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DERIVED REPRESENTATION SCHEMES, NAKAJIMA QUIVER VARIETIES & NONCOMMUTATIVE DERIVED POISSON REDUCTION

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Abstract

This thesis investigates some geometric properties of representation schemes of associative unital algebras, broadly speaking schemes whose geometric points correspond to equivalence classes of finite-dimensional representations of the chosen algebra. These schemes provide insights into "noncommutative geometry", in that they reflect many structures or properties of the algebra to analogous structures or properties of the representation schemes ('Kontsevich-Rosenberg principle'), e.g. associated vector bundles, symplectic and Poisson structures. However, the translation of noncommutative structures on algebras into commutative structures of their representation spaces seems to work correctly only when the algebras are smooth, a problem which seems to be related with the nonexactness of the representation functor, and consequently can be approached by introducing a refined derived version of the functor, using techniques from homological algebra (enrichment to differential graded objects and exploitation of their model structures). The thesis consists of two papers:

- **Derived representation schemes and Nakajima quiver varieties.**

  The first aspect to be explored is the relationship between derived representation schemes and symplectic resolutions of a more classical nature (GIT quotients) in the case of Nakajima quiver varieties, which are representation spaces for framed, preprojective algebras of quivers. An explicit model for the derived representation scheme is exhibited, via a minimal cofibrant resolution of the afore-mentioned preprojective algebra. The main conceptual result of the paper is a necessary and sufficient condition for the two resolutions to be equivalent. The equivalence of the two resolutions yields several interesting results, e.g. the cohomologies of tautological sheaves on Nakajima varieties can be

computed as the isotypical components of the derived representation schemes. The push-forward in K-theory towards a point of the above-mentioned equalities has as a very simple corollary some integral formulas which have appeared numerous times in the mathematical (Jeffrey-Kirwan residue formulas) and physical (Nekrasov partition function, the instanton part of the Seiberg-Witten prepotential for N = 2 supersymmetric 4-dimensional and 5-dimensional quiver gauge theories) literature in recent years.

- **Noncommutative derived Poisson reduction.**

  The paper deals with aspects of noncommutative Poisson and Hamiltonian geometry which come play a role in the framework of Nakajima quiver varieties. More specifically, a procedure of 'noncommutative derived Poisson reduction' is formalized, in the form of a functor which, via the representation functor, yields commutative derived Poisson reduction. The final result is a noncommutative version of the quotient stack of derived representation schemes under their natural action of the associated general linear groups. Some classes of examples are explicitly explained, in particular one of them seems to give more insight into some combinatorial conjectures involving the scheme of commuting matrices of size $n \times n$ and similar, providing a simplified proof of a conjecture for the case $n = 2$.

Riassunto

Questa tesi investiga alcune proprietà geometriche degli schemi di rappresentazioni di algebre associative unitali, in linea generale degli schemi i cui punti geometrici corrispondono a classi di equivalenza di rappresentazioni finito-dimensionali dell'algebra scelta. Questi schemi forniscono intuizioni sulla "geometria noncommutativa", in quanto riflettono molte strutture o proprietà dell'algebra a strutture o proprietà analoghe degli schemi di rappresentazione ('principio di Kontsevich-Rosenberg'), per esempio fibrati vettoriali associati, strutture di Poisson e simplettiche. Tuttavia la traduzione di strutture noncommutative su algebre in strutture commutative dei loro spazi di rappresentazione sembra funzionare correttamente soltanto nel caso in cui le algebre sono lisce, un problema questo che sembra essere relazionato con la non esattezza del funtore di rappresentazione, e di conseguenza può essere risolto con l'introduzione di una versione più raffinata del funtore, utilizzando tecniche di algebra omologica (arricchimento ad oggetti differenziali graduati e sfruttamento delle loro strutture modello). La tesi consiste di due articoli:

- **Derived representation schemes and Nakajima quiver varieties.**

  Il primo aspetto ad essere esplorato è la relazione tra schemi di rappresentazioni derivati e risoluzioni simplettiche di natura più classica (quozienti GIT) nel caso delle varietà dei quiver di Nakajima, che sono spazi di rappresentazioni di algebre preproiettive con 'incorniciatura' di quiver. Viene esibito un modello esplicito per lo schema di rappresentazione derivato, tramite una risoluzione cofibrante minima dell'algebra preproiettiva con framing del quiver. Il principale risultato concettuale dell'articolo è una condizione necessaria e sufficiente affinché le due risoluzioni siano equivalenti. L'equivalenza delle due

risoluzioni produce numerosi risultati interessanti, per esempio le coomologie dei fasci tautologici sulle varietà di Nakajima possono essere calcolati come le componenti isotipiche degli schemi di rappresentazione derivati. Il push-forward in K-teoria verso il punto delle eguaglianze di cui sopra ha come semplice corollario alcune formule integrali che sono apparse numerose volte nella letteratura matematica (formule dei residui di Jeffrey-Kirwan) e fisica (funzione di partizione di Nekrasov, la parte di istantonni del prepotenziale di Seiberg-Witten per teorie di dimensioni 4 e 5 supersimmetriche di gauge di quiver per $N = 2$) degli ultimi anni.

- **Noncommutative derived Poisson reduction.**

  L'articolo tratta di aspetti di geometria noncommutativa di Poisson e Hamiltoniana che giocano un ruolo nel quadro delle varietà di quiver di Nakajima. Più precisamente viene formalizzata una procedura di 'riduzione derivata noncommutativa di Poisson', nella forma di un funtore che, tramite il funtore di rappresentazione, produce la riduzione derivata commutativa di Poisson. Il risultato finale è una versione noncommutativa di stack quoziente degli schemi di rappresentazione derivati sotto la loro naturale azione dei gruppi generali lineari associati. Alcune classi di esempi vengono esplicitamente spiegate, in particolare una di loro sembra dare maggiore comprensione di alcune congetture combinatoriche che riguardano schemi di matrici $n \times n$ che commutano e simili, fornendo una dimostrazione semplificata di una congettura per il caso $n = 2$.

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La tesi è dedicata ai miei genitori, che mi hanno sempre dato il loro supporto incondizionato.

Ai miei genitori

“C’est dire que s’il y a une chose en mathématique qui (depuis toujours sans doute) me fascine plus que toute autre, ce n’est ni «le nombre», ni «la grandeur», mais toujours la forme. Et parmi les mille-et-un visages que choisit la forme pour se révéler à nous, celui qui m’a fasciné plus que tout autre et continue à me fasciner, c’est la structure cachée dans les choses mathématiques."

— Alexandre Grothendieck,
Récoltes et semailles

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Chapter 1

Introduction

1.1 Few words on notation and some general conventions

The word that appears most frequently in this thesis is 'algebra', so it seems appropriate to start there. By algebra we always mean an associative and unitary algebra, i.e. an object A of a monoidal category $(\mathcal{V}, \otimes, 1)$ (in most cases of our interest $\mathcal{V} = \mathtt{Vect}_k$ is the category of vector spaces over a field k) endowed with morphisms:

$$m : A \otimes A \to A, \qquad e : 1 \to A \tag{1.1.1}$$

which satisfy the properties of associativity and unity:

$$\begin{cases} m \circ (m \otimes \mathrm{id}_A) = m \circ (\mathrm{id}_A \otimes m)\,, \\ m \circ (e \otimes \mathrm{id}_A) = \mathrm{id}_A = m \circ (\mathrm{id}_A \otimes e)\,, \end{cases} \tag{1.1.2}$$

having implicitly used the associator and unitors in the above equations.

Categories are usually written in the mathematical monospace font, for example $\mathtt{Sets}$, $\mathtt{Grp}$, $\mathtt{Ab}$ are the categories of sets, groups and abelian groups, while generic categories are written in the calligraphic font, for example $\mathcal{A}, \mathcal{B}, \mathcal{C}$, and so on. For a category $\mathcal{A}$ and two objects $X, Y \in \mathcal{A}$ we denote by either $\mathrm{Hom}_{\mathcal{A}}(X, Y)$ or $\mathcal{A}(X, Y)$ the collection of morphisms between them, and such collection is usually a set as most of our categories are assumed to be locally small. An adjunction between functors $\mathsf{F} : \mathcal{A} \to \mathcal{B}$ and $\mathsf{G} : \mathcal{B} \to \mathcal{A}$, with F left adjoint to G, is denoted by $\mathsf{F} \dashv \mathsf{G}$.

1.2 Geometry and algebra

There is a series of results of the type of (anti)equivalence of categories that we can say falls under the generic name of duality between geometry and algebra. It is difficult to argue that instances of this duality are among the most profound results in the history of mathematics, up to the present day.

The duality often associates to a geometric space some algebra of functions on it, and conversely constructs a geometric space whose points correspond to ideals or subalgebraic structures of some kind. Exactly what kind of functions are considered depends on the geometric context, but typically, the collection of functions inherits algebraic structure from the algebraic structure of the object in which the functions have values, via pointwise defined operations.

More specifically, but still informally: there is a category of geometric objects $\mathcal{G}$ and a category of algebraic objects of some kind $\mathcal{A}$, with a specific object $\mathbb{A}$ that lives[1] both in $\mathcal{G}$ and in $\mathcal{A}$. The representable presheaf $\mathcal{G}(-, \mathbb{A}) : \mathcal{G}^{\mathrm{op}} \to \mathtt{Sets}$ can in fact, due to the algebraic structure on $\mathbb{A}$, be lifted to a functor $\mathcal{G}(-, \mathbb{A}) : \mathcal{G}^{\mathrm{op}} \to \mathcal{A}$. For a simple reason of conventions we consider the opposite functor:

$$\mathcal{G}(-, \mathbb{A})^{\mathrm{op}} : \mathcal{G} \to \mathcal{A}^{\mathrm{op}} , \tag{1.2.1}$$

which is often the duality between geometry and algebra we are looking for.

It follows that if we consider a category $\mathcal{A}$ of algebraic structures whose product is commutative, we obtain commutative algebraic structures on the set of functions, and indeed this is the form in which most classical dualities between geometry and algebra take shape. A few examples include:

- $\mathcal{G} = \mathtt{Top}_{\mathrm{Haus,\,cpt}}$ is the category of Hausdorff, compact topological spaces, $\mathcal{A} = \mathtt{C^*\text{-}Alg}_{\mathbb{C}}^{\mathrm{comm}}$ is the category of commutative, unital C*-

[1] One precise way of saying this is that both categories $\mathcal{G}$ and $\mathcal{A}$ have a forgetful functor towards $\mathtt{Sets}$ and that $\mathbb{A}$ really is a set with specific lifts to both categories. Another possibility is that $\mathbb{A}$ is an algebra of type $\mathcal{A}$ in the category $\mathcal{G}$ (provided that $\mathcal{G}$ has the necessary categorical properties needed to define the axiom of algebras of type $\mathcal{A}$ in it).

algebras over the complex numbers. The equivalence is given by $\mathtt{Top}(-,\mathbb{C})^{\mathrm{op}}$ and goes by the name of Gelfand duality ([28]).

- $\mathcal{G} = \mathtt{Mfld}$ is the category of smooth manifolds and $\mathcal{A} = \mathtt{CommAlg}_{\mathbb{R}}$ is the category of commutative algebras over the real numbers. The functor $\mathtt{Mfld}(-,\mathbb{R})^{\mathrm{op}}$ is fully faithful by Milnor's exercise ([48, Problem 1-C], proof in [39, Corollary 35.9]), hence it defines an equivalence of categories once we restrict it to land in its essential image.
- $\mathcal{G} = \mathtt{Aff}$ is the category of affine schemes, as a full subcategory of the category of locally ringed topological spaces, and $\mathcal{A} = \mathtt{CommRing}$ is the category of commutative rings. The equivalence of categories is realised by the global sections functor $\Gamma^{\mathrm{op}} \cong \mathtt{Aff}(-,\mathbb{A}^1_{\mathbb{Z}})^{\mathrm{op}}$, and by the Zariski spectrum in the other direction.
- ([45], [35], [3]) $\mathcal{G} = \mathtt{Top}_{\mathrm{sober}}$ is the category of sober topological spaces, and $\mathcal{A} = \mathtt{Frm}$ is the category of frames (distributive lattices with infinite joins, satisfying the infinite distributive law: $x \wedge (\vee_i y_i) = \vee_i (x \wedge y_i)$). We denote by $\mathbb{S} = \{0,1\}$ the Sierpiński space, the topological space with only proper nonempty open subset $\{1\}$, and also a frame with only nontrivial relation $0 \leqslant 1$. The functor $\mathtt{Top}(-,\mathbb{S})^{\mathrm{op}} : \mathtt{Top}_{\mathrm{sober}} \to \mathtt{Locale} := \mathtt{Frm}^{\mathrm{op}}$ is the functor that sends a topological space to the locale corresponding to the frame of open subsets (a continuous function $f : X \to \mathbb{S}$ is uniquely defined by the open $f^{-1}(1)$), and it is fully faithful. Hence it gives an equivalence of categories once we restrict it to land in its essential image (the category of 'locales with enough points').

The last example is perhaps the least known, however it is in some sense the most fundamental, and it also provides an explanation as to why algebraic structures corresponding to 'classical' geometric spaces are commutative. In fact, if by 'classical' geometric spaces we mean topological spaces plus additional structure, then we find out that commutativity corresponds to the elementary property that the intersection of open subsets is commutative: $U \cap V = V \cap U$ (intersection is the product in the frame of open subsets).

The skeptical reader might object that this case has little to do with the other geometric examples discussed above, and might ask, for example, how is this last example connected to classical algebraic geometry? It is a fact that every affine scheme is a sober topological space, hence the spectrum $\mathrm{Spec}(R)$ of a ring R is uniquely determined (modulo isomorphism) as a topological space by the corresponding frame of opens. There is a construction, due to A. Joyal ([36]), that exploits this idea and presents the spectrum of a ring as a frame, precisely as the frame generated by the symbols $D(f)$ with $f \in R$, and relations:

$$D(0) = 0, \quad D(1) = 1, \quad D(f+g) \leqslant D(f) \vee D(g), \quad D(fg) = D(f) \wedge D(g)\,. \tag{1.2.2}$$

One should in some sense interpret $D(f)$ as the fundamental open in the Zariski topology defined by $\{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$. The last equation shows why any attempt of producing a similar construction out of an arbitrary non(-necessarily)-commutative ring will fail. In fact, even if the product of two elements is not commutative, this would not be detectable at the level of fundamental opens:

$$D(fg) = D(f) \wedge D(g) = D(g) \wedge D(f) = D(gf)\,, \tag{1.2.3}$$

as a result, making topology an inadequate tool of study of such structures.

1.3 Noncommutative geometry and representation schemes

In the last section we learned that if we want to consider geometric spaces corresponding to noncommutative algebraic structures of some kind we probably need to forget about classical topology. Noncommutative geometry is the study of categories of supposedly noncommutative geometric spaces, in the sense that they have some duality with categories of algebraic objects that are not necessarily commutative. The above folkloric definition of noncommutative geometry can in fact be specialised to definitions of appropriate categories of noncommutative spaces.

For example one can define a category of noncommutative Hausdorff compact topological spaces as $\mathtt{NTop}_{\mathrm{Haus,cpt}} := \mathtt{C^*\text{-}Alg}_{\mathbb{C}}^{\mathrm{op}}$, the opposite category of the category of all unital C*-algebras over the complex numbers. The case of most interest for us is affine algebraic geometry, so we give the definition of noncommutative affine schemes (and noncommutative affine schemes over a field k) as:

$$\mathtt{NAff} := \mathtt{Ring}^{\mathrm{op}} \qquad (\mathtt{NAff}_k := \mathtt{Alg}_k^{\mathrm{op}})\,, \tag{1.3.1}$$

the opposite category of rings (and algebras over k).

Now that we have at least a reasonable formal definition of basic objects of noncommutative algebraic geometry we are tempted to try and give other meaningful reasonable geometric definitions on them. For example, what should a vector bundle on $X \in \mathtt{NAff}_k$ be? What about differential forms? Or a Poisson/symplectic structure? To answer these questions it is opportune to introduce the main objects of study of the thesis: representation schemes.

Let us first look back at the geometry of the spectrum of a commutative algebra $R \in \mathtt{CommAlg}_k$. The k-points of the affine scheme $\mathrm{Spec}(R)$ are:

$$\mathtt{Aff}_k(\mathrm{Spec}(k), \mathrm{Spec}(R)) \cong \mathtt{CommAlg}_k(R, k)\,, \tag{1.3.2}$$

homomorphisms of k-algebras $R \to k$, also known as 1-dimensional representations of the algebra R. When R is replaced by an arbitrary algebra $A \in \mathtt{Alg}_k$ with a non-necessarily commutative product, it is not reasonable to expect that the geometry of whatever its associated space is $X =$ "$\mathrm{Spec}_{nc}(A)$" $\in \mathtt{NAff}_k$, is determined by the same set of 1-dimensional representations. In fact all these representations factor through the ideal of commutators of the algebra, and we would simply obtain the spectrum of the abelianisation $A_{ab} \in \mathtt{CommAlg}_k$. We have a better hope of understanding the noncommutative geometry of A by considering the spaces of all finite dimensional representations, for each $n \geqslant 1$:

$$\mathrm{Rep}_n(A) := \text{"}\mathtt{Alg}_k(A, \mathrm{Mat}_{n\times n}(k))\text{"}\,, \tag{1.3.3}$$

because $\mathrm{Mat}_{n\times n}(k)$ is a noncommutative algebra for $n > 1$.

The reason why we write the right-hand side of (1.3.3) in quotation marks is that really this set has the structure of an affine scheme over k, which is what we want to denote by the symbol $\mathrm{Rep}_n(A) \in \mathtt{Aff}_k$. Its ring of functions is given by the representation functor

$$(-)_n = \mathcal{O}\mathrm{Rep}_n : \mathtt{Alg}_k \to \mathtt{CommAlg}_k\,, \tag{1.3.4}$$

which is left adjoint to the matrix functor $\mathrm{Mat}_{n\times n}(-) : \mathtt{CommAlg}_k \to \mathtt{Alg}_k$. The idea that the noncommutative geometry of A is encoded in the family of representation schemes was formalised by M. Kontsevich and A. Rosenberg, who suggested the following informal definition of noncommutative geometric structures ([41]):

MetaDefinition. ("Kontsevich-Rosenberg principle") Any noncommutative geometric structure on A should induce the corresponding commutative geometric analogue on each representation scheme $\mathrm{Rep}_n(A)$, for all $n \geqslant 1$.

Let us show a possible instance of the above MetaDefinition. The category of quasi-coherent sheaves on an affine scheme $\mathrm{Spec}(R)$ is, by the affine Serre's theorem:

$$\mathtt{QCoh}(\mathrm{Spec}(R)) \cong R\text{-}\mathtt{Mod}. \tag{1.3.5}$$

equivalent to the category of R-modules, via the global sections functor. Let us now consider a smooth algebra[2] $A \in \mathtt{Alg}_k$, and the corresponding Van den Bergh's functor ([69, Sections 2.1 and 3.3]):

$$(-)_n : A\text{-}\mathtt{Bimod} \to A_n\text{-}\mathtt{Mod}. \tag{1.3.7}$$

The notation is justified by the fact that the Van den Bergh's functor, applied to the algebra A viewed as a bimodule over itself, recovers $A_n = \mathcal{O}\mathrm{Rep}_n(A)$ the ring of functions on the representation scheme, as a module over itself.

It follows that, according to the Kontsevich-Rosenberg princple, the category of bimodules is a reasonable candidate for a category of noncommutative quasi-coherent sheaves, because the Van den Bergh's functor:

$$(-)_n : \mathrm{QCoh}_{nc}(\text{"}\mathrm{Spec}_{nc}(A)\text{"}) \to \mathrm{QCoh}(\mathrm{Rep}_n(A)) \tag{1.3.8}$$

now would send a noncommutative quasi-coherent sheaf on A to a quasi-coherent sheaf on $\mathrm{Rep}_n(A)$, for all $n \geqslant 1$.

The above construction has, as a requirement, the smoothness of the algebra A, and this is not a special feature of this example. What often

[2] An algebra is called smooth if it is finitely generated and formally smooth. The property of formal smoothness, or equivalently infinitesimal lifting property, is the noncommutative version of its commutative counterpart (which is equivalent to smoothness of the corresponding affine scheme). Namely, an algebra A is formally smooth if for any algebra B and any 2-sided nilpotent ideal $I \subset B$, the map induced by the projection $B \to B/I$:

$$\mathtt{Alg}_k(A, B) \to \mathtt{Alg}_k(A, B/I) \tag{1.3.6}$$

is surjective.

happens when the algebra is not formally smooth is that the resulting representation schemes are singular, hence the translation of geometric structures from A to $\mathrm{Rep}_n(A)$ starts to fail. A possible solution was firstly suggested in [9] with the introduction of a derived version of representation schemes $\mathrm{DRep}_n(A)$, which makes use of the dg-enhancement of the representation functor and the projective model structure on dg-algebras.

In Chapter 2 we study derived representation schemes for a family of framed, preprojective quiver path algebras which classically produce Nakajima quiver varieties. We relate these derived resolutions with the usual symplectic resolutions of affine Nakajima quiver varieties via GIT quotients, and prove a series of equalities between invariants coming from the two different approaches.

In Chapter 3 we formalise a noncommutative procedure of derived Poisson reduction, in the spirit of the Kontsevich-Rosenberg principle. We define a category of noncommutative Hamiltonian spaces as double Poisson algebras with an extra property that guarantees that the natural action of the general linear group on the representation scheme is Hamiltonian. For algebras in this category we define a noncommutative counterpart of the Chevalley-Eilenberg complex of the Koszul complex of the representation scheme. We show various examples of the above constructions, in particular the motivating example, which is the one of quiver varieties.

1.4 Derived Representation Schemes and Nakajima Quiver Varieties

This section is not intended to be a summary of Chapter 2, but rather an explanation of how this research project came about. In fact, we simply want to show a toy example of the fundamental result, which is hopefully understandable even without many prerequisites, and whose generalisation was then the main motivation for the project.

Let us consider symplectic reduction of the cotangent bundle $X = \mathbb{T}^*\mathbb{C}^2$ with respect to the Hamiltonian action of $\mathbb{C}^\times$ induced by the linear representation of $G = \mathbb{C}^\times$ on $\mathbb{C}^2$ with weights $(-1, -1)$ (dually on the ring of functions weight $(1, 1)$). Explicitly we have a moment map:

$$\begin{aligned} \mu : X &\to \mathfrak{g}^* \cong \mathbb{C} \\ (i, j) &\longmapsto i_1 j_1 + i_2 j_2 \end{aligned} \tag{1.4.1}$$

where $j \in \mathbb{C}^2$, while i plays the role of the cotangent vector, after the obvious trivialisation of the cotangent bundle. There is an additional 3-dimensional torus $T = (\mathbb{C}^\times)^2 \times \mathbb{C}^\times$ acting via the standard representation of $(\mathbb{C}^\times)^2$ on $\mathbb{C}^2$ (hence on its cotangent bundle) plus an additional rescaling of the cotangent fiber. We give directly the (dual) action on the generators of the ring of functions:

$$(t_\alpha, \hbar) \cdot (i_\alpha, j_\alpha) = (\hbar t_\alpha i_\alpha, t_\alpha^{-1} j_\alpha) \qquad (\alpha = 1, 2)\,. \tag{1.4.2}$$

The torus T is useful because it commutes with the G-action, hence it provides multi-gradings for the formulas we want to compute, both for X and for its symplectic reduction.

The affine symplectic reduction is the affine algebro-geometric quotient of the preimage of zero via the moment map for the action of G:

$$M_0 := \mu^{-1}(0)/G = \operatorname{Spec}(\mathcal{O}(\mu^{-1}(0)^G) \cong \operatorname{Spec}(\mathbb{C}[a, b, c]/(a^2 + bc))\,. \tag{1.4.3}$$

a, b, c are the generators of the G-invariant functions, respectively $a = i_1 j_1$, $b = i_1 j_2$, and $c = i_2 j_1$, hence they carry weights of T as follows:

$$\begin{cases} a \mapsto \hbar a \\ b \mapsto \hbar \frac{t_1}{t_2} b \\ c \mapsto \hbar \frac{t_2}{t_1} c \end{cases} \tag{1.4.4}$$

This last identification provides an easy computation of the Hilbert-Poincaré series of $\mathcal{O}(M_0)$ as:

$$\mathrm{ch}_T(\mathcal{O}(M_0)) = \frac{1 + h}{(1 - \hbar \frac{t_1}{t_2})(1 - \hbar \frac{t_2}{t_1})}\,, \tag{1.4.5}$$

corresponding to the decomposition $\mathcal{O}(M_0) \cong \mathbb{C}[b, c] \oplus \mathbb{C}[b, c]a$. Another way of obtaining the same result, without explicitly finding the generators of the invariant ring, is using Weyl's integral formula:

$$\mathrm{ch}_T(\mathcal{O}(M_0)) = \mathrm{ch}_T(\mathcal{O}(\mu^{-1}(0))^G) = \int_{U(1)} \mathrm{ch}_{T \times G}(\mathcal{O}(\mu^{-1}(0))) \frac{dx}{2\pi i x}\,. \tag{1.4.6}$$

Because the zero locus of the moment map is a complete intersection, the Hilbert-Poincaré series of its ring of functions is given by the series of the corresponding Koszul complex:

$$\begin{aligned}
&\mathrm{ch}_{T\times G}(\mathcal{O}(\mu^{-1}(0))) = \mathrm{ch}_{T\times G}(\mathcal{O}(X)\otimes_{\mathcal{O}(\mathfrak{g}^*)}\mathcal{O}(pt)) = \\
&\mathrm{ch}_{T\times G}(\mathcal{O}(X)\otimes^{\mathbf{L}}_{\mathcal{O}(\mathfrak{g}^*)}\mathcal{O}(pt)) = \frac{(1-\hbar)}{(1-\hbar x^{-1}t_1)(1-\hbar x^{-1}t_2)(1-xt_1^{-1})(1-xt_2^{-1})},
\end{aligned} \tag{1.4.7}$$

which once plugged in equation (1.4.6) and computed via residues gives back (1.4.5).

The quasi-projective symplectic reduction is the GIT quotient, which once chosen a nontrivial character $\chi : G \to \mathbb{C}^\times$ can be identified with the cotangent bundle of the projective line:

$$M := \mu^{-1}(0)/_{\chi}G \cong \{(i,j)\,|\,j\neq 0, i_1j_1+i_2j_2=0\}/G \cong \mathbb{T}^*\mathbb{P}^1\,. \tag{1.4.8}$$

There is a proper morphism $p : M \to M_0$ which is a symplectic resolution of singularities. The Hilbert-Poincaré series of the ring of functions on M can be computed via pushforward in T-equivariant K-theory, because higher cohomologies of the structure sheaf vanish. In turn, such pushforward can be computed using localisation formula:

$$\mathrm{ch}_T(M) = \sum_{m\in M^T}\frac{1}{\mathrm{ch}_T\left(\Lambda_{-1}\mathbb{T}^*_m M\right)}\,. \tag{1.4.9}$$

Because of the nontrivial rescaling of the cotangent direction, fixed points can be only in the base $\mathbb{P}^1$, and the generic cotangent fiber of M at a point $m\in\mathbb{P}^1\subset M$ is:

$$\mathbb{T}^*_m M = \mathbb{T}^*_m(\mathbb{T}^*\mathbb{P}^1)\cong \mathbb{T}^*_m\mathbb{P}^1\oplus\mathbb{T}_m\mathbb{P}^1\,, \qquad (\mathbb{T}_m\mathbb{P}^1 = \mathrm{Hom}_{\mathbb{C}}(m,\mathbb{C}^2/m))\,. \tag{1.4.10}$$

Fixed points in this case are only two, the "north pole" and the "south pole" $n,s\in\mathbb{P}^1\subset\mathbb{T}^*\mathbb{P}^1$, which as T-modules have characters, respectively,

$\mathrm{ch}_T(n) = t_1$, $\mathrm{ch}_T(s) = t_2$, hence:

$$\begin{aligned}
&\mathrm{ch}_T(\Lambda_{-1}\mathbb{T}^*_n M) = \left(1 - \frac{t_1}{t_2}\right)\left(1 - \hbar\frac{t_2}{t_1}\right) \\
&\mathrm{ch}_T(\Lambda_{-1}\mathbb{T}^*_s M) = \left(1 - \frac{t_2}{t_1}\right)\left(1 - \hbar\frac{t_1}{t_2}\right) \\
\Rightarrow \qquad &\mathrm{ch}_T(\mathcal{O}(M)) = \frac{1}{\left(1 - \frac{t_1}{t_2}\right)\left(1 - \hbar\frac{t_2}{t_1}\right)} + \frac{1}{\left(1 - \frac{t_2}{t_1}\right)\left(1 - \hbar\frac{t_1}{t_2}\right)} = \\
&= \cdots = \mathrm{ch}_T(\mathcal{O}(M_0))\,.
\end{aligned} \tag{1.4.11}$$

In conclusion we have two resolutions of the singular affine scheme M_0: the classical symplectic resolution via GIT M, and the derived resolution via Koszul complex (see (1.4.7)). All of them have the same ring of functions, as explicitly shown in the previous computations (this is an instance of Theorem 2.4.3.1). Now we can observe that M_0 is the character scheme of the representation scheme of a certain algebra A: the path algebra of the quiver with two vertices v, w and two arrows $j : v \to w, i : w \to v$, modulo the noncommutative counterpart of the Hamiltonian relation $ij = 0$ (with dimensions 1 and 2 for v and w respectively). The Koszul complex is the ring of functions on the derived representation scheme, and it is quasi-isomorphic to the zero locus. Hence we have the following picture:

$$M \xrightarrow{p} M_0 = \mathrm{Rep}_{1,2}(A)/G \xrightarrow{\simeq} \mathrm{DRep}_{1,2}(A)/G\,, \tag{1.4.12}$$

where p is a classical symplectic resolution of singularities and the other morphism is a quasi-isomorphism between the characters scheme and its derived version. A similar result holds also when we substitute the structure sheaf with other appropriate quasi-coherent sheaves on M and on the other hand the ring on functions on the derived representation scheme with the corresponding isotypical component (Theorem 2.4.4.1).

The simple quiver we mentioned has nothing special, and in fact an analogous result holds true for all framed quivers that produce the classes of symplectic resolutions $M \to M_0$ known as Nakajima quiver varieties. It is formalising and exploiting these ideas that motivated the project in Chapter 2.

1.5 Noncommutative derived Poisson reduction

The research project in Chapter 3 is born from the observation that derived Poisson reduction of representation schemes under the natural action of the general linear group can be already formalised at the level of noncommutative structures. We briefly describe the procedure in the case of algebras over a field k, which is then formalised and extended in Chapter 3.

The conjugation action

The representation functor $(-)_n = \mathcal{O}\mathrm{Rep}_n : \mathtt{Alg}_k \to \mathtt{CommAlg}_k$ is a left adjoint, hence it preserves all small colimits. In particular it sends cogroups to cogroups (dually, affine algebraic groups in $\mathtt{NAff}_k$ to affine algebraic groups in $\mathtt{Aff}_k$). The simplest and most notable example of this correspondence being the affine algebraic group $\mathrm{GL}_n(k)$, which is the representation scheme of the algebra $U = k[g^{\pm 1}]$ with cogroup structure:

$$\begin{aligned} \Delta : k[g^{\pm1}] &\to k\langle g_1^{\pm1}, g_2^{\pm1}\rangle \qquad & S : k[g^{\pm1}] &\to k[g^{\pm1}] \qquad & \epsilon : k[g^{\pm1}] &\to k \\ g &\longmapsto g_1 g_2 & g &\longmapsto g^{-1} & g &\longmapsto 1 \end{aligned} \tag{1.5.1}$$

Left adjointness of the representation functor also implies that cogroup coactions on algebras correspond to group actions on the representation schemes. In particular the action by conjugation $\mathrm{GL}_n(k) \curvearrowright \mathrm{Rep}_n(A)$ can be seen as the coaction:

$$\begin{aligned} \alpha : A &\to A\langle g^{\pm1}\rangle \\ a &\longmapsto g a g^{\pm1} \end{aligned} \tag{1.5.2}$$

Poisson structures and Hamiltonian actions

It was W. Crawley-Boevey ([20]) who first introduced a noncommutative structure on algebras that induces a Poisson structure on the moduli space of representations $\mathrm{Rep}_n(A)/\mathrm{GL}_n(k)$. In Chapter 3 we call this structure an H_0-Poisson structure because essentially it is a Lie structure on the zeroth Hochschild homolology $HH_0(A) = A_{cyc} = A/[A, A]$ (with an additional lifting property encoding the derivation property of Poisson structurees). It was then M. Van den Bergh who introduced a notion of double Poisson structures ($\{\!\{-,-\}\!\} : A^{\otimes 2} \to A^{\otimes 2}$), which are the appropriate notion to induce Poisson structures on the representation schemes $\mathrm{Rep}_n(A)$. The group $U = k[g^{\pm 1}]$ has a 'noncommutative' Lie algebra $k[x]$, with double Poisson structure on generators: $\{\!\{x, x\}\!\} = x \otimes 1 - 1 \otimes x$. This induces the standard Poisson structure on

$$\mathrm{Sym}(\mathfrak{gl}_n(k)) \cong \mathcal{O}(\mathfrak{gl}_n^*(k)) \cong \mathcal{O}(\mathrm{Rep}_n(k[x]))\,. \tag{1.5.3}$$

Suppose now that we have a noncommutative Poisson scheme meaning, dually, an algebra A and a double Poisson structure on it. As a consequence every representation scheme $\mathrm{Rep}_n(A)$ is a Poisson scheme, and it is natural to ask when the action of $\mathrm{GL}_n(k)$ is Hamiltonian (admits a (co)moment map). Because everything has a noncommutative origin (both the group action and the Poisson structure) it is to be expected that also the answer can be formulated at the noncommutative level. A comoment map for this action would be a Poisson morphism:

$$\mathcal{O}(\mathrm{Rep}_n(k[x])) \to \mathcal{O}(\mathrm{Rep}_n(A))\,, \tag{1.5.4}$$

hence induced by a double Poisson morphism $k[x] \to A$, which is nothing else than an element $\delta \in A$ (the image of x) with the property that $\{\!\{\delta, \delta\}\!\} = \delta \otimes 1 - 1 \otimes \delta$. The Hamiltonian property becomes that δ acts on all of A as the universal double derivation (which is the noncommutative counterpart of the infinitesimal conjugaction action on representation schemes):

$$\{\!\{\delta, a\}\!\} = a \otimes 1 - 1 \otimes a\,, \qquad \forall a \in A\,. \tag{1.5.5}$$

Classical and derived Poisson reduction

The zero locus of the moment map is the pullback of the moment map $\mu : \mathrm{Rep}_n(A) \to \mathfrak{gl}_n^*(k)$ with the zero value $0 : \mathrm{pt} \to \mathfrak{gl}_n^*(k)$:

$$\mathrm{Rep}_n(A) \times_{\mathfrak{gl}_n^*(k)} \mathrm{pt} = \mathrm{Spec}(A_n \otimes_{\mathrm{Sym}(\mathfrak{gl}_n(k))} k)\,. \tag{1.5.6}$$

The initial object in the category of algebras $k \in \mathtt{Alg}_k$ obviously has representation schemes $\mathrm{Rep}_n(k) = \mathrm{pt}$ for every n, meaning that the above ring of functions on the zero locus can be written also as:

$$A_n \otimes_{(k[x])_n} k_n \cong (A \amalg_{k[x]} k)_n = \mathcal{O}\mathrm{Rep}_n(A \amalg_{k[x]} k)\,. \tag{1.5.7}$$

In other words the zero locus is the representation scheme of a 'noncommutative zero locus' $A \amalg_{k[x]} k \cong A/\langle\delta\rangle$. Classical Poisson reduction is then just the character scheme of the noncommutative zero locus:

$$\mu^{-1}(0)/\mathrm{GL}_n(k) = \mathrm{Rep}_n(A/\langle\delta\rangle)/\mathrm{GL}_n(k)\,. \tag{1.5.8}$$

Derived Poisson reduction consists in replacing the zero locus with a derived zero locus:

$$\mathrm{Rep}_n(A) \times^{\mathbf{R}}_{\mathfrak{gl}_n^*(k)} \mathrm{pt} \cong \mathrm{Spec}(A_n \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{gl}_n(k))} k)\,, \tag{1.5.9}$$

and the quotient with a derived quotient (derived intersection of quotient stacks):

$$[\mathrm{Rep}_n(A)/\mathrm{GL}_n(k)] \times^{\mathbf{R}}_{[\mathfrak{gl}_n^*(k)/\mathrm{GL}_n(k)]} [\mathrm{pt}/\mathrm{GL}_n(k)] \tag{1.5.10}$$

Both procedures have a noncommutative counterpart, respectively a derived noncommutative intersection in the form of a generalised Shafarevich complex, and a noncommutative version of the Chevalley-Eilenberg complex. It is formalising the above ideas and constructions and working out a few examples (such as various versions of quiver varieties) that motivated the research project in Chapter 3.

Chapter 2

Derived Representation Schemes and Nakajima Quiver Varieties

Mathematics Subject Classification Primary 14D21 • 16G20;
Secondary 16E05 • 16E45 • 19L47.

Abstract

We introduce a derived representation scheme associated with a quiver, which may be thought of as a derived version of a Nakajima variety. We exhibit an explicit model for the derived representation scheme as a Koszul complex and by doing so we show that it has vanishing higher homology if and only if the moment map defining the corresponding Nakajima variety is flat. In this case we prove a comparison theorem relating isotypical components of the representation scheme to equivariant K-theoretic classes of tautological bundles on the Nakajima variety. As a corollary of this result we obtain some integral formulas present in the mathematical and physical literature since a few years, such as the formula for Nekrasov partition function for the moduli space of framed instantons on S^4. On the technical side we extend the theory of relative derived representation schemes by introducing derived partial character schemes associated with reductive subgroups of the general linear group and constructing an equivariant version of the derived representation functor for algebras with a rational action of an algebraic torus.

2.1 Introduction

Nakajima quiver varieties are certain Poisson varieties constructed from linear representations of a quiver. They were firstly introduced by Nakajima ([54], [53]) as a geometric tool to study representations of Kac-Moody algebras. They are also interesting from a purely geometric point of view, being a large class of examples of algebraic symplectic manifolds, many of which have been objects of study on their own (for example flag manifolds, framed moduli spaces of torsion free sheaves on $\mathbb{P}^2$, or a Lie algebra version of the character variety of a Riemann surface — see [5]). More recent studies have also supported the idea that symplectic resolutions, and in particular hyperkähler reductions such as Nakajima quiver varieties, provide a bridge between enumerative geometry, representation theory and integrable systems ([1], [57], [58], [60], [61]).

Quiver varieties are varieties of representations of a quiver: one fixes a vector space on each vertex of the quiver and then consider the linear space of representations obtained by associating to each arrow of the quiver a linear map. Kronheimer and Nakajima ([42]) have first introduced a framed version, which amounts to doubling the set of vertices and drawing a new arrow from each new vertex to its corresponding old one. One of the reasons for considering framed representations is that they appear naturally in the ADHM construction ([4]) of solutions of self-dual or antiself-dual Yang-Mills equations on S^4. They are also interesting from the point of view of representation theory of Lie algebras because dimension vectors of the framed vertices appear as highest weights of the representations ([52]). The framing is equivalent to a simpler operation of adding just one vertex with dimension vector 1, together with as many arrows to each vertex as the framing dimension (as pointed out in [19]), however in this paper we consider the framed version of Nakajima quiver varieties.

The framed quiver is then doubled, which means that each arrow gets doubled by an arrow that goes in the opposite direction: the linear space of representations becomes now a linear cotangent bundle $\mathrm{M}(Q, \boldsymbol{v}, \boldsymbol{w}) := \mathbb{T}^*\mathrm{L}(Q^{\mathrm{fr}}, \boldsymbol{v}, \boldsymbol{w})$ (where $\boldsymbol{v}, \boldsymbol{w}$ are dimension vectors for, respectively, the original and framing vertices). The gauge group is a general linear group on the original vertices $G = G_{\boldsymbol{v}}$ and there is a moment map

$$\mu : \mathrm{M}(Q, \boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{g}^*$$

in the form of a generalised ADHM equation. Nakajima quiver varieties are defined as Hamiltonian reductions of this action $G \curvearrowright M(Q, \boldsymbol{v}, \boldsymbol{w})$: either affine Hamiltonian reductions, $\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}) = \mu^{-1}(0) /\!\!/ G$, or quasi-projective $\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w}) = \mu^{-1}(0) /\!\!/_\chi G$, with the usual tools of geometric invariant theory ([50]). For each choice of a (nontrivial) character $\chi : G \to \mathbb{C}^\times$ there is a proper Poisson morphism

$$p : \mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}) \,, \tag{2.1.1}$$

which is often, but not always, a symplectic resolution of the singularities of $\mathfrak{M}^0$.

2.1.1 Outline and results

In this paper we link these varieties with some (derived) representation schemes. The idea of considering representation schemes is certainly not new, in fact it is motivated by the very first algebraic origin of these varieties (see, for example, representation schemes of preprojective algebras in [21] and [24]). However the derived version of representation schemes introduces some new invariants in a natural way.

The theory of representation schemes is recalled in detail in § 2.2.1. To a (unital, associative) algebra $A \in \mathtt{Alg}_k$ one associates $\mathrm{Rep}_V(A)$, the scheme of finite dimensional representations into a fixed vector space V. There is a relative version in which the algebra A comes with a fixed structure $\iota : S \to A$ of algebra over another algebra S with a fixed representation $\rho : S \to \mathrm{End}(V)$ and it is natural to define $\mathrm{Rep}_V(A)$ as the scheme of only those finite dimensional representations which are compatible with ρ.

General definitions and results on representation schemes work well over any field k of characteristic zero, but it is necessary to specialise to $k = \mathbb{C}$ in order to relate them to (Nakajima) quiver varieties, which are algebraic varieties over the complex numbers. The (complex) linear space of representations of a quiver Q is a representation scheme of the form $\mathrm{Rep}_V(A)$, where $A = \mathbb{C}Q$ is the path algebra of the quiver. This fact is a consequence of one of the basic results in the theory of representations of quivers:

There is an equivalence of categories between the category of $\mathbb{C}$-linear representations of a quiver Q and the category of left $\mathbb{C}Q$-modules.

The construction can be easily adapted to include the framing and the doubling of the quiver, and also the operation of taking the fiber of zero through the moment map. In other words it is possible to write the scheme $\mu^{-1}(0)$ as a representation scheme for the path algebra of the framed, doubled quiver, modulo the ideal $\mathcal{I}_\mu$ defined by the moment map:

$$\mu^{-1}(0) = \mathrm{Rep}_{\mathbb{C}^{v} \oplus \mathbb{C}^{w}}(A), \qquad A = \mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{I}_\mu\,,$$

where $\mathbb{C}^{v} = \oplus_a \mathbb{C}^{v_a}$ is the direct sum of the vector spaces placed on the original vertices of the quiver and $\mathbb{C}^{w} = \oplus_a \mathbb{C}^{w_a}$ is the one on the framing. We denote this representation scheme also simply by $\mathrm{Rep}_{v,w}(A)$. The gauge group by which we take the quotient is $G = G_v := \prod_a \mathrm{GL}_{v_a}(\mathbb{C}) \subset G_v \times G_w$. This group also arises naturally in the context of representation functors. It is possible to construct an invariant subfunctor by the group G and by doing so we obtain the affine Nakajima variety as the partial character variety

$$\mathfrak{M}^0(Q, v, w) = \mu^{-1}(0) /\!\!/ G = \mathrm{Rep}^G_{v,w}(A)\,.$$

Now that we have such a model for this singular scheme we can try to resolve it using the machinery of model categories and in particular the theory of derived representation schemes ([7], [9]): we consider the derived scheme

$$\mathrm{DRep}_{v,w}(A) \cong \mathrm{Rep}_{v,w}(A_{\mathrm{cof}})\,,$$

where $A_{\mathrm{cof}} \xrightarrow{\sim} A$ is a cofibrant replacement in the category of differential graded algebras. It is (the homotopy class of) a differential graded scheme of the form $X = (X_0, \mathcal{O}_{X,\bullet})$, where $X_0 \cong M(Q, v, w)$ is the vector space of linear representations of the framed, doubled quiver, and $\mathcal{O}_{X,\bullet}$ is a sheaf of dg-algebras whose zero homology gives:

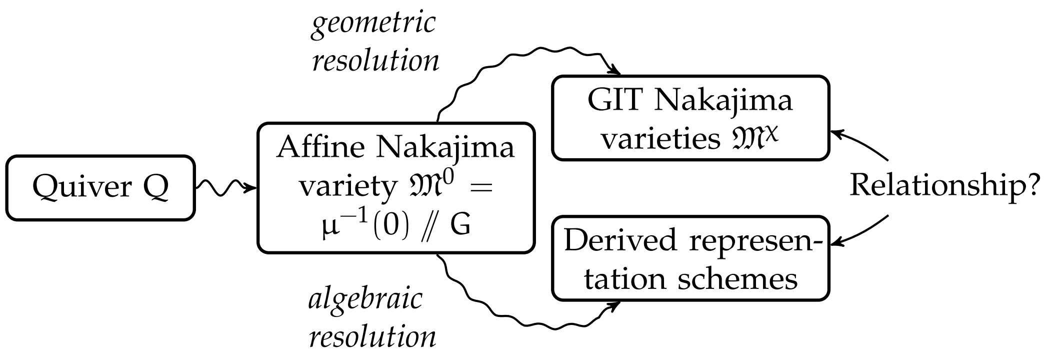
$$\pi_0(X) = \mathrm{Spec}\big(H_0(\mathcal{O}_{X,\bullet})\big) = \mu^{-1}(0)\,.$$

We exhibit an explicit (minimal) resolution $A_{\mathrm{cof}} \xrightarrow{\sim} A$ for which this derived representation scheme is a well-known object when it comes to studying resolutions of a singular locus:

Theorem (2.3.5.2 in § 2.3.5)**.** *There is a cofibrant resolution* $A_{\mathrm{cof}} \xrightarrow{\sim} A \in \mathtt{DGA}_S$ *which gives a model for the derived representation scheme as the (spectrum of the) Koszul complex on the moment map:*

$$\mathrm{DRep}_{v,w}(A) \cong \mathrm{Rep}_{v,w}(A_{\mathrm{cof}}) = \mathrm{Spec}\big(\mathcal{O}(M(Q, v, w)) \otimes \Lambda^\bullet \mathfrak{g}\big)\,. \qquad (2.1.2)$$

A somewhat natural question is whether or not there is any relationship between Nakajima resolutions (2.1.1) and these derived schemes, and if it is possible to obtain informations about one of the two from the other:



(2.1.3)

A first answer is a close relationship (an equivalence) between the condition of flatness for the moment map (which assures that $\mathfrak{M}^\chi \to \mathfrak{M}^0$ is indeed a resolution, for well-behaved characters χ), and the vanishing condition for higher homologies of derived representation schemes:

Theorem (2.4.1.4 in § 2.4.1)**.** *The derived representation scheme* $\mathrm{DRep}_{v,w}(A)$ *has vanishing higher homologies if and only if* $\mu^{-1}(0) \subset M(Q, v, w)$ *is a complete intersection, which happens if and only if the moment map is flat.*

We remark that in general it might not be easy to compute homologies of derived representation schemes, and even just to predict until which degree the homology is nontrivial. Nevertheless, in this special situation it is possible to give a sufficient and necessary condition for the vanishing of higher homologies based on a geometric property (flatness) of the moment map. The importance of Theorem 2.1.1 is that there is a combinatorial criterium on the dimension vectors v, w (proved by Crawley-Boevey, [19], based on the canonical decomposition of Kac, [37]) for the flatness of the moment map for representations of quivers.

A second answer to the question in (2.1.3) comes when we compare some invariants associated with the derived representation schemes with others associated with the varieties $\mathfrak{M}^\chi$. A natural choice is to consider tautological sheaves on the GIT quotient $\mathfrak{M}^\chi$ constructed with the usual machinery developed by Kirwan (§ 2.4.2). Because of reductiveness of the gauge group G we restrict to consider only tautological sheaves of the form $\mathcal{V}_\lambda$ induced from irreducible representations V_λ of G. The push-forward of these sheaves in the K-theory of the affine Nakajima variety through the map (2.1.1) computes their (T-)equivariant Euler characteristics:

$$\chi_T(\mathfrak{M}^\chi, \mathcal{V}_\lambda) \in K_T(\mathfrak{M}^0)\,, \tag{2.1.4}$$

where $T = T_{\boldsymbol{w}} \times T_\hbar$ is the product of the standard maximal torus in the other general linear group on the framing vertices $T_{\boldsymbol{w}} \subset G_{\boldsymbol{w}}$ and a 2-dimensional torus $T_\hbar$ rescaling the symplectic form and the cotangent direction.

On the other hand also the representation homology $H_\bullet(A, \boldsymbol{v}, \boldsymbol{w})$ (the homology of the derived representation scheme) is naturally a G-module and therefore decomposes into the direct sum of its isotypical components:

$$H_\bullet(A, \boldsymbol{v}, \boldsymbol{w}) = \bigoplus_\lambda \operatorname{Hom}_G\big(V_\lambda, H_\bullet(A, \boldsymbol{v}, \boldsymbol{w})\big) \otimes V_\lambda\,. \tag{2.1.5}$$

The isotypical components $\operatorname{Hom}_G(V_\lambda, H_\bullet(A, \boldsymbol{v}, \boldsymbol{w}))$ are modules over the G-invariant zeroth homology $H_0(A, \boldsymbol{v}, \boldsymbol{w})^G = \mathcal{O}(\mu^{-1}(0))^G$ and therefore their Euler characteristics define invariants in

$$\chi_T^\lambda(A, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i \geqslant 0} (-1)^i \big[\operatorname{Hom}_G(V_\lambda, H_i(A, \boldsymbol{v}, \boldsymbol{w}))\big] \in K_T(\mathfrak{M}^0)\,. \tag{2.1.6}$$

It is tempting to compare the invariants defined in (2.1.4) and (2.1.6), and the main results of this paper go in this direction. First of all, when we consider the trivial representation $V_\lambda = \mathbb{C}$, we prove that if the moment map is flat, then the two invariants are indeed equal:

Theorem (2.4.3.1 in § 2.4.3)**.** *Let $\boldsymbol{v}, \boldsymbol{w}$ be dimension vectors for which the moment map is flat and let χ such that $\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w})$ is a smooth variety (and therefore a resolution of $\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w})$). Then we have*

$$p_*\big([\mathcal{O}_{\mathfrak{M}^\chi(Q,\boldsymbol{v},\boldsymbol{w})}]\big) = [\mathcal{O}_{\mathfrak{M}^0(Q,\boldsymbol{v},\boldsymbol{w})}] = \chi_T^G(A, \boldsymbol{v}, \boldsymbol{w}) \in K_T(\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}))\,. \tag{2.1.7}$$

When we consider the Hilbert-Poincaré series of (2.1.7) we obtain an integral formula for the T-character of the ring of functions on the GIT quotient $\mathfrak{M}^\chi$, that has the following form

$$\mathrm{ch}_T(\mathcal{O}(\mathfrak{M}^\chi)) = \mathrm{ch}_T(\mathcal{O}(\mathfrak{M}^0)) = \frac{1}{|W|}\int_{T_v} \frac{\prod_i (1-\hbar_1\hbar_2 r_i)}{\prod_j (1-s_j)} \Delta(x) dx\,, \tag{2.1.8}$$

where $r_i = r_i(x)$ and $s_j = s_j(x,t)$ are characters for T_v and $T_v \times T$, respectively, $\Delta(x)$ is the Weyl factor for G_v and the integration is over the compact real form of T_v (see § 2.4.3 for a more detailed explanation).

Integral formulas of similar flavours already appear under different names, both in the mathematical literature (Jeffrey-Kirwan integral/residue formula for GIT quotients — [34]) and in the physical literature (integral formula for Nekrasov partition function — [55], [56] — proven, for example, in Appendix A in [25]). We could say that this is not a coincidence, in fact recognising the right-hand side of (2.1.8) in the known example of the Jordan quiver (Nekrasov partition function) as the Euler characteristic of the representation homology was one of the motivations of this project.

For what concerns other tautological sheaves $\mathcal{V}_\lambda$ an equality of the same flavour of (2.1.7) is true only for large enough λ, where the definition of largeness depends on the quiver, the dimension vectors $\boldsymbol{v}, \boldsymbol{w}$ and, perhaps more importantly, also on the GIT parameter χ (see § 2.4.4):

Theorem (2.4.4.1 in § 2.4.4). *Let $\boldsymbol{v}, \boldsymbol{w}$ be dimension vectors for which the moment map is flat, and χ a character for which $\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w})$ is smooth. For λ large enough (Definition 2.4.4.1) we have*

$$p_*([\mathcal{V}_\lambda]) = [H^0(\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w}), \mathcal{V}_\lambda)] = \chi_T^{\lambda^*}(A, \boldsymbol{v}, \boldsymbol{w}) \in K_T(\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}))\,. \tag{2.1.9}$$

Once again by taking the Hilbert-Poincaré series of (2.1.9) we obtain a second integral formula for tautological sheaves on the GIT quotient:

$$\mathrm{ch}_T(\chi_T(\mathfrak{M}^\chi, \mathcal{V}_\lambda)) = \mathrm{ch}_T(H^0(\mathfrak{M}^\chi, \mathcal{V}_\lambda)) = \frac{1}{|W|}\int_{T_v} \frac{\prod_i (1-\hbar_1\hbar_2 r_i)}{\prod_j (1-s_j)} f_\lambda(x) \Delta(x) dx\,, \tag{2.1.10}$$

where $f_\lambda(x) = \mathrm{ch}_{T_v}(V_\lambda)$ is a product of Schur polynomials.

2.1.2 Layout of the paper

In § 2.2 we introduce the general theory of (derived) representation schemes of an algebra. First we recall the theory of representation schemes with some examples, in particular the linear space of representations of a quiver as a representation scheme for its path algebra. Then we recall the derived version introduced by [9] and [7]. We introduce a more general way to take invariant subfunctors and an equivariant version of derived representation schemes for an action of an algebraic torus which is useful for our purposes. We decompose the representation homology in isotypical components and define new invariants in the K-theory of the classical character scheme.

In § 2.3 we recall the construction of Nakajima quiver varieties and we show how to view the affine Nakajima variety $\mathfrak{M}^0$ as a partial character scheme (a quotient of a representation scheme) for the algebra $A := \mathbb{C}\overline{Q^{fr}}/\mathcal{I}_\mu$. We construct the derived scheme associated to it and we use the invariants defined in § 2.2 to decompose the representation homology into classes in the K-theory of $\mathfrak{M}^0$. In § 2.3.4 we construct an explicit cofibrant resolution $A_{cof} \xrightarrow{\sim} A$ that gives a concrete model for the derived representation scheme as the (spectrum of the) Koszul complex on the moment map. Therefore we recall some classical properties of the Koszul complex and commutative complete intersections.

In § 2.4 we explain the main results of this paper. First we observe that, using the model found in § 2.3.4, the derived representation scheme has vanishing higher homologies if and only if the moment map is flat, which is a combinatorical condition on the dimension vectors of the quiver ([19]). We recall the definition of tautological sheaves on GIT quotients by the Kirwan map and prove results that compare them with the isotypical components of the representation homology ((2.1.7) and (2.1.9)). In particular we obtain some interesting integral formulas ((2.1.8) and (2.1.10)).

In § 2.5 we show some concrete examples, such as the quiver A_1 for which Nakajima varieties are cotangent spaces of Grassmannians, the Jordan quiver for which we obtain framed moduli space of torsion free sheaves on $\mathbb{P}^2$, and the quiver A_{n-1} with some special dimension vectors for which we obtain the symplectic dual $(\mathbb{T}^*\mathbb{P}^{n-1})^\vee$, and compute some of the integral formulas that we proved before.

In Appendix A we construct a model structure on equivariant dg-algebras that we need in § 2.2.5, and in Appendix B we recall the theory of irreducible representations for a product of general linear groups as multipartitions, and set some notation that we need in § 2.4.4.

Notation. Throughout the paper we denote categories by the standard monospace font: `Sets`, `Grp`, $\mathtt{Vect}_k$, $\mathtt{Alg}_k$, ... The notation used is often both standard and self-explanatory, and when this is not the case we usually recall it in the main body of the paper.

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2.2 Derived representation schemes of an algebra

The family of schemes of finite dimensional representations $\{\mathrm{Rep}_n(A)\}_{n\geqslant 1}$ of an algebra A has been object of study for many years (see for example the early work of Procesi, [62]). With the development of noncommutative geometry, they have been seen in a new light when Kontsevich and Rosenberg ([41]) proposed the following principle:

> "*Any noncommutative structure of some kind on A should give an analogous commutative structure on all the representation schemes* $\mathrm{Rep}_n(A)$, $n \geqslant 1$".

This principle seems to work well for (formally) smooth algebras, for which the representation schemes are smooth, but fails in general. The solution proposed in [9] is to find a smoothening of representation schemes by extending representation schemes to differential graded algebras, and using the general machinery of model categories to derive them. The purpose of this section is to recall in main details the construction of this derived version of representation schemes from [9] and [7], and describe some generalisations that are useful to our purposes.

2.2.1 Classical representation schemes

Let k be an algebraically closed field of characteristic zero (later we fix $k = \mathbb{C}$). Let $A \in \mathtt{Alg}_k$ be a unital, associative algebra and $V \in \mathtt{Vect}_k$ a finite dimensional vector space. We consider the functor on unital commutative algebras:

$$\begin{aligned} \mathrm{Rep}_V(A) : \mathtt{CommAlg}_k (= \mathtt{Aff}_k^{\mathrm{op}}) &\to \mathtt{Sets} \\ B &\longmapsto \mathrm{Hom}_{\mathtt{Alg}_k}\big(A, \mathrm{End}(V) \otimes_k B\big) . \end{aligned} \tag{2.2.1}$$

This functor is (co)-representable, by the commutative algebra $A_V := \big(\sqrt[V]{A}\big)_{\natural\natural}$. The two functors $\sqrt[V]{-}$ and $(-)_{\natural\natural}$ are, respectively, the matrix reduction functor and the abelianisation functor, which are left adjoints to the followings:

$$\begin{tikzcd} \mathtt{Alg}_k \arrow[r, shift left, "\sqrt[V]{-}"] \arrow[r, phantom, "\perp"] & \mathtt{Alg}_k \arrow[l, shift left, "\mathrm{End}(V)\otimes_k(-)"] \end{tikzcd} , \qquad \begin{tikzcd} \mathtt{Alg}_k \arrow[r, shift left, "(-)_{\natural\natural}"] \arrow[r, phantom, "\perp"] & \mathtt{CommAlg}_k \arrow[l, shift left, "U"] \end{tikzcd} . \tag{2.2.2}$$

Explicit formulas for them are $\sqrt[V]{A} = \big(\mathrm{End}(V) *_k A\big)^{\mathrm{End}(V)}$ and $(C)_{\natural\natural} = C/\langle [C, C] \rangle$, where $\langle [C, C] \rangle$ is the 2-sided ideal generated by the commutators. By combining the two adjunctions in (2.2.2) we get an adjunction for the representation functor:

$$\begin{tikzcd} \mathtt{Alg}_k \arrow[r, shift left, "(-)_V"] \arrow[r, phantom, "\perp"] & \mathtt{CommAlg}_k \arrow[l, shift left, "\mathrm{End}(V)\otimes_k(-)"] \end{tikzcd} , \tag{2.2.3}$$

so that the commutative algebra A_V is uniquely defined by the natural isomorphisms:

$$\mathrm{Hom}_{\mathtt{CommAlg}_k}(A_V, B) \cong \mathrm{Hom}_{\mathtt{Alg}_k}(A, \mathrm{End}(V) \otimes_k B), \quad \forall B \in \mathtt{CommAlg}_k . \tag{2.2.4}$$

Definition 2.2.1.1. The affine scheme associated to $A_V \in \mathtt{CommAlg}_k$ is the *representation scheme* $\mathrm{Rep}_V(A) = \mathrm{Spec}(A_V) \in \mathtt{Aff}_k$ (strictly speaking we identify it with its functor of points as we originally defined it $\mathrm{Rep}_V(A) \in \mathtt{Fun}(\mathtt{Aff}_k^{\mathrm{op}}, \mathtt{Sets})$ in (2.2.1)). We recover $A_V = \mathcal{O}(\mathrm{Rep}_V(A))$ as the algebra of functions on the representation scheme.

We can assume that $V = k^n$ and write simply $\mathrm{Rep}_n(A) = \mathrm{Spec}(A_n)$ instead of $\mathrm{Rep}_V(A) = \mathrm{Spec}(A_V)$. Let us show some examples:

Examples 1. (0) If $A \in \mathtt{CommAlg}_k \subset \mathtt{Alg}_k$ is a commutative algebra then clearly from (2.2.4):

$$A_1 = A \quad \leftrightarrow \quad \mathrm{Rep}_1(A) = \mathrm{Spec}(A)\,.$$

1. The free algebra in m generators $A = F_m = k\langle x_1, \dots, x_m\rangle$ has no relations and therefore $\mathrm{Rep}_n(F_m)$ is the scheme of m-tuples of $n \times n$ matrices:
$$\mathrm{Rep}_n(F_m) = M_{n\times n}(k)^m\,.$$

2. The polynomial algebra $A = k[x_1, \dots, x_m]$ can be expressed as the free algebra in m generators modulo the ideal generated by all commutators $[x_i, x_j]$, therefore its representation scheme is the closed subscheme of m-tuples of $n \times n$ matrices that pairwise commute:
$$\mathrm{Rep}_n(A) = C(m,n) := \left\{(X_1, \dots, X_m) \in M_{n\times n}(k)^m \,|\, [X_i, X_j] = 0\, \forall i, j\right\}.$$

3. The algebra of dual numbers $A = k[x]/(x^2)$ gives the scheme of square-zero matrices:
$$\mathrm{Rep}_n(A) = \left\{X \in M_{n\times n}(k) \,|\, X^2 = 0\right\}.$$

4. The algebra of differential operators on the affine line $A = \mathrm{Diff}(\mathbb{A}^1_k) = k\langle x, d\rangle/([d,x] = 1)$ has no finite-dimensional representations because if $X, D \in M_{n\times n}(k)$ are matrices satisfying $[D, X] = \mathbb{1}_n$, then taking traces we would get $0 = n$, which is absurd:
$$\mathrm{Rep}_n\big(\mathrm{Diff}(\mathbb{A}^1_k)\big) = \emptyset\,.$$

5. The algebra of Laurent polynomials in m variables $A = k[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is similar to the example of commuting matrices, except that now the matrices are required to be invertible:
$$\operatorname{Rep}_n(A) = \left\{(X_1, \dots, X_m) \in GL_n(k)^m \mid [X_i, X_j] = 0 \,\forall i, j\right\}.$$

6. More generally writing any finitely generated algebra as a free algebra modulo some relations
$$A = F_m/\langle r_1, \dots, r_s\rangle, \qquad r_1, \dots, r_s \in F_m = k\langle x_1, \dots, x_m\rangle,$$
then its representation scheme is identified with the closed subscheme
$$\begin{aligned}\operatorname{Rep}_n(A) = \left\{(X_1, \dots, X_m) \in M_{n\times n}(k)^m \mid r_i(X_1, \dots, X_m) = 0 \ \forall i\right\} \subset \\ \subset \operatorname{Rep}_n(F_m)\end{aligned}$$
of m-tuples of $n \times n$ matrices defined by the equations $r_1, \dots, r_s$.

Another fundamental example is that of path algebras of (finite) quivers. These algebras come with an additional structure of algebras over the finite dimensional algebras of their empty paths on the vertices, which is crucial when considering their representations, therefore we need to consider a *relative* version of representation schemes. Formally we fix an algebra $S \in \mathtt{Alg}_k$ and we consider the under category $S \downarrow \mathtt{Alg}_k$ (also denoted by $\mathtt{Alg}_S$ following the notation of [7] and [9]) which is the category of algebras $A \in \mathtt{Alg}_k$ together with a fixed morphism $S \to A$. We also fix a representation $\rho : S \to \operatorname{End}(V)$.

With these ingredients it is natural to consider only those representations $A \to \operatorname{End}(V)$ that agree with ρ on S. In terms of functor of points this corresponds to
$$\begin{aligned}\operatorname{Rep}_V(A) : \mathtt{CommAlg}_k &\to \mathtt{Sets} \\ B &\longmapsto \operatorname{Hom}_{\mathtt{Alg}_S}\big(A, \operatorname{End}(V) \otimes_k B\big).\end{aligned} \tag{2.2.5}$$
This functor is also (co)representable, by the commutative algebra A_V defined as before except for $*_k$ substituted by $*_S$, the coproduct in $\mathtt{Alg}_S$. Letting A vary we obtain a relative version of the representation functor $(-)_V$, and a similar adjunction
$$\begin{tikzcd} \mathtt{Alg}_S \arrow[r, shift left, "(-)_V"] \arrow[r, phantom, "\perp"] & \mathtt{CommAlg}_k \arrow[l, shift left, "\operatorname{End}(V)\otimes_k(-)"] \end{tikzcd}. \tag{2.2.6}$$

Example 2 (Path algebra of a quiver)**.** Let Q be a finite quiver and $A = \mathbb{C}Q \in \mathtt{Alg}_{\mathbb{C}}$ its path algebra over the complex numbers. What follows works well for any field k of characteristic zero but later we are interested only in $k = \mathbb{C}$. We recall that the path algebra is the free vector space on the admissible paths in the quiver, with product given by concatenation of paths. It has a set of orthogonal idempotents $\{e_i\}_{i\in Q_0} \subset A$:

$$e_i e_j = \delta_{ij} e_j \,,$$

which are the empty paths on the vertices, and their sum is the unit of the algebra: $\sum_{i\in Q_0} e_i = 1 \in A$. We can then consider the subalgebra generated by these idempotents

$$S = \langle e_i \rangle_{i\in Q_0} = \mathrm{Span}_{\mathbb{C}}\{e_i\}_{i\in Q_0} \,,$$

with the natural inclusion $\iota : S \to A$. We now fix a dimension vector $\boldsymbol{v} \in \mathbb{N}^{Q_0}$ and we consider the linear space of representations of the quiver Q with the complex vector space $\mathbb{C}^{v_i}$ placed at the vertex $i \in Q_0$:

$$L(Q, \boldsymbol{v}) := \bigoplus_{\gamma\in Q_1} \mathrm{Hom}_{\mathbb{C}}\big(\mathbb{C}^{v_{s(\gamma)}}, \mathbb{C}^{v_{t(\gamma)}}\big) \,. \tag{2.2.7}$$

where $s, t : Q_1 \to Q_0$ are the source and target maps of the quiver. From the algebraic point of view we fix the following representation of S in the vector space $\mathbb{C}^{\boldsymbol{v}} := \oplus_i \mathbb{C}^{v_i}$:

$$\begin{aligned} \rho = \rho_{\boldsymbol{v}} : S &\to \oplus_i \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{v_i}) \subset \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{v}}) \\ e_i &\longmapsto E_i := 0 \oplus \cdots \oplus \underbrace{\mathbb{1}_{\mathbb{C}^{v_i}}}_{\text{i-th factor}} \oplus \cdots \oplus 0 \,. \end{aligned} \tag{2.2.8}$$

Proposition 2.2.1.1. *The linear space of representations of the quiver Q with fixed dimension vector $\boldsymbol{v}$ is isomorphic to the (relative) representation scheme of its path algebra:*

$$L(Q, \boldsymbol{v}) \cong \mathrm{Rep}_{\mathbb{C}^{\boldsymbol{v}}}(\mathbb{C}Q) \,. \tag{2.2.9}$$

Proof. Let us consider the complex vector space with basis given by the set of arrows of the quiver $M := \mathrm{Span}_{\mathbb{C}}\{x_\gamma\}_{\gamma\in Q_1}$. It has the structure of an S-bimodule, and its tensor algebra is the path algebra of the quiver:

$$A = \mathbb{C}Q = T_S M := S \oplus M \oplus (M \otimes_S M) \oplus \dots \,.$$

For a dimension vector $\boldsymbol{\nu} \in \mathbb{N}^{Q_0}$ we consider the graded vector space $\mathbb{C}^{\boldsymbol{\nu}} = \oplus_i \mathbb{C}^{\nu_i}$, whose endomorphism algebra $\mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{\nu}})$ is an S-bimodule via the map (2.2.8). By the universal property of the tensor algebra, giving a representation $T_S M \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{\nu}})$ that agrees with ρ on S, is equivalent to give a S-bimodule map $M \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{\nu}})$:

$$\begin{aligned}&\mathrm{Hom}_{\mathtt{Alg}_S}(A, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{\nu}})) \cong \mathrm{Hom}_{S-\mathtt{Bimod}}(M, \mathrm{End}_{\mathbb{C}}(\mathbb{C}^{\boldsymbol{\nu}})) \cong \\ &\cong \bigoplus_{\gamma \in Q_1} \mathrm{Hom}_{\mathbb{C}}\left(\mathbb{C}^{\nu_{s(\gamma)}}, \mathbb{C}^{\nu_{t(\gamma)}}\right) = L(Q, \boldsymbol{\nu}) .\end{aligned}$$

□

2.2.2 Derived representation schemes

As already anticipated in the introduction of this section, the noncommutative geometry principle of transferring a geometric property on an algebra A (e.g. complete intersection, Cohen-Macaulay, etc.) on the corresponding commutative one on $\mathrm{Rep}_V(A)$ might fail when A is not a (formally) smooth algebra. This seems to be related to the fact that the functor $\mathrm{Rep}_V(-)$ is not exact.

We discuss the following derived version of representation schemes firstly introduced in [9]. The idea is to "resolve" the singularities of the representation schemes by using the tools of homological algebra, in the sense of Quillen's derived functors on model categories.

We enlarge the category of algebras to the one of differential graded algebras $\mathtt{DGA}_k$ (in our conventions differentials have always degree -1), and as before we consider the under category $\mathtt{DGA}_S := S \downarrow \mathtt{DGA}_k$ of dg-algebras A with a fixed morphism $S \to A$.

We also fix a differential graded vector space $V \in \mathtt{DGVect}_k$ of finite total dimension, and denote by $\underline{\mathrm{End}}(V) \in \mathtt{DGA}_k$ the differential graded algebra of endomorphisms, with differential

$$df = d_V \circ f - (-1)^i f \circ d_V, \quad f \in \underline{\mathrm{End}}(V)_i . \tag{2.2.10}$$

Moreover we need to fix a representation of S in V, that is a dga morphism $\rho : S \to \underline{\mathrm{End}}(V)$, which makes $\underline{\mathrm{End}}(V)$ an object of $\mathtt{DGA}_S$. With these ingredients we can define a differential graded version of the representation functor for $A \in \mathtt{DGA}_S$ as the functor from commutative dg-algebras:

$$\begin{aligned} \mathrm{Rep}_V(A) : \mathtt{CDGA}_k &\to \mathtt{Sets} \\ B &\longmapsto \mathrm{Hom}_{\mathtt{DGA}_S}(A, \underline{\mathrm{End}}(V) \otimes_k B)\,. \end{aligned} \tag{2.2.11}$$

Remark 2.2.2.1. We use the same notation as in the non-graded case because in the particular case of S, A, V being concentrated in degree zero we recover the same functor as before (when restricted to $\mathtt{Alg}_k \subset \mathtt{DGA}_k$).

This functor is also (co)-representable, by the object $A_V := \left(\sqrt[V]{A}\right)_{\natural\natural}$ constructed in the same way as before, with

$$\sqrt[V]{A} = (\underline{\mathrm{End}}(V) *_S A)^{\underline{\mathrm{End}}(V)}\ , \tag{2.2.12}$$

where $*_S$ is the free product over S, the categorical coproduct in $\mathtt{DGA}_S$. As before we obtain a pair of adjoint functors

$$\begin{tikzcd} \mathtt{DGA}_S \arrow[r, shift left, "(-)_V"] \arrow[r, phantom, "\perp"] & \mathtt{CDGA}_k \arrow[l, shift left, "\underline{\mathrm{End}}(V)\otimes_k(-)"] \end{tikzcd}\ . \tag{2.2.13}$$

These categories have model structures for which this adjunction is a Quillen adjunction, and therefore produces a total right-derived functor $\mathbf{R}\big(\underline{\mathrm{End}}(V) \otimes_k (-)\big)$, but more importantly a left-derived functor $\mathbf{L}(-)_V$ that we use to define the derived representation scheme.

We consider on $\mathtt{DGA}_k$ and $\mathtt{CDGA}_k$ the so-called projective model structures for which weak equivalences are quasi-isomorphisms of complexes and fibrations are degree-wise surjective maps (Theorem 4 in [7]). It is useful for later purposes to consider also the categories $\mathtt{DGA}_k^+$ and $\mathtt{CDGA}_k^+$, which are the categories of non-negatively graded differential graded and commutative differential graded algebras, respectively, and with their projective model structures with the only difference that now fibrations are degree-wise surjective maps in all (strictly) *positive* degrees. All these categories are fibrant (every object is fibrant), with initial object k and final object 0.

The category $\mathtt{DGA}_S$ is an example of an under category (category in which objects are objects of the original category coming with a fixed morphism from the object S in this case). As such it comes with a forgetful functor $\mathtt{DGA}_S \to \mathtt{DGA}_k$ and the model structure on $\mathtt{DGA}_S$ is the one in which weak-equivalences, fibrations and cofibrations are exactly the maps which are sent to weak-equivalences, fibrations and cofibrations via the forgetful functor. Clearly also the under category $\mathtt{DGA}_S$ is fibrant, with final object still 0 (viewed as an object of $\mathtt{DGA}_S$ via the unique map $S \to 0$), and initial object S (viewed as an object of $\mathtt{DGA}_S$ via the identity map $\mathrm{id}_S : S \to S$).

For a model category $\mathtt{C}$, we denote by $\mathtt{Ho}(\mathtt{C})$ its homotopy category and by $\gamma : \mathtt{C} \to \mathtt{Ho}(\mathtt{C})$ the canonical functor.

Theorem 2.2.2.1 (Theorem 7 in [7]). *(i) The pair of functors in* (2.2.13) *form a Quillen pair.*

(ii) The representation functor $(-)_V$ has a total left derived functor given by

$$\mathbf{L}(-)_V : \mathtt{Ho}(\mathtt{DGA}_S) \to \mathtt{Ho}(\mathtt{CDGA}_k)$$
$$\begin{cases} A \longmapsto \big(A_{\mathrm{cof}}\big)_V \\ \gamma f \longmapsto \gamma(\tilde{f})_V \end{cases} \tag{2.2.14}$$

where $A_{\mathrm{cof}} \xrightarrow{\sim} A$ is a cofibrant replacement in $\mathtt{DGA}_S$, and for a morphism $f : A \to B$, the morphism $\tilde{f} : A_{\mathrm{cof}} \to B_{\mathrm{cof}}$ is a lifting of f between the cofibrant replacements.

(iii) For any $A \in \mathtt{DGA}_S$ and any $B \in \mathtt{CDGA}_k$ there is a canonical isomorphism:

$$\mathrm{Hom}_{\mathtt{Ho}(\mathtt{CDGA}_k)}(\mathbf{L}(A)_V, B) \cong \mathrm{Hom}_{\mathtt{Ho}(\mathtt{DGA}_S)}(A, \underline{\mathrm{End}}(V) \otimes_k B)\,. \tag{2.2.15}$$

Definition 2.2.2.1. For $S \in \mathtt{Alg}_k$ concentrated in degree 0, the following composite functor

$$\begin{array}{ccc} \mathtt{Alg}_S \to \mathtt{Ho}(\mathtt{DGA}_S) & \xrightarrow{\mathbf{L}(-)_V} & \mathtt{Ho}(\mathtt{CDGA}_k) \\ A & \longmapsto & \mathbf{L}(A)_V \end{array} \tag{2.2.16}$$

is called *derived representation functor*. The homology of the (homotopy class of the) commutative differential graded algebra $\mathbf{L}(A)_V \in \mathtt{Ho}(\mathtt{CDGA}_k)$ depends only on $A \in \mathtt{Alg}_S$ and V. It is called the *representation homology* of A with coefficients in V:

$$\mathrm{H}_\bullet(A, V) := \mathrm{H}_\bullet(\mathbf{L}(A)_V)\,. \tag{2.2.17}$$

Remark 2.2.2.2. By its definition, the zero-th homology recovers the classical representation scheme (see Theorem 9 in [7]):

$$\mathrm{H}_0\big(A, V\big) \cong A_V = \mathcal{O}\big(\mathrm{Rep}_V(A)\big)\,. \tag{2.2.18}$$

As we anticipated before, we are interested in a slightly different version of this story: if we start from a vector space V concentrated in degree 0 and $S \in \mathtt{Alg}_k$ then the previous pair (2.2.13) restricts to a pair of functors

$$\begin{tikzcd} \mathtt{DGA}_S^+ \arrow[r, shift left, "(-)_V"] \arrow[r, phantom, "\perp"] & \mathtt{CDGA}_k^+ \arrow[l, shift left, "\underline{\mathrm{End}}(V)\otimes_k(-)"] \end{tikzcd} , \tag{2.2.19}$$

which is still a Quillen pair, and the analogous result of Theorem 2.2.2.1 holds. We give a second definition of:

Definition 2.2.2.2. The *derived representation functor* is the following functor:

$$\mathtt{Alg}_S \to \mathtt{Ho}(\mathtt{DGA}_S^+) \xrightarrow{\mathbf{L}(-)_V} \mathtt{Ho}(\mathtt{CDGA}_k^+) . \tag{2.2.20}$$

The *representation homology* of the relative algebra $A \in \mathtt{Alg}_S$ is the homology of $\mathbf{L}(A)_V \in \mathtt{Ho}(\mathtt{CDGA}_k^+)$.

Remark 2.2.2.3. Definition 2.2.2.1 and 2.2.2.2 are not really different. In fact, there is an adjunction between the categories $\mathtt{DGA}_S^+$ and $\mathtt{DGA}_S$

$$\begin{tikzcd} \mathtt{DGA}_S^+ \arrow[r, shift left, "\iota"] \arrow[r, phantom, "\perp"] & \mathtt{DGA}_S \arrow[l, shift left, "\tau"] \end{tikzcd} , \tag{2.2.21}$$

where the functor ι is the obvious inclusion and the functor τ is the one that sends an unbounded differential graded algebra $A \in \mathtt{DGA}_S$ to its truncation:

$$\tau(A) = [\cdots \to A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} \ker(d_0)] \in \mathtt{DGA}_S^+ .$$

It is straightforward to see that τ preserves fibrations and weak equivalences, and dually the map ι preserves cofibrations and weak equivalences, in particular it sends cofibrant objects to cofibrant objects. Now let $A \in \mathtt{Alg}_S$ and choose a cofibrant replacement $Q \xrightarrow{\sim} A \in \mathtt{DGA}_S^+$. A priori this map is only surjective in positive degrees, but because A is concentrated in degree 0, we have $A = H_0(A)$, and the isomorphism in homology $H_0(Q) \cong H_0(A)$ proves that it is surjective also in degree 0, so still a fibration in $\mathtt{DGA}_S$. In other words the cofibrant replacement $Q \xrightarrow{\sim} A$ is still a cofibrant replacement in $\mathtt{DGA}_S$ and therefore it can be used to compute the derived representation functor (2.2.16), showing that Definition 2.2.2.1 is equivalent to Definition 2.2.2.2.

Remark 2.2.2.4 (The dual language of dg-schemes). Another reason for considering the category $\mathtt{CDGA}_k^+$ instead of $\mathtt{CDGA}_k$ is that it is anti-equivalent to the category of differential graded schemes, as introduced by Ciocan-Fontanine and Kapranov in [18]. We recall their definition of dg-schemes (over k) as a pair $X = (X_0, \mathcal{O}_{X,\bullet})$, where X_0 is an ordinary scheme over k and $\mathcal{O}_{X,\bullet}$ is a sheaf of non-negatively graded commutative dg-algebras on X_0 such that the degree zero is $\mathcal{O}_{X,0} = \mathcal{O}_{X_0}$ the structure sheaf of the classical scheme X_0 and each $\mathcal{O}_{X,i}$ is quasicoherent over $\mathcal{O}_{X,0}$. A morphism of dg-schemes over k is just a morphism of dg-ringed spaces $f : X = (X_0, \mathcal{O}_{X,\bullet}) \to Y = (Y_0, \mathcal{O}_{Y,\bullet})$, and this makes $\mathtt{DGSch}_k$ into a category. A dg-scheme X is called *affine* if the underlying classical scheme X_0 is affine. The full subcategory of dg-affine schemes $\mathtt{DGAff}_k \subset \mathtt{DGSch}_k$ is antiequivalent to the category $\mathtt{CDGA}_k^+$, via the the equivalence of categories:

$$\begin{tikzcd} \mathtt{DGAff}_k^{op} \arrow[r, shift left, "\Gamma(-)"] \arrow[r, phantom, "\perp"] & \mathtt{CDGA}_k^+ \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}, \tag{2.2.22}$$

where $\Gamma(-)$ is the functor taking a dg-affine X into the global sections of the sheaf $\mathcal{O}_{X,\bullet}$ (degreewise), and Spec is the dg-spectrum sending a commutative dg-algebra A to the classical scheme $X_0 = \mathrm{Spec}(A_0)$ together with the quasicoherent sheaves $\mathcal{O}_{X,i}$ associated to the modules A_i via the correspondence $\mathtt{QCoh}_{X_0} \cong \mathtt{Mod}_{A_0}$. These names are motivated by the fact that the previous equivalence restricts to the classical equivalence of categories

$$\begin{tikzcd} \mathtt{Aff}_k^{op} \arrow[r, shift left, "\Gamma(-)"] \arrow[r, phantom, "\perp"] & \mathtt{CommAlg}_k \arrow[l, shift left, "\mathrm{Spec}"] \end{tikzcd}. \tag{2.2.23}$$

This definition of dg-affine schemes coincides with Toën-Vezzosi's definition of derived schemes $\mathtt{d\,Aff}_k^{op} = \mathtt{sCommAlg}_k$ as simplicial commutative algebras ([66]) because over a field k of characteristic zero they are equivalent to commutative dg-algebras.

The equivalence of categories (2.2.22) can be trivially used to transfer the projective model structure on commutative dg-algebras to the category of dg-affine schemes. Obviously the pair $(\Gamma(-), \mathrm{Spec})$ becomes a Quillen equivalence, i.e. an equivalence on the homotopy categories:

$$\begin{tikzcd} \mathtt{Ho}(\mathtt{DGAff}_k^{op}) \arrow[r, shift left, "\mathbf{L}\Gamma(-)"] \arrow[r, phantom, "\perp"] & \mathtt{Ho}(\mathtt{CDGA}_k^+) \arrow[l, shift left, "\mathbf{R}\mathrm{Spec}"] \end{tikzcd} . \tag{2.2.24}$$

Moreover because every object in $\mathtt{CDGA}_k^+$ is fibrant, the derived spectrum $\mathbf{R}$Spec actually coincides with the underived Spec on the objects.

Definition 2.2.2.3. The *derived representation scheme* of the relative algebra $A \in \mathtt{Alg}_S$ in a vector space V is the object $\mathrm{DRep}_V(A) \in \mathtt{Ho}(\mathtt{DGAff}_k)$ obtained applying to A the following composition of functors:

$$\mathrm{DRep}_V(-) : \mathtt{Alg}_S \to \mathtt{Ho}(\mathtt{DGA}_S^+) \xrightarrow{\mathbf{L}(-)_V} \mathtt{Ho}(\mathtt{CDGA}_k^+) \xrightarrow{\mathbf{R}\mathrm{Spec}} \mathtt{Ho}(\mathtt{DGAff}_k) . \tag{2.2.25}$$

This definition differs from the one given in [9] and [7] only from the last composition with the derived spectrum functor. The reason we do so is to be consistent with the notation for the classical representation scheme $\mathrm{Rep}_V(A) \in \mathtt{Aff}_k$.

Remark 2.2.2.5. Because every object in $\mathtt{CDGA}_k^+$ is fibrant, the derived representation scheme $\mathrm{DRep}_V(A)$ is simply

$$\mathrm{DRep}_V(A) = \mathbf{R}\mathrm{Spec}(\mathbf{L}(A)_V) = \mathrm{Spec}(\mathbf{L}(A)_V) = \mathrm{Spec}((A_{cof})_V) = \mathrm{Rep}_V(A_{cof}) , \tag{2.2.26}$$

where $A_{cof} \xrightarrow{\sim} A \in \mathtt{DGA}_S^+$ is a cofibrant replacement. Different choices of cofibrant replacements give different models to $\mathrm{DRep}_V(A)$, which are weakly equivalent to each other. In what follows we choose one specific model for $\mathrm{DRep}_V(A)$ obtained through a choice of a preferred cofibrant replacement. Strictly speaking in (3.3.22) we should write $\mathrm{DRep}_V(A) = \gamma\mathrm{Rep}_V(A_{cof}) \in \mathtt{Ho}(\mathtt{DGAff}_k)$ to remember that we are considering the homotopy class, but we make an abuse of notation by dropping γ.

Examples 3. In the following examples we describe explicit cofibrant resolutions for some of the algebras in the Examples 1 and give a model for their derived representation schemes with value in a vector space V concentrated in degree 0 (therefore we still use the notation $\mathrm{DRep}_n(-) = \mathrm{DRep}_V(-)$ for $V = k^n$).

1. The free algebra in m generators $A = F_m$ is already a cofibrant object in $\mathtt{DGA}_k^+$ because it is free, therefore
$$\mathrm{DRep}_n(F_m) \cong \mathrm{Rep}_n(F_m) \cong M_{n\times n}(k)^m .$$

2. The commutative algebra in two variables $A = k[x, y]$ is not cofibrant because of the relation $[x, y] = 0$. It turns out that it suffices to add one variable ϑ in homological degree 1 that kills this relation ($d\vartheta = [x, y]$) to obtain a cofibrant replacement:
$$A_{cof} := k\langle x, y, \vartheta\rangle \overset{\sim}{\twoheadrightarrow} A = k[x, y] ,$$
and therefore the derived representation scheme is the nothing else but the (spectrum of the) Koszul complex for the scheme of $n \times n$ commuting matrices:
$$\begin{aligned} \mathrm{DRep}_n(A) &\cong \mathrm{Rep}_n(A_{cof}) = \mathrm{Spec}\big(k[x_{ij}, y_{ij}, \vartheta_{ij}]_{i,j=1}^n\big) , \\ d\vartheta_{ij} &= \sum_k x_{ik}y_{kj} - y_{ik}x_{kj} . \end{aligned}$$

3. Calabi–Yau algebras of dimension 3 (see [29, § 1.3]). Consider the free algebra F_m and its commutator quotient space of cyclic words: $(F_m)_{cyc} = F_m/[F_m, F_m]$. M. Kontsevich introduced linear maps $\partial_i : (F_m)_{cyc} \to F_m$ for each $i = 1, \dots, m$ which we can use, together with a potential $\Phi \in (F_m)_{cyc}$, to define the algebra
$$A = \mathfrak{U}(F_m, \Phi) := F_m/(\partial_i\Phi)_{i=1,\dots,m} , \tag{2.2.27}$$
which is the quotient of the free algebra F_m by the two-sided ideal generated by the partial derivatives of the potential Φ. For example when $m = 3$, $F_3 = k\langle x, y, z\rangle$ and observe that the partial derivatives for the potential $\Phi = xyz - yxz$ give the commutators, therefore $A = k[x, y, z]$ is the polynomial ring in 3 variables. For an algebra defined by a potential as above in (2.2.27) we define the following dg-algebra:
$$\begin{aligned} &\mathfrak{D}(F_m, \Phi) := k\langle x_1, \dots, x_m, \vartheta_1, \dots, \vartheta_m, t\rangle , \\ &(\deg(x_i, \vartheta_i, t) = (0, 1, 2)) \quad d\vartheta_i = \partial_i\Phi, \quad dt = \sum_{i=1}^m [x_i, \vartheta_i] . \end{aligned} \tag{2.2.28}$$

Ginzburg explains in [29] how Calabi–Yau algebras of dimension 3 are all of the form (2.2.27) and they are exactly those for which a suitable completion of $\mathfrak{D}(F, \Phi)$ is a cofibrant resolution. This is in particular true for the example of polynomials in 3 variables (see example 6.3.2. in [7]), for which no completion is needed and:

$$\begin{aligned} \mathrm{DRep}_n(k[x,y,z]) &\cong \mathrm{Rep}_n(k[x,y,z,\xi,\vartheta,\lambda,t]) = \\ &= \mathrm{Spec}\big(k[x_{ij},y_{ij},z_{ij},\xi_{ij},\vartheta_{ij},\lambda_{ij},t_{ij}]_{i,j=1}^n\big)\,, \end{aligned}$$

where the variables ξ, ϑ, λ are the ones we called $\vartheta_1, \vartheta_2, \vartheta_3$ in (2.2.28).

2.2.3 G-invariants and isotypical components

On the (derived) representation scheme there is a natural action of the general linear group $\mathrm{GL}(V)$ by which one can consider the associated character scheme of invariants. Later we consider only invariants by a subgroup $G \subset \mathrm{GL}(V)$, therefore we propose the following theory of partial invariant subfunctors by G that generalises the theory introduced in [9, § 2.3.5] and in [7, § 3.4] in the absolute case $S = k$. However we point out that the results of this section are strongly inspired by [9] and [7], which already contain most of the material needed.

Suppose that both V and S are concentrated in degree 0, $\rho : S \to \mathrm{End}(V)$ is a fixed representation and consider

$$G_S := \{g \in \mathrm{GL}(V) \,|\, g^{-1}\rho(s)g = \rho(s) \ \ \forall s \in S\},$$

the subgroup of ρ-preserving transformations. Observe that in the absolute case $S = k$ then $G_S = \mathrm{GL}(V)$. Now consider any reductive subgroup $G \subset G_S$, whose right action on $\mathrm{End}(V)$ extends to the functor[1]:

$$\mathrm{End}(V) \otimes_k (-) : \mathtt{CDGA}_k \to \mathtt{DGA}_S$$

[1]We recall the notion of a categorical group action. Let G be a group (in the category of sets) and $X \in \mathcal{A}$ be an object of an essentially small category. A categorical (*left*) group action of G on X, denoted by $G \curvearrowright X$, is a morphism of groups $\rho : G \to \mathrm{Aut}_{\mathcal{A}}(X)$ (while a *right* group action is a left action of the opposite group G^{op}). If we apply this definition to the category $\mathcal{A} = \mathrm{Fun}(\mathcal{C}, \rfloor\mathcal{D})$ of functors between two categories (and natural transformations), then a group action on a functor X is a family of categorical group actions $\{\rho(\cdot)_c : G \to \mathrm{Aut}_{\rfloor\mathcal{D}}(Xc)\}_{c\in\mathcal{C}}$ with the property that for any morphism $f : c \to c'$, the induced morphism $Xf : Xc \to Xc'$ is a G-equivariant morphism.

(for this we need that G consists of transformations which all preserve ρ). Consequently we obtain a left action on $(-)_V : \mathtt{DGA}_S \to \mathtt{CDGA}_k$ and we can consider the invariant subfunctor

$$\begin{aligned} (-)^G_V : \mathtt{DGA}_S &\to \mathtt{CDGA}_k \\ A &\longmapsto A^G_V . \end{aligned} \tag{2.2.29}$$

As explained in [9], unlike $(-)_V$, the functor $(-)^G_V$ does not seem to have a right adjoint, so we cannot prove that it has a left derived functor from Quillen's adjunction theorem. Nevertheless we can prove that such a left derived functor exists:

Theorem 2.2.3.1. *(a)* $(-)^G_V : \mathtt{DGA}_S \to \mathtt{CDGA}_k$ *has a total left derived functor* $\mathbf{L}(-)^G_V$.

(b) For every $A \in \mathtt{DGA}_S$ *there is a natural isomorphism:*

$$H_\bullet[\mathbf{L}(A)^G_V] \cong H_\bullet(A, V)^G . \tag{2.2.30}$$

To prove this theorem it is convenient to recall a few notions/results. Let $\Omega = k[t] \oplus k[t]dt$ be the algebraic de Rham complex of the affine line $\mathbb{A}^1_k$ (in our conventions differentials have degree -1 and therefore dt has the wrong degree $|dt| = -1$). We define a *polynomial homotopy* between $f, g : A \to B \in \mathtt{DGA}_S$ as a morphism $h : A \to B \otimes \Omega \in \mathtt{DGA}_S$, such that $h(0) = f$ and $h(1) = g$, where for each $a \in k$, $h(a)$ is the following composite map:

$$h(a) : A \xrightarrow{h} B \otimes \Omega \xrightarrow{\pi} B \otimes \Omega/(t - a) \cong B \otimes k = B .$$

The reason why polynomial homotopy is equivalent to the homotopy equivalence relation in $\mathtt{DGA}_S$ is explained in Proposition B.2. in [9].

Lemma 2.2.3.1. *Let* $h : A \to B \otimes \Omega \in \mathtt{DGA}_S$ *be a polynomial homotopy between* $f, g : A \to B$. *Then:*

1. *There is a homotopy* $h_V : A_V \to B_V \otimes \Omega \in \mathtt{CDGA}_k$ *between* $h_V(0) = f_V$ *and* $h_V(1) = g_V$.

2. h_V *restricts to a morphism* $h^G_V : A^G_V \to B^G_V \otimes \Omega \in \mathtt{CDGA}_k$.

Remark 2.2.3.1. It is important to observe that, despite the misleading notation, the map h_V in part (1) is *not* the map obtained applying the functor $(-)_V$ to the map h. The latter would in fact be a map $A_V \to (B \otimes \Omega)_V \neq B_V \otimes \Omega$. The same thing applies for the map h_V^G in part (2), which is *not* the map obtained applying the functor $(-)_V^G$ to the map h.

Proof. We omit the proof because it is analogous to the proof of Lemma 2.5 in [9]. □

Proof of Theorem 2.2.3.1. The same proof used in Theorem 2.6 in [9] works, using Lemma 2.2.3.1 instead of Lemma 2.5. □

An analogous result holds also for the functor restricted on non-negatively graded objects, and it can be actually obtained as a corollary of Theorem 2.2.3.1:

Corollary 2.2.3.1. *(a)* $(-)_V^G : \mathtt{DGA}_S^+ \to \mathtt{CDGA}_k^+$ *has a total left derived functor* $\mathbf{L}(-)_V^G$.

(b) For every $A \in \mathtt{DGA}_S^+$ *there is a natural isomorphism:*

$$\mathrm{H}_\bullet[\mathbf{L}(A)_V^G] \cong \mathrm{H}_\bullet(A, V)^G . \tag{2.2.31}$$

Proof. Using Brown's lemma we just need to prove that $(-)_V^G$ sends a trivial cofibrations between cofibrant objects $A \overset{\sim}{\hookrightarrow} B$ to weak equivalences. We consider the following commutative diagram:

$$\begin{array}{ccc} \mathtt{DGA}_S^+ & \xrightarrow{(-)_V^G} & \mathtt{CDGA}_k^+ \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} \\ \mathtt{DGA}_S & \xrightarrow{(-)_V^G} & \mathtt{CDGA}_k \end{array} \tag{2.2.32}$$

and observe that $\iota(A \overset{\sim}{\hookrightarrow} B)$ is still a trivial cofibration between cofibrant objects in $\mathtt{DGA}_S$, according to Remark 2.2.2.3. Now we can use the proof of Theorem 2.2.3.1 to conclude that the functor $(-)_V^G : \mathtt{DGA}_S \to \mathtt{CDGA}_k$ sends this map to a weak equivalence:

$$(\iota A)_V^G = \iota(A_V^G) \xrightarrow{\sim} (\iota B)_V^G = \iota(B_V^G) \in \mathtt{CDGA}_k .$$

Finally from the very construction of ι we have that this map is a weak equivalence if and only if the map $A_V^G \xrightarrow{\sim} B_V^G \in \mathtt{CDGA}_k^+$ is a weak equivalence. This concludes the proof of (a), while (b) follows from (a) as in Theorem 2.2.3.1. $\square$

Now we derive also the other isotypical components of the representation functor. Let us fix any irreducible, finite-dimensional representation U_λ of the reductive group G. We consider the following functor:

$$\begin{aligned} (-)^G_{\lambda,V} : \mathtt{DGA}_S &\to \mathtt{DGVect}_k \\ A &\longmapsto \left(U_\lambda^* \otimes_k A_V\right)^G . \end{aligned} \tag{2.2.33}$$

which is the invariant subfunctor of the functor $(-)_{\lambda,V} := U_\lambda^* \otimes_k (-)_V$. Then we can prove the following analogue to Theorem 2.2.3.1:

Theorem 2.2.3.2. *(a) The functor* (2.2.33) *has a total left derived functor* $\mathbf{L}(-)^G_{\lambda,V}$.
(b) For every $A \in \mathtt{DGA}_S$ *there is a natural isomorphism:*

$$H_\bullet[\mathbf{L}(A)^G_{\lambda,V}] \cong \left(U_\lambda^* \otimes H_\bullet(A,V)\right)^G . \tag{2.2.34}$$

To prove it we need the following analogue of Lemma 2.2.3.1:

Lemma 2.2.3.2. *Let* $h : A \to B \otimes \Omega \in \mathtt{DGA}_S$ *be a polynomial homotopy between* $f, g : A \to B$. *Then:*

1. *There is a homotopy* $h_{\lambda,V} : A_{\lambda,V} \to B_{\lambda,V} \otimes \Omega \in \mathtt{DGVect}_k$ *between* $h_{\lambda,V}(0) = f_{\lambda,V}$ *and* $h_{\lambda,V}(1) = g_{\lambda,V}$.
2. $h_{\lambda,V}$ *restricts to a morphism* $h^G_{\lambda,V} : A^G_{\lambda,V} \to B^G_{\lambda,V} \otimes \Omega \in \mathtt{DGVect}_k$.

Proof. It is essentially a corollary of Lemma 2.2.3.1. In fact, we can define $h_{\lambda,V}$ to be

$$h_{\lambda,V} : A_{\lambda,V} = U_\lambda^* \otimes A_V \xrightarrow{\mathrm{id}_{U_\lambda^*} \otimes h_V} U_\lambda^* \otimes B_V \otimes \Omega = B_{\lambda,V} \otimes \Omega ,$$

where h_V is the map from part (1) of Lemma 2.2.3.1. The map h_V was G-equivariant, and therefore also $h_{\lambda,V} = \mathrm{id}_{U_\lambda^*} \otimes h_V$, from which part (2) follows. $\square$

Proof of Theorem 2.2.3.2. The proof works exactly as the proof of Theorem 2.2.3.1, using Lemma 2.2.3.2 instead of Lemma 2.2.3.1. □

The analogous results in the non-negative case also hold:

Corollary 2.2.3.2. *(a) The functor* $(-)^G_{\lambda,V} : \mathtt{DGA}^+_S \to \mathtt{DGVect}^+_k$ *has a total left derived functor* $\mathbf{L}(-)^G_{\lambda,V}$.

(b) For every $A \in \mathtt{DGA}^+_S$ *there is a natural isomorphism:*

$$H_\bullet[\mathbf{L}(A)^G_{\lambda,V}] \cong \big(U^*_\lambda \otimes H_\bullet(A,V)\big)^G . \tag{2.2.35}$$

Proof. The proof follows from Theorem 2.2.3.2 in the same way as the proof of Corollary 2.2.3.1 followed from Theorem 2.2.3.1. □

2.2.4 K-theoretic classes

We use the classical G-invariant subfunctor $(-)^G_V : \mathtt{Alg}_S \to \mathtt{CommAlg}_k$ to define

Definition 2.2.4.1. The *partial character scheme* of an algebra $A \in \mathtt{Alg}_S$ in a vector space V, relative to a subgroup $G \subset G_S$, is the affine quotient of the representation scheme:

$$\mathrm{Rep}^G_V(A) := \mathrm{Rep}_V(A) /\!\!/ G = \mathrm{Spec}(A^G_V) \in \mathtt{Aff}_k . \tag{2.2.36}$$

The name is motivated by the fact that in the absolute case $S = k$ and $G = GL(V)$ the full group, we would obtain the classical scheme of characters $\mathrm{Rep}^{GL(V)}_V(A)$. The derived version is:

Definition 2.2.4.2. The *derived partial character scheme* of $A \in \mathtt{Alg}_S$ in a vector space V, relative to a subgroup $G \subset G_S$, is the affine quotient of the derived representation scheme:

$$\mathrm{DRep}^G_V(A) := \mathrm{DRep}_V(A) /\!\!/ G = \mathbf{R}\mathrm{Spec}\big(\mathbf{L}(A)^G_V\big) \in \mathtt{Ho}(\mathtt{DGAff}_k) . \tag{2.2.37}$$

Let us recall that the obvious inclusion $\mathtt{Sch}_k \to \mathtt{DGSch}_k$ has for right adjoint the truncation functor $\pi_0 : \mathtt{DGSch}_k \to \mathtt{Sch}_k$ that associates to a dg-scheme $X = (X_0, \mathcal{O}_{X,\bullet})$ the closed subscheme $\pi_0(X) := \mathrm{Spec}(H_0(\mathcal{O}_{X,\bullet})) \subset X_0$:

$$\begin{tikzcd} \mathtt{Sch}_k \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathtt{DGSch}_k \arrow[l, shift left, "\pi_0"] \end{tikzcd} . \tag{2.2.38}$$

Because the differential $d : \mathcal{O}_{X,i} \to \mathcal{O}_{X,i-1}$ is $\mathcal{O}_{X_0}$-linear, the homologies $H_i(\mathcal{O}_{X,\bullet})$ are quasicoherent sheaves on X_0, and also on the closed subscheme $\pi_0(X) \subset X_0$. We can put these data together in a dg-affine scheme:

$$X_h := \big(\pi_0(X), H_\bullet(\mathcal{O}_{X,\bullet})\big) \in \mathtt{DGAff}_k\,,$$

which in the affine case $X = \mathrm{Spec}(A)$ is nothing but $\mathrm{Spec}(H_\bullet(A))$.

Definition 2.2.4.3 (Definition 2.2.6. in [18]). A dg-scheme X is *of finite type* if X_0 is a scheme of finite type and each $\mathcal{O}_{X,i}$ is a coherent sheaf on X_0.

Let now come to the case of our interest, a dg-affine scheme of finite type $X = \mathrm{Spec}(B)$, for which the sheaves $H_i(\mathcal{O}_{X,\bullet})$ are coherent both over X_0 and over $\pi_0(X) = \mathrm{Spec}(H_0(B))$, therefore they define a class in the algebraic K-theory[2]

$$[H_i(\mathcal{O}_{X,\bullet})] \in K(\pi_0(X))\,. \tag{2.2.39}$$

We first consider the derived scheme $X = \mathrm{DRep}_V(A)$. Let us assume that A is an algebra such that, for each vector space V, the following two conditions are satisfied:

1. The derived representation scheme $X = \mathrm{DRep}_V(A)$ is of finite type.
2. The structure sheaf $\mathcal{O}_{X,\bullet}$ of the derived representation scheme is bounded, in the sense that $\mathcal{O}_{X,i} = 0$ for $i \gg 0$.

This is true for all algebras that we consider in this article, as we show in § 2.3.4 and § 2.3.5. The truncated scheme obtained from the derived representation scheme is the classical representation scheme, as explained in Remark 2.2.2.2:

$$\pi_0(\mathrm{DRep}_V(A)) = \mathrm{Rep}_V(A)\,.$$

By condition (1) each homology defines a coherent sheaf on $\pi_0(X) = \mathrm{Rep}_V(A)$ and therefore a class

$$\big[H_i(A, V)\big] \in K(\mathrm{Rep}_V(A))\,.$$

By condition (2) there is only a finite number of them nonzero, therefore in particular the following definition makes sense, because the sum in (2.2.40) is bounded:

[2]By algebraic K-theory of a scheme we mean the Grothendieck ring of the abelian category of coherent sheaves on it.

Definition 2.2.4.4. The *virtual fundamental class* — or Euler characteristic — of the derived representation scheme $X = \mathrm{DRep}_V(A)$ is the following invariant in the K-theory of the classical representation scheme:

$$[X]^{\mathrm{vir}} = \chi(A, V) := \sum_{i=0}^{\infty} (-1)^i \big[H_i(A, V)\big] \in K(\mathrm{Rep}_V(A)) = K(\pi_0(X)) \,. \tag{2.2.40}$$

This virtual fundamental class carries an action of the group G, which is reductive, and therefore it decomposes into a direct sum of its irreducible components. To formalise this we first consider the quotient by derived partial character scheme $X^G = \mathrm{DRep}_V^G(A)$, whose truncation is $\pi_0(X^G) = \mathrm{Rep}_V^G(A)$. For each finite-dimensional irreducible representation U_λ of G we proved the existence of the derived functor of the corresponding component $\mathbf{L}(-)^G_{\lambda,V} : \mathtt{Ho}(\mathtt{DGA}_S^+) \to \mathtt{Ho}(\mathtt{DGVect}_k^+)$ and observed that

$$H_i\big(\mathbf{L}(A)^G_{\lambda,V}\big) \cong \big(U_\lambda^* \otimes H_i(A, V)\big)^G \in \mathtt{Mod}_{H_0(A,V)^G} \,,$$

and therefore they define coherent sheaves on $\mathrm{Rep}_V^G(A)$.

Definition 2.2.4.5. The Euler characteristic of the U_λ-irreducible component of the derived partial character scheme is

$$\chi^\lambda(A, V) := \sum_{i=0}^{\infty} (-1)^i [H_i(\mathbf{L}(A)^G_{\lambda,V})] \in K(\mathrm{Rep}_V^G(A)) \,. \tag{2.2.41}$$

We observe that the irreducible component corresponding to the trivial representation $U_0 = k$ is the virtual fundamental class of the derived partial character scheme, which we denote by

$$\chi^G(A, V) = \sum_{i=0}^{\infty} (-1)^i [H_i(A, V)^G)] = [X^G]^{\mathrm{vir}} \in K(\mathrm{Rep}_V^G(A)) \,.$$

2.2.5 T-equivariant enrichment

So far we have worked only with a group $G \subset G_S \subset GL(V)$ that acts on the representation scheme $\mathrm{Rep}_V(A)$ because of the standard action on the vector space V. However, often the algebra A itself comes with an action of some algebraic torus T which helps when calculating its invariants (for example the corresponding decomposition of A might consist of finite dimensional weight spaces, allowing a graded dimensions count). In this section we explain how such an action $T \curvearrowright A$ induces a well-defined group scheme action $T \curvearrowright \mathrm{DRep}_V(A)$, in the sense that different models for the derived representation scheme are linked by T-equivariant quasi-isomorphism, and therefore their homologies (and all the other invariants, as the Euler characteristics introduced in § 2.2.4) carry a well-defined induced T-action.

First we give a notion of a *rational* T-action, for an algebraic group $T \in \mathtt{Grp}_k$ on any (dg,commutative) algebra.

Definition 2.2.5.1. Let $\mathtt{C}$ be any of the following categories: $\mathtt{DGVect}_k$, $\mathtt{DGA}_S$, $\mathtt{CDGA}_k$ or their non-negatively graded versions. A *rational action* of an algebraic group T over k on an object $A \in \mathtt{C}$ is a morphism of groups $\rho : T \to \mathrm{Aut}_{\mathtt{C}}(A)$ with the additional property that every element $a \in A$ is contained in a finite dimensional T-stable vector subspace $a \in V \subset A$ on which the induced action $T \to \mathrm{GL}_k(V)$ is a morphism of algebraic groups over k. We denote by $\mathtt{C}^T$ the category with objects the objects in $\mathtt{C}$ with a rational T-action and morphisms the equivariant morphisms.

This definition is motivated by the fact that the equivalence of categories (2.2.22) enriches to an equivalence of categories between $(\mathtt{CDGA}_k^+)^T$ and the (opposite) category of dg-affine schemes with a group scheme action of T.

Remark 2.2.5.1. If we denote by a monospace font $\mathtt{T}$ the one-object groupoid associated to the group T, then a rational action on an object in $\mathtt{C}$ is just a functor $\mathtt{T} \to \mathtt{C}$ with some additional properties, and a T-equivariant morphism is a natural transformation of functors. Another way to say this is that we can view the category $\mathtt{C}^T \subset [\mathtt{T}, \mathtt{C}]$ as a full subcategory of the category of functors. If $\mathtt{C}, \mathtt{D}$ are two among the categories mentioned in 2.2.5.1, and $\mathtt{F} : \mathtt{C} \to \mathtt{D}$ is any functor between them, then we can consider the induced functor on the functor categories $\mathtt{F}_* = \mathtt{F} \circ (-) : [\mathtt{T}, \mathtt{C}] \to [\mathtt{T}, \mathtt{D}]$. If this induced functor sends objects of $\mathtt{C}^T \subset [\mathtt{T}, \mathtt{C}]$ into objects of $\mathtt{D}^T$, then it restricts to a functor that we denote by $\mathtt{F}^T : \mathtt{C}^T \to \mathtt{D}^T$. This is true whenever $\mathtt{F}$ is defined purely in "algebraic terms"[3], which is the case of all the functors

[3] We leave intentionally this as an intuitive, not well-defined, notion.

we considered so far. The induced functor F^T is an *enrichment* of the functor F in the sense that we can recover F under the natural forgetful functors:

$$\begin{tikzcd} \mathtt{C}^T \arrow[r, "F^T"] \arrow[d, "u"] & \mathtt{D}^T \arrow[d, "u"] \\ \mathtt{C} \arrow[r, "F"] & \mathtt{D} \end{tikzcd} \tag{2.2.42}$$

It is easy to see from the definition of the representation functor that a rational action $T \curvearrowright A$ induces (as explained in Remark 2.2.5.1), an action $T \curvearrowright A_V$ which is still rational, and therefore a group scheme action $T \curvearrowright \mathrm{Rep}_V(A)$. To summarise the adjunction (2.2.19) enriches to an adjunction:

$$\left(\mathtt{DGA}_S^+\right)^T \underset{\mathrm{End}(V)\otimes_k(-)}{\overset{(-)_V}{\underset{\perp}{\rightleftarrows}}} \left(\mathtt{CDGA}_k^+\right)^T \ . \tag{2.2.43}$$

We do not add a superscript $(-)^T$ to the enriched functors in this diagram in order to avoid confusion with the same symbols used with a different meaning in § 2.2.3.

From now on we restrict ourselves to the case of our interest in this paper of an algebraic torus $T = (k^\times)^r$. To do what we promised to do in the beginning of this section we need to prove that, roughly speaking, any T-equivariant algebra admits an equivariant cofibrant replacement in the model category $\mathtt{DGA}_S^+$, and that any two such equivariant cofibrant replacements produce quasi-isomorphic representation schemes. To do it we introduce a model structure on the category $(\mathtt{DGA}_S^+)^T$ compatible with the model structures on $\mathtt{DGA}_S^+$ under the forgetful functor (in the following Theorem we explain in which sense these model structures are compatible). We recall that $\mathtt{DGA}_S^+$ is equipped with the projective model structure in which weak equivalences are quasi-isomorphisms and fibrations are surjections in positive degrees. We also observe that actually the category of T-equivariant dg-algebras over S is $(\mathtt{DGA}_S^+)^T = S \downarrow (\mathtt{DGA}_k^+)^T$ nothing else but the under category of T-equivariant dg-algebras over k receiving a map from S if we give S the trivial action, and therefore we only need to give a model structure in the absolute case $S = k$.

Theorem 2.2.5.1. *There exists a model structure on* $(\mathtt{DGA}_k^+)^T$ *with the following properties:*

1. *Weak equivalences / fibrations are exactly the maps that are weak equivalences / fibrations under the forgetful functor* $U : (\mathtt{DGA}_k^+)^T \to \mathtt{DGA}_k^+$ *(and cofibrations are the maps with the left-lifting property with respect to acyclic fibrations defined in this way).*

2. *The forgetful functor preserves cofibrations.*

Proof. We refer the reader to Appendix A for the proof of this Theorem. □

As a corollary of this result we can naturally equip the derived representation scheme of a T-equivariant algebra with a group scheme action of T. In fact, let $S \in \mathtt{Alg}_k$ and $(A \in \mathtt{Alg}_S)^T = S \downarrow (\mathtt{Alg}_k)^T$ be a T-equivariant algebra.

Corollary 2.2.5.1. *There is a well-defined action* $T \curvearrowright \mathrm{DRep}_V(A)$ *which is compatible with the one on* $\mathrm{Rep}_V(A) \cong \pi_0(\mathrm{DRep}_V(A))$ *induced by* $T \curvearrowright A$.

Proof. First of all, we can pick up a T-equivariant cofibrant replacement $Q \overset{\sim}{\twoheadrightarrow} A \in (\mathtt{DGA}_S^+)^T$ using the model structure we just defined. Because of Theorem 2.2.5.1 (1) and (2), when we forget the T-action we still have a cofibrant replacement for A, therefore we can use this Q as a model for $\mathrm{DRep}_V(A) = \mathrm{Rep}_V(Q)$. There is a natural T-action on this dg-scheme induced by $T \curvearrowright Q$, which is compatible with the one on its truncation $\pi_0(\mathrm{Rep}_V(Q)) \cong \mathrm{Rep}_V(A)$.

To prove that the previous definition is well posed, we show that if $Q' \overset{\sim}{\twoheadrightarrow} A$ is any another T-equivariant cofibrant replacement, then there is a T-equivariant quasi-isomorphisms of dg-schemes $\mathrm{Rep}_V(Q) \overset{\sim}{\to} \mathrm{Rep}_V(Q')$. In fact by the general machinery of model categories we can lift the identity map $1_A : A \to A$ to a T-equivariant (weak equivalence) between the two cofibrant replacements $f : Q \overset{\sim}{\to} Q'$. When we forget the T-action, this is still a weak equivalence, therefore giving an isomorphism γf in the homotopy category $\mathtt{Ho}(\mathtt{DGA}_S^+)$ and therefore $\mathbf{L}(\gamma f)_V$ is an isomorphism in $\mathtt{Ho}(\mathtt{CDGA}_k^+)$. But because both domain and codomain are cofibrant, $\mathbf{L}(\gamma f)_V = \gamma f_V$, and therefore $f_V : Q_V \to (Q')_V$ is a T-equivariant isomorphism of commutative dg-algebras, which dually gives the desired T-equivariant map $\mathrm{Rep}_V(Q) \overset{\sim}{\to} \mathrm{Rep}_V(Q')$. □

As a final consequence, the representation homology of a T-equivariant algebra, and all the other invariants defined in § 2.2.4, enrich to T-equivariant invariants. For example we can define the T-equivariant virtual fundamental class of the derived representation scheme $X = \mathrm{DRep}_V(A)$ as the following object in the equivariant K-theory[4] of the classical representation scheme:

$$[X]^{\mathrm{vir}} = \chi_T(A, V) := \sum_{i=0}^{\infty} (-1)^i [H_i(A, V)] \in K_T(\mathrm{Rep}_V(A)), \tag{2.2.44}$$

and also all the other U_λ-irreducible components for a reductive group G by which we take the quotient (see § 2.2.4) as

$$\chi_T^\lambda(A, V) := \sum_{i=0}^{\infty} [H_i(\mathbf{L}(A)^G_{\lambda,V})] \in K_T(\mathrm{Rep}_V^G(A)) = K_T(\mathrm{Rep}_V^G(A). \tag{2.2.45}$$

In particular for $U_0 = k$ the trivial representation, we obtain an equivariant version of the virtual fundamental class of the derived partial character scheme $X^G = \mathrm{DRep}_V^G(A)$, which we denote by:

$$\chi_T^G(A, V) = \sum_{i=0}^{\infty} [H_i(A, V)^G] = [X^G]^{\mathrm{vir}} \in K_T(\mathrm{Rep}_V^G(A)). \tag{2.2.46}$$

2.3 The case of Nakajima quiver varieties

In this section we first recall the construction of Nakajima quiver varieties and secondly we construct some derived representation schemes related to them.

2.3.1 Nakajima quiver varieties

We already recalled in Example 2 that a finite quiver is a finite directed graph defined by its sets of vertices and edges $Q = (Q_0, Q_1)$ with two maps (source and target of an arrow) $s, t : Q_1 \to Q_0$.

[4]By equivariant K-theory of a scheme X with an algebraic group action $T \curvearrowright X$ we mean the Grothendieck ring of the abelian category of T-equivariant coherent sheaves on X.

We first frame the quiver, this means that we add a new vertex for each old one with a new arrow from the new to the old. Then we double the framed quiver, in order to obtain a cotangent (symplectic) space when we consider its representations. We denote this quiver by $\overline{Q^{fr}}$. To consider

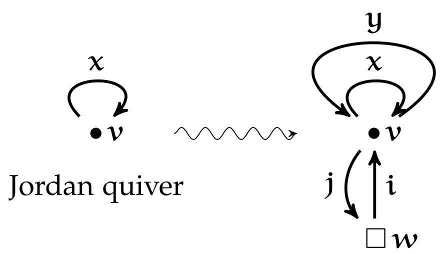

Figure 2.1: Example: framing and doubling the Jordan quiver. The framed vertices are usually denoted by a square symbol.

representations of a framed (doubled) quiver, we need to fix two dimension vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^{Q_0}$, and usually one assumes that (at least one of the components of) the framing vector is nonzero: $\boldsymbol{w} \neq 0$.

Notation. We denote the linear representations of the doubled, framed quiver by

$$\mathrm{M}(Q, \boldsymbol{v}, \boldsymbol{w}) := \mathrm{L}\left(\overline{Q^{fr}}, \boldsymbol{v}, \boldsymbol{w}\right) \cong \mathbb{T}^* \mathrm{L}\big(Q^{fr}, \boldsymbol{v}, \boldsymbol{w}\big)\,. \tag{2.3.1}$$

Explicitely it is the following cotangent linear space:

$$\mathrm{M}(Q, \boldsymbol{v}, \boldsymbol{w}) = \mathbb{T}^* \left(\bigoplus_{\gamma \in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{v_{s(\gamma)}}, \mathbb{C}^{v_{t(\gamma)}}) \oplus \bigoplus_{a \in Q_0} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{w_a}, \mathbb{C}^{v_a}) \right). \tag{2.3.2}$$

We denote elements of this space by quadruples $(X, Y, I, J) = (X_\gamma, Y_\gamma, I_a, J_a)_{\gamma, a}$, where $X_\gamma, I_a \in \mathrm{L}(Q^{fr}, \boldsymbol{v}, \boldsymbol{w})$ are elements of the representation space of the framed quiver, and (Y_γ, J_a) are cotangent vectors to them. The gauge group is the general linear group on the set of vertices of the original quiver Q:

$$G = G_{\boldsymbol{v}} := \prod_{a \in Q_0} \mathrm{GL}_{v_a}(\mathbb{C}) \subset \mathrm{GL}(\mathbb{C}^{\boldsymbol{v}} \oplus \mathbb{C}^{\boldsymbol{w}})\,, \tag{2.3.3}$$

which acts by conjugation in a Hamiltonian fashion on $M(Q, \boldsymbol{v}, \boldsymbol{w})$. The moment map for this action is

$$\begin{aligned} \mu : M(Q, \boldsymbol{v}, \boldsymbol{w}) &\to \mathfrak{g}_{\boldsymbol{v}}^* \cong \mathfrak{g}_{\boldsymbol{v}} \quad \text{(via trace)} \\ (X, Y, I, J) &\longmapsto [X, Y] + IJ\,, \end{aligned} \tag{2.3.4}$$

where in the above equation $[X, Y] + IJ$ is a shortened symbol for

$$[X, Y] + IJ = \left(\sum_{\gamma : t(\gamma) = a} X_\gamma Y_\gamma - \sum_{\gamma : s(\gamma) = a} Y_\gamma X_\gamma + I_a J_a \right)_{a \in Q_0} \in \bigoplus_{a \in Q_0} \mathfrak{gl}_{v_a}(\mathbb{C}) = \mathfrak{g}_{\boldsymbol{v}}\,. \tag{2.3.5}$$

Nakajima varieties are defined as symplectic reductions of $M(Q, \boldsymbol{v}, \boldsymbol{w})$ by this action. The affine Nakajima quiver variety is the geometric quotient:

$$\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}) := \mu^{-1}(0) /\!\!/ G = \operatorname{Spec}\big(\mathcal{O}(\mu^{-1}(0))^G\big)\,. \tag{2.3.6}$$

The GIT Nakajima variety is instead given by the choice of a character $\chi \in \operatorname{Hom}_{\mathtt{Grp}_{\mathbb{C}}}(G, \mathbb{C}^\times)$ as the proj of the graded ring of χ-quasiinvariant functions on $\mu^{-1}(0)$:

$$\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w}) = \mu^{-1}(0) /\!\!/_\chi G = \operatorname{Proj}\big(\mathcal{O}(\mu^{-1}(0))^{G,\chi}\big) \tag{2.3.7}$$

(elements of degree $n \geqslant 0$ of $\mathcal{O}(\mu^{-1}(0)^{G,\chi})$ are functions $f \in \mathcal{O}(\mu^{-1}(0))$ with the property $f(g \cdot p) = \chi^n(g) f(p)$ for all $g \in G$ and $p \in \mu^{-1}(0)$). The inclusion of G-invariant functions as degree zero elements of the graded ring of χ-quasiinvariant functions $\mathcal{O}(\mu^{-1}(0))^G \subset \mathcal{O}(\mu^{-1}(0))^{G,\chi}$ induces a projective morphism:

$$p : \mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w})\,, \tag{2.3.8}$$

which is often a symplectic resolution of singularities. Sometimes we denote these varieties simply by $\mathfrak{M}^\chi, \mathfrak{M}^0$ implicitly fixing the quiver Q, and the dimension vectors $\boldsymbol{v}, \boldsymbol{w}$.

2.3.2 Derived representation schemes models

In Proposition 2.2.1.1 we showed how the linear space of representations of a quiver is isomorphic to the representation scheme for its path algebra.

The same thing holds for the doubled, framed quiver so that

$$M(Q, v, w) = L\left(\overline{Q^{\mathrm{fr}}}, v, w\right) \cong \mathrm{Rep}_{v,w}\left(\mathbb{C}\overline{Q^{\mathrm{fr}}}\right). \tag{2.3.9}$$

To obtain the zero locus of the moment map, we consider the 2-sided ideal $\mathcal{J}_\mu \subset \mathbb{C}\overline{Q^{\mathrm{fr}}}$ generated by the $|Q_0|$-elements of the path algebra described in (2.3.4), and consider the quotient algebra

$$A := \mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{J}_\mu \in \mathtt{Alg}_S, \tag{2.3.10}$$

relative to the subalgebra $S \subset A$ of idempotents, with fixed representation $\rho = \rho_{v,w} : S \to \mathrm{End}_{\mathbb{C}}(\mathbb{C}^v \oplus \mathbb{C}^w)$ (as in (2.2.8)). The following result is an immediate consequence of the fact that taking the quotient by some ideal amounts simply to impose these new relations in the representation scheme (see Examples 1.(6)):

Proposition 2.3.2.1. *The zero locus of the moment map μ is the (relative) representation scheme for the path algebra of the framed, doubled quiver, modulo the Hamiltonian relation:*

$$\mu^{-1}(0) \cong \mathrm{Rep}_{v,w}(\mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{J}_\mu). \tag{2.3.11}$$

Notation. We denote the corresponding derived representation scheme and representation homology by:

$$\begin{aligned} \mathrm{DRep}_{v,w}(A) &= \mathrm{Spec}\big(\mathbf{L}(A)_{v,w}\big) \in \mathtt{Ho}(\mathtt{DGAff}_{\mathbb{C}}), \\ \mathrm{H}_\bullet(A, v, w) &= \mathrm{H}_\bullet\big(\mathbf{L}(A)_{v,w}\big) \in \mathtt{CDGA}^+_{\mathbb{C}}. \end{aligned} \tag{2.3.12}$$

The representation homology $\mathrm{H}_\bullet(A, v, w)$ is a graded commutative algebra, so when we view it in $\mathtt{CDGA}^+_{\mathbb{C}}$ we mean that the differential is zero.

Remark 2.2.2.2, together with Proposition 2.3.2.1 tells us that the π_0 of this derived scheme $X = \mathrm{DRep}_{v,w}(A)$ is the zero locus of the moment map:

$$\pi_0(X) = \mathrm{Spec}\big(\mathrm{H}_0(A, v, w)\big) \cong \mu^{-1}(0). \tag{2.3.13}$$

In particular when we consider the invariant subfunctor only by the gauge group on the original vertices G (2.3.3):

Corollary 2.3.2.1. *The* π_0 *of the partial character scheme* $X^G = \mathrm{DRep}^G_{v,w}(A)$ *is the affine Nakajima variety* $\mathfrak{M}^0$*:*

$$\pi_0(X^G) = \pi_0(\mathrm{DRep}^G_{v,w}(A)) \cong \mathfrak{M}^0\,. \tag{2.3.14}$$

Proof. It follows directly from the previous observation (2.3.13) and the Theorem 2.2.3.1. More precisely:

$$\pi_0(X^G) \cong \mathrm{Spec}\big(H_0(A, v, w)^G\big) \cong \mathrm{Spec}\big(\mathcal{O}(\mu^{-1}(0))^G\big) = \mu^{-1}(0) /\!\!/ \mathsf{G} = \mathfrak{M}^0\,.$$

$\square$

2.3.3 K-theoretic classes in the affine Nakajima variety

In § 2.3.4 we describe an explicit cofibrant resolution for our algebra $A = \mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{I}_\mu$, $A_{\mathrm{cof}} \xrightarrow{\sim}\!\!\!\!\rightarrow A$ and therefore a model for the derived representation scheme $\mathrm{DRep}_{v,w}(A) = \mathrm{Rep}_{v,w}(A_{\mathrm{cof}})$, but we can already use Corollary 2.3.2.1 to define some meaningful invariants in the K-theory of $\mathfrak{M}^0 = \mathrm{Rep}^G_{v,w}(A)$. Throughout this section we denote by $X = \mathrm{DRep}_{v,w}(A)$ the derived representation scheme and by $X^G = \mathrm{DRep}^G_{v,w}(A)$ the corresponding partial character scheme, whose $\pi_0(X^G) = \mathfrak{M}^0$ is the affine Nakajima variety.

There is a torus, the (standard) maximal torus of the gauge group on the framing vertices $T_w \subset G_w$ acting on the linear space of representations $\mathrm{Rep}_{v,w}(A)$, and therefore as explained in § 2.2.4 it induces an action $T_w \curvearrowright \mathrm{DRep}_{v,w}(A)$ and on its quotient by the gauge group G_v: $T_w \curvearrowright \mathrm{DRep}^{G_v}_{v,w}(A)$. There is an additional (2-dimensional) torus

$$T_\hbar = (\mathbb{C}^\times)^2 \curvearrowright A$$

acting rationally on the path algebra of the doubled framed quiver. This action can be described by assigning, respectively, the following $\mathbb{Z}^2$-weights to the arrows $(x_\gamma, y_\gamma, i_a, j_a)$ (see § 2.3.1 to recall the name of the arrows): $(1,0)$, $(0,1)$, $(1,1)$, $(0,0)$, or explicitly as

$$x_\gamma \mapsto \hbar_1 x_\gamma\,, \quad y_\gamma \mapsto \hbar_2 y_\gamma\,, \quad i_a \mapsto \hbar_1\hbar_2 i_a\,, \quad j_a \mapsto j_a\,.$$

As explained in § 2.2.5, also this torus induces actions $T_\hbar \curvearrowright \mathrm{DRep}_{\boldsymbol{v},\boldsymbol{w}}(A)$, $\mathrm{DRep}^{G_{\boldsymbol{v}}}_{\boldsymbol{v},\boldsymbol{w}}(A)$. In other words, the whole torus $T := T_{\boldsymbol{w}} \times T_\hbar$ acts on the derived representation scheme $X = \mathrm{DRep}_{\boldsymbol{v},\boldsymbol{w}}(A)$ and its partial character scheme $X^{G_{\boldsymbol{v}}} = \mathrm{DRep}^{G_{\boldsymbol{v}}}_{\boldsymbol{v},\boldsymbol{w}}(A)$.

Using the definitions we gave in § 2.2.4 and § 2.2.5 we obtain the following invariants in the (equivariant) K theory of the affine Nakajima variety $\mathfrak{M}^0 = \mathrm{Rep}^{G_{\boldsymbol{v}}}_{\boldsymbol{v},\boldsymbol{w}}(A)$, for example the virtual fundamental class

$$\left[X^{G_{\boldsymbol{v}}}\right]^{\mathrm{vir}} = \sum_{i=0}^{\infty} (-1)^i \left[H_i(A, \boldsymbol{v}, \boldsymbol{w})^{G_{\boldsymbol{v}}}\right] \in K_T\left(\mathfrak{M}^0\right). \tag{2.3.15}$$

More generally for each irreducible representation U_λ of $G_{\boldsymbol{v}}$, the Euler characterstic of the corresponding isotypical component as

$$\chi_T^\lambda(A, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i=0}^{\infty} (-1)^i \left[H_i(\mathbf{L}(A))^{G_{\boldsymbol{v}}}_{\lambda,\boldsymbol{v},\boldsymbol{w}}\right] \in K_T\left(\mathfrak{M}^0\right). \tag{2.3.16}$$

2.3.4 Explicit cofibrant resolution

In this section we describe an explicit cofibrant resolution for the S-algebra A constructed in the previous section. Let us recall that

$$A = \mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{I}_\mu \in \mathtt{Alg}_S \hookrightarrow \mathtt{DGA}_S^+, \tag{2.3.17}$$

where S is the subalgebra generated by the idempotents of the path algebra of the framed quiver. The main obstruction for this object to be cofibrant is the Hamiltonian relation described by the ideal $\mathcal{I}_\mu$. The simplest idea is then to add one more variable for each of the generating relations in $\mathcal{I}_\mu$ which kills the relation itself. This technique in general might not work due to higher homologies, but we prove that this case is one of the well-behaved cases. We construct the following quiver Q^ϑ, which is obtained by adding to the framed, doubled quiver $\overline{Q^{\mathrm{fr}}}$, one loop called ϑ_a on each original vertex $a \in Q_0$.

In the path algebra $\mathbb{C}Q^\vartheta$ we assign homological degree 0 to the original arrows, and homological degree 1 to the new arrows ϑ_a. The differential is induced by the moment map (equations as in (2.3.5))

$$d\vartheta_a = \mu_a(x, y, i, j).$$

We denote the resulting differential graded algebra by

$$A_{\mathrm{cof}} := (\mathbb{C}Q^{\vartheta}, d) \in \mathtt{DGA}^{+}_{\mathbb{C}}\,. \tag{2.3.18}$$

It sits in the following diagram

$$\begin{tikzcd} & S \arrow[dl, "\iota"'] \arrow[dr, "\varphi"] & \\ A_{\mathrm{cof}} \arrow[rr, "\pi"] & & A \end{tikzcd} \tag{2.3.19}$$

where π is the composition of the following two obvious projections:

$$\pi : A_{\mathrm{cof}} \to (A_{\mathrm{cof}})_0 = \mathbb{C}\overline{Q^{\mathrm{fr}}} \to \mathbb{C}\overline{Q^{\mathrm{fr}}}/\mathcal{J}_{\mu} = A\,.$$

Theorem 2.3.4.1. *A_{cof} is a cofibrant replacement for A in $\mathtt{DGA}^{+}_{S}$.*

This amounts to prove that, in the diagram (2.3.19), the map π is an acyclic fibration, and ι is a cofibration.

Lemma 2.3.4.1. *The map $\pi : A_{\mathrm{cof}} \to A$ is an acyclic fibration in $\mathtt{DGA}^{+}_{\mathbb{C}}$.*

Proof. We need to prove that:

(i) π is degreewise surjective in degrees $\geqslant 1$ (this is obvious, because A is concentrated in degree 0).
(ii) $H_i(\pi) : H_i(A_{\mathrm{cof}}) \to H_i(A)$ is an isomorphism for each $i \geqslant 0$, which becomes proving that

$$\begin{cases} H_0(\pi) : H_0(A_{\mathrm{cof}}) \xrightarrow{\sim} A\,, \\ H_i(A_{\mathrm{cof}}) = 0, \quad i \geqslant 1\,. \end{cases}$$

$H_0(\pi)$ is an isomorphism, this is evident from the construction of A_{cof}. We are left to prove that A_{cof} has no higher homologies.

Using the orthogonal idempotents $\{e_a\}_{a \in Q_0^{\vartheta}}$ we decompose A_{cof} as a direct sum of the dg-submodules of paths starting and ending at fixed vertices:

$$A_{\mathrm{cof}} = \bigoplus_{a,b \in Q_0^{\vartheta}} P_{a,b}\,, \qquad P_{a,b} = e_b \cdot A_{\mathrm{cof}} \cdot e_a\,. \tag{2.3.20}$$

Let us introduce also $\widetilde{A}_{\mathrm{cof}} = \mathbb{C}\widetilde{Q}^{\vartheta}$, where $\widetilde{Q}^{\vartheta}$ is the quiver obtained from $\overline{Q}$ by only adding one loop of degree zero c_a (instead of the pair of arrows i_a, j_a) and one loop ϑ_a of degree 1 on each vertex $a \in Q_0$. The differential on this new graded algebra is given by the analogous formula obtained by substituting in the previous formula the product $i_a j_a$ with c_a: "$d\vartheta = [x, y] + c$" (componentwise). We can also decompose this dg-algebra in analogous dg-submodules

$$\widetilde{A}_{\mathrm{cof}} = \bigoplus_{a,b \in Q_0} \widetilde{P}_{a,b}\,, \tag{2.3.21}$$

only that this time the direct sum runs over pairs of vertices in the quiver $\widetilde{Q}^{\vartheta}$, which are the same as the vertices of the original quiver Q.

<u>Claim:</u> If $H_i(\widetilde{A}_{\mathrm{cof}}) = 0$ for all $i > 0$, then also A_{cof} has no higher homologies.

Proof of the claim: The decompositions (2.3.20) and (2.3.21) are decompositions in dg-submodules, therefore also the homologies decompose accordingly. For a vertex $a \in Q_0$ we denote by $\overline{a} \in Q_0^{\vartheta}$ the corresponding framing vertex. For $a, b \in Q_0$, we have four cases:

$$\begin{cases} P_{a,b} \cong \widetilde{P}_{a,b} \\ P_{a,\overline{b}} = j_b \cdot P_{a,b} \cong P_{a,b} (\cong \widetilde{P}_{a,b}) \\ P_{\overline{a},b} = P_{a,b} \cdot i_a \cong P_{a,b} (\cong \widetilde{P}_{a,b}) \\ P_{\overline{a},\overline{b}} = j_b \cdot P_{a,b} \cdot i_a \cong P_{a,b} (\cong \widetilde{P}_{a,b}) \end{cases} \tag{2.3.22}$$

The first isomorphism is realised by sending the cycles of the form $i_s j_s$ to the loops c_s. This is possible because $P_{a,b}$ is made of paths starting and ending at vertices of the original quiver, therefore the only form in which i_s or j_s can appear is through their product $i_s j_s$. Analogous considerations show the other three cases. The claim then follows, because $\widetilde{A}_{\mathrm{cof}}$ has no higher homologies if and only if all the dg-submodules $\widetilde{P}_{a,b}$ have no higher homologies, and by the previous isomorphisms, which are isomorphisms of dg-vector spaces, neither A_{cof} does. □

To prove the lemma we are left to show that all the $\widetilde{P}_{a,b}$ have no higher homology. We consider the following filtration:

$$F^p(\widetilde{P}_{a,b}) := \mathrm{Span}_{\mathbb{C}}\{\text{ paths in } \widetilde{P}_{a,b} \text{ with } \#x + \#y \geqslant 2p\}.$$

Remember that the differential has the form "$d\vartheta = [x, y] + c$", so that the associated graded has differential of the form "$d_{gr}\vartheta = c$", which involves only loops on the vertices $a \in Q_0$.

Claim 2: The associated graded has no higher homologies.

Proof of claim 2: We denote by $G_{a,b}$ the associated graded of $\widetilde{P}_{a,b}$ under the afore-mentioned filtration. We consider the linear map $\pi : G_{a,b} \to G_{a,b}$ that sends a path γ to the path $\pi(\gamma)$ which is obtained by substituting any c_s or ϑ_s with e_s, the idempotent of the corresponding vertex. If we denote by $\Gamma_{a,b}$ the set of paths in $G_{a,b}$ containing only x's and y's, then we can decompose:

$$G_{a,b} = \bigoplus_{w \in \Gamma_{a,b}} G_w \,, \qquad G_w = \pi^{-1}(\mathbb{C}w)\,.$$

An easy inspection shows that the differential preserves this decomposition, and that each G_w is isomorphic, as a dg-vector space, to a tensor product of elementary dg-algebras of the form

$$G_w \cong L^{\otimes(\text{length}(w)+1)}\,, \qquad L = \big(\mathbb{C}\langle\vartheta, c\rangle, d\vartheta = c\big)\,,$$

where $\text{length}(w)$ is the number of collective x's and y's present in w. Finally we observe that L has no higher homologies[5], and this concludes the proof of claim 2. $\square$

The lemma then follows. $\square$

Lemma 2.3.4.2. $\iota : S \to A_{\text{cof}}$ *is a cofibration in* $\mathtt{DGA}^+_{\mathbb{C}}$*, or equivalently* A_{cof} *is a cofibrant object in* $\mathtt{DGA}^+_S$*.*

Proof. We need to prove that ι has the left lifting property with respect to acyclic fibrations.

$$\begin{tikzcd} S \arrow[r] \arrow[d, "\iota"'] & B \arrow[d, two heads, "\sim"] \\ A_{\text{cof}} \arrow[r] \arrow[ur, dashed, "\exists"] & C \end{tikzcd} \tag{2.3.23}$$

[5] An elementary argument is to observe that the derivation defined by the formula $h(c) = \vartheta$ and $h(c) = 0$, is a homotopy between the 0 map and the map $\text{length}(-) \cdot \text{Id}$, which is an isomorphism in (homological) degrees $\geqslant 1$. This implies that $H_i(L) = 0$ for $i \geqslant 1$.

Let us observe that because $A_{\mathrm{cof}} = \mathbb{C}Q^{\vartheta}$ is the (dg) path algebra of a quiver with idempotents S, we can view $A_{\mathrm{cof}} = T_S M := S \oplus M \oplus (M \otimes_S M) \dots$ as the tensor algebra of the S-(dg)bimodule

$$M := \mathrm{Span}_{\mathbb{C}}\{\text{arrows in } Q^{\vartheta}\}.$$

But then find a lifting in the diagram (2.3.23) amounts to simply give a (linear) lifting of the (dg) vector space M, which is possible for the surjectivity of the map $B \overset{\sim}{\twoheadrightarrow} C$ (acyclic fibrations are surjective in every homological degree).

□

2.3.5 Koszul complex and complete intersections

Theorem 2.3.4.1 tells us that a model for the derived representation scheme for the algebra A is the representation scheme of the cofibrant replacement A_{cof}. In this section we recognise it as the Koszul complex for the moment map, and in order to do so, we first recall a few classical notions and results about the latter.

The Koszul complex can be thought of as one of the main examples of derived intersections of subschemes of a scheme. Classically, affine varieties are the simplest examples of intersections, being zero loci of some simultaneous polynomial equations $f_1, \dots, f_m \in \mathcal{O}(\mathbb{A}^n_{\mathbb{C}}) = \mathbb{C}[x_1, \dots, x_n]$:

$$(X, \mathcal{O}) = \mathrm{Spec}\big(R/(f_1) \otimes_R \cdots \otimes_R R/(f_m)\big), \qquad R = \mathbb{C}[x_1, \dots, x_n]. \tag{2.3.24}$$

Then the associated *derived intersection* can be defined as the derived scheme

$$(X, \mathcal{O}_\bullet) = \mathrm{Spec}\big(R/(f_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R R/(f_m)\big), \tag{2.3.25}$$

where $\otimes^{\mathbf{L}}_R$ is the derived tensor product of R-modules. The algebra of functions on this derived scheme is the *Koszul complex*:

$$K = R/(f_1) \otimes^{\mathbf{L}}_R \cdots \otimes^{\mathbf{L}}_R R/(f_m) \in \mathtt{CDGA}^+_{\mathbb{C}}.$$

A more concrete way to describe it is the following: we can view the collection of functions $f = (f_1, \dots, f_m)$ as a map of affine schemes

$$f : \mathbb{A}^n_{\mathbb{C}} \to V := \mathbb{A}^m_{\mathbb{C}},$$

and consider its dual map

$$\mathcal{O}(f) : \mathcal{O}(V) = \mathrm{Sym}(V^*) \to \mathcal{O}(\mathbb{A}^n_{\mathbb{C}}) = R\,.$$

Then the Koszul complex is the commutative dg-algebra $K = (R \otimes_{\mathbb{C}} \Lambda^\bullet(V^*), d)$, where R is in homological degree 0, the vector space V^* is in homological degree 1, and the differential

$$d := \mathcal{O}(f)_{|V^*} : V^* \hookrightarrow \mathrm{Sym}(V^*) \to R\,.$$

An useful classical result on the Koszul complex is

Theorem 2.3.5.1 ([46]). *The following are equivalent:*

1. $\dim_{\mathbb{C}}(\mathrm{Spec}(H_0(K))) = n - m$.
2. *The sequence* $f_1, \ldots, f_m \in R$ *is a regular sequence.*
3. $H_1(K) = 0$.
4. $H_i(K) = 0$ *for all* $i \geqslant 1$.

Let us turn back to the case of our interest, in which we want to recognise

$$(A_{\mathrm{cof}})_{\boldsymbol{v},\boldsymbol{w}} = (\mathbb{C}Q^{\vartheta})_{\boldsymbol{v},\boldsymbol{w}}$$

as the Koszul complex on the moment map. We recall that the quiver Q^{ϑ} is constructed from the quiver $\overline{Q^{\mathrm{fr}}}$ by adding a new loop in homological degree 1 on each of the original vertices of the quiver Q. Therefore, a representation of the path algebra $\mathbb{C}Q^{\vartheta}$ is just a representation of the subalgebra $\mathbb{C}\overline{Q^{\mathrm{fr}}}$ (an element of the vector space $M(Q, \boldsymbol{v}, \boldsymbol{w})$), together with a family of endomorphisms

$$\Theta = (\Theta_a)_{a \in Q_0} \in \bigoplus_{a \in Q_0} \mathfrak{gl}_{v_a}(\mathbb{C}) = \mathfrak{g}_{\boldsymbol{v}}$$

in homological degree 1. Putting everything together we obtain

$$(\mathbb{C}Q^{\vartheta})_{\boldsymbol{v},\boldsymbol{w}} = \mathcal{O}(M(Q, \boldsymbol{v}, \boldsymbol{w})) \otimes_{\mathbb{C}} \Lambda^\bullet \mathfrak{g}_{\boldsymbol{v}} \in \mathtt{CDGA}^+_{\mathbb{C}}\,,$$

which is nothing else but the Koszul complex for the zero locus defined by the moment map
$$\mu : M(Q, \boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{g}_{\boldsymbol{v}}^* .$$
Its spectrum is a model for our derived representation scheme, as the derived intersection of the moment map equations:

Theorem 2.3.5.2. *The cofibrant resolution* $A_{\mathrm{cof}} \xrightarrow{\sim} A$ *in* $\mathtt{DGA}_S^+$ *gives a model for the derived representation scheme as the (spectrum of the) Koszul complex on the moment map:*

$$\mathrm{DRep}_{\boldsymbol{v},\boldsymbol{w}}(A) \cong \mathrm{Rep}_{\boldsymbol{v},\boldsymbol{w}}(A_{\mathrm{cof}}) = \mathrm{Spec}\big(\mathcal{O}(M(Q, \boldsymbol{v}, \boldsymbol{w})) \otimes \Lambda^\bullet \mathfrak{g}\big) . \tag{2.3.26}$$

In particular we can observe that this is a derived scheme of finite type (Definition 2.2.4.3) and that the Koszul complex is bounded, Therefore all the invariants defined in § 2.3.3 ((2.3.15), (2.3.16)) make sense, because the sums are bounded (by the dimension of the Lie algebra $\dim_{\mathbb{C}} \mathfrak{g}_{\boldsymbol{v}} = \boldsymbol{v}^2 = \boldsymbol{v} \cdot \boldsymbol{v}$).

Remark 2.3.5.1. In § 2.3.4 we gave a self-contained proof of why the resolution provided by the path algebra of the quiver Q^ϑ obtained by adding one loop on each vertex in which the corresponding component of the moment map is considered (i.e. the original vertices) works. In § 2.3.5 we explained why the resulting representation scheme is the Koszul complex on the moment map. We remark that the same results can be explained in a slightly different flavour through the theory of noncommutative complete intersections (NCCI) and partial preprojective algebras ([21], [24]).

2.4 Comparison theorems and integral formulas

2.4.1 Flat moment map and vanishing representation homology

In this section we recall some classical results on the flatness for the moment map of Nakajima quiver varieties which are useful for our purposes. We show how flatness is equivalent to the condition of vanishing of higher representation homologies for the corresponding algebra.

Remember that for each quiver Q and for each fixed dimensions $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{N}^{Q_0}$ we have the corresponding Nakajima varieties $\mathfrak{M}^0$ (affine) and $\mathfrak{M}^\chi$

(quasiprojective), where $\chi \in \mathrm{Hom}_{\mathtt{Grp}_{\mathbb{C}}}(G_{\boldsymbol{v}}, \mathbb{C}^\times)$ is a given (nontrivial) character. We also recall that the group of all characters of the gauge group $G = G_{\boldsymbol{v}} = \prod_{a \in Q_0} GL_{v_a}(\mathbb{C})$ is isomorphic to the lattice

$$\mathbb{Z}^{Q_0} \cong \mathrm{Hom}_{\mathtt{Grp}_{\mathbb{C}}}(G, \mathbb{C}^\times)\,,$$

via the assignment

$$\theta \mapsto \chi_\theta(g) = \prod_{a \in Q_0} \det(g_a)^{\theta_a}\,.$$

In this section we use the parameter θ for the characters and denote $\mathfrak{M}^{\chi_\theta}$ simply by $\mathfrak{M}^\theta$.

We recall that the *Cartan matrix* of the quiver Q is the matrix $C_Q = 2 \cdot \mathrm{Id} - A_{\overline{Q}}$, where $A_{\overline{Q}}$ is the adjacency matrix of the doubled quiver $\overline{Q}$. For a fixed dimension vector $\boldsymbol{v} \in \mathbb{N}^{Q_0}$, a vector $\theta \in \mathbb{Z}^{Q_0}$ is called $\boldsymbol{v}$-*regular*, if for each $\alpha \in \mathbb{Z}^{Q_0} \backslash \{0\}$ such that $C_Q \alpha \cdot \alpha \leqslant 2$ and $0 \leqslant \alpha \leqslant \boldsymbol{v}$ (component-wise) then

$$\sum_{i \in Q_0} \theta_i \alpha_i \neq 0\,.$$

The subset of $\mathbb{R}^{Q_0}$ of $\boldsymbol{v}$-regular vectors is the complement of some hyperplanes. Its connected components are called *chambers*, and the variety $\mathfrak{M}^\theta$ depends only on the chamber of θ.

Theorem 2.4.1.1 (Theorem 5.2.2. in [30]). *Let $\boldsymbol{v} \in \mathbb{N}^{Q_0}$ be a dimension vector and $\theta \in \mathbb{Z}^{Q_0}$ be $\boldsymbol{v}$-regular, then any θ-semistable point in $\mu^{-1}(0)$ is θ-stable and $\mathfrak{M}^\theta$ is a smooth, connected, complex symplectic variety of dimension*

$$\dim \mathfrak{M}^\theta = 2\boldsymbol{v} \cdot \boldsymbol{w} - C_Q \boldsymbol{v} \cdot \boldsymbol{v}\,,$$

(with the convention that $\mathfrak{M}^\theta = \emptyset$ when this dimension is negative).

Remark 2.4.1.1. Observe that the dimension counting is what we would expect. In fact

$$\dim(M(Q, \boldsymbol{v}, \boldsymbol{w})) = 2\boldsymbol{v} \cdot \boldsymbol{w} + A_{\overline{Q}} \boldsymbol{v} \cdot \boldsymbol{v} = 2\boldsymbol{v} \cdot \boldsymbol{w} - C_Q \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{v}\,.$$

When we take the zero locus by μ we expect to decrease the dimension by the number of equations of μ, which is $\boldsymbol{v} \cdot \boldsymbol{v}$ and then again by $\boldsymbol{v} \cdot \boldsymbol{v}$ when taking the $G_{\boldsymbol{v}}$-quotient.

Let us consider for some $\boldsymbol{v}$-regular θ the natural affinisation morphism

$$\varphi : \mathfrak{M}^\theta \to \operatorname{Spec}\big(\mathcal{O}(\mathfrak{M}^\theta)\big)\,. \tag{2.4.1}$$

This morphism is a Poisson morphism[6] (obviously, because φ^* is the identity) and it is a resolution of singularities (i.e. projective and birational) ([12]). The variety $\mathfrak{M}^\theta$ depends, a priori on the chamber of θ, but actually its affinisation $\operatorname{Spec}(\mathcal{O}(\mathfrak{M}^\theta))$ is independent of the choice of $\boldsymbol{v}$-regular θ. We can call this variety simply $\mathfrak{M}$ and we obtain a diagram of the following form

$$\begin{tikzcd} \mathfrak{M}^\theta \arrow[d, "\varphi"] \arrow[dr, "p"] & \\ \mathfrak{M} \arrow[r, "\psi"] & \mathfrak{M}^0 \end{tikzcd} \tag{2.4.2}$$

which is the so-called Stein factorisation ([65]) of the proper morphism p. The pre-image of the point $0 \in \mathfrak{M}^0$ through ψ is always $0 \in \mathfrak{M}$. In particular the fiber $p^{-1}(0)$ is equal to the central fiber $\varphi^{-1}(0)$ of the affinisation morphism

$$p^{-1}(0) = (\psi \circ \varphi)^{-1}(0) = \varphi^{-1}(\psi^{-1}(0)) = \varphi^{-1}(0)\,,$$

and therefore is a homotopy retract of the variety $\mathfrak{M}^\theta$.

Theorem 2.4.1.2 ([12])**.** *If the moment map $\mu : M(Q, \boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{g}_{\boldsymbol{v}}^*$ is flat, then ψ is an isomorphism, and in particular $\mathcal{O}(\mathfrak{M}^\theta) \cong \mathcal{O}(\mathfrak{M}^0)$.*

The combinatorial criterium for the flatness of the moment map proved in [19] is given in the setting of a non-framed quiver Γ. For any dimension vector $\boldsymbol{\alpha} \in \mathbb{N}^{\Gamma_0}$ we consider the linear space of representations of the doubled quiver $L(\overline{\Gamma}, \boldsymbol{\alpha})$. The gauge group acting a priori in a non-trivial way is now $G_{\boldsymbol{\alpha}}/\mathbb{C}^\times$ because, without the framing, the diagonal torus $\mathbb{C}^\times \subset G_{\boldsymbol{\alpha}}$ acts trivially on the linear space of representations. The Lie algebra of this group can be identified with the subalgebra $\mathfrak{g}_{\boldsymbol{\alpha}}^\natural \subset \mathfrak{g}_{\boldsymbol{\alpha}} = \oplus_i \mathfrak{gl}_{\alpha_i}(\mathbb{C})$ of matrices

[6]The Poisson structure on Nakajima varieties comes from the general formalism of Hamiltonian reduction, and coincides with the one induced by the symplectic form on the regular locus.

with sum of their traces equal to zero (the notation $\mathfrak{g}_{\alpha}^{\natural}$ is borrowed from [24]). The moment map is now

$$\begin{aligned} \mu_{\alpha} : L(\overline{\Gamma}, \boldsymbol{\alpha}) &\to \mathfrak{g}_{\alpha}^{\natural} \\ x &\longmapsto [x, x^*] . \end{aligned}$$

Let us denote by p the following function

$$p : \mathbb{N}^{\Gamma_0} \to \mathbb{Z}, \qquad p(\boldsymbol{\alpha}) := 1 + \sum_{\gamma \in \Gamma_1} \alpha_{s(\gamma)} \alpha_{t(\gamma)} - \boldsymbol{\alpha} \cdot \boldsymbol{\alpha} .$$

Theorem 2.4.1.3 (Theorem 1.1 in [19])**.** *The following are equivalent:*

1. μ_{α} *is a flat morphism.*
2. $\mu_{\alpha}^{-1}(0)$ *has dimension* $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha} - 1 + 2p(\boldsymbol{\alpha})$ *(*$= \dim L(\overline{\Gamma}, \boldsymbol{\alpha}) - \dim \mathfrak{g}_{\alpha}^{\natural}$*).*
3. $p(\boldsymbol{\alpha}) \geqslant \sum_{t=1}^{r} p(\boldsymbol{\beta}^{(t)})$ *for each decomposition* $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} + \cdots + \boldsymbol{\beta}^{(r)}$ *with each* $\boldsymbol{\beta}^{(t)}$ *positive root.*
4. $p(\boldsymbol{\alpha}) \geqslant \sum_{t=1}^{r} p(\boldsymbol{\beta}^{(t)})$ *for each decomposition* $\boldsymbol{\alpha} = \boldsymbol{\beta}^{(1)} + \cdots + \boldsymbol{\beta}^{(r)}$ *with each* $\boldsymbol{\beta}^{(t)} \in \mathbb{N}^{\Gamma_0} \setminus \{0\}$.

In a remark in § 1 in [19], Crawley-Boevey explains how to adapt this setting to the situation of a framed quiver. From a quiver Q and a framing vector $\boldsymbol{w}$ we can construct a new quiver $\Gamma := Q^{\infty}$, which is obtained by adding only one new vertex, denoted by ∞, together with a number of w_a arrows towards each vertex $a \in Q_0$. If we fix now a dimension vector $\boldsymbol{v} \in \mathbb{N}^{Q_0}$ and define the new vector $\boldsymbol{\alpha} := (\boldsymbol{v}, 1) \in \mathbb{N}^{\Gamma_0}$, then

$$L(\overline{\Gamma}, \boldsymbol{\alpha}) \cong L\big(\overline{Q^{\mathrm{fr}}}, \boldsymbol{v}, \boldsymbol{w}\big) = M(Q, \boldsymbol{v}, \boldsymbol{w}) , \tag{2.4.3}$$

by splitting the $v_a \times w_a$ matrices in $M(Q, \boldsymbol{v}, \boldsymbol{w})$ in columns and the $w_a \times v_a$ matrices in rows. The two gauge groups are also isomorphic: $G_{\alpha}/\mathbb{C}^{\times} \cong G_{\boldsymbol{v}}$, and under this isomorphism their actions on $L(\overline{\Gamma}, \boldsymbol{\alpha}) \cong M(Q, \boldsymbol{v}, \boldsymbol{w})$ are the same. Therefore also the moment maps are identified:

$$\begin{tikzcd} L(\overline{\Gamma}, \boldsymbol{\alpha}) \arrow[r, "\mu_{\alpha}"] \arrow[d, "\sim"] & \mathfrak{g}_{\alpha}^{\natural} \arrow[d, "\sim"] \\ M(Q, \boldsymbol{v}, \boldsymbol{w}) \arrow[r, "\mu"] & \mathfrak{g}_{\boldsymbol{v}} \end{tikzcd} \tag{2.4.4}$$

and we have the following criterium:

Corollary 2.4.1.1. *Consider the quiver* $\overline{Q^{\mathrm{fr}}}$ *with dimension vectors* $v, w \in \mathbb{N}^{Q_0}$, *and the quiver* $\Gamma = Q^{\infty}$ *with* $\alpha = (v, 1)$. *Then the following are equivalent:*

1. μ *is flat.*

2. μ_α *is flat.*

For the condition (2) now we can use the combinatorical test given by Theorem 2.4.1.3, and using this result, we can prove that the derived representation scheme has vanishing higher homologies if and only if the moment map μ is flat:

Theorem 2.4.1.4. *The representation homology* $H_\bullet(A, v, w)$ *for the algebra* A *as in* (2.3.17) *vanishes if and only if the moment map* μ *is flat.*

Proof. Because of the diagram (2.4.4) the moment map μ is flat if and only if μ_α is flat and by Theorem 2.4.1.3, condition (2), this happens if and only if

$$\begin{aligned} \dim \mu^{-1}(0) = \dim \mu_\alpha^{-1}(0) = \dim L(\overline{\Gamma}, \alpha) - \dim \mathfrak{g}_\alpha^\natural = \\ = \dim M(Q, v, w) - \dim \mathfrak{g}_v \,. \end{aligned} \tag{2.4.5}$$

The representation homology is the homology of the Koszul complex

$$H_\bullet(A, v, w) = H_\bullet\left(\mathcal{O}\big(M(Q, v, w)\big) \otimes \Lambda^\bullet \mathfrak{g}_v\right) ,$$

and therefore, by Theorem 2.3.5.1, it vanishes in degrees $i \geqslant 1$ if and only if the dimension condition (2.4.5) is satisfied. □

In the following examples we use Theorem 2.4.1.3 for some quivers and we find the combinatorical condition on the dimension vectors for the moment map to be flat. It is convenient to observe that for the quiver $\Gamma = Q^\infty$ the map p is, for vectors of the form $(\beta, 1)$ or $(\beta, 0)$ (that is the only type of vectors that we need to decompose the dimension vector $\alpha = (v, 1)$):

$$\begin{aligned} p(\beta, 1) &= \sum_{\gamma \in Q_1} \beta_{s(\gamma)} \beta_{t(\gamma)} + \beta \cdot w - \beta \cdot \beta \,, \\ p(\beta, 0) &= p(\beta, 1) + 1 \,. \end{aligned}$$

Examples 4. 1. The first example is that of a single-vertex quiver $Q = A_1$ with no arrows, whose $\Gamma = Q^\infty$ becomes a quiver with 2 vertices and w arrows going from one to the other. We need to test for which v it holds that for each decomposition

$$(v, 1) = (\beta_0, 1) + (\beta_1, 0) + \cdots + (\beta_r, 0), \qquad \beta_t \geqslant 0.$$

the following inequality holds:

$$v(w - v) \geqslant \beta_0(w - \beta_0) + r - \beta_1^2 - \cdots - \beta_r^2.$$

We can observe that actually all $\beta_1, \ldots, \beta_r \geqslant 1$ and therefore the function $r - \beta_1^2 - \cdots - \beta_r^2$ reaches its maximum for $\beta_1 = \cdots = \beta_r = 1$ for which it is 0. So we just need to test that

$$v(w - v) \geqslant \beta_0(w - \beta_0), \qquad \forall \beta_0 = 0, \ldots, v - 1$$

The inequality can also be rewritten as

$$\cancel{(v - \beta_0)}w \geqslant \cancel{(v - \beta_0)}(v + \beta_0), \quad \forall \beta_0 = 0, \ldots, v - 1 \quad \Leftrightarrow \quad w \geqslant 2v - 1.$$

2. The second example is a quiver with one vertex and m loops ($m \geqslant 1$). In particular the Jordan quiver for $m = 1$ described in Figure 2.1. We show that for each choice of $v \geqslant 0$ and $w \geqslant 1$ the moment map is flat. The quiver $\Gamma = Q^\infty$ still has 2 vertices, the first one with m loops and w arrows connecting the 2^{nd} to the 1^{st}, so that:

$$p(\alpha_1, \alpha_2) = 1 + m\alpha_1^2 + w\alpha_1\alpha_2 - \alpha_1^2 - \alpha_2^2.$$

We need to test that for each decomposition

$$(v, 1) = (\beta_0, 1) + (\beta_1, 0) + \cdots + (\beta_r, 1), \qquad \beta_1, \ldots, \beta_r \geqslant 1,$$

the following inequality holds

$$(m - 1)v^2 + vw \geqslant (m - 1)\beta_0^2 + \beta_0 w + r + (m - 1)(\beta_1^2 + \cdots + \beta_r^2),$$

which is actually true component-wise because

$$\begin{cases} (m - 1)v^2 \geqslant (m - 1)(\beta_0^2 + \cdots + \beta_r^2) \\ vw = (\beta_0 + \cdots + \beta_r)w \geqslant \beta_0 w + rw \geqslant \beta_0 w + r. \end{cases}$$

Therefore the moment map is always flat.

3. The third example is the quiver $Q = A_{n-1}$ with the following particular choice of vectors $\boldsymbol{v} = (1, \dots, 1)$ and $w_a = \delta_{a,1} + \delta_{a,n-1}$ (for which the Nakajima variety is the symplectic dual of $\mathbb{T}^*\mathbb{P}^{n-1}$, as explained in the next section). The resulting quiver $\Gamma = Q^\infty$ is the cyclic quiver with n vertices and dimension vector $\boldsymbol{\alpha} = (1, \dots, 1)$ constant to 1, for which it is easy to check that the moment map is flat. In fact $p(\boldsymbol{\alpha}) = 1$ while for any other $\boldsymbol{\beta} \in \mathbb{N}^n$, $0 \neq \boldsymbol{\beta} \neq \boldsymbol{\alpha}$ we have $p(\boldsymbol{\beta}) \leqslant 0$ so that condition (4) of Theorem 2.4.1.3 is satisfied.

2.4.2 Kirwan map and tautological sheaves

Let $\mathfrak{M}^\chi = \mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w})$ be a smooth Nakajima quiver variety (so $\chi = \chi_\theta$ with θ being $\boldsymbol{v}$-regular, see Theorem 2.4.1.1), then the locus of χ-semistable points coincides with the locus of χ-stable points, on which the action is free, and

$$\mathfrak{M}^\chi = \mu^{-1}(0) /\!\!/_\chi G = \mu^{-1}(0)^{\chi\text{-st}}/G \,.$$

The equivariant Kirwan map (in cohomology) is the map

$$\kappa_T : H^\bullet_{G\times T}\left(\mu^{-1}(0)\right) \to H^\bullet_T(\mathfrak{M}^\chi)\,, \tag{2.4.6}$$

obtained by composing the natural pullback for the inclusion $\mu^{-1}(0)^{\chi\text{-st}} \overset{\iota}{\subset} \mu^{-1}(0)$ with the isomorphism $H^\bullet_{G\times T}\left(\mu^{-1}(0)^{\chi\text{-st}}\right) \cong H^\bullet_T(\mathfrak{M}^\chi)$ due to the fact that the G-action on the χ-stable locus is free:

$$H^\bullet_{G\times T}\left(\mu^{-1}(0)\right) \xrightarrow{\iota^\bullet} H^\bullet_{G\times T}\left(\mu^{-1}(0)^{\chi\text{-st}}\right) \cong H^\bullet_T\left(\mu^{-1}(0)^{\chi\text{-st}}/G\right) = H^\bullet_T(\mathfrak{M}^\chi)\,.$$

McGerty and Nevins have recently shown that the Kirwan map (2.4.6) is surjective ([47, Corollary 1.5]), and that the same holds for other generalised cohomology theories such as K-theory and elliptic cohomology. We are particularly interested in the K-theory, so the Kirwan map is

$$\kappa_T : K_{G\times T}\left(\mu^{-1}(0)\right) \to K_T\left(\mathfrak{M}^\chi\right)\,. \tag{2.4.7}$$

Moreover the zero locus of the moment map $\mu^{-1}(0)$ is equivariantly contractible[7]:

$$K_{G\times T}\left(\mu^{-1}(0)\right) \cong K_{G\times T}(\mathrm{pt}) = \mathcal{R}(G\times T) \cong \mathcal{R}(G)\otimes\mathcal{R}(T)\,,$$

where $\mathcal{R}(-)$ is the representation ring (over $\mathbb{C}$), so the Kirwan map has the form:

$$\kappa_T : \mathcal{R}(G)\otimes\mathcal{R}(T) \to K_T\big(\mathfrak{M}^\chi\big)\,, \tag{2.4.8}$$

and it is a surjective map of $\mathcal{R}(T)$-modules. $K_T(\mathfrak{M}^\chi)$ is therefore generated by *tautological classes*, because they come from classes of topologically trivial vector bundles: if U is a $G\times T$-module, and $[U]\in\mathcal{R}(G\times T)$ is its class, then

$$\kappa_T([U]) = \left[\big(\mu^{-1}(0)^{\chi\text{-st}}\times U\big)/G\right] \in K_T\big(\mu^{-1}(0)^{\chi\text{-st}}/G\big) = K_T(\mathfrak{M}^\chi)\,. \tag{2.4.9}$$

Moreover the map (2.4.8) is a map of $\mathcal{R}(T)$-modules, so the only non-trivial part consist in its image on vector spaces U that are only representations of G. For $U = V_\lambda$ irreducible representation of G, we denote by a calligraphic $\mathcal{V}_\lambda$ the sheaf whose K-theoretic class is $[\mathcal{V}_\lambda] = \kappa_T([V_\lambda]) \in K_T(\mathfrak{M}^\chi)$. We can use these tautological classes to define invariants in the K-theory of the affine Nakajima variety by using the pushforward under the map p:

$$\mathcal{R}(G)\otimes\mathcal{R}(T) \xrightarrow{\kappa_T} K_T(\mathfrak{M}^\chi) \xrightarrow{p_*} K_T(\mathfrak{M}^0)\,. \tag{2.4.10}$$

It is important to recall that in general the push-forward of a proper map p in K-theory is given by the alternate sums of right-derived functors of p_*. In this particular case the target variety $\mathfrak{M}^0$ is affine, therefore this alternate sum calculates the Euler characteristic of a sheaf $\mathcal{F}$ on $\mathfrak{M}^\chi$, under the natural identifications:

$$p_*([\mathcal{F}]) = \chi_T(\mathfrak{M}^\chi,\mathcal{F}) \in K_T(\mathcal{O}(\mathfrak{M}^0)-\mathtt{Mod}) \cong K_T(\mathfrak{M}^0)\,. \tag{2.4.11}$$

The structure of $\mathcal{O}(\mathfrak{M}^0)$-module comes from the fact that the cohomologies $H^i(\mathfrak{M}^\chi,\mathcal{F})$ have a structure of $\mathcal{O}(\mathfrak{M}^\chi)$-modules and the map $p:\mathfrak{M}^\chi\to\mathfrak{M}^0$ gives to the latter a structure of $\mathcal{O}(\mathfrak{M}^0)$-module.

[7] This is because the moment map is equivariant and given by a homogeneous equation (in this particular case of degree 2, but the degree does not matter). Hence one can define an equivariant algebraic homotopy between the zero locus and the point $0\in\mu^{-1}(0)$.

For an irreducible representation $U = V_\lambda$ of G the composition (2.4.10) gives the Euler characteristic of the corresponding tautological sheaf $\mathcal{V}_\lambda$:

$$p_*\big(\kappa_T([V_\lambda])\big) = p_*\big([\mathcal{V}_\lambda]\big) = \chi_T(\mathfrak{M}^\chi, \mathcal{V}_\lambda) \in K_T(\mathfrak{M}^0)\,. \tag{2.4.12}$$

The notable special case of $U = V_0$ the trivial 1-dimensional representation of G, has image under the Kirwan map the (K-theoretic class of the) sheaf of functions on the GIT quotient $\mathcal{V}_0 = \mathcal{O}_{\mathfrak{M}^\chi}$, and its Euler characteristic:

$$p_*(\kappa_T([V_0])) = p_*([\mathcal{O}_{\mathfrak{M}^\chi}]) = \chi_T(\mathfrak{M}^\chi, \mathcal{O}_{\mathfrak{M}^\chi}) \in K_T(\mathfrak{M}^0)\,. \tag{2.4.13}$$

2.4.3 Comparison theorem and first integral formula

In § 2.3.3 we defined the virtual fundamental classes of the isotypical components of the derived character scheme

$$\chi_T^\lambda(A, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i=0}^{\infty} (-1)^i \left[(V_\lambda^* \otimes H_i(A, \boldsymbol{v}, \boldsymbol{w})])^G\right] \in K_T(\mathfrak{M}^0)\,, \tag{2.4.14}$$

and in particular for $V_\lambda = V_0 = \mathbb{C}$:

$$\chi_T^0(A, \boldsymbol{v}, \boldsymbol{w}) = \chi_T^G(A, \boldsymbol{v}, \boldsymbol{w}) = \sum_{i=0}^{\infty} (-1)^i \left[H_i(A, \boldsymbol{v}, \boldsymbol{w})^G\right] \in K_T(\mathfrak{M}^0)\,. \tag{2.4.15}$$

Theorem 2.4.3.1. *Let $\boldsymbol{v}, \boldsymbol{w}$ be dimension vectors for which the moment map is flat, and let $\chi = \chi_\theta$ with θ $\boldsymbol{v}$-regular, so that $\mathfrak{M}^\chi(Q, \boldsymbol{v}, \boldsymbol{w})$ is smooth. Then we have the following equality in the equivariant* K*-theory of the affine Nakajima variety :*

$$p_*([\mathcal{O}_{\mathfrak{M}^\chi(Q,\boldsymbol{v},\boldsymbol{w})}]) = [\mathcal{O}_{\mathfrak{M}^0(Q,\boldsymbol{v},\boldsymbol{w})}] = \chi_T^G(A, \boldsymbol{v}, \boldsymbol{w}) \in K_T\left(\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w})\right)\,. \tag{2.4.16}$$

Proof. The first equality is a somewhat classical result. Firstly, the (derived) pushforward in K-theory coincides with the underived pushforward

$$p_*([\mathcal{O}_{\mathfrak{M}^\chi}]) = \chi_T(\mathfrak{M}^\chi, \mathcal{O}_{\mathfrak{M}^\chi}) = \sum_{i \geqslant 0} (-1)^i [H^i(\mathfrak{M}^\chi, \mathcal{O}_{\mathfrak{M}^\chi})] = \left[\mathcal{O}_{\mathfrak{M}^\chi}\right] \in K_T(\mathfrak{M}^0)\,,$$

because of the vanishing of higher cohomologies. In fact $\mathfrak{M}^{\chi}$ is a smooth complex symplectic variety (Theorem 2.4.1.1), therefore the top-power of the symplectic form is nowhere-vanishing and it provides a trivialisation of the canonical sheaf $\mathcal{K}_{\mathfrak{M}^{\chi}} \cong \mathcal{O}_{\mathfrak{M}^{\chi}}$. Hence the Grauert-Riemenschneider theorem, saying that higher direct images of the canonical bundle under a proper birational morphism vanish, can be applied (see [33] or [43, Theorem 4.3.9.] for the Grauert-Riemenschneider theorem[8]). Moreover when the moment map is flat and $\mathfrak{M}^{\chi}$ is smooth we can use Theorem 2.4.1.2:

$$\left[\mathcal{O}_{\mathfrak{M}^{\chi}}\right] = [\mathcal{O}_{\mathfrak{M}^0}] \in K_T(\mathfrak{M}^0)$$

Finally by Theorem 2.4.1.4 the representation homology $H_{\bullet}(A, \boldsymbol{v}, \boldsymbol{w})$ vanishes in positive degrees, so that the Euler characteristic of its G-invariant part (2.4.15) is:

$$\chi_T^G(A, \boldsymbol{v}, \boldsymbol{w}) = [H_0(A, \boldsymbol{v}, \boldsymbol{w})^G] \overset{\text{Cor } 2.3.2.1}{=} [\mathcal{O}_{\mathfrak{M}^0}]\,.$$

□

Remark 2.4.3.1. In light of the previous explanations that we gave during the course of the paper, the result stated in Theorem 2.4.3.1 is not entirely surprising:

1. On one hand we have a symplectic resolution of singularities $p : \mathfrak{M}^{\chi} \to \mathfrak{M}^0$ therefore it is expected that functions on the smooth variety $\mathfrak{M}^{\chi}$ are equal to functions on the singular $\mathfrak{M}^0$.

2. On the other hand $\mu^{-1}(0)$ is a complete intersection in the linear space of representations $M(Q, \boldsymbol{v}, \boldsymbol{w})$, therefore the Koszul complex $\mathcal{O}(\mathrm{DRep}_{\boldsymbol{v},\boldsymbol{w}}(A)) \cong \mathcal{O}\left(M(Q, \boldsymbol{v}, \boldsymbol{w})\right) \otimes \Lambda^{\bullet}\mathfrak{g}_{\boldsymbol{v}}$ is a resolution of $\mathcal{O}(\mu^{-1}(0))$:

$$\begin{gathered} H_i(A, \boldsymbol{v}, \boldsymbol{w}) = \begin{cases} \mathcal{O}(\mu^{-1}(0))\,, & i = 0 \\ 0\,, & i \geqslant 1 \end{cases} \\ \left(\implies \quad \chi_T(A, \boldsymbol{v}, \boldsymbol{w}) = \mathcal{O}(\mu^{-1}(0))\right) \end{gathered} \tag{2.4.17}$$

[8] Or [30, proof of Proposition 1.2.2.] and [12, Corollary 2.4.(1)] for the application of Grauert-Riemenschneider in this specific situation.

and the subcomplex of G-invariants is a resolution of the functions on $\mathfrak{M}^0$:

$$
\begin{aligned}
H_i(A,\boldsymbol{v},\boldsymbol{w})^G = \begin{cases} \mathcal{O}(\mu^{-1}(0))^G\,, & i=0 \\ 0\,, & i\geqslant 1 \end{cases} \\
\left(\Longrightarrow \quad \chi_T^G(A,\boldsymbol{v},\boldsymbol{w}) = \mathcal{O}(\mathfrak{M}^0)\right)\,.
\end{aligned}
\tag{2.4.18}
$$

As a corollary of Theorem 2.4.3.1, we can take Hilbert-Poincaré series (character for the torus) of the equality in (2.4.16) and obtain a equality between numerical (power) series counting the graded dimensions. Formally, if $\mathfrak{M}^0$ were compact, the Hilbert-Poincaré would be the pushforward to the point: $\mathrm{ch}_T : K_T(\mathfrak{M}^0) \to K_T(\mathrm{pt}) = \mathcal{R}(T)$, instead in general we land in the field of fractions (see, for example, §4 in [55])

$$
\mathrm{ch}_T : K_T(\mathfrak{M}^0) \to \mathrm{Frac}(\mathcal{R}(T)) =: \mathcal{Q}(T)\,.
$$

Remark 2.4.3.2. If we consider the only fixed point for the torus action $0 \in \mathfrak{M}^0$, and denote its inclusion by $\iota_0 : \{0\} \to \mathfrak{M}^0$, then by functoriality we have $\mathrm{ch}_T = (\iota_{0,*})^{-1}$, and this tells us that is not really necessary to invert all non-zero elements in $\mathcal{R}(T)$, but only the ones of the form $1-t^\beta$ for non-zero weights β, so that we actually land in the following smaller localisation (see §2.1 and §2.3 in [59]):

$$
\mathcal{R}(T)_{,\mathrm{loc}} := \mathbb{C}\left[t^\alpha, \frac{1}{1-t^\beta}\right]\,,
$$

where α,β run over all weights of T and $\beta \neq 0$.

Let us denote by $x \in T_{\boldsymbol{v}} \subset G$ the variables in the maximal torus of the gauge group (*Kähler variables*) and by $t=(a,\hbar) \in T = T_{\boldsymbol{w}} \times T_\hbar$ the *equivariant variables*. Then we have, by Weyl's integral formula:

$$
\begin{aligned}
\mathrm{ch}_T\left(\chi_T^G(A,\boldsymbol{v},\boldsymbol{w})\right) &= \frac{1}{|G|}\int_G \mathrm{ch}_{G\times T}\left(\chi_T(A,\boldsymbol{v},\boldsymbol{w})\right)(g,t)\,dg = \\
&= \frac{1}{|W|}\int_{T_{\boldsymbol{v}}} \mathrm{ch}_{T_{\boldsymbol{v}}\times T}\left(\chi_T(A,\boldsymbol{v},\boldsymbol{w})\right))(x,t)\Delta(x)\,dx\,,
\end{aligned}
\tag{2.4.19}
$$

(W is the Weyl group of G, $\Delta(x)$ is the Weyl factor,
and integrations are over the compact real forms of $G, T_{\boldsymbol{v}}$)

Moreover, because the Euler characteristic of the homology of a complex is equal to the Euler characteristic of the complex itself, we have

$$\mathrm{ch}_{T_{\boldsymbol{v}}\times T}(\chi_T(A,\boldsymbol{v},\boldsymbol{w})) = \mathrm{ch}_{T_{\boldsymbol{v}}\times T}\big(\mathcal{O}(M(Q,\boldsymbol{v},\boldsymbol{w}))\otimes\Lambda^\bullet\mathfrak{g}\big) = \frac{\prod_i(1-\hbar_1\hbar_2 r_i)}{\prod_j(1-s_j)}\,. \tag{2.4.20}$$

where s_j are the weights of $M(Q,\boldsymbol{v},\boldsymbol{w})^*$ and r_i are the weights of $\mathfrak{g}$:

$$\mathrm{ch}_{T_{\boldsymbol{v}}\times T}(M(Q,\boldsymbol{v},\boldsymbol{w})) = \sum_j s_j^{-1}\,, \qquad \mathrm{ch}_{T_{\boldsymbol{v}}}(\mathfrak{g}) = \sum_i r_i\,.$$

To summarise:

Corollary 2.4.3.1. *Under the same conditions of Theorem 2.4.3.1, and with the notation used in the previous equations (in particular* (2.4.20)*), we have the following equality of Poincaré-Hilbert series in the field of fractions* $\mathcal{Q}(T)$*:*

$$\mathrm{ch}_T\mathcal{O}(\mathfrak{M}^\chi(Q,\boldsymbol{v},\boldsymbol{w})) = \mathrm{ch}_T\mathcal{O}(\mathfrak{M}^0(Q,\boldsymbol{v},\boldsymbol{w})) = \frac{1}{|W|}\int_{T_{\boldsymbol{v}}}\frac{\prod_i(1-\hbar_1\hbar_2 r_i)}{\prod_j(1-s_j)}\Delta(x)\,dx\,. \tag{2.4.21}$$

We calculate the above expression (2.4.21) in some concrete examples in § 2.5.

Remark 2.4.3.3. The right-hand side of (2.4.21) does not depend on the GIT parameter χ, while the left-hand side a priori does. By picking different $\boldsymbol{v}$-regular χ,χ' we obtain a combinatorical identity

$$\mathrm{ch}_T\mathcal{O}(\mathfrak{M}^\chi(Q,\boldsymbol{v},\boldsymbol{w})) = \mathrm{ch}_T\mathcal{O}(\mathfrak{M}^{\chi'}(Q,\boldsymbol{v},\boldsymbol{w}))\,,$$

which we will show to be non-trivial, also in simplest quiver cases (see § 2.5, specifically Remark 2.5.1.1 in § 2.5.1).

2.4.4 Other isotypical components and second integral formula

In this section we prove a result similar to Theorem 2.4.3.1 to relate other tautological sheaves with the corresponding isotypical components.

Let us recall that to define $\mathfrak{M}^{\chi}$ we fixed a character $\chi \in \mathrm{Hom}_{\mathtt{Grp}_{\mathbb{C}}}(G, \mathbb{C}^{\times})$. This character defines a 1-dimensional representation $\mathbb{C}_{\chi}$ of G, whose image under the Kirwan map is the Serre twisting sheaf

$$\kappa_T\left([\mathbb{C}_{\chi}]\right) = \left[\mathcal{O}_{\mathfrak{M}^{\chi}}(1)\right] \in K_T(\mathfrak{M}^{\chi}) . \tag{2.4.22}$$

For each V_{λ} irreducible representation of G, we have a tautological sheaf $\mathcal{V}_{\lambda}$ in the K-theory of $\mathfrak{M}^{\chi}$. By Serre vanishing theorem when we twist

$$\mathcal{V}_{\lambda}(m) := \mathcal{V}_{\lambda} \otimes \mathcal{O}_{\mathfrak{M}^{\chi}}(m) , \tag{2.4.23}$$

by a sufficiently large power $m \gg 0$ of the twisting sheaf, higher cohomology vanish, so that

$$\chi_T(\mathfrak{M}^{\chi}, \mathcal{V}_{\lambda}(m)) = H^0(\mathfrak{M}^{\chi}, \mathcal{V}_{\lambda}(m)) . \tag{2.4.24}$$

Moreover, more or less by definition of the GIT quotient $\mathfrak{M}^{\chi}$, this is equal to the G-invariant global sections of the trivial vector bundle $\underline{V_{\lambda} \otimes \mathbb{C}_{\chi^m}}$ over the stable locus:

$$H^0(\mathfrak{M}^{\chi}, \mathcal{V}_{\lambda}(m)) = \Gamma\left(\mu^{-1}(0)^{\chi\text{-st}}, \underline{V_{\lambda} \otimes \mathbb{C}_{\chi^m}}\right)^G . \tag{2.4.25}$$

Finally for $m \gg 0$ large enough, the following natural restriction map becomes an isomorphism (see for example the proof of Lemma 3 in Appendix A of [1]):

$$\Gamma\left(\mu^{-1}(0), \underline{V_{\lambda} \otimes \mathbb{C}_{\chi^m}}\right)^G \xrightarrow{\sim} \Gamma\left(\mu^{-1}(0)^{\chi\text{-st}}, \underline{V_{\lambda} \otimes \mathbb{C}_{\chi^m}}\right)^G , \tag{2.4.26}$$

but the left-hand side is nothing else but

$$\Gamma\left(\mu^{-1}(0), \underline{V_{\lambda} \otimes \mathbb{C}_{\chi^m}}\right)^G = \left(\mathcal{O}(\mu^{-1}(0)) \otimes V_{\lambda} \otimes \mathbb{C}_{\chi^m}\right)^G . \tag{2.4.27}$$

It is worth noticing at this point that irreducible representations V_{λ} of G are labelled by collections of partitions $\lambda = (\lambda^{(1)}, \ldots, \lambda^{(n)})$ and that the representation $V_{\lambda} \otimes \mathbb{C}_{\chi^m}$ is still a irreducible representation of G, corresponding to the shifted collection of partitions:

$$V_\lambda \otimes \mathbb{C}_{\chi^m} = V_{\widetilde{\lambda}}\,, \qquad \widetilde{\lambda} := \lambda + m\underline{\theta} = (\lambda^{(1)} + m\underline{\theta_1}, \ldots, \lambda^{(n)} + m\underline{\theta_n}) \quad (\chi = \chi_\theta)\,, \tag{2.4.28}$$

(see Appendix B for the notation). We give the following definition:

Definition 2.4.4.1. We say that an irreducible representation $V_{\widetilde{\lambda}}$ is *large enough* if $\widetilde{\lambda} = \lambda + m\underline{\theta}$ (see (2.4.28)) with $m \gg 0$ large enough for both (2.4.24) and (2.4.26) to be true. This notion depends on the quiver Q, on the dimension vectors $\boldsymbol{v}, \boldsymbol{w}$ and on the $\boldsymbol{v}$-regular $\chi = \chi_\theta$.

Denoting by $\widetilde{\lambda}^*$ the partition corresponding to the dual representation, we can continue equation (2.4.27) to recognise:

$$\Big(\mathcal{O}(\mu^{-1}(0)) \otimes V_{\widetilde{\lambda}}\Big)^G = \Big(\mathcal{O}(\mu^{-1}(0)) \otimes V^*_{\widetilde{\lambda}^*}\Big)^G = H_0(A, \boldsymbol{v}, \boldsymbol{w})^G_{\widetilde{\lambda}^*}\,, \tag{2.4.29}$$

the isotypical component of $\widetilde{\lambda}^*$ of the (zeroth) representation homology. Finally if we observe that with flat moment map, higher homologies vanish, we obtain the following result:

Theorem 2.4.4.1. *Let $\boldsymbol{v}, \boldsymbol{w}$ be dimension vectors for which the moment map is flat, and fix $\chi = \chi_\theta$ with θ $\boldsymbol{v}$-regular. For λ large enough (in the sense of Definition 2.4.4.1) we have*

$$p_*([\mathcal{V}_\lambda]) = [H^0(\mathfrak{M}^\chi, \mathcal{V}_\lambda)] = \chi_T^{\lambda^*}(A, \boldsymbol{v}, \boldsymbol{w}) \in K_T(\mathfrak{M}^0(Q, \boldsymbol{v}, \boldsymbol{w}))\,. \tag{2.4.30}$$

The analogous integral formula to obtained by taking characters is

Corollary 2.4.4.1. *Under the same conditions of Theorem 2.4.4.1, and with the notation used in* (2.4.20)*, we have the following equality of Poincaré-Hilbert series in the field of fractions $\mathcal{Q}(T)$:*

$$\mathrm{ch}_T(\chi_T(\mathfrak{M}^\chi, \mathcal{V}_\lambda)) = \mathrm{ch}_T(H^0(\mathfrak{M}^\chi, \mathcal{V}_\lambda)) = \frac{1}{|W|}\int_{T_v} \frac{\prod_i (1 - \hbar_1\hbar_2 r_i)}{\prod_j (1 - s_j)} f_\lambda(x)\Delta(x)dx\,. \tag{2.4.31}$$

where $f_\lambda(x) = \mathrm{ch}_{T_v}(V_\lambda)$ (it is the product of Schur polynomials associated to the partitions in λ).

2.5 Examples

In this section we explain some concrete examples, mainly from the easiest quivers already considered in the previous sections. We see how such elementary quivers still produce varieties of great interest in various fields of mathematics.

2.5.1 Cotangent bundle of Grassmannian

The quiver $Q = A_1$ with only one vertex and no arrows. The framed, doubled quiver has two vertices and two arrows connecting them in opposite directions.

$$\begin{tikzcd}
\bullet v \arrow[r, rightsquigarrow] & \bullet v \arrow[d, bend right, "j"'] \\
\text{Single-vertex quiver} & \square w \arrow[u, "i"']
\end{tikzcd}$$

Figure 2.2: Framing and doubling the single-vertex quiver.

Therefore:

$$\mu^{-1}(0) = \{(I, J) \in \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^w, \mathbb{C}^v) \oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^v, \mathbb{C}^w) \,|\, I \circ J = 0\}.$$

Because we have only one vertex we have to choose the GIT parameter $\theta \in \mathbb{Z}$, and it is easy to check that the v-regularity condition means simply $\theta \neq 0$ (independently from v). For $\theta \neq 0$ we have the following identifications of the semistable locus:

$$\theta\text{-semistable points} = \begin{cases} J \ \text{injective}, & \theta < 0 \\ I \ \text{surjective}, & \theta > 0 \end{cases}$$

and the GIT quotient is isomorphic to the cotangent bundle $\mathbb{T}^*\mathbb{Gr}(v, w)$ of v-planes in $\mathbb{C}^w$ in the case $\theta < 0$ and to $\mathbb{T}^*\mathbb{Gr}(w - v, w)$ in the case $\theta > 0$. The two varieties are isomorphic to each other, but we have the following different identifications of the points in the Grassmannian:

$$\theta < 0: \quad \mathrm{im}(J) \in \mathbb{Gr}(v, w) \ ,$$

$$\theta > 0: \quad \ker(I) \in \mathbb{Gr}(w-v, w)\ .$$

The affine quotient can be identified (using some version of the fundamental theorem of invariant theory):

$$\mathfrak{M}^0 = \operatorname{Spec}\big(\mathcal{O}(\mu^{-1}(0))^{GL_v}\big) \cong \{A \in M_{w\times w}(\mathbb{C}) \,|\, A^2 = 0,\ \operatorname{rk}(A) \leqslant v\},$$

where A represents the composition $J \circ I : \mathbb{C}^w \to \mathbb{C}^w$. The condition on the rank is due to the fact that $A : W \to V \to W$ factorises through V, but sometimes it is superfluous. In fact in general $A^2 = 0$ forces already $\operatorname{rk}(A) \leqslant \lfloor w/2 \rfloor$. The moment map is flat if and only if $2v - 1 \leqslant w$ (see Examples 4), and only in this cases the projective morphism

$$p : \mathbb{T}^*\mathbb{Gr}(v, w) \to \mathfrak{M}^0$$

is a resolution of singularities.

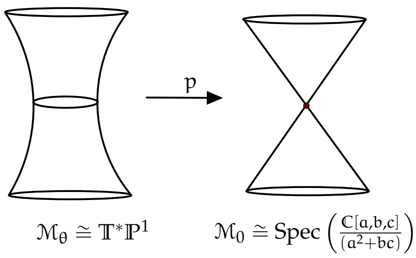

Figure 2.3: The (real) picture of the case $(v, w) = (1, 2)$: this is also known as Springer resolution of the nilpotent cone of $\mathfrak{sl}_2(\mathbb{C})$.

In this case in the torus $T = T_w \times T_\hbar$ only the product $\hbar_1\hbar_2$ appears and we denote it by $\hbar$. We can use (2.4.21) for $\chi = \chi_{-1}$ for which $\mathfrak{M}^\chi = \mathbb{T}^*\mathbb{Gr}(v, w)$ and obtain a formula for the character of the ring of functions on the cotangent bundle of Grassmannian:

$$\operatorname{ch}_T(\mathcal{O}(\mathbb{T}^*\mathbb{Gr}(v, w))) =$$

$$= \frac{1}{v!}\cdot \oint_{|x_\alpha|=1} \frac{\prod_{\alpha,\beta}(1 - \hbar x_\alpha^{-1} x_\beta)}{\prod_{\alpha,\gamma}(1 - \hbar x_\alpha^{-1} a_\gamma)(1 - x_\alpha a_\gamma^{-1})} \cdot \overbrace{\prod_{\alpha\neq\beta}(1 - x_\alpha^{-1} x_\beta)}^{\Delta(x)} \overbrace{\prod_\alpha \frac{dx_\alpha}{2\pi i x_\alpha}}^{dx}, \tag{2.5.1}$$

where in the above $x = (x_\alpha) = (x_1, \dots, x_v)$ and $a = (a_\gamma) = (a_1, \dots, a_w)$.

The integral in the right-hand side can be computed by iterated residues, and by doing so we can recognise the localisation formula in equivariant K-theory as a sum over the fixed points $p \in (\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w))^{\mathsf{T}}$ of the inverse of the K-theoretic Euler class of the tangent space at that point:

$$
\begin{aligned}
\mathrm{ch}_{\mathsf{T}}(\mathcal{O}(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w))) &= \sum_{\substack{B \subset \{1,\dots,w\} \\ \#B = v}} \frac{1}{\prod_{\substack{\beta \in B \\ \gamma \notin B}} \left(1 - \frac{a_\beta}{a_\gamma}\right)\left(1 - \hbar\frac{a_\gamma}{a_\beta}\right)} = \\
&= \sum_{p \in (\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w))^{\mathsf{T}}} \frac{1}{\mathrm{ch}_{\mathsf{T}}\left(\Lambda_{-1}\mathbb{T}_p^*\left(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w)\right)\right)}\,.
\end{aligned}
\tag{2.5.2}
$$

For what concerns other sheaves, let us consider the standard representation $V = \mathbb{C}^v$ of $G = GL_v(\mathbb{C})$. The associated tautological sheaf $\mathcal{V}$ on $\mathfrak{M}^{\chi_{-1}} = \mathbb{T}^*\mathbb{G}\mathrm{r}(v,w)$ is indeed the usual tautological sheaf of rank v. Irreducible representations are labelled by Schur functors $V_\lambda = \mathbb{S}_\lambda(V)$ where $\lambda = (\lambda_1 \geqslant \dots \geqslant \lambda_v)$ is a integer partition of v parts, and we consider the corresponding tautological sheaves $\mathcal{V}_\lambda$. For example the (standard) tautological sheaf itself is $\mathcal{V} = \mathcal{V}_{(1,0,\dots,0)}$, or powers of the Serre twisting sheaf are:

$$
\mathcal{O}_{\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w)}(m) = \det{}^{-m}(\mathcal{V}) = \mathcal{V}_{(-m,\dots,-m)}\,. \tag{2.5.3}
$$

A partition λ becomes large (Definition 2.4.4.1) in the sense that we can apply Theorem 2.4.3.1 when all its components are negative enough (because the character $\chi = \chi_{-1}$ is negative), and it turns out that it suffices to have $\lambda_1 \leqslant 0$, that is equivalent to say that the partition is made of non-positive terms (an example is (2.5.3), in which for $m > 0$ the partition is negative and the corresponding sheaf has vanishing higher cohomologies). In this range we have

$$
\begin{aligned}
&\mathrm{ch}_{\mathsf{T}} H^0\left(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w), \mathcal{V}_\lambda\right) = \\
&\frac{1}{v!}\cdot \oint_{|x|=1} \frac{\left(\prod_{\alpha,\beta}(1 - \hbar x_\alpha^{-1}x_\beta)\right) s_\lambda(x)}{\prod_{\alpha,\gamma}(1 - \hbar x_\alpha^{-1}a_\gamma)(1 - x_\alpha a_\gamma^{-1})} \cdot \prod_{\alpha \neq \beta}(1 - x_\alpha^{-1}x_\beta) \prod_\alpha \frac{dx_\alpha}{2\pi i x_\alpha}\,,
\end{aligned}
\tag{2.5.4}
$$

where $s_\lambda(x) = \mathrm{ch}_{\mathsf{T}_v}(V_\lambda)$ is the Schur polynomial associated to the partition λ. Again, the integral in the right-hand side can be computed by means of iterated residues, giving the localisation formula for the corresponding tautological sheaf:

$$\begin{aligned} \mathrm{ch}_T H^0\big(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w),\mathcal{V}_\lambda\big) &= \sum_{\substack{B\subset\{1,\dots,w\}\\ \#B=v}} \frac{s_\lambda(a_B)}{\prod\limits_{\substack{\beta\in B\\ \gamma\notin B}}\left(1-\frac{a_\beta}{a_\gamma}\right)\left(1-\hbar\frac{a_\gamma}{a_\beta}\right)} = \\ &= \sum_{p\in(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w))^T} \frac{\mathrm{ch}_T(\mathcal{V}_\lambda)_{|p}}{\mathrm{ch}_T\big(\Lambda_{-1}\mathbb{T}^*_p\left(\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w)\right)\big)}, \end{aligned} \tag{2.5.5}$$

where the expression $s_\lambda(a_B)$ means that we are evaluating the Schur polynomial $s_\lambda(x_1,\dots,x_v)$ in the point $x=(a_\beta)_{\beta\in B}$.

Remark 2.5.1.1. As already observed in Remark 2.4.3.3 the right-hand side of the integral formula (2.4.21) does not depend on the character χ, while a priori the left-hand side does. In (2.5.1) we used the character $\chi=\chi_{-1}$ for which $\mathfrak{M}^\chi=\mathbb{T}^*\mathbb{G}\mathrm{r}(v,w)$. If we use $\chi'=\chi_1$ we have $\mathfrak{M}^{\chi'}=\mathbb{T}^*\mathbb{G}\mathrm{r}(w-v,w)$. The fixed point formula for the first variety (2.5.2) can be compared with the one for the second variety, and it gives a non-trivial combinatorical identity:

$$\sum_{\substack{B\subset\{1,\dots,w\}\\ \#B=v}} \frac{1}{\prod\limits_{\substack{\beta\in B\\ \gamma\notin B}}\left(1-\frac{a_\beta}{a_\gamma}\right)\left(1-\hbar\frac{a_\gamma}{a_\beta}\right)} = \sum_{\substack{B\subset\{1,\dots,w\}\\ \#B=v}} \frac{1}{\prod\limits_{\substack{\beta\in B\\ \gamma\notin B}}\left(1-\frac{a_\gamma}{a_\beta}\right)\left(1-\hbar\frac{a_\beta}{a_\gamma}\right)}. \tag{2.5.6}$$

2.5.2 Framed moduli space of torsion free sheaves on $\mathbb{P}^2$

This is the case of the Jordan quiver, the quiver with one vertex and one loop (Figure 2.1). Therefore zero locus of the moment map $\mu^{-1}(0)$ is identified with quadruples satisfying the ADHM equation:

$$\{(X,Y,I,J)\in \mathrm{End}_{\mathbb{C}}(\mathbb{C}^v)^{\oplus 2}\oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^w,\mathbb{C}^v)\oplus \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^v,\mathbb{C}^w)\;\big|\;[X,Y]+IJ=0\}.$$

For GIT paramater $\theta\in\mathbb{Z}$:

$$\theta\text{-semistable points} = \\ = \begin{cases} \nexists 0\neq S\subset V \text{ s.t. } \mathbb{C}\langle X,Y\rangle(S)\subset S \text{ and } S\subset\ker(J), & \theta<0\\ \nexists S\subsetneq V \text{ s.t. } \mathbb{C}\langle X,Y\rangle(S)\subset S \text{ and } \mathrm{im}(I)\subset S, & \theta>0\end{cases}$$

In both cases we have an identification between the Nakajima variety $\mathfrak{M}^\theta$ and $M(w,v)$, the (framed) moduli space of torsion free sheaves on $\mathbb{CP}^2$ of rank w, second Chern class $c_2 = v$, and fixed trivialisation at the line at ∞. The affine Nakajima variety is $\mathfrak{M}^0 \cong M_0(w,v)$ the framed moduli space of ideal instantons on $S^4 = \mathbb{C}^2 \cup \{\infty\}$. The map $p : M(w,v) \to M_0(w,v)$ is always a resolution of singularities because the moment map is always flat.

When the framing is $w = 1$ we obtain the Hilbert-Chow morphism from the Hilbert scheme of v points on $\mathbb{C}^2$ to the symmetric v-power:

$$p : \mathrm{Hilb}_v(\mathbb{C}^2) \to \mathrm{Sym}^v(\mathbb{C}^2)\,.$$

For general v and w the integral formula looks like:

$$\mathrm{ch}_T \mathcal{O}(M(w,v)) = \frac{1}{v!} \cdot \oint_{|x|=1} I(x,a,\hbar) \cdot \prod_{\alpha\neq\beta} (1 - x_\alpha^{-1} x_\beta) \prod_\alpha \frac{dx_\alpha}{2\pi i x_\alpha}\,, \tag{2.5.7}$$

where

$$I(x,t,\hbar) = \\ = \frac{\prod_{\alpha,\beta}(1-\hbar_1\hbar_2 x_\alpha^{-1}x_\beta)}{\prod_{\alpha,\beta}(1-\hbar_1 x_\alpha^{-1}x_\beta)(1-\hbar_2 x_\alpha^{-1}x_\beta)\cdot\prod_{\alpha,\gamma}(1-\hbar_1\hbar_2 x_\alpha^{-1}t_\gamma)(1-x_\alpha t_\gamma^{-1})}\,,$$

and it is also known as the integral formula for Nekrasov partition function (proved for example in Appendix A of [25]).

For other isotypical components, let us say that we fixed $\chi = \chi_1$. Again we have a tautological sheaf of rank v, $\mathcal{V}$, and other sheaves associated to irreducible representations are labelled by Schur functors $\mathcal{V}_\lambda$ where λ is an integer partition of v parts. In this case the largeness condition indeed means that the partition is big enough, and it turns out that it suffices for it to be non-negative $\lambda_1 \geqslant \cdots \geqslant \lambda_v \geqslant 0$. In this range we have:

$$\mathrm{ch}_T H^0(M(w,v),\mathcal{V}_\lambda) = \frac{1}{v!} \oint_{|x|=1} I(x,a,\hbar) \cdot s_\lambda(x) \cdot \prod_{\alpha\neq\beta}(1-x_\alpha^{-1}x_\beta) \prod_\alpha \frac{dx_\alpha}{2\pi i x_\alpha}\,. \tag{2.5.8}$$

For $\lambda \geqslant 0$ the Schur polynomial $s_\lambda(x)$ is indeed an actual polynomial (and not a Laurent polynomial), and therefore with (2.5.8) we recover the integral formula for Nekrasov partition function with matter fields (the matter field is represented by the sheaf $\mathcal{V}_\lambda$ in this case) which was proved for example in [49].

2.5.3 Symplectic dual of $\mathbb{T}^*\mathbb{P}^{n-1}$

$X = \mathbb{T}^*\mathbb{G}\mathrm{r}(k,n)$ has a symplectic dual[9], $X^{\vee}$, which for the choice of parameters $2k \leqslant n$ can be shown to be also a Nakajima quiver variety ([63]). Specifically it is the Nakajima variety associated to the following A_{n-1} quiver, with dimension vectors:

$$\begin{cases} \boldsymbol{v} = (1,2,\dots,k-1, \underbrace{k,\dots,k}_{(n-2k+1)\text{-times}}, k-1,\dots,2,1)\,, \\ \boldsymbol{w} = (w_1,\dots,w_{n-1}) \qquad w_i = \delta_{i,k} + \delta_{i,n-k}\,. \end{cases}$$

We restrict to the case $k = 1$, for which dimension vectors are

$$\begin{cases} \boldsymbol{v} = (1,\dots,1)\,, \\ \boldsymbol{w} = (1,0,\dots,0,1)\,, \end{cases} \tag{2.5.9}$$

and the corresponding Nakajima quiver variety is the symplectic dual of $\mathbb{T}^*\mathbb{P}^{n-1}$. For $n = 2$ we go back to the A_1 case with dimensions $v = 1$ and $w = 2$, so we find that $\mathbb{T}^*\mathbb{P}^1$ is symplectic dual to itself. Let us study the other cases $n \geqslant 3$ which are different.

As usual we denote the arrows in the quiver by $x_1,\dots,x_{n-2}$, their dual by $y_1,\dots,y_{n-2}$ and then we have i_1, j_1 and i_{n-1}, j_{n-1} because of the non-trivial framing at the vertices 1 and $n-1$. The zero locus of the moment map is the following algebraic variety in a $2n$-dimensional affine space

$$\mu^{-1}(0) \cong \mathrm{Spec}\left(\frac{\mathbb{C}[x_1,y_1,\dots,x_{n-2},y_{n-2},i_1,j_1,i_{n-1},j_{n-1}]}{i_1j_1 = x_1y_1 = x_2y_2 = \dots = x_{n-2}y_{n-2} = -i_{n-1}j_{n-1}}\right)\,.$$

The gauge group is a $n-1$-dimensional torus $G_{\boldsymbol{v}} = GL_1(\mathbb{C})^{n-1} = (\mathbb{C}^\times)^{n-1}$, and the affine Nakajima variety is identified with the ADE singularity of type A_{n-1}:

$$\mathfrak{M}^0 \cong \mathrm{Spec}\left(\frac{\mathbb{C}[x,y,z]}{xy = z^n}\right) \cong \mathbb{C}^2/\mathbb{Z}_n\,, \tag{2.5.10}$$

where $x = x_1\cdots x_{n-2}i_1j_{n-1}$, $y = y_1\cdots y_{n-2}i_{n-1}j_1$, $z = x_1y_1$. We recall that the action $\mathbb{Z}_n \curvearrowright \mathbb{C}^2$ that gives the corresponding ADE singularity of type

[9] We do not give a definition of symplectic dual because it is beyond the scope of this thesis. We refer the interested reader to [14].

A_{n-1} is given by the embedding $\mathbb{Z}_n \subset SL_2(\mathbb{C})$ in which a n-th root of unity $\xi \in \mathbb{Z}_n$ becomes the matrix $\mathrm{diag}(\xi, \xi^{-1}) \in SL_2(\mathbb{C})$.

We fix GIT parameter $\chi = \chi_{\theta_+}$ with $\theta_+ = (1, 1, \ldots, 1)$. The corresponding smooth Nakajima quiver variety is a consecutive ($n-1$ times) blowup of the singular point $x = y = z = 0$ in (2.5.10):

$$p : \mathfrak{M}^{\chi_+} = \widetilde{\mathbb{C}^2/\mathbb{Z}_n} \longrightarrow \mathfrak{M}^0 = \mathbb{C}^2/\mathbb{Z}_n\,, \tag{2.5.11}$$

with exceptional fiber $p^{-1}(0)$ given by $n-1$ copies of Riemann spheres $\mathbb{P}^1$ intersecting in such a way that their underlying intersection graph is A_{n-1} (see [22]), as shown in Figure 2.4.

Figure 2.4: Every sphere is replaced by a vertex and two vertices are linked by as many arrows as intersection points of the corresponding spheres.

The associated derived representation scheme is

$$\mathrm{DRep}_{v,w} = \mathrm{Spec}\left(\mathbb{C}[x_1, y_1, x_2, y_2, \ldots, x_{n-2}, y_{n-2}, i_1, j_1, i_{n-1}, j_{n-1}, \vartheta_1, \ldots, \vartheta_{n-1}]\right)\,,$$

where ϑ_i have homological degree 1 and differential

$$\begin{cases} d\vartheta_1 = -y_1 x_1 + i_1 j_1\,, \\ d\vartheta_k = x_{k-1} y_{k-1} - y_k x_k\,, \quad (k = 2, \ldots, n-2)\,, \\ d\vartheta_{n-1} = x_{n-2} y_{n-2} + i_{n-1} j_{n-1}\,, \end{cases} \tag{2.5.12}$$

and they are invariants under the gauge group $G_v = GL_1^{n-1}$, so that the associated character scheme is simply

$$\mathrm{DRep}_{v,w}^{G_v} \cong \mathrm{Spec}\left(\left(\frac{\mathbb{C}[x, y, z_1, \ldots, z_{n-2}, z_{n-1}, z_n]}{xy = z_1 \cdots z_n}\right)[\vartheta_1, \ldots, \vartheta_{n-1}]\right)\,,$$

where x, y are the same classes as before in (2.5.10), $z_k = x_k y_k$ for $k = 1, \dots, n-2$, $z_{n-1} = i_1 j_1$, $z_n = i_{n-1} j_{n-1}$. We denote the variables in the equivariant torus $T = T_w \times T_\hbar$ by $(a, \tilde{a}, \hbar_1, \hbar_2)$ (where a is on the vertex 1 and $\tilde{a}$ on the vertex $n-1$) and we have:

$$\begin{aligned} &\mathrm{ch}_T \mathcal{O}\left(\widetilde{\mathbb{C}^2/\mathbb{Z}_n}\right) = \mathrm{ch}_T \mathcal{O}\left(\mathbb{C}^2/\mathbb{Z}_n\right) = \\ &= \mathrm{ch}_T\left(\chi_T(\mathrm{DRep}^{G_v}_{v,w})\right) = \frac{1 + \hbar_1\hbar_2 \cdots + \hbar_1^{n-1}\hbar_2^{n-1}}{\left(1 - \hbar_1^{n-1}\hbar_2 \frac{a}{\tilde{a}}\right)\left(1 - \hbar_1\hbar_2^{n-1}\frac{\tilde{a}}{a}\right)}. \end{aligned} \tag{2.5.13}$$

Chapter 3

Noncommutative Derived Poisson Reduction

Mathematics Subject Classification Primary 53D30; Secondary 14A22.

Abstract

In this paper we propose a procedure for a noncommutative derived Poisson reduction, in the spirit of the Kontsevich-Rosenberg principle: "a noncommutative structure of some kind on A should give an analogous commutative structure on all schemes $\mathrm{Rep}_n(A)$". We use double Poisson structures as noncommutative Poisson structures and noncommutative Hamiltonian spaces — as first introduced by M. Van den Bergh — to define (derived) zero loci of Hamiltonian actions and a noncommutative Chevalley-Eilenberg and BRST constructions, showing how we recover the corresponding commutative constructions using the representation functor. In a dedicated final short section we highlight how the categorical properties of the representation functor lead to the natural introduction of new interesting notions, such as noncommutative group schemes, group actions, or Poisson-group schemes, which could help to understand the previous results in a different light, and in future research generalise them into a broader, clearer correspondence between noncommutative and commutative equivariant geometry.

3.1 Introduction

A known principle in noncommutative geometry ([41]) says that every geometrically meaningful structure on an (associative, unital) algebra $A \in \mathtt{Alg}_k$ should induce the corresponding geometric structure on the scheme of representations $\mathrm{Rep}_n(A)$ in a n-dimensional vector space. Noncommutative Poisson geometry was worked out first by W. Crawley-Boevey who defined a Poisson structure on the character scheme $\mathrm{Rep}_n(A) /\!\!/ \mathrm{GL}_n$ through his definition of H_0-Poisson structures ([20]) and then by M. Van den Bergh in [68], who made the observation that a Poisson bracket on the full representation scheme $\mathrm{Rep}_n(A)$ shall be defined on the generators $\{a_{ij}, b_{kl}\}$, and because it depends on four indices, it is natural to assume that it comes from a double bracket $\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$, with some properties that ensure that the induced bracket on the representation scheme is indeed a Poisson bracket.

M. Van den Bergh also defined, using double Poisson structures, a noncommutative version of Hamiltonian spaces, essentially double Poisson algebras with a distinguished 'gauge' element $\delta \in A$ that acts via the double Poisson bracket as the universal derivation on the algebra (3.3.19). This ensures that the corresponding action of the gauge group $\mathrm{GL}_n \curvearrowright \mathrm{Rep}_n(A)$ on the representation scheme is a Hamiltonian action, with moment map $\mu_n : \mathrm{Rep}_n(A) \to \mathfrak{gl}_n^{(*)}$ described as the evaluation of a representation on the element δ. One can define a noncommutative version of Poisson reduction then, by considering the quotient algebra $A/\langle\delta\rangle$ by the two sided ideal generated by δ, which is a noncommutative counterpart of the zero locus: $\mathrm{Rep}_n(A/\langle\delta\rangle) = \mu_n^{-1}(0)$, so that its GL_n-quotient is the Poisson reduction $\mu_n^{-1}(0) /\!\!/ \mathrm{GL}_n$. These ideas appear in various forms in the first papers on noncommutative symplectic and Poisson geometry, such as [40, 32, 13, 31, 21, 24, 68, 69].

In this paper, we elaborate an idea from V. Ginzburg (who first defined some 'noncommutative BRST complexes' in [29]) and we work out in details a possible procedure to do noncommutative Poisson reduction in a derived fashion: we add variables in positive homological degrees to kill relations instead of considering quotients, and we add other variables in negative homological degrees as some sort of Chevalley-Eilenberg generators instead of considering invariants. In order to clarify our definitions involved in this 'derived Poisson reduction' in the noncommutative world, it is conveninet to first recall briefly the commutative construction of derived Poisson reduction, in the style of [16, 64].

We start from a Poisson algebra B, a Hamiltonian group scheme action of a reductive group $G \curvearrowright X = \mathrm{Spec}(B)$, with (co)moment Poisson map: $\mathrm{Sym}(\mathfrak{g}) \to B$ (the Poisson structure on $\mathrm{Sym}(\mathfrak{g})$ is the natural extension of the Lie bracket). We first define the derived zero locus of the corresponding map of schemes $\mu : X \to \mathfrak{g}^*$ as the homotopy pull-back in the category of dg schemes (dually, the homotopy push-out diagram in the category of commutative dg algebras over):

$$B \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{g})} k \quad \Big(\leftrightarrow \quad X \times^h_{\mathfrak{g}^*} \mathrm{pt}\Big) \,. \tag{3.1.1}$$

We then apply the Chevalley-Eilenberg functor $C(\mathfrak{g}, -) = \mathrm{Hom}_k(\mathrm{Sym}(\mathfrak{g}[1]), -)$ to the derived zero locus and obtain the classical BRST complex

$$\begin{aligned} C(\mathfrak{g}, B \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{g})} k) &\simeq C(\mathfrak{g}, B) \otimes^{\mathbf{L}}_{C(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}))} C(\mathfrak{g}, k) \\ &\Big(\leftrightarrow \quad [X/G] \times^h_{[\mathfrak{g}^*/G]} [\mathrm{pt}/G]\Big) \end{aligned} \tag{3.1.2}$$

as a derived model for the algebra of functions on the reduced space $\mu^{-1}(0) /\!\!/ G$.

3.1.1 Summary of results

In the noncommutative context we consider a dg algebra $A \in \mathtt{DGA}_S$ over S, a finite dimensional algebra of orthogonal idempotents $S = kI$ (path algebra of a quiver with vertex set I and no arrows). When we consider representations over S in a vector space V we need to specify a dimension vector $\underline{n} \in \mathbb{N}^I$ (of total dimension $= \dim V$) or in other words fix the representation $\rho_{\underline{n}} : S \to \mathrm{End}(V)$ sending the i-th orthogonal idempotent to the corresponding one according to the decomposition given by the dimension vector $\underline{n}$. The representation functor has the form:

$$\begin{aligned} (-)_{\underline{n}} : \mathtt{DGA}_S &\to \mathtt{CDGA}_k \\ A &\longmapsto A_{\underline{n}} = \mathcal{O}\mathrm{Rep}_{\underline{n}}(A) \,, \end{aligned} \tag{3.1.3}$$

where $\mathrm{Rep}_{\underline{n}}(A)$ denotes the scheme of representations of A that agree with $\rho_{\underline{n}}$ through the structure map $S \to A$. When A is equipped with a double

Poisson algebra structure, there is an induced Poisson structure on $A_{\underline{n}}$ defined on its generators by:

$$\{a_{ij}, b_{kl}\} = \{\!\{a, b\}\!\}'_{kj}\{\!\{a, b\}\!\}''_{il}\,. \tag{3.1.4}$$

In other words the representation functor enriches to a functor between the categories of (dg) double Poisson algebras, and commutative (dg) Poisson algebras:

$$(-)_{\underline{n}} : \mathtt{DGPPA}_S \to \mathtt{CDGPA}_k\,. \tag{3.1.5}$$

Let now $A \in \mathtt{PPAlg}_S$ be a (ungraded) double Poisson algebra. The representation scheme $\mathrm{Rep}_{\underline{n}}(A)$ comes with a natural action $G_S \curvearrowright \mathrm{Rep}_{\underline{n}}(A)$ of the gauge group of S-preserving automorphisms $G_S \subset \mathrm{GL}(V)$, which in this case is just a product of general linear groups $\mathrm{GL}_{\underline{n}} := \prod_i \mathrm{GL}_{n_i}$. The corresponding Lie algebra $\mathfrak{g} = \mathfrak{gl}_{\underline{n}}$ can be obtained as the representation scheme of the path algebra of a quiver with vertex set I and one loop t_i on each vertex, $T_S(L)$ (L is the linear span of the loops). As it turns out there is a natural double Poisson structure on $T_S(L)$ such that the induced Poisson structure on

$$(T_S(L))_{\underline{n}} = \mathcal{O}(\mathfrak{gl}_{\underline{n}}) = \mathrm{Sym}(\mathfrak{gl}^*_{\underline{n}}) \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \tag{3.1.6}$$

is the standard extension of the Lie algebra structure on $\mathfrak{gl}_{\underline{n}}$ (where in the last identification we used the canonical isomorphism induced by the trace). Once we know this, we see that the action $\mathrm{GL}_{\underline{n}} \curvearrowright \mathrm{Rep}_{\underline{n}}(A)$ is Hamiltonian exactly when A comes with a morphism of double Poisson algebras $T_S(L) \to A$, with the additional property that the image of a loop $t_i \mapsto \delta_i$ has Poisson bracket with any $a \in A$:

$$\{\!\{\delta_i, a\}\!\} = ae_i \otimes e_i - e_i \otimes e_i a\,, \tag{3.1.7}$$

(the i-th component of the universal derivation). In other words it is possible to define of a category of noncommutative Hamiltonian spaces as a full subcategory (objects with structure map having the property (3.1.7)) of the under category $\mathtt{PPAlg}^{H}_{T_S(L)} \subset \mathtt{DGPPA}_{T_S(L)}$ in such a way that:

Theorem (§3.4.2)**.** *The representation functor enriches to a functor between the category of noncommutative Hamiltonian spaces and commutative Hamiltonian*

$\mathfrak{gl}_{\underline{n}}$*-spaces:*

$$\begin{tikzcd} \mathtt{PPAlg}^{H}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] \arrow[d] & \mathtt{CPAlg}^{H}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \arrow[d] \\ \mathtt{DGA}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] & \mathtt{CDGA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \end{tikzcd} \tag{3.1.8}$$

where the vertical functors forget the Poisson structures and view the algebras as dg algebras placed in degree zero. Moreover the representation functor at the level $(-)_{\underline{n}} : \mathtt{DGA}_{T_S(L)} \to \mathtt{CDGA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})}$ *is cocontinuous (preserves small colimits), so in particular it preserves coproducts.*

Once this is clear it is natural to give the necessary definitions and a procedure to do noncommutative derived Poisson reduction, simply by substituting the constructions in (3.1.1) and (3.1.2) by the corresponding noncommutative ones.

The noncommutative analogue of the zero locus of the Poisson moment map is the coproduct over $T_S(L)$ of the Hamiltonian algebra $T_S(L) \to A$ and S (viewed as a $T_S(L)$-algebra via the standard projection that sends L to zero):

$$A/\langle L\rangle = A \amalg_{T_S(L)} S \overset{(-)_{\underline{n}}}{\longmapsto} A_{\underline{n}} \otimes_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} k\,. \tag{3.1.9}$$

We can therefore define a noncommutative derived zero locus substituting the coproduct with the derived coproduct, in such a way that we recover the classical derived zero locus:

Theorem (§3.4.2)**.** *The following model of noncommutative derived zero locus corresponds, under the representation functor, to the classical derived zero locus - the Koszul complex:*

$$A \amalg^{\mathbf{L}}_{T_S(L)} S \cong A \amalg_S T_S(L[1]) \overset{(-)_{\underline{n}}}{\longmapsto} A_{\underline{n}} \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} k \cong A_{\underline{n}} \otimes_k \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1])\,. \tag{3.1.10}$$

We denote this specific model for the noncommutative derived zero locus by $\mathrm{Sh}(A) = A \amalg_S T_S(L[1])$, because it is some sort of a generalised 'Shafarevich complex'. The next step for doing Poisson reduction is a non-commutative Chevalley-Eilenberg construction, which we define to be the following coproduct (with twisted differential, as in the commutative case — details in §3.3.4):

$$\begin{aligned} \mathrm{CE} : \mathtt{DGPPA}_{\mathrm{T_S(L)}} &\to \mathtt{DGA}_{\mathrm{T_S(L \oplus L^*[-1])}} \\ A &\longmapsto A \amalg_{\mathrm{T_S(L)}} \mathrm{T_S}(\mathrm{L} \oplus \mathrm{L}^*[-1])\,. \end{aligned} \tag{3.1.11}$$

The reader who is wondering why we momentaneously forget the Poisson structure is encouraged to read the details of this construction in §3.3.4, especially Remark 3.3.4.1. When we start from a noncommutative Hamiltonian space $A \in \mathtt{PPAlg}^{\mathrm{H}}_{\mathrm{T_S(L)}}$ and apply the Chevalley-Eilenberg construction to the Shafarevich complex we obtain a noncommutative version of the BRST complex:

$$\mathrm{BRST}(A) := \mathrm{CE}(\mathrm{Sh}(A)) \cong A \amalg_{\mathrm{S}} \mathrm{T_S}(\mathrm{L}[1] \oplus \mathrm{L}^*[-1])\,, \tag{3.1.12}$$

now equipped with the natural double Poisson structure which comes from A and the natural pairing between L, L^*. We obtain the following result:

Theorem (§3.4.3)**.** *There is a commutative diagram between the noncommutative and the commutative BRST construction:*

$$\begin{tikzcd} \mathtt{PPAlg}^{\mathrm{H}}_{\mathrm{T_S(L)}} \arrow[r, "(-)_{\underline{\mathfrak{n}}}"] \arrow[d, "\mathrm{BRST}"] & \mathtt{CPAlg}^{\mathrm{H}}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{\mathfrak{n}}})} \arrow[d, "\mathrm{BRST}"] \\ \mathtt{DGPPA}_{\mathrm{T_S(L \oplus L[1] \oplus L^*[-1])}} \arrow[r, "(-)_{\underline{\mathfrak{n}}}"] & \mathtt{CDGPA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{\mathfrak{n}}} \oplus \mathfrak{gl}_{\underline{\mathfrak{n}}}[1] \oplus \mathfrak{gl}^*_{\underline{\mathfrak{n}}}[-1])} \end{tikzcd} \tag{3.1.13}$$

We conclude this introduction by providing a ‘dictionary’, a summary of the above-mentioned noncommutative constructions and their commutative counterparts.

Table 3.1: Dictionary between noncommutative and commutative geometry ($\mathfrak{g} = \mathfrak{gl}_{\underline{n}}$ in the table).

'Dictionary'	Noncommutative geometry$^{(\mathrm{op})}$	Commutative geometry$^{(\mathrm{op})}$
Base scheme	S	k
Derived affine schemes	$\mathtt{DGA}_S$	$\mathtt{CDGA}_k$
Derived Poisson schemes	$\mathtt{DGPPA}_S$	$\mathtt{CDGPA}_k$
Gauge algebra	$T_S(L)$	$\mathrm{Sym}(\mathfrak{g})$
Hamiltonian spaces	$\mathtt{PPAlg}^{H}_{T_S(L)} \subset \mathtt{DGPPA}_{T_S(L)}$	$\mathtt{CPAlg}^{H}_{\mathrm{Sym}(\mathfrak{g})} \subset \mathtt{CDGPA}_{\mathrm{Sym}(\mathfrak{g})}$
Derived zero locus (Koszul complex)	$A \amalg^{\mathbf{L}}_{T_S(L)} S \cong A \amalg_S T_S(L[1])$	$B \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{g})} k \cong B \otimes_k \mathrm{Sym}(\mathfrak{g}[1])$
Quotient stack (Chevalley-Eilenberg complex)	$T_S(L^*[-1]) \amalg_S -$	$\mathrm{Hom}_k(\mathrm{Sym}(\mathfrak{g}[1]), -)$
Derived Poisson reduction (BRST complex)	$T_S(L^*[-1]) \amalg_S \left(A \amalg^{\mathbf{L}}_{T_S(L)} S\right)$	$\mathrm{Hom}_k \left(\mathrm{Sym}(\mathfrak{g}[1]), B \otimes^{\mathbf{L}}_{\mathrm{Sym}(\mathfrak{g})} k\right)$

After the theoretical part, we show the details of noncommutative derived Poisson reduction for some concrete well-known algebras such as cotangent bundles of smooth algebras, and in particular path algebras of doubled quivers, obtaining a BRST model for Nakajima-type quiver varieties.

We give a proof of the somewhat classical result that the (commutative) BRST homology is the tensor product of the $GL_{\underline{n}}$-invariant homology of the corresponding Koszul complex with the Lie algebra (co)homology of $\mathfrak{gl}_{\underline{n}}$:

Theorem (§3.5.1)**.** *Let* A *be a noncommutative Hamiltonian space and, for a fixed dimension* $\underline{n}$*, let* $\mathcal{B}_{\underline{n}}(A), \mathcal{K}_{\underline{n}}(A)$ *the associated (commutative) BRST and Koszul complexes, respectively. Then we have*

$$H_\bullet(\mathcal{B}_{\underline{n}}(A)) \cong H_\bullet(\mathcal{K}_{\underline{n}}(A))^{GL_{\underline{n}}} \otimes_k H^{-\bullet}(\mathfrak{gl}_{\underline{n}}, k)\,. \tag{3.1.14}$$

Then we show a couple of examples of path algebras of quivers such as the quiver with one vertex and g loops, corresponding to a Lie algebra version of the character variety of a Riemann surface of genus g, and in particular the commuting scheme for $g = 1$. Finally we pick two more examples different from path algebras of a quiver, which correspond to a Lie group-Lie algebra and a Lie group-Lie group version of the commuting scheme (which would be the case Lie algebra-Lie algebra).

3.1.2 Layout of the paper and instructions for the reader

§3.2 explains the theory of double Poisson algebras (as introduced by M. Van den Bergh), with a particular emphasis on the differential graded case and a categorical meaning of these structures. In §3.3 we formalise the constructions contained in the noncommutative side of the 'dictionary' (Figure 3.1) together with a few structural results that these definitions are indeed well-posed and well-behaved. In §3.4 we prove our main results, showing in which sense the noncommutative side of the 'dictionary' corresponds to the commutative side. In §3.5 we discuss the large class of examples of cotangent bundles, in particular path algebras of doubled quivers (together with a computation of the commutative BRST homology in this context) and various versions of the commuting scheme. In the last, short §3.6 we introduce the notions of noncommutative analogues of more general group schemes, group actions and Poisson-group schemes, which are a possible direction in which we can generalise the results of the paper.

Notations and conventions

$\mathtt{k}$ denotes an algebraically closed field of characteristic zero. We denote categories by the standard monospace font: $\mathtt{Vect}_\mathtt{k}$, $\mathtt{Alg}_\mathtt{k}$, $\mathtt{DGA}_\mathtt{k}$, ... We always work in chain complexes, so for us a differential graded object (algebra, vector space, ...) has differential of degree -1, differential graded is often shortened by "dg", and commutative differential graded by "cdg". For a category $\mathtt{C}$ and an object $S \in \mathtt{C}$ we denote by $S \downarrow \mathtt{C}$ the under category (in the case of dg algebras we denote this also by $S \downarrow \mathtt{DGA}_\mathtt{k} = \mathtt{DGA}_S$). The coproduct in the under category $S \downarrow \mathtt{C}$ is the push-out in $\mathtt{C}$ of diagrams $\bullet \leftarrow S \rightarrow \bullet$, and denoted by $- \amalg_S -$. Left and right derived functors of a functor between model category $\mathtt{F} : \mathtt{C} \rightarrow \mathtt{D}$, when they exist, are denoted by $\mathbf{L}(\mathtt{F}), \mathbb{R}(\mathtt{F})$ and by them we mean the total left/right derived functors between the homotopy categories. In the case of schemes, we denote the derived pull-back also by the more traditional symbol: $X \times^{\mathbf{R}}_Z Y = X \times^h_Z Y$ ('h' stands for 'homotopy' pull-back).

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3.2 Double Poisson algebras

In [68] M. Van den Bergh introduced double Poisson brackets as the main candidates for noncommutative Poisson structures according to the Kontsevich-Rosenberg principle (indeed if one wants a Poisson bracket on representation schemes, needs a bracket with values in $A \otimes A$). In this Section we recall the main definitions and results from M. Van den Bergh in order to set up the notation and adapt them slightly to better suit our purposes. Mainly we discuss the differential graded version of double Poisson brackets (which is already sketched in [68], and studied in [27] in relation to cyclic A_∞-algebras) and by doing so we consider the category of dg double Poisson algebras, which is the natural category in which we should do derived Poisson reduction. Finally we introduce a special class of dg double Poisson algebras whose differential is given by the induced single Poisson bracket with a distinguished (double) Maurer-Cartan element which we call "noncommutative charge", because the induced differential on representation schemes is obtained as the Poisson bracket by the trace of this element. The expert reader can skip this Section entirely, or just come back when some notion from this Section is used in the following part of the paper.

3.2.1 Graded objects

Let $\mathtt{DGA}_k$ be the category of differential graded algebras over k (the differentials have degree -1 in our conventions). We recall that for a differential graded algebra A, the tensor product $A \otimes A$ has two natural graded bimodule structures over A:

$$\begin{aligned}
&\text{(outer)} \quad a \cdot (u \otimes v) \cdot b := au \otimes vb \,, \\
&\text{(inner)} \quad a * (u \otimes v) * b := (-1)^{|a||b|+|a||u|+|b||v|} ub \otimes av \,.
\end{aligned}$$

The two structures commute with each other (with a sign):

$$\begin{cases} a \cdot (a' * (u \otimes v) * b') \cdot b = (-1)^{|a||a'|+|b||b'|} a' * (a \cdot (u \otimes v) \cdot b) * b' \,, \\ a' * (a \cdot (u \otimes v) \cdot b) * b' = (-1)^{|a||a'|+|b||b'|} a \cdot (a' * (u \otimes v) * b') \cdot b \,. \end{cases} \tag{3.2.1}$$

For each $n = 1, 2, \dots$ and each permutation $\sigma \in \Sigma_n$ we denote by $\tau_\sigma : A^{\otimes n} \to A^{\otimes n}$ the isomorphism:

$$\tau_\sigma(a_1 \otimes \cdots \otimes a_n) = (-1)^s a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)} \,,$$

where s is the sign that counts all the swappings involved in σ:

$$s = \sum_{\substack{i<j: \\ \sigma^{-1}(i)>\sigma^{-1}(j)}} |a_{\sigma^{-1}(i)}||a_{\sigma^{-1}(j)}| \,.$$

The permutation $(-)^\circ := \tau_{(12)} : A^{\otimes 2} \to A^{\otimes 2}$ intertwines the two bimodule structure:

$$\begin{cases} (a \cdot (u \otimes v) \cdot b)^\circ = a * ((u \otimes v)^\circ) * b\,, \\ (a * (u \otimes v) * b)^\circ = a \cdot ((u \otimes v)^\circ) \cdot b\,. \end{cases} \tag{3.2.2}$$

For a graded object A (a dg algebra, a graded vector space, ...) and an integer $m \in \mathbb{Z}$ we denote its m-shifted object by $A[m]$:

$$(A[m])_i := A_{i-m}\,, \tag{3.2.3}$$

so that if A is concentrated in degree zero, then $A[m]$ is concentrated in degree m, and a homogeneous map $A \to A[m]$ is a map $A_i \to A_{i-m}$ (shifted of degree $-m$).

3.2.2 Multi-brackets on differential graded algebras

Definition 3.2.2.1. An n-*bracket* on a differential graded algebra $A \in \mathtt{DGA}_k$ is a map $\{\!\{-, \dots, -\}\!\} : A^{\otimes n} \to A^{\otimes n}$ (of degree 0) with the following properties:

1. (derivation) The map $\{\!\{a_1, \dots, a_{n-1}, -\}\!\} : A \to A^{\otimes n}$ is a graded derivation (for the outer bimodule structure on $A^{\otimes n}$) of degree $p := |a_1| + \cdots + |a_{n-1}|$, that is

$$\{\!\{a_1, \dots, a_{n-1}, bc\}\!\} = \{\!\{a_1, \dots, a_{n-1}, b\}\!\} \cdot c + (-1)^{|b|p} b \cdot \{\!\{a_1, \dots, a_{n-1}, c\}\!\}\,, \tag{3.2.4}$$

2. (cyclic invariance)

$$\{\!\{-, \dots, -\}\!\} = (-1)^{n+1} \tau_{(12\dots n)} \circ \{\!\{-, \dots, -\}\!\} \circ \tau^{-1}_{(12\dots n)}\,, \tag{3.2.5}$$

3. (compatibility between bracket and differential)

$$d \circ \{\!\{-, \dots, -\}\!\} = \{\!\{-, \dots, -\}\!\} \circ d\,. \tag{3.2.6}$$

Definition 3.2.2.2. If $A \in \mathtt{DGA}_S := S \downarrow \mathtt{DGA}_k$ is a dg algebra over S, then an n-bracket is called S-*linear* if it vanishes when its last argument is in the image of S under the structure map $S \to A$ (and consequently, by the cyclic invariance, if any argument is in the image of S).

Remark 3.2.2.1. There is a more general notion of $(-m)$-shifted n-bracket which is a bracket $\{\!\{-,\dots,-\}\!\} : A^{\otimes n} \to A^{\otimes n}[m]$ which satisfies the corresponding shifted properties analogous to (1),(2),(3) (see [27]). All the results in this Section hold also for m-shifted brackets, however, in this paper we do not need these structures, therefore we discuss only the 0-shifted (homogeneous) case, which shortens the length of the signs involved in the formulas.

In the particular cases $n = 2$ and $n = 3$ such a structure is called, respectively, a double or a triple bracket. A double bracket is a map that satisfies

1. $\{\!\{a, bc\}\!\} = \{\!\{a, b\}\!\} \cdot c + (-1)^{|a||b|} b \cdot \{\!\{a, c\}\!\}$,
2. $\{\!\{a, b\}\!\} = -(-1)^{|a||b|} \{\!\{b, a\}\!\}^{\circ}$,
3. $d\{\!\{a, b\}\!\} = \{\!\{da, b\}\!\} + (-1)^{|a|} \{\!\{a, db\}\!\}$,

and, because of (3.2.2), once property (1) is fixed, property (2) is equivalent to ask that the bracket is a (graded) derivation in the first argument, for the inner bimodule structure:

$$(2^*)\ \{\!\{ab, c\}\!\} = a * \{\!\{b, c\}\!\} + (-1)^{|b||c|} \{\!\{a, c\}\!\} * b\ .$$

Given a binary operation $\{\!\{-,-\}\!\}$ (which does not have to be necessarily a double bracket) we define the following operation $A^{\otimes 3} \to A^{\otimes 3}$:

$$\{\!\{a, u \otimes v\}\!\}_L := \{\!\{a, u\}\!\} \otimes v\,, \tag{3.2.7}$$

and using this we define the following triary operation

$$\begin{aligned} \{\!\{a, b, c\}\!\} &:= \{\!\{a, \{\!\{b, c\}\!\}\}\!\}_L + (-1)^{|a|(|b|+|c|)} \tau_{(123)} \{\!\{b, \{\!\{c, a\}\!\}\}\!\}_L \\ &+ (-1)^{|c|(|a|+|b|)} \tau_{(132)} \{\!\{c, \{\!\{a, b\}\!\}\}\!\}_L\,, \end{aligned} \tag{3.2.8}$$

or more abstractly

$$\{\!\{-,-,-\}\!\} = \sum_{i=0}^{2} \tau^{i}_{(123)} \circ \{\!\{-, \{\!\{-,-\}\!\}\}\!\}_L \circ \tau^{-i}_{(123)}\,, \tag{3.2.9}$$

which makes it clear that it is cyclically invariant. Moreover one can prove that if $\{\!\{-,-\}\!\}$ is a double bracket, then it is also a graded derivation in its last argument, so that:

Lemma 3.2.2.1 ([68]). *If $\{\!\{-,-\}\!\}$ is a double bracket then the associated triary operation $\{\!\{-,-,-\}\!\}$ is a triple bracket.*

Proof. Let us show the super-derivation property in its last argument. The same calculations of the ungraded version of this Lemma ([68, Proposition 2.3.1]), if we keep track of signs, yield for the three summands of $\{\!\{a,b,cc'\}\!\}$:

$$\underbrace{\{\!\{a,\{\!\{b,cc'\}\!\}\}\!\}_L}_{=:1} + \underbrace{(-1)^{s_1}\tau_{(123)}\{\!\{b,\{\!\{cc',a\}\!\}\}\!\}_L}_{=:2} + \underbrace{(-1)^{s_2}\tau_{(132)}\{\!\{cc',\{\!\{a,b\}\!\}\}\!\}_L}_{=:3},$$

where $s_1 = |a|(|b|+|c|+|c'|)$ and $s_2 = (|c|+|c'|)(|a|+|b|)$.

$$1 = \{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L \cdot c' + (-1)^{|b||c|}\{\!\{a,c\}\!\} \cdot \{\!\{b,c\}\!\} + (-1)^{(|a|+|b|)|c|}c \cdot \{\!\{a,\{\!\{b,c'\}\!\}\}\!\}_L,$$
$$2 = (-1)^{s_1+|b||c|}c \cdot \tau_{(123)}\{\!\{b,\{\!\{c',a\}\!\}\}\!\}_L +$$
$$(-1)^{s_1+|a||c'|}\tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L \cdot c' + -(-1)^{|b||c|}\{\!\{a,c\}\!\} \cdot \{\!\{b,c'\}\!\},$$
$$3 = (-1)^{s_2}c \cdot \tau_{(132)}\{\!\{c',\{\!\{a,b\}\!\}\}\!\}_L + (-1)^{|c|(|a|+|b|)}\tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L \cdot c',$$

where, in lines 1 and 2, by $(x \otimes y) \cdot (u \otimes v)$ we mean $x \otimes yu \otimes v$. Summing the three expressions we obtain

$$\{\!\{a,b,cc'\}\!\} = \{\!\{a,b,c\}\!\} \cdot c' + (-1)^{(|a|+|b|)|c|}c \cdot \{\!\{a,b,c'\}\!\}.$$

As for the compatibility between $\{\!\{-,-,-\}\!\}$ and the differential, this follows from the fact that both

$$\{\!\{-,\{\!\{-,-\}\!\}\}\!\}_L = (1_A \otimes \{\!\{-,-\}\!\}) \circ (\{\!\{-,-\}\!\} \otimes 1_A)$$

and τ_σ, for any permutation σ, commute with the differential. □

3.2.3 Double Poisson brackets

Definition 3.2.3.1 ([68]). A double bracket $\{\!\{-,-\}\!\}$ on a dg algebra $A \in \mathtt{DGA}_S$ is called a *double Poisson bracket* if the associated triple bracket is zero: $\{\!\{-,-,-\}\!\} = 0$. The identity $\{\!\{a,b,c\}\!\} = 0$ is called (graded) double Jacobi identity. The pair $(A, \{\!\{-,-\}\!\})$ is called a *differential graded double Poisson algebra*.

Definition 3.2.3.2. A *morphism* of differential graded double Poisson algebras is a morphism $\varphi : A \to B$ of dg algebras over S, such that the induced map $\varphi^{\otimes 2} : A^{\otimes 2} \to B^{\otimes 2}$ intertwines the two double Poisson structures. We denote the (so obtained) category of dg double Poisson algebras by $\mathtt{DGPPA}_S$.

Remark 3.2.3.1. Differently from $\mathtt{DGA}_S = S \downarrow \mathtt{DGA}_k$ which denotes the under category, the category of dg double Poisson algebras over S is not the under category $\mathtt{DGPPA}_S \neq S \downarrow \mathtt{DGPPA}_k$ (with S equipped with the zero double Poisson structure). In fact an object of the former has the property that the bracket with any element in the image of S vanishes, while for an object of the former the bracket vanishes a priori only if both variables belong to the image of S.

Notation. We denote the full subcategory of dg double Poisson algebras consisting of algebras concentrated in degree zero by $\mathtt{PPAlg}_S \subset \mathtt{DGPPA}_S$.

Let us denote the multiplication map by $m : A^{\otimes 2} \to A$ and, given a double bracket $\{\!\{-,-\}\!\}$, let us consider the associated single bracket

$$\{-,-\} : A^{\otimes 2} \xrightarrow{\{\!\{-,-\}\!\}} A^{\otimes 2} \xrightarrow{m} A\,.$$

Proposition 3.2.3.1 ([68]). *If $\{\!\{-,-\}\!\}$ is a double bracket then the following equation holds in $A^{\otimes 2}$:*

$$\begin{aligned} &\{a,\{\!\{b,c\}\!\}\} - \{\!\{\{a,b\},c\}\!\} - (-1)^{|a||b|}\{\!\{b,\{a,b\}\}\!\} = \\ &= (m\otimes 1)\{\!\{a,b,c\}\!\} - (-1)^{|a||b|}(1\otimes m)\{\!\{b,a,c\}\!\}\,, \end{aligned} \tag{3.2.10}$$

where $\{a,-\}$ acts on tensors $u\otimes v$ by $\{a,u\otimes v\} = \{a,u\}\otimes v + (-1)^{|a||u|}u\otimes\{a,v\}$.

Proof. With a few intermediate calculations one proves that

$$\begin{aligned} \{a,\{\!\{b,c\}\!\}\} &= (m\otimes 1)\{\!\{a,\{\!\{b,c\}\!\}\}\!\}_L + (-1)^{|b||c|}(1\otimes m)\tau_{(123)}\{\!\{a,\{\!\{c,b\}\!\}\}\!\}_L\,,\\ \{\!\{\{a,b\},c\}\!\} &= -(-1)^{|c|(|a|+|b|)}(m\otimes 1)\tau_{(132)}\{\!\{c,\{\!\{a,b\}\!\}\}\!\}_L + \\ &\qquad (-1)^{|a||b|+|b||c|+|a||c|}(1\otimes m)\tau_{(132)}\{\!\{c,\{\!\{b,a\}\!\}\}\!\}_L\,,\\ \{\!\{b,\{a,c\}\}\!\} &= -(-1)^{|a||c|}(m\otimes 1)\tau_{(123)}\{\!\{b,\{\!\{c,a\}\!\}\}\!\}_L + (1\otimes m)\{\!\{b,\{\!\{a,c\}\!\}\}\!\}_L\,, \end{aligned}$$

from which equation (3.2.10) follows. □

Let us denote by $[A, A] \subset A$ the linear subspace spanned by graded commutators and the quotient by $A_\natural = A/[A, A]$. For an element $a \in A$ we denote by $\overline{a} \in A_\natural$ its class modulo $[A, A]$.

Lemma 3.2.3.1 **([68]).** *Let $\{\!\{-,-\}\!\}$ be a double bracket. Then the associated single bracket $\{-,-\} : A^{\otimes 2} \to A$ has the following properties:*

1. $\{[A, A], -\} = 0$,
2. $\{a, -\}$ *is a graded derivation of degree* $|a|$,
3. $\overline{\{a, b\}} = -(-1)^{|a||b|}\overline{\{b, a\}}$,
4. $d \circ \{-,-\} = \{-,-\} \circ d$,
5. *If $\{\!\{-,-\}\!\}$ is a double Poisson bracket, then the following "Leibniz property" (a version of the Jacobi identity) holds in A:*

$$\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\}. \tag{3.2.11}$$

Proof. Using the super-derivation property in the first argument:

$$\begin{aligned}\{\!\{[a, b], c\}\!\} &= a * \{\!\{b, c\}\!\} - (-1)^{|a|(|b|+|c|)}\{\!\{b, c\}\!\} * a+\\ &+ (-1)^{|b||c|}(\{\!\{a, c\}\!\} * b - (-1)^{|b|(|a|+|c|)} b * \{\!\{a, c\}\!\}) ,\end{aligned}$$

and in general for $a \in A$ and $\omega \in A^{\otimes 2}$: $m(a * \omega) = (-1)^{|a||\omega|}m(\omega * a)$, from which (1) follows:

$$\{[a, b], c\} = m\{\!\{[a, b], c\}\!\} = 0 .$$

(2) is obvious. For (3) it is enough to observe that for any $\alpha \in A^{\otimes 2}$, $m(\alpha - \alpha^\circ) \in [A, A]$, and apply this to $\alpha = \{\!\{a, b\}\!\}$. (4) is:

$$d \circ \{-,-\} = d \circ m \circ \{\!\{-,-\}\!\} = m \circ d \circ \{\!\{-,-\}\!\} = m \circ \{\!\{-,-\}\!\} \circ d = \{-,-\} \circ d ,$$

where by the same symbol d we mean both the differential on A and the induced one on $A^{\otimes 2}$. (5) follows from applying the multiplication map to (3.2.10).

□

By (1) and (2) the map $\{-,-\}$ induces a well-defined map $\{-,-\}_\natural : A_\natural^{\otimes 2} \to A_\natural$ which, by properties (3) (4) and (5), makes $A_\natural$ into a differential graded Lie algebra $(A_\natural, \{-,-\}_\natural) \in \mathtt{DGLA}_k$:

Lemma 3.2.3.2. *If $(A, \{\!\{-,-\}\!\})$ is a double Poisson algebra, the induced bracket $\{-,-\}_\natural$ on $A_\natural$ makes it a differential graded Lie algebra.*

Proof. Indeed if we use antisymmetry of the induced bracket on $A_\natural$, the Leibniz identity (3.2.11) becomes the (graded) Jacobi identity in its usual form:

$$(-1)^{|a||c|}\{a,\{b,c\}_\natural\}_\natural + (-1)^{|a||b|}\{b,\{c,a\}_\natural\}_\natural + (-1)^{|b||c|}\{c,\{a,b\}_\natural\}_\natural = 0\,.$$

The compatibility between the differential and $\{-,-\}_\natural$ follows from the compatibility between the differential and $\{-,-\}$ and the projection $A \to A_\natural$. □

The bracket $\{-,-\}_\natural$ is slightly more than simply a (dg) Lie structure, in fact for each element $a \in A$, the map $\{\overline{a},-\}_\natural : A_\natural \to A_\natural$ is induced by a (graded) derivation (of degree $|a|$) $\partial_a = \{a,-\}$. This is what is called a (differential graded) H_0-Poisson structure on A:

Definition 3.2.3.3 ([20] — ungraded version)**.** A (differential graded) H_0-*Poisson structure* on a (differential graded) algebra $A \in \mathtt{DGA}_S$ is a (differential graded) Lie bracket $\{-,-\}_\natural$ on $A_\natural$ with the property that for each homogeneous element $a \in A$, the map $\{\overline{a},-\}_\natural$ is induced by a graded, S-linear derivation $\partial_a : A \to A$ of degree $|a|$. We call the pair $(A, \{-,-\}_\natural)$ an H_0-*Poisson algebra (over S)*.

Definition 3.2.3.4. A *morphism* of H_0-Poisson algebras is a morphism $\varphi : A \to B$ of dg algebras over S such that the induced map $\varphi_\natural : A_\natural \to B_\natural$ is a morphism of dg Lie algebras. We denote the (so obtained) category of dg H_0-Poisson algebras by $\mathtt{DGPA}_S$.

Lemma 3.2.3.3. *There is a natural forgetful functor $\mathtt{DGPPA}_S \to \mathtt{DGPA}_S$ which sends a double Poisson algebra $(A, \{\!\{-,-\}\!\})$ to the H_0-Poisson algebra $(A, \{-,-\}_\natural)$, where $\{-,-\}_\natural$ is the bracket on $A_\natural$ induced by the single bracket associated to $\{\!\{-,-\}\!\}$.*

Remark 3.2.3.2. H_0-Poisson structures were first introduced by W. Crawley-Boevey in [20]. They are the natural structure to consider if one wants an induced ordinary Poisson structure on the GL_n-invariant part of the representation scheme $A_n^{GL_n}$. In fact one proves that there is only one induced Poisson structure on $A_n^{GL_n}$ with the property that the trace map $\mathrm{tr} : A_\natural \to A_n^{GL_n}$ is a map of dg Lie algebras:

$$\mathrm{tr}\{a, b\} = \{\mathrm{tr}(a), \mathrm{tr}(b)\}. \tag{3.2.12}$$

The reason why such a 'single' noncommutative Poisson structure is enough if one wants a Poisson structure on the GL_n-invariant subalgebra is that the latter is generated by traces, therefore actually the Poisson structure depends only on 2 indices, and not 4. If one wants a Poisson structure on the whole A_n, is forced to consider double Poisson structures.

3.2.4 Building new double Poisson structures from old

Proposition 3.2.4.1 ([68])**.** *If* $(A, \{\!\{-,-\}\!\}_A)$ *and* $(B, \{\!\{-,-\}\!\}_B)$ *are double Poisson algebras over* S*, their free product* $A *_S B$ *(coproduct in the category of dg algebras over* S*) has an induced natural double Poisson structure over* S*, defined uniquely by the formulas:*

$$\{\!\{a, b\}\!\} := 0\,, \quad \{\!\{a, a'\}\!\} := \{\!\{a, a'\}\!\}_A\,, \quad \{\!\{b, b'\}\!\} := \{\!\{b, b'\}\!\}_B\,, \tag{3.2.13}$$

for each $a, a' \in A$ *and* $b, b' \in B$.

Proof. As the ungraded version of this Proposition ([68, Proposition 2.5.1]) the proof is left to the reader, because fairly easy. In fact the induced structure on the coproduct does not mix the two structures, therefore its properties (super-derivation, cyclic invariance, compatibility with the differential) follow from the corresponding properties of the double Poisson brackets on A and B. □

Proposition 3.2.4.2. *The free product construction in Proposition 3.2.4.1 is the coproduct in the category* $\mathtt{DGPPA}_S$*. The algebras* S *and* 0*, both with zero double Poisson structure, are respectively, the inital and final object in* $\mathtt{DGPPA}_S$*.*

Proof. Let $C \in \mathtt{DGPPA}_S$ and $\varphi : A \to C$, $\psi : B \to C$ morphisms of $\mathtt{DGPPA}_S$. Because morphisms of dg double Poisson are morphisms of dg algebras with additional properties, in particular we have a unique morphism of dg algebras $F : A *_S B \to C$ that extends φ and ψ. The fact that $F^{\otimes 2}$ commutes with the double Poisson bracket defined in (3.2.13) can be easily tested on the generators:

$$\begin{aligned}
&F^{\otimes 2}(\{\!\{a, b\}\!\}) = 0 = \{\!\{\varphi(a), \psi(b)\}\!\} = \{\!\{F(a), F(b)\}\!\},\\
&F^{\otimes 2}(\{\!\{a, a'\}\!\}) = \varphi^{\otimes 2}(\{\!\{a, a'\}\!\}_A) = \{\!\{\varphi(a), \varphi(a')\}\!\}_A = \{\!\{F(a), F(a')\}\!\},\\
&F^{\otimes 2}(\{\!\{b, b'\}\!\}) = \psi^{\otimes 2}(\{\!\{b, b'\}\!\}_B) = \{\!\{\psi(b), \psi(b')\}\!\}_B = \{\!\{F(b), F(b')\}\!\}.
\end{aligned}$$

The fact that S and 0 are the initial and final objects in $\mathtt{DGPPA}_S$ follows from the fact that they are such objects in $\mathtt{DGA}_S$ and that for any $A \in \mathtt{DGPPA}_S$ the initial and terminal map $S \to A \to 0$ are maps of double Poisson algebras (because of the assumption of S-linearity on the double Poisson bracket on A). □

Another construction that will be useful later is the particular case of differential graded algebras A with a double Poisson bracket and such that their differential is of the very special form

$$d = \{\gamma, -\}$$

where $\{-, -\}$ is the associated single bracket on A. If we start simply from a graded algebra with a double Poisson bracket without differential we give the following definition:

Definition 3.2.4.1. Let A be a graded algebra and $\{\!\{-, -\}\!\}$ a double Poisson bracket on it. We call an element γ of degree $|\gamma| = -1$ a *noncommutative charge* if the single bracket of it by itself lies in the (graded) commutators subspace: $\{\gamma, \gamma\} \in [A, A]$.

Proposition 3.2.4.3. *If γ is a noncommutative charge on $(A, \{\!\{-, -\}\!\})$, then $d = \{\gamma, -\}$ is a differential on A which is compatible with the double Poisson bracket, so it gives A the structure of a differential graded double Poisson algebra.*

Proof. Obviously d is a linear map of degree -1 satisfying $d(ab) = d(a)b + (-1)^{|a|}ad(b)$ (from Lemma 3.2.3.1, (2)). If we apply the Leibniz property

(Lemma 3.2.3.1, (5)) to $a = b = \gamma$ we obtain:

$$d^2 = \frac{1}{2}\{\{\gamma, \gamma\}, -\},$$

so that $d^2 = 0$ follows from $\{\gamma, \gamma\} \in [A, A]$ (because of Lemma 3.2.3.1, (1)). From Proposition 3.2.3.1 applied to $a = \gamma$ we have

$$d\{\!\{b, c\}\!\} = \{\!\{db, c\}\!\} + (-1)^{|b|}\{\!\{b, dc\}\!\},$$

which is the desired compatibility between the bracket and the differential.
□

3.3 Derived noncommutative Poisson reduction

The aim of this Section is to introduce the noncommutative constructions mentioned in the left side of the 'dictionary' (Figure 3.1), together with some proofs of a few structural results that these definitions are well-behaved and satisfy some good properties. The impatient reader who wants to know why we give such definitions is encouraged to jump from time to time from this Section to §3.4, which hopefully clarifies everything.

3.3.1 Crash course in noncommutative geometry

This first Section is a crash course in noncommutative geometry. We recall a few definitions and basic results in the theory, but we try to stick to the minimum required in order to understand the following parts of the paper. There is obviously much more to be said, and the interested reader can consult the more foundational references [21, 31, 32].

Throughout this Section we consider algebras $A \in \mathtt{Alg}_S$ for which the structure map $i : S \to A$ is injective. The space of noncommutative 1-forms, or Kähler differentials, is the A-bimodule $\Omega^1_S A := \ker(m)$, kernel of $m : A \otimes_S A \to A$, the multiplication map (over S). The universal (S-linear) derivation is the map $d \in \mathrm{Der}_S(A, \Omega^1_S A)$ defined by

$$d(a) = a \otimes 1 - 1 \otimes a\,. \tag{3.3.1}$$

The couple $(\Omega^1_S A, d)$ is a universal A-bimodule equipped with a derivation in the sense that for each A-bimodule M, there is a natural isomorphism

$$\mathrm{Der}_S(A, M) \xrightarrow{\sim} \mathrm{Hom}_{A-\mathtt{Bimod}}(\Omega^1_S A, M)\,, \quad \partial \mapsto \varphi_\partial\,, \tag{3.3.2}$$

where φ_∂ is the only morphism such that $\varphi_\partial \circ d = \partial$. Because $\Omega^1_S A$ is spanned by the elements adb, for $a, b \in A$, we have $\varphi_\partial(adb) = a\partial(b)$.

Let us consider the contravariant (duality):

$$(-)^\vee := \mathrm{Hom}_{A-\mathtt{Bimod}}(-, A \otimes A) : A - \mathtt{Bimod} \to A - \mathtt{Bimod}\,, \tag{3.3.3}$$

where we view $A \otimes A$ with the outer A-bimodule structure, but the inner structure survives and gives, for each $M \in A - \mathtt{Bimod}$, the above-claimed A-bimodule structure on $M^\vee$. In particular if we consider $M = \Omega^1_S A$ we obtain, by (3.3.2), a natural isomorphism between the noncommutative vector fields $\Theta^1_S A := (\Omega^1_S A)^\vee$ and the space of double (S-linear) derivations:

$$\mathbb{D}\mathrm{er}_S(A) := \mathrm{Der}_S(A, A \otimes A) \cong \mathrm{Hom}_{A-\mathtt{Bimod}}(\Omega^1_S A, A \otimes A) =: \Theta^1_S A \tag{3.3.4}$$

Definition 3.3.1.1. The *noncommutative cotangent bundle* of A is the tensor algebra of the A-bimodule of noncommutative vector fields, and it is denoted by

$$T^*A := T_A\left(\Theta^1_S A\right) = T_A\left(\mathbb{D}\mathrm{er}_S(A)\right)\,. \tag{3.3.5}$$

Remark 3.3.1.1. This could be considered either as graded algebra or as an ungraded algebra placed all in degree zero. In the first case if we apply the representation functor to it we obtain the shifted cotangent bundle of the representation scheme of A, and in the second case the ordinary (unshifted) cotangent bundle. In this paper we consider mainly the second version (the ungraded one), for which we reserve the symbol $T^*A(= T_A\mathbb{D}\mathrm{er}_S(A))$, while when we really want to specify the graded version, we write $T^*_{\mathrm{odd}}A(= T_A(\mathbb{D}\mathrm{er}_S(A)[1]))$.

The setting in which later we do noncommutative Poisson reduction is the case in which the algebra S is the vector space generated by a finite set $S = kI$, consisting of orthogonal idempotents: $e_i e_j = \delta_{ij} e_j$. In this case we can define the following distinguished double derivation $\Delta \in \mathbb{D}\mathrm{er}_S(A)$:

$$\Delta(a) = \sum_i (ae_i \otimes e_i - e_i \otimes e_i a)\,, \tag{3.3.6}$$

which is a preferred lifting of the universal derivation in the sense that $d = \pi \circ \Delta$, where $\pi : A \otimes A \to A \otimes_S A$ denotes the canonical projection. This distinguished double derivation can be divided into its components over I:

$$\Delta = \sum_i \Delta_i, \quad \Delta_i(a) = ae_i \otimes e_i - e_i \otimes e_i a\,. \tag{3.3.7}$$

3.3.2 Natural double Poisson structure on cotangent bundles

M. Van den Bergh introduced a natural (-1)-shifted (see Remark 3.2.2.1) graded double Poisson structure on $T^*_{\mathrm{odd}}A$ which is called double Schouten-Nijenhuis structure ([68, §3.2]). In this Section we recall his construction and we show that the same definition (on generators) actually gives (0-shifted) double Poisson structure on T^*A, which — with a minus sign — is the one we are interested in.

Throughout this Section we suppose that A is finitely generated (all the examples we want to consider in this paper are of this form). In this case there is a natural identification between triple derivations and the tensor product of double derivations with A itself:

$$\begin{aligned} \mathbb{D}\mathrm{er}_S(A) \otimes A &\xrightarrow{\sim} \mathrm{Der}_S(A, A^{\otimes 3}) \\ \partial \otimes a &\longmapsto \Psi_{\partial \otimes a}(b) = \partial(b)' \otimes a \otimes \partial(b)''\,, \end{aligned} \tag{3.3.8}$$

where we use the (sumless) Sweedler notation for $\partial(b) = \partial(b)' \otimes \partial(b)''$. This follows from the identification between derivations and bimodules (3.3.2) and the finitely generatedness condition. Because of the form of (3.3.8), if we start from a triple derivation $\Psi \in \mathrm{Der}_S(A, A^{\otimes 3})$ and we compose it with either $\tau_{(23)}$ or $\tau_{(12)}$ we obtain, respectively:

$$\begin{aligned} \tau_{(23)} \circ \Psi &\in \mathbb{D}\mathrm{er}_S(A) \otimes A\,, \\ \tau_{(12)} \circ \Psi &\in A \otimes \mathbb{D}\mathrm{er}_S(A)\,. \end{aligned} \tag{3.3.9}$$

Proposition 3.3.2.1 ([68, §3.2]). *Let* $\delta, \partial \in \mathbb{D}\mathrm{er}_S(A)$. *Then*

$$\begin{aligned} \{\!\{\delta, \partial\}\!\}^{\sim}_l &:= (\delta \otimes 1_A) \circ \partial - (1_A \otimes \partial) \circ \delta\,, \\ \{\!\{\delta, \partial\}\!\}^{\sim}_r &:= (1_A \otimes \delta) \circ \partial - (\partial \otimes 1_A) \circ \delta = -\{\!\{\partial, \delta\}\!\}^{\sim}_l\,, \end{aligned} \tag{3.3.10}$$

define elements of $\mathrm{Der}_S(A, A^{\otimes 3})$.

When we compose them with the above-mentioned permutations ((3.3.9)) we can view them as elements of

$$\begin{aligned} \{\!\{\delta, \partial\}\!\}_l &:= \tau_{(23)} \circ \{\!\{\delta, \partial\}\!\}_r^{\sim} \in \mathbb{D}\mathrm{er}_S(A) \otimes A \subset (T^*A)^{\otimes 2}, \\ \{\!\{\delta, \partial\}\!\}_r &:= \tau_{(12)} \circ \{\!\{\delta, \partial\}\!\}_r^{\sim} \in A \otimes \mathbb{D}\mathrm{er}_S(A) \subset (T^*A)^{\otimes 2}, \end{aligned} \tag{3.3.11}$$

and they are related one to the other by swapping the first and the second component: $\{\!\{\delta, \partial\}\!\}_r = -\{\!\{\partial, \delta\}\!\}_l^{\circ}$.

Theorem 3.3.2.1 ([68, Theorem 3.2.2])**.** *There is a natural* (-1)*-shifted graded double Poisson structure (over* S*) on the noncommutative cotangent bundle* $T^*_{\mathrm{odd}}A$ *of a finitely generated algebra* A*, called double Schouten-Nijenhuis structure. It is defined uniquely by the formulas:*

$$\{\!\{a, b\}\!\} = 0, \quad \{\!\{\delta, a\}\!\} = \delta(a), \quad \{\!\{\delta, \partial\}\!\} = \{\!\{\delta, \partial\}\!\}_l + \{\!\{\delta, \partial\}\!\}_r, \tag{3.3.12}$$

for $a, b \in A$, $\delta, \partial \in \mathbb{D}\mathrm{er}_S(A)$.

Proposition 3.3.2.2. *The same definitions on the generators give a* 0*-shifted double Poisson structure on the ungraded cotangent bundle* T^*A*, which — with a minus — we call the "natural" double Poisson structure:*

$$\{\!\{a, b\}\!\} = 0, \quad \{\!\{\delta, a\}\!\} = -\delta(a), \quad \{\!\{\delta, \partial\}\!\} = -\{\!\{\delta, \partial\}\!\}_l - \{\!\{\delta, \partial\}\!\}_r. \tag{3.3.13}$$

Proof. Let us consider the following general situation. Let $A \in \mathtt{Alg}_S$ and a bimodule $\mathbb{D} \in A-\mathtt{Bimod}$ consider its tensor algebra in the two following versions: odd $T_A(\mathbb{D}[1])$ and even $T_A\mathbb{D}$. A (-1)-shifted double Poisson bracket on $T_A(\mathbb{D}[1])$ is uniquely determined by giving:

$$\{\!\{a, b\}\!\} = 0, \quad \{\!\{\delta, a\}\!\} \in A^{\otimes 2}, \quad \{\!\{\delta, \partial\}\!\} \in \mathbb{D} \otimes A + A \otimes \mathbb{D},$$

for $a, b \in A$, $\delta, \partial \in \mathbb{D}$ with the following properties:

1. $\{\!\{\delta, -\}\!\} : A \to A^{\otimes 2}$ is a double (S-linear) derivation in the second argument and it is compatible with the A-bimodule structure in the first argument.

2. $\{\!\{\delta, \partial\}\!\} = -\{\!\{\partial, \delta\}\!\}^{\circ}$ for each $\delta, \partial \in \mathbb{D}$.

3. The double Jacobi identity is satisfied on the generators, and this can be tested only in the following two situations:

   (i) two elements of $\mathbb{D}$ and one of A:

$$\{\!\{\delta, \partial, a\}\!\} = \{\!\{\delta, \{\!\{\partial, a\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{\partial, \{\!\{a, \delta\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{a, \{\!\{\delta, \partial\}\!\}\}\!\}_L = 0\,.$$

   (ii) three elements of $\mathbb{D}$:

$$\{\!\{\delta, \partial, d\}\!\} = \{\!\{\delta, \{\!\{\partial, d\}\!\}\}\!\}_L + \tau_{(123)}\{\!\{\partial, \{\!\{d, \delta\}\!\}\}\!\}_L + \tau_{(132)}\{\!\{d, \{\!\{\delta, \partial\}\!\}\}\!\}_L = 0\,.$$

The crucial observation is that the signs involved in the properties to be tested on the generators are the same as in the ungraded case (essentially because we only need to test them on particular choices of elements of degree zero and one). Therefore the same definitions actually give a (0-shifted) double Poisson structure on $T_A\mathbb{D}$ which coincides with the (-1)-shifted on $T_A(\mathbb{D}[1])$ on the generators. Now we apply this in the case of $\mathbb{D} = \mathbb{D}\mathrm{er}_S(A)$ with the natural A-bimodule structures and definitions (3.3.12) which, by [68, Theorem 3.2.2], define a (-1)-shifted double Poisson structure on $T^*_{\mathrm{odd}}A$. □

3.3.3 Noncommutative Hamiltonian spaces

In this Section we restrict to the case $S = kI$, a finite dimensional algebra made of orthogonal idempotents. We want to define a noncommutative version of Hamiltonian spaces, and in order to do so we should first introduce a noncommutative version of the Lie algebra $\mathfrak{gl}_{\underline{n}}$.

We consider the path algebra of the quiver with vertices I and with simply one loop t_i on each vertex $i \in I$. It is the tensor algebra over S of the S-bimodule $L = \mathrm{Span}_k\{t_i\}$ (isomorphic to S itself, but obviously we call its basis elements with different names to distinguish them from the orthogonal idempotents $e_i \in I$ in the path algebra):

$$T_S(L) = \mathrm{PathAlg}_k \left(\begin{tikzcd} 1 \arrow[loop above, "t_1"] & 2 \arrow[loop above, "t_2"] & \cdots & {|I|} \arrow[loop above, "t_{|I|}"] \end{tikzcd} \right). \quad (3.3.14)$$

It has a natural double Poisson structure defined uniquely by

$$\{\!\{t_i, t_j\}\!\} = \delta_{ij}(t_i \otimes e_i - e_i \otimes t_i)\,. \quad (3.3.15)$$

The verification that (3.3.15) actually defines a double Poisson is straightforward.

As explained in the introduction the role of $T_S(L)$ in the noncommutative world is analogous to the one played by the Lie algebra of the gauge group $\mathfrak{gl}_{\underline{n}} = \mathrm{Lie}(GL_{\underline{n}})$ acting on representation schemes (in fact the standard Poisson structure on $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})$ is the one induced by the above-mentioned double Poisson structure). Any double Poisson algebra $A \in \mathtt{PPAlg}_S$ with a map $T_S(L) \to A$, has an induced map $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \to A_{\underline{n}}$ via the representation functor, which is a Poisson morphism guaranteed that the map $T_S(L) \to A$ is a map of double Poisson algebra. In order to obtain a Hamiltonian space $A_{\underline{n}}$, we need a compatibility condition between the $\mathfrak{gl}_{\underline{n}}$-action and the induced Poisson bracket on $A_{\underline{n}}$, and this is given by the condition (3.1.7).

Definition 3.3.3.1. Let $A \in \mathtt{PPAlg}_S$ a double Poisson algebra over S. A *noncommutative Hamiltonian action* on it is a morphism $f : T_S(L) \to A$ of double Poisson algebras, such that, for each $i \in I$ and for each $a \in A$:

$$\{\!\{f(t_i), a\}\!\} = ae_i \otimes e_i - e_i \otimes e_i a \,(= \Delta_i(a))\,. \quad (3.3.16)$$

We call such a double Poisson A equipped with a noncommutative Hamiltonian action a *noncommutative Hamiltonian space*.

Remark 3.3.3.1. This definition agrees with [68, Definition 2.6.4.] of *Hamiltonian algebra*. The map $T_S(L) \to A$ is also called a *noncommutative moment map*.

Definition 3.3.3.2. A *morphism of noncommutative Hamiltonian spaces* is just a morphism of double Poisson algebras $A \to B$ compatible with the structure maps $T_S(L) \to A, B$. We denote the category of noncommutative Hamiltonian spaces by $\mathtt{PPAlg}^{H}_{T_S(L)}$.

Remark 3.3.3.2. In other words the category of noncommutative Hamiltonian spaces can be seen as a full subcategory of the under category $\mathtt{PPAlg}^{H}_{T_S(L)} \subset \mathtt{DGPPA}_{T_S(L)} := T_S(L) \downarrow \mathtt{DGPPA}_S$ (with an abuse of notation with respect to the previous notation $\mathtt{DGPPA}_S$ which was *not* the under category of $\mathtt{DGPPA}_k$ with respect to S).

Notation. From now on we will often denote the images of the loops under a Hamiltonian action $T_S(L) \to A$ by the symbols:

$$t_i \longmapsto f(t_i) = \delta_i \in A\,, \tag{3.3.17}$$

where the choice of the letter (small) 'delta' is motivated by the fact that their Poisson brackets on A give the action of the (big) deltas, the components of the chosen lifting of the universal derivation:

$$\{\!\{\delta_i, a\}\!\} = ae_i \otimes e_i - e_i \otimes e_i a = \Delta_i(a)\,. \tag{3.3.18}$$

We also denote their sum by $\delta = \sum_i \delta_i$ and observe that a Hamiltonian action is nothing else but a choice of a diagonal element $\delta \in \oplus_i e_i A e_i$ such that

$$\{\!\{\delta, a\}\!\} = \sum_i ae_i \otimes e_i - e_i \otimes e_i a = \Delta(a)\,. \tag{3.3.19}$$

The elements $\{\delta_i\}$ play the role of *noncommutative gauge elements*.

If we consider the two-sided ideal generated by the image of the Hamiltonian action $\mathcal{I} = \langle \delta \rangle \subset A$, we obtain as a quotient the noncommutative analogue of the zero locus of the Hamiltonian map: $A/\mathcal{I}$. We can also view it as the coproduct $A/\mathcal{I} = A \amalg_{T_S(L)} S \in \mathtt{Alg}_{T_S(L)}$, where S is viewed as a $T_S(L)$-algebra through the obvious projection $T_S(L) \to S$ that sends L to zero.

Definition 3.3.3.3. The *noncommutative zero locus* of a Hamiltonian space $A \in \mathtt{PPAlg}^{H}_{T_S(L)}$ is the algebra $A/\mathcal{I} = A \amalg_{T_S(L)} S \in \mathtt{Alg}_{T_S(L)}$.

Remark 3.3.3.3 (Poisson structure on the zero locus)**.** As for the Poisson structure, the double Poisson structure on A does not descend on the zero locus, however it is easy to verify that the following formula defines an induced H_0-Poisson structure on $A/\mathcal{I}$:

$$\begin{aligned} (A/\mathcal{I})_\natural^{\otimes 2} &\xrightarrow{\{-,-\}_\natural} (A/\mathcal{I})_\natural \\ \overline{a+\mathcal{I}} \otimes \overline{b+\mathcal{I}} &\longmapsto \overline{\{a,b\}+\mathcal{I}} \end{aligned}$$

where $\overline{a+\mathcal{I}} \in (A/\mathcal{I})_\natural$ denotes the class modulo commutator of the element $a+\mathcal{I} \in A/\mathcal{I}$, and for $a, b \in A$, $\{a, b\} \in A$ is the single bracket associated to the double Poisson structure on A.

Following the construction done in the commutative setting we define a derived version of the zero locus by using the total left-derived functor of the coproduct (we refer the interested reader to Appendix C for explanations on the derived coproduct).

Definition 3.3.3.4. The *derived noncommutative zero locus* of a Hamiltonian space $A \in \mathtt{PPAlg}_{T_S(L)}$ is the (homotopy class of the) dg algebra

$$A \amalg^{\mathbf{L}}_{T_S(L)} S \in \mathtt{Ho}(\mathtt{DGA}_{T_S(L)})\,. \tag{3.3.20}$$

In what follows however, we will often consider a specific model for it, obtained by choosing the following cofibrant replacement of S in the category $\mathtt{DGA}_{T_S(L)}$ (see Appendix C on why we can replace only one and not both variables of the coproduct) as the Shafarevich complex:

$$T_S(L) \hookrightarrow T_S(L \oplus L[1]) \xrightarrow{\sim} S\,,$$

where the differential in $T_S(L \oplus L[1])$ is $d\vartheta_i = t_i$ ($\vartheta_i \in L[1]$ basis).

Lemma 3.3.3.1. *The projection map* $T_S(L \oplus L[1]) \to S$ *is a quasi-isomorphism.*

Proof. For any $s \in S$ we denote the corresponding elements in $L, L[1]$ by t_s, ϑ_s, respectively. We define a super derivation $h : T_S(L \oplus L[1])_\bullet \to T_S(L \oplus L[1])_{\bullet+1}$ on the generators of the algebra by:

$$h(s) = 0\,, \quad h(t_s) = \vartheta_s\,, \quad h(\vartheta_s) = 0\,.$$

Then h is an homotopy between the zero map and the 'length' map l:

$$dh + hd = l\,,$$

where on elementary monomial words $w \in T_S(L \oplus L[1])$, $l(w) = \#(w)w$ is the word itself multiplied by its length (in the natural grading give by the tensor algebra, so it is counting only the number of t and ϑ in the word w). The map l is an isomorphism in homological degrees $\geqslant 1$, and this proves that $H_{\geqslant 1}(T_S(L \oplus L[1])) = 0$. $\square$

Thus a model for the noncommutative zero locus is a sort of generalised Shafarevich complex

$$A \amalg^{\mathbf{L}}_{T_S(L)} S \cong A \amalg_{T_S(L)} T_S(L \oplus L[1]) \cong A \amalg_S T_S(L[1])\,, \quad d\vartheta_i = \delta_i\,. \tag{3.3.21}$$

Notation. We denote this complex by $\mathrm{Sh}(A) := A \amalg_S T_S(L[1])$.

Remark 3.3.3.4. The zero-th homology of the derived zero locus recovers the underived zero locus

$$H_0\,(\mathrm{Sh}(A)) \cong A \amalg_{T_S(L)} S\,.$$

If the higher homologies vanish we could say that the couple $(A, \mathcal{J})$ is a generalised *noncommutative complete intersection* (see [**EG**, 7] for the case of noncommutative complete intersection of the form $A = T_k V$ a tensor algebra). Suppose that this is true and, in addition, the structure map $S \hookrightarrow A$ is a cofibration. In this case the Shafarevich complex is a cofibrant resolution of the underived zero locus $A/\mathcal{J}$ in the category $\mathtt{DGA}_S$ and it can be used to compute the derived representation functor at the level $\mathbf{L}(-)_{\underline{n}} : \mathtt{Ho}(\mathtt{DGA}_S) \to \mathtt{Ho}(\mathtt{CDGA}_k)$,

$$\mathbf{L}(A/\mathcal{J})_{\underline{n}} \cong (A \amalg_S T_S(L[1]))_{\underline{n}} = A_{\underline{n}} \otimes_k \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1])\,. \tag{3.3.22}$$

We direct the interested reader to [7, 9] for details on the derived representation scheme and to § 2.3.4 and § 2.3.5 for details on its relationship with the Koszul complex. In particular Theorem 2.3.5.2 shows (3.3.22) for partial preprojective algebras corresponding geometrically to Nakajima quiver varieties.

One particular class of examples of Hamiltonian spaces are cotangent bundles: $T^*A = T_A \mathbb{D}er_S(A)$. In (3.3.6) we defined the distinguished double derivation $\Delta \in \mathbb{D}er_S(A)$ which is our preferred lifting of the universal derivation $d : A \to \Omega^1_S A$. The elements obtained by decomposing it as a direct sum of its I-graded components (3.3.7):

$$\Delta_i = e_i \Delta e_i \in T^*A\,, \tag{3.3.23}$$

are the natural candidate as gauge elements (with a minus sign):

Lemma 3.3.3.2. *With the double Poisson structure on* T^*A *given in Proposition 3.3.2.2, the map*

$$\begin{aligned} T_S(L) &\to T^*A \\ t_i &\longmapsto \delta_i := -\Delta_i \end{aligned} \tag{3.3.24}$$

is a noncommutative Hamiltonian action.

Proof. We need to show that for any $i \in I$ and any $\omega \in T^*A$, we have

$$\{\!\{\delta_i, \omega\}\!\} = \omega e_i \otimes e_i - e_i \otimes e_i \omega\,,$$

or equivalently, denoting by $\{\!\{-,-\}\!\}_{\mathrm{odd}}$ the double Schouten-Nijunhuis bracket on $T^*_{\mathrm{odd}}A$, that:

$$\{\!\{\Delta_i, \omega\}\!\}_{\mathrm{odd}} = \omega e_i \otimes e_i - e_i \otimes e_i \omega\,. \tag{3.3.25}$$

This can be tested only on the generators of T^*A, that is either elements $a \in A$ or double derivations $\partial \in \mathbb{D}er_S(A)$. For elements $a \in A$, equation (3.3.25) is exactly the definition of the double Poisson bracket on $T^*_{\mathrm{odd}}A$. So we only need to prove that

$$\{\!\{\Delta_i, \partial\}\!\}_{\mathrm{odd}} = \partial e_i \otimes e_i - e_i \otimes e_i \partial \in (T^*A)^{\otimes 2}\,.$$

This is the content of [68, Proposition 3.3.1]. □

3.3.4 Noncommutative Chevalley-Eilenberg and BRST

In this last Section we define a noncommutative version of the Chevalley-Eilenberg functor and BRST complex needed for the derived Poisson reduction.

For the moment we start simply from an object in the under category $A \in \mathtt{DGPPA}_{T_S(L)}$ (we do not require the action to be Hamiltonian) and define

Definition 3.3.4.1. The *noncommutative Chevalley-Eilenberg complex* is

$$\mathrm{CE}(A) := A \amalg_{T_S(L)} (T_S(L \oplus L^*[-1])) \cong A \amalg_S T_S(L^*[-1])\,, \tag{3.3.26}$$

equipped with the following twisted differential (on the generators $a \in A$ and $\eta_i \in L^*[-1]$):

$$\begin{cases} da = d_{\mathrm{old}}a - [\sum_i \eta_i, a]\,, \\ d\eta_i = -\eta_i^2 = -\frac{1}{2}[\eta_i, \eta_i]\,, \end{cases} \tag{3.3.27}$$

where d_{old} is the original differential on A.

In fact, we can write the differential on the Chevalley-Eilenberg complex as the sum of two super-commuting differentials (Lemma 3.3.4.1):

$$d = d_{\mathrm{old}} + d_{\mathrm{CE}}\,, \tag{3.3.28}$$

where d_{old} is the old differential on A extended to zero elsewhere and d_{CE} is the Chevalley-Eilenberg differential on $T_S(L^*[-1])$ ($d_{\mathrm{CE}}\eta_i = -\eta_i^2$) extended by commutators as indicated in (3.3.27) on A.

Lemma 3.3.4.1. *The two maps* d_{old}, d_{CE} *defined in* (3.3.28) *are indeed (super-commuting) differentials on* $\mathrm{CE}(A)$. *This proves that* $d = d_{\mathrm{old}} + d_{\mathrm{CE}}$ *is also a differential.*

Proof. It is obvious that $(d_{\mathrm{old}})^2 = 0$ because it is just the old differential, extended to zero elsewhere. For what concerns d_{CE}, let us call $\eta = \sum_i \eta_i$, and observe that on the generators $d_{\mathrm{CE}}^2 = 0$:

$$d_{\mathrm{CE}}^2(a) = d_{\mathrm{CE}}(-[\eta, a]) = [\eta^2, a] - [\eta, [\eta, a]] = \frac{1}{2}[[\eta, \eta], a] - [\eta, [\eta, a]] = 0\,,$$

$$d_{\mathrm{CE}}^2(\eta_i) = d_{\mathrm{CE}}(-\eta_i^2) = \eta_i\eta_i^2 - \eta_i^2\eta_i = 0\,.$$

Moreover the two differentials super-commute with each other:

$$(d_{\mathrm{old}}d_{\mathrm{CE}} + d_{\mathrm{CE}}d_{\mathrm{old}})(a) = d_{\mathrm{old}}(-[\eta, a]) - [\eta, d_{\mathrm{old}}a] = [\eta, d_{\mathrm{old}}a] - [\eta, d_{\mathrm{old}}a] = 0\,,$$

$$(d_{\mathrm{old}}d_{\mathrm{CE}} + d_{\mathrm{CE}}d_{\mathrm{old}})(\eta_i) = d_{\mathrm{old}}(-\eta_i^2) = 0\,.$$

If follows that also $d = d_{\mathrm{old}} + d_{\mathrm{CE}}$ is a differential:

$$d^2 = d_{\mathrm{old}}^2 + d_{\mathrm{CE}}^2 + (d_{\mathrm{old}}d_{\mathrm{CE}} + d_{\mathrm{CE}}d_{\mathrm{old}}) = 0\,.$$

□

For the moment we could view this Chevalley-Eilenberg construction as a functor:

$$\mathrm{CE} : \mathtt{DGPPA}_{T_S(L)} \to \mathtt{DGA}_{T_S(L \oplus L^*[-1])}\,, \tag{3.3.29}$$

where we momentarily forget the double Poisson structure.

Remark 3.3.4.1. In the commutative case, the Chevalley-Eilenberg complex $C(\mathfrak{g}, M)$ can be defined for any $\mathfrak{g}$-module M, without any requirement of a algebra structure on M, nor a Poisson structure on M compatible with the $\mathfrak{g}$ action. However we are interested only in the class of examples of $M = \mathcal{O}(X)$, where X is a Hamiltonian $\mathfrak{g}$-space, so there is a moment map $X \to \mathfrak{g}^*$ with dual map a Poisson morphism $\mathrm{Sym}(\mathfrak{g}) \to \mathcal{O}(X)$, so that the action of $x \in \mathfrak{g} \curvearrowright M$ is the Poisson bracket in M with the element $x \in \mathfrak{g} \subset \mathrm{Sym}(\mathfrak{g}) \to M$.

Analogously, in the noncommutative case we can define a more general Chevalley-Eilenberg complex for objects A which do not have necessarily a algebra structure, nor a double Poisson structure (essentially only a $T_S(L)$-bimodule structure), but we are only interested in this case for derived Poisson reduction.

When we start from a noncommutative Hamiltonian space $A \in \mathtt{PPAlg}^{H}_{T_S(L)}$ and we apply the Chevalley-Eilenberg construction to the Shafarevich model for the derived zero locus (3.3.21) we obtain a noncommutative version of the BRST complex:

$$\mathrm{CE}(\mathrm{Sh}(A)) \cong A \amalg_S T_S(L[1] \oplus L^*[-1])\,. \tag{3.3.30}$$

which now we can equip with a natural double Poisson structure. It is the free product of the double Poisson structure on A together with the obvious one on $T_S(L[1] \oplus L^*[-1])$ (with only non-trivial double brackets between generators $\{\!\{\vartheta_i, \eta_j\}\!\} = \delta_{ij} e_i \otimes e_i$).

Definition 3.3.4.2. We define the *noncommutative BRST construction* as the functor:

$$\begin{aligned} \mathrm{BRST} : \mathtt{PPAlg}^{H}_{T_S(L)} &\to \mathtt{DGPPA}_{T_S(L \oplus L[1] \oplus L^*[-1])} \\ A &\longmapsto A \amalg_S T_S(L[1] \oplus L^*[-1])\,. \end{aligned} \tag{3.3.31}$$

Remark 3.3.4.2. Analogously to the commutative case, the differential on $\mathrm{BRST}(A)$ is induced by the following noncommutative charge (in the sense of Definition 3.2.4.1)

$$\gamma = \underbrace{\sum_i \eta_i \delta_i}_{\gamma_{\mathrm{Ham}}} - \underbrace{\sum_i \eta_i^2 \vartheta_i}_{\gamma_{\mathrm{CE}}} . \tag{3.3.32}$$

where the term $\gamma_{\mathrm{Ham}} \in L^*[-1] \otimes A$ represents the Hamiltonian action, or Shafarevich differential $L[1] \to A$, and the term $\gamma_{\mathrm{CE}} \in L^*[-1]^{\otimes 2} \otimes L[1]$ represents the multiplication map $S \otimes S \to S$ (once we identify $L \cong S$ and shift the degrees). Indeed one can easily verify that $\{\gamma, \gamma\}$ lies in the commutators subspace and that the differential is obtained by taking the single Poisson bracket with γ:

$$d_{\mathrm{BRST}} = \{\gamma, -\}. \tag{3.3.33}$$

We conclude this Section with a remark that there are natural maps of dg algebras linking the objects involved in the 2-step derived noncommutative Poisson reduction (from the zero locus to the derived zero locus to the BRST complex):

$$\mathrm{BRST}(A) \xrightarrow{\mathrm{ev}_{\eta=0}} \mathrm{Sh}(A) \twoheadrightarrow A/\mathcal{I}, \tag{3.3.34}$$

where the map $\mathrm{ev}_{\eta=0}$ is the map sending all the Chevalley-Eilenberg generators $\eta_i \to 0$, and the map $\mathrm{Sh}(A) \twoheadrightarrow A/\mathcal{I}$ is the map sending first the Shafarevich generators $\vartheta_i \to 0$ and then taking the quotient by the two-sided ideal $\mathcal{I}$ generated by their differentials ($d\vartheta_i = \delta_i$). The fact that the map $\mathrm{ev}_{\eta=0}$ is a map of dg algebras follows essentially from the fact that once we set $\eta_i = 0$ the Chevalley-Eilenberg differential becomes just the old differential on $\mathrm{Sh}(A)$.

3.4 Representation schemes

In this section we want to show that when we apply the (opportune version of the) representation functor to the various objects that we defined in Section 3.2 and 3.3 we obtain the corresponding objects in the commutative world, therefore justifying our definitions (Hamiltonian spaces, derived zero loci, Chevalley-Eilenberg functor, ...) according to the general Kontsevich-Rosenberg principle ([41]). The ground algebra is always $S = kI$, a finite dimensional algebra of orthogonal idempotents. We fix a dimension vector $\underline{n} = (n_i)_{i\in I}$ and consider the representation $\rho_{\underline{n}} : S \to E_{\underline{n}} := \mathrm{End}(k^n)$, $n = \sum_i n_i$, corresponding to the S-bimodule structure

$$(E_{\underline{n}})_{ij} = \mathrm{Hom}_k(k^{n_i}, k^{n_j}) .$$

Notation. Because now we need another set of indices running from 1 to n, to avoid confusion we denote by $i, j \in I$, and by $r, s, u, v, (\dots) \in \{1, \dots, n\}$.

Given $A \in \mathtt{DGA}_S$, we denote by $\mathrm{Rep}_{\underline{n}}(A) \in \mathtt{DGAff}_k$ the differential graded affine scheme of S-representations of A in k^n, and by $A_{\underline{n}} = \mathcal{O}\mathrm{Rep}_{\underline{n}}(A) \in \mathtt{CDGA}_k$ its commutative graded algebra of global functions. There is a group scheme action on the representation scheme $\mathrm{Rep}_{\underline{n}}(A)$ by the group of automorphisms of k^n which preserves the representation $\rho_{\underline{n}} : S \to E_{\underline{n}}$, which in this case

$$G_S := \{g \in \mathrm{Aut}(k^n) \,|\, g\rho_{\underline{n}}(s) = \rho_{\underline{n}}(s)g \;\forall s \in S\} \cong \prod_{i\in I} \mathrm{GL}_{n_i} =: \mathrm{GL}_{\underline{n}} ,$$

is a product of general linear groups.

We recall that $A_{\underline{n}}$ is linearly spanned by elements a_{rs}, with $a \in A$, and indices $r, s = 1, \dots, n$, with relations:

$$(ab)_{rs} = \sum_{u=1}^{n} a_{ru} b_{us} , \quad (\lambda a + \mu b)_{rs} = \lambda a_{rs} + \mu b_{rs} , \quad (\lambda, \mu \in k) . \tag{3.4.1}$$

A more abstract way to view $A_{\underline{n}}$ is $A_{\underline{n}} \cong (\sqrt[\underline{n}]{A})_{\natural\natural}$ (see for example [7, 9], or § 2.2.1), where

$$\sqrt[\underline{n}]{A} = \left(E_{\underline{n}} *_S A\right)^{E_{\underline{n}}} = \{x \in E_{\underline{n}} *_S A \,|\, x \cdot e = e \cdot x \;\forall e \in E_{\underline{n}}\}$$

and $(\sqrt[\underline{n}]{A})_{\natural\natural} = (\sqrt[\underline{n}]{A})/\langle[\sqrt[\underline{n}]{A}, \sqrt[\underline{n}]{A}]\rangle$ is the commutative dg algebra obtained when taking the quotient by the two-sided ideal generated by (graded) commutators.

3.4.1 Representation schemes of double Poisson algebras

If A is a double Poisson algebra and we regard $E_{\underline{n}}$ as a double Poisson algebra with the zero bracket, we have an induced free product double Poisson structure on $E_{\underline{n}} *_S A$, which restricts to a double Poisson structure on $\sqrt[n]{A}$ (easy to verify), and descends to a well-defined double Poisson structure on $A_{\underline{n}}$ (see [68]). In terms of the generators $\{a_{rs}\}$:

Theorem 3.4.1.1 ([68]). *If $A \in \mathtt{DGPPA}_S$, the algebra of functions on its representation scheme $A_{\underline{n}}$ has a natural induced double Poisson bracket, which is defined on its generators by*

$$\{\!\{a_{rs}, b_{uv}\}\!\} = \{\!\{a, b\}\!\}'_{us} \otimes \{\!\{a, b\}\!\}''_{rv} . \tag{3.4.2}$$

Remark 3.4.1.1. Let us denote the trace map by:

$$\mathrm{tr} : A_\natural = A/[A, A] \to A_{\underline{n}}^{\mathrm{GL}_{\underline{n}}} . \tag{3.4.3}$$

In the special double Poisson algebras with differential $d = \{\gamma, -\}$ given by a noncommutative charge, the induced differential on $A_{\underline{n}}$ is obtained as the Poisson bracket with the trace of the charge: $d = \{\mathrm{tr}(\gamma), -\}$. This can be easily verified on the generators:

$$\begin{aligned} d(a_{rs}) &= (da)_{rs} = (\{\gamma, a\})_{rs} = (\{\!\{\gamma, a\}\!\}'\{\!\{\gamma, a\}\!\}'')_{rs} = \\ &= \sum_u \{\!\{\gamma, a\}\!\}'_{ru}\{\!\{\gamma, a\}\!\}''_{us} = \sum_u \{\gamma_{uu}, a_{rs}\} = \{\mathrm{tr}(\gamma), a_{rs}\} . \end{aligned}$$

The natural question after reading Theorem 3.4.1.1 is whether a morphism of double Poisson algebras induces a Poisson morphism between the induced structures on representation schemes, and the answer is yes:

Theorem 3.4.1.2. *If $f : A \to B$ is a morphism of double Poisson algebras, the induced map $f_{\underline{n}} : A_{\underline{n}} \to B_{\underline{n}}$ is a morphism of double Poisson algebras.*

Proof. It is straightforward to prove on the generators $a_{rs} \otimes a'_{uv} \in A_{\underline{n}}^{\otimes 2}$:

$$\begin{aligned} &f_{\underline{n}}^{\otimes 2}\left(\{\!\{a_{rs}, a'_{uv}\}\!\}\right) = f_{\underline{n}}^{\otimes 2}\left(\{\!\{a, a'\}\!\}'_{us} \otimes \{\!\{a, a'\}\!\}''_{rv}\right) = f(\{\!\{a, a'\}\!\}')_{us} \otimes f(\{\!\{a, a'\}\!\}'')_{rv} = \\ &\{\!\{f(a), f(a')\}\!\}')_{us} \otimes \{\!\{f(a), f(a')\}\!\}''_{rv} = \{\!\{f(a)_{rs}, f(a')_{uv}\}\!\} = \{\!\{f_{\underline{n}}(a_{rs}), f_{\underline{n}}(a'_{uv})\}\!\} . \end{aligned}$$

$\square$

Theorem 3.4.1.1 and 3.4.1.2 prove that the representation functor at the level of double Poisson algebras is indeed a functor. Because in the commutative world we are interested only in the single Poisson structure, we can view this 'Poisson representation functor' as a functor $(-)_{\underline{n}} : \mathtt{DGPPA}_S \to \mathtt{CDGPA}_k$, an enrichment of the classical (dg)-representation functor:

$$\begin{tikzcd} \mathtt{DGPPA}_S \arrow[r] \arrow[d] & \mathtt{CDGPA}_k \arrow[d] \\ \mathtt{DGA}_S \arrow[r] & \mathtt{CDGA}_k \end{tikzcd} \tag{3.4.4}$$

(the vertical arrows are the natural forgetful functors).

Remark 3.4.1.2. A natural question that arises in the mind of the reader who knows that the classical representation functor $(-)_{\underline{n}} : \mathtt{DGA}_S \to \mathtt{CDGA}_k$ has a total left derived functor $\mathbf{L}(-)_{\underline{n}} : \mathtt{Ho}(\mathtt{DGA}_S) \to \mathtt{Ho}(\mathtt{CDGA}_k)$ (see [9, 7] or § 2.2.2) is whether or not also the functor at the level of double Poisson algebras can be derived. As pointed out by Y. Berest, it is not known whether such categories of noncommutative Poisson algebras (double Poisson in our case, H_0-Poisson in the case of [6]) have a model structure compatible with the standard (projective) model structure on dg algebras. We can make the following observations

1. It is possible to define a "homotopy category" $\mathtt{Ho}^*(\mathtt{DGPPA}_S)$ with objects the double Poisson algebras which are also cofibrant as dg algebras over S, and morphisms the homotopy classes of morphisms (where the notion of homotopy is a double Poisson version of polynomial homotopy of dg algebras, as explained in [6, Remark 5.1.1]). With these definitions the representation functor at the level of double Poisson algebras preserves polynomial homotopies, and therefore admits a total left derived functor

$$\mathbf{L}(-)_{\underline{n}} : \mathtt{Ho}^*(\mathtt{DGPPA}_S) \to \mathtt{Ho}(\mathtt{CDGA}_k)\,,$$

which is computed as $\mathbf{L}(A)_{\underline{n}} = (QA)_{\underline{n}}$, where $S \hookrightarrow QA \overset{\sim}{\twoheadrightarrow} A$ is a cofibrant replacement in the category of dg algebras over S and the maps are all maps of double Poisson algebras.

2. Analogously to what suggested in [6] for H_0-Poisson algebras one could consider first the infinite-dimensional limit (to eliminate the dependence on the dimension vector $\underline{n}$):

$$(-)_\infty : \mathtt{DGA}_S \to \mathtt{CDGA}_k\,,$$

which has a total left derived functor $\mathbf{L}(-)_\infty$, and use the construction of *homotopy pull-back of model categories along functors with total left derived functors* (see [67]) — in this case along the forgetful functor $\mathtt{CDGPA}_k \to \mathtt{CDGA}_k$ — to obtain a model category $\mathtt{DGA}_S \times^h_{\mathtt{CDGA}_k} \mathtt{CDGPA}_k$, which because of the diagram (3.4.4) comes with a functor from double Poisson algebras:

$$\mathtt{DGPPA}_S \to \mathtt{DGA}_S \times^h_{\mathtt{CDGA}_k} \mathtt{CDGPA}_k\,,$$

which is homotopy invariant and conjecturally it induces an equivalence of categories: $\mathtt{Ho}^*(\mathtt{DGPPA}_S) \to \mathtt{Ho}(\mathtt{DGA}_k \times^h_{\mathtt{CDGA}_k} \mathtt{CDGPA}_k)$. We do not know whether this is true or not, and it could be part of future work.

3.4.2 Hamiltonian spaces

In this Section we show how we obtain classical Hamiltonian $\mathfrak{gl}_{\underline{n}}$-spaces by applying the representation functor to noncommutative Hamiltonian spaces defined in §3.3.3. Let us start by considering the algebra $T_S(L)$ from (3.3.14), a free product over S of copies of polynomials in 1 variable, with double Poisson structure:

$$\{\!\{t_i, t_j\}\!\} = \delta_{ij}(t_i \otimes e_i - e_i \otimes t_i)\,. \tag{3.4.5}$$

For some dimension vector $\underline{n}$, the representation scheme of $T_S(L)$ is just a product of general linear algebras, in fact:

$$\begin{aligned}\mathrm{Rep}_{\underline{n}}(T_S(L)) &\cong \prod_{i\in I} \mathrm{Rep}_{\underline{n}}(k[t_i]) \cong \prod_{i\in I} \mathrm{Hom}_{S-\mathtt{Bimod}}(k\cdot t_i, \mathfrak{gl}_n) \cong \\ &\cong \prod_{i\in I} \mathfrak{gl}_{n_i}(k) =: \mathfrak{gl}_{\underline{n}}\,.\end{aligned} \tag{3.4.6}$$

We identify its algebra of functions with $\mathcal{O}(\mathfrak{gl}_{\underline{n}}) = \mathrm{Sym}(\mathfrak{gl}^*_{\underline{n}}) \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}})$ using the trace and we can show that the Poisson bracket induced from the double Poisson bracket on $T_S(L)$ is the natural Poisson structure on $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})$:

Proposition 3.4.2.1. *The induced Poisson structure on* $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \cong T_S(L)_{\underline{n}}$ *is the natural extension of the Lie bracket.*

Proof. Because of the decomposition in (3.4.6) we can just prove the case with one vertex: $|I| = 1$. This case is explained in [68, Example 7.5.3.]. □

From now on we will make implicit use of the trace isomorphism and therefore identify $T_S(L)_{\underline{n}} \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}})$ without saying it explicitly again. Other examples of interest later are when we shift the S-bimodule L to either L[1] or L[−1]. In this case we obtain:

$$\begin{aligned} T_S(L[1])_{\underline{n}} &= \mathcal{O}(\mathfrak{gl}_{\underline{n}}[-1]) \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1])\,, \\ T_S(L[-1])_{\underline{n}} &= \mathcal{O}(\mathfrak{gl}_{\underline{n}}[1]) \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[-1])\,, \end{aligned} \tag{3.4.7}$$

where $\mathrm{Sym}(-) : \mathtt{DGVect}_k \to \mathtt{CDGA}_k$ is the graded commutative one, so it gives the symmetric algebra for even degrees and the antisymmetric algebra for odd degrees $\mathrm{Sym}(V) = S(V_{\mathrm{even}}) \otimes \Lambda(V_{\mathrm{odd}})$.

Now let us consider a noncommutative Hamiltonian space $A \in \mathtt{PPAlg}^{H}_{T_S(L)}$, whose structure map $T_S(L) \to A$ induces a $GL_{\underline{n}}$-equivariant map of Poisson algebras $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \to A_{\underline{n}}$. Because of the property (3.3.16) this corresponds to the infinitesimal action of $\mathfrak{gl}_{\underline{n}}$ coming from the natural action of $GL_{\underline{n}}$ by conjugation — this is essentially [68, Proposition 7.11.1] — so that $A_{\underline{n}}$ is a Hamiltonian $\mathfrak{gl}_{\underline{n}}$-space. Dually, the map of schemes $\mu_{\underline{n}} : \mathrm{Rep}_{\underline{n}}(A) \to \mathfrak{gl}^*_{\underline{n}}$ is a Poisson moment map for the canonical action of $GL_{\underline{n}} \curvearrowright \mathrm{Rep}_{\underline{n}}(A)$. In other words, we obtain the following enriched version of the diagram (3.4.4):

Theorem 3.4.2.1. *The representation functor enriches to a functor between noncommutative Hamiltonian spaces and commutative Hamiltonian spaces:*

$$\begin{tikzcd} \mathtt{PPAlg}^{H}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] \arrow[d] & \mathtt{CPAlg}^{H}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \arrow[d] \\ \mathtt{DGA}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] & \mathtt{CDGA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \end{tikzcd} \tag{3.4.8}$$

where the vertical arrows are the natural forgetful functors. Moreover, the representation functor $(-)_{\underline{n}} : \mathtt{DGA}_{T_S(L)} \to \mathtt{CDGA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})}$ *is cocontinuous, so in particular it preserves coproducts.*

Proof. The previous considerations prove the upper part of the diagram. As for the cocontinuity, it is a simple consequence of the fact that it is the "under category version" of the cocontinuous functor $(-)_{\underline{n}} : \mathtt{DGA}_S \to \mathtt{CDGA}_k$ and, the fact that colimits in the under category are computed in the original category. □

If we denote by $\mathcal{I}$ the two-sided ideal in A of the image of the Hamiltonian action $T_S(L) \to A$, it follows from the property of cocontinuity of the representation functor (in particular, coproducts) that the zero locus $A/\mathcal{I} = A \amalg_{T_S(L)} S$ corresponds to the classical zero locus of the induced moment map $\mu_{\underline{n}} : \mathrm{Rep}_{\underline{n}}(A) \to \mathfrak{gl}_{\underline{n}}^*$:

$$\begin{tikzcd} T_S(L) \arrow[r] \arrow[d] & S \arrow[d] \\ A \arrow[r] & A \amalg_{T_S(L)} S \end{tikzcd} \quad \xrightarrow{(-)_{\underline{n}}} \quad \begin{tikzcd} \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \arrow[r] \arrow[d] & k \arrow[d] \\ A_{\underline{n}} \arrow[r] & A_{\underline{n}} \otimes_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} k = \mathcal{O}(\mu_{\underline{n}}^{-1}(0)) \end{tikzcd} \tag{3.4.9}$$

Remark 3.4.2.1 (Poisson structure on the zero locus)**.** As for the Poisson structure we observe that, by Remark 3.3.3.3, $A \amalg_{T_S(L)} S$ has an induced H_0-Poisson structure, which induces (Remark 3.2.3.2) a Poisson structure on the ordinary Poisson reduction

$$\mathrm{Rep}_{\underline{n}}(A/\mathcal{I}) /\!\!/ \mathrm{GL}_{\underline{n}} = \mathrm{Spec}(\mathcal{O}(\mu_{\underline{n}}^{-1}(0))^{\mathrm{GL}_{\underline{n}}}) .$$

Finally when we consider the specific chosen model representing the derived zero locus as the Shafarevich complex $\mathrm{Sh}(A) = A \amalg_{T_S(L)} T_S(L \oplus L[1])$ we obtain:

Lemma 3.4.2.1. *The Shafarevich complex (a chosen model for the noncommutative derived zero locus) is sent under the representation functor to the standard model for the commutative derived zero locus, the Koszul complex:*

$$(\mathrm{Sh}(A))_{\underline{n}} \cong A_{\underline{n}} \otimes_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \mathrm{Sym}(\mathfrak{gl}_{\underline{n}} \oplus \mathfrak{gl}_{\underline{n}}[1]) = A_{\underline{n}} \otimes_k \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1]) . \tag{3.4.10}$$

Notation. For a noncommutative Hamiltonian space A, we denote the associated commutative Koszul complex of dimension $\underline{n}$ by

$$\mathcal{K}_{\underline{n}}(A) := (\mathrm{Sh}(A))_{\underline{n}} \cong A_{\underline{n}} \otimes \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1]) . \tag{3.4.11}$$

Last but not least — because it represents the largest class of examples of Hamiltonian spaces — we show the correspondence between noncommutative and commutative cotangent bundles:

Theorem 3.4.2.2. *If A is smooth, there is a natural isomorphism between the representation scheme of the noncommutative cotangent bundle of A and the ordinary cotangent bundle of the representation scheme of A:*

$$\mathrm{Rep}_{\underline{n}}(\mathsf{T}^*A) \cong \mathsf{T}^*\mathrm{Rep}_{\underline{n}}(A)\,. \tag{3.4.12}$$

The Poisson structure induced on $\mathsf{T}^*\mathrm{Rep}_{\underline{n}}(A)$ by the double Poisson structure on T^*A is the Poisson structure coming from its standard symplectic structure.

Proof. Let us consider the Van den Bergh's functor $(-)_{\underline{n}} : A-\mathtt{Bimod} \to A_{\underline{n}}-\mathtt{Mod}$, which has the property that for $\mathbb{D} \in A-\mathtt{Bimod}$ and any positive shifting $r = 0, 1, \ldots$:

$$(\mathsf{T}_A\mathbb{D}[r])_{\underline{n}} \cong \mathrm{Sym}_{A_{\underline{n}}}(\mathbb{D}_{\underline{n}}[r])\,.$$

When algebra A is smooth, the Van den Bergh's functor sends the A-bimodule of double derivations to the $A_{\underline{n}}$-module of its derivations: $\mathbb{D}\mathrm{er}_S(A) \to \mathrm{Der}(A_{\underline{n}})$ ([69, Proposition 3.3.4]). As a consequence the representation functor sends odd and even cotangent bundle to the odd and even cotangent bundles of the representation scheme:

$$\begin{aligned}
(\mathsf{T}^*_{\mathrm{odd}}A)_{\underline{n}} &\cong \mathrm{Sym}_{A_{\underline{n}}}(\mathrm{Der}(A_{\underline{n}})[1]) = \mathcal{O}(\mathsf{T}^*[-1]\mathrm{Rep}_{\underline{n}}(A))\,,\\
(\mathsf{T}^*A)_{\underline{n}} &\cong \mathrm{Sym}_{A_{\underline{n}}}(\mathrm{Der}(A_{\underline{n}})) = \mathcal{O}(\mathsf{T}^*\mathrm{Rep}_{\underline{n}}(A))\,.
\end{aligned}$$

[68, Proposition 7.6.1] proves that the induced bracket on $\mathsf{T}^*[-1]\mathrm{Rep}_{\underline{n}}(A)$ is the Schouten-Nijenhuis bracket. But then, because of the relationship between the natural Poisson structure on $\mathsf{T}^*\mathrm{Rep}_{\underline{n}}(A)$ and the Schouten-Nijenhuis one (it is — with a minus sign[1] — the same on the generators), and the relationship between the double Poisson structures on $\mathsf{T}^*_{\mathrm{odd}}A$ and T^*A (Theorem 3.3.2.1 and Proposition 3.3.2.2) the result follows. □

[1]This depends also on the conventions on what is the 'natural symplectic form on a cotangent bundle'. Here we take the one for which the Poisson bracket $\{x, y\} = 1$ if x denotes a coordinate on the basis and y on the fiber.

3.4.3 Chevalley-Eilenberg and BRST

In this Section we complete the last step of the two-step derived Poisson reduction by considering the Chevalley-Eilenberg functor and its composite with the derived zero locus, the BRST construction. We show that the noncommutative definitions in §3.3.4 give the corresponding one in the commutative world.

Let us start from the Chevalley-Eilenberg functor (3.3.26) which is the coproduct over $T_S(L)$ with $T_S(L \oplus L^*[-1])$, with a twisted differential. By the discussion in (3.4.7) this differential graded algebra ($d\eta_i = -\eta_i^2$) is sent under the representation functor to the commutative Chevalley-Eilenberg complex for the module $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})$:

$$T_S(L \oplus L^*[-1])_{\underline{n}} \cong \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \otimes_k \mathrm{Sym}(\mathfrak{gl}^*_{\underline{n}}[-1]) = C(\mathfrak{gl}_{\underline{n}}, \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}))\,, \quad (3.4.13)$$

As a consequence (using the monoidal properties of the representation functor) it is easy to see that for any noncommutative Hamiltonian space $A \in \mathtt{DGPPA}_{T_S(L)}$ the Chevalley-Eilenberg complex corresponds to the usual one:

$$\mathrm{CE}(A)_{\underline{n}} \cong A_{\underline{n}} \otimes_k \mathrm{Sym}(\mathfrak{gl}^*_{\underline{n}}[-1]) = C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}})\,. \quad (3.4.14)$$

In fact the twisted differential on $C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}})$ is obtained by adding the terms commutators of Chevalley-Eilenberg generators with elements of $A_{\underline{n}}$, and it is induced exactly from the twisting defined on the noncommutative Chevalley-Eilenberg complex (3.3.27).

In other words we have the following result:

Lemma 3.4.3.1. *There is a commutative diagram*

$$\begin{tikzcd} \mathtt{DGPPA}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] \arrow[d, "\mathrm{CE}"] & \mathtt{CDGPA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \arrow[d, "C(\mathfrak{gl}_{\underline{n}}, -)"] \\ \mathtt{DGA}_{T_S(L \oplus L^*[-1])} \arrow[r, "(-)_{\underline{n}}"] & \mathtt{CDGA}_{C(\mathfrak{gl}_{\underline{n}}, \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}))} \end{tikzcd} \quad (3.4.15)$$

As a consequence also the noncommutative BRST construction in (3.3.31) becomes the commutative one:

$$(\mathrm{BRST}(A))_{\underline{n}} = (A \amalg_S T_S(L[1] \oplus L^*[-1]))_{\underline{n}} \cong A_{\underline{n}} \otimes_k \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1] \oplus \mathfrak{gl}^*_{\underline{n}}[-1])\,. \quad (3.4.16)$$

The double Poisson structure on $\mathrm{BRST}(A)$ (free product of the one on A and the canonical one on $T_S(L[1] \oplus L^*[-1])$), induces the Poisson structure on (3.4.16) which is the tensor product of the one on $A_{\underline{n}}$ and the canonical Poisson structure on $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1] \oplus \mathfrak{gl}_{\underline{n}}^*[-1])$ (the extension of the natural pairing $\mathfrak{gl}_{\underline{n}} \otimes \mathfrak{gl}_{\underline{n}}^* \to k$).

Theorem 3.4.3.1. *There is a commutative diagram between the noncommutative and the commutative BRST constructions:*

$$\begin{tikzcd}
\mathtt{PPAlg}^{H}_{T_S(L)} \arrow[r, "(-)_{\underline{n}}"] \arrow[d, "\mathrm{BRST}"] & \mathtt{CPAlg}^{H}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}})} \arrow[d, "\mathrm{BRST}"] \\
\mathtt{DGPPA}_{T_S(L \oplus L[1] \oplus L^*[-1])} \arrow[r, "(-)_{\underline{n}}"] & \mathtt{CDGPA}_{\mathrm{Sym}(\mathfrak{gl}_{\underline{n}} \oplus \mathfrak{gl}_{\underline{n}}[1] \oplus \mathfrak{gl}_{\underline{n}}^*[-1])}
\end{tikzcd} \tag{3.4.17}$$

Notation. For a noncommutative Hamiltonian space A, we denote the associated commutative BRST complex of dimension $\underline{n}$ (the object obtained in the right-bottom side of the diagram (3.4.17) by following either of the two paths):

$$\mathcal{B}_{\underline{n}}(A) := \mathrm{BRST}(A)_{\underline{n}} \cong A_{\underline{n}} \otimes \mathrm{Sym}(\mathfrak{gl}_{\underline{n}}[1] \oplus \mathfrak{gl}_{\underline{n}}^*[-1])\,. \tag{3.4.18}$$

Remark 3.4.3.1. Remember that the differential on the noncommutative BRST complex is induced by a charge $d_{\mathrm{BRST}} = \{\gamma, -\}$ (Remark 3.3.4.2). As a consequence the induced differential on the commutative BRST complex $\mathcal{B}_{\underline{n}}(A)$ is also induced by a charge, obtained as the trace of the noncommutative charge (see Remark 3.4.1.1):

$$d_{\mathcal{B}_{\underline{n}}(A)} = \{\mathrm{tr}(\gamma), -\}\,.$$

It is easy to verify that this is indeed the usual BRST charge associated to the $\mathfrak{gl}_{\underline{n}}$ action on $A_{\underline{n}}$.

Finally, we notice that from the noncommutative dg algebra maps (3.3.34) linking the BRST complex, the Shafarevich complex and the zero locus, we obtain the analogous well known (commutative) dg algebra maps:

$$\mathcal{B}_{\underline{n}}(A) \xrightarrow{(\mathrm{ev}_{\eta=0})_{\underline{n}}} \mathcal{K}_{\underline{n}}(A) \twoheadrightarrow \mathcal{O}(\mu_{\underline{n}}^{-1}(0))\,. \tag{3.4.19}$$

3.5 Some homological computations and examples

We devote this last Section to some homological computations (commutative BRST homology) and some examples, such as noncommutative cotangent bundles of path algebras of quivers, and in particular the example of the Jordan quiver — the scheme of commuting matrices — and similar associated schemes.

3.5.1 Computation of the Chevalley-Eilenberg (co)homology

In this Section we compute the (commutative) Chevalley-Eilenberg homology for representation algebras. Let us recall first that in our conventions (differentials of degree -1) the Chevalley-Eilenberg complex for a $\mathfrak{g}$-module M is the following chain complex concentrated in non-positive degrees:

$$C(\mathfrak{g}, M) := \mathrm{Hom}_k(\mathrm{Sym}(\mathfrak{g}[1]), M) = [M \longrightarrow \mathfrak{g}^* \otimes M \longrightarrow \Lambda^2\mathfrak{g}^* \otimes M \longrightarrow \dots]\,. \tag{3.5.1}$$

Its homology is essentially the Lie algebra cohomology (the cohomology of the usual cochain complex version of Chevalley-Eilenberg):

$$H_\bullet(C(\mathfrak{g}, M)) = H^{-\bullet}(\mathfrak{g}, M)\,.$$

Let us start from a double Poisson algebra $A \in \mathtt{DGPPA}_{T_S(L)}$ and, for a dimension vector $\underline{n}$, the induced Poisson morphism $\mathrm{Sym}(\mathfrak{gl}_{\underline{n}}) \to A_{\underline{n}}$ and the associated Chevalley-Eilenberg complex:

$$C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}})\,.$$

The calculation of its homology is a somewhat classical result:

Theorem 3.5.1.1. *Let $A \in \mathtt{DGPPA}_{T_S(L)}$ and $\underline{n}$ a dimension vector. The Chevalley-Eilenberg homology of the $\mathfrak{gl}_{\underline{n}}$-module $A_{\underline{n}}$ is*

$$H_\bullet(C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}})) \cong H_\bullet(A_{\underline{n}})^{GL_{\underline{n}}} \otimes H^{-\bullet}(\mathfrak{gl}_{\underline{n}}, k)\,. \tag{3.5.2}$$

Proof. We recall that the Chevalley-Eilenberg complex for $A_{\underline{n}}$ is obtained applying the representation functor to the noncommutative one: $C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}}) \cong$

$(CE(A))_{\underline{n}}$. The differential on $C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}})$ is the sum of the two super-commuting differentials (induced from the two super-commuting differentials (3.3.28)):

$$d = d_1 + d_2 ,$$

where d_1 is the old differential on $A_{\underline{n}}$ extended to zero elsewhere and d_2 is the Chevalley-Eilenberg differential extended to commutators on $A_{\underline{n}}$ (the $\mathfrak{gl}_{\underline{n}}$ action on $A_{\underline{n}}$ induced from $GL_{\underline{n}} \curvearrowright A_{\underline{n}}$). To compute the homology we can use the 'classical' technique consisting of the following steps:

1. We consider the filtration
$$C(\mathfrak{gl}_{\underline{n}}, A_{\underline{n}}) = F_0 \supset F_1 \supset F_2 \supset \cdots \supset F_N \supset F_{N+1} = 0 ,$$
$$\Big(N = \dim(\mathfrak{gl}_{\underline{n}}) = \sum_i n_i^2\Big)$$
where F_p is the linear span of monomials containing at least p terms η, and consider the associated spectral sequence $\{E^r_{p,q}\}$.

2. The differential on the associated graded (zero page) contains only d_1, therefore:
$$E^1 \cong \mathrm{Sym}(\mathfrak{gl}^*_{\underline{n}}[-1]) \otimes H_\bullet(A_{\underline{n}}) = C\big(\mathfrak{gl}_{\underline{n}}, H_\bullet(A_{\underline{n}})\big) .$$
with $d^{(1)} : E^1 \to E^1$ being the Chevalley-Eilenberg differential for the $\mathfrak{gl}_{\underline{n}}$ graded module $H_\bullet(A_{\underline{n}})$ (now without differential).

3. Because $\mathfrak{gl}_{\underline{n}}$ is a finite dimensional reductive Lie algebra, the Chevalley-Eilenberg homology of the graded module $H_\bullet(A_{\underline{n}})$ is
$$E^2 \cong H_\bullet(A_{\underline{n}})^{GL_{\underline{n}}} \otimes H^{-\bullet}(\mathfrak{gl}_{\underline{n}}, k) .$$

4. By (bi)degree inspection the differential on the second page is zero $d^{(2)} = 0$. Therefore the sequence collapses at $E^2 = E^3 = \cdots = E^\infty$ and the second page computes the homology.

$\square$

Now let us start from a noncommutative Hamiltonian space $A \in \mathtt{PPAlg}^{H}_{T_S(L)}$, and consider for some dimension $\underline{\mathfrak{n}}$ the associated commutative BRST complex, which is the Chevalley-Eilenberg complex for the Koszul complex:

$$\mathrm{BRST}(A_{\underline{n}}) = \mathcal{B}_{\underline{n}}(A) = C(\mathfrak{gl}_{\underline{n}}, \mathcal{K}_{\underline{n}}(A))\,. \tag{3.5.3}$$

As a corollary of the previous Theorem (using $\mathrm{Sh}(A)$ instead of A):

Corollary 3.5.1.1. *Let $A \in \mathtt{PPAlg}^{H}_{T_S(L)}$. The homology of the BRST complex for $A_{\underline{n}}$ is the tensor product of the $\mathrm{GL}_{\underline{n}}$-invariant homology of the Koszul complex with the Lie algebra (co)homology of $\mathfrak{gl}_{\underline{n}}$:*

$$H_\bullet(\mathcal{B}_{\underline{n}}(A)) \cong H_\bullet(\mathcal{K}_{\underline{n}}(A))^{\mathrm{GL}_{\underline{n}}} \otimes H^{-\bullet}(\mathfrak{gl}_{\underline{n}}, k)\,. \tag{3.5.4}$$

The goal would be to actually compute $H_\bullet(\mathcal{K}_{\underline{n}}(A))^{\mathrm{GL}_{\underline{n}}}$ which in fact, even in the easier cases is not known. For example when $S = k$ (one vertex), and $A = T^*k[x] \cong k\langle x, y\rangle$, the zero locus $A/\mathcal{J} = k\langle x,y\rangle/\langle [x,y]\rangle = k[x,y]$ is a polynomial algebra in two variables, and the homology $H_\bullet(\mathcal{K}_n(A))^{\mathrm{GL}_n}$ is the GL_n-invariant part of the Koszul homology for the scheme of commuting matrices. There is a conjecture ([8, Conjecture 1]) saying that the following diagonal restriction map is an isomorphism:

$$H_\bullet(\mathcal{K}_n(A))^{\mathrm{GL}_n} = H_\bullet\left(k[x_{uv}, y_{uv}, \vartheta_{uv}]\right)^{\mathrm{GL}_n} \to k[x_u, y_u, \vartheta_u]^{S_n}\,.$$

We are not able to fully compute the homology $H_\bullet(\mathcal{K}_n(A))^{\mathrm{GL}_n}$ but, using the Poisson structure on $\mathcal{B}_n(A)$ and trace maps, we can give another proof to a decomposition of $H_\bullet(\mathcal{K}_n(A))^{\mathrm{GL}_n}$ which was not previously known to us. We postpone the result to §3.5.4, after we have introduced the class of examples which are cotangent bundles of quiver path algebras.

3.5.2 Path algebras of quivers

In this Section we work out derived noncommutative Poisson reduction for cotangent bundles of path algebras of quivers T^*kQ. The reader who wants to have a more detailed introduction on the noncommutative geometry of quiver path algebras can read the main references [21],[24].

We fix $S = kI$ and we consider the path algebra kQ of a quiver $Q = (Q_0 = I, Q_1)$ with vertex set I. Path algebras are the simplest examples of algebras over S, because they are in fact free algebras over S, as we can write

$$kQ = T_S(M)\,,$$

where $M = kQ_1$ is the S-bimodule which, as a vector space, is freely generated by the arrows. The space of double derivations for kQ is rather easy to compute. For each arrow of the quiver $x \in Q_1$ we consider the following double derivation $\partial_x \in \mathbb{D}\mathrm{er}_S(kQ)$ defined on generators by

$$\partial_x(y) = \delta_{x,y} e_{t(x)} \otimes e_{s(x)}\,, \tag{3.5.5}$$

where $s, t : Q_1 \to I$ are, respectively, the source and target map (we use the convention of concatenation of paths *from right to left*). It is easy to check that $\mathbb{D}\mathrm{er}_S(kQ)$ is the free kQ-bimodule with generators $\{\partial_x\}_{x \in Q_1}$. We recall that the kQ-bimodule structure on $\mathbb{D}\mathrm{er}_S(kQ)$ is induced by the inner bimodule structure on $(kQ)^{\otimes 2}$, explicitely:

$$(a \cdot \partial \cdot b)(c) = a * (\partial(c)) * b, \quad a, b, c \in A, \partial \in \mathbb{D}\mathrm{er}_S(kQ)\,.$$

Because $S \subset kQ$ we can view $\mathbb{D}\mathrm{er}_S(kQ)$ also as a S-bimodule, that is a $I \times I$-graded vector space, and it is easy to see that for each arrow $x \in Q_1$ going from $i = s(x)$ to $j = t(x)$, the double derivation $\partial_x \in \mathbb{D}\mathrm{er}_S(kQ)_{ji}$ is an "arrow" going in the opposite direction, from j to i, in fact:

$$(e_k \cdot \partial_x \cdot e_l)(y) = e_k * (\delta_{x,y} e_j \otimes e_i) * e_l = \delta_{x,y} e_j e_l \otimes e_k e_i = \delta_{jl}\delta_{ik}\partial_x(y)\,, \quad \forall y \in Q_1\,. \tag{3.5.6}$$

If we consider the doubled quiver $\overline{Q}$, obtained from Q by adding for each arrow $x \in Q_1$ a dual arrow x^* going in the opposite direction, we have:

Proposition 3.5.2.1. *There is a natural isomorphism between the noncommutative cotangent bundle of* kQ*, and the path algebra of the double quiver* $k\overline{Q}$*, given by:*

$$\begin{gathered} k\overline{Q} \xrightarrow{\sim} T^*(kQ) = T_{kQ}(\mathbb{D}\mathrm{er}_S(kQ)) \\ \begin{cases} x \longmapsto x\,, \\ x^* \longmapsto \partial_x\,. \end{cases} \end{gathered} \tag{3.5.7}$$

We remark that the above isomorphism is an isomorphism of kQ-bimodules, and in particular of S-bimodules, that is $I \times I$-graded vector spaces (as shown in (3.5.6)). In terms of the kQ-bimodule basis $\{\partial_x\}_{x \in Q_1} \subset \mathbb{D}\mathrm{er}_S(kQ)$, the distinguished derivation is

$$\Delta = \sum_{x \in Q_1} [\partial_x, x]\,. \tag{3.5.8}$$

In fact it is sufficient to prove the equality (3.5.8) on each arrow $y \in Q_1$ of the quiver:

$$\sum_{x \in Q_1} [\partial_x, x](y) = \sum_{x \in Q_1} \partial_x(y) * x - x * \partial_x(y) =$$
$$= (e_{t(y)} \otimes e_{s(y)}) * y - y * (e_{t(y)} \otimes e_{s(y)}) = y \otimes e_{s(y)} - e_{t(y)} \otimes y = \Delta(y)\,.$$

By the general construction explained in §3.3.2, the cotangent bundle $T^*(kQ)$ carries a natural double Poisson structure.

Proposition 3.5.2.2. *Under the isomorphism* (3.5.7) *the induced double Poisson structure on the path algebra of the doubled quiver* $k\overline{Q}$ *with only non-zero brackets the one pairing an arrow* x *and its dual* x^* *by:*

$$\{\!\{x, x^*\}\!\} = e_{s(x)} \otimes e_{t(x)}\,, \quad \left(\{\!\{x^*, x\}\!\} = -e_{t(x)} \otimes e_{s(x)}\right)\,. \tag{3.5.9}$$

The noncommutative moment map $T_S(L) \to k\overline{Q}$ *is the one sending* $t_i \mapsto \delta_i = e_i \left(\sum_x [x, x^*]\right) e_i$.

Proof. Given $x, y \in Q_1$ we have to show that the double bracket $\{\!\{\partial_x, \partial_y\}\!\}$ given in (3.3.12) is zero. This follows from the fact that any elementary double derivation ∂_x sends any arrow $z \in Q_1$ into $S \otimes S$, therefore the triple derivation

$$\{\!\{\partial_x, \partial_y\}\!\}^{\sim}_l = (\partial_x \otimes 1) \circ \partial_y - (1 \otimes \partial_y) \circ \partial_x$$

vanishes on the arrows, and so it vanishes everywhere. □

Remark 3.5.2.1. The double Poisson on $k\overline{Q}$ induces a H_0-Poisson structure on $(k\overline{Q})_\natural$ (the space of "cyclic paths"), which is the well-known Necklace Lie algebra structure ([13]).

The noncommutative zero locus is:

$$T^*(kQ)/\langle\delta\rangle \cong k\overline{Q}/\left\langle \sum_{x\in Q_1}[x,x^*]\right\rangle =: \Pi(Q)\,, \tag{3.5.10}$$

the so called *preprojective algebra* of the quiver Q. The Shafarevich complex is

$$\mathrm{Sh}(kQ) \cong k\overline{Q} \amalg_S T_S(L[1]) \cong kQ^{\vartheta}\,, \tag{3.5.11}$$

where we denote by Q^{ϑ} the quiver obtained from the doubled quiver $\overline{Q}$ by adding a new loop (ϑ_i) in homological degree 1 on each vertex, and kQ^{ϑ} is its (graded) path algebra. The BRST complex becomes

$$\mathrm{BRST}(kQ) \cong k\overline{Q} *_S T_S(L[1]\oplus L^*[-1]) \cong k\widehat{Q}\,, \tag{3.5.12}$$

the path algebra of $\widehat{Q}$, which is the quiver obtained from the doubled quiver $\overline{Q}$ adding two new loops on each vertex, one in homological degree 1 (ϑ_i) and one in degree -1 (η_i). The BRST differential is, on the generators:

$$\begin{cases} dx = -[\eta,x]\,, \qquad (\eta = \sum_i \eta_i) \\ dx^* = -[\eta,x^*]\,, \\ d\vartheta_i = \delta_i - [\eta_i,\vartheta_i]\,, \\ d\eta_i = -\eta_i^2 = -\frac{1}{2}[\eta_i,\eta_i]\,. \end{cases} \tag{3.5.13}$$

3.5.3 The scheme of commuting matrices and similar

Let us consider one particular class of examples of path algebras where the quiver has only one vertex. In this case we can only have arrows that are loops on this vertex, so the quiver is uniquely determined by the number of loops, and we denote by Q_g the quiver with g loops. We denote by $x_1,\dots x_g$ its loops and by $y_1,\dots,y_g$ their dual loops, so that its cotangent bundle is a free algebra on $2g$ generators.

$$k\overline{Q_g} = k\langle x_1,y_1,\dots,x_g,y_g\rangle\,.$$

The image of t under the moment map $k[t]\to k\overline{Q_g}$ is $\delta = \sum_a [x_a,y_a]$. Underived commutative Poisson reduction gives the affine GL_n-quotient of the zero locus of the moment map $\mu_n : (\mathfrak{gl}_n)^{\times 2g} \to \mathfrak{gl}_n^{(*)}$:

$$\begin{aligned}\mu_n^{-1}(0) /\!\!/ \mathrm{GL}_n &= \mathrm{Rep}_n\left(k\overline{Q_g}/\mathcal{I}\right) /\!\!/ \mathrm{GL}_n = \\ &= \left\{(X_a, Y_a) \in (\mathfrak{gl}_n)^{\times 2g} \mid \sum_a [X_a, Y_a] = 0\right\} /\!\!/ \mathrm{GL}_n\,,\end{aligned} \tag{3.5.14}$$

a linearised (Lie algebra) version of the GL_n-character variety of a Riemann surface of genus g: $\mathrm{Hom}_{\mathtt{Grp}}(\pi_1(\Sigma_g), \mathrm{GL}_n) /\!\!/ \mathrm{GL}_n$. The noncommutative Shafarevich complex and the BRST complex are, respectively:

$$\begin{aligned}\mathrm{Sh}(k\overline{Q_g}) &= k\langle x_1, y_1, \ldots, x_g, y_g, \vartheta\rangle,, \qquad d\vartheta = \sum_a [x_a, y_a]\,, \\ \mathrm{BRST}(k\overline{Q_g}) &= k\langle x_1, y_1, \ldots, x_g, y_g, \vartheta, \eta\rangle\,, \qquad d = d_{\mathrm{Sh}} + d_{\mathrm{CE}}\,.\end{aligned}$$

For $g \geqslant 2$ the scheme (3.5.14) is a complete intersection in $(\mathfrak{gl}_n)^{\times 2g}$, so that the homology of the Koszul complex is only in degree zero, and the homology of the commutative BRST complex is essentially just the ring of functions on (3.5.14):

$$\begin{aligned}H_\bullet(\mathcal{B}_n(k\overline{Q_g})) &\cong H_\bullet(\mathcal{K}_n(k\overline{Q_g}))^{\mathrm{GL}_n} \otimes H^{-\bullet}(\mathfrak{gl}_n, k) = \\ &= \mathcal{O}(\mu^{-1}(0))^{\mathrm{GL}_n} \otimes H^{-\bullet}(\mathfrak{gl}_n, k)\,,\end{aligned}$$

(considering that $H^{-\bullet}(\mathfrak{gl}_n, k) = k[\mathrm{tr}(\eta), \mathrm{tr}(\eta^3), \ldots, \mathrm{tr}(\eta^{2n-1})]$ is an exterior algebra, so essentially 'invertible').

The case $g = 1$ instead corresponds to the scheme of commuting matrices, for which the homology of the Koszul complex in concentrated in degrees $0, 1, \ldots, n$, and it is not known. There is a conjecture ([8, Conjecture 1]), that the diagonal restriction to multisymmetric polynomials is a quasi-isomorphism:

$$(\mathcal{K}_n(kQ_1))^{\mathrm{GL}_n} = k[x_{ij}, y_{ij}, \vartheta_{ij}]^{\mathrm{GL}_n} \to k[x_i, y_i, \vartheta_i]^{S_n}\,. \tag{3.5.15}$$

Using (3.5.4) we can rewrite this conjecture by saying that the following map is an isomorphism

$$H_\bullet(\mathcal{B}_n(kQ_1)) = H_\bullet(k[x_{ij}, y_{ij}, \vartheta_{ij}, \eta_{ij}]) \to k[x_i, y_i, \vartheta_i]^{S_n}[\mathrm{tr}(\eta), \ldots, \mathrm{tr}(\eta^{2n-1})]\,. \tag{3.5.16}$$

We conclude this Section with two more examples similar to the commuting scheme:

1. First we consider the cotangent bundle of the ring of Laurent polynomials: $A = T^*B = T^*(k[x^{\pm 1}])$. Any double derivation $B \to B \otimes B$ is uniquely determined by its value on x, and by using the B-bimodule structure we can obtain any value starting from the following derivation $\partial_x(x) = 1 \otimes 1$. $\mathbb{D}\mathrm{er}(B)$ is a free B-bimodule generated by ∂_x, which we call y, and we obtain:

$$A = T^*B \cong k\langle x^{\pm 1}, y\rangle\,, \quad \{\!\{x, y\}\!\} = 1 \otimes 1\,,$$

It is easy to verify on the generators that the gauge element in T^*B is still $\delta = xy - yx = [x, y]$, so that the zero locus is $A/\langle\delta\rangle = k[x^{\pm 1}, y]$, and the corresponding commutative zero locus is a group-Lie algebra version of the commuting scheme:

$$\mathrm{Rep}_n(A/\langle\delta\rangle) = \{(X, Y) \in GL_n \times \mathfrak{gl}_n \,|\, [X, Y] = 0\}\ .$$

The Shafarevich complex is obtained by adding one more variable ϑ whose differential is $[x, y]$. However we can observe that in this case, because x is invertible, we can rewrite $[x, y] = (xyx^{-1} - y)x$ and at the level of matrices $XYX^{-1} = \mathrm{Ad}_X(Y)$ is the adjoint action of GL_n on $\mathfrak{gl}_n$. Thus the Koszul complex, which a priori is the following homotopy pull-back diagram:

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{K}_n(A)) \arrow[r] \arrow[d] & GL_n \times \mathfrak{gl}_n \arrow[d] \\
\mathrm{pt} \arrow[r, "0"] & \mathfrak{gl}_n
\end{tikzcd} \qquad (d\Theta = [X, Y])$$

can be rewritten in a more intrinsic way as the following homotopy pull-back diagram, generalisable to other Lie algebras:

$$\begin{tikzcd}
\mathfrak{gl}_n \times^h_{(\mathfrak{gl}_n \times \mathfrak{gl}_n)} (GL_n \times \mathfrak{gl}_n) \arrow[r] \arrow[d] & GL_n \times \mathfrak{gl}_n \arrow[d, "(\mathrm{Ad},1)"] \\
\mathfrak{gl}_n \arrow[r, "\mathrm{diag}"] & \mathfrak{gl}_n \times \mathfrak{gl}_n
\end{tikzcd} \qquad (d\Theta = \mathrm{Ad}_X(Y) - Y)$$

We could write an analogous conjecture to the Lie algebra-Lie algebra case by saying that:

Conjecture 3.5.3.1. *The following diagonal restriction map is a quasi-isomorphism:*

$$k[x_{ij}, y_{ij}, \vartheta_{ij}, \det(X)^{-1}]^{GL_n} \xrightarrow{\sim} k[x_i^{\pm}, y_i, \vartheta_i]^{S_n}\,. \qquad (3.5.17)$$

2. Next we consider the following example $A = k\langle x^{\pm 1}, y^{\pm 1}\rangle$, which is not a cotangent bundle. However we can still define a double Poisson structure by setting $\{\!\{x, y\}\!\} = 1 \otimes 1$. We can easily verify on the generators that the element $\delta = [x, y]$ defines a Hamiltonian action, that is:

$$\{\!\{\delta, a\}\!\} = a \otimes 1 - 1 \otimes a, \quad \forall a \in A\,.$$

The zero locus is $A/\langle\delta\rangle = k[x^{\pm 1}, y^{\pm 1}]$, and the corresponding commutative zero locus is a group-group version of the commuting scheme:

$$\mathrm{Rep}_n(A/\langle\delta\rangle) = \left\{(X, Y) \in GL_n^{\times 2} \,|\, [X, Y] = 0\right\}\,.$$

As in the previous example, using that now both matrices are invertible, we can rewrite the relation as $xyx^{-1}y^{-1} = 1$, so that we can identify the Hamiltonian reduction with the GL_n-character variety of the Riemann surface with genus $g = 1$:

$$\mathrm{Rep}_n(A/\langle\delta\rangle) /\!\!/ GL_n \cong \mathrm{Hom}_{\mathtt{Grp}}(\pi_1(\Sigma_1), GL_n) /\!\!/ GL_n\,. \qquad (3.5.18)$$

Remark 3.5.3.1 (The Poisson structure on the character variety is *not* the standard one)**.** We need to remark that the above identification (3.5.3.1) holds only at the level of affine schemes while the induced Poisson structure that we have on $\mathrm{Rep}_n(A/\langle\delta\rangle) /\!\!/ GL_n$ does not coincide with the one on the character variety of Σ_1. In fact, the latter is obtained by quasi-Hamiltonian reduction of the quasi-Hamiltonian space $GL_n \curvearrowright GL_n^{\times 2}$, equipped with the canonical 2-form ω on the double of a Lie group ([2]), which is different from the one induced by $GL_n^{\times 2} \subset \mathfrak{gl}_n^{\times 2} \cong \mathfrak{gl}_n \times \mathfrak{gl}_n^* \cong T^*\mathfrak{gl}_n$. If we would like to obtain the standard Poisson structure on the character variety we would need a noncommutative analogue of the quasi-Hamiltonian formalism and Lie-group valued moment maps of [2], which at least in the case of the standard action of GL_n by conjugation was developed in [69].

The corresponding conjecture relating the corresponding Koszul complex with its diagonal part would be (and it already appears in [11, Conjecture 1]):

Conjecture 3.5.3.2. *The following diagonal restriction map is a quasi-isomorphism:*

$$k[x_{ij}, y_{ij}, \vartheta_{ij}, \det(X)^{-1}, \det(Y)^{-1}]^{GL_n} \xrightarrow{\sim} k[x_i^{\pm}, y_i^{\pm 1}, \vartheta_i]^{S_n}\,. \tag{3.5.19}$$

3.5.4 Decomposition of the homology of the commuting scheme

Let $A = k\langle x, y\rangle$. Consider the following maps on the BRST complex $\mathcal{B}_n(A) = k[x_{ij}, y_{ij}, \vartheta_{ij}, \eta_{ij}]$:

$$\begin{aligned} \varphi_\bullet &= \{\mathrm{tr}(\eta), -\} : \mathcal{B}_n(A)_\bullet \to \mathcal{B}_n(A)_{\bullet-1}\,, \\ \psi_\bullet &= \mathrm{tr}(\vartheta)\cdot(-) : \mathcal{B}_n(A)_\bullet \to \mathcal{B}_n(A)_{\bullet+1}\,. \end{aligned} \tag{3.5.20}$$

Proposition 3.5.4.1. *The following relations are satisfied:*

$$\varphi_{\bullet+1}\psi_\bullet + \psi_{\bullet-1}\varphi_\bullet = n1\,, \qquad \varphi_{\bullet-1}\varphi_\bullet = 0\,, \qquad \psi_{\bullet+1}\psi_\bullet = 0\,. \tag{3.5.21}$$

Moreover the maps $\varphi_\bullet, \psi_\bullet$ preserve boundaries and cycles of the BRST differential, so they induce maps on the homology $H_\bullet(\mathcal{B}_n(A))$, satisfying the same relations.

Proof. For any element $\alpha \in \mathcal{B}_n(A)$ we have:

$$\{\mathrm{tr}(\eta), \mathrm{tr}(\vartheta)\alpha\} = \{\mathrm{tr}(\eta), \mathrm{tr}(\vartheta)\}\alpha - \mathrm{tr}(\vartheta)\{\mathrm{tr}(\eta), \alpha\} = \mathrm{tr}\{\eta, \vartheta\}\alpha - \mathrm{tr}(\vartheta)\{\mathrm{tr}(\eta), \alpha\} = \mathrm{tr}1\alpha - \mathrm{tr}(\vartheta)\{\mathrm{tr}(\eta), \alpha\} = n\alpha - \mathrm{tr}(\vartheta)\{\mathrm{tr}(\eta), \alpha\}\,,$$

which proves the first relation. The second follows from the Jacobi identity and the fact that $\{\mathrm{tr}(\eta), \mathrm{tr}(\eta)\} = \mathrm{tr}(\{\eta, \eta\}) = \mathrm{tr}(0) = 0$. The third one is because $\mathrm{tr}(\vartheta)$ is an odd element, therefore $\mathrm{tr}(\vartheta)^2 = 0$. They preserve boundaries and cycles essentially because the differentials $d\vartheta = [x, y] - [\eta, \vartheta]$ and $d\eta = -\frac{1}{2}[\eta, \eta]$ are commutators, therefore $d\mathrm{tr}(\vartheta) = d\mathrm{tr}(\eta) = 0$. □

Moreover in the decomposition $H_\bullet(\mathcal{B}_n(A)) \cong H_\bullet(\mathcal{K}_n(A))^{GL_n} \otimes H^{-\bullet}(\mathfrak{gl}_n, k)$ they preserve the submodule $H_\bullet(\mathcal{K}_n(A))^{GL_n} \subset H_\bullet(\mathcal{B}_n(A))$, and we denote by the same symbol the induced maps on the Koszul homology:

$$\begin{aligned} \varphi_\bullet &: H_\bullet(\mathcal{K}_n(A))^{GL_n} \to H_{\bullet-1}(\mathcal{K}_n(A))^{GL_n}\,, \\ \psi_\bullet &: H_\bullet(\mathcal{K}_n(A))^{GL_n} \to H_{\bullet+1}(\mathcal{K}_n(A))^{GL_n}\,, \end{aligned} \tag{3.5.22}$$

which satisfy the same relations (3.5.21). From which it follows that:

Corollary 3.5.4.1. *The Koszul homology decomposes as two copies of the same graded module which are obtained one from the other by a degree* 1 *shift, as follows:*

$$H_\bullet(\mathcal{K}_n(A))^{GL_n} = \underbrace{\ker(\varphi_\bullet)}_{\bullet=0,1,\dots,n-1} \oplus \underbrace{\mathrm{im}(\psi_{\bullet-1})}_{\bullet=1,2,\dots,n}, \tag{3.5.23}$$

and the isomorphism is provided by $\psi_\bullet|_{\ker(\varphi_\bullet)} : \ker(\varphi_\bullet) \xrightarrow{\cong} \mathrm{im}(\psi_\bullet)$.

Proof. This follows precisely from the relations that the induced maps (3.5.22) satisfy (which are the same as (3.5.21)). In fact from the first and the second equation we have that any homology class $\alpha \in H_\bullet(\mathcal{K}_n(A))^{GL_n}$ can be written as a sum of two elements:

$$\alpha = \frac{1}{n}\underbrace{(\varphi_{\bullet+1}\psi_\bullet)}_{\in\ker(\varphi_\bullet)} + \frac{1}{n}\underbrace{(\psi_{\bullet-1}\varphi_\bullet)}_{\in\mathrm{im}(\psi_{\bullet-1})}\,.$$

The intersection of the two submodules is zero because if $\alpha = \psi_{\bullet-1}\beta$ and $\varphi_\bullet\alpha = 0$, by the third property:

$$n\alpha = \varphi_{\bullet+1}\psi_\bullet \underbrace{\alpha}_{\psi_{\bullet-1}\beta} + \psi_{\bullet-1}\varphi_\bullet\alpha = 0\,.$$

Finally $\psi_\bullet|_{\ker(\varphi_\bullet)}$ is injective because it is the left inverse of (n times) the identity: $\varphi_{\bullet+1}\psi_\bullet|_{\ker(\varphi_\bullet)} = n1$, and it is surjective on $\mathrm{im}(\psi_\bullet)$ because $\psi_\bullet$ is zero on the complement of $\ker(\varphi_\bullet)$. □

Remark 3.5.4.1. The decomposition (3.5.23) can be explained also without using the trace maps in the following way. The homology of the Koszul complex $\mathcal{K}_n(A)$ is essentially only due to the diagonal elements $([x, y]_{ii})$, because the other ones form a regular sequence ([38]). Moreover of these diagonal elements one is superfluous, because their sum is zero:

$$\operatorname{tr}([x, y]) = \sum_i [x, y]_{ii} = 0\,.$$

It follows that the homology decomposes as the tensor product of the reduced homology (the one obtained by removing the superfluous element $\operatorname{tr}(\vartheta)$) and the antisymmetric algebra on the 1-dimensional vector space generated by $\operatorname{tr}(\vartheta)$:

$$H_\bullet(\mathcal{K}_n(A)) = \widetilde{H}_\bullet \oplus \widetilde{H}_{\bullet-1} \cdot \operatorname{tr}(\vartheta)\,.$$

This decomposition is the same as (3.5.23) so, incidentally, we find another interpretation of the reduced Koszul homology of the commuting scheme as $\widetilde{H}_\bullet \cong \ker(\varphi_\bullet)$: the classes whose Poisson bracket with $\operatorname{tr}(\eta)$ vanishes.

3.6 Noncommutative group actions and Poisson-group schemes

In this final, short section we formalise the notions of noncommutative analogues of group schemes, group actions and Poisson-group schemes. The purpose is two-fold:

- On the one hand these seems to be rather natural definitions that, at least to our knowledge, did not appear yet in the literature and can help understand from a more intrinsic, coordinate-free way, many results in the paper.
- On the other hand these notions could be used to define generalisations of the above story for noncommutative Hamiltonian actions different from the standard one (that induces the ordinary conjugation action of GL_n).

3.6.1 Noncommutative group schemes and actions

First we recall a couple of general notions about categorical groups (and cogroups), associated functors, and categorical group actions.

Groups: To a cartesian monoidal category $\mathtt{C}$ (category with binary products and terminal object $1_{\mathtt{C}} \in \mathtt{C}$), we can associate the category $\mathtt{Grp}(\mathtt{C})$ of group objects in $\mathtt{C}$: quadruples (G, m, ι, e) with $G \in \mathtt{C}$ an object, a 'multiplication morphism' $m : G \times G \to G$, an 'inverse morphism' $\iota : G \to G$, and a 'unit morphism' $e : 1_{\mathtt{C}} \to G$ satisfying the group axioms – and morphisms the morphisms of underlying objects preserving multiplication. A (internal) group action $G \overset{\alpha}{\curvearrowright} X$ of a group $G \in \mathtt{Grp}(\mathtt{C})$ on $X \in \mathtt{C}$ is a morphism $\alpha : G \times X \to X$ satisfying the usual two conditions of group actions. We can define a category of actions on $\mathtt{C}$, which we denote by (a perhaps unconventional notation) $\mathtt{Act}(\mathtt{C})$: objects are group actions $G \curvearrowright X$, and morphisms from $G \overset{\alpha}{\curvearrowright} X$ to $H \overset{\beta}{\curvearrowright} Y$ are couples (φ, f) consisting of a morphism of groups $\varphi : G \to H$ and a morphism $f : X \to Y$ such that $\beta \circ (\varphi \times f) = f \circ \alpha$. We can also fix the group G (and $\varphi = \mathrm{id}_G$) and consider only the category of G-equivariant objects, which we denote by $\mathtt{Act}_G(\mathtt{C})$. To a cartesian functor $F : \mathtt{C} \to \mathtt{D}$ between cartesian monoidal categories we have induced functors between all the categories introduced above:

1. $\mathtt{Grp}(F) : \mathtt{Grp}(\mathtt{C}) \to \mathtt{Grp}(\mathtt{D})$
2. $\mathtt{Act}(F) : \mathtt{Act}(\mathtt{C}) \to \mathtt{Act}(\mathtt{D})$
3. $\mathtt{Act}_G(F) : \mathtt{Act}_G(\mathtt{C}) \to \mathtt{Act}_{F(G)}(\mathtt{D})$

Cogroups: If $\mathtt{A}$ has binary coproducts and initial object $\emptyset_{\mathtt{A}} \in \mathtt{A}$ (that is, $\mathtt{A}^{\mathrm{op}}$ is cartesian monoidal), then we can define the category $\mathtt{CoGrp}(\mathtt{A})$ of cogroup objects in $\mathtt{A}$: quadruples (A, Δ, S, ϵ) with $A \in \mathtt{A}$ an object, a 'comultiplication' $\Delta : A \to A \coprod A$, a 'coinverse' $S : A \to A$, and a 'counit' $\epsilon : A \to \emptyset_{\mathtt{A}}$ satisfying the cogroup axioms – and morphisms the comultiplication preserving ones. Obviously $\mathtt{CoGrp}(\mathtt{A}) \cong (\mathtt{Grp}(\mathtt{A}^{\mathrm{op}}))^{\mathrm{op}}$. Dualising the previous constructions we can define the category of cogroup coactions $\mathtt{CoAct}(\mathtt{A})$ $(\cong (\mathtt{Act}(\mathtt{A}^{\mathrm{op}}))^{\mathrm{op}})$ and for some fixed cogroup $A \in \mathtt{CoGrp}(\mathtt{A})$, the category of A-coequivariant objects $\mathtt{CoAct}_A(\mathtt{A})$ $(\cong (\mathtt{Act}_{A^{\mathrm{op}}}(\mathtt{A}^{\mathrm{op}}))^{\mathrm{op}})$. To any cocartesian functor $F : \mathtt{A} \to \mathtt{B}$ we have induced three functors:

1. $\mathtt{CoGrp}(F)$ $(\cong \mathtt{Grp}(F^{\mathrm{op}})^{\mathrm{op}}) : \mathtt{CoGrp}(\mathtt{A}) \to \mathtt{CoGrp}(\mathtt{B})$
2. $\mathtt{CoAct}(F)$ $(\cong \mathtt{Act}(F^{\mathrm{op}})^{\mathrm{op}}) : \mathtt{CoAct}(\mathtt{A}) \to \mathtt{Act}(\mathtt{B})$

3. $\mathtt{CoAct}_{\mathtt{A}}(\mathtt{F})\ (\cong \mathtt{Act}_{\mathtt{A}^{\mathrm{op}}}(\mathtt{F}^{\mathrm{op}})^{\mathrm{op}}) : \mathtt{CoAct}_{\mathtt{A}}(\mathtt{A}) \to \mathtt{CoAct}_{\mathtt{F}(\mathtt{A})}(\mathtt{B})$

Example 5. Let $\mathtt{C} = \mathtt{Aff}_k$ the category of affine k-schemes. The category $\mathtt{Grp}(\mathtt{Aff}_k)$ is the category of affine group schemes over k (we denote it simply by $\mathtt{Grp}_k$), $\mathtt{Act}(\mathtt{Aff}_k)$ is the category of affine group scheme actions (we denote it by $\mathtt{Act}_k$), and $\mathtt{Act}_G(\mathtt{Aff}_k)$ the category of G-equivariant affine schemes over k (we denote it by $G - \mathtt{Aff}_k$).

Example 6. (= **Example** 5^{op}). $\mathtt{A} = \mathtt{CommAlg}_k = \mathtt{Aff}_k^{\mathrm{op}}$. The category

$$\mathtt{CoGrp}(\mathtt{CommAlg}_k) \cong \mathtt{CHopf}_k$$

is the category of commutative Hopf algebras over k (notice, in fact, that the antipode map of a Hopf algebra is in general an antihomomorphism of algebras, but when they are commutative, it is a homomorphism, and it corresponds to the coinverse of the cogroup structure). The category $\mathtt{CoAct}(\mathtt{CommAlg}_k)$ is the category of coactions of commutative Hopf algebras on commutative algebras over k, and $\mathtt{CoAct}_A(\mathtt{CommAlg}_k)$ the category of A-coequivariant commutative algebras over k.

Definition 3.6.1.1 (Noncommutative version of Example 5)**.** The category of *noncommutative affine group schemes* is the category of group objects in the cartesian monoidal category of noncommutative affine schemes $\mathtt{NAff}_k := Alg_k^{\mathrm{op}}$. We denote it by $\mathtt{NGrp}_k := \mathtt{Grp}(\mathtt{NAff}_k)$. The category of *noncommutative affine group scheme actions* is the category of actions on noncommutative affines: $\mathtt{NAct}_k := \mathtt{Act}(\mathtt{NAff}_k)$. For a fixed noncommutative affine group scheme Q we denote the category of Q-equivariant objects by $Q - \mathtt{NAff}_k := \mathtt{Act}_Q(\mathtt{NAff}_k)$.

Proposition 3.6.1.1. *The (opposite) representation functor* $\mathrm{Rep}_n = (-)_n^{\mathrm{op}} : \mathtt{NAff}_k \to \mathtt{Aff}_k$ *is a cartesian functor, therefore it induces the following three functors, which by an abuse of notation we denote by the same symbol:*

1. $\mathrm{Rep}_n : \mathtt{NGrp}_k \to \mathtt{Grp}_k$ *, which enriches the usual one* $\mathrm{Rep}_n : \mathtt{NAff}_k \to \mathtt{Aff}_k$ *in the sense that there is a commutative diagram linking these two functors under the natural forgetful functors* $\mathtt{NGrp}_k \to \mathtt{NAff}_k$*,* $\mathtt{Grp}_k \to \mathtt{Aff}_k$*.*

2. $\mathrm{Rep}_n : \mathtt{NAct}_k \to \mathtt{Act}_k$ *,*

3. $\mathrm{Rep}_n : Q - \mathtt{NAff}_k \to \mathrm{Rep}_n(Q) - \mathtt{Aff}_k$,

therefore justifying Definition 3.6.1.1, according to the Kontsevich-Rosenberg principle.

Remark 3.6.1.1. For an algebra $A \in \mathtt{Alg}_k$ we denote by $\mathrm{Sp}(A) \in \mathtt{NAff}_k$ the corresponding object in the opposite category (and analogously for $A \in \mathtt{CoGrp}(\mathtt{Alg}_k)$ we denote by $\mathrm{Sp}(A) \in \mathtt{NGrp}_k$ the corresponding noncommutative affine group scheme). If we do so the notation we used for the functor Rep_n in the previous Proposition is in slight contradiction with the previous notation that we used in the introduction of the paper, in which we evaluated $\mathrm{Rep}_n(A)$ on algebras A. Now instead, to be precise, we should (and will) say $\mathrm{Rep}_n(\mathrm{Sp}(A))$.

Example 7 (Noncommutative additive group $\mathbb{NG}_a$)**.** This is $\mathbb{NG}_a := \mathrm{Sp}(k[x])$, where the cogroup structure on $k[x]$ is:

$$\begin{aligned} \Delta :& k[x] \to k\langle x_1, x_2\rangle \qquad & S :& k[x] \to k[x] \qquad & \epsilon :& k[x] \to k \\ & x \longmapsto x_1 + x_2 & & x \longmapsto -x & & x \longmapsto 0 \end{aligned}$$

The corresponding (abelian) affine group scheme is $\mathrm{Rep}_n(\mathbb{NG}_a) \cong \mathfrak{gl}_n(k)$ (with respect to the sum).

Example 8 (Noncommutative multiplicative group $\mathbb{NG}_m$)**.** This is $\mathbb{NG}_m := \mathrm{Sp}(k[g^{\pm 1}])$, where the cogroup structure on $k[g^{\pm 1}]$ is:

$$\begin{aligned} \Delta :& k[g^{\pm 1}] \to k\langle g_1^{\pm 1}, g_2^{\pm 1}\rangle \qquad & S :& k[g^{\pm 1}] \to k[g^{\pm 1}] \qquad & \epsilon :& k[g^{\pm 1}] \to k \\ & g \longmapsto g_1 g_2 & & g \longmapsto g^{-1} & & g \longmapsto 1 \end{aligned}$$

It is interesting to observe that every noncommutative affine scheme $X = \mathrm{Sp}(A)$ has a somewhat natural $\mathbb{NG}_m$-action (the precise statement is that this gives a functor $\gamma : \mathtt{NAff}_k \to \mathbb{NG}_m - \mathtt{NAff}_k$), described dually as the coaction:

$$\begin{aligned} \alpha :& A \to A *_k k[g] = A\langle g\rangle \\ & a \longmapsto g a g^{-1} \end{aligned}$$

The corresponding action of the affine group scheme $\mathrm{Rep}_n(\mathbb{NG}_m) \cong \mathrm{GL}_n(k)$ on $\mathrm{Rep}_n(X)$ is the standard conjugation action on representations. The composition of the functor associating to each X the natural $\mathbb{NG}_m$-action and the representation functor of Proposition 3.6.1.1(3) is a factorisation of the representation functor with target category the $GL_n(k)$-equivariant affine schemes:

$$\begin{tikzcd} \mathtt{NAff}_k \arrow[r, "\gamma"] \arrow[rr, bend left, "\mathrm{Rep}_n"] & \mathbb{NG}_m - \mathtt{NAff}_k \arrow[r, "\mathrm{Rep}_n"] & GL_n(k) - \mathtt{Aff}_k \end{tikzcd}$$

Remark 3.6.1.2. As anticipated in the beginning of this section, the explanation of the ordinary GL_n-action in these new terms tells us what we should do in case we would like to consider different actions of GL_n: we consider simply $\mathrm{Rep}_n : \mathbb{NG}_m - \mathtt{NAff}_k \to GL_n(k) - \mathtt{Aff}_k$, without precomposing it with the functor γ. This means that we start from a space $\mathrm{Sp}(A)$ with a $\mathbb{NG}_m$-action that is not necessarily the usual one, and in this way obtain all possible actions of $GL_n \curvearrowright \mathrm{Rep}_n(A)$ that arise from noncommutative geometry.

3.6.2 Noncommutative Poisson-group schemes

First we give the definition of affine Poisson-group schemes, which is the obvious algebro-geometric notion analogous to the differential-geometric one of Poisson-Lie groups ([17]). First we recall that the category $\mathtt{CPAlg}_k$ of commutative Poisson algebras over k has binary coproducts (the underlying algebra is the tensor product) and initial object the algebra k with trivial Poisson structure. Dually the category of affine Poisson schemes $\mathtt{PAff}_k := \mathtt{CPAlg}_k^{op}$ is a cartesian monoidal category with final object $* = \mathrm{Spec}(k)$ the point. Recall that we have two forgetful functors $\mathtt{PAff}_k \to \mathtt{Aff}_k$, $\mathtt{Grp}_k \to \mathtt{Aff}_k$: one forgets the Poisson structure and the other the group structure.

Definition 3.6.2.1. The category $\mathtt{PGAff}_k$ of *(affine) Poisson-group schemes* (over k) is the full subcategory of the pullback[2] category $\mathtt{PAff}_k \times_{\mathtt{Aff}_k} \mathtt{Grp}_k$ consisting of those objects whose multiplication map is a Poisson map.

[2]We mean the strict pullback in the 1-category of categories $\mathtt{Cat}$. In other words an object in $\mathtt{PAff}_k \times_{\mathtt{Aff}_k} \mathtt{Grp}_k$ is a pair of a Poisson scheme and a group scheme which are *equal* under the forgetful functors, which is just a fancy way of saying that we have an affine scheme equipped with a group structure as well as a Poisson structure.

Remark 3.6.2.1. We remark that this is *not* the same as the category of group objects in Poisson schemes $\mathtt{Grp}(\mathtt{PAff}_k)$. In fact objects of this category are Poisson schemes equipped with a group structure for which multiplication, inverse, and unit are *all* Poisson maps, while $\mathtt{PGAff}_k$ only requires the multiplication to be a Poisson map. Moreover, one can show that a Poisson-group schemes has inverse map being a antiPoisson-homomorphism, therefore every group object in Poisson schemes has trivial (zero) Poisson structure, in other words $\mathtt{Grp}(\mathtt{PAff}_k) \cong \mathtt{Grp}_k$ is trivially just the category of group schemes.

Now we have everything we need to define the noncommutative analogues of such structures. First we recall that the category $\mathtt{PPAlg}_k$ of double Poisson algebras over k has binary coproducts (Proposition 3.2.4.2) and initial object the algebra k with trivial double Poisson structure. Dually, the category of *noncommutative affine Poisson schemes* $\mathtt{NPAff}_k := \mathtt{PPAlg}_k^{op}$ is a cartesian monoidal category with final object $* = \mathrm{Sp}(k)$, the 'point'. We have two forgetful functors $\mathtt{NPAff}_k \to \mathtt{NAff}_k$, $\mathtt{NGrp}_k \to \mathtt{NAff}_k$: one forgets the (double) Poisson structure and the other the group structure.

Definition 3.6.2.2. The category $\mathtt{NPGAff}_k$ of *noncommutative (affine) Poisson-group schemes* (over k) is the full subcategory of the pullback category $\mathtt{NPAff}_k \times_{\mathtt{Aff}_k} \mathtt{NGrp}_k$ consisting of those objects whose multiplication map is a Poisson map.

Remark 3.6.2.2 (Dually, algebraic side)**.** A noncommutative (affine) Poisson-group scheme $X = \mathrm{Sp}(A) \in \mathtt{NPGAff}_k$ is, dually, an algebra A equipped with a double Poisson structure and a cogroup structure for which the comultiplication map $\Delta : A \to A *_k A$ is a double Poisson map.

Remark 3.6.2.3. We have the following ingredients:

(i) $\mathrm{Rep}_n : \mathtt{NAff}_k \to \mathtt{Aff}_k$, the ordinary representation functor.
(ii) $\mathrm{Rep}_n : \mathtt{NPAff}_k \to \mathtt{PAff}_k$, enriching (i) under the natural forgetful functors (3.4.4).
(iii) $\mathrm{Rep}_n : \mathtt{NGrp}_k \to \mathtt{Grp}_k$, enriching (i) under the natural forgetful functors (Proposition 3.6.1.1(1)).

It follows that we have an induced pullback functor:

$$\mathrm{Rep}_n : \mathtt{NPAff}_k \times_{\mathtt{Aff}_k} \mathtt{NGrp}_k \to \mathtt{PAff}_k \times_{\mathtt{Aff}_k} \mathtt{Grp}_k \tag{3.6.1}$$

Theorem 3.6.2.1. *The functor* (3.6.1) *restricts to the full subcategories of noncommutative Poisson-group schemes and Poisson-group schemes, respectively. In other words we have a factorisation:*

$$\begin{tikzcd} \mathtt{NPGAff}_k \arrow[r, "\mathrm{Rep}_n"] \arrow[d, hook] & \mathtt{PGAff}_k \arrow[d, hook] \\ \mathtt{NPAff}_k \times_{\mathtt{Aff}_k} \mathtt{NGrp}_k \arrow[r, "\mathrm{Rep}_n"] & \mathtt{PAff}_k \times_{\mathtt{Aff}_k} \mathtt{Grp}_k \end{tikzcd} \tag{3.6.2}$$

justifying, once again, Definition (3.6.2.2).

Proof. Dually, we need to show that if A is a double Poisson algebra with a cogroup structure such that the comultiplication map $\Delta : A \to A *_k A$ is a morphism of double Poisson algebras, then the induced comultiplication $\Delta_n : A_n \to (A *_k A)_n \cong A_n \otimes_k A_n$ is a morphism of Poisson algebras. This is granted once we show that the equality $(A *_k A)_n \cong A_n \otimes_k A_n$ holds not only at the level of algebras, but also at the level of double Poisson algebras, or in other words that the functor $\mathrm{Rep}_n : \mathtt{NPAff}_k \to \mathtt{PAff}_k$ (Remark 3.6.2.3(ii)) is a cartesian functor. This follows straightforwardly from the identification $(A *_k B)_n \cong A_n \otimes_k B_n$ and the trivial verification of the equality of the two Poisson structures on the generators. $\square$

Appendix A

Projective model structure on T-equivariant dg-algebras

In this Appendix we give a proof of Theorem 2.2.5.1 that gives a projective-like model structure on the category of T-equivariant dg-algebras $(\mathtt{DGA}_k^+)^T$, for an algebraic torus $T = (k^\times)^r$. We use the same strategy used in [10], in which the authors prove that the category of *bigraded* dg-algebras $\mathtt{BiDGA}_k$ has a projective-like model structure[1]. The key observation is to recognise that $\mathtt{BiDGA}_k$ being the category of dg-algebras with an additional non-negative (polynomial) compatible grading, is equivalent to the category of T-equivariant dg-algebras with a polynomial torus action (i.e. weight spaces are only for non-negative weights), and that the polynomial condition can be dropped, and substituted by the rational condition, in which weights can be arbitrary integers.

More precisely, weight spaces for a torus $T = (k^\times)^r$ are r-tuples of integers $n \in \mathbb{Z}^r$, and we observe that the category of dg-algebras with a rational T-action $(\mathtt{DGA}_k^+)^T$ is equivalent to the category of dg-algebras $A \in \mathtt{DGA}_k^+$ with:

1. An additional grading of the underlying chain complex $A = \oplus_{n \in \mathbb{Z}^r} A(n)$. This means that each $A(n)$ is a complex of vector spaces preserved by the differential in A: $dA(n) \subset A(n)$.

[1] Which ultimately follows the explicit proofs of the existence of the projective model structure on $\mathtt{DGA}_k^+$ by [51] or [26].

2. The grading is compatible with the multiplication in A: $A(n) \cdot A(m) \subset A(n+m)$.

In fact, on one hand if $A \in (\mathtt{DGA}_k^+)^T$ then for $n \in \mathbb{Z}^r$ we define

$$A(n) = \{a \in A \mid t \cdot a = t^n a,\ \forall t \in T\}$$

as the corresponding weight space and the above 2 conditions are satisfied thanks to the rationality of the action (recall, Definition 2.2.5.1). On the other hand, obviously if we have such a decomposition we define the T-action on A accordingly by $t \cdot a := \sum_n t^n a(n)$, where $a = \sum_n a(n)$, and the resulting T-action is rational.

The observation that $(\mathtt{DGA}_k^+)^T$ is equivalent to the category of dg-algebras with an additional grading as described above will be also useful later, and we will use indifferently one or the other property, according to what is more convenient from time to time.

Let us also denote by $k[T] = \mathcal{O}(T)$, a Laurent polynomial ring in r variables and observe that

Lemma A.0.0.1. *The forgetful functor $U : (\mathtt{DGA}_k^+)^T \to \mathtt{DGA}_k^+$ is left-adjoint to the "free T-equivariant extension" functor:*

$$\begin{tikzcd} (\mathtt{DGA}_k^+)^T \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathtt{DGA}_k^+ \arrow[l, shift left, "k[T]\otimes(-)"] \end{tikzcd} . \tag{A.0.1}$$

Proof. The adjunction is given by the natural isomorphisms:

$$\mathrm{Hom}_{\mathtt{DGA}_k^+}(UA, B) \cong \mathrm{Hom}_{(\mathtt{DGA}_k^+)^T}(A, k[T] \otimes B)\,,$$

where to a T-equivariant morphism $\varphi : A \to k[T] \otimes B$ we assign the composition with the evaluation map at $1 \in T$:

$$A \xrightarrow{\varphi} k[T] \otimes B \xrightarrow{ev_1 \otimes 1_B} k \otimes B \cong B\,.$$

Conversely if we start from a map $f : UA \to B$ which is not necessarily T-equivariant, we can construct a T-equivariant map $\varphi : A \to k[T] \otimes B$ by decomposing:

$$\varphi : \bigoplus_{n \in \mathbb{Z}^r} A(n) \to \bigoplus_{n \in \mathbb{Z}^r} B \cdot t^n\,,$$

and defining $\varphi_{|_{A(n)}} : A(n) \to B \cdot t^n$ as $f_{|_{A(n)}}(-) \cdot t^n$. $\square$

In order to prove Theorem 2.2.5.1 we need a few definitions and lemmas. Throughout this section of the Appendix we denote by $\mathtt{C} = \mathtt{DGA}_k^+$ and by $\mathtt{C}^T = (\mathtt{DGA}_k^+)^T$.

Notation. We denote by $\mathcal{C}of, \mathcal{WE}, \mathcal{F}ib$ the collection of cofibrations, weak equivalences, and fibrations in the projective model structure on $\mathtt{C}$. So $\mathcal{F}ib$ are surjective maps in positive homological degrees, $\mathcal{WE}$ are the quasi-isomorphisms, and $\mathcal{C}of = {}^{\boxslash}(\mathcal{WE} \cap \mathcal{F}ib)$, where ${}^{\boxslash}(-)$ denotes the collection of morphisms with the left lifting property with respect to another collection of morphisms. Finally recall that a fibration which is also a quasi-isomorphism is actually surjective in *all* homological degrees, so that $\mathcal{WE} \cap \mathcal{F}ib$ consists of surjective quasi-isomorphisms.

Definition A.0.0.1. A morphism $i : S \to R \in \mathtt{C}^T$ is a T-*equivariant noncommutative Tate extension* (also simply a Tate extension) if there is a (possibly infinite) sequence $V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \ldots$ of (homologically) graded, T-equivariant vector spaces such that

1. Each $S *_k T(V^{(i)})$ has a differential and a compatible embedding $S *_k T(V^{(i)}) \subset R$ such that at the limit $V = \cup_i V^{(i)}$:
$$S *_k TV = \varinjlim_i S *_k T(V^{(i)}) = R\,.$$
2. Each differential has the property that $d(V^{(i)}) \subset S *_k T(V^{(i-1)})$ (and for $i = 0$, $d(V^{(0)}) \subset S$).

We denote the collecion of such morphism by $\mathcal{TE} \subset \mathrm{Mor}(\mathtt{C}^T)$.

Lemma A.0.0.2. *(i) Every morphism $S \to A$ in $\mathtt{C}^T$ has a factorisation of the form $S \xrightarrow{i} R \xrightarrow{p} A$ where $i \in \mathcal{TE}$ and $p \in U^{-1}(\mathcal{WE} \cap \mathcal{F}ib)$ (is a surjective quasi-isomorphism).*

(ii) Every Tate extension has the left lifting property with respect to morphisms that are surjective quasi-isomorphisms: $\mathcal{TE} \subset {}^{\boxslash}(U^{-1}(\mathcal{WE} \cap \mathcal{F}ib))$.

For the proof one can check that the proof of Proposition 3.1 (which relies on Proposition 2.1(ii)) of [26] can be used also in this case of T-equivariant (i.e. additionally graded) objects.

Now let x be a variable of positive homological degree as well as of some weight $n \in \mathbb{Z}^r$ for the torus T, and set $V_x := [0 \to k \cdot x \to k \cdot dx \to 0]$, and its tensor algebra $T(V_x) \in C^T$. Extensions by objects of this form play another important role:

Definition A.0.0.2. A morphism in C^T of the form $S \to S *_k \coprod_{i \in I} T(V_{x_i})$, where I is any, possibly uncountably infinite, indexing set, is called a *special extension*. We denote the collection of special extensions by $\mathcal{SE} \subset \mathrm{Mor}(C^T)$.

Lemma A.0.0.3. *(i) Every morphism $S \to A$ in C^T has a factorisation of the form $S \xrightarrow{i} R \xrightarrow{p} A$ where $i \in \mathcal{SE}$ and $p \in U^{-1}(\mathcal{F}ib)$.*

(ii) $\mathcal{SE} \subset {}^{\boxslash}(U^{-1}(\mathcal{F}ib))$.

(iii) $\mathcal{SE} \subset U^{-1}(\mathcal{WE})$.

Proof. (i) It suffices to consider the set of elements of A of positive homological degrees as well as of some weight for the torus action: $I := \{a \in A(n)_i \,|\, n \in \mathbb{Z}^r,\ i > 0\}$. For each $a \in I$ we consider the obvious $TV_{x_a} \xrightarrow{p_a} A$ given by $p_a(x_a) = a$ (and consequently $p_a(dx_a) = da$). Then

$$S \xrightarrow{i} S *_k \coprod_{a \in I} T(V_{x_a}) \xrightarrow{f *_k \coprod_{a \in I} p_a} A$$

is the desired factorisation. (ii) and (iii) are quite obvious. $\square$

Now we have everything we need to prove that the following definition yields the desired model structure on C^T:

Definition A.0.0.3. We define weak equivalences, fibrations and cofibrations in C^T as:

$$\begin{aligned} &\mathcal{WE}^T := U^{-1}(\mathcal{WE}), \quad \mathcal{F}ib^T := U^{-1}(\mathcal{F}ib), \\ &\mathcal{C}of^T := {}^{\boxslash}(\mathcal{WE}^T \cap \mathcal{F}ib^T) = {}^{\boxslash}(U^{-1}(\mathcal{WE} \cap \mathcal{F}ib)). \end{aligned} \tag{A.0.2}$$

We observe that, by Lemma A.0.0.2(ii), Tate extensions are cofibrations: $\mathcal{TE} \subset \mathcal{C}of^T$, and by Lemma A.0.0.3, special extensions are acyclic cofibrations: $\mathcal{SE} \subset \mathcal{WE}^T \cap \mathcal{C}of^T$. In fact, it is useful to observe that

Proposition A.0.0.1. *Every acyclic cofibration in* $\mathtt{C}^{\mathsf{T}}$ *is a retract of a special extension.*

Proof. Let $i : A \to B$ be an acyclic cofibration and let us factor it as $A \xrightarrow{\widetilde{i}} R \xrightarrow{q} B$ where $\widetilde{i}$ is a special extension and q is a fibration, according to Lemma A.0.0.3(i). q is also a weak equivalence, because of the 2-out-of-3 property (see (MC2) in the proof of the next Theorem), therefore it is an acyclic fibration, and we can find a lift of the diagram:

$$\begin{tikzcd} A \arrow[r, "\widetilde{i}"] \arrow[d, "i"'] & R \arrow[d, "q"] \\ B \arrow[r, "1_B"'] \arrow[ur, dashed, "\exists l"] & B \end{tikzcd} \tag{A.0.3}$$

which proves that i is a retract of the special extension $\widetilde{i}$:

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[d, "i"'] & A \arrow[r, "1_A"] \arrow[d, "\widetilde{i}"] & A \arrow[d, "i"] \\ B \arrow[r, "l"] \arrow[rr, bend right, "1_B"'] & R \arrow[r, "q"] & B \end{tikzcd} \tag{A.0.4}$$

□

Theorem (2.2.5.1). *1. Definition A.0.0.3 defines a model structure on* $\mathtt{C}^{\mathsf{T}}$.

2. The forgetful functor $U : \mathtt{C}^{\mathsf{T}} \to \mathtt{C}$ *preserves cofibrations.*

Proof. (1) (MC1) (notation of Definition 3.3 of [23]): finite limits and colimits exist in $\mathtt{C}^{\mathsf{T}}$ because equalizers, coequalizers, finite product and finite coproducts exist (the same constructions as in $\mathtt{C}$ work in the equivariant setting). (MC2) $\mathcal{WE}^{\mathsf{T}}$ has the 2-out-of-3 property because it is $U^{-1}(\mathcal{WE})$ with $\mathcal{WE}$ having the 2-out-of-3 property. (MC3) $\mathcal{WE}^{\mathsf{T}}$ and $\mathcal{F}ib^{\mathsf{T}}$ are closed under retracts because, again, defined as U^{-1} of classes closed under retracts. $\mathcal{C}of^{\mathsf{T}}$ are closed under retracts because they are defined as the morphisms with the left lifting property with respect to some class ${}^{\boxslash}\mathcal{A}$, and this is always closed under retracts (it does not matter what $\mathcal{A}$ is). (MC4) We need to prove that for a diagram in $\mathtt{C}^{\mathsf{T}}$ of the following form:

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle p} \\ B & \xrightarrow{g} & D \end{array} \tag{A.0.5}$$

a lift exists in the following situations: (i) i is a cofibration and p is an acyclic fibration, (ii) i is an acyclic cofibration and p is a fibration. (i) is obviously true by the definition of cofibrations. To prove that a lift exists in the case (ii), thanks to Proposition A.0.0.1 we only need to find a lift when i is a special extension, but this is true by Lemma A.0.0.3(ii). (MC5) We need to prove that each morphism $S \to A$ in $\mathtt{C}^{\mathsf{T}}$ has factorisations of the form: (i) cofibration followed by an acyclic fibration, (ii) acyclic cofibration followed by a fibration. (i) follows from Lemma A.0.0.2(i), and (ii) follows from Lemma A.0.0.3(i).

(2) This follows from the fact that U is left adjoint to $k[T] \otimes (-)$ (Lemma A.0.0.1), and the latter preserves weak equivalences and fibrations, therefore U preserves cofibrations. $\square$

Appendix B

Representation theory of $G = G_\nu$

In this Appendix we recall the theory of irreducible representations of (a product of) general linear groups and we fix the notation. Polynomial irreducible representations of $GL_\nu(\mathbb{C})$ are labelled by ordinary (non-negative) partitions $\lambda = (\lambda_1, \dots, \lambda_\nu)$. More precisely, they are obtained by applying the Schur functors $\mathbb{S}_\lambda(-) : \mathtt{Vect}_\mathbb{C} \to \mathtt{Vect}_\mathbb{C}$ to the standard representation $V = \mathbb{C}^\nu$:

$$\mathbb{S}_\lambda(V)\,. \tag{B.0.1}$$

Their characters, the Schur polynomials, form a linear basis of the ring of symmetric polynomials in ν variables:

$$s_\lambda(x) := \mathrm{ch}(\mathbb{S}_\lambda(V)) \in \mathbb{Z}[x_1, \dots, x_\nu]^{\Sigma_\nu}\,. \tag{B.0.2}$$

Examples are

1. $V = \mathbb{C}^\nu$ itself is $\mathbb{S}_{(1,0,\dots,0)}(V)$ and $s_{(1,0,\dots,0)}(x) = x_1 + \cdots + x_\nu$.

2. More generally $\mathbb{S}_{(d,0,\dots,0)}(V) = \mathrm{Sym}^d(V)$ and $s_{(d,0,\dots,0)}(x) = h_d(x)$ the complete symmetric polynomial.

3. For $\lambda = (1, 1, \dots, 1, 0, \dots 0)$ with 1 repeated d-times, $\mathbb{S}_\lambda(V) = \Lambda^d(V)$ and $s_\lambda(x) = e_d(x)$ the elementary symmetric polynomial.

4. 1-dimensional representations are given by $\underline{m} := (m, m, \dots, m)$, for which $\mathbb{S}_{\underline{m}}(V) = \det(V)^m$, and $s_\lambda(x) = e_\nu(x)^m = x_1^m \cdots x_\nu^m$.

If we shift a partition λ to $\lambda + \underline{m} := (\lambda_1 + m, \dots, \lambda_\nu + m)$ we have

$$\mathbb{S}_{\lambda+\underline{m}}(V) = \mathbb{S}_\lambda(V) \otimes \det(V)^m\,, \tag{B.0.3}$$

which allows to extend the definition of Schur functors to partitions made possibly of some negative parts $\lambda \in \mathcal{P}_\nu := \{\lambda \in \mathbb{Z}^\nu \,|\, \lambda_1 \geqslant \cdots \geqslant \lambda_\nu\}$ as

$$\mathbb{S}_\lambda(V) := \mathbb{S}_{\lambda-\underline{\lambda_\nu}}(V) \otimes \det(V)^{\lambda_\nu}\,. \tag{B.0.4}$$

All irreducible rational representations of $\mathrm{GL}_\nu(\mathbb{C})$ are of the form (B.0.4) for some integer-valued partition $\lambda \in \mathcal{P}_\nu$. Their characters are generalised Schur polynomials and they form a linear basis of the ring of symmetric Laurent polynomials in ν variables:

$$s_\lambda(x) = \mathrm{ch}(\mathbb{S}_\lambda(V)) \in \mathbb{Z}[x_1, x_1^{-1}, \dots, x_\nu, x_\nu^{-1}]^{\Sigma_\nu}\,. \tag{B.0.5}$$

If now $\boldsymbol{\nu} = (\nu_1, \dots, \nu_n)$ is a dimension vector and $G_{\boldsymbol{\nu}} = \prod_i \mathrm{GL}_{\nu_i}(\mathbb{C})$ is a product of general linear groups, then its irreducible rational representations are labelled by n-tuples of partitions $\lambda = (\lambda^{(1)}, \dots, \lambda^{(n)}) \in \prod_i \mathcal{P}_{\nu_i}$, as the external tensor product of Schur modules:

$$V_\lambda := \mathbb{S}_{\lambda^{(1)}}(\mathbb{C}^{\nu_1}) \boxtimes \cdots \boxtimes \mathbb{S}_{\lambda^{(n)}}(\mathbb{C}^{\nu_n})\,. \tag{B.0.6}$$

Their characters are products of (generalised) Schur polynomials and we denote them by (the same notation as in (2.4.31)):

$$f_\lambda(x) := \mathrm{ch}(V_\lambda) = s_{\lambda^{(1)}}(x^{(1)}) \cdots s_{\lambda^{(n)}}(x^{(n)})\,, \tag{B.0.7}$$

where $x = (x^{(1)}, \dots, x^{(n)})$ and each $x^{(i)}$ is a set of ν_i variables: $x^{(i)} = (x_1^{(i)}, \dots, x_{\nu_i}^{(i)})$.

Appendix C

Derived coproducts

Let $\mathtt{C}$ be a model category and $S \downarrow \mathtt{C}$ the under category with respect to a fixed object S. The natural forgetful functor is right adjoint to the coproduct by S:

$$\begin{tikzcd} \mathtt{C} \arrow[r, shift left, "S\amalg -"] \arrow[r, phantom, "\perp"] & S \downarrow \mathtt{C} \arrow[l, shift left, "U"] \end{tikzcd} .$$

The model structure on $S \downarrow \mathtt{C}$ is the one with cofibrations, weak equivalences, and fibrations the preimage of the corresponding classes under the forgetful functor (therefore making the pair $(S \amalg -, U)$ a Quillen pair). The initial object in $S \downarrow \mathtt{C}$ is S with identity map as structure map $S \to S$, while the final object is the final object in $\mathtt{C}$, with structure map the unique map $S \to *$. Push-outs and pull-backs in $S \downarrow \mathtt{C}$ are computed as in $\mathtt{C}$ (with structure maps coming from the additional structure maps in the push-out/pull-back data). In particular the coproduct in $S \downarrow \mathtt{C}$ is the push-out in $\mathtt{C}$ of the diagram $\bullet \leftarrow S \to \bullet$, and we denote this coproduct by the standard symbol $A \amalg_S B$. The coproduct in a model category (in this case $S \downarrow \mathtt{C}$) is left adjoint to the diagonal functor:

$$\begin{tikzcd} (S \downarrow \mathtt{C})^{\times 2} \arrow[r, shift left, "-\amalg_S -"] \arrow[r, phantom, "\perp"] & S \downarrow \mathtt{C} \arrow[l, shift left, "\Delta"] \end{tikzcd} , \tag{C.0.1}$$

which obviously preserves all classes of maps (cofibrations, weak equivalences, fibrations). As a consequence the above pair is a Quillen pair and the coproduct has a total left derived functor, which we denote by

$\mathbf{L}(-\amalg_S -) = -\amalg^{\mathbf{L}}_S -$, and is a priori computed by picking cofibrant replacements for both variables. However, in the case of a left proper model category (weak equivalences are preserved by pushout along cofibrations) ordinary pushouts along cofibrations compute homotopy pushouts ([44, Proposition A.2.4.4.(ii)]), therefore the derived coproduct is computed by picking a cofibrant replacement of only one of the two variables:

$$A \amalg^{\mathbf{L}}_S B \cong A \amalg^{\mathbf{L}}_S QB\,, \tag{C.0.2}$$

where QB is a cofibrant replacement of B in the under category $S \downarrow \mathtt{C}$, or equivalently, a diagram $S \hookrightarrow QB \overset{\sim}{\twoheadrightarrow} B$ in $\mathtt{C}$. Finally we remark that the categories we are interested in, such as dg algebras or commutative dg algebras over a field $\mathtt{C} = \mathtt{DGA}_k, \mathtt{CDGA}_k$ (with coproduct being, respectively, the free product and the tensor product) are left proper model categories (see [15, Remark 2.15]).

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