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ARTICLE

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On coloring digraphs with forbidden induced subgraphs

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Abstract

We prove a conjecture by Aboulker, Charbit, and Naserasr by showing that every oriented graph in which the out-neighborhood of every vertex induces a transitive tournament can be partitioned into two acyclic induced subdigraphs. We prove multiple extensions of this result to larger classes of digraphs defined by a finite list of forbidden induced subdigraphs. We thereby resolve several special cases of an extension of the famous Gyárfás–Sumner conjecture to directed graphs stated by Aboulker et al.

KEYWORDS

directed graphs, graph coloring, Gyárfás–Sumner conjecture, induced subgraphs

1 | INTRODUCTION

All graphs and digraphs considered in this paper are simple. We say that a digraph is an *oriented graph* if it does not contain directed cycles of length two (digons) and we call it *bioriented* if each of its arcs belongs to a digon. Given a digraph D, an *acyclic k-coloring* of D, also referred to as a *k-dicoloring*, is an assignment $c : V(D) \to S$ of colors from a color set S of size k to the vertices such that every *color class* $c^{-1}(i), i \in S$ induces an *acyclic* subdigraph of D. The *dichromatic number* $\vec{\chi}(D)$ as introduced by Erdős [6] and Neumann-Lara [8] is the smallest integer k for which an acyclic k-coloring of D exists. Given a set of (di)graphs F, we denote by $\mathrm{Forb_{ind}}(F)$ the set of (di)graphs which do not contain an induced sub(di)graph isomorphic to a member of F. Given a class of (di)graphs $\mathcal{C}$, we denote by $\vec{\chi}(\mathcal{C})$ (or $\chi(\mathcal{C})$, respectively) the maximum (di) chromatic number of (di)graphs in the class (∞ if the latter is unbounded). We say that a finite set

F of (di)graphs is *heroic*, if $\mathrm{Forb}_{\mathrm{ind}}(F)$ has bounded (di)chromatic number. For $k \in \mathbb{N}$, we denote by $\vec{K}_k$ the transitive tournament of order k.

A classical research topic in the theory of graph coloring is to study the chromatic number of graphs with forbidden induced subgraphs, we refer to [9] for a recent survey article summarizing important results on this topic. Extending this area of research to digraphs, Aboulker, Charbit, and Naserasr [2] recently initiated the systematic study of the relation between excluded induced subdigraphs and the dichromatic number and asked the following intriguing question.

Problem 1.1. Characterize the inclusionwise minimal heroic sets of digraphs.

In the following, let us mention a few related questions and results from the literature.

- Maybe the most important open problem for coloring undirected graphs with forbidden induced subgraphs, namely, the *Gyárfás–Sumner Conjecture* [7, 10], can be restated in digraph terminology as follows:

Conjecture 1.2. *If a minimal heroic set F of digraphs includes $\vec{K}_2$ (the oriented edge), then F consists of at most three members, namely, $\vec{K}_2$, a biorientation of a forest, and a biorientation of a clique.*

Note that in the above setting, $\mathrm{Forb}_{\mathrm{ind}}(F)$ contains only bioriented graphs, whose chromatic number coincides with their dichromatic number. Hence, the above is the same as saying that the minimal heroic sets of graphs consist of a forest and a clique, which is the classical phrasing of the Gyárfás–Sumner Conjecture.

- In [4] Berger, Choromanski, Chudnovsky, Fox, Loebl, Scott, Seymour, and Thomassé studied the dichromatic number of tournaments which exclude a *single* fixed tournament H as an (induced) subdigraph. They defined a *hero* as a tournament H such that the tournaments excluding isomorphic copies of H have bounded dichromatic number. In other words, a digraph H is a hero if the set $\{\overleftrightarrow{K}_2, \overline{K}_2, H\}$ is heroic, where $\overline{K}_2$ is the anticlique of order 2. The main result of Berger et al. in [4] is a recursive characterization of all heroes, it implies in particular that all tournaments on at most four vertices are heroes.

Motivated by Problem 1.1, Aboulker et al. [2] made the following conjecture for oriented graphs which exclude the oriented out- (or in-) star S_2^+ (or S_2^-) with two leaves, as well as the directed triangle $\vec{C}_3$.

Conjecture 1.3 (cf. Aboulker et al. [2, Conjecture 15]).

$$\vec{\chi}\left(\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K}_2, S_2^+, \vec{C}_3\right)\right) = \vec{\chi}\left(\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K}_2, S_2^-, \vec{C}_3\right)\right) = 2.$$

Note that by the symmetry of reversing all arcs, it suffices to prove Conjecture 1.3 for the out-star S_2^+. The digraphs in $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K}_2, S_2^+, \vec{C}_3\right)$ are exactly the oriented graphs with no directed triangle such that the out-neighborhood of every vertex induces a tournament. As the first main result of this paper, we prove Conjecture 1.3.

In fact, it will be convenient to prove the following stronger result involving the hero W_3^+, consisting of a vertex connected by three arcs to the vertices of a directed triangle. It directly implies Conjecture 1.3 via $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, \vec{C_3}\right) \subseteq \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$.

Theorem 1. $\vec{\chi}\left(\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)\right) = 2.$

Note that the members of $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ are exactly the oriented graphs in which the out-neighborhood of every vertex induces a transitive tournament. To prove Theorem 1, in Section 2, we will exhibit and exploit some structural properties of digraphs in this class.

A more general conjecture of Aboulker et al. (cf. Conjecture 11 in [2]) would imply that for *every* hero H the triple $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ is heroic. Motivated by this open problem, we extend Theorem 1 to more triples $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ for certain heroes H. In [4] it was shown that every hero can be generated using three types of recursive construction steps. In our second main result we show that one of these three steps, namely, adding a dominating sink to a hero, preserves the heroicness of the triple $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$.

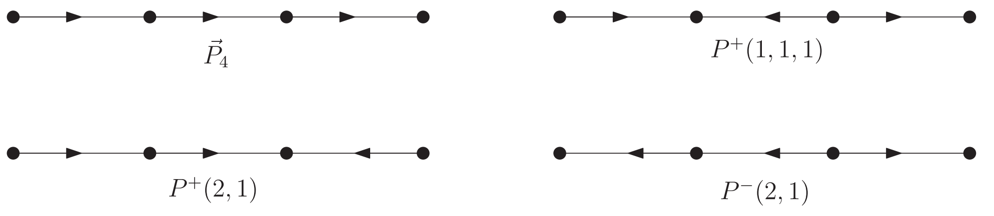
Theorem 2. *Let H be a hero and let H^- be the hero obtained from H by adding a dominating sink. If $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ is heroic, then so is $\left\{\overleftrightarrow{K_2}, S_2^+, H^-\right\}$.*

The above in particular implies that the triple $\left\{\overleftrightarrow{K_2}, S_2^+, W_3^-\right\}$ is heroic, where W_3^- denotes the 4-vertex tournament consisting of a directed triangle and a dominating sink. Members of $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^-\right)$ correspond to the oriented graphs in which the out-neighborhood of each vertex induces a tournament, while the in-neighborhood of every vertex does not contain directed triangles.

Our last new result in this paper concerns another conjecture of Aboulker et al., which can be stated as follows ($\vec{K_k}$ denotes the transitive tournament on k vertices).

Conjecture 1.4 (cf. Aboulker et al. [2, Conjecture 11]). *For every orientation of a forest F and every $k \in \mathbb{N}$ the triple $\{\overleftrightarrow{K_2}, F, \vec{K_k}\}$ is heroic.*

As noted in [2], Conjecture 1.4 holds true for forests on at most three vertices. The first open cases therefore appear when F is an orientation of P_4. Aboulker et al. considered the directed path $\vec{P_4}$ and showed in one of their main results that the set $\{\overleftrightarrow{K_2}, \vec{P_4}, \vec{K_3}\}$ is heroic. There are three other oriented paths on four vertices. Two of them, which are called $P^+(2, 1)$ and $P^-(2, 1)$ in [2], consist of two oppositely oriented dipaths of length two and one, respectively.



As noted by Aboulker et al., Chudnovsky, Scott, and Seymour proved in [5] that for every $k \in \mathbb{N}$, digraphs in the set $\mathrm{Forb}_{\mathrm{ind}}(\overleftrightarrow{K_2}, P, \overrightarrow{K_k})$ have underlying graphs with bounded chromatic number (and thus bounded dichromatic number) for $P \in \{P^+(2, 1), P^-(2, 1)\}$. Hence, Conjecture 1.4 holds for these two orientations of P_4. This leaves open the remaining orientation of P_4, denoted $P^+(1, 1, 1)$ in [2]. Here we complement the result of Aboulker et al. [2] concerning $\overrightarrow{P_4}$, showing that also the set $\{\overleftrightarrow{K_2}, P^+(1, 1, 1), \overrightarrow{K_3}\}$ is heroic.

Theorem 3. $\vec{\chi}(\mathrm{Forb}_{\mathrm{ind}}(\overleftrightarrow{K_2}, P^+(1, 1, 1), \overrightarrow{K_3})) = 2$.

1.1 | Structure of the paper

In Section 2 we investigate the structure of digraphs in the class $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ and use these insights to prove Theorem 1. In Section 3 we prove Theorem 2. Finally, in Section 4 we prove Theorem 3 and we conclude with final comments in Section 5.

1.2 | Note

After the submission of this manuscript, independently discovered proofs both for Theorem 1 and Remark 5.2 (which appears in the conclusion) have appeared in the arXiv-preprint [1] by Aboulker, Aubian, and Charbit. Their proof of Theorem 1 is quite different, as they obtain and use a full structural characterization of the class $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ of digraphs. In our work, however, the approach was not to obtain such a structural result, but instead to generate 2-colorings of these oriented graphs directly using local reductions.

1.3 | Notation

Given a digraph D, we denote by $V(D)$ its vertex-set and by $A(D) \subseteq V(D) \times V(D)$ its set of arcs. We put $v(D) := |V(D)|$, $a(D) := |A(D)|$. Arcs are denoted as (u, v), where u is the tail of the arc and v is its head. For $v \in V(D)$ we denote by $N_D^+(u), N_D^-(u)$ the sets of out- and in-neighbors of v in D, respectively. We generalize this notation to vertex subsets by putting $N_D^+(X) := \bigcup_{x \in X} N_D^+(x) \backslash X$ and $N_D^-(X) := \bigcup_{x \in X} N_D^-(x) \backslash X$ for all $X \subseteq V(D)$. We further denote by $D[X]$ the subdigraph with vertex-set X and arc-set $(X \times X) \cap A(D)$. Any digraph of the form $D[X]$ with $\emptyset \neq X \subseteq V(D)$ is called an *induced subdigraph* of D. Given a set X of vertices or arcs, we denote by $D - X$ the digraph obtained from D by deleting X.

2 | $\{S_2^+, W_3^+\}$-FREE ORIENTED GRAPHS

In this section, we will prove Theorem 1 and thereby show that $\left\{\overleftrightarrow{K_2}, S_2^+, W_3^+\right\}$ is a heroic set. An important subclass of $\text{Forb}_{\text{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ is the so-called *round digraphs*, cf. [2, 3]. A strongly connected digraph is called *round* if it admits a cyclical vertex-ordering $(v_1, \ldots, v_n)$ such that for every arc $(v_i, v_j) \in A(D)$ and each $i < k < j$, both (v_i, v_k) and (v_k, v_j) are also arcs (indices ordered cyclically). For these more special digraphs, 2-colorability has been noted already in [2], we refer to Figure 1 for examples from this class.

Given $D \in \text{Forb}_{\text{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, we define $F(D)$ to be the spanning subdigraph of D consisting of the arcs $(x, y) \in A(D)$ such that y is the source in the transitive tournament induced by the out-neighborhood of x in D. From the definition of $F(D)$ we immediately obtain:

Observation 2.1. Let $D \in \text{Forb}_{\text{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ and $(x, y) \in A(F(D))$. Then we have

$$N_D^+(x) \subseteq N_D^+(y) \cup \{y\}.$$

A useful notion for our proof is *out-modules* in digraphs, as defined next.

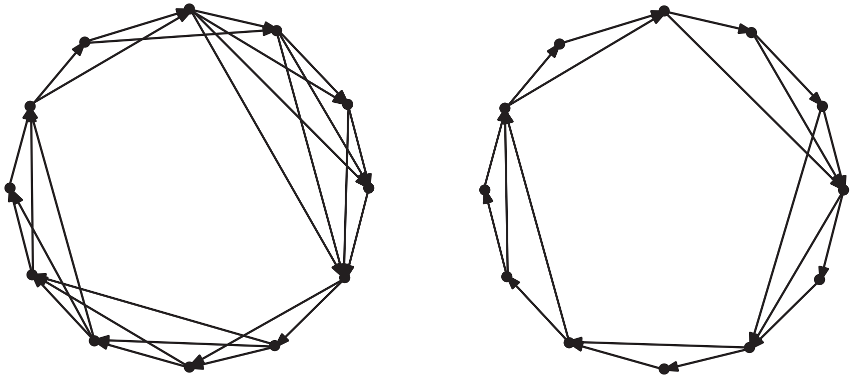


FIGURE 1 Two examples of oriented graphs with transitive out-neighborhoods. The digraph on the left is a round digraph, while the right one is not. In both cases, $F(D)$ consists of the arcs of the outer directed cycle.

Definition 2.2. Let D be a digraph, and $\emptyset \neq M \subseteq V(D)$. We say that M is an *out-module* in D if it holds that $(x, z) \in A(D) \Rightarrow (y, z) \in A(D)$ for every $x, y \in M$ and $z \in V(D) \backslash M$. Equivalently, $N_D^+(x) \backslash M = N_D^+(y) \backslash M$ for all $x, y \in M$.

We remark the following simple fact for later use.

Observation 2.3. Let D be a digraph and $M \subseteq V(D)$ such that $D[M]$ is strongly connected. If $(x, y) \in A(D) \wedge (x, z) \in A(D) \Rightarrow (y, z) \in A(D)$ for every $x, y \in M$ and $z \in V(D) \backslash M$, then M is an out-module in D.

Proof. To verify that $(x, z) \in A(D) \Rightarrow (y, z) \in A(D)$ for every $x, y \in M$ and $z \in V(D) \backslash M$, it suffices to consider a directed x–y path $x = x_1, \ldots, x_\ell = y$ in $D[M]$ and the logical chain $(x_1, z) \in A(D) \Rightarrow (x_2, z) \in A(D) \Rightarrow \cdots \Rightarrow (x_\ell, z) \in A(D)$. □

For a nonempty vertex-set U in a digraph D, we denote by D/U the digraph obtained by *identifying* U, that is, the digraph with vertex-set $(V(D) \backslash U) \cup \{x_U\}$, where $x_U \notin V(D)$ is some newly added vertex representing U, and the following arcs: the arcs of D inside $V(D) \backslash U$, the arc (x_U, v) for each $v \in N_D^+(U)$, and the arc (v, x_U) for each $v \in N_D^-(U)$.

In the following we prepare the proof of Theorem 1 with a set of useful lemmas, starting with two operations for digraphs which preserve the containment in the class $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$.

Lemma 2.4. *For every* $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$ *and for every out-module* $U \subseteq V(D)$ *it holds that* $D/U \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$.

Proof. Suppose towards a contradiction that D/U contains a $\overleftrightarrow{K_2}$ spanned by vertices x, y. Clearly we need to have $x_U \in \{x, y\}$, say $x = x_U$. By definition of D/U there are $u_1, u_2 \in U$ such that $(u_1, y), (y, u_2) \in A(D)$. Since U is an out-module, the vertices u_2, y span a $\overleftrightarrow{K_2}$ in D, which is impossible. Next, suppose there are vertices x, y, z spanning an S_2^+ in D/U, where x is the central vertex. Clearly we need to have $x_U \in \{x, y, z\}$. If $y = x_U$ or $z = x_U$, we find an S_2^+ in D by replacing x_U with an out-neighbor of x in U. If $x = x_U$, then by definition of D/U and since U is an out-module every vertex $u \in U$ together with y, z spans an S_2^+ in D, in each case a contradiction. Finally, suppose that there is a copy of W_3^+ in D/U, which clearly must contain x_U. If x_U is the source vertex of the W_3^+, then since U is an out-module any vertex $u \in U$ together with the remaining three vertices induces a W_3^+ in D. If x_U is not the source vertex of the W_3^+, then let x, y, z denote the other three vertices of the W_3^+, such that x is the source vertex, and $(x_U, y), (y, z), (z, x_U) \in A(D/U)$. Let $u \in U$ be a vertex such that $(x, u) \in A(D)$. Since U is an out-module, we must have $(u, y) \in A(D)$. Then u, y, z are three out-neighbors of x in D and $(u, y), (y, z) \in A(D)$. Hence, by transitivity (recall that $D\left[N_D^+(x)\right]$ is a transitive tournament) $(u, z) \in A(D)$. However, this means that z, x_U form a $\overleftrightarrow{K_2}$ in D/U, which is impossible as shown above. □

Lemma 2.5. *Let $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, and let $(x, y) \in A(F(D))$. Let $z \in N_D^+(y)$ such that $(x, z), (z, x) \notin A(D)$. Then the digraph $D^+(x, z)$ obtained from D by adding the arc (x, z) is contained in $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$.*

Proof. As z is not connected to x, adding (x, z) does not create a $\overleftrightarrow{K_2}$. Suppose it creates an S_2^+ and let t be the vertex in the copy of S_2^+ different from x, z. Then $(x, t) \in A(D)$. But then $(y, t) \in A(D)$ (since $(x, y) \in A(F(D))$) and thus $\{y, z, t\}$ induces an S_2^+ in D. Now suppose a W_3^+ is created. First consider the case that x is the source vertex of this copy of W_3^+ and let z, a, b be the other three vertices such that $(z, a), (a, b), (b, z) \in A(D)$. If $y \notin \{a, b\}$, then since $(x, y) \in A(F(D))$ the vertices $\{y, z, a, b\}$ induce a W_3^+ in D. If $y = a$, then $\{a, z\}$ induces a $\overleftrightarrow{K_2}$ in D. Similarly, if $y = b$, then $\{a, b\}$ induces a $\overleftrightarrow{K_2}$ in D. If x is not the source vertex of the copy of W_4^+, then the source vertex together with x and z induces an S_2^+. Obtaining a contradiction in each case, we conclude the proof. □

The next lemma shows the existence of out-modules with special properties.

Lemma 2.6. *Let $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, and let $v \in V(D)$. If $N_D^-(v) \neq \emptyset$, then there exists an out-module M in D such that $D[M]$ is strongly connected, $M \subseteq N_D^-(v)$ and $N_D^+(M) \subseteq N_D^+(v) \cup \{v\}$.*

Proof. Let M be the set of vertices of a strong component of $D\left[N_D^-(v)\right]$ such that no arc leaves it in $D\left[N_D^-(v)\right]$.[1] Then M satisfies the required properties: Consider any $u \in N_D^+(M) \setminus \{v\}$ and let $t \in M$ be such that $(t, u) \in A(D)$. Then u, v are distinct out-neighbors of t and thus have to be adjacent in D. But we cannot have $u \in N_D^-(v)$, since no arc in $D\left[N_D^-(v)\right]$ leaves M, thus we have $u \in N_D^+(v)$. This verifies $N_D^+(M) \subseteq N_D^+(v) \cup \{v\}$. To see why M is an out-module, we use Observation 2.3. So let $x, y \in M$ and $z \in V(D) \setminus M$ such that $(x, y), (x, z) \in A(D)$. If $z = v$, then trivially $(y, z) \in A(D)$, so assume $z \in N_D^+(v)$. Then y, v, z are three distinct vertices in the transitive tournament induced by $N_D^+(x)$, and since $(y, v), (v, z) \in A(D)$ it follows that $(y, z) \in A(D)$ by transitivity, as desired. □

Lemma 2.7. *Let $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, let $M \subseteq V(D)$ be an out-module in D and let $v \in V(D) \setminus M$. Let T be the set of vertices defined by*

$$T := \{t \in M \mid \exists\, u \in V(D) \setminus M : (v, u), (u, t) \in A(D)\}.$$

Then $D[T]$ is a (possibly empty) transitive tournament.

[1]One may obtain such a component by contracting all strong components and selecting a component corresponding to a sink in the resulting acyclic digraph.

Proof. We prove the assertion by showing that $D[T]$ is a tournament and contains no directed triangle. First, suppose towards a contradiction there are nonadjacent members $t_1 \neq t_2$ of T. For $i = 1, 2$ let $u_i \in V(D) \backslash M$ be a vertex such that $(v, u_i), (u_i, t_i) \in M$. If $u_1 = u_2$, then u_1, t_1, t_2 span an S_2^+ in D, so $u_1 \neq u_2$. Then u_1, u_2 as out-neighbors of v must be adjacent, w.l.o.g. $(u_1, u_2) \in A(D)$. This implies further that t_1, u_2, as out-neighbors of u_1, are adjacent. However, we cannot have $(t_1, u_2) \in A(D)$, for then also $(t_2, u_2) \in A(D)$ and t_2, u_2 would form a digon in D. Hence, $(u_2, t_1) \in A(D)$ and therefore u_2, t_1, t_2 induce an S_2^+ in D, a contradiction.

Next, suppose towards a contradiction that some vertices $t_1, t_2, t_3 \in T$ form a directed triangle. Let $u_i \in V(D) \backslash M, i = 1, 2, 3$ be vertices such that $(v, u_i), (u_i, t_i) \in A(D)$. First note that $(t_i, u_j) \notin A(D)$ for every $i, j \in \{1, 2, 3\}$, for otherwise M being an out-module would imply that t_j and u_j span a $\overleftrightarrow{K_2}$ in D. Next notice that not all of u_1, u_2, u_3 can be equal, for otherwise u_1, t_1, t_2, t_3 would induce a W_3^+ in D. If exactly two of them are equal, say $u_1 \neq u_2 = u_3$, then u_1 and $u_2 = u_3$ are adjacent. If $(u_1, u_2) \in A(D)$, then t_1 and u_2 are adjacent and hence by our initial remark we have $(u_2, t_1) \in A(D)$. However, now $u_2 = u_3, t_1, t_2, t_3$ span a W_3^+ in D, a contradiction. If on the other hand $(u_2, u_1) = (u_3, u_1) \in A(D)$, then u_1 must be adjacent to both t_2 and t_3, hence u_1 sees all of t_1, t_2, t_3, so u_1, t_1, t_2, t_3 span a W_3^+, again a contradiction.

Finally consider the case that u_1, u_2, u_3 are pairwise distinct. Since they are contained in the transitive tournament induced by $N_D^+(v)$, we may assume w.l.o.g. that $(u_1, u_2), (u_1, u_3), (u_2, u_3) \in A(D)$. Then u_3 must be adjacent to both t_1 and t_2, and by our initial remark this means that $(u_3, t_1), (u_3, t_2) \in A(D)$. Hence, u_3, t_1, t_2, t_3 form a W_3^+ in this case, a final contradiction which concludes the proof. □

We are now sufficiently prepared to give the proof of Theorem 1. In fact, to make our inductive proof work we state a stronger version of the claim, which allows one to enforce a monochromatic coloring on the closed out-neighborhood of a vertex.

Theorem 4. *Let* $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, *and* $v \in V(D)$. *Then there exists an acyclic coloring* $c : V(D) \to \{1, 2\}$ *of* D *such that* $c(u) = c(v)$ *for every* $u \in N_D^+(v)$.

Proof. Suppose towards a contradiction that the claim is wrong, and let (D, v) be a counterexample to the claim minimizing $v(D)$. □

Claim 1. D is strongly connected.

Proof. Suppose not, then there is a partition of $V(D)$ into nonempty parts X, Y such that no arc in D starts in X and ends in Y. This property together with the fact that $N_D^+(v)$ induces a transitive tournament implies that there exist vertices $x \in X, y \in Y$ such that $\{v\} \cup N_D^+(v) \subseteq \left(x \cup N_{D[X]}^+(x)\right) \cup \left(y \cup N_{D[Y]}^+(y)\right)$. By minimality of (D, v), there exist acyclic 2-colorings c_X and c_Y of $D[X]$ and $D[Y]$ so that the closed out-neighborhoods of x and y in $D[X]$ and $D[Y]$, respectively, are monochromatic. Possibly after permuting colors in c_Y and putting these colorings together yields an acyclic 2-coloring of D in which the closed out-neigborhood of v is monochromatic, a contradiction (note that we do not create a monochromatic directed cycle in the process, as such a cycle would have to traverse an arc from X to Y). □

Note that Claim 1 implies that $N_D^-(v) \neq \emptyset$. Hence we may apply Lemma 2.6 to the vertex v of D and find an out-module $M \subseteq N_D^-(v)$ in D such that $D[M]$ is strongly connected and $N_D^+(M) \subseteq N_D^+(v) \cup \{v\}$. Let $T \subseteq M$ be the set of vertices $t \in M$ for which there exists $u \in N_D^+(v)$ such that $(u, t) \in A(D)$. Since $N_D^+(v) \cap M = \emptyset$, the definition of T here coincides with the one in Lemma 2.7. Now, Lemma 2.7 implies that $D[T]$ is a (possibly empty) transitive tournament.

Claim 2. The digraph $D[M]$ admits an acyclic 2-coloring $c_M : M \to \{1, 2\}$ satisfying $c_M(t) = 2$ for all $t \in T$.

Proof. Since $v(D[M]) < v(D)$, the minimality of (D, v) implies that $D[M]$ satisfies the assertion of the theorem. If $T = \emptyset$, Claim 2 is satisfied by an arbitrary choice of an acyclic 2-coloring of $D[M]$. If $T \neq \emptyset$, let $t_0 \in T$ be the source of the transitive tournament $D[T]$. Applying the assertion of the theorem to $D[M]$ and the vertex t_0, we find an acyclic 2-coloring of $D[M]$ in which t_0 has the same color as all its out-neighbors. Without loss of generality we may choose this color to be 2, and since $\{t_0\} \cup N_{D[M]}^+(t_0) \supseteq T$, the claim follows. □

Claim 3. $D[M]$ contains a directed cycle.

Proof. Since $D[M]$ is strongly connected, it suffices to rule out $|M| = 1$. Towards a contradiction suppose that $M = \{m\}$ is a single vertex. Then $N_D^+(m) = N_D^+(M) \subseteq \{v\} \cup N_D^+(v)$. Let $D' := D - m$. By minimality of (D, v), we know that D' admits an acyclic 2-coloring $c' : V(D)\backslash M \to \{1, 2\}$ in which $c'(v) = c'(u) = 1$ for every $u \in N_D^+(v)$. Let c be the extension of c' to $V(D)$ obtained by assigning color 2 to m. Then c is an acyclic coloring of D: Any newly created directed cycle must use an out-arc of m, however, we have $c(m) = 2$ and $c(x) = c'(x) = 1$ for every $x \in N_D^+(m) \subseteq \{v\} \cup N_D^+(v)$, so such a cycle has both colors. This is a contradiction, since we assumed that D does not admit an acyclic 2-coloring in which the closed out-neighborhood of v is monochromatic. □

Claim 3 in particular implies that $|M| \geq 3$ and $M\backslash T \neq \emptyset$.

Let us further note that since M forms an out-module in D, $M\backslash T \neq \emptyset$ is an out-module in the digraph $D - T \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$, and hence by Lemma 2.4 we also have $D_0 := (D - T)/(M\backslash T) \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, W_3^+\right)$. Also note that since $T \subseteq M \subseteq N_D^-(v)$, we have $N_{D_0}^+(v) = N_D^+(v)$.

Claim 4. $(x_{M\backslash T}, v) \in A(F(D_0))$, and $(u, x_{M\backslash T}) \notin A(D_0)$ for every $u \in N_{D_0}^+(v)$.

Proof. We have $M\backslash T \subseteq N_D^-(v)$ and $N_D^+(M) \subseteq \{v\} \cup N_D^+(v)$. This directly implies that $(x_{M\backslash T}, v) \in A(D_0)$ and that $N_{D_0}^+(x_{M\backslash T}) \subseteq N_D^+(M) \subseteq N^+(v) \cup \{v\} = N_{D_0}^+(v) \cup \{v\}$. Hence,

$v \in N^+_{D_0}(x_{M\setminus T})$ has an out-arc to every other out-neighbor of $x_{M\setminus T}$ in D_0, and this shows (by definition) that $(x_{M\setminus T}, v) \in A(F(D_0))$.

For the second claim, suppose towards a contradiction that there exists $u \in N^+_{D_0}(v)$ such that $(u, x_{M\setminus T}) \in A(D_0)$. By definition of D_0, this means that $u \in N^+_D(v)$ and that there exists a vertex $m \in M\setminus T$ such that $(u, m) \in A(D)$. By definition of T, this however shows that $m \in T$, a contradiction. $\square$

In the following, let $\overline{D}$ be the digraph defined by

$$V(\overline{D}) := V(D_0), A(\overline{D}) := A(D_0) \cup \left\{(x_{M\setminus T}, u) | u \in N^+_{D_0}(v)\right\}$$

(see Figure 2 for an illustration).

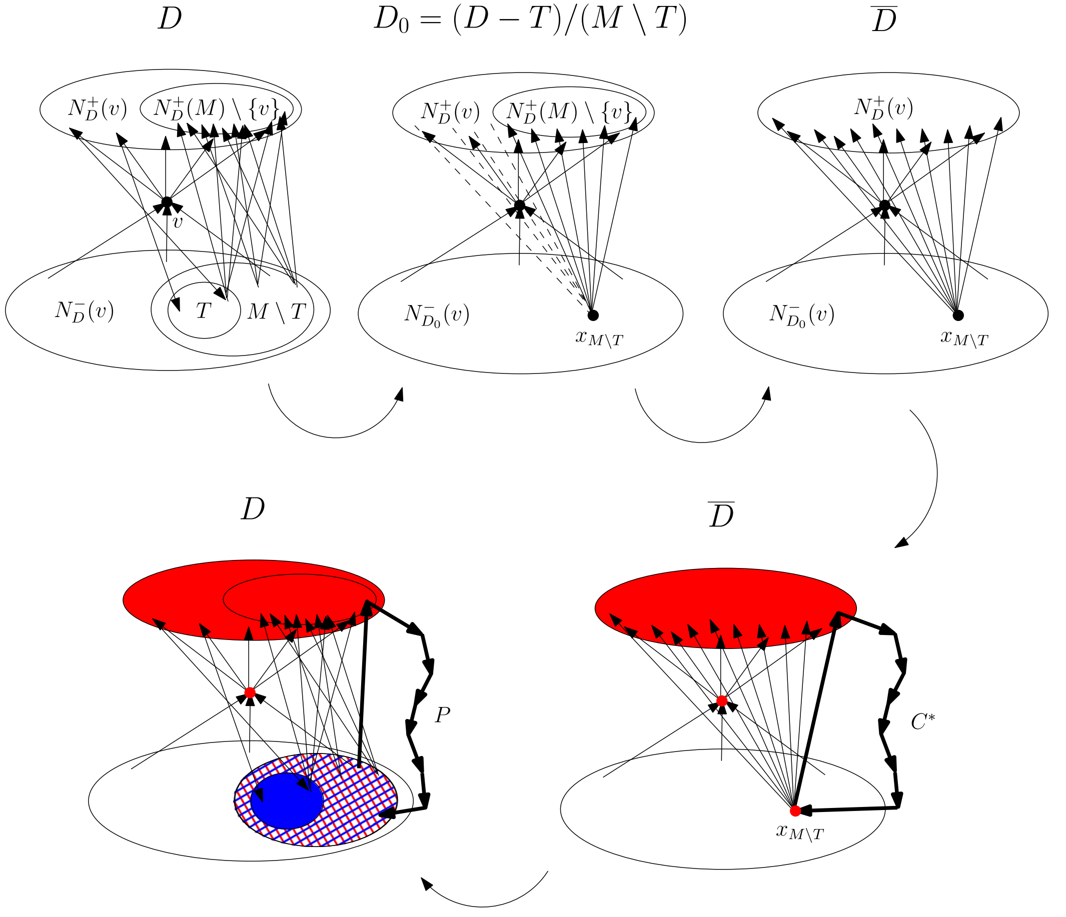


FIGURE 2 Schematic illustration of the construction of the digraph $\overline{D}$ from D (top row), and how the coloring $\bar{c}$ of $\overline{D}$ is combined with c_M to obtain an acyclic coloring c of D. The correspondence of the segment P of the monochromatic cycle C in D and the monochromatic cycle C^* in $\overline{D}$ is also indicated. Dashed edges indicate that the endpoints are not connected, while colors red and blue correspond to colors 1 and 2 in the proof.

Claim 5. $\overline{D} \in \text{Forb}_{\text{ind}}\left(\overleftrightarrow{K}_2, S_2^+, W_3^+\right)$.

Proof. Let $e_i = (x_{M\setminus T}, u_i)$, $i = 1, ..., k$ be a list of the arcs contained in $A(\overline{D})\setminus A(D_0)$ for some $k \geq 0$. For $0 \leq i \leq k$ let D_i denote the digraph defined by $V(D_i) := V(D_0)$ and $A(D_i) := A(D_0) \cup \{e_1, ..., e_i\}$. Note that $D_k = \overline{D}$.

Let us show inductively that $D_i \in \text{Forb}_{\text{ind}}\left(\overleftrightarrow{K}_2, S_2^+, W_3^+\right)$ and $(x_{M\setminus T}, v) \in A(F(D_i))$ for every $i \in \{0, 1, ..., k\}$. The claim then follows via $\overline{D} = D_k$.

For $i = 0$ the claim holds true by the previous discussions and Claim 4. Now let $1 \leq i \leq k$ and suppose we know that the claim holds for D_{i-1}.

Note that $D_i = D_{i-1}^+(x_{M\setminus T}, u_i)$, where $u_i \in N_{D_0}^+(v) = N_{D_{i-1}}^+(v)$, $(x_{M\setminus T}, v) \in A(F(D_{i-1}))$. Note that $e_i \notin A(D_{i-1})$, as well as $(u_i, x_{M\setminus T}) \notin A(D_{i-1})$ by Claim 4. Therefore Lemma 2.5 applied to D_{i-1} with $x = x_{M\setminus T}, y = v, z = u_i$ implies that $D_i \in \text{Forb}_{\text{ind}}\left(\overleftrightarrow{K}_2, S_2^+, W_3^+\right)$. To verify $(x_{M\setminus T}, v) \in A(F(D_i))$, note that the only new out-neighbor of $x_{M\setminus T}$ in D_i compared to D_{i-1} is the vertex $u_i \in N_D^+(v)$, which is seen by the vertex v and hence v still forms the source of the transitive tournament induced by the out-neighbors of $x_{M\setminus T}$ in D_i. This concludes the proof by induction. □

The number of vertices of $\overline{D}$ satisfies

$$v(\overline{D}) = v(D_0) = v(D) - |T| - (|M\setminus T| - 1) \leq v(D) - (|M| - 1) \leq v(D) - 2 < v(D)$$

since $|M| \geq 3$ by Claim 3. Hence, the minimality of D implies that the assertion of the theorem holds for $\overline{D}$. Applying this assertion to the vertex $x_{M\setminus T}$ in $\overline{D}$, we find that there exists an acyclic 2-coloring $\bar{c} : V(\overline{D}) \to \{1, 2\}$ of $\overline{D}$ such that $\bar{c}(x_{M\setminus T}) = 1 = \bar{c}(u)$ for every $u \in N_{\overline{D}}^+(x_{M\setminus T})$. Using the facts $N_{D_0}^+(x_{M\setminus T}) \subseteq N_D^+(v) \cup \{v\}$, $N_{D_0}^+(v) = N_D^+(v)$ and $(x_{M\setminus T}, v) \in A(D_0)$, the definition of $\overline{D}$ yields that $N_{\overline{D}}^+(x_{M\setminus T}) = N_D^+(v) \cup \{v\}$. Hence, we have $\bar{c}(x_{M\setminus T}) = \bar{c}(v) = \bar{c}(u) = 1$ for every $u \in N_D^+(v)$.

Let $c : V(D) \to \{1, 2\}$ be the coloring of D defined by $c(x) := c_M(x)$ for every $x \in M$, and $c(x) := \bar{c}(x)$ for every $x \in V(D)\setminus M$. We note that $c(v) = c(u)$ for all $u \in N_D^+(v)$. Hence, by the initial assumption on D, the coloring c cannot be acyclic, that is, there is a directed cycle C in D which is monochromatic in the coloring c. Since c_M is an acyclic coloring, we must have $V(C)\setminus M \neq \emptyset$. Analogously, we have $V(C) \cap M \neq \emptyset$ since otherwise C would be a directed cycle in $D - M \subseteq (D - T)/(M\setminus T) = D_0 \subseteq \overline{D}$, contradicting that $\bar{c}$ is an acyclic coloring. Hence there must be an arc $(x, y) \in A(C)$ such that $x \in M$ and $y \notin M$. However, this means that $y \in N_D^+(M) \subseteq \{v\} \cup N_D^+(v)$, and hence $c(y) = 1$. Thus C is a cycle in color 1, and since $c(t) = c_M(t) = 2$ for every $t \in T$, it cannot intersect T. Let z be the first vertex of M we meet when traversing C in the forward direction, starting at y. Then $z \in M\setminus T$. Let P be the subpath of C from x to z. Now $(V(P)\setminus\{x, z\}) \cup \{x_{M\setminus T}\}$ forms the vertex-set of a directed cycle C^* in $(D - T)/(M\setminus T) = D_0 \subseteq \overline{D}$, and it is monochromatic w.r.t. $\bar{c}$: Every vertex $x \in V(C^*)\setminus\{x_{M\setminus T}\}$ is contained in P and thus has color $\bar{c}(x) = c(x) = 1$, and also $\bar{c}(x_{M\setminus T}) = 1$ by definition. This is a contradiction to the fact that $\bar{c}$ is an acyclic coloring of $\overline{D}$. This shows that a (smallest) counterexample D to the claim of the theorem cannot exist, and concludes the proof of the theorem.

3 | ADDING A DOMINATING SINK TO A HERO

In this section our goal is to prove Theorem 2. Let us first prove the following lemma.

Lemma 3.1. *Let $D \in \mathrm{Forb_{ind}}\left(\overleftrightarrow{K_2}, S_2^+\right)$ and let $C \in \mathbb{N}$ be such that $\vec{\chi}\left(D\left[N_D^-(x)\right]\right) \leq C$ for every $x \in V(D)$. Let $u, v \in V(D)$ and let P be a shortest u–v-dipath in D. Let $X := V(P) \cup N_D^-(V(P))$. Then $\vec{\chi}(D[X]) \leq 3C + 2$.*

Proof. Let $u = x_0, x_1, ..., x_{\ell-1}, x_\ell = v$ be the vertex-trace of P and consider the partition $(A_i)_{i=1}^{\ell}$ of $N_D^-(V(P))$, where $A_i := N_D^-(x_i) \setminus (V(P) \cup \bigcup_{1 \leq j<i} A_j)$, $i = 0, ..., \ell$. □

Claim. Let $0 \leq i < j \leq \ell$ with $j - i \geq 3$. Then there exists no arc in D starting in A_i and ending in A_j.

Proof. Suppose towards a contradiction that there are vertices $x \in A_i$, $y \in A_j$ with $(x, y) \in A(D)$. Then x_i and y as out-neighbors of x must be adjacent in D. By definition of A_j we have $A_j \cap N_D^-(x_i) = \emptyset$ and hence $(x_i, y) \in A(D)$. However, now the directed path described by the vertices $u = x_0, x_1, ..., x_i, y, x_j, ..., x_\ell = v$ is a u–v-dipath in D shorter than P, a contradiction. This proves the claim. □

For every $0 \leq i \leq \ell$ we have $\vec{\chi}(D[A_i]) \leq \vec{\chi}\left(D\left[N_D^-(x_i)\right]\right) \leq C$. Let us define the set $B_r := \bigcup\{A_i | i \equiv r \pmod 3\}$ for every $r \in \{0, 1, 2\}$. From the above claim it follows that no directed cycle in $D[B_r]$ intersects two different sets A_i, A_j. Hence, we have

$$\vec{\chi}(D[B_r]) \leq \max\{\vec{\chi}(D[A_i]) | i \equiv r \pmod 3\} \leq C$$

for $r = 0, 1, 2$. Further note that the two sets

$$V_0 := \{x_i | i \in \{0, ..., \ell\} \text{ even}\}, V_1 := \{x_i | i \in \{0, ..., \ell\} \text{ odd}\}$$

both induce acyclic subdigraphs of D, for otherwise D would not be a shortest u–v-dipath in D. Since X is the disjoint union of B_0, B_1, B_2, V_0, V_1, we conclude

$$\vec{\chi}(D[X]) \leq \vec{\chi}(D[B_0]) + \vec{\chi}(D[B_1]) + \vec{\chi}(D[B_2]) + \vec{\chi}(D[V_0]) + \vec{\chi}(D[V_1]) \leq 3C + 2,$$

as required.

Proof of Theorem 2. Let $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ be heroic and $C := \vec{\chi}\left(\mathrm{Forb_{ind}}\left(\overleftrightarrow{K_2}, S_2^+, H\right)\right)$.

We claim that every digraph $D \in \mathrm{Forb_{ind}}\left(\overleftrightarrow{K_2}, S_2^+, H^-\right)$ admits an acyclic coloring with $C^- := v(H)(C + 1) + 3C + 2$ colors.

Suppose towards a contradiction that there exists some $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, H^-\right)$ with $\vec{\chi}(D') > C'$, and choose such a D minimizing $v(D)$. Then we have $\vec{\chi}(D) > C' \geq C$ and hence there is $Y \subseteq V(D)$ such that $D[Y]$ is isomorphic to H. The minimality of $v(D)$ implies that D is strongly connected, for the dichromatic number of D equals the maximum of the dichromatic numbers of its strong components.

Let $S \supseteq Y$ denote a set of vertices in D defined as follows:

If $D[Y]$ (resp., H) is strongly connected, put $S := Y$. Otherwise, let $Y_1, \ldots, Y_t$ be a partition of Y into the $t \geq 2$ strong components of $D[Y]$ such that all arcs between Y_i and Y_j start in Y_i and end in Y_j, for any $1 \leq i < j \leq t$ (note that since $D[Y]$ is a tournament all elements of $Y_i \times Y_j$ are arcs of $D[Y]$ for $1 \leq i < j \leq t$). Now pick $u \in Y_t, v \in Y_1$ arbitrarily, let P be a shortest u–v-dipath in D and put $S := V(P) \cup Y$. Let us note that in any case, $D[S]$ is strongly connected.

Let $Z := S \cup N_D^-(S)$. Then we have $Z = X \cup Y \cup N_D^-(Y)$, where X is defined as $X := \emptyset$ if $S = Y$, and as $X := V(P) \cup N_D^-(V(P))$ otherwise. For every $x \in V(D)$ we know that since D is H^--free, the digraph $D\left[N_D^-(x)\right]$ is contained in $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, H\right)$, and hence $\vec{\chi}\left(D\left[N_D^-(x)\right]\right) \leq C$. Using Lemma 3.1 we obtain that $\vec{\chi}(D[X]) \leq 3C + 2$. Putting it all together, we find that

$$\vec{\chi}(D[Z]) \leq \sum_{y \in Y} \underbrace{\vec{\chi}\left(D\left[\{y\} \cup N_D^-(y)\right]\right)}_{\leq C+1} + \vec{\chi}(D[X]) \leq v(H)(C+1) + 3C + 2 = C'.$$

□

Claim. No arc in D leaves Z.

Proof. We first note that it suffices to argue that there is no arc in D from S to $V(D) \backslash Z$: For every vertex $x \in Z \backslash S$ there is some $s \in S$ such that $x \in N_D^-(s) \subseteq Z$. Since D is S_2^+-free this implies that $N_D^+(x) \backslash Z \subseteq N_D^+(s) \backslash Z$. Hence if $s \in S$ does not see a vertex in $V(D) \backslash Z$, the same holds for x.

So suppose there exists an arc $(s, w) \in S \times (V(D) \backslash Z)$. We claim that then also $(s', w) \in A(D)$ for every $s' \in S$. Consider $s' \in S$ arbitrarily. Since $D[S]$ is strongly connected, there exist vertices $s = s_0, s_1, \ldots, s_k = s'$ in S such that $(s_{i-1}, s_i) \in A(D)$, $i = 1, \ldots, k$. But now we can deduce that $(s', w) \in A(D)$ from the logical chain $(s_0, w) \in A(D) \Rightarrow (s_1, w) \in A(D) \Rightarrow \cdots \Rightarrow (s_k, w) = (s, w) \in A(D)$, where in each step we have $(s_{i-1}, w) \Rightarrow (s_i, w)$ since s_i and w are adjacent as distinct out-neighbors of s_{i-1} and since $w \notin Z \supseteq N_D^-(s_i)$. This shows that indeed $(s', w) \in A(D)$ for all $s' \in S$. Hence $D[Y \cup \{w\}]$ is an induced subdigraph of D isomorphic to H^-, a contradiction to $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, H^-\right)$. This concludes the proof. □

Since D is strongly connected, it follows that $Z = V(D)$, and hence that $\vec{\chi}(D) = \vec{\chi}(D[Z]) \leq C'$, a contradiction which concludes the proof of the theorem.

4 | ORIENTED 4-VERTEX-PATHS

In this section we establish that $\{\overleftrightarrow{K_2}, \overrightarrow{K_3}, P^+(1,1,1)\}$ is heroic, proving Theorem 3.

Proof of Theorem 3. We prove inductively that every $D \in \text{Forb}_{\text{ind}}(\overleftrightarrow{K_2}, \overrightarrow{K_3}, P^+(1,1,1))$ is 2-colorable. The claim trivially holds for $v(D) = 1$, so suppose that $v(D) \geq 2$ and every digraph in $\text{Forb}_{\text{ind}}(\overleftrightarrow{K_2}, \overrightarrow{K_3}, P^+(1,1,1))$ having less than $v(D)$ vertices is 2-colorable. Pick some $x \in V(D)$ and define a sequence $X_0, X_1, X_2, \ldots$ of subsets of $V(D)$ as follows:

$$X_i := \begin{cases} \{x\} & \text{if } i = 0, \\ N^+(X_{i-1}) \setminus \bigcup_{j=0}^{i-1} X_j & \text{if } i \text{ odd}, \\ N^-(X_{i-1}) \setminus \bigcup_{j=0}^{i-1} X_j & \text{if } i \geq 2 \text{ even}. \end{cases}$$

The sets $(X_i)_{i \geq 0}$ are by definition pairwise disjoint, and so there exists $k \geq 1$ such that $X_1, \ldots, X_k \neq \emptyset$ and $X_i = \emptyset$ for all $i > k$. For an illustration see Figure 3. □

Claim. X_i is an independent set of D for every $i \geq 0$.

Proof. We prove the claim by induction on i. The claim trivially holds for $i = 0$ since $X_0 = \{x\}$, and since D does not contain a $\overrightarrow{K_3}$, also $X_1 = N^+(x)$ must be an independent set in D. Now let $i \geq 2$ and suppose that we already established that $X_0, \ldots, X_{i-1}$ are independent. Now suppose that there is an arc $(x, y) \in A(D[X_i])$. Let $x_1, y_1 \in X_{i-1}$ and $x_2, y_2 \in X_{i-2}$ be such that $(x_1, x_2), (x_1, x), (y_1, y_2), (y_1, y) \in A(D)$ if i is odd, respectively, (x_2, x_1), (x, x_1), (y_2, y_1), $(y, y_1) \in A(D)$ if i is even. We have $x_1 \neq y_1$, as

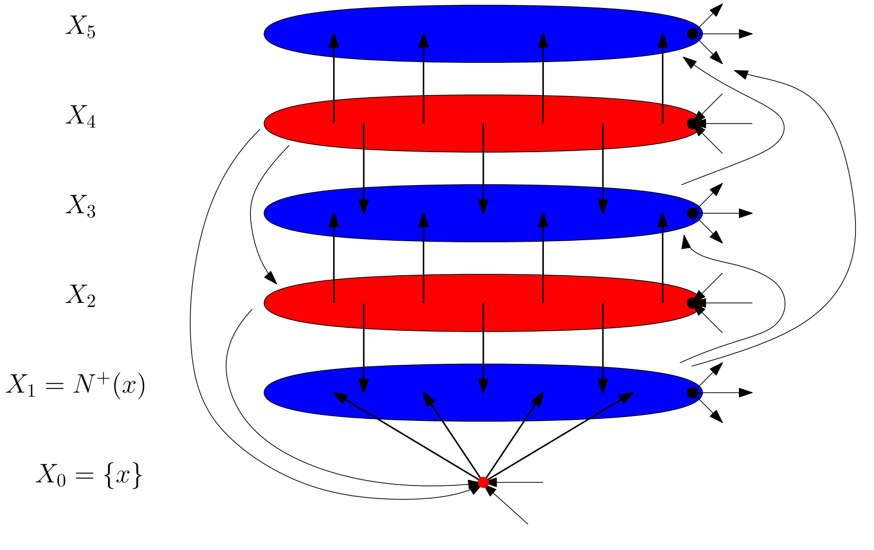


FIGURE 3 Illustration of the definition of the X_i, equipped with an alternating coloring of the sets with two colors. The additional arcs indicate possible directions of connections between different X_i-sets of the same color, as well as possible connections from these sets to the rest of the digraph.

otherwise $x_1 = y_1, x, y$ form a $\overrightarrow{K_3}$ in D. Next consider the oriented path P consisting of $x, (x, y), y, (y_1, y), y_1, (y_1, y_2), y_2$ if i is odd, and of $x_2, (x_2, x_1), x_1, (x, x_1), x, (x, y), y$ if i is even. In order for this path not to be an induced $P^+(1, 1, 1)$ two nonconsecutive vertices of the path must be adjacent. However, since D does not contain $\overrightarrow{K_3}$, this is only possible if x and y_2 (i odd), respectively, x_2 and y (i even) are adjacents. Since $x \notin X_{i-1}$, we have $x \notin N^-(X_{i-2})$ if i is odd and $y \notin N^+(X_{i-2})$ if i is even. Since $x_2, y_2 \in X_{i-2}$ we conclude that $(y_2, x) \in A(D)$ if i is odd and $(y, x_2) \in A(D)$ if i is even. In both cases we conclude that $x_2 \neq y_2$, since otherwise the vertices $x_2 = y_2, x_1, x$, respectively, $x_2 = y_2, y_1, y$ would induce a $\overrightarrow{K_3}$ in D. Now consider the oriented path Q in D defined as $Q = y_2, (y_2, x), x, (x_1, x), x_1, (x_1, x_2), x_2$ if i is odd and as $Q = y_2, (y_2, y_1), y_1, (y, y_1), y, (y, x_2), x_2$ if i is even. In order for Q not to be an induced $P^+(1, 1, 1)$ the endpoints x_2 and y_2 of Q must be adjacent. This contradicts the induction hypothesis that X_{i-2} is an independent set. Hence, our assumption was wrong, X_i is indeed independent. This concludes the proof of the claim. $\square$

Let $X := X_0 \cup \cdots \cup X_k$ and $D' := D - X$. By the induction hypothesis D' admits an acyclic coloring $c' : V(D') \to \{1, 2\}$. Let us now define $c : V(D) \to \{1, 2\}$ by $c(x) := c'(x)$ for $x \in V(D) \backslash X$, $c(x) := 1$ for $x \in X_i$ such that i is even, and $c(x) := 2$ for $x \in X_i$ such that i is odd. We claim that D defines an acyclic coloring of D: Suppose there is a monochromatic directed cycle C in D. Since c' is an acyclic coloring, we must have $V(C) \cap X \neq \emptyset$. By definition of the sets $(X_i)_{i \geq 0}$ we have $N^+(\bigcup_{i\,\text{even}} X_i), N^-(\bigcup_{i\,\text{odd}} X_i) \subseteq X$. Hence, there are no arcs from $c^{-1}(\{1\}) \cap X$ to $V(D) \backslash X$ or from $V(D) \backslash X$ to $c^{-1}(\{2\}) \cap X$. Since $V(C) \subseteq c^{-1}(t)$ for some $t \in \{1, 2\}$, the strong connectivity of C shows that in fact $V(C) \subseteq c^{-1}(t) \cap X$ for some $t \in \{1, 2\}$. Let $i_0 \geq 0$ be the smallest such that $X_{i_0} \cap V(C) \neq \emptyset$. Let $u \in X_{i_0} \cap V(C) \neq \emptyset$, and let $u^-, u^+ \in V(C)$ be such that $(u^-, u), (u, u^+) \in A(C)$. We have $u^-, u^+ \in \bigcup_{j > i_0} X_j$ since X_{i_0} is an independent set. Thus $u^+ \in N^+(X_{i_0}) \backslash \bigcup_{j=0}^{i_0-1} X_j = X_{i_0+1}$ if i_0 is even and $u^- \in N^-(X_{i_0}) \backslash \bigcup_{j=0}^{i_0-1} X_j = X_{i_0+1}$ if i_0 is odd, in both cases yielding that C cannot be monochromatic. This contradiction shows that c is an acyclic coloring and $\overrightarrow{\chi}(D) \leq 2$, concluding the proof.

5 | CONCLUSION

In the first two sections of this paper we have proved that set $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ is heroic for several small heroes H, and in particular we resolved Conjecture 1.3. It would be interesting to prove that in fact, for any hero H, $\left\{\overleftrightarrow{K_2}, S_2^+, H\right\}$ is heroic, as this would be a broad generalization of the main result of Berger et al. [4] from tournaments to *locally out-complete* oriented graphs, that is, oriented graphs in which the out-neighborhood of every vertex induces a tournament. This class of digraphs has been thoroughly studied in the past, see, for instance, [3] for a survey of results on locally complete digraphs.

The smallest open case of this problem would be to show that $\left\{\overleftrightarrow{K_2}, S_2^+, \overrightarrow{K_4}^s\right\}$ is heroic, where $\overrightarrow{K_4}^s$ denotes the unique strong tournament on four vertices. It seems that already for this case a new method is required. We do however believe that the following is true.

Conjecture 5.1. $\vec{\chi}\left(\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, \vec{K}_4^s\right)\right) = 3.$

Here, a tight lower bound would be provided by the following construction: Take a threefold blow-up of a directed four-cycle (every arc being replaced by an oriented $K_{3,3}$) and connect each of the three blow-up triples by a directed triangle. This oriented graph is contained in $\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, \vec{K}_4^s\right)$ and has dichromatic number 3.

Let us further remark at this point that there exists a very simple proof that if we exclude *both* S_2^+ and S_2^-, that is, we consider *locally complete* oriented graphs (where the out- *and* in-neigborhood of every vertex induces a tournament), then we can show that the exclusion of any hero indeed bounds the dichromatic number as follows.

Remark 5.2. For any hero H, we have

$$\vec{\chi}\left(\mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, S_2^-, H\right)\right) \leq 2\vec{\chi}\,(\mathrm{Forb}_{\mathrm{ind}}(\overleftrightarrow{K_2}, \overline{K_2}, H)) < \infty.$$

Proof. By the result of Berger et al. [4] we have $C_0 := \vec{\chi}\,(\mathrm{Forb}_{\mathrm{ind}}(\overleftrightarrow{K_2}, \overline{K_2}, H)) < \infty$. Towards a contradiction suppose that $\vec{\chi}\,(D) > 2C_0$ for some $D \in \mathrm{Forb}_{\mathrm{ind}}\left(\overleftrightarrow{K_2}, S_2^+, S_2^-, H\right)$, and choose D such that $v(D)$ is minimum. Pick some $v \in V(D)$ and consider $D' := D - (\{v\} \cup N_D(v))$. Since $v(D') < v(D)$, there exists an acyclic $2C_0$-coloring $c' : V(D') \to \{1, \ldots, 2C_0\}$ of D'. Since D is S_2^+, S_2^--free, we further know that $D^+ := D\left[\{v\} \cup N_D^+(v)\right]$ and $D^- := D\left[N_D^-(v)\right]$ are tournaments excluding H. Hence there exist acyclic C_0-colorings $c^+ : V(D^+) \to \{1, \ldots, C_0\}$ of D^+ and $c^- : V(D^-) \to \{C_0 + 1, \ldots, 2C_0\}$ of D^-. Let c be the $2C_0$-coloring of D obtained by piecing together c', c^+, c^-. We claim that this is an acyclic coloring, which will contradict our assumption $\vec{\chi}\,(D) > 2C_0$ and thus conclude the proof. To verify this, note that D contains no S_2^+ centered at an in-neighbor of v and no S_2^- centered at an out-neighbor of v, and hence there is no arc leaving $\{v\} \cup N_D(v)$ that starts in $N_D^-(v)$ and nor arc entering $\{v\} \cup N_D(v)$ that ends in $N_D^+(v)$. Thus every directed cycle in D is either disjoint from $\{v\} \cup N_D(v)$, contained in $D[\{v\} \cup N_D(v)]$ or it intersects both $N_D^-(v)$ and $N_D^+(v)$. In all cases it cannot be monochromatic, since c' is an acyclic coloring and since c^+ and c^- are acyclic colorings with disjoint color sets. □

In the last section of this paper we investigated oriented graphs excluding the antidirected 4-vertex-path $P^+(1, 1, 1)$. It would certainly be very interesting and insightful to generalize both Theorem 3 as well as the result of Aboulker et al. concerning $\vec{P}_4$ by proving that $\{\overleftrightarrow{K_2}, \vec{P}_4, \vec{K}_k\}$ and $\{\overleftrightarrow{K_2}, P^+(1, 1, 1), \vec{K}_k\}$ are heroic for all $k \geq 4$.

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