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ARTICLE

Improved bound for improper colourings of graphs with no odd clique minor

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Abstract

Strengthening Hadwiger's conjecture, Gerards and Seymour conjectured in 1995 that every graph with no odd K_t-minor is properly $(t-1)$-colourable. This is known as the *Odd Hadwiger's conjecture*. We prove a relaxation of the above conjecture, namely we show that every graph with no odd K_t-minor admits a vertex $(2t-2)$-colouring such that all monochromatic components have size at most $\lceil \frac{1}{2}(t-2)\rceil$. The bound on the number of colours is optimal up to a factor of 2, improves previous bounds for the same problem by Kawarabayashi (2008, *Combin. Probab. Comput.* **17** 815–821), Kang and Oum (2019, *Combin. Probab. Comput.* **28** 740–754), Liu and Wood (2021, arXiv preprint, arXiv:1905.09495), and strengthens a result by van den Heuvel and Wood (2018, *J. Lond. Math. Soc.* **98** 129–148), who showed that the above conclusion holds under the more restrictive assumption that the graph is K_t-minor-free. In addition, the bound on the component-size in our result is much smaller than those of previous results, in which the dependency on t was given by a function arising from the graph minor structure theorem of Robertson and Seymour. Our short proof combines the method by van den Heuvel and Wood for K_t-minor-free graphs with some additional ideas, which make the extension to odd K_t-minor-free graphs possible.

Keywords: Odd Hadwiger's conjecture; graph minors; odd minors; clustered coloring; defective coloring; improper coloring

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1. Introduction

Given an integer $t \geq 1$, a K_t*-model* is a graph F consisting of t vertex-disjoint trees $(T_s)_{s=1}^t$, each two of them joined by exactly one additional edge. We say that a graph G *contains* K_t *as a minor*, or that it *contains a* K_t*-minor*, if G contains a subgraph which is a K_t-model.

The work in this paper is motivated by the famous graph colouring conjecture of Hadwiger.

Conjecture 1.1. (Hadwiger's conjecture, 1943, [7]). *For every integer $t \geq 2$, if G is a graph not containing a K_t-minor, then G is properly $(t-1)$-colourable.*

A lot of work in graph theory has been inspired by and built around Hadwiger's conjecture. A survey of results and open problems covering the state of the art up until roughly 3 years ago was written by Seymour [25]. Hadwiger's conjecture has been proved for all values $t \leq 6$ (see Robertson, Seymour and Thomas [24]). It remains open starting from $t = 7$. For a long time, the best asymptotic upper bound on the chromatic number of graphs with no K_t-minor has remained $O(t\sqrt{\log t})$ as established independently by Kostochka and Thomason [15, 27]. However, this

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bound was improved considerably recently, see [2, 18, 19, 21, 22]. The current best bound of $O(t \log\log t)$ was obtained roughly a year ago by Delcourt and Postle [2].

A K_t-model F is said to be *odd* if there exists a 2-colouring c of $V(F)$ such that the restriction of c to any single tree T_s forms a proper colouring of that tree, while every edge joining two distinct trees is monochromatic with respect to c, i.e., has the same colours at its endpoints. We say that a graph G *contains an odd K_t-minor* or that it contains K_t *as an odd minor* if G has a subgraph which is an odd K_t-model. Gerards and Seymour (see [6], Section 6.5) proposed the following strengthening of Hadwiger's conjecture, called *Odd Hadwiger's conjecture*.

Conjecture 1.2. (Odd Hadwiger's conjecture, 1995, [6]). *For every integer $t \geq 2$, if G is a graph not containing an odd K_t-minor, then G is properly $(t-1)$-colourable.*

To see that this conjecture indeed considerably strengthens Hadwiger's conjecture, consider for example $t = 3$. While Hadwiger's conjecture in this case amounts to saying that forests (the K_3-minor-free graphs) are 2-colourable, the Odd Hadwiger's conjecture captures the more general statement that all graphs without odd cycles (the odd K_3-minor-free graphs) are 2-colourable. In general, every K_t-minor-free graph is also odd K_t-minor-free, but there are odd K_3-minor-free (i.e., bipartite) graphs which contain arbitrarily large clique minors.

The above conjecture has been verified for $t \leq 4$ by Catlin [1], and a solution for the case $t = 5$ was announced by Guenin (cf. [25]). For $t \geq 6$ the conjecture remains wide open. As for Hadwiger's conjecture, asymptotic upper bounds on the chromatic number of odd K_t-minor-free graphs have been studied. An upper bound of $O(t\sqrt{\log t})$ was proved by Geelen, Gerards, Reed, Seymour and Vetta in [5] (see also [11]). Recently this has been improved in [2, 20, 23, 26], with the current best bound being $O(t \log\log t)$ from [26]. For more results around the Odd Hadwiger's conjecture, we refer the interested reader to Section 7 of the survey [25].

Given a (not necessarily proper) colouring $c: V(G) \to S$ of a graph G, a subset of vertices is called a *monochromatic component*, if it is a component of the induced subgraph $G[c^{-1}(s)]$ for some $s \in S$. For instance, a colouring is proper iff all its monochromatic components have size 1.

The purpose of this paper is to prove the following relaxation of the Odd Hadwiger's conjecture, in which we allow our colouring to be improper, but instead require a constant bound (depending only on t) for the maximum size of its monochromatic components. In return, our colouring uses much fewer colours than the known results for proper colourings.

Theorem 1.3. *Let $t \geq 3$ be an integer. Then every graph G without an odd K_t-minor admits a (not necessarily proper) vertex colouring using $2t-2$ colours such that all monochromatic components have size at most $\lceil \frac{1}{2}(t-2) \rceil$.*

Theorem 1.3 lines up with a wide set of results on so-called *improper colourings* of graphs with excluded minors. Instead of giving a long list of the individual results, let us just point to the comprehensive 70 page-survey on improper colourings written recently by Wood [29] as well as to Section 6 of Seymour's survey [25]. Two main variants of improper colourings have been studied: *clustered* and *defective* colourings. Given a graph G and integers k, c, d, we say that a k-vertex colouring of G has *clustering c* if all monochromatic components have size at most c, and we say that it has *defect d* if the maximum degree of all monochromatic components is bounded by d. Clearly, every k-colouring with clustering c also has defect $c-1$. We may therefore rephrase Theorem 1.3 by saying that for $t \geq 3$, every odd K_t-minor-free graph is $2(t-1)$-colourable with clustering $\lceil \frac{1}{2}(t-2) \rceil$ and defect $\lceil \frac{1}{2}(t-4) \rceil$. The number of colours in our result improves upon previous results for this problem by Kawarabayashi [10], Kang and Oum [9] and Liu and Wood [17], summarized in Table 1 below. It is optimal up to a factor of 2, as it was shown in [4, 9] that (odd) K_t-minor-free graphs in general do not admit $(t-2)$-colourings with clustering bounded as a function of t.

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Table 1. Bounds for improper colourings of odd K_t-minor-free graphs. The functions $f_1(t), \dots, f_4(t)$ are used to indicate that the bound guaranteed on the defect or clustering is only dependent on t. These functions depend on bounds from the graph minor structure theorem of Robertson and Seymour

	Number of colours	Clustering	Defect
Kawarabayashi [10]	$496t$	$f_1(t)$	$f_1(t) - 1$
Kang and Oum [9]	$6t - 9$	–	$f_2(t)$
Kang and Oum [9]	$10t - 13$	$f_3(t)$	$f_3(t) - 1$
Liu and Wood [17]	$8t - 12$	$f_4(t)$	$f_4(t) - 1$
This paper	$2t - 2$	$\lceil \frac{1}{2}(t-2) \rceil$	$\lceil \frac{1}{2}(t-4) \rceil$

As an additional advantage, our result also improves the dependency of the size of the clustering on t. Namely, the bounds on the clustering from [9, 10, 17] were functions of t depending on bounds for the graph minor structure theorem of Robertson and Seymour.

Clustered and defective colourings of K_t-minor-free graphs have also been extensively studied, see [4, 8, 12, 16, 17, 28]. Here the state of the art bounds are as follows: For defective colouring it was shown by Edwards, Kang, Kim, Oum and Seymour [4] that K_t-minor-free graphs can be $(t-1)$-coloured with bounded defects. Van den Heuvel and Wood [8] proved that the defect can be bounded by $t-2$. For clustered colouring, it has been proved that K_t-minor-free graphs can be $(t+1)$-coloured with bounded clustering by Liu and Wood [17]. An optimal bound of $t-1$ colours was announced in 2017 by Dvořák and Norin [3]. A weaker bound on the number of colours, however with an explicit bound on the clustering, was previously shown by van den Heuvel and Wood, namely that every K_t-minor-free graph is $(2t-2)$-colourable with clustering $\lceil \frac{1}{2}(t-2) \rceil$. Theorem 1.3 extends this result by van den Heuvel and Wood to odd K_t-minor-free graphs.

In the remainder of this paper, we give the proof of Theorem 1.3. Our proof follows closely a method introduced by van den Heuvel and Wood in [8] to first establish a decomposition of the considered graphs into nicely structured disjoint subgraphs, from which a clustered colouring can then easily be obtained. Our decomposition result (Theorem 2.3) is similar to a corresponding result for K_t-minor-free graphs by van den Heuvel and Wood, but enhances it by some additional features, through which the extension from K_t-minor-free graphs to odd K_t-minor-free graphs becomes possible.

2. Proof of Theorem 1.3

We need the following lemma proved by van den Heuvel and Wood in [8].

Lemma 2.1. (cf. Lemma 8, item (4) in [8]). *Let G be a connected graph, and let $S \subseteq V(G)$ be such that $|S| = k \geq 1$. Let $H \subseteq G$ be an induced connected subgraph with a minimum number of vertices such that $S \subseteq V(H)$.*

Then H admits a partition of its vertex-set into two disjoint (possibly empty) subsets A and B such that both $G[A]$ and $G[B]$ have all their connected components of size at most $\lceil \frac{k}{2} \rceil$.

The main idea of our proof is the following modified version of the above lemma, which will be useful for constructing odd minors. We use the following notation: Given a graph H and a partition of its vertex-set into subsets A and B, we denote by $H[A, B]$ the spanning bipartite subgraph containing all the edges with one endpoint in A and one endpoint in B.

Lemma 2.2. *Let G be a connected graph, and let $S \subseteq V(G)$ be such that $|S| = k \geq 1$. Then there exists a connected induced subgraph $H \subseteq G$ with $S \subseteq V(H)$ such that the following hold:*

1. *H admits a partition of its vertex-set into two disjoint subsets A and B such that both $G[A]$ and $G[B]$ have maximum component-size at most $\lceil \frac{k}{2} \rceil$.*
2. *$H[A, B]$ is connected.*
3. *For every vertex $v \in V(G) \setminus V(H)$ which is adjacent in G to at least one vertex in $V(H)$, there exist vertices $a \in A$ and $b \in B$ such that $av, bv \in E(G)$.*

Proof. By Lemma 2.1, there exists at least one connected induced subgraph H_0 of G such that $S \subseteq V(H_0)$ and a partition of $V(H_0)$ into subsets A_0, B_0 such that both $G[A_0], G[B_0]$ have maximum component-size at most $\lceil \frac{k}{2} \rceil$.

Now, let (H, A, B) be a triple consisting of a connected induced subgraph $H \subseteq G$ with $S \subseteq V(H)$ and a partition $V(H) = A \cup B$ of its vertex-set such that $G[A]$ and $G[B]$ have maximum component-size at most $\lceil \frac{k}{2} \rceil$, chosen such that the number of edges in $H[A, B]$ is maximized among all possible choices of such triples.

We now claim that H with the partition A, B satisfies all three properties required by the lemma. Statement (1) follows directly by our choice of the triple. To verify (2), suppose towards a contradiction that $H[A, B]$ is disconnected. This would mean that there exists a partition of $V(H)$ into non-empty sets X, Y such that there are no edges between $X \cap A$ and $Y \cap B$, and no edges between $X \cap B$ and $Y \cap A$ in H. Now define a new partition of $V(H)$ by $A' := (X \cap A) \cup (Y \cap B)$, and $B' := (X \cap B) \cup (Y \cap A)$. It is easy to see that since no edges in G connect $X \cap A$ and $Y \cap B$ or $X \cap B$ and $Y \cap A$, every component of $G[A']$ or $G[B']$ is fully contained in either A or B. Hence it is contained in a component of either $G[A]$ or $G[B]$, and hence has size at most $\lceil \frac{k}{2} \rceil$. However, since H is connected, there exists at least one edge $e \in E(H)$ with endpoints in X and Y. This edge must then connect $X \cap A$ and $Y \cap A$ or $X \cap B$ and $Y \cap B$. In each case, e is contained in the bipartite subgraph of H spanned between A' and B'. Also, every edge of $H[A, B]$ has exactly one endpoint in A' and in B'. Hence, (H, A', B') is a triple satisfying all required properties which has strictly more edges between different sets in the partition than (H, A, B). This is a contradiction to our choice of (H, A, B), and proves (2).

Finally, let us verify (3). Towards a contradiction, suppose that there exists a vertex $v \in V(G) \setminus V(H)$ such that v is adjacent in G to at least one vertex in $V(H)$, but it does not have neighbours both in A and in B. Then, w.l.o.g. (renaming A and B if necessary) we may assume that v has no neighbours in A. Now, let $H' := G[V(H) \cup \{v\}]$ and put $A' := A \cup \{v\}$ and $B' := B$. Then H' is a connected induced subgraph of G with $S \subseteq V(H) \subseteq V(H')$. Since v has no neighbours in A, every component in $G[A']$ or $G[B']$ is either a component of $G[A]$ or $G[B]$ and hence has size at most $\lceil \frac{k}{2} \rceil$, or is equal to $\{v\}$ and has size $1 \leq \lceil \frac{k}{2} \rceil$.

Furthermore, the number of edges in H' spanned between A' and B' is strictly greater than the number of edges in H spanned between A and B, since in addition to these edges we have the edges incident to v in H'. This again shows that (H', A', B') is a triple satisfying all required properties with more edges between different sets in the partition than (H, A, B), contradicting our maximality assumption. This shows that also (3) is satisfied for (H, A, B) and concludes the proof of the lemma. □

We next use the above lemma to prove the following decomposition result, which resembles a corresponding decomposition theorem proved by van den Heuvel and Wood in [8] for K_t-minor-free graphs (compare Theorem 11 in [8]). It extends part of the latter result with some additional features that will allow us to relate to odd minor containment instead of ordinary minor containment when building the decomposition of our graph. Once the decomposition theorem (Theorem 2.3 below) is established, Theorem 1.3 will follow easily. In the following, given a graph

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G and two vertex-disjoint subgraphs H_1 and H_2 of G, we say that H_1 and H_2 are *adjacent* (in G) if there exist vertices $x \in V(H_1), y \in V(H_2)$ such that $xy \in E(G)$.

Theorem 2.3. *Let $t \geq 3$ be an integer, and let G be a connected graph without an odd K_t-minor. Then there exists $\ell \in \mathbb{N}$ and a collection $H_1 \ldots, H_\ell$ of vertex-disjoint induced connected subgraphs of G with $V(H_1) \cup \cdots \cup V(H_\ell) = V(G)$ such that all of the following properties are satisfied for every $i \in [\ell]$:*

1. *H_i admits a partition of its vertex-set into two disjoint parts A_i and B_i such that in each of $G[A_i]$, $G[B_i]$, the maximum component-size is at most $\lceil \frac{t-2}{2} \rceil$.*
2. *$H_i[A_i, B_i]$ is connected.*
3. *For every vertex $v \in V(G) \setminus (V(H_1) \cup \cdots \cup V(H_i))$ which is adjacent in G to at least one vertex in $V(H_i)$, there exist vertices $a \in A_i, b \in B_i$ such that $av, bv \in E(G)$.*
4. *For every connected component C of $G - (V(H_1) \cup \cdots \cup V(H_i))$, at most $(t-2)$ among the subgraphs $H_1, \ldots, H_i$ are adjacent to C, and these subgraphs are pairwise adjacent to each other.*

Proof. We construct the induced connected subgraphs $H_1, \ldots, H_\ell$ iteratively, maintaining the properties $(1)-(4)$ for all already constructed subgraphs in the sequence during the process.

Let $\mathcal{Z}$ denote the collection of all vertex-subsets $Z \subseteq V(G)$ such that $G[Z]$ is bipartite and connected (note that $\mathcal{Z} \neq \emptyset$, since every singleton-set in $V(G)$ belongs to $\mathcal{Z}$). Let now $X \in \mathcal{Z}$ be an inclusion-wise maximal member of $\mathcal{Z}$. Define $H_1 := G[X]$. By choice of X, the subgraph H_1 of G is induced, bipartite and connected. Let us further verify that the invariants $(1)-(4)$ are satisfied: To verify (1), we can simply let A_1, B_1 be the colour classes of a bipartition of H_1. Item (2) is satisfied trivially, since H_1 is connected and all edges of H_1 join A_1 and B_1. For item (3), consider any vertex $v \in V(G) \setminus X$ which has a neighbour in X. We have $X \cup \{v\} \notin \mathcal{Z}$ by our choice of X, and hence, $G[X \cup \{v\}]$ is non-bipartite. This means that v must have neighbours both in A_1 and B_1, for otherwise either $(A_1 \cup \{v\}, B_1)$ or $(A_1, B_1 \cup \{v\})$ would form a bipartition of $G[X \cup \{v\}]$. Finally, this implies that there are neighbours $a \in A_1, b \in B_1$ of v, as required. Finally, item (4) is trivially satisfied, since $t-2 \geq 1$.

Next, suppose that for some integer $i \geq 1$ we have already constructed disjoint induced connected subgraphs $H_1, \ldots, H_i$ of G, each satisfying the invariants $(1)-(4)$, but such that $V(H_1) \cup \cdots \cup V(H_i) \neq V(G)$. Now, pick (arbitrarily) a connected component C of the graph $G - (V(H_1) \cup \cdots \cup V(H_i))$. Let $Q_1, \ldots, Q_k$ be the (ordered) sublist of $H_1, \ldots, H_i$, containing exactly those subgraphs which are adjacent to $G[C]$. Since G is connected, we have $k \geq 1$. By the invariant (4) we furthermore know that $k \leq t-2$ and that $Q_1, \ldots, Q_k$ are pairwise adjacent to each other. For every index $j \in [k]$, by definition there exists a vertex $v_j \in C$ such that v_j has a neighbour in Q_j. Let $S := \{v_1, \ldots, v_k\}$. Now, apply Lemma 2.2 to the connected graph $G[C]$ and the set S. We conclude that there exists an induced and connected subgraph H of G such that $S \subseteq V(H) \subseteq C$, equipped with a partition of its vertex-sets into subsets A and B such that

- all components of $G[A]$ and $G[B]$ have size at most $\left\lceil \frac{|S|}{2} \right\rceil \leq \lceil \frac{k}{2} \rceil \leq \lceil \frac{t-2}{2} \rceil$,
- $H[A, B]$ is connected,
- every vertex $v \in C \setminus V(H)$ which is adjacent to a vertex in $V(H)$ has neighbours both in A and in B.

We now finally define $H_{i+1} := H$ and $A_{i+1} := A, B_{i+1} := B$, and claim that the extended sequence $H_1, \ldots, H_i, H_{i+1}$ still satisfies the invariants $(1)-(4)$. That the invariants (1) and (2) remain valid is an immediate consequence of the first two properties of H listed above. Let us now verify that invariants (3) and (4) hold (and clearly, these need only be checked for the index $i+1$, since the claim is satisfied for smaller indices by assumption).

For invariant (3), let a vertex $v \in V(G) \setminus (V(H_1) \cup \cdots \cup V(H_i) \cup V(H_{i+1}))$ be given arbitrarily, and suppose that v has at least one neighbour in H_{i+1}. Note that this implies that $v \in C$, since C is a connected component of $G - (V(H_1) \cup \cdots \cup V(H_i))$ and $V(H_{i+1}) = V(H) \subseteq C$. Therefore, by the third property of H listed above, we conclude that v has neighbours both in $A_{i+1} = A$ and in $B_{i+1} = B$. This verifies that the invariant (3) remains satisfied.

Finally, let us consider invariant (4). For this purpose, let a connected component C' of the graph $G - (V(H_1) \cup \cdots \cup V(H_i) \cup V(H_{i+1}))$ be given to us arbitrarily. Let $\mathcal{Q} \subseteq \{H_1, \ldots, H_i, H_{i+1}\}$ contain all the subgraphs adjacent to $G[C']$ in G. In order to verify invariant (4) for C', we need to show that $|\mathcal{Q}| \le t-2$ and that the members of $\mathcal{Q}$ are pairwise adjacent to each other.

Since C is a connected component of $G - (V(H_1) \cup \cdots \cup V(H_i))$, we must either have $C' \cap C = \emptyset$ or $C' \subseteq C$, for otherwise $C \cup C'$ would induce a connected subgraph of $G - (V(H_1) \cup \cdots \cup V(H_i))$ and strictly contain C, a contradiction. For the same reason, if $C' \cap C = \emptyset$ then there is no edge in G connecting C to C', and hence C' in particular forms a connected component also of the graph $G - (V(H_1) \cup \cdots \cup V(H_i))$, and $H_{i+1} \notin \mathcal{Q}$. Therefore, in the case $C' \cap C = \emptyset$ the facts that $|\mathcal{Q}| \le t-2$ and that the members of $\mathcal{Q}$ are pairwise adjacent to each other follow from invariant (4) for index i, which is satisfied by our initial assumptions.

Moving on, suppose that $C' \subseteq C$. Then we clearly must have $\mathcal{Q} \subseteq \{Q_1, \ldots, Q_k, H_{i+1}\}$. Note that by invariant (4) for index i (applied with the component C), the subgraphs $Q_1, \ldots, Q_k$ are pairwise adjacent in G. Furthermore, since $S = \{v_1, \ldots, v_k\} \subseteq V(H) = V(H_{i+1})$ by our choice of H, we know that H_{i+1} is adjacent to each of $Q_1, \ldots, Q_k$ in G. Hence, the members of $\mathcal{Q}$ are pairwise adjacent to each other. It remains to be shown that $|\mathcal{Q}| \le t-2$. Towards a contradiction, suppose that $|\mathcal{Q}| \ge t-1$. We have $k \le t-2$, and therefore this is only possible if $k = t-2$ and $\mathcal{Q} = \{Q_1, \ldots, Q_{t-2}, H_{i+1}\}$.

We will now obtain the desired contradiction to the above assumption by constructing an odd K_t-model which is a subgraph of G (clearly this does not exist by assumption on G). Let us denote by $i_1 < i_2 < \cdots < i_{t-1} = i+1$ the sequence of indices such that $\{Q_1, \ldots, Q_{t-2}, H_{i+1}\} = \{H_{i_1}, \ldots, H_{i_{t-1}}\}$. By invariant (2) for $H_1, \ldots, H_i, H_{i+1}$, for every $s \in [i+1]$ we know that $H_s[A_s, B_s]$ is connected, and therefore admits a spanning tree T_s. This is a spanning tree of H_s which uses only edges spanned between A_s and B_s, for every $s \in [i+1]$. Furthermore, let T be any fixed spanning tree of the connected graph $G[C']$. Finally, consider a 2-colour-assignment $c\colon \left(\bigcup_{j=1}^{t-1} V(T_{i_j})\right) \cup V(T) \to \{1,2\}$ to the vertices in the t disjoint trees $T_{i_1}, \ldots, T_{i_{t-1}}, T$ by piecing together proper 2-colourings of the individual trees. To finish the construction of the odd K_t-model, we need the following claim.

($*$) Any pair of two distinct trees from the collection $T_{i_1}, \ldots, T_{i_{t-1}}, T$ is joined by at least one edge $xy \in E(G)$ satisfying $c(x) = c(y)$.

Proof of ($*$). Consider first the case that the pair of trees is of the form T_{s_1}, T_{s_2} with $s_1 < s_2$ and $s_1, s_2 \in \{i_1, \ldots, i_{t-1}\}$. Then, since $H_{s_1}, H_{s_2} \in \mathcal{Q}$ are adjacent, there exists a vertex $y \in V(T_{s_2}) = V(H_{s_2})$ which is adjacent to a vertex in $V(T_{s_1}) = V(H_{s_1})$. By invariant (3), applied for the index s_1 and the vertex y, we find that y must have neighbours $a \in A_{s_1}$ and $b \in B_{s_1}$ in G. Note that since c restricted to $V(T_{s_1})$ is a proper colouring, we must have $c(a) \neq c(b)$. Hence, there exists $x \in \{a, b\}$ with $c(x) = c(y)$, and the edge $xy \in E(G)$ connecting T_{s_1} and T_{s_2} verifies ($*$) in this case.

Secondly, consider the case that the pair of trees is of the form T_s, T for some $s \in \{i_1, \ldots, i_{t-1}\}$. Since $G[C']$ by definition is adjacent to every member of $\mathcal{Q}$, which includes H_s, analogous to the previous case there exists a vertex $y \in V(T)$ which is joined to H_s. Applying the invariant (3) with the index s and the vertex y now yields that there are neighbours $a \in A_s, b \in B_s$ of y. As above, we conclude that since $c(a) \neq c(b)$ there exists $x \in \{a, b\}$ with $c(x) = c(y)$. The edge xy is monochromatic and connects T_s and T, thus ($*$) is verified also in the second case.

This proves ($*$).

Now the collection of the t disjoint trees $T_{i_1}, \ldots, T_{i_{t-1}}, T$ in G, the colouring c as well as the monochromatic edges guaranteed between each pair of trees by ($*$) certify that G contains an odd K_t-model. This is a contradiction to the assumption that G is odd K_t-minor-free, and hence, our above assumption that $|\mathcal{Q}| \geq t-1$ was wrong. This concludes the proof that also the invariant (4) remains satisfied after extending the sequence $H_1, \ldots, H_i$ of subgraphs by H_{i+1}.

Finally, since all the subgraphs $H_1, H_2, \ldots$ as defined above are non-empty, after finitely many steps the union of the subgraphs will cover all vertices of G, i.e., we will find an integer $\ell \geq 1$ such that $V(H_1) \cup \cdots \cup V(H_\ell) = V(G)$ forms a partition of G, with all four invariants $(1) - (4)$ satisfied for each index $i \in [\ell]$. This concludes the proof of the theorem. $\square$

After having done the main bulk of work in the previous proof, we can now easily conclude Theorem 1.3.

Proof of Theorem 1.3. Let $t \geq 3$ be an integer an let G be any given odd K_t-minor-free graph. W.l.o.g. we may assume that G is connected. We apply Theorem 2.3 to obtain a collection $H_1, \ldots, H_\ell$ of connected induced subgraphs of G such that

- $V(H_1), \ldots, V(H_\ell)$ forms a partition of $V(G)$,
- every graph H_i with $i \in [\ell]$ admits a 2-colouring $f_i{:}V(H_i) \to \{1, 2\}$ with monochromatic components of size at most $\lceil \frac{t-2}{2} \rceil$ (by property (1) in Theorem 2.3), and
- for every $1 \leq i < \ell$ the subgraph H_{i+1} is adjacent in G to at most $t-2$ among the subgraphs $H_1, \ldots, H_i$ (by property (4) in Theorem 2.3, applied to the connected component of $G - (V(H_1) \cup \cdots \cup V(H_i))$ which contains $V(H_{i+1})$).

Now define an auxiliary simple graph on the vertex-set $[\ell]$, in which two indices $i, j \in [\ell]$ are made adjacent if and only if the subgraphs H_i and H_j are adjacent in G. By the third item above, this graph is $(t-2)$-degenerate, and hence, it has chromatic number at most $(t-2)+1 = t-1$. Fix a proper $(t-1)$-colouring $f{:}[\ell] \to [t-1]$ of this auxiliary graph. Now consider the product colouring $g{:}V(G) \to [t-1] \times \{1, 2\}$, defined by $g(x) := (f(i), f_i(x))$ for every $x \in V(H_i)$. From the definition of the auxiliary graph and since f is a proper colouring we have that every monochromatic component in G with respect to the colouring g must be fully included in $V(H_i)$ for some $i \in [\ell]$. But then it is a monochromatic component also of the colouring f_i of H_i. Hence, by the second item above it cannot be of size more than $\lceil \frac{t-2}{2} \rceil$. Since g uses a colour-set of size $2(t-1)$, this proves the claim of the theorem. $\square$

References

[1] Catlin, P. (1979) Hajós' graph colouring conjecture: variations and counterexamples. *J. Combin. Theory Ser. B* **26**(2) 268–274.

[2] Delcourt, M. and Postle, L. (2021) Reducing linear Hadwiger's conjecture to coloring small graphs. arXiv preprint, arXiv: 2108.01633.

[3] Dvořák, Z. and Norin, S. (2017) Islands in minor-closed classes. I. Bounded treewidth and separators. arXiv preprint, arXiv: 1710.02727.

[4] Edwards, K., Kang, D. Y., Kim, J., Oum, S. and Seymour, P. (2015) A relative of Hadwiger's conjecture. *SIAM J. Discrete Math.* **29**(4) 2385–2388.

[5] Geelen, J., Gerards, B., Reed, B., Seymour, P. and Vetta, A. (2009) On the odd-minor variant of Hadwiger's conjecture. *J. Combin. Theory Ser. B* **99**(1) 20–29.

[6] Jensen, T. and Toft, B. (1995) *Graph Coloring Problems*. Wiley, 115.

[7] Hadwiger, H. (1943) Über eine Klassifikation der Streckenkomplexe. *Vierteljahrsschr Naturforsch. Ges. Zür* **88** 133–142.

[8] van den Heuvel, J. and Wood, D. R. (2018) Improper colourings inspired by Hadwiger's conjecture. *J. Lond. Math. Soc.* **98**(1) 129–148.

[9] Kang, D. Y. and Oum, S. (2019) Improper colouring of graphs with no odd clique minor. *Combin. Probab. Comput.* **28** 740–754, .

[10] Kawarabayashi, K. (2008) A weakening of the Odd Hadwiger's conjecture. *Combin. Probab. Comput.* **17** 815–821.

[11] Kawarabayashi, K. (2009) Note on coloring graphs without odd K_k-minors. *J. Combin. Theory Ser. B* **99** 728–731.

https://doi.org/10.1017/S0963548322000268 Published online by Cambridge University Press

[12] Kawarabayashi, K. and Mohar, B. (2007) A relaxed Hadwiger's conjecture for list colorings. *J. Combin. Theory Ser. B* **97** 647–651.
[13] Kawarabayashi, K. and Reed, B. (2008) Fractional coloring and the odd Hadwiger's conjecture. *Eur. J. Combin.* **29** 411–417.
[14] Kawarabayashi, K. and Song, Z.-X. (2007) Some remarks on the odd Hadwiger's conjecture. *Combinatorica* **27** 429–438.
[15] Kostochka, A. V. (1984) Lower bound on the Hadwiger number of graphs by their average degree. *Combinatorica* **4** 307–316.
[16] Liu, C.-H. and Oum, S. (2018) Partitioning H-minor-free graphs into three subgraphs with no large components. *J. Combin. Theory Ser. B* **128** 114–133.
[17] Liu, C.-H. and Wood, D. R. (2021) Clustered coloring of graphs excluding a subgraph and a minor. arXiv preprint, arXiv: 1905.09495.
[18] Norin, S., Postle, L. and Song, Z.-X. (2019) Breaking the degeneracy barrier for coloring graphs with no K_t minor. arXiv preprint, arXiv: 1910.09378.
[19] Norin, S. and Postle, L. (2020) Connectivity and choosability of graphs with no K_t minor. *J. Combin. Theory Ser. B* **1**(2) 107. doi:10.1016/j.jctb.2021.02.001.
[20] Norin, S. and Song, Z.-X. (2021) A new upper bound on the chromatic number of graphs with no odd K_t minor. *Combinatorica*, doi:10.1007/s00493-021-4390-3.
[21] Postle, L. (2020) Further progress towards Hadwiger's conjecture. arXiv preprint, arXiv: 2006.11798.
[22] Postle, L. (2020) An even better density increment theorem and its application to Hadwiger's conjecture. arXiv preprint, arXiv: 2006.14945.
[23] Postle, L. (2020) Further progress towards the list and odd versions of Hadwiger's conjecture. arXiv preprint, arXiv: 2010.05999.
[24] Robertson, N., Seymour, P. and Thomas, R. (1993) Hadwiger's conjecture for K_6-free graphs. *Combinatorica* **13** 279–361.
[25] Seymour, P. (2016) *Hadwiger's Conjecture, Open Problems in Mathematics*. Springer, 417–437, https://web.math.princeton.edu/.
[26] Steiner, R. (2022) Asymptotic equivalence of Hadwiger's conjecture and its odd minor-variant. *J. Combin. Theory Ser. B* **155**(1) 45–51.
[27] Thomason, A. (1984) An extremal function for contractions of graphs. *Math. Proc. Cambridge Philos. Soc.* **95**(2) 261–265.
[28] Wood, D. R. (2010) Contractibility and the Hadwiger's conjecture. *Eur. J. Combin.* **31**(8) 2102–2109.
[29] Wood, D. R. (2018) Defective and clustered graph colouring. *Electron. J. Combin. Dyn. Surv.* DS23.

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