# Coercive combined field integral equations

*J. Numer. Math.*, Vol. 11, No. 2, pp. 115–134 (2003)


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*Received 24 March, 2003*

**Abstract** — Many boundary integral equations for exterior Dirichlet and Neumann boundary value problems for the Helmholtz equation suffer from a motorious instability for wave numbers related to interior resonances. The so-called combined field integral equations are not affected.

This article presents combined field integral equations on two-dimensional closed surfaces that possess coercivity in canonical trace spaces. For the exterior Dirichlet problem the main idea is to use suitable regularizing operators in the framework of an indirect method. This permits us to apply the classical convergence theory of conforming Galerkin methods.

**Keywords:** acoustic scattering, indirect boundary integral equations, combined field integral equations (CFIE), coercivity, boundary element methods, Galerkin schemes

## 1. INTRODUCTION

The propagation of time-harmonic sound waves in a homogeneous isotropic medium that occupies the domain $\Omega \subset \mathbb{R}^3$ is governed by the Helmholtz equation, which, in non-dimensional form, reads

$$-\Delta U - \varkappa^2 U = 0. \tag{1.1}$$

Here, $U$ designates the complex amplitude of either the density or of a velocity potential (see Section 2.1 of [5]) and $\varkappa > 0$ stands for a fixed wave number. In acoustic scattering $\Omega$ is the complement of a bounded scatterer $\Omega^-$ and will be denoted by $\Omega^+ := \mathbb{R}^3 \setminus \bar{\Omega}^-$. In this case Sommerfeld radiation conditions (see Definition 9.5 of [11]):

$$\frac{\partial U}{\partial r}(\mathbf{x}) - i\varkappa U(\mathbf{x}) = o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty \tag{1.2}$$

have to be imposed 'at $\infty$', whereas on $\Gamma := \partial\Omega^-$ we prescribe either Dirichlet boundary conditions

$$U = g \quad \text{on } \Gamma \quad \text{for some } g \in H^{1/2}(\Gamma) \tag{1.3}$$

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or Neumann boundary condition

$$\mathbf{grad}\, U \cdot \mathbf{n} = \varphi \quad \text{on } \Gamma \quad \text{for some } \varphi \in H^{-1/2}(\Gamma). \tag{1.4}$$

We take for granted that the boundary $\Gamma$ is Lipschitz continuous. Thus, it will possess an exterior unit normal vectorfield $\mathbf{n} \in L^\infty(\Gamma)$ pointing from $\Omega^-$ into $\Omega^+$ . Numerical approximation in mind, we will even assume that $\Gamma$ is a curvilinear Lipschitz polyhedron in the parlance of [7]. This will cover most geometric arrangements that occur in practical simulations. We emphasize that non-smooth geometries are the main focus of this paper.

It is well known that the above exterior boundary value problems possess unique solutions (see Theorem 9.10 of [11]):

**Theorem 1.1.** *The exterior Dirichlet problem* (1.1) *and* (1.3)*, and the exterior Neumann problem* (1.1) *and* (1.4)*, respectively, for the Helmholtz equation have at most one solution satisfying the Sommerfeld radiation conditions* (1.2).

Integral equation methods are particularly suited for the numerical treatment of exterior scattering problems, because they reduce the problem to equations on the bounded surface $\Gamma$. A variety of schemes is conceivable, among them direct and indirect methods. However, those that can be derived from an integral representation formula for Helmholtz solutions in a straightforward fashion display a worrisome instability: if $\varkappa^2$ agrees with a Dirichlet or Neumann eigenvalue (resonant frequency) of the Laplacian in $\Omega^-$, then the integral equations fail to possess a unique solution. In light of Theorem 1.1 this may be called a spurious resonance phenomenon.

Spurious resonances are particularly distressing for numerical procedures based on the integral equations, because whenever $\varkappa^2$ is close to an interior resonant frequency the resulting linear systems of equations will be extremely ill-conditioned. A wonderful remedy is offered by the Combined Field Integral Equations (CFIE), which owe their name to the presence of both single layer and double layer potential in the trial expression for the Helmholtz solution. This trick was independently be discovered by Brakhage and Werner [1], Leis [10], and Panich [12] in 1965. Since then, it has become the foundation for numerous numerical methods in direct and inverse acoustic and electromagnetic scattering (see Chapters 3 and 6 of [5]).

In terms of mathematical analysis many combined field integral equations are challenging. This is particularly true for non-smooth surfaces, for which the double layer integral operator is no longer a compact perturbation of the identity in $L^2(\Gamma)$. Thus, in the case of the exterior Dirichlet problem, Fredholm theory can no longer be used to settle the issue of existence and uniqueness of solutions of the traditional CFIE. Hence, modified CFIE involving a regularizing operator have been suggested for theoretical purposes [5,12].

Many options are available for the discretization of combined field integral equations. We will only consider Galerkin schemes, because they seem to be the only approach amenable to a rigorous theoretical treatment so far. However, the very lack of coercivity of combined field integral equations mentioned above turns out to be a major obstacle to obtaining convergence results for Galerkin methods.

Hence, in this paper we also take the cue from the idea to introduce regularizing operators. We derive variational formulations that are coercive in natural trace spaces, which guarantees asymptotically quasi-optimal convergence of Galerkin boundary element solutions.

## 2. COERCIVITY

In this section we briefly review the abstract theory of coercive bilinear forms and its implications for Galerkin discretization. In general these results are well known (cf. Chapter 2 of [11]), but they will be supplied for the sake of completeness. Below $V$ stands for a reflexive Banach space over the field $\mathbb{C}$. This space has to support an involutory, anti-linear mapping $\bar{\ } : V \mapsto V$ (related to complex conjugation). By $V'$ we denote the dual space, and by $\langle \cdot, \cdot \rangle_{V' \times V}$ the duality pairing.

Let $d : V \times V \mapsto \mathbb{C}$ be a bilinear form, which is supposed to feature

- continuity, that is $\quad \exists C > 0 : \quad |d(u,v)| \leqslant C \|u\|_V \|v\|_V \quad \forall u,v \in V;$ (2.1)
- $V$-ellipticity, that is $\quad \exists c > 0 : \quad |d(u,\bar{u})| \geqslant c \|u\|_V^2 \quad \forall u \in V.$ (2.2)

Therefore, we can associate a bounded operator $D : V \mapsto V'$ to $d(\cdot,\cdot)$ by

$$\langle Du, v \rangle_{V' \times V} := d(u,v) \quad \forall u,v \in V.$$

**Lemma 2.1.** *If there is a continuous and $V$-elliptic bilinear form $d$ on $V$, then $\|v\|_V = \|\bar{v}\|_V$ for all $v \in V$.*

**Proof.** As ‘complex conjugation’ is an involution, we have $\bar{\bar{u}} = u$. Thus

$$c \|\bar{u}\|_V^2 \leqslant |d(\bar{u},\bar{\bar{u}})| = |d(\bar{u},u)| \leqslant C \|\bar{u}\|_V \|u\|_V .$$

This means $\|\bar{u}\|_V \leqslant C/c \|u\|_V$, which implies $\|v\|_V = \|\bar{v}\|_V$. □

**Theorem 2.1.** *Given the above properties* (2.1) *and* (2.2) *of $d(\cdot,\cdot)$, the operator $D$ is an isomorphism.*

**Proof.** By the definition of the norm in $V'$ we have

$$\|Du\|_{V'} = \sup_{v \neq 0} \frac{|d(u,v)|}{\|v\|_V} \geqslant \frac{|d(u,\bar{u})|}{\|\bar{u}\|_V} \geqslant c \|u\|_V \quad \forall u \in V.$$

This implies that $D$ is injective and has closed range. Assume that $D(V) \neq V'$. Since $D(V) \subset V'$ is closed, the Hahn-Banach theorem confirms the existence of $v^* \in V'' = V$, $v^* \neq 0$, such that $\langle Du, v^* \rangle_{V' \times V} = 0$ for all $u \in V$. In particular $d(\bar{v}^*, v^*) = 0$, which yields a contradiction. Altogether, $D$ has to be surjective. □

**Definition 2.1.** A bilinear form $a : V \times V \mapsto \mathbb{C}$ is called coercive, if it satisfies a Gårding-type inequality

$$\exists c > 0 : \quad |a(u,\bar{u}) + \langle Ku, \bar{u}\rangle_{V'\times V}| \geqslant c\,\|u\|_V^2 \quad \forall u \in V$$

with a compact operator $K : V \mapsto V'$.

**Theorem 2.2.** *The operator $A : V \mapsto V'$ associated with a continuous bilinear form $a : V \times V \mapsto \mathbb{C}$ through $\langle Au, v\rangle_{V'\times V} = a(u,v)$, $u,v \in V$, is Fredholm of index zero.*

**Proof.** Set

$$d(u,v) := a(u,v) + \langle Ku, v\rangle_{V'\times V}, \quad u,v \in V.$$

It is clear that the bilinear form $d$ is continuous. By Theorem 2.1 and (1.1) its associated operator $D : V \mapsto V'$ is an isomorphism. By definition of $d$ we have

$$D = A + K \iff A = D - K.$$

Hence, $A$ is a compact perturbation of an isomorphism. According to Theorem 2.26 of [11] This implies that $A$ is Fredholm of index 0. □

**Lemma 2.2.** *If $a : V \times V \mapsto \mathbb{C}$ is a continuous coercive bilinear form for which $a(u,v) = 0$ for all $v \in V$ implies $u = 0$, then there is $c_s > 0$ such that*

$$\sup_{v\in V} \frac{|a(u,v)|}{\|v\|_V} \geqslant c_s\,\|u\|_V, \quad \sup_{v\in V} \frac{|a(v,u)|}{\|v\|_V} \geqslant c_s\,\|u\|_V \quad \forall u \in V.$$

**Proof.** The assumption of the theorem means that the operator $A : V \mapsto V$ related to $a(\cdot,\cdot)$ is injective. By Theorem 2.2 $A$ is bijective and the inf-sup conditions are a consequence of the open mapping theorem and of the fact that the norms of an operator and of its adjoint agree (see Theorem 4.15 of [13]). □

Next, we consider a sequence of closed subspaces $V_n \subset V$, $n \in \mathbb{N}$. The $V_n$ must be stable under conjugation. We assume that there is an associated sequence of bounded linear operators $P_n : V \mapsto V_n$ that converges to zero strongly, i.e.,

$$\forall u \in V : \quad \lim_{n\to\infty} \|u - P_n u\|_V = 0. \tag{2.3}$$

If $V$ is a Hilbert space and $\{V_n\}_{n\in\mathbb{N}}$ is a family of nested finite-dimensional subspaces such that $\bigcup_n V_n \subset V$ is dense, then $P_n$ can be chosen as orthogonal projection onto $V_n$.

Now, we consider the variational problem

$$u \in V : \quad a(u,v) = \langle \varphi, v\rangle_{V'\times V} \quad \forall v \in V \tag{2.4}$$

with $\varphi \in V'$. For the remainder of this section, $u$ will always stand for its solution.

The following theorem is the main tool in proving convergence for conforming Galerkin approximations of coercive variational problems. A first version was discovered by A. Schatz [14] (see also [16]).

**Theorem 2.3.** *If the bilinear form $a : V \times V \mapsto \mathbb{C}$ is coercive, continuous, and injective (i.e. $a(u,v) = 0$ for all $v \in V$ implies $u = 0$), then there is an $N \in \mathbb{N}$ such that the variational problems*

$$u_h \in V_n : \quad a(u_h, v_h) = \langle \varphi, v_h \rangle_{V' \times V} \quad \forall v_h \in V_n$$

*have unique solutions $u_h \in V_n$ for all $n > N$. Those are asymptotically quasi-optimal in the sense that there is a constant $C > 0$ such that*

$$\|u - u_h\|_V \leqslant C \inf_{v_h \in V_n} \|u - v_h\|_V .$$

**Proof.** We define the operator $S : V \mapsto V$ by

$$a(v, S\bar{w}) = \langle Kw, \bar{v} \rangle_{V' \times V} \quad \forall v \in V.$$

Please note that Lemma 2.2 guarantees the existence of $A^{-1}$. Also by Lemma 2.2 $S$ is continuous and we find $S = (A^*)^{-1}\bar{K}$. Hence, $S$ inherits compactness from $K$. Remember that compact operators convert strong convergence into uniform convergence (see Corollary 10.4 of [9]), which means

$$\lim_{n \to \infty} \|(P_n - I)S\|_V = 0. \tag{2.5}$$

Pick some $u_h \in V_n$ and estimate

$$\begin{aligned} |a(u_h, (Id + P_n S)\bar{u}_h)| &\geqslant |a(u_h, (Id + S)\bar{u}_h)| - |a(u_h, (P_n - Id)S\bar{u}_h)| \\ &\geqslant |a(u_h, \bar{u}_h) + \langle Ku_h, \bar{u}_h \rangle_{V' \times V}| - \|a\| \, \|(P_n - Id)S\|_V \, \|u_h\|_V^2 \\ &\geqslant (c_G - \|a\| \, \|(P_n - Id)S\|_V) \, \|u_h\|_V^2 . \end{aligned}$$

Thanks to (2.5) it is possible to choose $N \in \mathbb{N}$ such that $\|a\| \, \|(P_n - Id)S\|_V < c_G/2$ for all $n > N$. Then, with $v_h := (Id + P_n S)\bar{u}_h \in V_n$,

$$|a(u_h, v_h)| \geqslant \tfrac{1}{2} c_G \|u_h\|_V^2 .$$

Making use of the (uniform) continuity of $P_n$ and $S$, this yields the inf-sup condition

$$\sup_{v_h \in V_n} \frac{|a(u_h, v_h)|}{\|v_h\|_V} \geqslant c_d \|u_h\|_V \quad \forall u_h \in V_n, \; n > N. \tag{2.6}$$

Using (2.6) and Galerkin orthogonality we get for any $v_h \in V_n$, $n > N$,

$$\begin{aligned}\|u-u_h\|_V &\leqslant \|u-v_h\|_V + \|v_h-u_h\|_V \\ &\leqslant \|u-v_h\|_V + \frac{1}{c_d}\sup_{w_h\in V_n}\frac{|a(v_h-u_h,w_h)|}{\|v_h\|_V} \\ &\leqslant \left(1+\frac{\|a\|}{c_d}\right)\|u-v_h\|_V .\end{aligned}$$

This is the asserted asymptotic quasi-optimality with $C := 1 + \|a\|/c_d$. □

## 3. BOUNDARY INTEGRAL OPERATORS

In this section we review important properties of boundary integral operators related to Helmholtz' equation. The main reference is the textbook [11] and the pioneering work by M. Costabel [6].

Without further explanation we will use Sobolev spaces $H^s$, $s \in \mathbb{R}$ , on domains and boundaries, in particular $H^1(\Omega)$, $H^{1/2}(\Gamma)$, and $H^{-1/2}(\Gamma)$ (cf. Chapter 2 of [11]). Here, we merely recall the definition of the Sobolev–Slobodeckij norm

$$\|u\|^2_{H^{1/2}(\Gamma)} := \|u\|^2_{L^2(\Gamma)} + |u|^2_{H^{1/2}(\Gamma)}, \quad |u|^2_{H^{1/2}(\Gamma)} := \int_\Gamma\int_\Gamma \frac{(u(\mathbf{x})-u(\mathbf{y}))^2}{|\mathbf{x}-\mathbf{y}|^2}\,\mathrm{d}S(\mathbf{x},\mathbf{y}).$$

The corresponding Frechet spaces on unbounded domains will be tagged by a subscript loc, e.g. $H^1_{\mathrm{loc}}(\Omega)$. Their associated dual spaces will carry the subscript 'comp' to illustrate that they contain compactly supported distributions.

Writing

$$H(\Delta,\Omega) := \{U \in H^1_{\mathrm{loc}}(\Omega),\, \Delta U \in L^2_{\mathrm{loc}}(\Omega)\}$$

for the domain of the Laplacian, we have continuous and surjective trace operators (cf. Lemma 3.2 of [6]):

$$\begin{aligned}\text{Dirichlet trace} \quad & \gamma_D : H^1_{\mathrm{loc}}(\Omega) \mapsto H^{1/2}(\Gamma) \\ \text{Neumann trace} \quad & \gamma_N : H(\Delta,\Omega) \mapsto H^{-1/2}(\Gamma)\end{aligned}$$

that generalize the restrictions for smooth $U \in C^\infty(\bar{\Omega})$

$$(\gamma_D U)(\mathbf{x}) = U(\mathbf{x}), \quad (\gamma_N U)(\mathbf{x}) = \mathbf{grad}\, U(\mathbf{x})\cdot\mathbf{n}(\mathbf{x}), \quad \mathbf{x}\in\Gamma$$

respectively.

So far $\Omega \subset \mathbb{R}^3$ has been a generic domain. Returning to our particular setting, superscripts '+' and '-' will tag traces from $\Omega^-/\Omega^+$. Jumps are defined as

$$[\gamma_D U]_\Gamma = \gamma_D^+ U - \gamma_D^- U, \qquad [\gamma_N U]_\Gamma = \gamma_N^+ U - \gamma_N^- U.$$

Averages are denoted by

$$\{\gamma_D U\}_\Gamma = \tfrac{1}{2}(\gamma_D^+ U + \gamma_D^- U), \qquad \{\gamma_N U\}_\Gamma = \tfrac{1}{2}(\gamma_N^+ U + \gamma_N^- U).$$

We recall that the bilinear symmetric pairing

$$\langle \varphi, v \rangle_\Gamma := \int_\Gamma uv \, \mathrm{d}S, \qquad \varphi, v \in L^2(\Gamma)$$

can be extended to the duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. Thanks to the definition of the Neumann trace we have the integration by parts formulas

$$\int_{\Omega^-} \mathbf{grad}\, U \cdot \mathbf{grad}\, V + \Delta U\, V \, \mathrm{d}\mathbf{x} = \langle \gamma_N^- U, \gamma_D^- V \rangle_\Gamma \tag{3.1}$$

$$-\int_{\Omega^+} \mathbf{grad}\, U \cdot \mathbf{grad}\, V + \Delta U\, V \, \mathrm{d}\mathbf{x} = \langle \gamma_N^+ U, \gamma_D^+ V \rangle_\Gamma \tag{3.2}$$

for $U \in H_{\mathrm{loc}}(\Delta, \Omega^\pm)$, $V \in H^1_{\mathrm{loc}}(\Omega^\pm)$. We will also need spaces with 'vanishing average'

$$H_*^{1/2}(\Gamma) := \{u \in H^{1/2}(\Gamma), \langle \mathbf{1}, u \rangle_\Gamma = 0\}$$
$$H_*^{-1/2}(\Gamma) := \{\varphi \in H^{-1/2}(\Gamma), \langle \varphi, \mathbf{1} \rangle_\Gamma = 0\}$$

where $\mathbf{1} \in H^{1/2}(\Gamma)$ means the constant function $\equiv 1$ on $\Gamma$, whereas $\mathbf{1} \in H^{-1/2}(\Gamma)$ refers to the functional $v \mapsto \int_\Gamma v \, \mathrm{d}S$.

**Lemma 3.1.** *The spaces $H_*^{1/2}(\Gamma)$ and $H_*^{-1/2}(\Gamma)$ are dual to each other with respect to the pairing $\langle \cdot, \cdot \rangle_\Gamma$.*

**Proof.** For $w \in H^{1/2}(\Gamma)$ denote by $w^*$ the average $w^* := \int_\Gamma w \, \mathrm{d}S \cdot \mathbf{1}$. We point out that

$$\|w - w^*\|^2_{H^{1/2}(\Gamma)} = \|w\|^2_{L^2(\Gamma)} - \|w^*\|^2_{L^2(\Gamma)} + |w|^2_{H^{1/2}(\Gamma)} \leqslant \|w\|^2_{H^{1/2}(\Gamma)} .$$

Therefore, for $\varphi \in H_*^{-1/2}(\Gamma)$

$$\begin{aligned} \|\varphi\|_{H^{-1/2}(\Gamma)} &= \sup_{w \in H^{1/2}(\Gamma)} \frac{|\langle \varphi, w \rangle_\Gamma|}{\|w\|_{H^{1/2}(\Gamma)}} \leqslant \sup_{w \in H^{1/2}(\Gamma)} \frac{|\langle \varphi, w - w^* \rangle_\Gamma|}{\|w - w^*\|_{H^{1/2}(\Gamma)}} \\ &= \sup_{w \in H_*^{1/2}(\Gamma)} \frac{|\langle \varphi, w \rangle_\Gamma|}{\|w\|_{H^{1/2}(\Gamma)}} . \end{aligned}$$

This amounts to the assertion of the theorem. □

For fixed wavenumber $\varkappa > 0$ a distribution $U$ is called a *radiating Helmholtz solution*, if

$$\begin{aligned} \Delta U + \varkappa^2 U &= 0 \quad \text{in } \Omega^- \cup \Omega^+ \\ \frac{\partial U}{\partial r}(\mathbf{x}) - i\varkappa U(\mathbf{x}) &= o(r^{-1}) \quad \text{uniformly as } r := |\mathbf{x}| \to \infty. \end{aligned} \tag{3.3}$$

Based on the Helmholtz kernel

$$\Phi_\varkappa(\mathbf{x},\mathbf{y}) := \frac{\exp(i\varkappa|\mathbf{x}-\mathbf{y}|)}{4\pi|\mathbf{x}-\mathbf{y}|}$$

we can state the transmission formula for radiating Helmholtz solution $U$ (see Theorem 6.10 of [11]):

$$U = -\Psi^\varkappa_{\mathrm{SL}}([\gamma_N U]_\Gamma) + \Psi^\varkappa_{\mathrm{DL}}([\gamma_D U]_\Gamma) \tag{3.4}$$

with potentials

$$\begin{aligned} \text{single layer potential:} \quad & \Psi^\varkappa_{\mathrm{SL}}(\lambda)(\mathbf{x}) = \int_\Gamma \Phi_\varkappa(\mathbf{x},\mathbf{y})\lambda(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \\ \text{double layer potential:} \quad & \Psi^\varkappa_{\mathrm{DL}}(u)(\mathbf{x}) = \int_\Gamma \frac{\partial \Phi_\varkappa(\mathbf{x},\mathbf{y})}{\partial \mathbf{n}(\mathbf{y})} u(\mathbf{y})\,\mathrm{d}S(\mathbf{y}). \end{aligned}$$

The potentials themselves provide radiating Helmholtz solutions, that is

$$(\Delta + \varkappa^2)\Psi^\varkappa_{\mathrm{SL}} = 0, \qquad (\Delta + \varkappa^2)\Psi^\varkappa_{\mathrm{DL}} = 0 \quad \text{in } \Omega^- \cup \Omega^+. \tag{3.5}$$

Moreover, they describe continuous mappings (see Theorem 6.12 of [11]):

$$\begin{aligned} \Psi^\varkappa_{\mathrm{SL}} &: H^{-1/2}(\Gamma) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3) \cap H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+) \\ \Psi^\varkappa_{\mathrm{DL}} &: H^{1/2}(\Gamma) \mapsto H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+). \end{aligned}$$

This means that we can apply the trace operators to the potentials. This will yield the following four continuous boundary integral operators (cf. Theorem 7.1 of [11] and [8]):

$$\begin{aligned} &\mathsf{V}_\varkappa : H^s(\Gamma) \mapsto H^{s+1}(\Gamma), && -1 \leqslant s \leqslant 0, && \mathsf{V}_\varkappa := \{\gamma_D \Psi^\varkappa_{\mathrm{SL}}\}_\Gamma \\ &\mathsf{K}_\varkappa : H^s(\Gamma) \mapsto H^s(\Gamma), && 0 \leqslant s \leqslant 1, && \mathsf{K}_\varkappa := \{\gamma_D \Psi^\varkappa_{\mathrm{DL}}\}_\Gamma \\ &\mathsf{K}^*_\varkappa : H^s(\Gamma) \mapsto H^s(\Gamma), && -1 \leqslant s \leqslant 0, && \mathsf{K}^*_\varkappa := \{\gamma_N \Psi^\varkappa_{\mathrm{SL}}\}_\Gamma \\ &\mathsf{D}_\varkappa : H^s(\Gamma) \mapsto H^{s-1}(\Gamma), && 0 \leqslant s \leqslant 1, && \mathsf{D}_\varkappa := -\{\gamma_N \Psi^\varkappa_{\mathrm{DL}}\}_\Gamma. \end{aligned}$$

By the *jump relations* (see Theorem 6.11 of [11]):

$$\begin{aligned} &[\gamma_D \Psi^\varkappa_{\mathrm{SL}}(\lambda)]_\Gamma = 0, && [\gamma_N \Psi^\varkappa_{\mathrm{SL}}(\lambda)]_\Gamma = -\lambda && \forall \lambda \in H^{-1/2}(\Gamma) \\ &[\gamma_D \Psi^\varkappa_{\mathrm{DL}}(u)]_\Gamma = u, && [\gamma_N \Psi^\varkappa_{\mathrm{DL}}(u)]_\Gamma = 0 && \forall u \in H^{1/2}(\Gamma) \end{aligned}$$

we find

$$
\begin{aligned}
\gamma_D^- \Psi_{\mathrm{DL}}^{\varkappa} &= \mathsf{K}_{\varkappa} - \tfrac{1}{2} Id, \qquad & \gamma_D^+ \Psi_{\mathrm{DL}}^{\varkappa} &= \mathsf{K}_{\varkappa} + \tfrac{1}{2} Id \\
\gamma_N^- \Psi_{\mathrm{SL}}^{\varkappa} &= \mathsf{K}_{\varkappa}^* + \tfrac{1}{2} Id, \qquad & \gamma_N^+ \Psi_{\mathrm{SL}}^{\varkappa} &= \mathsf{K}_{\varkappa}^* - \tfrac{1}{2} Id.
\end{aligned} \tag{3.6}
$$

Besides, the Newton potential

$$
(\mathsf{N}_{\varkappa} f)(\mathbf{x}) = \int_{\mathbb{R}^3} \varphi_{\varkappa}(\mathbf{x}, \mathbf{y})\, f(\mathbf{y})\, \mathrm{d}\mathbf{y}, \qquad \mathsf{N}_{\varkappa} : H^{-1}_{\mathrm{comp}}(\mathbb{R}^3) \mapsto H^1_{\mathrm{loc}}(\mathbb{R}^3)
$$

can be used to get the concise representations

$$\mathsf{V}_{\varkappa} = \gamma_D \circ \mathsf{N}_{\varkappa} \circ \gamma_D^* \tag{3.7}$$
$$\mathsf{K}_{\varkappa} = \{\gamma_D\}_{\Gamma} \circ \mathsf{N}_{\varkappa} \circ \gamma_N^* \tag{3.8}$$
$$\mathsf{K}^*_{\varkappa} = \{\gamma_N\}_{\Gamma} \circ \mathsf{N}_{\varkappa} \circ \gamma_D^* \tag{3.9}$$
$$\mathsf{D}_{\varkappa} = \gamma_N \circ \mathsf{N}_{\varkappa} \circ \gamma_N^*. \tag{3.10}$$

Here, an '$*$' labels the dual adjoint operator. These expressions immediately show the symmetries (see Theorems 6.15 and 6.17 of [11]):

$$\langle \psi, \mathsf{V}_{\varkappa}\varphi \rangle_{\Gamma} = \langle \varphi, \mathsf{V}_{\varkappa}\psi \rangle_{\Gamma} \quad \forall \varphi, \psi \in H^{-1/2}(\Gamma) \tag{3.11}$$
$$\langle \varphi, \mathsf{K}_{\varkappa} u \rangle_{\Gamma} = \langle \mathsf{K}^*_{\varkappa}\varphi, u \rangle_{\Gamma} \quad \forall \varphi \in H^{-1/2}(\Gamma), u \in H^{1/2}(\Gamma) \tag{3.12}$$
$$\langle \mathsf{D}_{\varkappa} u, v \rangle_{\Gamma} = \langle \mathsf{D}_{\varkappa} v, u \rangle_{\Gamma} \quad \forall u, v \in H^{1/2}(\Gamma). \tag{3.13}$$

Crucial will be the ellipticity of boundary integral operators in the natural trace norms (see Corollary 8.13 and Theorem 8.21 of [11]):

$$\langle \bar{\varphi}, \mathsf{V}_0 \varphi \rangle_{\Gamma} \geqslant c_V \, \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma) \tag{3.14}$$
$$\langle \mathsf{D}_0 v, \bar{v} \rangle_{\Gamma} \geqslant c_D \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}_*(\Gamma). \tag{3.15}$$

Therefore, $\mathsf{V}_0 : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ and $\mathsf{D}_0 : H^{1/2}_*(\Gamma) \mapsto H^{-1/2}_*(\Gamma)$ are isomorphisms and we conclude that for all $\varphi \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$

$$\|\mathsf{V}_0 \varphi\|_{H^{1/2}(\Gamma)} \approx \|\varphi\|_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}(\Gamma) \tag{3.16}$$
$$\langle \mathsf{V}_0^{-1} v, \overline{v} \rangle_{\Gamma} \geqslant \widetilde{c}_V \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma) \tag{3.17}$$
$$\|\mathsf{D}_0 v\|_{H^{-1/2}_*(\Gamma)} \approx \|v\|_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}_*(\Gamma) \tag{3.18}$$
$$\langle \varphi, \mathsf{D}_0^{-1} \overline{\varphi} \rangle_{\Gamma} \geqslant \widetilde{c}_D \, \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}_*(\Gamma). \tag{3.19}$$

Here, $\approx$ designates equality up to constants that only depend on $\Gamma$.

From Theorem 9.15 of [11] we learn that for $u, v \in H^1(\Gamma)$

$$\langle \mathsf{D}_\varkappa u, v \rangle_\Gamma = \langle \mathsf{V}_\varkappa \mathbf{curl}_\Gamma u, \mathbf{curl}_\Gamma v \rangle_\Gamma - \varkappa^2 \langle \mathsf{V}_\varkappa (u \cdot \mathbf{n}), v \cdot \mathbf{n} \rangle_\Gamma \tag{3.20}$$

where $\mathsf{V}_\varkappa$ has to be read as vectorial single layer potential, and $\mathbf{curl}_\Gamma : H^1(\Gamma) \mapsto L_\mathbf{t}(\Gamma)$ is the surface rotation, which agrees with rotated surface gradient. It can be extended to a mapping $\mathbf{curl}_\Gamma : H^{1/2}(\Gamma) \mapsto (H^{-1/2}(\Gamma))^3$ (see [3]).

**Lemma 3.2.** *The operators* $\mathsf{V}_\varkappa - \mathsf{V}_0 : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$, $\mathsf{K}_\varkappa - \mathsf{K}_0 : H^{1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$, *and* $\mathsf{D}_\varkappa - \mathsf{D}_0 : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$ *are compact.*

**Proof.** Note that $\widetilde{\Phi}(r) := \exp(i\varkappa r) - 1/4\pi r$ is an analytic function on $\mathbb{R}$. Therefore the integral operator

$$(\widetilde{\mathsf{N}}_\varkappa f)(\mathbf{x}) := \int_{\mathbb{R}^3} \widetilde{\Phi}(|\mathbf{x} - \mathbf{y}|) f(\mathbf{y}) \, d\mathbf{y}$$

has a continuous kernel with bounded derivatives and weakly singular second derivatives. This means that $\widetilde{\mathsf{N}}_\varkappa$ is an operator of order $+4$, continuous $\widetilde{\mathsf{N}} : H^{-2}_{\mathrm{comp}}(\mathbb{R}^3) \mapsto H^2_{\mathrm{loc}}(\mathbb{R}^3)$. Therefore we conclude the continuity of

$$\mathsf{V}_\varkappa - \mathsf{V}_0 = \gamma_D \circ \widetilde{\mathsf{N}}_\varkappa \circ \gamma_D : H^{-1/2}(\Gamma) \mapsto H^1(\Gamma).$$

The compact embedding $H^1(\Gamma) \hookrightarrow H^{1/2}(\Gamma) \hookrightarrow H^{-1/2}(\Gamma)$ finishes the proof of the first assertion.

To confirm the second, we point out that

$$\gamma_N^* : H^{1/2}(\Gamma) \mapsto H^{-2}_{\mathrm{comp}}(\Omega^- \cup \Omega^+)$$

is continuous due to the continuous embedding $H^2_{\mathrm{loc}}(\Omega^- \cup \Omega^+) \subset H_{\mathrm{loc}}(\Delta, \Omega^- \cup \Omega^+)$. Then, the identity

$$\mathsf{K}_\varkappa - \mathsf{K}_0 = \{\gamma_D\}_\Gamma \circ \widetilde{\mathsf{N}}_\varkappa \circ \gamma_N^*$$

combined with the compact embedding $H^2_{\mathrm{loc}}(\Omega^- \cup \Omega^+) \subset H^1_{\mathrm{loc}}(\Omega^- \cup \Omega^+)$ gives the result.

To confirm the assertion for the hypersingular operator, we appeal to the formula (3.20) and the compactness of $\mathsf{V}_\varkappa - \mathsf{V}_0$ that carries over to the vectorial single layer potential operator. Further the multiplication with $\mathbf{n}$ is an isometry $L^2(\Gamma) \mapsto L^2(\Gamma)$ such that the second term in (3.20) is readily seen to be a compact perturbation. $\square$

## 4. CLASSICAL CFIE

We recall that indirect methods start from a potential representation for (exterior) radiating Helmholtz solutions in $\Omega^+$. By virtue of (3.5) we may set

$$U = \Psi^{\varkappa}_{\mathrm{SL}}(\varphi),\ \varphi \in H^{-1/2}(\Gamma) \quad \text{or} \quad U = \Psi^{\varkappa}_{\mathrm{DL}}(u),\ u \in H^{1/2}(\Gamma). \tag{4.1}$$

Applying $\gamma_D^+$ to (3.6) we obtain the following integral equations for the exterior Dirichlet problem:

$$\mathsf{V}_{\varkappa}(\varphi) = g \quad \text{or} \quad (\mathsf{K}_{\varkappa} + \tfrac{1}{2}Id)u = g.$$

Similarly, the resulting boundary integral equations for the Neumann problem are

$$(\mathsf{K}^*_{\varkappa} - \tfrac{1}{2}Id)\varphi = \varphi \quad \text{or} \quad -\mathsf{D}_{\varkappa}\varphi = \varphi.$$

However, these boundary integral equations are haunted by the problem of 'resonant frequencies' (see Section 7.7 of [4]): if $\varkappa^2$ is a Dirichlet eigenvalue of $-\Delta$ in $\Omega^-$, then the Neumann traces of the corresponding eigenfunctions will belong to the kernel of $\mathsf{V}_{\varkappa}$ and $\mathsf{K}^*_{\varkappa} - \frac{1}{2}Id$. Conversely, if $\varkappa^2$ is a Neumann eigenvalue, the Dirichlet traces of the eigenfunctions form the kernel of $\mathsf{D}_{\varkappa}$ and $\mathsf{K}_{\varkappa} + \frac{1}{2}Id$. This fact destroys injectivity of the operators in the boundary integral equations and bars us from applying the powerful Fredholm theory outlined in Section 2.

As pointed out in the introduction, this awkward situation led to the development of the classical combined field integral equations (cf. Section 3.2 of [5]). They can be obtained from an indirect approach starting from the trial expression

$$U = \Psi^{\varkappa}_{\mathrm{DL}}(u) + i\eta\Psi^{\varkappa}_{\mathrm{SL}}(u) \tag{4.2}$$

with real $\eta \neq 0$. Applying the exterior Dirichlet trace results in the boundary integral equation

$$g = \left(\tfrac{1}{2}Id + \mathsf{K}_{\varkappa}\right)u + i\eta\mathsf{V}_{\varkappa}u \tag{4.3}$$

whereas the exterior Neumann problem leads to

$$\varphi = -\mathsf{D}_{\varkappa}u + i\eta\left(\mathsf{K}^*_{\varkappa} - \tfrac{1}{2}Id\right)u. \tag{4.4}$$

To begin with, we discuss (4.4) and set

$$\mathsf{C}_{\varkappa} := -\mathsf{D}_{\varkappa} + i\eta\left(\mathsf{K}^*_{\varkappa} - \tfrac{1}{2}Id\right).$$

**Lemma 4.1.** *The operator* $\mathsf{C}_{\varkappa} : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$ *is injective.*

**Proof.** Let $u \in H^{1/2}(\Gamma)$ be a solution of $\mathsf{C}_{\varkappa} u = 0$. Then $U$ given by (4.2) is a Helmholtz solution that satisfies $\gamma_N^+ U = 0$. Thus, the unique solvability of the exterior Neumann problem according to Theorem 1.1 enforces $U = 0$ in $\Omega^+$.

By the jump conditions we conclude

$$\gamma_D^- U = -u, \qquad \gamma_N^- U = \mathrm{i}\eta u.$$

As a consequence of the integration by parts formula

$$\mathrm{i}\eta \int_\Gamma |u|^2 \,\mathrm{d}S = \langle \gamma_N^- U, \gamma_D^- \bar{U} \rangle_\Gamma = \int_{\Omega^-} |\mathbf{grad}\, U|^2 - \varkappa^2 |U|^2 \,\mathrm{d}\mathbf{x}.$$

Since $\eta \in \mathbb{R} \setminus \{0\}$, this involves $u = 0$. □

The equation (4.4) is set in the space $H^{-1/2}(\Gamma)$. Hence, the natural test space is $H^{1/2}(\Gamma)$, which perfectly matches the space for the unknown $u$. We arrive at the variational problem: find $u \in H^{1/2}(\gamma)$ with

$$\langle \mathsf{C}_{\varkappa} u, v \rangle_\Gamma = \langle \varphi, v \rangle_\Gamma \quad \forall v \in H^{1/2}(\Gamma). \tag{4.5}$$

The next result shows that the assumptions of the abstract theory of Section 2 is satisfied for (4.5).

**Lemma 4.2.** *The bilinear form* $\langle \mathsf{C}_{\varkappa} \cdot, \cdot \rangle_\Gamma : H^{1/2}(\Gamma) \times H^{1/2}(\Gamma) \mapsto \mathbb{C}$ *is coercive.*

**Proof.** We can split

$$\langle \mathsf{C}_{\varkappa} u, v \rangle_\Gamma = -\langle \mathsf{D}_0 u, v \rangle_\Gamma + \langle (\mathsf{D}_0 - \mathsf{D}_{\varkappa}) u, v \rangle_\Gamma + \mathrm{i}\eta \left\langle \left( \mathsf{K}_{\varkappa}^* - \tfrac{1}{2} Id \right) u, v \right\rangle_\Gamma.$$

The last term is compact since $\mathsf{K}_{\varkappa}^* - \frac{1}{2} Id : L^2(\Gamma) \mapsto L^2(\Gamma)$ is continuous and the embedding $H^{1/2}(\Gamma) \hookrightarrow L^2(\Gamma)$ is compact. The second term is compact by Lemma 3.2. The $H_*^{1/2}(\Gamma)$-ellipticity of the first term according to (3.15) completes the proof.
□

Summing up, we conclude existence and uniqueness of solutions of (4.4). In addition we get asymptotic quasi-optimality for any conforming Galerkin boundary element discretization. The discussion of actual convergence will be postponed until Section 6.

The situation is much worse in the case of the exterior Dirichlet problem and the associated CFIE (4.3). Actually, the equation is set in $H^{1/2}(\Gamma)$ and the density $u$ should be sought in $H^{-1/2}(\Gamma)$. For obvious reasons, this is not possible, unless we use a pairing in $H^{-1/2}(\Gamma)$ to convert the equation into weak form. Yet, this will introduce products of non-local operators, which render the equations unsuitable for numerical purposes. The fundamental difficulty is that, unlike in the case of the exterior Neumann problem, we cannot use matching trial and test spaces, because the potentials involved in (4.1) require arguments with different regularity. What remains is to lift the equation (4.3) into $L^2(\Gamma)$ and seek the unknown density $u$ in $L^2(\Gamma)$, too.

A key argument in the theoretical treatment of (4.3) in $L^2(\Gamma)$ is the compactness of the double layer potential operator $\mathsf{K}_\varkappa : L^2(\Gamma) \mapsto L^2(\Gamma)$ on smooth surfaces, which renders the boundary integral operator associated with (4.3) a compact perturbation of the identity. On non-smooth surfaces this argument is not available. This prompted us to explore the regularized formulation presented in the next section.

## 5. REGULARIZED CFIE

The idea is to introduce a regularizing operator into the argument of the single layer potential in the trial expression (4.2). However, this operator has to be chosen carefully in order to permit us to prove uniqueness of solutions along the lines of the proof of Lemma 4.1. Crucial is the following result (cf. Section 5 of [15]):

**Lemma 5.1.** *With a constant $c_1 > 0$ we have*

$$\left\langle \mathsf{D}_0 v, (\tfrac{1}{2} Id + \mathsf{K}_0)\bar{v} \right\rangle_\Gamma \geqslant c_1 \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}_*(\Gamma).$$

**Proof.** Using integration by parts (3.2) and $\Delta \Psi^0_{\mathrm{DL}} = 0$ in $\Omega^+$, we get for $v \in H^{1/2}_*(\Gamma)$

$$\begin{aligned}\left\langle \mathsf{D}_0 v, (\tfrac{1}{2} Id + \mathsf{K}_0)\bar{v} \right\rangle_\Gamma &= -\left\langle \gamma_N^+ \Psi^0_{\mathrm{DL}}(v), \gamma_D^+ \Psi^0_{\mathrm{DL}}(\bar{v}) \right\rangle_\Gamma \\ &= \left\| \mathbf{grad}\, \Psi^0_{\mathrm{DL}}(v) \right\|^2_{L^2(\Omega^+)} \geqslant c \left\| \gamma_N^+ \Psi^0_{\mathrm{DL}}(v) \right\|^2_{H^{-1/2}_*(\Gamma)} \\ &\geqslant c \left\| \mathsf{D}_0(v) \right\|^2_{H^{-1/2}_*(\Gamma)} \geqslant c \|v\|^2_{H^{1/2}(\Gamma)} .\end{aligned}$$

Here, we have also used the continuity of $\gamma_N$, the estimate (3.18), and the ellipticity of $\mathsf{D}_0$. □

Setting $v := \mathsf{D}_0^{-1} \varphi$, using (3.18) and the symmetry properties of the boundary integral operators, we conclude from Lemma 5.1 that there is $c_N > 0$ such that

$$\left\langle \varphi, \mathsf{D}_0^{-1} (\tfrac{1}{2} Id + \mathsf{K}_0^*) \overline{\varphi} \right\rangle_\Gamma \geqslant c_N \|\varphi\|^2_{H^{-1/2}(\Gamma)} \quad \forall \varphi \in H^{-1/2}_*(\Gamma).$$

Note that $(\tfrac{1}{2} Id + \mathsf{K}_0^*)\varphi \in H^{-1/2}_*(\Gamma)$ for all $\varphi \in H^{-1/2}(\Gamma)$. Thus, owing to Theorem 2.1 and Lemma 3.1, the operator

$$\mathsf{R} := \mathsf{D}_0^{-1} (\tfrac{1}{2} Id + \mathsf{K}_0^*) : H^{-1/2}_*(\Gamma) \mapsto H^{1/2}_*(\Gamma)$$

is an isomorphism.

We still have to deal with the constant functions that are in the kernel of $\mathsf{D}_0$.

**Lemma 5.2.** *We have* $\|\mathbf{1}\|_{H^{1/2}(\Gamma)} = |\Gamma|^{1/2}$ *and* $\|\mathbf{1}\|_{H^{-1/2}(\Gamma)} = |\Gamma|^{1/2}$.

**Proof.** Using the definition of the Sobolev–Slobodeckij norm $\|\cdot\|_{H^{1/2}(\Gamma)}$, the statement about $\|\mathbf{1}\|_{H^{1/2}(\Gamma)}$ is trivial. To compute $\|\mathbf{1}\|_{H^{-1/2}(\Gamma)}$ consider the variational problem

$$\inf\left\{\tfrac{1}{2}\|v\|^2_{H^{1/2}(\Gamma)}\,,\ \int_\Gamma v\,\mathrm{d}S = 1\right\}$$

which gives rise to the saddle point problem: seek $v \in H^{1/2}(\Gamma)$

$$\begin{aligned} (v,q)_{H^{1/2}(\Gamma)} \ + \ \lambda\int_\Gamma q\,\mathrm{d}S \ &= 0 \qquad \forall q \in H^{1/2}(\Gamma) \\ \int_\Gamma v\,\mathrm{d}S \qquad\qquad &= 1. \end{aligned}$$

Its unique solution is $v \equiv |\Gamma|^{-1}$. Then

$$\|\mathbf{1}\|_{H^{-1/2}(\Gamma)} = \sup_{v\in H^{1/2}(\Gamma)} \frac{\int_\Gamma v\,\mathrm{d}S}{\|v\|_{H^{1/2}(\Gamma)}} = \frac{\int_\Gamma \mathbf{1}\,\mathrm{d}S}{\|\mathbf{1}\|_{H^{1/2}(\Gamma)}} = |\Gamma|^{1/2}$$

where we have used the definition of the dual norm. □

For $\nu > 0$ we define

$$\tilde{\mathsf{R}}\varphi := \mathsf{R}(\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|) + \nu\langle\varphi,\mathbf{1}\rangle_\Gamma \mathbf{1} : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma). \tag{5.1}$$

Since $\mathsf{R}$ maps into $H^{1/2}_*(\Gamma)$, this implies that for all $\varphi \in H^{-1/2}(\Gamma)$

$$\begin{aligned} \langle\overline{\varphi},\tilde{\mathsf{R}}\varphi\rangle_\Gamma &= \langle\overline{\varphi},\mathsf{R}(\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|) + \nu\langle\varphi,\mathbf{1}\rangle_\Gamma\mathbf{1}\rangle_\Gamma \\ &= \langle\overline{\varphi} - \overline{\varphi}(\mathbf{1})\mathbf{1}/|\Gamma|, \mathsf{R}(\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|)\rangle_\Gamma + \nu|\langle\varphi,\mathbf{1}\rangle_\Gamma|^2 \\ &\geqslant c_N\,\|\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|\|^2_{H^{-1/2}(\Gamma)} + \nu|\langle\varphi,\mathbf{1}\rangle_\Gamma|^2 \\ &\geqslant c_N\left(\|\varphi\|_{H^{-1/2}(\Gamma)} - |\varphi(\mathbf{1})|/|\Gamma|\,\|\mathbf{1}\|_{H^{-1/2}(\Gamma)}\right)^2 + \nu|\langle\varphi,\mathbf{1}\rangle_\Gamma|^2 \\ &\geqslant c_N\left(\|\varphi\|_{H^{-1/2}(\Gamma)} - |\langle\varphi,\mathbf{1}\rangle_\Gamma||\Gamma|^{-1/2}\right)^2 + \nu|\langle\varphi,\mathbf{1}\rangle_\Gamma|^2 \\ &\geqslant \tfrac{1}{2}c_N\,\|\varphi\|^2_{H^{-1/2}(\Gamma)} + (\nu - 2c_N/|\Gamma|)|\langle\varphi,\mathbf{1}\rangle_\Gamma|^2. \end{aligned}$$

In the sequel we assume $\nu > 2c_N/|\Gamma|$. Then, $\mathsf{\breve{R}}$ turns out to be $H^{-1/2}(\Gamma)$-elliptic. Thus, according to Theorem 2.1, $\mathsf{\breve{R}} : H^{-1/2}(\Gamma) \mapsto H^{1/2}(\Gamma)$ is an isomorphism and for some $\tilde{c} > 0$

$$\left\langle \mathsf{\breve{R}}^{-1} v, \overline{v} \right\rangle_\Gamma \geqslant \tilde{c} \, \|v\|^2_{H^{1/2}(\Gamma)} \quad \forall v \in H^{1/2}(\Gamma). \tag{5.2}$$

The new combined field integral equation (CFIE) arises from an indirect boundary integral approach to the exterior Dirichlet problem (1.1) and (1.3) using the special trial expression

$$U = \Psi^{\varkappa}_{\mathrm{DL}}(u) + i\eta \Psi^{0}_{\mathrm{SL}}(\mathsf{\breve{R}}^{-1} u) \quad u \in H^{1/2}(\Gamma). \tag{5.3}$$

By (3.5), this is a radiating Helmholtz solution in $\Omega^- \cup \Omega^+$. As before, applying the Dirichlet trace to (5.3) yields the boundary integral equation

$$g = (\tfrac{1}{2} Id + \mathsf{K}_\varkappa) u + i\eta (\mathsf{V}_\varkappa \circ \mathsf{\breve{R}}^{-1})(u) \quad \text{in } H^{1/2}(\Gamma). \tag{5.4}$$

For the sake of brevity, we introduce the boundary integral operator

$$\mathsf{B}_\varkappa := (\tfrac{1}{2} Id + \mathsf{K}_\varkappa) + i\eta \mathsf{V}_\varkappa \circ \mathsf{\breve{R}}^{-1} : H^{1/2}(\Gamma) \mapsto H^{1/2}(\Gamma).$$

**Lemma 5.3.** *The boundary integral operator $\mathsf{B}_\varkappa$ is injective.*

**Proof.** We adapt the proof of Lemma 4.1. Let $v \in H^{1/2}(\Gamma)$ be a solution of $\mathsf{B}_\varkappa u = 0$. Set $U := \Psi^{\varkappa}_{\mathrm{DL}}(u) + i\eta \Psi^{0}_{\mathrm{SL}}(\mathsf{\breve{R}}^{-1} u)$, whose restriction to $\Omega^+$ is a radiating exterior Helmholtz solution with $\gamma_D^+ U = 0$. From Theorem 1.1 we conclude $U = 0$ in $\Omega^+$. Thus, by the jump relations,

$$-\gamma_D^- U = [\gamma_D U]_\Gamma = u, \qquad \gamma_N^- U = -[\gamma_N U]_\Gamma = -i\eta \mathsf{\breve{R}}^{-1} u$$

the integration by parts formula (3.1) yields

$$i\eta \left\langle \mathsf{\breve{R}}^{-1} u, \overline{u} \right\rangle_\Gamma = \left\langle \gamma_N^{-1} U, \gamma_D^{-1} \overline{U} \right\rangle_\Gamma = \|\mathbf{grad}\, U\|^2_{L^2(\Omega^-)} - \varkappa^2 \|U\|^2_{L^2(\Omega^-)}.$$

Thanks to (5.2) and $\eta > 0$ the left hand side is purely imaginary, whereas the right hand side is real. Necessarily, $\left\langle \mathsf{\breve{R}} u, \overline{u} \right\rangle_\Gamma = 0$, which, by (5.2), implies $u = 0$. □

A Galerkin discretization cannot deal with the products of boundary integral operators occurring in the definition of $\mathsf{B}_\varkappa$. The usual trick to avoid operator products is to switch to a mixed formulation. Here, this is done by introducing the new unknown $\lambda := \mathsf{\breve{R}}^{-1} u \in H^{-1/2}(\Gamma)$ and gives us

$$\begin{array}{rcccll} i\eta \mathsf{V}_\varkappa(\lambda) & + & (\tfrac{1}{2} Id + \mathsf{K}_\varkappa) u & = & g & \text{in } H^{1/2}(\Gamma) \\ \mathsf{\breve{R}} \lambda & - & u & = & 0 & \text{in } H^{1/2}(\Gamma). \end{array} \tag{5.5}$$

These equations are equivalent to (5.4), as $\mathsf{R}$ is an isomorphism. However, a product of integral operators is still concealed in the definition of $\breve{\mathsf{R}}$. Fortunately, it involves the inverse of the boundary integral operator $\mathsf{D}_0$, which suggests plain multiplication of the second equation of (5.5) with $\mathsf{D}_0$. Yet, $\mathsf{D}_0$ is not an isomorphism and this simple approach is not feasible, unless we take care of the kernel of $\mathsf{D}_0$: for $\xi > 0$ define

$$\breve{\mathsf{D}}_0 v := \mathsf{D}_0(v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma|) + \xi \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}, \quad v \in H^{1/2}(\Gamma)$$

which, due to (3.15) and Lemma 5.2, satisfies

$$\begin{aligned}\langle \breve{\mathsf{D}}_0 v, \overline{v}\rangle_\Gamma &= \langle \mathsf{D}_0(v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma|), \overline{v} - \langle \mathbf{1}, \overline{v}\rangle_\Gamma \mathbf{1}/|\Gamma|\rangle_\Gamma + \xi |\langle \mathbf{1}, v\rangle_\Gamma|^2 \\ &\geqslant c_D \left\| v - \langle \mathbf{1}, v\rangle_\Gamma \mathbf{1}/|\Gamma| \right\|^2_{H^{1/2}(\Gamma)} + \xi |\langle \mathbf{1}, v\rangle_\Gamma|^2 \\ &\geqslant \tfrac{1}{2} \|v\|^2_{H^{1/2}(\Gamma)} + (\xi - 2c_D/|\Gamma|)|\langle \mathbf{1}, v\rangle_\Gamma|^2.\end{aligned}$$

If $\xi > 2c_D/|\Gamma|$, then $\breve{\mathsf{D}}_0$ is $H^{1/2}(\Gamma)$-elliptic and gives rise to an isomorphism $\breve{\mathsf{D}}_0 : H^{1/2}(\Gamma) \mapsto H^{-1/2}(\Gamma)$. This choice of the parameter will be assumed, henceforth.

As illustrated by the following lemma, we can now get rid of all products of integral operators by multiplying the second equation of (5.5) with $\breve{\mathsf{D}}_0$.

**Lemma 5.4.** *We have*

$$\breve{\mathsf{D}}_0 \breve{\mathsf{R}} \varphi = (\tfrac{1}{2} Id + \mathsf{K}_0^*)(\varphi) + \mathsf{T}\varphi$$

*where*

$$\mathsf{T}\varphi := -\langle \varphi, \mathbf{1}\rangle_\Gamma / |\Gamma| (\tfrac{1}{2} Id + \mathsf{K}_0^*)(\mathbf{1}) + \nu \xi |\Gamma| \langle \varphi, \mathbf{1}\rangle_\Gamma \mathbf{1}.$$

**Proof.** Obviously $\breve{\mathsf{D}}_0 \mathbf{1} = \xi |\Gamma| \mathbf{1}$ and $\breve{\mathsf{D}}_0 v = \mathsf{D}_0 v$, if $v \in H^{1/2}_*(\Gamma)$. This means

$$\begin{aligned}\breve{\mathsf{D}}_0 \breve{\mathsf{R}} \varphi &= \breve{\mathsf{D}}_0 \mathsf{R}(\varphi - \varphi(\mathbf{1})\mathbf{1}/|\Gamma|) + \nu \xi |\Gamma| \langle \varphi, \mathbf{1}\rangle_\Gamma \mathbf{1} \\ &= (\tfrac{1}{2} Id + \mathsf{K}_0^*)(\varphi) - \langle \varphi, \mathbf{1}\rangle_\Gamma / |\Gamma| (\tfrac{1}{2} Id + \mathsf{K}_0^*)(\mathbf{1}) + \nu \xi |\Gamma| \langle \varphi, \mathbf{1}\rangle_\Gamma \mathbf{1}\end{aligned}$$

where (5.1) has been employed. □

Hence, applying the isomorphism $\breve{\mathsf{D}}_0$ to the second line of (5.5) gives

$$\begin{array}{rcccl} i\eta \mathsf{V}_\varkappa \lambda & + & (\tfrac{1}{2} Id + \mathsf{K}_\varkappa) u & = & g \\ (\tfrac{1}{2} Id + \mathsf{K}_0^*)\lambda + \mathsf{T}\lambda & - & \breve{\mathsf{D}}_0 u & = & 0. \end{array} \tag{5.6}$$

We remark that the $u$-component of any solution of (5.6) instantly yields a solution of $\mathsf{B}_\varkappa u = g$. Therefore, Lemma 5.3 also asserts the uniqueness of solutions of (5.6).

The first equation of (5.6) is set in $H^{1/2}(\Gamma)$, the second in $H^{-1/2}(\Gamma)$. Thus the duality of these spaces gives rise to the natural weak form of (5.6): seek $\lambda \in H^{-1/2}(\Gamma)$, $u \in H^{1/2}(\Gamma)$ such that for all $\mu \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$

$$\begin{array}{rcccl} i\eta\langle\mu, \mathsf{V}_\varkappa(\lambda)\rangle_\Gamma & + & \langle\mu, (\frac{1}{2}Id + \mathsf{K}_\varkappa)u\rangle_\Gamma & = & \langle g, \mu\rangle_\Gamma \\ -\langle(\frac{1}{2}Id + \mathsf{K}_0^*)\lambda, v\rangle_\Gamma - \langle\mathsf{T}\lambda, v\rangle_\Gamma & + & \langle\tilde{\mathsf{D}}_0 u, v\rangle_\Gamma & = & 0. \end{array} \tag{5.7}$$

The bilinear form $a : (H^{-1/2}(\Gamma) \times (H^{1/2}(\Gamma)) \times (H^{-1/2}(\Gamma) \times (H^{1/2}(\Gamma)) \mapsto \mathbb{C}$ ) associated with (5.7) reads

$$a\left(\begin{pmatrix}\lambda\\u\end{pmatrix}, \begin{pmatrix}\mu\\v\end{pmatrix}\right) := i\eta\langle\mu, \mathsf{V}_\varkappa\lambda\rangle_\Gamma + \langle\mu, (\tfrac{1}{2}Id + \mathsf{K}_\varkappa)u\rangle_\Gamma \\ - \overline{\langle(\tfrac{1}{2}Id + \mathsf{K}_0^*)\lambda, v\rangle_\Gamma} - \overline{\langle\mathsf{T}\lambda, v\rangle_\Gamma} + \overline{\langle\tilde{\mathsf{D}}_0 u, v\rangle_\Gamma}.$$

Now, we alter this bilinear form by adding compact terms. First, we drop $\langle\mathsf{T}\lambda, v\rangle_\Gamma$, which is obviously compact since the range of $\mathsf{T}$ has dimension two. Next, we invoke Lemma 3.2 to replace $\mathsf{V}_\varkappa$ and $\mathsf{K}_\varkappa$ with $\mathsf{V}_0$ and $\mathsf{K}_0$, respectively. Ultimately, we end up with the perturbed bilinear form

$$\tilde{a}\left(\begin{pmatrix}\lambda\\u\end{pmatrix}, \begin{pmatrix}\mu\\v\end{pmatrix}\right) := i\eta\langle\mu, \mathsf{V}_0\lambda\rangle_\Gamma + \langle\mu, (\tfrac{1}{2}Id + \mathsf{K}_0)u\rangle_\Gamma \\ - \overline{\langle(\tfrac{1}{2}Id + \mathsf{K}_0^*)\lambda, v\rangle_\Gamma} + \overline{\langle\tilde{\mathsf{D}}_0 u, v\rangle_\Gamma}.$$

The symmetry (3.12) permits us to cancel cross terms and confirms $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$-ellipticity

$$\left|a\left(\begin{pmatrix}\lambda\\u\end{pmatrix}, \begin{pmatrix}\overline{\lambda}\\\overline{u}\end{pmatrix}\right)\right| = \left|i\eta\langle\overline{\lambda}, \mathsf{V}_0\lambda)\rangle_\Gamma + \langle\tilde{\mathsf{D}}_0 u, \overline{u}\rangle_\Gamma\right| \\ \geqslant \frac{1}{\sqrt{2}}\left(\eta c_V \|\lambda\|^2_{H^{-1/2}(\Gamma)} + c_D \|u\|^2_{H^{1/2}(\Gamma)}\right).$$

This means that the bilinear form associated with (5.7) is coercive in $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$. In addition we have established uniqueness of solutions. Therefore, we have verified all assumptions of Lemma 2.2 and Theorem 2.3 and reap all the desirable consequences for Galerkin discretization discussed at the end of the previous section.

**Remark 5.1.** The reader has to be aware that the choice of the regularizing operator $\tilde{\mathsf{R}}$ is tightly constrained by the essential cancellation of the cross terms of $\tilde{a}$. This forces us to incorporate $\frac{1}{2}Id + \mathsf{K}_0^*$ into $\mathsf{R}$. In addition, $\mathsf{R}$ has to be $H^{-1/2}(\Gamma)$-elliptic, see (5.2), and it is by no means obvious, how a choice different from (5.1) can comply with both requirements.

**Remark 5.2.** The product of the parameters *nu* and $\xi$ will enter the final variational formulation (5.7). It is important to note that uniqueness of solutions will be squandered, if $\nu\xi$ is chosen too small. Conversely, a large value for $\nu\xi$ might delay the onset of asymptotic phase, that is, in terms of the statement of Thm. (2.6) $N$ will become very large. The necessity to pick parameters is definitely a drawback of this regularized formulation.

## 6. GALERKIN DISCRETIZATION

Conforming boundary element spaces for the approximation of functions in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, are standard. First, we equip $\Gamma$ with a family $\{\mathscr{T}_h\}_h$ of triangulations comprising (curved) triangles and/or quadrilaterals. The meshes $\mathscr{T}_h$ have to resolve the shape of the curvilinear polyhedron $\Omega^-$ in the sense that none of their elements may reach across an edge of $\Omega^-$. Then, the boundary element spaces $\mathscr{S}_h \subset H^{1/2}(\Gamma)$ and $\mathscr{Q}_h \subset H^{-1/2}(\Gamma)$ will contain piecewise polynomials of total/maximal degree $k$, $k \in \mathbb{N}_0$. Further, functions in $\mathscr{S}_h$ have to be continuous so that $k \geqslant 1$ is required in this case.

Let $h$ denote the meshwidth of $\mathscr{T}_h$ and assume uniform shape-regularity, which, sloppily speaking, imposes a uniform bound on the distortion of the elements. Then we can find constants $C_s, C_q > 0$ such that for all $0 \leqslant t \leqslant k+1$ (see Section 4.4 of [2]):

$$\inf_{\varphi_h \in \mathscr{Q}_h} \|\varphi - \varphi_h\|_{H^{-1/2}(\Gamma)} \leqslant C_q h^{t+1/2} \|\varphi\|_{H^t(\Gamma)} \quad \forall \varphi \in H^t(\Gamma), h \tag{6.1}$$

$$\inf_{v_h \in \mathscr{S}_h} \|v - v_h\|_{H^{1/2}(\Gamma)} \leqslant C_s h^{t-1/2} \|v\|_{H^t(\Gamma)} \quad \forall v \in H^t(\Gamma), h. \tag{6.2}$$

Thus, the quantitative investigation of convergence boils down to establishing the Sobolev regularity of the continuous solutions. We will embark on this for the variational boundary integral equations (4.5) and (5.7).

It is useful to characterize the lifting properties of Neumann-to-Dirichlet maps for the interior/exterior Helmholtz problem by means of two real numbers $\alpha^+/\alpha^-$. In particular, let $\alpha^-/\alpha^+$ be the largest real number such that for an interior/exterior Helmholtz solution $\gamma_N^\pm U \in H^{s-1/2}(\Gamma)$ implies $\gamma_D^\pm \in H^{s+1/2}(\Gamma)$ for all $s \leqslant \alpha^\pm$. It is known from Theorem 4.24 of [11] that for mere Lipschitz domains $\alpha^-, \alpha^+ \geqslant 1/2$.

We first examine equation (4.5) and assume that the Neumann data $\varphi$ belong to $H^{-1/2+\sigma}(\Gamma)$, $\sigma > 0$. According to the definition of $\alpha^+$ this implies $\gamma_D^+ \in H^{1/2+\min\{\sigma,\alpha^+\}}(\Gamma)$. Now, let $u \in H^{1/2}(\Gamma)$ stand for the unique solution of (4.5) and let the Helmholtz solution $U$ be given by (4.2). By the jump relations

$$[\gamma_N U]_\Gamma = -i\eta u, \qquad [\gamma_D U]_\Gamma = u \tag{6.3}$$

we conclude that $U_{|\Omega^-}$ satisfies the inhomogeneous Robin-type boundary conditions

$$\gamma_N^- U - i\eta\gamma_D^- U = \varphi - i\eta\gamma_D^+ U. \tag{6.4}$$

This will endow the Neumann data with extra regularity and we can crank up the machine of a bootstrap argument that confirms higher and higher regularity for Neumann and Dirichlet data in turns. A limit will be set by the lifting exponents $\alpha^-, \alpha^+$: the best we can get is

$$u \in H^{1/2+\min\{\sigma,\alpha^-,\alpha^+\}}(\Gamma).$$

For piecewise linear continuous boundary elements on a sequence of shape regular surface meshes this will mean $O(h^{\min\{\sigma,\alpha^+,\alpha^-\}})$ convergence in $H^{1/2}(\Gamma)$.

The bad news is that in the case of the single layer regularization (5.7) of Section 5 the lifting arguments will fail. Please note that for $U$ from (5.3), where $u \in H^{1/2}(\Gamma)$ is the solution of (5.4), the following interior Robin-type boundary conditions hold:

$$\gamma_N^- U - i\eta \breve{\mathsf{R}}^{-1}(\gamma_D^- U) = \gamma_N^- U - i\eta \breve{\mathsf{R}}^{-1} g. \tag{6.5}$$

In contrast to (6.4), we cannot infer any enhanced regularity of either $\gamma_N^- U$ or $\gamma_D^- U$ from (6.5). Hence, no quantitative rate of convergence can be obtained for a Galerkin boundary element discretization of (5.7). Due to the density of the boundary element spaces on infinite sequences of ever finer meshes in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively, the method will converge for $h \to 0$, but convergence could be extremely slow.

## 7. CONCLUSION

We found that the classical combined field integral equation for the exterior Neumann problem for Helmholtz' equation leads to a $H^{1/2}(\Gamma)$-coercive variational problem. Satisfactory rates of convergence can be deduced for conforming Galerkin BEM schemes. Conversely, the analysis of the CFIE for the exterior Dirichlet problem has to rely on a special regularizing operator. However, the use of this operator destroys lifting properties needed to conclude enhanced regularity of the unknown density. Hence, quantitative estimates of convergence remain elusive for this method.

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