Lipschitz Chain Approximation of Metric Integral Currents

Journal Article

Author(s):
Goldhirsch, Tommaso

Publication date:
2022-01

Permanent link:
https://doi.org/10.3929/ethz-b-000575372

Originally published in:
Analysis and Geometry in Metric Spaces 10(1), https://doi.org/10.1515/agms-2022-0140

Anal. Geom. Metr. Spaces 2022; 10:40–49 DE GRUYTER

Research Article Open Access

Tommaso Goldhirsch*

Lipschitz Chain Approximation of Metric Integral Currents

https://doi.org/10.1515/agms-2022-0140
Received June 4, 2022; accepted August 10, 2022

Abstract: Every integral current in a locally compact metric space X can be approximated by a Lipschitz chain with respect to the normal mass, provided that Lipschitz maps into X can be extended slightly.

Keywords: metric spaces; integral currents; polyhedral chains; Lipschitz chains

MSC: 49Q15, 58A25, 53C23

1 Introduction

In [6], after proving their celebrated deformation theorem, Federer and Fleming show that every integral current admits an approximation by Lipschitz chains. More precisely, for each current $T \in \mathbf{I}_{n,c}(\mathbb{R}^N)$ and $\varepsilon > 0$ there is a Lipschitz chain $P \in \mathbf{L}_{n,c}(\mathbb{R}^N)$ such that $\mathbf{N}(T - P) < \varepsilon$, see Theorem 5.8 in [6].

In this paper we prove an analogue of this result for a locally compact metric space X with the property that every Lipschitz map into X can be extended to a neighborhood of its domain. In fact, we need this property to hold only locally and for Lipschitz maps with compact domains. We work with metric *integer rectifiable currents* with finite mass $\mathcal{I}_n(X)$, with compact support $\mathcal{I}_{n,c}(X)$, and metric *integral currents* with compact support $\mathbf{I}_{n,c}(X)$. Each current T has *boundary* ∂T, *mass* $\mathbf{M}(T)$, and *normal mass* $\mathbf{N}(T) = \mathbf{M}(T) + \mathbf{M}(\partial T)$. Every singular Lipschitz chain in X with integer coefficients induces a *Lipschitz chain* $P \in \mathbf{L}_{n,c}(X)$, which is an element of $\mathbf{I}_{n,c}(X)$ (see Section 2).

We briefly describe Federer and Fleming's approach in [6]. As a consequence of the deformation theorem (see Theorem 5.5 in [6]), they first prove that if a current $T \in \mathbf{I}_{n,c}(\mathbb{R}^N)$ has boundary $\partial T \in \mathbf{L}_{n-1,c}(\mathbb{R}^N)$, it can be deformed into a Lipschitz chain (compare with Lemma 5.7 in [6]).

Lemma 1.1. *Let $N > n \geq 1$ and let $T \in \mathbf{I}_{n,c}(\mathbb{R}^N)$ with $\partial T \in \mathbf{L}_{n-1,c}(\mathbb{R}^N)$. Then for every $\eta > 0$ there is $R \in \mathbf{I}_{n+1,c}(\mathbb{R}^N)$ with $T - \partial R \in \mathbf{L}_{n,c}(\mathbb{R}^N)$, $\mathbf{N}(R) \leq \eta$ and $\operatorname{spt}(R) \subset \operatorname{spt}(T)_\eta$.*

Here $\operatorname{spt}(T)_\eta$ denotes the open η-neighborhood of $\operatorname{spt}(T)$. Applying Lemma 1.1 twice, once in dimension $n - 1$ and once in dimension n, they finally prove the $\mathbf{N}$-approximation theorem for $\mathbb{R}^N$. This argument does not apply directly in our setting, as there is no analogue of the deformation theorem for general metric spaces, although a recent paper [3] proves a Federer-Fleming type deformation theorem for quasiconvex metric spaces of finite Nagata dimension admitting Euclidean isoperimetric inequalities.

We solve this issue taking inspiration from the strategy used in [5], [13] and [14]. We embed a compact neighborhood K of the support $\operatorname{spt}(T)$ of $T \in \mathbf{I}_{n,c}(X)$ into $l^\infty(\mathbb{N})$ and exploit the metric approximation property of $l^\infty(\mathbb{N})$ to project it onto a finite dimensional vector subspace, in which we can apply Lemma 1.1. To go back to the original current T in X we assume that Lipschitz maps into X can be extended slightly.

***Corresponding Author: Tommaso Goldhirsch:** Department of Mathematics, ETH Zürich, Zürich, Switzerland,
E-mail: tommaso.goldhirsch@math.ethz.ch

Definition 1.2. A metric space X has *property L* if the following holds. For every metric space Y, every compact subset $K \subset Y$, and every 1-Lipschitz map $g\colon K \to X$, there exist $\varepsilon = \varepsilon(g) > 0$ and $L = L(g) \geq 1$ such that g admits an L-Lipschitz extension $\overline{g}\colon K_\varepsilon \to X$ to the open ε-neighborhood K_ε of K. A metric space X has *local property L* if every point in X has a neighborhood with property L.

Without loss of generality we can set $Y = l^\infty(\mathbb{N})$. If a metric space X has property L, then every open subset has property L. In particular property L implies local property L. Both conditions imply that X is semi-locally quasi-convex. Examples of spaces having property L include absolute Lipschitz neighborhood retracts, metric spaces locally bi-Lipschitz equivalent to $\mathbb{R}^n$ or to a finite piecewise Euclidean polyhedral complex. A more detailed discussion can be found in Section 3.

To expand on the argument above, suppose that X has property L and let $\iota\colon K \to l^\infty(\mathbb{N})$ be an isometric embedding with image $K' := \iota(K)$. By the metric approximation property of $l^\infty(\mathbb{N})$ there is a finite dimensional subspace $V \subset l^\infty(\mathbb{N})$ arbitrarily close to K', and a 1-Lipschitz projection $\pi\colon K' \to V$. In particular, we can choose V close enough such that the extension of $\iota^{-1}\colon K' \to X$, say g, provided by property L is defined on $\pi(K')$.

There is a push-forward $T'' = (\pi \circ \iota)_\# T \in \mathbf{I}_{n,\mathrm{c}}(V)$ of T in $\pi(K')$. Ideally, we could apply Federer and Fleming's $\mathbf{N}$-approximation theorem in V and then map back to X using g. The issue is that by doing so we do not control the difference between the original current $T = (g \circ \iota)_\# T$ and $g_\# T''$. We keep track of the difference between $\iota_\# T$ and T'' by constructing a "homotopy filling" between them in $l^\infty(\mathbb{N})$, and by using Lemma 1.1 instead of directly applying the approximation result for the $\mathbf{N}$-norm.

After a small decomposition argument (Lemma 4.1) we can relax the assumption from (global) property L to local property L, and prove the following analogue of Theorem 5.8 in [6], which is the main result of this paper.

Theorem 1.3 ($\mathbf{N}$-Approximation)**.** *Let $n \geq 1$, let X be a locally compact metric space with local property L, and let $T \in \mathbf{I}_{n,\mathrm{c}}(X)$. Then for every $\varepsilon > 0$ there is $P \in \mathbf{L}_{n,\mathrm{c}}(X)$ with $\mathbf{N}(T - P) < \varepsilon$ and* $\operatorname{spt}(P) \subset \operatorname{spt}(T)_\varepsilon$.

If $n = 1$, then we can find P with $\partial P = \partial T$. This is already known without assuming any Lipschitz extension property of X (see [2], [13] and [14]).

The strategy of proof outlined above is only necessary to approximate $T \in \mathbf{I}_{n,\mathrm{c}}(X)$ with respect to the normal mass $\mathbf{N}$. The argument for the mass is simpler and and does not involve passing through $l^\infty(\mathbb{N})$. Every compactly supported integer valued function $u \in \mathrm{L}^1(\mathbb{R}^n)$ induces an integer rectifiable current $[\![u]\!] \in \mathcal{I}_{n,\mathrm{c}}(\mathbb{R}^n)$ with $\mathbf{M}([\![u]\!]) = \|u\|_{\mathrm{L}^1}$. If $F\colon \mathbb{R}^n \to X$ is L-Lipschitz, then there is a push-forward $F_\#[\![u]\!] \in \mathcal{I}_{n,\mathrm{c}}(X)$, and $\mathbf{M}(F_\#[\![u]\!]) \leq L^n\,\mathbf{M}([\![u]\!])$.

Every current $T \in \mathcal{I}_n(X)$ can be written as $T = \sum_i T_i$, where the sum converges with respect to the mass norm and each $T_i \in \mathcal{I}_{n,\mathrm{c}}(X)$ is of the form $T_i = (F_i)_\#[\![u_i]\!]$ for some integer valued $u_i \in \mathrm{L}^1(\mathbb{R}^n)$, where $F_i\colon K_i \to F_i(K_i) \subset X$ is bi-Lipschitz and $K_i \subset \mathbb{R}^n$ is a compact set containing the support of u_i. By a purely measure-theoretic argument we approximate each u_i in the L^1-norm with a finite sum of characteristic functions corresponding to Borel subsets B_j contained in $\operatorname{spt}(u_i)$. Then, we approximate these Borel sets by cubes. This produces an approximation of $[\![u_i]\!]$ by Lipschitz chains with respect to the $\mathbf{M}$-norm in $\mathbb{R}^n$. However, these cubes may leak slightly outside of B_j and in particular outside of $\operatorname{spt}(u_i)$, so that their image in X via F_i is not defined.

This shows that if $T \in \mathcal{I}_n(X)$ has support contained in an open subset of X with property L, then it admits an approximation by Lipschitz chains with respect to the $\mathbf{M}$-norm. In general, however, if we assume that X has local property L, neither T nor the T_i's have support inside such an open subset of X and this argument needs a small refinement: each T_i can be decomposed as a finite sum $T_i = T_i^1 + \cdots + T_i^{N_i}$ with each T_i^j having support inside an open subset of X having property L (Lemma 4.1). Note that the domain of each F_i is a subset of $\mathbb{R}^n$ so that for this discussion we might as well assume that X is locally *Lipschitz* $(n-1)$*-connected*, that is, there is a constant c such that every L-Lipschitz map $f\colon S^k \to X$ with $0 \leq k \leq n-1$ admits a cL-Lipschitz extension $\overline{f}\colon B^{k+1} \to X$, where S^k and B^{k+1} denote the unit sphere and ball in $\mathbb{R}^{n+1}$, respectively. In this situation we can apply the partial result just outlined for each T_i^j and obtain the following.

Proposition 1.4 (**M**-Approximation). *Let $n \geq 1$, let X be a locally compact metric space, and let $T \in \mathcal{I}_n(X)$. Suppose that X has local property L or that X is locally Lipschitz $(n-1)$-connected. Then for every $\varepsilon > 0$ and open subset $U \subset X$ with $\operatorname{spt}(T) \subset U$ there is $P \in \mathbf{L}_{n,c}(X)$ with $\operatorname{spt}(P) \subset U$ and $\mathbf{M}(T-P) < \varepsilon$.*

We remark that property L and being Lipschitz n-connected for some n are not equivalent, in fact, the sphere S^n has property L but is not Lipschitz n-connected (see Section 3 for more details).

2 Preliminaries

Let $X = (X, d)$ be a metric space. We write $B_x(r) := \{y \in X : d(x, y) \leq r\}$ for the closed ball of radius $r \geq 0$ and center $x \in X$.

Given a subset $A \subset X$ and $\varepsilon > 0$ we denote by $A_\varepsilon := \{x \in X : d(A, x) < \varepsilon\}$ the open ε-neighborhood of A in X, where $d(A, x)$ is the infimum over all $d(a, x)$ with $a \in A$.

A map $f : X \to Y$ into another metric space $Y = (Y, d)$ is *L-Lipschitz*, for some constant $L \geq 0$, if $d(f(x), f(x')) \leq L d(x, x')$ for all $x, x' \in X$. The *Lipschitz constant* $\operatorname{Lip}(f)$ of f is the infimum over all such L. A map f is *Lipschitz* if it is L-Lipschitz for some L, it is *locally Lipschitz* if every point in X has a neighborhood on which f is Lipschitz. We denote by $\operatorname{Lip}_{\mathrm{loc}}(X)$ and $\operatorname{Lip}_{\mathrm{c}}(X)$ the spaces of functions $X \to \mathbb{R}$ which are locally Lipschitz or Lipschitz with compact support, respectively. A map f is a *bi-Lipschitz* embedding if it is injective and both f and f^{-1} are Lipschitz.

Metric Currents

Metric currents of finite mass were introduced by Ambrosio and Kirchheim in [1]. Here we will work with a variant of this theory for locally compact metric spaces, as described by Lang in [10]. In this section we provide some background on this theory and refer the reader to [10] for more details. We will assume throughout that the underlying metric space X is locally compact.

For every integer $n \geq 0$ let $\mathscr{D}^n(X)$ be the set of all $(f, \pi) := (f, \pi_1, \ldots, \pi_n)$ in $\operatorname{Lip}_{\mathrm{c}}(X) \times [\operatorname{Lip}_{\mathrm{loc}}(X)]^n$. We endow $\mathscr{D}^n(X)$ with the topology for which $(f^k, \pi^k) \to (f, \pi)$ if $f^k \to f$ and $\pi_i^k \to \pi_i$ pointwise on X with uniformly bounded Lipschitz constants on each compact set, and with $\bigcup_k \operatorname{spt}(f^k) \subset K$ for some compact set $K \subset X$. The idea is that $(f, \pi_1, \ldots, \pi_n) \in \mathscr{D}^n(X)$ represents the compactly supported differential n-form $f d\pi_1 \wedge \ldots \wedge d\pi_n$ if X is (an open subset of) $\mathbb{R}^N$ and the functions $f, \pi_1, \ldots, \pi_n$ are smooth; and roughly speaking, a current (with some additional properties defined below) is a map $\mathscr{D}^n(X) \to \mathbb{R}$ representing integration on a submanifold of $\mathbb{R}^N$.

Definition 2.1. An n-dimensional *current* T on X is an $(n+1)$-linear function $T : \mathscr{D}^n(X) \to \mathbb{R}$ such that $T(f^k, \pi^k) \to T(f, \pi)$ whenever $(f^k, \pi^k) \to (f, \pi)$ in $\mathscr{D}^n(X)$, and $T(f, \pi) = 0$ whenever one of the functions π_i is constant on a neighborhood of $\operatorname{spt}(f)$.

The vector space of all n-dimensional currents on X is denoted by $\mathscr{D}_n(X)$. Every function $u \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^n)$ induces a current $[\![u]\!] \in \mathscr{D}_n(\mathbb{R}^n)$ defined by

$$[\![u]\!](f, \pi_1, \ldots, \pi_n) := \int u f \det\left(\frac{\partial \pi_i}{\partial x^j}\right)_{i,j=1}^n \mathrm{d}x$$

for all $(f, \pi_1, \ldots, \pi_n) \in \mathscr{D}^n(\mathbb{R}^n)$, where the partial derivatives $\partial \pi_i / \partial x^j$ exist almost everywhere according to Rademacher's theorem. If $W \subset \mathbb{R}^n$ is a Borel set and χ_W is its characteristic function, we set $[\![W]\!] := [\![\chi_W]\!]$.

Support, Push-forward, and Boundary

Let $T \in \mathscr{D}_n(X)$ be an n-dimensional current. The *support* $\operatorname{spt}(T)$ of T is the smallest closed subset of X such that the value $T(f, \pi_1, \ldots, \pi_n)$ depends only on the restrictions of $f, \pi_1, \ldots, \pi_n$ to it.

For a proper Lipschitz map $F\colon \operatorname{spt}(T) \to Y$ into another locally compact metric space Y, the *push-forward* $F_\# T \in \mathscr{D}_n(Y)$ is defined by

$$F_\#(f, \pi_1, \ldots, \pi_n) := T(f \circ F, \pi_1 \circ F, \ldots, \pi_n \circ F)$$

for all $(f, \pi) \in \mathscr{D}^n(Y)$. It holds that $\operatorname{spt}(F_\# T) \subset F(\operatorname{spt}(T))$.

For $n \geq 1$, the *boundary* $\partial T \in \mathscr{D}_{n-1}(X)$ of T is defined by

$$(\partial T)(f, \pi_1, \ldots, \pi_{n-1}) = T(\sigma, f, \pi_1, \ldots, \pi_{n-1})$$

for $(f, \pi_1, \ldots, \pi_{n-1}) \in \mathscr{D}^{n-1}(X)$, where σ is any compactly supported Lipschitz function, that is identically 1 on $\operatorname{spt}(f) \cap \operatorname{spt}(T)$. It holds that $\partial \circ \partial = 0$, $\operatorname{spt}(\partial T) \subset \operatorname{spt}(T)$, and $F_\#(\partial T) = \partial(F_\# T)$ for F as above. (For more details, see Section 3 in [10].)

Mass

Let $T \in \mathscr{D}_n(X)$ be an n-dimensional current. For an open set $U \subset X$, the *mass* $\|T\|(U) \in [0, \infty]$ of T in U is defined as the supremum of $\sum_{i=1}^k T(f^i, \pi_1^i, \ldots, \pi_n^i)$ over all finite families $(f^i, \pi_1^i, \ldots, \pi_n^i)_{i=1}^k \subset \mathscr{D}^n(X)$ such that the restrictions of $\pi_1^i, \ldots, \pi_n^i$ to $\operatorname{spt}(f^i)$ are 1-Lipschitz for all i, $\bigcup_{i=1}^k \operatorname{spt}(f^i) \subset U$ and $\sum_{i=1}^k |f^i| \leq 1$.

This defines a regular Borel measure $\|T\|$ on X. The total mass $\|T\|(X)$ of T is denoted $\mathbf{M}(T)$ and is called the *mass* of T. If $S \in \mathscr{D}_n(X)$ is another current, then

$$\mathbf{M}(T + S) \leq \mathbf{M}(T) + \mathbf{M}(S).$$

The *normal mass* of T is defined as $\mathbf{N}(T) := \mathbf{M}(T) + \mathbf{M}(\partial T)$, and T is *normal* if $\mathbf{N}(T) < \infty$. If $\|T\|$ is locally finite, then for every Borel set $B \subset X$ there is a *restriction* $T \llcorner B \in \mathscr{D}_n(X)$ of T to B; the measure $\|T \llcorner B\|$ coincides with the restriction $\|T\| \llcorner B$ of the measure $\|T\|$. If $\mathbf{M}(T) < \infty$ and $F\colon \operatorname{spt}(T) \to Y$ is a proper L-Lipschitz map into a locally compact metric space Y, then

$$\mathbf{M}(F_\# T) \leq L^n\, \mathbf{M}(T).$$

We denote by $\mathbf{M}_n(X)$, $\mathbf{M}_{n,\mathrm{c}}(X)$, $\mathbf{N}_n(X)$, $\mathbf{N}_{n,\mathrm{c}}(X)$ the real vector spaces of all currents with finite mass, compact support, normal currents, and normal currents with compact support, respectively.

If $u \in \mathrm{L}^1(\mathbb{R}^n)$, then $\mathbf{M}([\![u]\!]) = \|u\|_{L^1}$, in particular, if $K \subset \mathbb{R}^n$ is a Borel set, then $\mathbf{M}([\![K]\!]) = \mathcal{L}^n(K)$ and $\mathbf{M}(\partial[\![K]\!]) < \infty$ whenever χ_K has finite variation. (For more details, see Sections 2, 4 and 7 of [10].)

Integral Currents

A current $T \in \mathbf{M}_n(X)$ is *integer rectifiable* if there are countably many Lipschitz maps $F_i\colon A_i \subset \mathbb{R}^n \to X$ such that $\|T\|$ is concentrated on $\bigcup_i F_i(A_i)$, and for every Borel set $B \subset X$ with compact closure and every Lipschitz map $\pi\colon X \to \mathbb{R}^n$ the push-forward $\pi_\#(T \llcorner B) \in \mathscr{D}_n(\mathbb{R}^n)$ is of the form $[\![u]\!]$, for some integer valued $u = u_{B,\pi} \in \mathrm{L}^1(\mathbb{R}^n)$.

The abelian group of *integer rectifiable n*-currents in X is denoted by $\mathcal{I}_n(X)$; it is closed under push-forwards and restrictions to Borel sets. We write $\mathcal{I}_{n,\mathrm{c}}(X)$ for the subgroup of integer rectifiable currents with compact support.

A current $T \in \mathcal{I}_{n,\mathrm{c}}(X)$ is an *integral current* with compact support, or simply an integral current, if whenever $n \geq 1$, its boundary ∂T is integer rectifiable as well. We denote the corresponding abelian groups by $\mathbf{I}_{n,\mathrm{c}}(X)$, and observe that they form a chain complex. By Theorem 8.7 (boundary rectifiability) in [10], $T \in \mathcal{I}_{n,\mathrm{c}}(X)$ is integral if $\mathbf{M}(\partial T) < \infty$.

If $K \subset \mathbb{R}^n$ is a bounded Borel set, then $[\![K]\!]$ is an element of $\mathcal{I}_{n,\mathrm{c}}(\mathbb{R}^n)$, and it is in $\mathbf{I}_{n,\mathrm{c}}(\mathbb{R}^n)$ whenever χ_K has finite variation.

An integral current T is a *cycle* whenever $\partial T = 0$ and we denote by $\mathbf{Z}_{n,\mathrm{c}}(X) \subset \mathbf{I}_{n,\mathrm{c}}(X)$ the subgroup of integral cycles. An element of $\mathbf{I}_{0,\mathrm{c}}(X)$ is an integer linear combination of currents of the form $[\![x]\!]$, where $x \in X$ and $[\![x]\!](f) = f(x)$ for all compactly supported Lipschitz functions $f \in \mathscr{D}^0(X)$. In this case $\mathbf{Z}_{0,\mathrm{c}}(X) \subset \mathbf{I}_{0,\mathrm{c}}(X)$

denotes the subgroup of integer linear combinations whose coefficients sum to zero. Note that $\partial\colon \mathbf{I}_{n,c}(X) \to \mathbf{Z}_{n-1,c}(X)$ for all $n \geq 1$, and if $F\colon X \to Y$ is a proper Lipschitz map into a locally compact metric space Y, then the push-forward $F_\#$ maps $\mathbf{I}_{n,c}(X)$ to $\mathbf{I}_{n,c}(Y)$ and $\mathbf{Z}_{n,c}(X)$ to $\mathbf{Z}_{n,c}(Y)$. Given $Z \in \mathbf{Z}_{n,c}(X)$ we call $V \in \mathbf{I}_{n+1,c}(X)$ a *filling* of Z if $\partial V = Z$. (For more details, see Section 8 of [10].)

In general, the restriction $T\llcorner B$ of an integral current T to an arbitrary Borel subset $B \subset X$ is not integral. However, for every $x \in X$ the restriction $T \llcorner B_x(r)$ is in $\mathbf{I}_{n,c}(X)$ for almost every $r \geq 0$. (See Section 6 and Theorem 8.5 in [10], and Section 2.6 in [8].)

Lipschitz Chains

An n-dimensional polyhedron K in $\mathbb{R}^n$, such as a (hyper-)cube or an n-simplex, is the convex hull of finitely many (non coplanar) points in $\mathbb{R}^n$. As noted above, $[\![K]\!]$ is in $\mathbf{I}_{n,c}(\mathbb{R}^n)$. A *Lipschitz n-chain* in X is a finite sum

$$L = \sum_{i=1}^{l} a_i(\varphi_i)_\#[\![D_i]\!],$$

where $a_i \in \mathbb{Z}$, $D_i \subset \mathbb{R}^n$ are n-dimensional polyhedra and $\varphi_i\colon D_i \to X$ are Lipschitz maps. We denote by $\mathbf{L}_{n,c}(X) \subset \mathbf{I}_{n,c}(X)$ the abelian group of Lipschitz n-chains in X.

There is a chain isomorphism $\mathbf{I}_{*,c}(\mathbb{R}^n) \to \mathbf{I}^{\mathrm{FF}}_{*,c}(\mathbb{R}^n)$ between (metric) integral currents in $\mathbb{R}^n$ and "classical" Federer-Fleming integral currents of [6] which is bi-Lipschitz with respect to the $\mathbf{M}$-norm with constants depending only on the dimensions, and which restricts to an isomorphism between the respective subchains of Lipschitz chains (see Theorem 5.5 in [10]). In particular we can apply Lemma 1.1 to metric integral currents in $\mathbf{I}_{n,c}(\mathbb{R}^n)$.

Finally, note that all 0-dimensional integral currents are by definition Lipschitz chains, that is, $\mathbf{I}_{0,c}(X) = \mathbf{L}_{0,c}(X)$, and therefore an approximation theorem for the $\mathbf{N}$-norm is not necessary in dimension 0.

Homotopies

We recall a useful technique to produce fillings of cycles, for more details we refer the reader to Theorem 2.9 in [15] and Section 2.7 of [8]. We use this in Proposition 4.2 and Proposition 4.4 to bridge the gap between the isometric image of a subset of a metric space and a finite dimensional subspace in $l^\infty(\mathbb{N})$. Let Y denote a normed vector space and $K \subset Y$ a compact subset. Let $\varphi, \psi\colon K \to Y$ be L-Lipschitz maps with $|\psi(x) - \varphi(x)| \leq D$ for all $x \in K$, and consider the affine homotopy $H\colon [0,1] \times K \to Y$ from φ to ψ, that is, $H(t,x) := t\psi(x) + (1-t)\varphi(x)$. If P is an element of $\mathbf{L}_{n,c}(K)$ with $\partial P = 0$, then the push-forward $H_\#([\![0,1]\!] \times P) \in \mathbf{L}_{n+1,c}(Y)$ has support contained in $H([0,1] \times K)$ and satisfies

$$\begin{gathered}\partial H_\#\big([\![0,1]\!] \times P\big) = \psi_\# P - \varphi_\# P, \\ \mathbf{M}\left(H_\#\big([\![0,1]\!] \times P\big)\right) \leq (n+1)L^n D\, \mathbf{M}(P).\end{gathered}$$

We call $H_\#([\![0,1]\!] \times P)$ the *affine (homotopy) filling* of $\psi_\# P - \varphi_\# P$.

Finite Dimensional Projections

As mentioned in the introduction, in order to exploit the deformation theorem we project integral currents defined on a metric space X into a finite dimensional vector space. This is done in two steps.

First, every metric space X embeds isometrically into the Banach space $l^\infty(X)$ of bounded maps on X via the map $x \mapsto d(x,\cdot) - d(x_0,\cdot)$, for any base point $x_0 \in X$. If X is compact, or more generally separable, and $(x_i)_{i\in\mathbb{N}} \subset X$ is a countable dense subset, then $x \mapsto (d(x_i,x)_i - d(x_i,x_0))_{i\in\mathbb{N}}$ is an isometric embedding into $l^\infty(\mathbb{N})$, and the second term $d(x_i,x_0)$ is not necessary if X is bounded. This allows us to embed a compact neighborhood of the support of $T \in \mathbf{I}_{n,c}(X)$ into $l^\infty(\mathbb{N})$.

Then, we find a finite dimensional subspace of $l^\infty(\mathbb{N})$ that is "close enough" to the image of the embedding. Recall that a Banach space V has the *bounded approximation property* if there exists $\lambda \geq 1$ such that the following holds. For every compact subset $K \subset V$ and $\varepsilon > 0$ there is a finite dimensional vector subspace $V' \leq V$ and a λ-Lipschitz map $\pi\colon K \to V'$ satisfying $|\pi(x) - x| \leq \varepsilon$ for all $x \in K$. We say that V has the *metric approximation property* in the case $\lambda = 1$. Conveniently, $l^\infty(\mathbb{N})$ has this property. For a detailed proof of this fact we refer to Lemma 5.7 in [13]. In the next section we discuss the property needed to go back from $l^\infty(\mathbb{N})$ to X.

3 Lipschitz Extensions

We briefly compare property L with other Lipschitz extension properties found in the literature.

Lemma 3.1. *Let X be a locally compact metric space with property L. Then X is semi-locally quasi-convex, that is, for every point $o \in X$ there are constants $r = r(o) > 0$ and $L = L(o) \geq 1$ such that any two points $x, y \in B_o(r)$ are joined by a curve of length $\leq Ld(x, y)$ contained in $B_o(2Lr)$.*

Suppose that X is a locally compact metric space with local property L. Then each point has an *open* neighborhood U which is locally compact and has property L and hence this lemma implies that X is semi-locally quasi-convex.

Proof. Let $o \in X$ and take $\delta > 0$ small enough such that $K := B_o(\delta)$ is compact. Consider the isometric embedding $\iota\colon K \to l^\infty(\mathbb{N})$ with image $K' := \iota(K)$. By assumption there exist $\varepsilon > 0$, $L \geq 1$ and an L-Lipschitz extension $g\colon K'_\varepsilon \to X$ of $\iota^{-1}\colon K' \to X$.

Let $r := \min\{\frac{\varepsilon}{2}, \delta\}$ and consider the possibly smaller ball $B_o(r)$. For $x, y \in B_o(r)$ let $\gamma\colon [0, 1] \to l^\infty(\mathbb{N})$ be the straight segment $\gamma(t) := \iota(x) + t(\iota(y) - \iota(x))$ from $\iota(x)$ to $\iota(y)$ of length $|\iota(x) - \iota(y)| = d(x, y) \leq 2r$. The image of γ is within distance at most r from $\{\iota(x), \iota(y)\} \subset K'$ and hence contained in K'_ε. Thus $g \circ \gamma\colon [0, 1] \to X$ is a curve from x to y of length at most $Ld(x, y)$ and contained in $B_o(r + Lr) \subset B_o(2Lr)$. □

A metric space X is an *absolute Lipschitz retract* if whenever $\iota\colon X \to Y$ is an isometric embedding into a metric space Y, then there exists a Lipschitz retraction $\pi\colon Y \to \iota(X)$. It is an *absolute Lipschitz neighborhood retract* if the retraction is defined only on a neighborhood W of $\iota(X)$ in Y.

Exploiting the isometric embedding of X into the injective space $l^\infty(X)$, one can prove that X is an absolute Lipschitz (neighborhood) retract if and only if for every metric space B and every subset $A \subset B$, every Lipschitz map $f\colon A \to X$ admits a Lipschitz extension to (a neighborhood of A in) X (compare with Proposition 2.2 in [9]). In particular, X has property L if it is an absolute Lipschitz neighborhood retract or even if every compact subset is contained in one.

The opposite implication need not be true because property L only extends Lipschitz maps defined on compact sets. Also, let K be a compact subset of X and consider the inclusion $K \hookrightarrow X$. If X is an absolute Lipschitz neighborhood retract, then the inclusion extends to a Lipschitz retraction onto K, while if X has property L, then the image of the extension need not be contained in K (in fact, it might as well be the identity on X).

Lang and Schlichenmaier [11] provide an instance in which X is an absolute Lipschitz retract and so has property L (see Corollary 1.8 in [11]).

Theorem 3.2. *Suppose that X is a metric space with finite Nagata dimension $\dim_N(X) \leq n < \infty$. Then X is an absolute Lipschitz retract if and only if X is complete and Lipschitz n-connected.*

The sphere S^n has property L but is not Lipschitz n-connected for all $n \geq 1$. The latter assertion follows because by Brouwer's Fixed Point Theorem the identity $\mathrm{id}\colon S^n \to S^n$ does not admit a continuous extension $B^{n+1} \to S^n$. To see that S^n has property L one can exploit the fact that $S^n \subset \mathbb{R}^{n+1}$ is a Lipschitz neighborhood retract, and $\mathbb{R}^{n+1}$ has property L by a well-known extension result due to McShane [12].

A similar argument shows that finite piecewise Euclidean complexes have property L, we refer to [4] for the relevant terminology and results. Let X be a finite piecewise Euclidean polyhedral complex, then X is isometric to a simplicial complex, which in turn is bi-Lipschitz homeomorphic to its affine realization X' in $\mathbb{R}^N$, equipped with the induced Euclidean distance or the induced length metric, where N denotes the number of vertices. Equipped with the Euclidean distance, X' is a Lipschitz neighborhood retract in $\mathbb{R}^N$ and the same argument as above implies property L.

4 N-Approximation

We begin this section with the decomposition lemma mentioned in the introduction. Property L can be replaced with Lipschitz $(n-1)$-connected without changing the argument.

Lemma 4.1. *Let $n \geq 1$, and let X be locally compact metric space with local property L. Every $T \in \mathcal{I}_{n,c}(X)$ admits a decomposition $T = T_1 + \cdots + T_k$ with $T_i \in \mathcal{I}_{n,c}(X)$ such that each* $\operatorname{spt}(T_i)$ *is contained in* $\operatorname{spt}(T)$ *and has a neighborhood with property L. Suppose in addition that $T \in \mathbf{I}_{n,c}(X)$, then each $T_i \in \mathbf{I}_{n,c}(X)$ as well.*

Proof. Suppose that $T \in \mathbf{I}_{n,c}(X)$; the argument for $T \in \mathcal{I}_{n,c}(X)$ is simpler but the one presented here applies as well. By assumption there exist finitely many points $x_1, \ldots, x_k \in \operatorname{spt}(T)$ and radii $r_1, \ldots, r_k > 0$ such that $\operatorname{spt}(T) \subset \bigcup_{i=1}^k B_{x_i}(\frac{r_i}{2})$ and each $B_{x_i}(r_i)$ has a neighborhood with property L.

Take $s_1 \in (\frac{r_1}{2}, r_1)$ such that $T_1 := T \llcorner B_{x_1}(s_1) \in \mathbf{I}_{n,c}(X)$, then $\operatorname{spt}(T_1) \subset \operatorname{spt}(T)$, $T - T_1 = T \llcorner (X \setminus B_{x_1}(s_1)) \in \mathbf{I}_{n,c}(X)$ has support in $\operatorname{spt}(T)$ and covered by $\bigcup_{i=2}^k B_{x_i}(\frac{r_i}{2})$. Then proceed analogously for $r_2, \ldots, r_k$. □

We now prove a version of the **N**-Approximation Theorem for an integral current whose boundary is already a Lipschitz chain.

Proposition 4.2. *Let $n \geq 1$, let X be a locally compact metric space, and let $U \subset X$ be an open subset with property L. Let $T \in \mathbf{I}_{n,c}(X)$ with $\partial T \in \mathbf{L}_{n-1,c}(X)$ and* $\operatorname{spt}(T) \subset U$. *Then for every $\varepsilon > 0$ there is $R \in \mathbf{L}_{n,c}(X)$ with $\mathbf{M}(T - R) < \varepsilon$, $\partial T = \partial R$ and* $\operatorname{spt}(R) \subset \operatorname{spt}(T)_\varepsilon$, *in particular $\mathbf{N}(T - R) < \varepsilon$.*

Since $\mathbf{Z}_{0,c}(X) \subset \mathbf{L}_{0,c}(X)$, it follows that any $T \in \mathbf{I}_{1,c}(X)$ automatically satisfies the assumptions of this proposition.

Proof. Let K denote the closed $\frac{\varepsilon}{2}$-neighborhood of $\operatorname{spt}(T)$ in X, without loss of generality we might assume that K is compact and that $\operatorname{spt}(T)_\varepsilon \subset U$. Let $\iota\colon K \to l^\infty(\mathbb{N})$ be an isometric embedding with compact image $K' := \iota(K)$. By property L there exist $\varepsilon_0 > 0$, $L \geq 1$ and an L-Lipschitz extension

$$g\colon K'_{\varepsilon_0} \to X$$

of $\iota^{-1} = g\big|_{K'}$ to the open ε_0-neighborhood of K' in $l^\infty(\mathbb{N})$. According to Proposition 1.4 (**M**-Approximation) we find $P \in \mathbf{L}_{n,c}(X)$ with $\operatorname{spt}(P) \subset \operatorname{spt}(T)_{\varepsilon/2} \subset K$ and $\mathbf{M}(T - P) < \frac{\varepsilon}{6L^n} < \frac{\varepsilon}{2}$.

By the metric approximation property of $l^\infty(\mathbb{N})$ there is a finite dimensional subspace $V \subset l^\infty(\mathbb{N})$ and a 1-Lipschitz projection $\pi\colon l^\infty(\mathbb{N}) \to V$, such that $|x - \pi(x)| \leq \frac{\delta}{2}$ for all $x \in K'$, where

$$\delta := \min\left\{\frac{\varepsilon_0}{2}, \frac{\varepsilon}{3nL^n\,\mathbf{M}(\partial T - \partial P)}, \frac{\varepsilon}{4L}\right\},$$

in particular, $K'' := \pi(K') \subset K'_\delta \subset K'_{\varepsilon_0/2} \cap K'_{\varepsilon/(4L)}$.

Now, consider

$$T' := \iota_\# T \in \mathbf{I}_{n,c}(l^\infty(\mathbb{N})), \qquad P' := \iota_\# P \in \mathbf{L}_{n,c}(l^\infty(\mathbb{N})),$$
$$T'' := \pi_\# T' \in \mathbf{I}_{n,c}(V), \qquad P'' := \pi_\# P' \in \mathbf{L}_{n,c}(V).$$

Note that $\partial T' \in \mathbf{L}_{n-1,c}(l^\infty(\mathbb{N}))$, $\partial T'' \in \mathbf{L}_{n-1,c}(V)$, $\mathbf{M}(T'' - P'') \leq \mathbf{M}(T' - P') = \mathbf{M}(T - P)$, the supports of T', P' are contained in K' and the supports of T'', P'' are contained in K''.

Let $H\colon [0,1] \times K' \to l^\infty(\mathbb{N})$ denote the affine homotopy between $\mathrm{id}_{K'}$ and $\pi\big|_{K'}$, and let $W := H_\#([\![0,1]\!] \times (\partial T' - \partial P')) \in \mathbf{L}_{n,c}(l^\infty(\mathbb{N}))$ be the affine filling of $(\partial T'' - \partial P'') - (\partial T' - \partial P')$ as defined in Section 2. Note that

$H(t,\cdot)\colon K' \to l^\infty(\mathbb{N})$ is 1-Lipschitz for all $t \in [0,1]$ and $H(\cdot,x)\colon [0,1] \to l^\infty(\mathbb{N})$ has length at most $\delta/2$ for all $x \in K'$. Therefore the support $\mathrm{spt}(W)$ of W is contained in $K'_\delta \subset K'_{\varepsilon_0/2} \cap K'_{\varepsilon/(4L)}$ and its mass is bounded by

$$\mathbf{M}(W) \leq n\frac{\delta}{2}\,\mathbf{M}(\partial T' - \partial P') \leq \frac{\varepsilon}{6L^n}.$$

As $\partial(T'' - P'') = \partial T'' - \partial P'' \in \mathbf{L}_{n-1,\mathrm{c}}(V)$, by Lemma 1.1 we find $S \in \mathbf{I}_{n+1,\mathrm{c}}(V)$ satisfying

$$\begin{aligned}
&\mathbf{N}(S) \leq \eta := \min\left\{\frac{\varepsilon_0}{2}, \frac{\varepsilon}{6L^n}\right\} < \frac{\varepsilon}{4L},\\
&\mathrm{spt}(S) \subset \mathrm{spt}(T'' - P'')_\eta \subset K''_\eta \subset K'_{\eta+\delta} \subset K'_{\varepsilon_0} \cap K'_{\varepsilon/(2L)},\\
&T'' - P'' - \partial S \in \mathbf{L}_{n,\mathrm{c}}(V),
\end{aligned}$$

(in fact, $\mathrm{spt}(S)$ is contained in the open η-neighborhood of $\mathrm{spt}(T'' - P'')$ in V).

Finally, note that $T'' - P'' - \partial S$ and W are both Lipschitz n-chains with supports in $K'_{\varepsilon_0} \cap K'_{\varepsilon/(2L)}$, so that $g_\#\big(T'' - P'' - \partial S - W\big) \in \mathbf{L}_{n,\mathrm{c}}(X)$ is well defined, has support in $g(K'_{\varepsilon/(2L)}) \subset K_{\varepsilon/2}$, mass

$$\mathbf{M}\left(g_\#\big(T'' - P'' - \partial S - W\big)\right) \leq L^n\left(\mathbf{M}(T'' - P'') + \mathbf{M}(\partial S) + \mathbf{M}(W)\right) \leq \frac{\varepsilon}{2}$$

and boundary

$$\begin{aligned}
\partial\big(g_\#\big(T'' - P'' - \partial S - W\big)\big) &= g_\#\big(\partial T'' - \partial P'' - \big(\partial T'' - \partial P'' - \partial T' + \partial P'\big)\big)\\
&= g_\#\big(\partial T' - \partial P'\big)\\
&= \partial T - \partial P,
\end{aligned}$$

where in the last equality we have used that $g\big|_{K'} = \iota^{-1}$. Overall, the Lipschitz n-chain

$$R := P + g_\#\big(T'' - P'' - \partial S - W\big) \in \mathbf{L}_{n,\mathrm{c}}(X)$$

satisfies $\mathbf{M}(T - R) \leq \mathbf{M}(T - P) + \mathbf{M}(R - P) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, $\partial R = \partial P + \partial T - \partial P = \partial T$ and $\mathrm{spt}(R) \subset \mathrm{spt}(P) \cup K_{\varepsilon/2} \subset \mathrm{spt}(T)_\varepsilon$. □

Corollary 4.3. *Let $n \geq 1$, let X be a locally compact metric space, and let $U \subset X$ be an open subset with property L. Then for every $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ with* $\mathrm{spt}(Z) \subset U$*, and every $\varepsilon > 0$, there is $R \in \mathbf{L}_{n,\mathrm{c}}(X)$ with $\partial R = 0$, $\mathbf{N}(Z - R) < \varepsilon$, and* $\mathrm{spt}(R) \subset \mathrm{spt}(Z)_\varepsilon$.

If X has local property L we are not able to prove statements like Proposition 4.2 and Corollary 4.3 for integral currents in X whose boundary is a Lipschitz chain but without restrictions on their supports. This is because even if $\partial T \in \mathbf{L}_{n-1,\mathrm{c}}(X)$ or $\partial T = 0$, the decomposition $T = T_1 + \cdots + T_k$ of Lemma 4.1 does not prevent $\partial T_i \notin \mathbf{L}_{n-1,\mathrm{c}}(X)$ or $\partial T_i \neq 0$ for some i.

Nonetheless, if $n = 1$ then we can improve the conclusion of Theorem 1.3. Indeed $\partial T_i \in \mathbf{Z}_{0,\mathrm{c}}(X) \subset \mathbf{L}_{0,\mathrm{c}}(X)$ for all i, and by Proposition 4.2 we obtain $R_1, \ldots, R_k \in \mathbf{L}_{1,\mathrm{c}}(X)$ with $\mathbf{M}(T_i - R_i) < \varepsilon/k$, $\partial R_i = \partial T_i$, and $\mathrm{spt}(R_i) \subset \mathrm{spt}(T_i)_\varepsilon$ for all i. Thus $P := \sum R_i \in \mathbf{L}_{1,\mathrm{c}}(X)$ satisfies $\partial P = \partial T$, $\mathbf{N}(T - P) < \varepsilon$ and $\mathrm{spt}(P) \subset \mathrm{spt}(T)_\varepsilon$.

If U is an open subset of X with property L and a $K \subset U$ is compact, then we can consider the isometric embedding $\iota\colon K \to l^\infty(\mathbb{N})$ and the Lipschitz extension $g\colon K'_{\varepsilon_0} \to X$ of $\iota^{-1} = g\big|_{K'}$. If $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ has support in K we can fill $\iota_\# Z$ in $l^\infty(\mathbb{N})$, and if $\mathbf{M}(Z)$ is small enough can push the filling back into X using g. The next proposition establishes this result precisely.

Proposition 4.4. *Let $n \geq 1$, let X be a locally compact metric space, and let $U \subset X$ be an open subset with property L. Then for every compact subset $K \subset U$ and $\varepsilon > 0$ there exists $M > 0$ such that every $Z \in \mathbf{Z}_{n,\mathrm{c}}(X)$ with* $\mathrm{spt}(Z) \subset K$ *and $\mathbf{M}(Z) < M$ possesses a filling $S \in \mathbf{I}_{n+1,\mathrm{c}}(X)$ with* $\mathrm{spt}(S) \subset \mathrm{spt}(Z)_\varepsilon$ *and $\mathbf{M}(S) < \varepsilon$.*

As in the proof of Proposition 4.2 we pass to a finite dimensional subspace V of $l^\infty(\mathbb{N})$, then we use that V admits a Euclidean isoperimetric inequality for $\mathbf{Z}_{n,\mathrm{c}}(V)$, and the existence of solutions to the Plateau problem.

Therefore every $Z \in \mathbf{Z}_{n,c}(V)$ admits a filling $S \in \mathbf{I}_{n+1,c}(V)$ with $\mathbf{M}(S) \leq C\,\mathbf{M}(Z)^{(n+1)/n}$ and support spt(S) within distance at most $(n+1)C\,\mathbf{M}(Z)^{1/n}$ from spt(Z), where C is a constant depending only on n. This was shown for classical integral currents in [6], later for Lipschitz cycles in Banach spaces [7], and holds more generally for metric currents (see [15] and [8]).

Proof. Let $\iota\colon K \to l^\infty(\mathbb{N})$ be an isometric embedding with compact image $K' := \iota(K)$. By property L there exist $\varepsilon_0 > 0$, $L \geq 1$ and an L-Lipschitz extension $g\colon K'_{\varepsilon_0} \to X$ of $\iota^{-1} = g\big|_{K'}$. We might assume that $\varepsilon \leq 1$ and $\varepsilon_0 < \varepsilon/L$ so that $g(K'_{\varepsilon_0}) \subset K_\varepsilon$. By the metric approximation property of $l^\infty(\mathbb{N})$, there exist a finite dimensional subspace $V \subset l^\infty(\mathbb{N})$ and a 1-Lipschitz map $\pi\colon l^\infty(\mathbb{N}) \to V$ such that $\left|x - \pi(x)\right| \leq \varepsilon_0/4$ for all $x \in K'$, in particular $K'' := \pi(K') \subset K'_{\varepsilon_0/2}$.

Let $C \geq 1$ be the constant from above, set

$$M := \min\left\{\left(\frac{\varepsilon_0}{2(n+1)C}\right)^n, \frac{\varepsilon}{(C + \frac{n+1}{4}\varepsilon_0)L^{n+1}}\right\}$$

and let $Z \in \mathbf{Z}_{n,c}(X)$ with spt$(Z) \subset K$ and $\mathbf{M}(Z) < M$.

Consider

$$Z' := \iota_\# Z \in \mathbf{Z}_{n,c}(l^\infty(\mathbb{N})), \qquad Z'' := \pi_\# Z' \in \mathbf{Z}_{n,c}(V),$$

which have supports in K' and K'', respectively, and satisfy $\mathbf{M}(Z'') \leq \mathbf{M}(Z') = \mathbf{M}(Z) < M$. Let $H\colon [0,1] \times \text{spt}(Z')' \to l^\infty(\mathbb{N})$ denote the affine homotopy between $\text{id}_{\text{spt}(Z')}$ and $\pi\big|_{\text{spt}(Z')}$, and let $Q := H_\#([\![0,1]\!] \times Z') \in \mathbf{I}_{n+1,c}(l^\infty(\mathbb{N}))$ be the affine filling of $Z'' - Z'$, as defined in Section 2. Note that $H(t,\cdot)\colon \text{spt}(Z') \to l^\infty(\mathbb{N})$ is 1-Lipschitz for all $t \in [0,1]$ and $H(\cdot,x)\colon [0,1] \to l^\infty(\mathbb{N})$ has length at most $\varepsilon_0/4$ for all $x \in \text{spt}(Z')$. Thus the support spt(Q) of Q is contained in $\text{spt}(Z')_{\varepsilon_0} \subset K'_{\varepsilon_0}$ and its mass is bounded by

$$\mathbf{M}(Q) \leq (n+1)\frac{\varepsilon_0}{4}\,\mathbf{M}(Z') < \frac{n+1}{4}\varepsilon_0 M.$$

As noted above Z'' possesses a filling $S'' \in \mathbf{I}_{n+1,c}(V)$ with mass

$$\mathbf{M}(S'') \leq C\,\mathbf{M}(Z'')^{\frac{n+1}{n}} < CM^{\frac{n+1}{n}} \leq CM$$

and support within distance at most $(n+1)C\,\mathbf{M}(Z'')^{1/n} < \varepsilon_0/2$ from $\text{spt}(Z'') \subset \pi(\text{spt}(Z'))$, in particular it is contained in $\pi(\text{spt}(Z'))_{\varepsilon_0/2} \subset \text{spt}(Z')_{\varepsilon_0} \subset K'_{\varepsilon_0}$.

Finally, S'' and Q have support in $\text{spt}(Z')_{\varepsilon_0} \subset K'_{\varepsilon_0}$ so that $S := g_\#(S'' - Q) \in \mathbf{I}_{n+1,c}(X)$ is well defined, has support in $g(\text{spt}(Z')_{\varepsilon_0}) \subset \text{spt}(Z)_\varepsilon$, has boundary $\partial S = g_\#(\partial S'' - \partial Q) = g_\#(Z'' - Z'' + Z') = Z$ and its mass is bounded by

$$\mathbf{M}(S) \leq L^{n+1}\left(\mathbf{M}(S'') + \mathbf{M}(Q)\right) < L^{n+1}\left(C + \tfrac{n+1}{4}\varepsilon_0\right)M \leq \varepsilon. \qquad \square$$

We can now upgrade Proposition 4.2 to any current $T \in \mathbf{I}_{n,c}(X)$.

Proposition 4.5. *Let $n \geq 1$, let X be a locally compact metric space, and let $U \subset X$ be an open subset with property L. Then for every $T \in \mathbf{I}_{n,c}(X)$ with* spt$(T) \subset U$, *and every $\varepsilon > 0$, there is $P \in \mathbf{L}_{n,c}(X)$ with $\mathbf{N}(T - P) < \varepsilon$ and* spt$(P) \subset \text{spt}(T)_\varepsilon$.

The case $n = 1$ holds already by Proposition 4.2, so that in this proof we can assume that $n \geq 2$ and apply Proposition 4.4 in dimension $n - 1 \geq 1$.

Proof. Suppose $n \geq 2$. Let K denote the closed $\frac{\varepsilon}{2}$-neighborhood of spt(T) in X, without loss of generality we might assume that K is compact and that $\text{spt}(T)_\varepsilon \subset U$. Let $M > 0$ be the constant of Proposition 4.4 for K and $\varepsilon/4$; up to decreasing it we might assume that $M \leq \varepsilon/4$.

Consider $T' := \partial T \in \mathbf{Z}_{n-1,c}(X)$ and note that $\text{spt}(T')_{\varepsilon/2} \subset \text{spt}(T)_{\varepsilon/2} \subset K \subset U$. By Proposition 4.2 we can find $P' \in \mathbf{L}_{n-1,c}(X)$ with $\partial P' = \partial T'(=0)$, $\mathbf{M}(T' - P') < M \leq \varepsilon/4$ and $\text{spt}(P') \subset \text{spt}(T')_{\varepsilon/4} \subset K$.

According to Proposition 4.4 and the choice of M, there exists a filling $S \in \mathbf{I}_{n,c}(X)$ of $T' - P'$ with $\mathbf{M}(S) < \varepsilon/4$ and $\text{spt}(S) \subset \text{spt}(T')_{\varepsilon/2} \subset U$.

Note that $T - S \in \mathbf{I}_{n,c}(X)$ has support contained in $\mathrm{spt}(T)_{\varepsilon/2} \subset U$ and boundary $\partial(T - S) = T' - (T' - P') = P' \in \mathbf{L}_{n-1,c}(X)$ so applying Proposition 4.2 a second time we find $P \in \mathbf{L}_{n,c}(X)$ with $\mathbf{M}(T - S - P) < \varepsilon/2$, $\partial P = \partial(T - S) = P'$, and $\mathrm{spt}(P) \subset \mathrm{spt}(T - S)_{\varepsilon/2} \subset \mathrm{spt}(T)_{\varepsilon}$.

Therefore P satisfies:

$$\mathbf{M}(T - P) \leq \mathbf{M}(T - S - P) + \mathbf{M}(S) < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} \leq \varepsilon,$$
$$\mathbf{M}(\partial T - \partial P) = \mathbf{M}(T' - P') < M < \varepsilon.$$

□

The proof of Theorem 1.3 (**N**-Approximation) now follows by combining Lemma 4.1 and Proposition 4.5.

Proof of Theorem 1.3. Let X be a locally compact metric space with local property L, $T \in \mathbf{I}_{n,c}(X)$ and $\varepsilon > 0$. By Lemma 4.1 we can write $T = T_1 + \cdots + T_k$ with each $T_i \in \mathbf{I}_{n,c}(X)$ having support contained in both $\mathrm{spt}(T)$ and in an open subset of X having property L. By Proposition 4.5 there exist $P_i \in \mathbf{L}_{n,c}(X)$ with $\mathbf{N}(T_i - P_i) < \varepsilon/k$ and $\mathrm{spt}(P_i) \subset \mathrm{spt}(T_i)_{\varepsilon} \subset \mathrm{spt}(T)_{\varepsilon}$, so that $P := P_1 + \cdots + P_k \in \mathbf{L}_{n,c}(X)$ is the desired Lipschitz approximation of T with $N(T - P) < \varepsilon$ and $\mathrm{spt}(P) \subset \mathrm{spt}(T)_{\varepsilon}$. □

Acknowledgements: I am indebted to Urs Lang for suggesting the problem and providing key insights while writing this paper. I also wish to thank Lisa Ricci for many helpful discussions. I thank an anonymous referee for many valuable comments. I gratefully acknowledge the support of the Swiss National Science Foundation.

Conflict of interest: The author states no conflict of interest.

References

[1] L. Ambrosio and B. Kirchheim. Currents in metric spaces. *Acta Mathematica* **185** (2000), 1–80.

[2] L. Ambrosio and S. Wenger. Rectifiability of flat chains in Banach spaces with coefficients in $\mathbb{Z}_p$. *Math. Z.* **268** (2011), 477–506.

[3] G. Basso, S. Wenger, and R. Young. Undistorted fillings in subsets of metric spaces (2021). arXiv: 2112.11905 [math.MG].

[4] M. R. Bridson and A. Haefliger. Metric Spaces of Non-Positive Curvature. Springer Berlin Heidelberg (1999).

[5] T. De Pauw. Approximation by polyhedral G chains in Banach spaces. *Z. Anal. Anwend.* **33** (2014), 311–334.

[6] H. Federer and W. H. Fleming. Normal and Integral Currents. *Annals of Mathematics* **72** (1960), 458–520.

[7] M. Gromov. Filling Rimannian Manifolds. *J. Differential Geom.* **18** (1983), 1–147.

[8] B. Kleiner and U. Lang. Higher rank hyperbolicity. *Inventiones mathematicae* **221** (2020), 597–664.

[9] U. Lang. Injective hulls of certain discrete metric spaces and groups. *J. Topol. Anal.* **5** (2013), 297–331.

[10] U. Lang. Local currents in metric spaces. *J. Geom. Anal.* **21** (2011), 683–742.

[11] U. Lang and T. Schlichenmaier. Nagata Dimension, Quasisymmetric Embeddings, and Lipschitz Extensions. *Internat. Math. Res. Notices* **58** (2005), 3625–3655.

[12] E. J. McShane. Extension of range of functions. *Bull. Amer. Math. Soc.* **40** (1934), 837 –842.

[13] E. Paolini and E. Stepanov. Decomposition of acyclic normal currents in a metric space. *J. Funct. Anal.* **263** (2012), 3358–3390.

[14] J. W. Portegies. Semicontinuity of eigenvalues under intrinsic flat convergence. *Calc. Var. Partial Differential Equations* **54** (2015), 1725–1766.

[15] S. Wenger. Isoperimetric inequalities of Euclidean type in metric spaces. *Geom. Funct. Anal.* **15** (2005), 534–554.