}_0.$$

Applying Lemma 3.13 allows us to infer a similar estimate for $\tilde{u}$:

$$\|\nabla^p\tilde{u}\|_{L^2(B_{R'}\cap\Omega)} \le CK'^p\max(p,\varepsilon^{-1})^p\,(1 + \|u_\varepsilon\|_{\varepsilon,B_{2R}\cap\Omega}) \qquad \forall p \in \mathbb{N}_0$$

where $B_{R'}$ is a ball or radius $R' > 0$ with center x_0 such that $B_{R'}\cap\Omega \subset \zeta^{-1}(G_{R/2})$. The constants C, $K' > 0$ depend again on R, f, g, and the point x_0 but are independent of ε.
A compactness argument allows us to conclude the proof of the theorem. □

The remainder of this section is devoted to the proof of Propositions 3.3 and 3.12 as they are at the heart of Theorem 3.1. As all the estimates of Propositions 3.3 and 3.12 are intrinsically local, we introduce the short hand

$$\begin{aligned} \|v\|_R &:= \|v\|_{L^2(B_R)} \qquad \text{for balls } B_R \text{ of radius } R, \\ \|v\|^2_{\varepsilon,R} &:= \varepsilon^2\|\nabla v\|^2_R + \|v\|^2_R, \\ \|v\|_{G_R} &:= \|v\|_{L^2(G_R)} \qquad \text{for semi discs } G_R \text{ of radius } R, \\ \|v\|^2_{\varepsilon,G_R} &:= \varepsilon^2\|\nabla v\|^2_{G_R} + \|v\|^2_{G_R}. \end{aligned}$$

By standard theory ([8], Chap. 5.7), we know that the solution u_ε of (1.1) is analytic on $\overline{\Omega}$ and our aim is merely to assert that the derivatives of u_ε grow indeed in the fashion indicated in (3.1). In fact, we will follow the proof of [8].

3.1 Interior Estimates

In this subsection we consider the following problem

$$-\varepsilon^2\Delta u + b(x,y)u = f \qquad \text{on a ball } B_R \text{ of radius } R \tag{3.3}$$

where b, f are analytic and satisfy the estimates

$$\|\nabla^p f\|_{L^\infty(B_R)} \leq C_f\gamma^p p! \qquad \forall p\in\mathbb{N}_0, \tag{3.4}$$

$$\|\nabla^p b\|_{L^\infty(B_R)} \leq C_b B^p p! \qquad \forall p\in\mathbb{N}_0 \tag{3.5}$$

for some constants C_f, C_b, γ, and $B\geq 0$.

Proposition 3.3 *Assume that u satisfies (3.3) and that f and b satisfy (3.4), (3.5). Then for $K\geq 1$ satisfying (3.8) which is independent of ε we have the estimate*

$$N_{R,p}(u) \leq K^{p+2}\frac{\max(p,\varepsilon^{-1})^{p+2}}{[p]!}\left(\|u\|_{\varepsilon,R}+1\right) \qquad \forall p\geq -2 \tag{3.6}$$

where $N_{R,p}$ is defined in (3.7) below.

Remark 3.4: If $f=0$, then the estimate (3.6) can be strengthened to yield

$$N_{R,p}(u) \leq K^{p+2}\frac{\max(p,\varepsilon^{-1})^{p+2}}{[p]!}\|u\|_{\varepsilon,R}.$$

In order to prove Proposition 3.3, we need to introduce some notation.

$$\begin{aligned} M_{R,p}(v) &:= \frac{1}{p!}\sup_{R/2\leq r<R}(R-r)^{2+p}\|\nabla^p v\|_r, \qquad p\in\mathbb{N}_0,\\ N_{R,p}(v) &:= \frac{1}{[p]!}\sup_{R/2\leq r<R}(R-r)^{2+p}\|\nabla^{p+2} v\|_r, \qquad p\in\mathbb{N}_0\cup\{-2,-1\},\\ [p]! &= \max(1,p)!. \end{aligned} \tag{3.7}$$

From standard elliptic theory, we have the following a-priori estimate

Lemma 3.5 *Let $f\in L^2(B_R)$, $u\in H^1(B_R)\cap H^2(B_r)$ for all $r<R$. Let u satisfy $\Delta u = f$ on B_R. Then there is a generic $C>0$ (i.e., independent of R, r, δ, and f) such that*

$$\int_{B_r}|\nabla^2 u|^2\,dxdy \leq C\int_{B_{r+\delta}}\left(|f|^2+\delta^{-2}|\nabla u|^2+\delta^{-4}|u|^2\right)\,dxdy$$

if $0<r<r+\delta<R$ and $0<\delta\leq r$. Similarly, if f and u are only defined on a semi-disc G_R and if $u=0$ on the straight part, then

$$\int_{G_r}|\nabla^2 u|^2\,dxdy \leq C\int_{G_{r+\delta}}\left(|f|^2+\delta^{-2}|\nabla u|^2+\delta^{-4}|u|^2\right)\,dxdy.$$

Proof: The proof can be found in [8], Lemma 5.7.1. □

Lemma 3.6 *Let u solve $\Delta u = f$ on B_R. Then there is $C_1 > 0$ independent of u, R, and f such that*

$$N_{R,p}(u) \leq C_1 \left[M_{R,p}(f) + N_{R,p-1}(u) + N_{R,p-2}(u)\right] \qquad \forall p \in \mathbb{N}_0.$$

Proof: The proof is based on Lemma 3.5 and can be found in [8], Lemma 5.7.3. □

Lemma 3.7 *Let u, $v \in C^p$. Then*

$$|\nabla^p(uv)| \leq \sum_{q=0}^{p} \binom{p}{q} |\nabla^q u|\, |\nabla^{p-q} u|.$$

Proof: See [8], Lemma 5.7.4. □

Lemma 3.8 *Let b, u be analytic and assume that b satisfies (3.5). Then*

$$M_{R,p}(bu) \leq C_b \sum_{q=0}^{p} \left(B\frac{R}{2}\right)^{p-q} \left(\frac{R}{2}\right)^2 \frac{[q-2]!}{q!} N_{R,q-2}(u).$$

Proof: By Lemma 3.7 we have

$$\begin{aligned}
M_{R,p}(bu) &\leq \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^p(bu)\|_r \\
&\leq \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \sum_{q=0}^{p} \binom{p}{q} \||\nabla^q u|\,|\nabla^{p-q} b|\|_r \\
&\leq C_b \sum_{q=0}^{p} \binom{p}{q} \frac{(p-q)!}{p!} B^{p-q} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^q u\|_r \\
&\leq C_b \sum_{q=0}^{p} B^{p-q} \left(\frac{R}{2}\right)^{p-q+2} \frac{1}{q!} \sup_{R/2 \leq r < R} (R-r)^{(q-2)+2} \|\nabla^{(q-2)+2} u\|_r \\
&\leq C_b \sum_{q=0}^{p} \left(\frac{BR}{2}\right)^{p-q} \left(\frac{R}{2}\right)^2 \frac{[q-2]!}{q!} N_{R,q-2}(u).
\end{aligned}$$

□

Proof of Proposition 3.3: Let C_1 be the generic constant of Lemma 3.6 and choose $2K > \max(2, R, \gamma R, BR)$ such that

$$C_1 C_f \sqrt{\pi} \left(\frac{R}{2}\right)^3 K^{-2} \left(\frac{\gamma R}{2K}\right)^p + C_1 \left(\frac{C_b R/(2K)}{1 - BR/(2K)} + K^{-1} + K^{-2}\right) \leq 1 \tag{3.8}$$

for all $p \in \mathbb{N}_0$. We will proceed by induction on p. As $K \geq \max(1, R/2)$, the claim holds for $p = -2$ and $p = -1$. Let us therefore assume that (3.6) holds for all $-2 \leq p' < p$. As $-\Delta u = \varepsilon^{-2}(f - bu)$, we get for $p \in \mathbb{N}_0$ using Lemma 3.6 and Lemma 3.8

$$\begin{aligned} N_{R,p}(u) &\leq C_1 \left\{\varepsilon^{-2} M_{R,p}(f - bu) + N_{R,p-1}(u) + N_{R,p-2}(u)\right\} \\ &\leq C_1 \Big\{\varepsilon^{-2} M_{R,p}(f) + \varepsilon^{-2} C_b \sum_{q=0}^{p} \left(\frac{BR}{2}\right)^{p-q} \left(\frac{R}{2}\right)^2 \frac{[q-2]!}{q!} N_{R,q-2}(u) + \\ &\qquad + N_{R,p-1}(u) + N_{R,p-2}(u)\Big\}. \end{aligned}$$

From the induction hypothesis (3.6) we obtain

$$\begin{aligned} N_{R,p}(u) &\leq C_1 \varepsilon^{-2} M_{R,p}(f) + C_1(\|u\|_{\varepsilon,R} + 1)\Big\{C_b \sum_{q=0}^{p} \left(\frac{BR}{2}\right)^{p-q} \left(\frac{R}{2}\right)^2 \varepsilon^{-2} K^q \frac{\max(q-2, \varepsilon^{-1})^q}{q!} + \\ &\qquad + K^{p+1} \frac{\max(p-1, \varepsilon^{-1})^{p+1}}{[p-1]!} + K^p \frac{\max(p-2, \varepsilon^{-1})^p}{[p-2]!}\Big\}. \end{aligned}$$

As we have the estimates

$$\begin{aligned} \varepsilon^{-2} \frac{1}{p!} \frac{p!}{q!} \max(q-2, \varepsilon^{-1})^q &\leq \frac{1}{p!} \max(p, \varepsilon^{-1})^{p+2}, \\ \frac{1}{p!} \frac{p!}{[p-1]!} \max(p-1, \varepsilon^{-1})^{p+1} &\leq \frac{1}{p!} \max(p, \varepsilon^{-1})^{p+2}, \\ \frac{1}{p!} \frac{p!}{[p-2]!} \max(p-2, \varepsilon^{-1})^p &\leq \frac{1}{p!} \max(p, \varepsilon^{-1})^{p+2}, \end{aligned}$$

we obtain

$$\begin{aligned} N_{R,p}(u) &\leq C_1 \varepsilon^{-2} M_{R,p}(f) + \frac{\max(p, \varepsilon^{-1})^{p+2}}{p!} K^{p+2}(\|u\|_{\varepsilon,R} + 1) \times \\ &\qquad \times C_1 \Big\{C_b \sum_{q=0}^{p} \left(\frac{BR}{2}\right)^{p-q} \left(\frac{R}{2}\right)^2 K^{q-p-2} + K^{-1} + K^{-2}\Big\} \\ &\leq C_1 \varepsilon^{-2} M_{R,p}(f) + \frac{\max(p, \varepsilon^{-1})^{p+2}}{p!} K^{p+2}(\|u\|_{\varepsilon,R} + 1) \times \\ &\qquad \times C_1 \Big\{C_b \frac{C_b\,(R/(2K))^2}{1 - BR/(2K)} + K^{-1} + K^{-2}\Big\}. \end{aligned}$$

Finally, as we have

$$M_{R,p}(f) \leq C_f \gamma^p \sqrt{\pi} \left(\frac{R}{2}\right)^{3+p},$$

we get

$$\begin{aligned} N_{R,p}(u) &\leq K^{p+2} \frac{\max(p, \varepsilon^{-1})^{p+2}}{p!} (\|u\|_{\varepsilon,R} + 1) \times \\ &\qquad \times \left[C_1 C_f \sqrt{\pi} \left(\frac{R}{2}\right)^3 K^{-2} \left(\frac{\gamma R}{2K}\right)^p + C_1 \Big\{C_b \frac{C_b\,(R/(2K))^2}{1 - BR/(2K)} + K^{-1} + K^{-2}\Big\}\right]. \end{aligned}$$

The fact that the bracketed expression is bounded by one by the choice of K in(3.8) concludes the induction argument. □

3.2 Estimates on Straight Parts of the Boundary

The strategy to get estimates near the boundary is to control first the derivatives in the tangential direction and then in a second step control the normal derivatives. We consider the problem

$$\begin{aligned} -\varepsilon^2 \Delta u + bu &= f \qquad \text{on } G_R, \\ u &= 0 \qquad \text{on } \Gamma_R \end{aligned} \tag{3.9}$$

where G_R is a semi-disc of radius R and Γ_R is the straight part of ∂G_R. Without loss of generality, we may assume that

$$G_R = \{(x,y) \,|\, x^2 + y^2 < R, \quad y > 0\}.$$

We assume that the functions f, b are analytic on G_R and satisfy the estimates

$$\|\nabla^p f\|_{L^\infty(G_R)} \leq C_f \gamma^p p! \qquad \forall p \in \mathbb{N}_0, \tag{3.10}$$

$$\|\nabla^p b\|_{L^\infty(G_R)} \leq C_b B^p p! \qquad \forall p \in \mathbb{N}_0. \tag{3.11}$$

Additionally, we introduce the notation

$$\begin{aligned} M'_{R,p}(v) &= \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\partial_x^p\, v\|_{G_r}, \\ N'_{R,p}(v) &= \begin{cases} \dfrac{1}{p!} \sup\limits_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^2 \partial_x^p\, v\|_{G_r} & \text{if } p \geq 0 \\ \sup\limits_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^{2+p} v\|_{G_r} & \text{if } p = -2, -1, \end{cases} \\ N'_{R,p,q}(v) &= \frac{1}{[p+q]!} \sup_{R/2 \leq r < R} (R-r)^{p+q+2} \|\partial_y^{q+2}\, \partial_x^p\, v\|_{G_r}, \qquad p \geq 0, q \geq -2, \\ \tilde{M}_{R,p}(v) &= \frac{1}{p!} \sup_{R/2 \leq r < R} (R-r)^{p+2} \|\nabla^p v\|_{G_r}. \end{aligned} \tag{3.12}$$

Note that we have $N'_{R,p,0} \leq N'_{R,p}$. We have

Lemma 3.9 *Let $u \in H^1(G_R)$ solve $\Delta u = f$ on G_R and assume that $u = 0$ on the straight part of G_R. Then there is a generic constant $C_2 > 0$ such that*

$$N'_{R,p}(u) \leq C_2 \left\{ M'_{R,p}(f) + N'_{R,p-1}(u) + N'_{R,p-2}(u) \right\}.$$

Proof: The proof follows from Lemma 3.5 and can be found in [8], Lemma 5.7.3′. □

Lemma 3.10 *Let u, $v \in C^{p+q}$. Then*

$$|\partial_y^p\, \partial_x^q\, (uv)| \leq \sum_{m=0}^{p} \sum_{n=0}^{q} \binom{p}{m} \binom{q}{n} |\partial_y^m\, \partial_x^n\, v|\, |\partial_y^{p-m}\, \partial_x^{q-n}\, u|.$$

Proof: Can be found in [8], Lemma 5.7.4′. □

Proposition 3.11 *Let $u \in H^1(G_R)$ solve (3.9) on G_R and assume that f and b satisfy (3.10), (3.11). Then there is $K_6 > 0$ independent of ε such that*

$$N'_{R,p}(u) \leq K_6^{p+2} \frac{\max(p, \varepsilon^{-1})^{p+2}}{[p]!} (1 + \|u\|_{\varepsilon,G_R}).$$

Proof: The proof is almost verbatim the same as the proof of Proposition 3.3. Instead of using Lemma 3.6 we make use of Lemma 3.9. In particular, the constant K_6 will be chosen such that $K_6 > \max(1, \gamma R/2, BR/2, \gamma R/2)$. □

Proposition 3.12 *Under the same hypotheses as in Proposition 3.11 there is $K_7 > 0$ given by (3.13) independent of ε such that*

$$N'_{R,p,q}(u) \leq K_6^{p+2} K_7^{q+2} \frac{\max(p+q, \varepsilon^{-1})^{p+q+2}}{[p]!} (1 + \|u\|_{\varepsilon,G_R}) \qquad p \geq 0, q \geq -2$$

where $K_6 \geq 1$ is the constant of Proposition 3.11 and $N'_{R,p,q}$ is defined in (3.12).

Proof: Choose $2K_7 > \max(2, BR, \gamma R, R)$ such that for all $p \geq 0$, $q \geq 0$

$$\left[C_f \sqrt{\pi/2} \left(\frac{R}{2}\right)^3 \left(\frac{\gamma R}{2K_6}\right)^p \left(\frac{\gamma R}{2K_7}\right)^q K_6^{-2} K_7^{-2} + K_6^2 K_7^{-2} + \frac{C_b (R/(2K_7))^2}{(1 - BR/(2K_6))(1 - BR/(2K_7))} \right] \leq 1. \tag{3.13}$$

We will proceed by induction on q. By Proposition 3.11 and our earlier observation that $N'_{R,p,0} \leq N'_{R,p}$, the claim is true for $q = 0$ and all $p \geq 0$, and it is easy to see that the claim is also true for $q = -2$, $q = -1$: We have for $q = -2$ that the claim holds for $p = 0$, $p = 1$. For $p \geq 2$, we have

$$\begin{aligned} N'_{R,p,-2}(u) &= \frac{1}{(p-2)!} \sup_{R/2 \leq r < R} (R-r)^p \|\partial_x^p u\|_{G_r} \leq \frac{1}{(p-2)!} \sup_{R/2 \leq r < R} (R-r)^{(p-2)+2} \|\nabla^2 \partial_x^{p-2} u\|_{G_r} \\ &\leq \left(\frac{R}{2}\right)^2 N'_{R,p-2}(u) \end{aligned}$$

which concludes the case $q = -2$. Similarly, for $q = -1$, we observe that the claim is true for $p = 0$, and obtain for $p \geq 1$

$$\begin{aligned} N'_{R,p,-1}(u) &= \frac{1}{(p-1)!} \sup_{R/2 \leq r < R} (R-r)^{p+1} \|\partial_y \partial_x^p u\|_{G_r} \leq \frac{1}{(p-1)!} \sup_{R/2 \leq r < R} (R-r)^{p+1} \|\nabla^2 \partial_x^{p-1} u\|_{G_r} \\ &\leq \left(\frac{R}{2}\right)^2 N'_{R,p-1}(u). \end{aligned}$$

Let us now proceed with the proof of the proposition. Let us assume that the induction hypothesis is proven for $-2 \leq q' < q$. We have

$$\begin{aligned} -\partial_y^2 u &= \partial_x^2 u + \varepsilon^{-2} (f - bu), \\ |\partial_x^p \partial_y^{q+2} u| &\leq |\partial_x^{p+2} \partial_y^q u| + \varepsilon^{-2} |\partial_x^p \partial_y^q f| + \varepsilon^{-2} |\partial_x^p \partial_y^q (bu)|. \end{aligned}$$

By Lemma 3.10, we obtain

$$\begin{aligned} |\partial_x^p\,\partial_y^q\,(bu)| &\leq \sum_{m=0}^{q}\sum_{n=0}^{p}\binom{p}{n}\binom{q}{m}|\partial_x^{p-n}\,\partial_y^{q-m}\,b|\,|\partial_x^n\,\partial_y^m\,u| \\ &\leq C_b\sum_{m=0}^{q}\sum_{n=0}^{p}\binom{p}{n}\binom{q}{m}B^{p+q-m-n}(p+q-m-n)!|\partial_x^n\,\partial_y^m\,u|. \end{aligned}$$

Hence, we obtain for $N'_{R,p,q}(u)$

$$\begin{aligned} N'_{R,p,q}(u) &= \frac{1}{(p+q)!}\sup_{R/2\leq r<R}(R-r)^{p+q+2}\|\partial_x^p\,\partial_y^{q+2}\,u\|_{G_r} \\ &\leq N'_{R,p+2,q-2}(u)+\varepsilon^{-2}\tilde{M}_{R,p+q}(f)+ \\ &+C_b\varepsilon^{-2}\sum_{m=0}^{q}\sum_{n=0}^{p}\binom{p}{n}\binom{q}{m}B^{p+q-m-n}\left(\frac{R}{2}\right)^{p+q-m-n+2}\frac{(p+q-m-n)![m-2+n]!}{(p+q)!}N'_{R,n,m-2}(u). \end{aligned}$$

By the induction hypothesis

$$[m-2+n]!N_{R,n,m-2}(u)\leq K_6^{n+2}K_7^m\max\,(m-2+n,\varepsilon^{-1})^{m+n}(1+\|u\|_{\varepsilon,G_R}),$$

and applying Lemma B.8 twice, we get

$$\begin{aligned} N'_{R,p,q}(u) &\leq K_6^{p+4}K_7^q\frac{\max\,(p+q,\varepsilon^{-1})^{p+q+2}}{(p+q)!}(1+\|u\|_{\varepsilon,G_R})+\varepsilon^{-2}\tilde{M}_{R,p+q}(f)+ \\ &+C_b(1+\|u\|_{\varepsilon,G_R})\varepsilon^{-2}\sum_{m=0}^{q}\sum_{n=0}^{p}B^{p+q-m-n}\left(\frac{R}{2}\right)^{p+q-m-n+2}K_6^{n+2}K_7^m\frac{\max\,(m-2+n,\varepsilon^{-1})^{m+n}}{(p+q)!} \\ &\leq K_6^{p+4}K_7^q\frac{\max\,(p+q,\varepsilon^{-1})^{p+q+2}}{(p+q)!}(1+\|u\|_{\varepsilon,G_R})+\varepsilon^{-2}\tilde{M}_{R,p+q}(f)+ \\ &+C_b(1+\|u\|_{\varepsilon,G_R})K_6^{p+2}K_7^q\left(\frac{R}{2}\right)^2\sum_{m=0}^{q}\sum_{n=0}^{p}\left(\frac{BR}{2}\right)^{p+q-m-n}K_6^{n-p}K_7^{m-q}\frac{\max\,(p+q,\varepsilon^{-1})^{p+q+2}}{(p+q)!} \\ &\leq \varepsilon^{-2}\tilde{M}_{R,p+q}(f)+K_6^{p+2}K_7^{q+2}\frac{\max\,(p+q,\varepsilon^{-1})^{p+q+2}}{(p+q)!}(1+\|u\|_{\varepsilon,G_R})\times \\ &\times\left[K_6^2K_7^{-2}+\frac{C_b(R/(2K_7))^2}{(1-BR/(2K_6))(1-BR/(2K_7))}\right]. \end{aligned}$$

Finally, as

$$\tilde{M}_{R,p+q}(f)\leq C_f\sqrt{\pi/2}\gamma^{p+q}\left(\frac{R}{2}\right)^{p+q+3},$$

we conclude

$$\begin{aligned} N'_{R,p,q}(u) &\leq K_6^{p+2}K_7^{q+2}\frac{\max\,(p+q,\varepsilon^{-1})^{p+q+2}}{(p+q)!}(1+\|u\|_{\varepsilon,G_R})\times \\ &\times\left[C_f\sqrt{\pi/2}\left(\frac{R}{2}\right)^3\left(\frac{\gamma R}{2K_6}\right)^p\left(\frac{\gamma R}{2K_7}\right)^q K_6^{-2}K_7^{-2}+K_6^2K_7^{-2}+\frac{C_b(R/(2K_7))^2}{(1-BR/(2K_6))(1-BR/(2K_7))}\right]. \end{aligned}$$

As the bracketed expression is bounded by one by the choice of K_7 in (3.13), the induction argument is completed. □

Lemma 3.13 *Let G, $G_1 \subset \mathbb{R}^2$ be bounded open sets. Assume that $g = (g_1, g_2) : \overline{G_1} \to \mathbb{R}^2$ is analytic and injective on $\overline{G}_1$, $\det g' \neq 0$ on $\overline{G}_1$, and satisfies $g(G_1) \subset G$. Let $f : \overline{G} \to \mathbb{C}$ be analytic on $\overline{G}$ and assume that it satisfies for some ε, C_f, $\gamma > 0$*

$$\|\nabla^p f\|_{L^2(G)} \leq C_f \gamma^p \max(p, \varepsilon^{-1})^p \qquad \forall p \in \mathbb{N}_0.$$

Then there are C, $K > 0$ depending only on C_f, γ, and the map g such that

$$\|\nabla^p (f \circ g)\|_{L^2(G_1)} \leq C K^p \max(p, \varepsilon^{-1})^p \qquad \forall p \in \mathbb{N}_0.$$

Proof: The growth conditions on the derivatives of f imply that f can be extended to a holomorphic function (also denoted f) on $\tilde{G} \subset \mathbb{C} \times \mathbb{C}$ with $\overline{G} \subset \tilde{G}$ and $\tilde{G}$ independent of $\varepsilon > 0$. First, we claim that there are δ_0, γ', and $C > 0$ depending only on γ and C_f such that

$$\|f(\cdot + z_1(\cdot), \cdot + z_2(\cdot))\|_{L^2(G)} \leq C e^{\gamma' \delta/\varepsilon} \tag{3.14}$$

for all continous functions z_1, $z_2 : G \to \mathbb{C}$ with $\|z_i\|_{L^\infty(G)} \leq \delta \leq \delta_0$, $i = 1, 2$. As f is holomorphic on $\tilde{G}$, there is $\delta_0 > 0$ such that for all $(x, y) \in \overline{G}$ the power series expansion of f about (x, y) converges on a ball of radius $2\delta_0$. For functions z_1, z_2 with $\|z_i\|_{L^\infty(G)} \leq \delta \leq \delta_0$ we obtain:

$$|f(x + z_1(x, y), y + z_2(x, y))| = \left| \sum_{\alpha \in \mathbb{N}_0^2} \frac{1}{\alpha!} D^\alpha f(x, y)(z_1, z_2)^\alpha \right| \leq \sum_{\alpha \in \mathbb{N}_0^2} \frac{1}{\alpha!} |D^\alpha f(x, y)| \, \delta^{|\alpha|}.$$

Therefore we get

$$\begin{aligned}
\|f(\cdot + z_1(\cdot), \cdot + z_2(\cdot))\|_{L^2(G)} &\leq \sum_{\alpha \in \mathbb{N}_0^2} \frac{1}{\alpha!} \|D^\alpha f\|_{L^2(G)} \delta^{|\alpha|} \\
&\leq \sum_{p=0}^{\infty} \sum_{|\alpha|=p} \left((p!)^{1/2} (\alpha!)^{-1/2} \|D^\alpha f\|_{L^2(G)}\right) \left((\alpha!)^{-1/2} p!^{-1/2} \delta^p\right) \\
&\leq \sum_{p=0}^{\infty} \|\nabla^p f\|_{L^2(G)} \left(\sum_{|\alpha|=p} \frac{1}{\alpha! p!} \delta^{2p} \right)^{1/2} = \sum_{p=0}^{\infty} \|\nabla^p f\|_{L^2(G)} \frac{1}{p!} 2^{p/2} \delta^p \\
&\leq C_f \sum_{0 \leq p \leq \varepsilon^{-1}} \frac{1}{p!} \left(\sqrt{2} \gamma \varepsilon^{-1} \delta\right)^p + C_f \sum_{p > \varepsilon^{-1}} \frac{p^p}{p!} \gamma^p 2^{p/2} \delta^p \\
&\leq C_f e^{\sqrt{2}\gamma\delta/\varepsilon} + C \sum_{p > \varepsilon^{-1}} \left(e \sqrt{2} \gamma \delta\right)^p \leq C_f e^{\sqrt{2}\gamma\delta/\varepsilon} + \frac{1}{1 - \sqrt{2}\gamma\delta_0} \leq C e^{\sqrt{2}\gamma\delta/\varepsilon}
\end{aligned}$$

where we used Stirling's formula in the form $p! \geq C p^p e^{-p}$ and made the tacit assumption that δ_0 is so small that $e\sqrt{2}\gamma\delta_0 < 1$ so that the second sum is finite. This proves (3.14).
As g is analytic on $\overline{G}_1$ there is a holomorphic extension (also denoted g) to $\tilde{G}_1 \subset \mathbb{C} \times \mathbb{C}$. Thus, there are η, $\delta_0' > 0$ such that for all $(x, y) \in \overline{G}_1$

$$|g_i(x + z_1, y + z_2) - g_i(x, y)| \leq \eta\delta, \qquad i = 1, 2, \quad z_1, z_2 \in \mathbb{C} \text{ with } |z_1|, |z_2| \leq \delta \leq \delta_0'. \tag{3.15}$$

For any $0 < \delta \leq \min(\delta_0', \delta_0/\eta)$ we obtain by Cauchy's integral theorem for derivatives for every $(x,y) \in G_1$ and every $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2$

$$\begin{aligned} D^\alpha (f \circ g)(x,y) &= -\frac{\alpha!}{4\pi^2} \int_{|z_1|=\delta} \int_{|z_2|=\delta} \frac{(f\circ g)(x+z_1, y+z_2)}{(-z_1)^{\alpha_1+1}(-z_2)^{\alpha_2+1}} \, dz_1 dz_2 \\ |D^\alpha (f \circ g)(x,y)|^2 &\leq \frac{\alpha!^2}{4\pi^2 \delta^{2|\alpha|+2}} \int_{|z_1|=\delta} \int_{|z_2|=\delta} \left| f\Big(g_1(x+z_1, y+z_2), g_2(x+z_1, y+z_2)\Big)\right|^2 \, |dz_1|\,|dz_2|. \end{aligned}$$

By (3.15), we can write

$$g_1(x+z_1, y+z_2) = g_1(x,y) + \zeta_1, \qquad g_2(x+z_1, y+z_2) = g_2(x,y) + \zeta_2$$

where ζ_1, ζ_2 are smooth functions of x, y, z_1, z_2, and $|\zeta_i| \leq \eta\delta$, $i = 1, 2$. Integrating over G_1, we obtain after the smooth change of variables $g(x,y) = (x', y')$ (note that $0 < c_1 \leq |\det g'| \leq c_2 < \infty$) and denoting ζ_1', ζ_2' the functions corresponding to ζ_1, ζ_2 after the change of variables

$$|D^\alpha (f\circ g)(x,y)|^2_{L^2(G_1)} \leq c_2 \frac{(\alpha!)^2}{4\pi^2\delta^{2|\alpha|+2}} \int_{|z_1|=\delta} \int_{|z_2|=\delta} \int_G |f(x'+\zeta_1', y'+\zeta_2')|^2 \, dx'dy'|dz_1|\,|dz_2|.$$

As $|\zeta_1'|$, $|\zeta_2'| \leq \eta\delta$ uniformly in $(x', y') \in G$, $|z_1|$, $|z_2| \leq \delta$, estimate (3.14) yields

$$\|D^\alpha (f \circ g)\|_{L^2(G_1)} \leq C \frac{\alpha!}{\delta^{|\alpha|}} e^{\gamma'\eta\delta/\varepsilon} \qquad \forall 0 < \delta \leq \min(\delta_0', \delta_0/\eta).$$

In order to extract from this estimate the claim of the lemma, we distinguish the cases $|\alpha|\varepsilon$ large and $|\alpha|\varepsilon$ small. If $|\alpha|\varepsilon/(\eta\gamma') < \min(\delta_0', \delta_0/\eta)$, choose $\delta := |\alpha|\varepsilon/(\eta\gamma')$ to get

$$\|D^\alpha (f\circ g)\|_{L^2(G_1)} \leq C(e\eta\gamma')^{|\alpha|}\varepsilon^{-|\alpha|}.$$

If $|\alpha|\varepsilon/(\eta\gamma') \geq \min(\delta_0', \delta_0/\eta)$, choose $\delta := \min(\delta_0', \delta_0/\eta)$ and observe that $\varepsilon^{-1} \leq |\alpha|/(\eta\gamma'\delta)$ to arrive at

$$\|D^\alpha (f\circ g)\|_{L^2(G_1)} \leq C\alpha! \; \delta^{-|\alpha|} e^{|\alpha|}$$

which completes the proof. □

A Derivative Estimates for the Inner Expansion

A.1 Preliminaries

In this Appendix we want to analyze the growth of the derivatives of the functions generated by the inner expansion. In order to do so, we need to consider first the following simple one dimensional boundary value problem.

$$\begin{aligned} -u'' + u &= f \qquad \text{on } (0,\infty), \\ u(0) &= g \in \mathbb{R}, \\ u &\to 0 \;\; \text{as } x \to \infty. \end{aligned} \tag{A.1}$$

Proposition A.1 *Let $0 \le \alpha < 1$ and $f \in H^0_\alpha$. Then there is a (unique) solution u of (A.1) which satisfies*

$$\|u\|_{1,\alpha} \le \frac{3}{1-\alpha}\left[\|f\|_{0,\alpha} + |g|\right]. \tag{A.2}$$

Proof: Let us first observe that the function ge^{-x} solves the homogeneous equation and satisfies the Dirichlet boundary condition at $x = 0$. As

$$\|ge^{-x}\|_{1,\alpha} \le |g|\frac{1}{\sqrt{1-\alpha}}$$

we may therefore restrict out attention to solving (A.1) with homogeneous Dirichlet data, i.e., $g = 0$. In order to do so, we introduce the Hilbert spaces

$$V_\alpha = \{u \in H^1_\alpha \,|\, u(0) = 0\}, \qquad V_{-\alpha} = \{u \in H^1_{-\alpha} \,|\, u(0) = 0\},$$

and define on $V_\alpha \times V_{-\alpha}$ the bilinear form

$$B(u,v) = \int_0^\infty (u'v' + uv)\, dx.$$

By Schwarz's inequality, this bilinear form is well-defined, and in fact we have

$$|B(u,v)| \le \|u\|_{1,\alpha}\|v\|_{1,-\alpha} \qquad \forall u \in V_\alpha, v \in V_{-\alpha}.$$

We claim now that B satisfies the inf-sup condition. Given $u \in V_\alpha$ choose $v(x) = e^{2\alpha x}u(x)$. We have $v'(x) = e^{2\alpha x}(2\alpha u + u')$ and therefore

$$\begin{aligned} \|v\|^2_{1,-\alpha} &= \int_0^\infty e^{-2\alpha x}\left(v'^2 + v^2\right) dx = \int_0^\infty e^{2\alpha x}\left((2\alpha u + u')^2 + u^2\right) dx \\ &\le \int_0^\infty e^{2\alpha x}\left((8\alpha^2+1)u'^2 + (4\alpha^2 + 3/2)u^2\right) dx \le 9\|u\|^2_{1,\alpha}. \end{aligned}$$

On the other hand,

$$\begin{aligned} B(u,v) &= \int_0^\infty e^{2\alpha x}\left((2\alpha u + u')u' + u^2\right) dx = \|u\|^2_{1,\alpha} + \int_0^\infty e^{2\alpha x} 2\alpha u' u \, dx \\ &\ge \|u\|^2_{1,\alpha} - \alpha \int_0^\infty e^{2\alpha x}\left[u'^2 + u^2\right] dx \\ &= (1-\alpha)\|u\|^2_{1,\alpha} \ge \frac{1-\alpha}{3}\|u\|_{1,\alpha}\|v\|_{1,-\alpha}. \end{aligned}$$

Furthermore, it is easily seen that for any $0 \ne v \in V_{-\alpha}$ we have $\sup_{u \in V_\alpha} B(u,v) \ne 0$. Therefore, the problem

$$\text{find } u \in V_\alpha \text{ such that } B(u,v) = \int_0^\infty f(x)v(x)\, dx \qquad \forall v \in V_{-\alpha}$$

has a unique solution u which satisfies

$$\|u\|_{1,\alpha} \le \frac{3}{1-\alpha}\|f\|_{0,\alpha}$$

as desired. □

A.2 Controlling the θ derivatives of the inner expansion: proof of Proposition 2.2

Proof of Proposition 2.2: Let us first see that Proposition 2.2 is true for $i=0$ and $i=1$. As $\widehat{F}_0=0$, we have $\widehat{U}_0=G_0(\theta)e^{-\widehat{\rho}}$. We have $\widehat{F}_1=\widehat{U}_0$ and therefore, $\widehat{U}_1=\frac{1}{2}\widehat{\rho}G_0(\theta)e^{-\widehat{\rho}}+G_1(\theta)e^{-\widehat{\rho}}$. $\widehat{U}_0$ and $\widehat{U}_1$ are analytic in both $\widehat{\rho}$ and θ. We will now proceed by induction on i for all $m\in\mathbb{N}_0$ simultaneously. The formulas for $\widehat{U}_0$, $\widehat{U}_1$ show that (2.11) is true for $i=0$ and $i=1$. Let us therefore consider the case $i\geq 2$ and assume that (2.11) holds true for all $m\in\mathbb{N}_0$ and all $j\leq i-1$.

As all the $\widehat{U}_i$ are analytic in θ, we may differentiate the differential equation with respect to θ. For each $m\in\mathbb{N}_0$ we therefore get that $\partial_\theta^m\widehat{U}_i$ satisfies

$$\begin{aligned}
-\partial_{\widehat{\rho}}^2\partial_\theta^m\widehat{U}_i+\partial_\theta^m\widehat{U}_i &= \partial_\theta^m\widehat{F}_i \qquad \text{on } (0,\infty),\\
\partial_\theta^m\widehat{U}_i(0,\theta) &= \partial_\theta^m G_i(\theta),\\
\partial_\theta^m\widehat{U}_i(\widehat{\rho},\theta) &\to 0 \qquad \text{as } \widehat{\rho}\to\infty.
\end{aligned} \tag{A.3}$$

Proposition A.1 gives the a-priori estimate

$$\begin{aligned}
\|\partial_\theta^m\widehat{U}_i\|_{1,\alpha,\infty} &\leq \frac{3}{1-\alpha}\left[\|\partial_\theta^m\widehat{F}_i\|_{0,\alpha,\infty}+\|D^mG_i\|_{L^\infty([0,L))}\right]\\
&\leq \frac{3}{1-\alpha}\left[\sum_{l=1}^{3}\|\partial_\theta^m\widehat{F}_i^l\|_{0,\alpha,\infty}+\|D^mG_i\|_{L^\infty([0,L))}\right].
\end{aligned} \tag{A.4}$$

Let us estimate each of the four terms on the right hand side separately. Let us first deal with $\partial_\theta^m\widehat{F}_i^1$. We have

$$\partial_\theta^m\widehat{F}_i^1=\sum_{j=0}^{i-1}\sum_{\mu=0}^{m}\binom{m}{\mu}D^\mu a_1^j\,\widehat{\rho}^j\partial_\theta^{m-\mu}\partial_{\widehat{\rho}}\widehat{U}_{i-1-j}.$$

By Lemma B.1, we have for each fixed θ and j

$$\begin{aligned}
&\|\widehat{\rho}^j\,\partial_\theta^{m-\mu}\partial_{\widehat{\rho}}\widehat{U}_{i-1-j}\|_{0,\alpha}\leq\delta^{-j}j^je^{-j}\|\partial_\theta^{m-\mu}\widehat{U}_{i-1-j}\|_{1,\alpha+\delta}\\
&\quad\leq\delta^{-j}j^je^{-j}C_UK_1^{m-\mu}K_2^{i-1-j}(m-\mu+i-1-j)^{m-\mu+i-1-j}(1-\alpha-\delta)^{-(i-1-j+1)}
\end{aligned}$$

where used the induction hypothesis. Lemma B.2 suggests now the choice $\delta=(1-\alpha)j/i$. We obtain

$$\begin{aligned}
\|\widehat{\rho}^j\,\partial_\theta^{m-\mu}\partial_{\widehat{\rho}}\widehat{U}_{i-1-j}\|_{0,\alpha,\infty} &\leq C_U\frac{K_1^{m-\mu}K_2^{i-1-j}e^{-j}(m-\mu+i-1-j)^{m-\mu+i-1-j}}{(1-\alpha)^i(i-j)^{i-j}}i^i\\
&\leq C_U\frac{K_1^{m-\mu}K_2^{i-1-j}e^{-j}(m-\mu+i-1-j+1)^{m-\mu+i-1-j}}{(1-\alpha)^i(i-j)^{i-j}}i^i
\end{aligned}$$

where the second estimate was justified by the fact that $m-\mu+i-1-j\geq 0$. This estimate together with the assumptions (2.8) implies

$$\begin{aligned}
&\|\partial_\theta^m F_i^1\|_{0,\alpha,\infty}\leq C_UC_A\frac{K_1^mK_2^i}{(1-\alpha)^i}K_2^{-1}\times\\
&\quad\times\sum_{j=0}^{i-1}\sum_{\mu=0}^{m}\binom{m}{\mu}\mu!\,A^\mu\tilde{A}^jK_1^{-\mu}K_2^{-j}e^{-j}(m-\mu+i-1-j+1)^{m-\mu+i-1-j}\frac{i^i}{(i-j)^{i-j}}.
\end{aligned}$$

Lemma B.6 with the particular choice $b = 1$, $a = i - 1 - j$ is applicable, and we obtain

$$\|\partial_\theta^m \widehat{F}_i^1\|_{0,\alpha,\infty} \leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^i} K_2^{-1} \times$$
$$\times \sum_{j=0}^{i-1} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m + i - 1 - j + 1)^{m+i-j-1} \frac{i^i}{(i-j)^{i-j}}.$$

Now, Lemma B.7 is applicable with $p = 0$ (implying $\nu = 0$), $a = m - 1$, and $b = 1$, i.e., $a \geq -1$, and $a + b = m \geq 0$ and the expression simplifies considerably:

$$\begin{aligned}\|\partial_\theta^m \widehat{F}_i^1\|_{0,\alpha,\infty} &\leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^i} K_2^{-1} \sum_{j=0}^{i-1} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m+i)^{m+i-1} e^{(1+\ln 2)j} \\ &\leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^i} (i+m)^{i+m-1} K_2^{-1} \frac{1}{1 - A/K_1} \frac{1}{1 - 2\tilde{A}/K_2}\end{aligned}$$

if A/K_1, $2\tilde{A}/K_2 < 1$. Let us now turn to $\partial_\theta^m \widehat{F}_i^2$. The estimates are completely analogous to those for $\partial_\theta^m \widehat{F}_i^1$. We have

$$\partial_\theta^m \widehat{F}_i^2 = \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} \binom{m}{\mu} D^\mu a_2^j \,\widehat{\rho}^j \partial_\theta^{m-\mu+2} \widehat{U}_{i-2-j}.$$

As before Lemma B.1 with $\delta = (1-\alpha)j/(i-1)$ and the induction hypothesis lead to

$$\|\widehat{\rho}^j \partial_\theta^{m-\mu+2} \widehat{U}_{i-2-j}\|_{0,\alpha,\infty} \leq C_U \frac{K_1^{m-\mu+2} K_2^{i-2-j} e^{-j} (m - \mu + i - j)^{m-\mu+i-j} (i-1)^{i-1}}{(1-\alpha)^{i-1} (i-1-j)^{i-1-j}}.$$

Therefore, we obtain for $\partial_\theta^m \widehat{F}_i^2$

$$\|\partial_\theta^m \widehat{F}_i^2\|_{0,\alpha,\infty} \leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1^2 K_2^{-2} \times$$
$$\times \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} \binom{m}{\mu} \mu!\, A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m - \mu + i - j)^{m-\mu+i-j} \frac{(i-1)^{i-1}}{(i-1-j)^{i-1-j}}.$$

As $i - j \geq 2$, Lemma B.6 is applicable with $a = i - j$, $b = 0$, and we get

$$\|\partial_\theta^m \widehat{F}_i^2\|_{0,\alpha,\infty} \leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1^2 K_2^{-2} \times$$
$$\times \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m + 1 + i - 1 - j)^{m+1+i-1-j} \frac{(i-1)^{i-1}}{(i-1-j)^{i-1-j}}.$$

Now choosing $p = 0$, $a = m + 1$, $b = 0$, and replacing i by $i - 1$ in Lemma B.7, we get

$$\begin{aligned}\|\partial_\theta^m \widehat{F}_i^2\|_{0,\alpha,\infty} &\leq C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1^2 K_2^{-2} \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m+i)^{m+i} e^{(1+\ln 2)j} \\ &\leq C_U C_A \frac{K_1^m K_2^i (m+i)^{m+i}}{(1-\alpha)^{i-1}} K_1^2 K_2^{-2} \frac{1}{1 - A/K_1} \frac{1}{1 - 2\tilde{A}/K_2}.\end{aligned}$$

Let us now deal with $\partial_\theta^m \widehat{F}_i^3$. We have

$$\partial_\theta^m \widehat{F}_i^3 = \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} \binom{m}{\mu} D^\mu a_3^j \,\widehat{\rho}^j \partial_\theta^{m-\mu+1} \widehat{U}_{i-2-j}$$

As before Lemma B.1 with $\delta = (1-\alpha)j/(i-1)$ and the induction hypothesis lead to

$$\|\widehat{\rho}^j \partial_\theta^{m-\mu+1} \widehat{U}_{i-2-j}\|_{0,\alpha,\infty} \le C_U \frac{K_1^{m-\mu+1} K_2^{i-2-j} e^{-j} (m-\mu+i-1-j)^{m-\mu+i-1-j} (i-1)^{i-1}}{(1-\alpha)^{i-1}(i-1-j)^{i-1-j}}.$$

Therefore, we obtain for $\partial_\theta^m \widehat{F}_i^3$

$$\|\partial_\theta^m \widehat{F}_i^2\|_{0,\alpha,\infty} \le C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1 K_2^{-2} \times$$
$$\times \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} \binom{m}{\mu} \mu!\, A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m-\mu+i-1-j)^{m-\mu+i-1-j} \frac{(i-1)^{i-1}}{(i-1-j)^{i-1-j}}.$$

As $i-1-j \ge 1$, Lemma B.6 is applicable with $a = i-1-j$, $b=0$, and we get

$$\|\partial_\theta^m \widehat{F}_i^3\|_{0,\alpha,\infty} \le C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1 K_2^{-2} \times$$
$$\times \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m+i-1-j)^{m+i-1-j} \frac{(i-1)^{i-1}}{(i-1-j)^{i-1-j}}.$$

Now choosing $p=0$, $a=m$, $b=0$, and replacing i by $i-1$ in Lemma B.7, we get

$$\begin{aligned}
\|\partial_\theta^m \widehat{F}_i^2\|_{0,\alpha,\infty} &\le C_U C_A \frac{K_1^m K_2^i}{(1-\alpha)^{i-1}} K_1 K_2^{-2} \sum_{j=0}^{i-2} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} e^{-j} (m+i-1)^{m+i-1} e^{(1+\ln 2)j} \\
&\le C_U C_A \frac{K_1^m K_2^i (m+i)^{m+i-1}}{(1-\alpha)^{i-1}} K_1 K_2^{-2} \frac{1}{1-A/K_1} \frac{1}{1-2\tilde{A}/K_2}.
\end{aligned}$$

Finally, we have by assumption

$$\|D^m G_i\|_{L^\infty([0,L))} \le C_G G^i \tilde{G}^m (i+m)^{i+m}.$$

The a-priori estimate (A.4) therefore gives

$$\|\partial_\theta^m \widehat{U}_i\|_{1,\alpha,\infty} \le C_U \frac{K_1^m K_2^i (i+m)^{i+m}}{(1-\alpha)^{i+1}} \times$$
$$\times \left[\left\{ K_2^{-1} + K_2^{-2} K_1^2 + K_2^{-2} K_1 \right\} \frac{3C_A}{(1-A/K_1)(1-2\tilde{A}/K_2)} + \frac{3C_G}{C_U} \left(\frac{G}{K_2}\right)^i \left(\frac{\tilde{G}}{K_1}\right)^m \right].$$

The definitions (2.12), (2.13), and (2.14) of C_U, K_1, and K_2 imply now that the bracketed expression is bounded by 1 which concludes the induction argument. □

A.3 Controlling all derivatives of the inner expansion: proof of Proposition 2.3

Proof of Proposition 2.3: We will proceed by induction on n for all i and m simultaneously. As $K_3 \geq 1$, Proposition 2.2 implies that (2.15) holds for $n = 0$ and $n = 1$. Let us therefore see that (2.15) is true for $n \geq 2$ under the assumption that the claim of the proposition is true for $\nu \leq n-1$. For notational convenience we introduce the new parameter

$$p = n - 2 \geq 0.$$

Differentiating (A.3) p times, we get

$$-\partial_{\hat\rho}^{n}\, \partial_\theta^m\, \widehat{U}_i = \partial_{\hat\rho}^{p}\, \partial_\theta^m\, \widehat{F}_i - \partial_{\hat\rho}^{p}\, \partial_\theta^m\, \widehat{U}_i. \tag{A.5}$$

Now, we proceed just as in the proof of Proposition 2.2. Let us first consider

$$\partial_{\hat\rho}^{p}\, \partial_\theta^m\, \widehat{F}_i^1 = \sum_{j=0}^{i-1} \sum_{\mu=0}^{m} \sum_{\nu=0}^{\min(p,j)} \binom{m}{\mu} \binom{p}{\nu} D^\mu a_1^j \binom{j}{\nu} \nu! \hat\rho^{j-\nu} \partial_\theta^{m-\mu}\, \partial_{\hat\rho}^{p-\nu+1}\, \widehat{U}_{i-1-j}.$$

Without loss of generality we may assume that $i \geq 1$, for otherwise $\widehat{F}_i \equiv 0$. By Lemma B.1 we have, together with the induction hypothesis (and the estimate $\max(p - \nu + 1 - 2, 0) \leq p + 1$), for $0 \leq \nu \leq \min(j, p)$ and each fixed θ

$$\begin{aligned}\|\hat\rho^{j-\nu} \partial_\theta^{m-\mu}\, \partial_{\hat\rho}^{p-\nu+1}\, \widehat{U}_{i-1-j}\|_{0,\alpha} &\leq \delta^{-(j-\nu)} (j-\nu)^{j-\nu} e^{-(j-\nu)} \|\partial_\theta^{m-\mu}\, \partial_{\hat\rho}^{p-\nu+1}\, \widehat{U}_{i-1-j}\|_{0,\alpha+\delta} \leq \\ &\leq C_U \frac{K_1^{m-\mu} K_2^{i-1-j} K_3^{p-\nu+1} e^{-(j-\nu)} (j-\nu)^{j-\nu} (m - \mu + i - 1 - j + p + 1)^{m-\mu+i-1-j}}{\delta^{j-\nu} (1 - \alpha - \delta)^{i-1-j+1}}.\end{aligned}$$

Lemma B.2 suggests the choice $\delta = (1-\alpha)(j-\nu)/(i-\nu)$ which leads to

$$\begin{aligned}&\|\hat\rho^{j-\nu} \partial_\theta^{m-\mu}\, \partial_{\hat\rho}^{p-\nu+1}\, \widehat{U}_{i-1-j}\|_{0,\alpha,\infty} \leq \\ &\leq C_U \frac{K_1^{m-\mu} K_2^{i-1-j} K_3^{p-\nu+1} e^{-(j-\nu)} (m - \mu + i - 1 - j + p + 1)^{m-\mu+i-1-j} (i-\nu)^{i-\nu}}{(1-\alpha)^{i-\nu} (i-j)^{i-j}}.\end{aligned}$$

Together with the assumptions (2.8), we arrive at

$$\begin{aligned}\|\partial_{\hat\rho}^{p}\, \partial_\theta^m\, \widehat{F}_i^1\|_{0,\alpha,\infty} &\leq C_U C_A \frac{K_1^m K_2^i K_3^{p+1}}{(1-\alpha)^i} K_2^{-1} \sum_{j=0}^{i-1} \sum_{\nu=0}^{\min(j,p)} \sum_{\mu=0}^{m} \binom{m}{\mu} \mu! \binom{p}{\nu} \binom{j}{\nu} \nu! \times \\ &\times A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} K_3^{-\nu} e^{-(j-\nu)} \frac{(m - \mu + i - 1 - j + p + 1)^{m-\mu+i-1-j} (i-\nu)^{i-\nu}}{(1-\alpha)^{-\nu} (i-j)^{i-j}}.\end{aligned}$$

As before, we apply first Lemma B.6 with $a = i - 1 - j \geq 0$, $b = p + 1 \geq 1$ and then Lemma B.7 with $a = m - 1 \geq -1$, $b = 1$ to obtain

$$\begin{aligned}\|\partial_{\hat\rho}^{p}\, \partial_\theta^m\, \widehat{F}_i^1\|_{0,\alpha,\infty} &\leq C_U C_A \frac{K_1^m K_2^i K_3^{p+1}}{(1-\alpha)^i} K_2^{-1} \times \\ &\times \sum_{j=0}^{i-1} \sum_{\nu=0}^{\min(j,p)} \sum_{\mu=0}^{m} A^\mu \tilde{A}^j K_1^{-\mu} K_2^{-j} K_3^{-\nu} e^{-(j-\nu)} (1-\alpha)^\nu (m + i + p)^{m+i-1} e^{(1+\ln 2)j}.\end{aligned}$$

We conclude

$$\|\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^1\|_{0,\alpha,\infty} \le$$
$$\le C_U \frac{K_1^m K_2^i K_3^{p+2}}{(1-\alpha)^i}(i+m+p)^{m+i-1}\frac{C_A K_2^{-1}K_3^{-1}}{(1-A/K_1)(1-2\tilde A/K_2)(1-e(1-\alpha)/K_3)}.$$

The terms $\widehat F_i^2$ and $\widehat F_i^3$ are treated similarly. Without loss of generality, we may assume that $i \ge 2$ for otherwise $\widehat F_i^2 = \widehat F_i^3 \equiv 0$. We have,

$$\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^2 = \sum_{j=0}^{i-2}\sum_{\mu=0}^{m}\sum_{\nu=0}^{\min(p,j)} \binom{m}{\mu}\binom{p}{\nu} D^\mu a_2^j \binom{j}{\nu}\nu!\hat\rho^{j-\nu}\partial_\theta^{m-\mu+2}\partial_{\hat\rho}^{p-\nu}\widehat U_{i-2-j}.$$

As before Lemma B.1 with $\delta = (1-\alpha)(j-\nu)/(i-1-\nu)$, the induction hypothesis, and the estimate $\max(p-\nu-2,0) \le p$, we get

$$\|\hat\rho^{j-\nu}\partial_\theta^{m-\mu+2}\partial_{\hat\rho}^{p-\nu}\widehat U_{i-2-j}\|_{0,\alpha,\infty} \le$$
$$\le C_U \frac{K_1^{m-\mu+2}K_2^{i-2-j}K_3^{p-\nu}e^{-(j-\nu)}(m-\mu+i-j+p)^{m-\mu+i-j}(i-1-\nu)^{i-1-\nu}}{(1-\alpha)^{i-1-\nu}(i-1-j)^{i-1-j}}.$$

Inserting this and the estimates (2.8) in the definition of $\widehat F_i^2$, we obtain for $\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^2$,

$$\|\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^2\|_{0,\alpha,\infty} \le C_U \frac{K_1^m K_2^i K_3^p}{(1-\alpha)^{i-1}} C_A K_1^2 K_2^{-2}\sum_{j=0}^{i-2}\sum_{\nu=0}^{\min(j,p)}\sum_{\mu=0}^{m}\binom{m}{\mu}\mu!\binom{p}{\nu}\binom{j}{\nu}\nu!\times$$
$$\times A^\mu \tilde A^j K_1^{-\mu}K_2^{-j}K_3^{-\nu}e^{-(j-\nu)}\frac{(m-\mu+1+i-1-j+p)^{m-\mu+1+i-1-j}(i-1-\nu)^{i-1-\nu}}{(i-1-j)^{i-1-j}}.$$

Applying Lemma B.6 with $a = i-1-j+1 \ge 2$ and $b = p \ge 0$ and then Lemma B.7 with $a = m+1$, $b = 0$, and i replaced with $i-1$ leads to

$$\|\partial_\theta^m \partial_{\hat\rho}^p \widehat F_i^2\|_{0,\alpha,\infty} \le C_U C_A \frac{K_1^m K_2^i K_3^p}{(1-\alpha)^{i-1}} K_1^2 K_2^{-2}\times$$
$$\times \sum_{j=0}^{i-2}\sum_{\nu=0}^{\min(j,p)}\sum_{\mu=0}^{m} A^\mu \tilde A^j K_1^{-\mu}K_2^{-j}K_3^{-\nu}e^{-(j-\nu)}(1-\alpha)^\nu (m+i+p)^{m+i}e^{(1+\ln 2)j}.$$

We conclude

$$\|\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^2\|_{0,\alpha,\infty} \le$$
$$\le C_U \frac{K_1^m K_2^i K_3^{p+2}}{(1-\alpha)^{i-1}}(i+m+p)^{m+i}\frac{C_A K_1^2 K_2^{-2}K_3^{-2}}{(1-A/K_1)(1-2\tilde A/K_2)(1-e(1-\alpha)/K_3)}.$$

Let us see that the term $\widehat F_i^3$ is also under control. Without loss of generality, we may assume that $i \ge 2$ for otherwise $\widehat F_i^3 \equiv 0$. We have,

$$\partial_{\hat\rho}^p \partial_\theta^m \widehat F_i^3 = \sum_{j=0}^{i-2}\sum_{\mu=0}^{m}\sum_{\nu=0}^{\min(p,j)} \binom{m}{\mu}\binom{p}{\nu} D^\mu a_3^j \binom{j}{\nu}\nu!\hat\rho^{j-\nu}\partial_\theta^{m-\mu+1}\partial_{\hat\rho}^{p-\nu}\widehat U_{i-2-j}.$$

As before Lemma B.1 with $\delta = (1-\alpha)(j-\nu)/(i-1-\nu)$, the induction hypothesis, and the estimate $\max(p-\nu-2,0) \le p$

$$\|\hat\rho^{j-\nu}\partial_\theta^{m-\mu+1}\partial_{\hat\rho}^{p-\nu}\hat U_{i-2-j}\|_{0,\alpha,\infty} \le$$
$$\le C_U \frac{K_1^{m-\mu+1}K_2^{i-2-j}K_3^{p-\nu}e^{-(j-\nu)}(m-\mu+i-1-j+p)^{m-\mu+i-1-j}(i-1-\nu)^{i-1-\nu}}{(1-\alpha)^{i-1-\nu}(i-1-j)^{i-1-j}}.$$

Therefore, inserting this and the estimates (2.8) in the definition of $\hat F_i^3$, we obtain for $\partial_{\hat\rho}^p\partial_\theta^m \hat F_i^3$,

$$\|\partial_{\hat\rho}^p\partial_\theta^m \hat F_i^3\|_{0,\alpha,\infty} \le C_U\frac{K_1^mK_2^iK_3^p}{(1-\alpha)^{i-1}}C_AK_1K_2^{-2}\sum_{j=0}^{i-2}\sum_{\nu=0}^{\min(j,p)}\sum_{\mu=0}^{m}\binom{m}{\mu}\mu!\binom{p}{\nu}\binom{j}{\nu}\nu!\times$$
$$\times A^\mu\tilde A^jK_1^{-\mu}K_2^{-j}K_3^{-\nu}e^{-(j-\nu)}\frac{(m-\mu+i-1-j+p)^{m-\mu+i-1-j}(i-1-\nu)^{i-1-\nu}}{(1-\alpha)^{-\nu}(i-1-j)^{i-1-j}}.$$

Applying Lemma B.6 with $a = i-1-j \ge 1$ and $b = p \ge 0$ and then Lemma B.7 with $a = m \ge 0$, $b = 0$, and i replaced with $i-1$ leads to

$$\|\partial_\theta^m\partial_{\hat\rho}^p \hat F_i^3\|_{0,\alpha,\infty} \le C_UC_A\frac{K_1^mK_2^iK_3^p}{(1-\alpha)^{i-1}}K_1K_2^{-2}\times$$
$$\times\sum_{j=0}^{i-2}\sum_{\nu=0}^{\min(j,p)}\sum_{\mu=0}^{m}A^\mu\tilde A^jK_1^{-\mu}K_2^{-j}K_3^{-\nu}e^{-(j-\nu)}(1-\alpha)^\nu(m+i-1+p)^{m+i-1}e^{(1+\ln 2)j}.$$

We conclude

$$\|\partial_{\hat\rho}^p\partial_\theta^m \hat F_i^3\|_{0,\alpha,\infty} \le$$
$$\le C_U\frac{K_1^mK_2^iK_3^{p+2}}{(1-\alpha)^{i-1}}(i-1+m+p)^{m+i-1}\frac{C_AK_1K_2^{-2}K_3^{-2}}{(1-A/K_1)(1-2\tilde A/K_2)(1-e(1-\alpha)/K_3)}.$$

Finally, by the induction hypothesis,

$$\begin{aligned}\|\partial_{\hat\rho}^p\partial_\theta^m \hat U_i\|_{0,\alpha,\infty} &\le C_U\frac{K_1^mK_2^iK_3^p}{(1-\alpha)^{i+1}}(i+m+\max(p-2,0))^{i+m}\\ &\le C_U\frac{K_1^mK_2^iK_3^{p+2}}{(1-\alpha)^{i+1}}(i+m+p)^{i+m}K_3^{-2},\end{aligned}$$

and we get therefore for $\|\partial_{\hat\rho}^n\partial_\theta^m \hat U_i\|_{0,\alpha,\infty}$

$$\begin{aligned}\|\partial_{\hat\rho}^n\partial_\theta^m \hat U_i\|_{0,\alpha,\infty} &\le \|\partial_{\hat\rho}^p\partial_\theta^m \hat F_i\|_{0,\alpha,\infty} + \|\partial_{\hat\rho}^p\partial_\theta^m \hat U_i\|_{0,\alpha,\infty}\\ &\le C_U\frac{K_1^mK_2^iK_3^{p+2}}{(1-\alpha)^{i+1}}(i+m+p)^{i+m}\times\\ &\quad\times\left[C_A\frac{K_2^{-1}K_3^{-1}+K_1^2K_2^{-2}K_3^{-2}+K_1K_2^{-2}K_3^{-2}}{(1-A/K_1)(1-2\tilde A/K_2)(1-e/K_3)}+K_3^{-2}\right].\end{aligned}$$

By the choice of K_3 in (2.16) we have that the bracketed expression is bounded by one which completes the induction argument. □

A.4 Controlling $L_\varepsilon u_M^{BL}$

Lemma A.2 *With C_U, K_1, K_2, and K_3 as in Proposition 2.3 there are constants $C(\alpha)$ and $K > 0$ depending only on the constants of Proposition 2.3 such that*

$$\begin{aligned} |\widehat{\rho}^j \partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{U}_i(\widehat{\rho}, \theta)| &\leq C(\alpha) p! m! K^m \left(\frac{j}{\alpha e}\right)^j \left(\frac{eK_2}{1-\alpha}\right)^i i^i \qquad \forall \widehat{\rho} \geq 0, \theta \in [0, L), \\ \|\widehat{\rho}^j \partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{U}_i\|_{0,\alpha,\infty} &\leq C_U \frac{K_1^m K_2^i K_3^p}{(1-\alpha)^{i+1+j}} e^{i+1} j^j \left[i + m + \max(p-2, 0)\right]^{i+m}. \end{aligned}$$

Proof: The first estimate follows from Corollary 2.4 (and Cauchy's integral theorem for the derivatives) and the fact that $|x^j e^{-\alpha x}| \leq (j/\alpha)^j e^{-j}$ for all x, $j \geq 0$. For the second estimate, we use Lemma B.1 to get

$$\|\widehat{\rho}^j \partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{U}_i\|_{0,\alpha,\infty} \leq \delta^{-j} j^j e^{-j} \|\partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{U}_i\|_{0,\alpha+\delta,\infty}.$$

Choosing $\delta = (1-\alpha)j/(i+1+j)$ as suggested by Lemma B.2 the desired estimate follows from Proposition 2.3 and the bound $(i+j+1)^{i+j+1} \leq (i+1)^{i+1} j^j e^{i+j+1}$. $\square$

From the proof of Proposition 2.2 we can extract the following lemma.

Lemma A.3 *With C_U, K_1, K_2, and K_3 as in Proposition 2.3 we have*

$$\forall i, p, m \in \mathbb{N}_0 \qquad \|\partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{F}_i\|_{0,\alpha,\infty} \leq \sum_{j=1}^{3} \|\partial_{\widehat{\rho}}^p \partial_\theta^m \widehat{F}_i^j\|_{0,\alpha,\infty} \leq C_U \frac{K_1^m K_2^i K_3^{p+2}}{(1-\alpha)^{i+1}} (i+m+p)^{i+m}.$$

Proof: Follows directly from the proof of Proposition 2.3. $\square$

Lemma A.4 *For every $M \in \mathbb{N}_0$ the function $u_M^{BL} = \sum_{i=0}^{2M+1} \varepsilon^i \widehat{U}_i$ satisfies for $0 < \rho \leq \rho_0$:*

$$\begin{aligned} L_\varepsilon u_M^{BL} = &\sum_{i=0}^{2M+1} \varepsilon^i \left[-\partial_{\widehat{\rho}}^2 \widehat{U}_i + \widehat{U}_i - \widehat{F}_i\right] - \varepsilon^{2M+2} \widehat{F}_{2M+2} - \varepsilon^{2M+3} \widehat{F}_{2M+3}^2 - \varepsilon^{2M+3} \widehat{F}_{2M+3}^3 \\ &- \varepsilon^{2M+3} \sum_{j=0}^{2M+1} \left[\sum_{i=0}^{\infty} \varepsilon^i \widehat{\rho}^{j+i+1} a_1^{j+i+1} \partial_{\widehat{\rho}} + \varepsilon^{i+1} \widehat{\rho}^{j+i+1} a_2^{j+i+1} \partial_\theta^2 + \varepsilon^{i+1} \widehat{\rho}^{j+i+1} a_3^{j+i+1} \partial_\theta\right] \widehat{U}_{2M+1-j}. \end{aligned}$$

Remark A.5: The functions $\widehat{U}_i$ are constructed in such a way that $-\partial_{\widehat{\rho}}^2 \widehat{U}_i + \widehat{U}_i - \widehat{F}_i = 0$ for all $i \in \mathbb{N}_0$.

Proof: The sum in the representation (2.6) of the differential operator L_ε has three parts. Let us consider each part separately. For $0 < \rho \leq \rho_0$, (i.e., for $0 < \widehat{\rho} \leq \rho_0/\varepsilon$) each part is an absolutely converging series, which may be reordered. We have

$$\begin{aligned}
\sum_{i=0}^{2M+1}\sum_{j=0}^{\infty}\widehat{\rho}^j a_1^j\varepsilon^{i+1+j}\partial_{\hat\rho}\widehat{U}_i &= \sum_{j=0}^{\infty}\sum_{i=j+1}^{j+1+2M+1}\widehat{\rho}^j a_1^j\varepsilon^i\partial_{\hat\rho}\widehat{U}_{i-1-j}\\
&= \sum_{i=1}^{2M+2}\varepsilon^i\sum_{j=0}^{i-1}\widehat{\rho}^j a_1^j\partial_{\hat\rho}\widehat{U}_{i-1-j} + \sum_{i=2M+3}^{\infty}\varepsilon^i\sum_{j=i-2M-2}^{i-1}\widehat{\rho}^j a_1^j\partial_{\hat\rho}\widehat{U}_{i-1-j}\\
&= \sum_{i=0}^{2M+2}\varepsilon^i\widehat{F}_i^1 + \sum_{i=2M+3}^{\infty}\varepsilon^i\sum_{j=0}^{2M+1}\widehat{\rho}^{j-2M-2+i}a_1^{j-2M-2+i}\partial_{\hat\rho}\widehat{U}_{2M+1-j}\\
&= \sum_{i=0}^{2M+2}\varepsilon^i\widehat{F}_i^1 + \varepsilon^{2M+3}\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{j+i+1}a_1^{j+i+1}\partial_{\hat\rho}\widehat{U}_{2M+1-j}.
\end{aligned}$$

Similarly, we obtain for the second part:

$$\begin{aligned}
\sum_{i=0}^{2M+1}\sum_{j=0}^{\infty}\widehat{\rho}^j a_2^j\varepsilon^{i+2+j}\partial_\theta^2\widehat{U}_i &= \sum_{j=0}^{\infty}\sum_{i=j+2}^{j+2M+3}\widehat{\rho}^j a_2^j\varepsilon^i\partial_\theta^2\widehat{U}_{i-2-j}\\
&= \sum_{i=2}^{2M+3}\varepsilon^i\sum_{j=0}^{i-2}\widehat{\rho}^j a_2^j\partial_\theta^2\widehat{U}_{i-2-j} + \sum_{i=2M+4}^{\infty}\varepsilon^i\sum_{j=i-2M-3}^{i-2}\widehat{\rho}^j a_2^j\partial_\theta^2\widehat{U}_{i-2-j}\\
&= \sum_{i=0}^{2M+3}\varepsilon^i\widehat{F}_i^2 + \sum_{i=2M+4}^{\infty}\varepsilon^i\sum_{j=0}^{2M+1}\widehat{\rho}^{i-2M-3+j}a_2^{i-2M-3+j}\partial_\theta^2\widehat{U}_{2M+1-j}\\
&= \sum_{i=0}^{2M+3}\varepsilon^i\widehat{F}_i^2 + \varepsilon^{2M+4}\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_2^{i+j+1}\partial_\theta^2\widehat{U}_{2M+1-j}.
\end{aligned}$$

And completely analogously, for the third part

$$\sum_{i=0}^{2M}\sum_{j=0}^{\infty}\widehat{\rho}^j a_3^j\varepsilon^{i+2+j}\partial_\theta\widehat{U}_i = \sum_{i=0}^{2M+3}\varepsilon^i\widehat{F}_i^3 + \varepsilon^{2M+4}\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_3^{i+j+1}\partial_\theta\widehat{U}_{2M+1-j}.$$

Inserting these three sums in the representation (2.6) yields the desired result. □

Lemma A.6 *Let ρ_0 be such that $\rho_0\tilde{A} =: q < 1$ where $\tilde{A}$ is given by (2.8). Then there are K, C, and $C(\alpha) > 0$ depending depending only on q and the constants of Proposition 2.2 ($C(\alpha)$ depends additionally on $\alpha \in [0,1)$), such that*

$$\begin{aligned}
\sup_{\theta\in[0,L)}\left\{\int_{\rho=0}^{\rho_0}e^{2\alpha\rho/\varepsilon}\left|L_\varepsilon u_M^{BL}(\rho,\theta)\right|^2 d\rho\right\}^{1/2} &\leq C(\alpha)\varepsilon^{1/2}\left(\frac{K\varepsilon(2M+2)}{1-\alpha}\right)^{2M+2},\\
|L_\varepsilon u_M^{BL}(\rho,\theta)| &\leq C\left(K\varepsilon(2M+2)\right)^{2M+2}, \qquad 0 \leq \rho \leq \rho_0.
\end{aligned}$$

Proof: From Lemma A.4 we get

$$\begin{aligned}
\left|L_\varepsilon u_M^{BL}(\widehat{\rho},\theta)\right| &\leq \varepsilon^{2M+2}\left|\widehat{F}_{2M+2}(\widehat{\rho},\theta)\right| + \varepsilon^{2M+3}\left|\widehat{F}^2_{2M+3}(\widehat{\rho},\theta)\right| + \varepsilon^{2M+3}\left|\widehat{F}^3_{2M+3}(\widehat{\rho},\theta)\right|\\
&+\varepsilon^{2M+3}\left|\sum_{j=0}^{2M+1}\left[\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{j+i+1}a_1^{j+i+1}\partial_{\hat\rho} + \varepsilon^{i+1}\widehat{\rho}^{j+i+1}a_2^{j+i+1}\partial_\theta^2 + \varepsilon^{i+1}\widehat{\rho}^{j+i+1}a_3^{j+i+1}\partial_\theta\right]\widehat{U}_{2M+1-j}\right|.
\end{aligned}$$

Let us first estimate $\widehat{F}_{2M+2}$. The change of variables $\rho = \widehat{\rho}\varepsilon$ and Lemma A.3 yields

$$\begin{aligned}\left\{\int_{\rho=0}^{\rho_0} e^{2\alpha\rho/\varepsilon}\left|\varepsilon^{2M+2}\widehat{F}_{2M+2}(\widehat{\rho},\theta)\right|^2 d\rho\right\}^{1/2} &\le \varepsilon^{1/2}\varepsilon^{2M+2}\|\widehat{F}_{2M+2}\|_{0,\alpha,\infty}\\ &\le C(\alpha)\varepsilon^{1/2}\left(\frac{K_2\varepsilon(2M+2)}{1-\alpha}\right)^{2M+2}.\end{aligned}$$

From Lemma A.3 and Lemma B.3, we obtain for $\alpha = 1/2$

$$\begin{aligned}\varepsilon^{2M+2}|\widehat{F}_{2M+2}(\widehat{\rho},\theta)| &\le C\varepsilon^{2M+2}\|\partial_{\widehat{\rho}}\widehat{F}_{2M+2}\|_{0,1/2,\infty} \le C\varepsilon^{2M+2}(2K_2)^{2M+2}(2M+3)^{2M+2}\\ &\le Ce\,(2K_2\varepsilon(2M+2))^{2M+2}.\end{aligned}$$

Completely analogously, we obtain the appropriate estimates for $\widehat{F}^2_{2M+3}$ and $\widehat{F}^3_{2M+3}$ if we observe that we have have by assumption $\varepsilon \le 1$ and that we can bound

$$(2M+3)^{2M+3} \le (2M+3)e(2M+2)^{2M+2} \le e2^{2M+2}(2M+2)^{2M+2}.$$

Let us now turn to estimating the double sum in the expression for $L_\varepsilon u_M^{BL}$. We will only consider the terms involving a_1^{i+j+1}, the others being handled completely analogously. From the assumptions (2.8) on the functions a_1^{i+j+1}, the assumption $\rho_0\tilde{A} = q < 1$, and the fact that we have $0 \le \rho \le \rho_0$, we obtain

$$\varepsilon^i\widehat{\rho}^{i+j+1}|a_1^{i+j+1}| \le C_A(\varepsilon\widehat{\rho}\tilde{A})^i(\widehat{\rho}\tilde{A})^{j+1} \le C_Aq^i(\widehat{\rho}\tilde{A})^{j+1}.$$

This leads to

$$\left|\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_1^{i+j+1}\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}\right| \le \frac{C_A}{1-q}\sum_{j=0}^{2M+1}\tilde{A}^{j+1}\widehat{\rho}^{j+1}\left|\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}\right|. \tag{A.6}$$

By means of the pointwise estimate in Lemma A.2 (choose $\alpha = 1/2$), we finally get the estimates

$$\begin{aligned}&\left|\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_1^{i+j+1}\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}\right|\\ &\quad\le \frac{C}{1-q}(2eK_2)^{2M+1}\sum_{j=0}^{2M+1}\left(\tilde{A}/(K_2e^2)\right)^j(j+1)^{j+1}(2M+1-j)^{2M+1-j}\\ &\quad\le C(q)K^{2M+1}(2M+2)^{2M+2}.\end{aligned}$$

for appropriately chosen $K > 0$. This estimate completes the argument for the pointwise estimate of Lemma A.6. In order to conclude the other estimate, we invoke Lemma A.2 again to get

$$\begin{aligned}&\int_{\rho=0}^{\rho_0} e^{2\alpha\rho/\varepsilon}\left|\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_1^{i+j+1}\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}\right|^2 d\rho\\ &\quad\le \frac{C_A^2}{(1-q)^2}\varepsilon\sum_{j=0}^{2M+1}\tilde{A}^{2(j+1)}\sum_{j=0}^{2M+1}\|\widehat{\rho}^{j+1}\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}\|^2_{0,\alpha,\infty}\\ &\quad\le \varepsilon C(\alpha,q)\left(\frac{K}{1-\alpha}\right)^{2(2M+2)}\sum_{j=0}^{2M+1}(j+1)^{2(j+1)}(2M+1-j)^{2(2M+1-j)}\\ &\quad\le \varepsilon C(\alpha,q)\left(\frac{K}{1-\alpha}\right)^{2(2M+2)}(2M+2)^{2(2M+2)}\end{aligned}$$

where $K > 0$ is again appropriately chosen. □

In a similar fashion higher derivatives of $L_\varepsilon u_M^{BL}$ may be estimated. Let us record one possible result of this form.

Lemma A.7 *Let $\rho_0 < \|\kappa\|_{L^\infty([0,L))}$. Then there are C, K_8, K_9, and K_{10} depending only on f, g, ρ_0, and κ, (i.e., the geometry of Ω) such that*

$$\left|\partial_\rho^p \partial_\theta^m L_\varepsilon u_M^{BL}(\rho,\theta)\right| \le Cp!m!K_8^p K_9^m \varepsilon^{-p}\left(K_{10}\varepsilon(2M+2)\right)^{2M+2} \qquad 0 \le \rho \le \rho_0, \quad \theta \in [0,L), \quad p,m \in \mathbb{N}_0$$

Proof: We use the same representation formula for $L_\varepsilon u_M^{BL}$ as in the proof of Lemma A.6 and estimate each term separately. We have by Lemma A.3 and Lemma B.3 with $\alpha = 1/2$

$$\begin{aligned}\varepsilon^{2M+2}\left|\partial_\rho^p \partial_\theta^m \widehat{F}_{2M+2}(\widehat{\rho},\theta)\right| &\le C\varepsilon^{2M+2}\varepsilon^{-p}\|\partial_{\widehat{\rho}}^{p+1}\partial_\theta^m \widehat{F}_{2M+2}\|_{0,1/2,\infty} \\ &\le C\varepsilon^{2M+2}\varepsilon^{-p}K_1^m(2K_2)^{2M+2}K_3^{p+3}(2M+2+m+p+1)^{2M+2+m} \\ &\le C\varepsilon^{2M+2}\varepsilon^{-p}(e^2K_1)^m(2eK_2)^{2M+2}(e^2K_3)^p m!\,(2M+2))^{2M+2}\end{aligned}$$

where we used the estimate $(2M+2+m+p+1)^{2M+2+m} \le (2M+2)^{2M+2}m^m e^{2M+2+m+2(p+1)} \le C(2M+2)^{2M+2}m!e^m e^{2M+2+2m+2(p+1)}$. Hence, the term involving $\widehat{F}_{2M+2}$ can be estimated as desired. The terms involving $\widehat{F}_{2M+3}^2$ and $\widehat{F}_{2M+3}^3$ can be controlled similarly.
Let us now turn our attention to the double sum. Just as in the proof of Lemma A.6, we will only consider the first term of the double sum as the the other two are treated similarly. Introduce the short-hand

$$S := \sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\widehat{\rho}^{i+j+1}a_1^{i+j+1}\partial_{\widehat{\rho}}\widehat{U}_{2M+1-j}.$$

We want to apply Cauchy's integral theorem for derivatives. In order to do so, let us choose $\rho_0' > 0$ such that $\rho_0 < \rho_0' < \|\kappa\|_{L^\infty([0,L))}$, and set $\delta := (\rho_0' - \rho_0)/2$. We observe that by the analyticity of the function κ we have the existence of $\delta' > 0$ and C_A', $\overline{A} > 0$ such that

$$\begin{aligned}\rho_0'\overline{A} &=: q' < 1 \\ \left|a_l^i(\theta+\zeta)\right| &\le C_A'\overline{A}^i \qquad i \in \mathbb{N}_0, \quad \theta \in [0,L), \quad \zeta \in \mathbb{C}, \quad |\zeta| \le \delta'.\end{aligned}$$

We obtain by Cauchy's integral theorem and Corollary 2.4

$$\begin{aligned}\left|\partial_\rho^p \partial_\theta^m S\right| &= \varepsilon^{-p}\frac{1}{4\pi^2}p!m!\left|\int_{|z|=\delta}\int_{|\zeta|=\delta'}\frac{S(\widehat{\rho}+z,\theta+\zeta)}{(-z)^{p+1}(-\zeta)^{m+1}}\,dzd\zeta\right| \\ &\le C\varepsilon^{-p}p!m!\delta^{-p}\delta'^{-m}\sum_{j=0}^{2M+1}\sum_{i=0}^{\infty}\varepsilon^i\left(\widehat{\rho}+\delta\right)^{i+j+1}\overline{A}^{i+j+1}e^{-\alpha\widehat{\rho}}\left(\frac{eK_2}{1-\alpha}\right)^{2M+1-j}(2M+1-j)^{2M+1-j}\end{aligned}$$

where $C > 0$ depends on C_A' and δ'. As we have by the assumptions $\rho_0 + \delta \le \rho_0'$ and $\rho_0'\overline{A} = q' < 1$, we have

$$\left|\partial_\rho^p \partial_\theta^m S\right| \le C\varepsilon^{-p}p!m!\delta^{-p}\delta'^{-m}\frac{1}{1-q'}\sum_{j=0}^{2M+1}\left(\widehat{\rho}+\delta\right)^{j+1}\overline{A}^{j+1}e^{-\alpha\widehat{\rho}}\left(\frac{eK_2}{1-\alpha}\right)^{2M+1-j}(2M+1-j)^{2M+1-j}$$

Finally, as we have the estimate

$$(\hat{\rho}+\delta)^{j+1}e^{-\alpha\hat{\rho}} \le \left(\frac{j+1}{\alpha}\right)^{j+1} e^{\alpha\delta-(j+1)}$$

we can conclude by fixing $\alpha \in (0,1)$ and reasoning as in the proof of Lemma A.6 that

$$\left|\partial_\rho^p\, \partial_\theta^m\, S\right| \le C\varepsilon^{-p} p! m! \delta^{-p} \delta'^{-m} (2M+2)^{2M+2} K^{2M+2}$$

for appropriately chosen $K > 0$. K is independent of ε and M. □

B Some Technical Lemmas

Lemma B.1 *Let $j \ge 0$. Assume that $\alpha \ge 0$, $\delta > 0$ and $f \in H^0_{\alpha+\delta}$. Then*

$$\|x^j f(x)\|_{0,\alpha} \le \delta^{-j} \left(\frac{j}{e}\right)^j \|f\|_{0,\alpha+\delta}.$$

We allow $\delta = 0$ for the case $j = 0$.

Proof: We write

$$\|x^j f(x)\|^2_{0,\alpha} = \int_0^\infty e^{2\alpha x} x^{2j} f^2(x)\, dx = \int_0^\infty e^{2(\alpha+\delta)x} f^2(x) e^{-2\delta x} x^{2j}\, dx.$$

As the function $x \mapsto x^{2j} e^{-2\delta x}$ attains its maximum at $x = j/\delta$, we get the desired estimate. □

Lemma B.2 *Let $\alpha \in (0,1)$ and $0 \le j < \beta$. Then the function $\delta \mapsto \delta^j (1-\alpha-\delta)^{\beta-j}$ defined on $[0, 1-\alpha]$ attains its maximum at*

$$\delta = (1-\alpha)\frac{j}{\beta},$$

and the maximal value is

$$(1-\alpha)^\beta j^j \beta^{-\beta} (\beta-j)^{\beta-j}.$$

Proof: It is convenient to consider the logarithm of the function, i.e., $\delta \mapsto j \ln\delta + (\beta - j)\ln(1 - \alpha - \delta)$. Computing the zero of the derivative of this function completes the argument. □

Lemma B.3 *Let $\alpha > 0$, $x \ge 0$, and $h \in H^1_\alpha$. Then*

$$\begin{aligned} \|h\|_{L^\infty(x,\infty)} &\le \frac{1}{\sqrt{2\alpha}} e^{-\alpha x} \|h'\|_{0,\alpha}, \\ \left\{\int_x^\infty |h'(t)|^2 + |h(t)|^2\, dt\right\}^{1/2} &\le e^{-\alpha x} \|h\|_{1,\alpha}. \end{aligned}$$

Proof: For the first estimate, we use the fact that for $\alpha > 0$ the function h decays at infinity which produces the representation

$$\begin{aligned} h(x) &= -\int_x^\infty h'(t)\,dt, \qquad x \in [0,\infty), \\ |h(x)| &\leq \left| \int_x^\infty e^{\alpha t} h'(t) e^{-\alpha t}\,dt \right| \leq \frac{1}{\sqrt{2\alpha}} e^{-\alpha x} \|h'\|_{0,\alpha}. \end{aligned}$$

This proves the first estimate. The second one is proved similarly. □

Lemma B.4 *Let $I := [a,b] \subset \mathbb{R}$ be a closed bounded interval, f, g be analytic on I. Then there are C, K_1, $K_2 > 0$ such that*

$$\|D^p f^n g\|_{L^\infty(I)} \leq C p! K_1^n K_2^p \qquad \forall n, p \in \mathbb{N}_0.$$

Moreover, the constant $K_1 > \|f\|_{L^\infty(I)}$ may be chosen arbitrarily close to $\|f\|_{L^\infty(I)}$.

Proof: The claim of the lemma follows by Cauchy's integral formula. As f and g are analytic on the closed interval I, there is a smooth Jordan curve $L \subset \mathbb{C}$ such that I is contained in the interior $\operatorname{Int}(L)$ of the curve L, $\operatorname{dist}(I,L) \geq d > 0$, and the functions f, g are analytic on $\operatorname{Int}(L) \cup L$. For any $z \in I$, Cauchy's integral theorem now yields

$$\begin{aligned} |D^p f^n(z) g(z)| &= \left| \frac{p!}{2\pi i} \int_L \frac{f^n(t) g(t)}{(z-t)^{p+1}} dt \right| \\ &\leq \frac{p!}{2\pi} \operatorname{length}(L) \|f\|^n_{L^\infty(\operatorname{Int}(L))} \|g\|_{L^\infty(\operatorname{Int}(L))} d^{-(p+1)}. \end{aligned}$$

Setting $K_1 = \|f\|_{L^\infty(\operatorname{Int}(L))}$, $K_2 = d^{-1}$, and $C = \operatorname{length}(L)\|g\|_{L^\infty(\operatorname{Int}(L))}/(2\pi d)$ completes the argument. □

Lemma B.5 *Denote ψ the logarithmic derivative of the Γ functions, i.e., $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$. Then*

(i) $\psi(x) \leq \ln x \qquad \forall x > 0$.

(ii) ψ *is monotonically increasing on* $(0,\infty)$.

(iii) $x \mapsto \psi(1+x) + x\psi'(1+x)$ *is monotonically increasing on* $(0,\infty)$.

(iv) $\psi(x) = \frac{1}{2}\psi(\frac{1}{2}x) + \frac{1}{2}\psi(\frac{1}{2}x + \frac{1}{2}) + \ln 2 \qquad \forall x > 0$.

(v) $-\lambda\psi(1+\lambda x) + \psi(1+x) - (1-\lambda)\psi(1+(1-\lambda)x) \leq \ln 2 \qquad \forall \lambda \in [0,1] x \geq 0$.

Proof: (i) follows immediately from 8.361.3 of [9]. (ii) follows immediately from the series representation of ψ' below. (iv) can be found as ex. 2.1 in 2.2.1 of [10]. For (iii), we use the series representations of ψ and ψ' (8.362.1, 8.363.8, 8.365.1 of [9]):

$$\begin{aligned}\psi(x) &= -\gamma - \frac{1}{x} + x\sum_{k=1}^{\infty}\frac{1}{k(x+k)},\\ \psi'(x) &= \sum_{k=0}^{\infty}\frac{1}{(x+k)^2},\\ \psi(1+x) &= \psi(x) + \frac{1}{x}\end{aligned}$$

where γ denotes Euler's constant. Therefore

$$\psi(1+x) + x\psi'(1+x) = -\gamma + \sum_{k=1}^{\infty}\frac{x}{k(k+x)} + \frac{x}{(k+x)^2}.$$

Termwise differentiation (which is justified by the uniform convergence of the termwise differentiated series) yields

$$(\psi(1+x) + x\psi'(1+x))' = \sum_{k=1}^{\infty}\frac{2k}{(k+x)^3} > 0 \qquad \text{for } x \geq 0$$

which proves (iii). Finally, for (v), we note that for fixed $x \geq 0$ the function $h(\lambda) = -\lambda\psi(1+\lambda x) + \psi(1+x) - (1-\lambda)\psi(1+(1-\lambda)x)$ is symmetric with respect to $\lambda = 1/2$. Moreover, on $(0, 1/2)$ the function h is monotonically increasing:

$$h'(\lambda) = -\left[\psi(1+\lambda x) + \lambda x\psi'(1+\lambda x)\right] + \left[\psi(1+(1-\lambda)x) + (1-\lambda)x\psi'(1+(1-\lambda)x)\right] \geq 0$$

the last estimate following from (iii) and the fact that for $\lambda \in (0, 1/2)$ we have $\lambda x \leq (1-\lambda)x$. Therefore, h is maximal for $\lambda = 1/2$. We conclude

$$\begin{aligned}&-\lambda\psi(1+\lambda x) + \psi(1+x) - (1-\lambda)\psi(1+(1-\lambda)x)\\ \leq\ & -\frac{1}{2}\psi(1+x/2) + \psi(1+x) - \frac{1}{2}\psi(1+x/2)\\ \leq\ & -\frac{1}{2}\psi(1+x/2) + \frac{1}{2}\psi(1/2+x/2) + \frac{1}{2}\psi(1+x/2) + \ln 2 - \frac{1}{2}\psi(1+x/2)\\ \leq\ & \ln 2 + \frac{1}{2}\left(\psi(1/2+x/2) - \psi(1+x/2)\right) \leq \ln 2\end{aligned}$$

by (ii) and (iv). □

Lemma B.6 *Let $m \geq 0$ and a, $b \geq 0$ such that $a + b \geq 1$. Then the function*

$$\mu \mapsto \frac{\Gamma(m+1)}{\Gamma(m-\mu+1)}(a+b+m-\mu)^{(a+m-\mu)}$$

is monotonically decreasing on $[0, m]$ and therefore attains its maximum at $\mu = 0$.

Proof: Consider the logarithm of the function in question, i.e.,

$$\begin{aligned} f(\mu) &:= \ln \Gamma(m+1) - \ln\Gamma(m-\mu+1) + (a+m-\mu)\ln(a+b+m-\mu), \\ f'(\mu) &= \psi(m-\mu+1) - \ln(a+b+m-\mu) - \frac{a+m-\mu}{a+b+m-\mu} \\ &\leq \ln(m-\mu+1) - \ln(a+b+m-\mu) - \frac{a+m-\mu}{a+b+m-\mu} \leq 0 \end{aligned}$$

where we used Lemma B.5, (i) and the assumption $a+b \geq 1$. □

Lemma B.7 *Let $p \in \mathbb{N}_0$, $i \in \mathbb{N}$, $a \geq -1$, and b such that $a+b+p \geq 0$. Then the function*

$$f : (\nu, j) \mapsto e^{-(1+\ln 2)j} \binom{p}{\nu} \binom{j}{\nu} \nu! \frac{(i-j+a+b+p)^{i-j+a}}{(i-j)^{i-j}} (i-\nu)^{i-\nu}$$

defined on

$$0 \leq j \leq i-1, \qquad 0 \leq \nu \leq \min(j,p)$$

attains its maximum at $\nu = j = 0$.

Proof: For $\lambda \in [0,1]$ let us consider the function

$$j \mapsto \ln f(\lambda j, j)$$

defined for $j \in [0, \min(i-1, p/\lambda)]$ where we assume that all the factorials in the definition of f are expressed in terms of the Γ function and we set $p/\lambda = \infty$ for $\lambda = 0$. We claim now that this function is monotonically decreasing in j for each λ which proves the claim of the lemma. For each fixed λ we have

$$\begin{aligned} &\ln f(\lambda j, j) = -(1+\ln 2)j + \ln p! - \\ &\quad - \ln\Gamma(p-\lambda j+1) - \ln\Gamma(\lambda j+1) + \ln\Gamma(j+1) - \ln\Gamma((1-\lambda)j+1) + \\ &\quad + (i-j+a)\ln(i-j+a+b+p) + (i-\lambda j)\ln(i-\lambda j) - (i-j)\ln(i-j). \end{aligned}$$

Taking the derivative with respect to j yields

$$\begin{aligned} \frac{d}{dj}\ln f(\lambda j, j) &= -(1+\ln 2) + \lambda\psi(p-\lambda j+1) - \\ &\quad \lambda\psi(\lambda j+1) + \psi(1+j) - (1-\lambda)\psi((1-\lambda)j+1) - \\ &\quad - \ln(i-j+a+b+p) - \frac{i-j+a}{i-j+a+b+p} - \lambda\ln(i-\lambda j) - \lambda + \ln(i-j) + 1. \end{aligned}$$

By Lemma B.5, (i) and (v), and the assumptions $a+b+p \geq 0$, $a \geq -1$, we have

$$\begin{aligned} \frac{d}{dj}\ln f(\lambda j, j) &\leq -(1+\ln 2) + \lambda\ln(p-\lambda j+1) + \ln 2 \\ &\quad - \ln(i-j+a+b+p) - \frac{i-j+a}{i-j+a+b+p} - \lambda\ln(i-\lambda j) - \lambda + \ln(i-j) + 1 \\ &\leq \lambda\ln(p-\lambda j+1) - \lambda\ln(i-\lambda j) - \\ &\quad - \ln(i-j+a+b+p) + \ln(i-j) - \frac{i-j+a}{i-j+a+b+p} \\ &\leq \lambda\ln(p-\lambda j+1) - \lambda\ln(i-\lambda j). \end{aligned}$$

In order to obtain an estimate independent of λ, let us maximize the function

$$g : \lambda \mapsto \lambda \ln(p - \lambda j + 1) - \lambda \ln(i - \lambda j).$$

We obtain for g'

$$g'(\lambda) = \ln(p - \lambda j + 1) - \ln(i - \lambda j) + \lambda j \left(\frac{1}{i - \lambda j} - \frac{1}{p - \lambda j + 1} \right)$$

We see that $g' \geq 0$ for $i \leq p + 1$ and $g' \leq 0$ for $i \geq p + 1$ and hence

$$g \leq \begin{cases} g(1) = \ln(p - j + 1) - \ln(i - j) \leq 0 & \text{if } i \leq p + 1 \\ g(0) = 0 & \text{if } i \geq p + 1. \end{cases}$$

Therefore, $\frac{d}{dj} \ln f(\lambda j, j) \leq 0$ for all $\lambda \in [0, 1]$ which completes the proof. □

Lemma B.8 *Let p, $n \in \mathbb{N}_0$. Then the function*

$$m \mapsto \binom{p}{m} (p + n - m)!$$

defined for integer $m \in [0, p]$ takes its maximum at $m = 0$.

Proof:

$$\binom{p}{m} (p + n - m)! = \frac{p!}{m!} \prod_{\nu=1}^{n} (p - m + \nu)$$

is clearly decreasing as m increases which proves the claim. □

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