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Singular modules for affine Lie algebras, and applications to irregular WZNW conformal blocks

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Abstract

We give a mathematical definition of irregular conformal blocks in the genus-zero WZNW model for any simple Lie algebra, using coinvariants of modules for affine Lie algebras whose parameters match up with those of moduli spaces of irregular meromorphic connections: the open de Rham spaces. The Segal–Sugawara representation of the Virasoro algebra is used to show that the spaces of irregular conformal blocks assemble into a flat vector bundle over the space of isomonodromy times à la Klarès, and we provide a universal version of the resulting flat connection generalising the irregular KZ connection of Reshetikhin and the dynamical KZ connection of Felder–Markov–Tarasov–Varchenko.

Keywords Affine Lie algebras · Conformal field theory · Irregular meromorphic connections · Integrable quantum systems · Isomonodromic deformations

Mathematics Subject Classification 81T40 · 17B38 · 17B10

Contents

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1 Introduction and main results

In this paper we pursue the viewpoint that a natural mathematical formulation of conformal field theory (CFT) lies within the geometry of moduli spaces of meromorphic connections, and we take a step in this direction.

The prototype are the Knizhnik–Zamolodchikov equations (KZ) [38], in the genus-zero Wess–Zumino–Novikov–Witten model (WZNW) for 2-dimensional CFT [43, 54, 55]. They were originally introduced as the partial differential equations satisfied by n-point correlators, and mathematically they amount to a flat connection on a vector bundle over the space of configurations of n-tuples of points in the complex plane [20].

The construction of the flat connection relies on representation-theoretic constructions for affine Lie algebras, and on the Segal–Sugawara representation of the Virasoro algebra on affine-Lie-algebra modules [39]. An alternative derivation is possible via deformation quantisation of the Hamiltonian system controlling isomonodromic deformations of Fuchsian systems on the Riemann sphere [30, 46], the Schlesinger system [48]. In particular the vector bundle where the KZ connection is defined comes from the quantisation of moduli spaces of meromorphic connections with tame/regular singularities (simple poles).

In this paper we develop a representation-theoretic setup for any simple finite-dimensional complex Lie algebra $\mathfrak{g}$, in order to go beyond the case of regular singularities and allow for irregular/wild ones. We will thus define a family of modules for $\mathfrak{g}$ and for the affine Lie algebra $\widehat{\mathfrak{g}}$ associated with $\mathfrak{g}$, which we call "singular" modules.[1] Their parameters match up with those of symplectic moduli spaces of (possibly irregular) meromorphic connections on the sphere, generalising Verma modules.

Indeed the regular case will correspond to "tame" modules $V_\lambda \subseteq \widehat{V}_\lambda$, which are standard Verma modules for $\mathfrak{g} \subseteq \widehat{\mathfrak{g}}$, whose defining representations depend on characters $\mathfrak{b}^+ \to \mathbb{C}$ and $\widehat{\mathfrak{b}}^+ \to \mathbb{C}$ for Borel subalgebras $\mathfrak{b}^+ \subseteq \widehat{\mathfrak{b}}^+$—corresponding to positive roots within the root system given by a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}^+$. Such characters are encoded by linear maps $\lambda \in \mathfrak{h}^\vee$, which in turn match up with local normal forms for (germs of) meromorphic connections around a simple pole via the natural residue-pairing $\mathcal{L}\mathfrak{g}dz \otimes \mathcal{L}\mathfrak{g} \to \mathbb{C}$, where $\mathcal{L}\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}(\!(z)\!)$ is the (formal) loop algebra of $\mathfrak{g}$. Moreover, if G is the connected simply-connected Lie group integrating $\mathfrak{g}$, then the G-action on the coadjoint G-orbit $\mathcal{O} \subseteq \mathfrak{g}^\vee$ of the character corresponds to the gauge action on meromorphic connections on a trivial principal G-bundle. Repeating this construction at $n \geqslant 1$ marked points on the sphere provides a finite-dimensional description of the moduli space of isomorphism classes of logarithmic connections with prescribed positions of the poles and residue orbits, defined on holomorphically trivial bundles: this is the open part $\mathcal{M}^*_{\mathrm{dR}} \subseteq \mathcal{M}_{\mathrm{dR}}$ of the de Rham space, that enters into the nonabelian Hodge correspondence on (complex) curves. The full de Rham space $\mathcal{M}_{\mathrm{dR}}$ is obtained by removing the requirement that the bundle be holomorphically trivial, rather just topologically trivial [8, Rem. 2.1].

Hence classically there is a complex symplectic reduction of a product of coadjoint G-orbits $\mathcal{O}_i \subseteq \mathfrak{g}^\vee$, the moduli space

$$\mathcal{M}^*_{\mathrm{dR}} = \Big(\prod_i \mathcal{O}_i \Big) /\!\!/_0\, G\,, \tag{1}$$

whose quantum counterpart is the vector space $\mathscr{H} = \mathcal{H}_{\mathfrak{g}}$ of $\mathfrak{g}$-coinvariants of the tensor product $\mathcal{H} = \bigotimes_i V_{\lambda_i}$ of tame modules: the space of WZNW *covacua*. If the level is integral, which we do not assume here, one would replace Verma modules by integrable ones; then dualising yields the space of *vacua*, i.e. the space of WZWN conformal blocks.

Now one very important feature are the deformations, both in the semiclassical and quantum setting. Namely as the positions of the simple poles vary the moduli spaces

[1] "Singularity modules" is also a fitting name, since they are attached to the singularity of a connection—at a point on Riemann surface.

assemble into a symplectic fibre bundle

$$\widetilde{\mathcal{M}}^*_{\mathrm{dR}} \longrightarrow \mathrm{Conf}_n(\mathbb{C}),$$

over the space $\mathrm{Conf}_n(\mathbb{C}) \subseteq \mathbb{C}^n$ of (ordered) configurations of points on the complex affine line, equipped with a flat symplectic Ehresmann connection: it is the *isomonodromy* connection [31], defined here by the integrable (nonautonomous) Schlesinger system [48]. The leaves of this (nonlinear) connection are *isomonodromic* families of (linear) meromorphic connections, i.e. the monodromy data are kept locally constant.[2] Hence classically we find a flat symplectic fibre bundle.

On the quantum side one thus looks for a (linear) flat connection on the conformal block bundle, to yield identifications of different fibres up to the braiding of the marked points, analogously to the symplectomorphisms defined by the nonlinear isomonodromy connection. This natural flat connection is precisely the KZ connection, which is intrinsically defined via the slot-wise action of the Sugawara operator $L_{-1} \in \mathfrak{Vir}$ on the tensor product $\widehat{\mathcal{H}} = \bigotimes_i \widehat{V}_{\lambda_i}$ of tame modules for the affine Lie algebra, where $\mathfrak{Vir}$ is the Virasoro algebra. The action is compatible with that of the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on the punctured sphere, hence induces a well defined connection on the bundle of coinvariants.

This is the picture that we wish to generalise on the side of the representation theory of affine Lie algebras. Namely to define generalisations of Verma modules we look at the symplectic geometry of moduli spaces of *irregular* meromorphic connections, which has been studied in much greater generality: for arbitrary genus, complex reductive structure group, polar divisor, and for any nongeneric/twisted irregular types [8–10, 13, 16]. Intrinsic definitions allow for the construction of symplectic local systems of wild character varieties generalising the above, also entering the *wild* nonabelian Hodge correspondence on (complex) curves [5]. We concern ourselves here with the case of genus zero, of a simple group, and untwisted irregular types; cf. [7, 14] for terminology and motivation.

Hence the open de Rham spaces $\mathcal{M}^*_{\mathrm{dR}}$ are still defined. Importantly one now considers isomorphism classes of connections with higher-order poles, which have local moduli at each pole parametrising the principal parts—beyond the residue term. This may be formalised in terms of "deeper" coadjoint orbits of the dual Lie algebra $\mathfrak{g}_p^\vee$, where

$$\mathfrak{g}_p = \mathfrak{g}[\![z]\!]/z^p\mathfrak{g}[\![z]\!] \simeq \bigoplus_{i=0}^{p-1} \mathfrak{g} \otimes z^i,$$

which is a Lie algebra of truncated $\mathfrak{g}$-currents, holomorphic at $z = 0$. Indeed the residue-pairing matches $\mathfrak{g}_p$ up with a space of meromorphic $\mathfrak{g}$-valued 1-forms, which

[2] These data are the G-conjugacy class of the monodromy representation of the fundamental group of the punctured sphere, with the poles removed. Indeed the isomonodromy connection is the pullback of the (nonabelian) Gauß–Manin connection on the associated family of character varieties along the Riemann–Hilbert correspondence, viz. the map taking monodromy data (which can thought as a "global" version of the exponential $\mathfrak{g} \to G$, cf. the abstract of [14]).

we see as principal parts of (germs of) meromorphic connections on a trivial principal G-bundle at a wild singularity. The upshot is that one still has the description (1): now however one considers coadjoint G_p-orbit $\mathcal{O}_i \subseteq \mathfrak{g}_p^\vee$, where

$$G_p = G\big(\mathbb{C}[\![z]\!]/z^p\mathbb{C}[\![z]\!]\big)\,.$$

This is the group of $(p-1)$-jets of bundle automorphisms of the trivial principal G-bundle on a (formal) disc, integrating $\mathfrak{g}_p$. The diagonal G-action corresponds to a change of global trivialisation of the bundle, as in the tame case, cf. the proof of [8, Proposition 2.1].

Hence we will define modules $W_\chi^{(p)} \subseteq \widehat{W}_\chi^{(p)}$ (at depth $p \geqslant 1$) for $\mathfrak{g}_p$ and $\widehat{\mathfrak{g}}$ respectively, whose defining representations depend on elements of $\mathfrak{h}_p^\vee \subseteq \mathfrak{g}_p^\vee$. In turn the latter correspond to characters for Lie subalgebras generalising the positive (affine) Borels, so that for $p = 1$ they reduce to the usual tame modules (else they are "wild/irregular"). This is done in Def. 3.1, which is a variations of similar definitions considered elsewhere, and which is the best suited to our viewpoint on the moduli spaces (1). For example (10) has a more general scope than the "confluent Verma modules" of [34, 42], since we allow for an arbitrary simple Lie algebra and for arbitrary irregular singularities (of arbitrary Poincaré rank)—and we do not use the viewpoint of confluence. Also we do not work in Liouville theory, i.e. we do not consider modules for the Virasoro algebra as in [41]. The approach in this paper is closer to the "level subalgebra" of [21] (cf. also Sect. 14), or rather to one of its "more reasonable" variants (see Rem. 4 of op. cit.). The other variant is used in [22, § 2.8]: in this setup the natural pairing (15) matches the parameter of the modules with *half* of principal parts of irregular meromorphic connections, contrary to (10).[3] The other two important differences with [22] is that we work at *noncritical* level, and that our $\mathfrak{g}_p$-modules are highest-weight, leading to finite-dimensional spaces of coinvariants.

The singular modules enjoy several natural generalisations of the standard properties of tame modules, some of which we gather here. We will refer to "affine" modules when $\widehat{\mathfrak{g}}$ is involved, and to "finite" modules when $\mathfrak{g}_p$ is.

Theorem 1
- *The singular modules admit explicit PBW-generators (Cor. 5.1 and Cor. 5.2).*
- *The singular modules are smooth (Lem. 5.2).*[4]
- *The singular modules are $\mathfrak{h}$-semisimple (Proposition 5.1), and the finite singular modules have finite-dimensional $\mathfrak{h}$-weight spaces (Proposition 5.2).*
- *The finite singular modules are highest-weight $\mathfrak{g}_p$-modules (Lem. 5.4).*
- *The singular modules are cyclically generated by a common eigenvector for the Sugawara operators $\{L_n\}_{n \geqslant p-1}$ (Proposition 7.1), which is a Gaiotto–Teschner irregular state of order $p-1$* [28].

[3] The viewpoint of op. cit. on meromorphic connections is different: at critical level $\kappa = -h^\vee$ one identifies quotients of the "universal Gaudin algebra" with algebras of functions on spaces of opers with prescribed singularities for the Langlands dual group ${}^L G$ of G, with a view towards the geometric Langlands correspondence for loop groups [25].

[4] Recall a $\mathfrak{g}[\![z]\!]$-module is smooth if every vector is annihilated by $z^N \mathfrak{g}[\![z]\!] \subseteq \mathfrak{g}[\![z]\!]$ for $N \gg 0$.

We also give a formula for the (finite) dimension of $\mathfrak{h}$-weight spaces of finite modules, generalising the usual Weyl characters of Verma modules, in (27). The combinatorial complexity still lies in the positive root lattice, so in the archetypal case of $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ there is a simple solution (see (29)).

After establishing these properties we consider tensor products of singular modules labeled by marked points on the Riemann sphere, and study their space $\mathscr{H}$ of coinvariants for the action of $\mathfrak{g}$-valued meromorphic functions with poles at the marked points. Introducing generalisations of the standard filtrations/gradings of tame modules we prove the following.

Theorem 2 • *The space $\mathscr{H}$ is canonically identified with the space of $\mathfrak{g}$-coinvariants for the tensor product of finite modules (Props.* 9.1*,* 9.2 *and* 10.1*).*

- *The space $\mathscr{H}$ is finite-dimensional if one module is tame (Cor.* 9.1*).*

To ensure nontriviality of the space of coinvariants we explore two options: either replacing one of the modules at the marked points with its associated contragredient representation (see Proposition 10.2), or restricting the action of rational function to the subalgebra of those which vanish at an unmarked point (see Rem. 9.1).

Finally we consider deformations of the marked points, i.e. variations of the *tame* isomonodromy times. This is not the full set of isomonodromy times, as in the most general setup one may also vary the irregular types/classes and give nonlinear differential equations for the invariance of *Stokes* data along these deformation. This goes beyond the isomonodromy times of [37], but also going the "generic" case of [33]. We briefly discuss one natural setup to introduce a space of irregular isomonodromy times in § 6.4, and we plan to pursue its quantum version in future work, which should be more closely related to to [21, 22] (cf. the outlook section below).[5]

Thus we allow for variations of marked points at finite distance on the sphere. Then we use the Sugawara operators to define a flat connection on the trivial vector bundle whose fibre is the tensor product of affine singular modules, and show this is compatible with the action of rational functions on the punctured sphere. Hence the spaces of coinvariants assemble into a flat vector bundle over the space of tame isomonodromy times, so in particular their dimension is a deformation-invariant—when finite.

Using the above results it is possible to give descriptions of the flat connections on the space $\mathscr{H}$ of coinvariants. Considering all possible cases of our setup we recover as expected:

(1) The KZ connection [38] (Sect. 9.2.1);
(2) A variation of the Cartan term of the dynamical KZ connection [23] (Sect. 9.2.2), and the very same Cartan term with a slightly different setup (Sect. 14);
(3) The general case of [46] (Sect. 9.2.3), which generalises the KZ connection;
(4) A generalisation of op. cit. with nontrivial action on the module at infinity (§ 9.2.4).

In particular the semiclassical limit of the flat connections indeed yields isomonodromy systems for irregular meromorphic connections on the sphere, as wanted.

[5] A recent series of papers gave an intrinsic description of irregular isomonodromy times: in full generality in the untwisted case [18, 19], and in the type-A twisted case [15].

Note the last two items in principle descend from a more general setup, where the point at infinity is not fixed, provided one can show how horizontal sections transform under the pull-back diagonal $\mathrm{PSL}(2,\mathbb{C})$-action. Going in this direction, in Sect. 13 we prove that horizontal sections of the bundle of coinvariants are naturally equivariant under the action of the subgroup of affine transformations of the affine line, with the explicit transformation (67).

Finally we abstract the formulæ for the reduced connections and define a family of *universal* ones: these are connections ∇_p on the trivial vector bundle with fibre $U(\mathfrak{g}_p)^{\otimes n}$ for $p \geqslant 1$, over the space of tame isomonodromy times, which induce the above connections on $\mathscr{H}$ by taking representations.[6] Since all induced connections are flat and well defined on $\mathfrak{g}$-coinvariants, it is natural to conjecture that the same holds for the universal connections *before* taking representations.

Theorem 3 (Thms. 12.1 and 12.2, and Proposition 12.1) *The connection ∇_p is flat, and descends to a connection on $\mathfrak{g}$-coinvariants of the tensor power $U(\mathfrak{g}_p)^{\otimes n}$.*

These results show the singular modules provide a solid mathematical notion of irregular conformal blocks in the genus-zero WNZW model [28, 56].

Recall irregular conformal blocks are central objects in the recent literature on the asymptotically-free extension of the Alday–Gaiotto–Tachikawa correspondence (AGT) [1, 26], which however is formulated in Virasoro/Liouville theory. Irregular extensions in Liouville theory have been obtained within the formalism of Whittaker modules, e.g. [24, 41]; in principle it should be possible to relate the latter with our construction for $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$, in view of the duality between Liouville theory and the H_3^+-WZNW model [47] (then in turn our construction should generalise [27] beyond $\mathfrak{sl}(2,\mathbb{C})$, which is compatible with the duality [47]).

2 Layout of the paper

In Sect. 3 we consider a *depth* $p \geqslant 1$ to introduce singular Lie algebras $\mathfrak{S}^{(p)} \subseteq \widehat{\mathfrak{g}}$, singular characters $\chi \colon \mathfrak{S}^{(p)} \to \mathbb{C}$, and affine/finite induced singular modules $W_\chi^{(p)} \subseteq \widehat{W}_\chi$.

In Sect. 4 we explicitly match up the data (p,χ) with the local moduli for the isomorphism class of (the germ of) an irregular meromorphic connection.

In Sect. 5 we introduce countable PBW-bases $\mathcal{B}_W \subseteq W$ of the finite singular modules, as well as gradings and filtrations on the finite and affine singular modules: notably gradings $\mathcal{F}_\bullet^+$ and $\widehat{\mathcal{F}}_\bullet^\pm$ for the degree in the variable "z", their associated filtrations, and then $\mathfrak{h}$-weight gradings.

In Sect. 6 we introduce left-module structures on (graded/restricted) dual vector spaces $\widehat{W}^* \twoheadrightarrow W^*$.

In Sect. 7 we introduce the Sugawara operators L_n for $n \in \mathbb{Z}$, and prove that the cyclic vector $w \in W \subseteq \widehat{W}$ is a common eigenvector for L_n with $n \geqslant p-1$. This concludes proving the properties of Thm. 1.

[6] Note the connection of [46] is given in universal terms: $\mathfrak{g}_p$-modules and (co)invariants are not discussed, nor are the irregular types of irregular meromorphic connections.

In Sect. 8 we define the spaces of irregular covacua $\mathscr{H}$. They are quotients of tensor products $\mathcal{H} \subseteq \widehat{\mathcal{H}}$ of finite/affine singular modules labeled by marked points on the Riemann sphere with respect to the action of $\mathfrak{g}$-valued meromorphic functions (and we globalise the action introducing suitable sheaves over the space of tame isomonodromy times).

In Sects. 9 and 10 we study coinvariants, and we prove Theorem 2 using the material of Sects. 5 and 6.

In Sect. 11 we introduce the flat connection on the bundle of coinvariants, using the Sugawara operator L_{-1} and fixing the point at infinity. In § 9.2 we give explicit formulæ for the reduced connection.

In Sect. 12 we introduce the universal connection ∇_p at depth $p \geqslant 1$, on the trivial vector bundle with fibre $U(\mathfrak{g}_p)^{\otimes n}$ over the (restricted) space of tame isomonodromy times, and we prove Theorem 3.

In Sect. 13 we introduce the action of Möbius transformations on horizontal sections of the bundle of coinvariants, and establish equivariance under affine transformations.

Finaly in Sect. 14 we slightly modify the setup of Sect. 3 to generalise the dynamical KZ connection, i.e. [23, Eq. 3].

Some standard notion is recalled in the appendix A, while some lengthy computations are gathered in B.

Affine spaces, vector spaces, vector bundles, associative/Lie algebras and tensor products are defined over $\mathbb{C}$—unless otherwise specified.

The end of a remark is signaled by a "$\triangle$".

3 Setup

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ a Cartan subalgebra. Let then $\mathcal{R}^+ \subseteq \mathcal{R} \subseteq \mathfrak{h}^\vee$ be a choice of positive roots within the root system $\mathcal{R} = \mathcal{R}(\mathfrak{g}, \mathfrak{h})$, and $\mathcal{R}^- := -\mathcal{R}^+$ the subset of negative roots. There is a triangular/Cartan decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ ,$$

where $\mathfrak{n}^\pm$ is the maximal positive/negative nilpotent subalgebra defined by the subset of positive/negative roots:

$$\mathfrak{n}^\pm := \bigoplus_{\alpha \in \mathcal{R}^\pm} \mathfrak{g}_\alpha , \quad \mathfrak{g}_\alpha := \left\{ X \in \mathfrak{g} \;\middle|\; \big(\mathrm{ad}_H - \alpha(H)\big) X = 0 \text{ for } H \in \mathfrak{h} \right\} .$$

Equip $\mathfrak{g}$ with the minimal nondegenerate $\mathrm{ad}_{\mathfrak{g}}$-invariant symmetric bilinear form $(\cdot \mid \cdot) \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$—so the highest root has length $\sqrt{2}$. Consider then the (formal) loop algebra

$$\mathcal{L}\mathfrak{g} = \mathfrak{g}(\!(z)\!) := \mathfrak{g} \otimes \mathbb{C}(\!(z)\!) ,$$

and let $\widehat{\mathfrak{g}}_{(\cdot|\cdot)} = \widehat{\mathfrak{g}} \simeq \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K$ be the associated affine Lie algebra. The Lie bracket of $\widehat{\mathfrak{g}}$ is defined by $K \in \mathfrak{Z}(\widehat{\mathfrak{g}})$ and

$$\big[X \otimes f, Y \otimes g\big]_{\widehat{\mathfrak{g}}} = \big[X, Y\big]_{\mathfrak{g}} \otimes fg + c(X \otimes f, Y \otimes g)K\,, \quad \text{for } f, g \in \mathbb{C}(\!(z)\!),\ X, Y \in \mathfrak{g}, \tag{2}$$

where $c\colon \mathcal{L}\mathfrak{g} \wedge \mathcal{L}\mathfrak{g} \to \mathbb{C}$ is the Lie-algebra cocycle defined by

$$c(X \otimes f, Y \otimes g) := (X \mid Y) \cdot \operatorname{Res}_{z=0}(g\,df)\,, \tag{3}$$

and where in turn $\operatorname{Res}_{z=0}\big(\omega\big) := f_{-1}$ for $\omega = \sum_i f_i z_i dz \in \mathbb{C}(\!(z)\!)dz$.

Then there is an analogous decomposition $\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}^- \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}^+$, where

$$\widehat{\mathfrak{n}}^+ := (\mathfrak{n}^+ \otimes 1) \oplus z\mathfrak{g}[\![z]\!]\,, \quad \widehat{\mathfrak{n}}^- := z^{-1}\mathfrak{g}\big[z^{-1}\big] \oplus (\mathfrak{n}^- \otimes 1)\,, \quad \widehat{\mathfrak{h}} := (\mathfrak{h} \otimes 1) \oplus \mathbb{C}K\,.$$

Finally let $\mathfrak{b}^\pm := \mathfrak{h} \oplus \mathfrak{n}^\pm$ be the positive/negative Borel subalgebras associated with the sets of positive/negative roots, and $\widehat{\mathfrak{b}}^\pm := (\mathfrak{b}^\pm \otimes 1) \oplus z\mathfrak{g}[\![z]\!] \oplus \mathbb{C}K$.

Hereafter we drop the "$\otimes 1$" from the notation for vector subspaces of the constant part $\mathfrak{g} \subseteq \mathcal{L}\mathfrak{g}$, and the subscripts from the Lie brackets.

Remark The dual Coxeter number $h^\vee$ of the quadratic Lie algebra $\big(\mathfrak{g}, (\cdot \mid \cdot)\big)$ is half of the eigenvalue for the adjoint action of the standard quadratic tensor on $\mathfrak{g}$ [35].

More precisely let $(X_k)_k$ be a basis of $\mathfrak{g}$, $(X^k)_k$ the $(\cdot \mid \cdot)$-dual basis, and define

$$\Omega := \sum_k X_k \otimes X^k \in \mathfrak{g}^{\otimes 2}\,,$$

i.e. intrinsically the element corresponding to $\mathrm{Id}_{\mathfrak{g}} \in \mathfrak{g} \otimes \mathfrak{g}^\vee$ in the duality $\mathfrak{g}^\vee \simeq \mathfrak{g}$ induced by $(\cdot \mid \cdot)$. The projection of Ω to the universal enveloping algebra is the quadratic Casimir

$$C = \sum_K X_k X^k \in U(\mathfrak{g})\,, \tag{4}$$

which is a central element—by the invariance of $(\cdot \mid \cdot)$. The adjoint action of C on $\mathfrak{g}$ is thus a homothety, and we define $h^\vee$ by

$$\operatorname{ad}_C X = \sum_k \big[X_k, [X^k, X]\big] = 2h^\vee X\,, \qquad \text{for } X \in \mathfrak{g}\,.$$

We will also need a generalisation of the standard quadratic tensor Ω. For $m, l \in \mathbb{Z}$ define

$$\Omega_{ml} := \sum_k X_k z^m \otimes X^k z^l \in \mathcal{L}\mathfrak{g}^{\otimes 2}\,, \tag{5}$$

with the shorthand notation $Xz^i = X \otimes z^i$ for $X \in \mathfrak{g}$ and $i \in \mathbb{Z}$. Then the identity $[C, X] = \sum_k \left[X_k X^k, X\right] = 0$, valid for all $X \in \mathfrak{g}$, also implies

$$\sum_k X_k z^m \cdot \left[X^k, X\right] z^l + \left[X_k, X\right] z^m \cdot X^k z^l = 0, \qquad \text{for } m, l \in \mathbb{Z}_{\geqslant 0}. \tag{6}$$

$\triangle$

3.1 Singular modules

For an integer $p \geqslant 1$ consider the *singular* Lie subalgebra $\mathfrak{S}^{(p)} \subseteq \widehat{\mathfrak{b}}^+$ (of depth p), defined by

$$\mathfrak{S}^{(p)} := \mathfrak{b}^+[\![z]\!] + z^p \mathfrak{g}[\![z]\!] \oplus \mathbb{C}K, \tag{7}$$

so that $\mathfrak{S}^{(1)} = \widehat{\mathfrak{b}}^+$.

Lemma 3.1 *There is an identification of abelian Lie algebras*

$$\mathfrak{S}^{(p)}_{\mathrm{ab}} \simeq \mathfrak{h}_p \oplus \mathbb{C}K. \tag{8}$$

Proof We can define a linear surjection $\pi \colon \mathfrak{S}^{(p)} \twoheadrightarrow \mathfrak{h}_p \oplus \mathbb{C}K$ with kernel

$$\left[\mathfrak{S}^{(p)}, \mathfrak{S}^{(p)}\right] = \mathfrak{n}^+[\![z]\!] + z^p \mathfrak{g}[\![z]\!], \tag{9}$$

by setting

$$\sum_{i=0}^{p-1} (H_i + X_i) \otimes z^i + z^p f + aK \longmapsto \sum_{i=0}^{p-1} H_i \otimes z^i + aK,$$

where $f \in \mathfrak{g}[\![z]\!]$, $a \in \mathbb{C}$, $H_i \in \mathfrak{h}$, and $X_i \in \mathfrak{n}^+$ for $i \in \{0, \ldots, p-1\}$. $\square$

Characters of (7) are coded by linear maps $\mathfrak{S}^{(p)}_{\mathrm{ab}} \to \mathbb{C}$, i.e. by elements of $\mathfrak{h}_p^\vee$ plus the choice of a *level* $\kappa \in \mathbb{C}$ for the central element—using (8). We split the notation: for $p = 1$ write $\lambda \in \mathfrak{h}^\vee$ the linear map, and for $p \geqslant 2$ write it $(\lambda, q) \in \mathfrak{h}_p^\vee$, where

$$q = (a_1, \ldots, a_{p-1}) \in \left(\mathfrak{h}_p / \mathfrak{h}\right)^\vee \simeq \bigoplus_{i=1}^{p-1} \left(\mathfrak{h} \otimes z^i\right)^\vee.$$

We will refer to $\chi = \chi(\lambda, q, \kappa) \colon \mathfrak{S}^{(p)} \to \mathbb{C}$ as a *singular* character (of depth p), and we denote $\mathbb{C}_\chi$ the 1-dimensional left $U\left(\mathfrak{S}^{(p)}\right)$-module defined by it. We also refer to λ as the *tame* part of the singular character, and to q as the *wild* part.

Remark This hints to the dictionary with irregular meromorphic connections on the Riemann sphere: λ corresponds to a semisimple formal residue at a simple pole (a tame/regular singularity), and q to an untwisted irregular type at a higher-order pole (a wild/irregular singularity), see Sect. 4.

We will use the uniform notation $\lambda = a_0$ when this distinction is not relevant. △

Definition 3.1 *(Affine singular modules)*

- The affine singular module (of depth p) for the singular character χ is

$$\widehat{W} = \widehat{W}_\chi^{(p)} := \operatorname{Ind}_{U(\mathfrak{S}^{(p)})}^{U(\widehat{\mathfrak{g}})} \mathbb{C}_\chi = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{S}^{(p)})} \mathbb{C}_\chi . \tag{10}$$

- We write $\widehat{V} = \widehat{V}_\chi := \widehat{W}_\chi^{(1)}$, and call it the tame affine module for the character $\chi = \chi(\lambda, \kappa) \colon \widehat{\mathfrak{b}}^+ \to \mathbb{C}$.

The latter item is the standard definition of an affine Verma module, and by definition these are level-κ modules.[7]

Now denote $w = [1 \otimes 1] \in \widehat{W}$ the canonical cyclic vector; then using (8) and (9) yields

$$\begin{aligned} z^p \mathfrak{g}[\![z]\!] w = (0) = \mathfrak{n}^+ [\![z]\!] w , \\ H z^i w = \langle a_i, H z^i \rangle w , \qquad \text{for } H \in \mathfrak{h},\ i \in \{ 0, \ldots, p-1 \} , \end{aligned} \tag{11}$$

plus $Kw = \kappa w$. This generalises the relations satisfied by the highest-weight vector in a tame module.

Consider now the subspace

$$\widehat{W}^- := U\big(\mathfrak{g}[z^{-1}]\big) w \subseteq \widehat{W} .$$

Because of (11) it equals $\widehat{W}^- = U(\widehat{\mathfrak{n}}^-) w$, so it is naturally a left $U(\widehat{\mathfrak{n}}^-)$-module with cyclic vector w—and it is canonically isomorphic to $U(\widehat{\mathfrak{n}}^-)$ as vector space. Further matching up cyclic vectors yields an isomorphism $\widehat{W}^- \simeq \widehat{V}$ of left $U\big(\mathfrak{g}[z^{-1}]\big)$-modules, regardless of $p \geqslant 1$ and $q \in (\mathfrak{h}_p / \mathfrak{h})^\vee$. Note we implicitly use a $\mathbb{C}$-basis of $U(\widehat{\mathfrak{g}})$ as provided by the Poincaré–Birkhoff–Witt theorem (PBW) for countable-dimensional Lie algebras.

Consider then the subspace

$$W := U\big(\mathfrak{g}[\![z]\!]\big) w \subseteq \widehat{W} , \tag{12}$$

which is naturally a left $U\big(\mathfrak{g}[\![z]\!]\big)$-module, and which will play a more important role. An inductive proof on the length of monomials—with base (11)—shows that $z^p \mathfrak{g}[\![z]\!] W = (0)$, so the $\mathfrak{g}[\![z]\!]$-action factorises through the finite-dimensional quotient

[7] Beware a "regular" Verma modules is a Verma module defined by a *dominant* weight $\lambda \in \mathfrak{h}$, so "tame" is better terminology here.

$\mathfrak{g}[\![z]\!] \twoheadrightarrow \mathfrak{g}_p$, so (12) is naturally a left $U(\mathfrak{g}_p)$-module. Further $W = U(\mathfrak{n}_p^-)w$ since $\mathfrak{b}_p^+ w = \mathbb{C}w$, so in particular $W \simeq U(\mathfrak{n}_p^-)$ as vector spaces, independently of χ.

Remark Here we use the triangular decomposition $\mathfrak{g}_p = \mathfrak{n}_p^- \oplus \mathfrak{h}_p \oplus \mathfrak{n}_p^+$ and the inclusion $\mathfrak{b}_p^+ = \mathfrak{n}_p^+ \oplus \mathfrak{h}_p \subseteq \mathfrak{g}_p$. △

One has $\mathfrak{n}_p^+ = \left[\mathfrak{b}_p^+, \mathfrak{b}_p^+\right]$ and $\left(\mathfrak{b}_p^+\right)_{\mathrm{ab}} \simeq \mathfrak{h}_p$, so by (11) there is a canonical identification

$$W \simeq \operatorname{Ind}_{U(\mathfrak{b}_p^+)}^{U(\mathfrak{g}_p)} \mathbb{C}_\chi = U(\mathfrak{g}_p) \otimes_{U(\mathfrak{b}_p^+)} \mathbb{C}_\chi , \tag{13}$$

where we keep the notation $\chi \colon \mathfrak{b}_p^+ \to \mathbb{C}$ for the character defined by $(\lambda, q) \in \mathfrak{h}_p^\vee$—the level κ is lost.

Definition 3.2 *(Finite singular modules)*

- We call $W = W_\chi^{(p)} \subseteq \widehat{W}$ the finite singular module (of depth p) for the singular character χ.
- We write $V = V_\chi = W_\chi^{(1)}$, and call it the tame finite singular module for the character $\chi = \chi(\lambda) \colon \mathfrak{b}^+ \to \mathbb{C}$.

The latter item is the standard definition of a finite Verma module. Analogously to the above, the finite tame module is canonically embedded as a $U(\mathfrak{g})$-submodule, namely as the subspace $\widehat{W}^- \cap W = U(\mathfrak{g})w \subseteq W$.

On the whole there is an identification of left $U(\widehat{\mathfrak{n}}^-)$-modules

$$\widehat{W} \simeq U(\widehat{\mathfrak{n}}^-) \otimes_{U(\mathfrak{n}^-)} U(\mathfrak{n}_p^-) , \tag{14}$$

independent of χ.

3.2 Algebricity

The structure of $\widehat{W}$ as left-module is controlled by algebraic elements, not by arbitrary formal power series.

More precisely define

$$\mathcal{L}\mathfrak{g}_{\mathrm{alg}} = \mathfrak{g}\left[z^{\pm 1}\right] := \mathfrak{g} \otimes \mathbb{C}\left[z^{\pm 1}\right] ,$$

and then $\widehat{\mathfrak{g}}_{\mathrm{alg}} \twoheadrightarrow \mathcal{L}\mathfrak{g}_{\mathrm{alg}}$ using the restriction of the cocycle (3). These are the *algebraic* loop algebra and the *algebraic* affine Lie algebra of $\mathfrak{g}$, respectively. Replacing "$\mathfrak{g}[\![z]\!]$" by "$\mathfrak{g}[z]$" in (10) then yields left $U(\widehat{\mathfrak{g}}_{\mathrm{alg}})$-modules, temporarily denoted $\widehat{W}_{\mathrm{alg}}$, generated by a cyclic vector $w_{\mathrm{alg}} \in \widehat{W}_{\mathrm{alg}}$.

On the other hand the modules $\widehat{W}$ are left $U(\widehat{\mathfrak{g}}_{\mathrm{alg}})$-modules via the inclusion $U(\widehat{\mathfrak{g}}_{\mathrm{alg}}) \hookrightarrow U(\widehat{\mathfrak{g}})$, and composing with the canonical projection

$$U(\widehat{\mathfrak{g}}) \twoheadrightarrow \widehat{W} \simeq U(\widehat{\mathfrak{g}}) / \operatorname{Ann}_{U(\widehat{\mathfrak{g}})}(w)$$

yields a linear map $\iota \colon \mathrm{U}\big(\widehat{\mathfrak{g}}_{\mathrm{alg}}\big) \to \widehat{W}$.

Lemma 3.2 *The map ι induces an isomorphism $\widehat{W}_{\mathrm{alg}} \simeq \widehat{W}$ of left $\mathrm{U}\big(\widehat{\mathfrak{g}}_{\mathrm{alg}}\big)$-modules.*

Proof By (11) the map ι is surjective, since $\widehat{W}$ is generated by the cyclic vector over $\mathrm{U}\big(\mathcal{L}\mathfrak{g}_{\mathrm{alg}}\big)$. Its kernel is

$$\mathrm{Ker}(\iota) = \mathrm{Ann}_{\mathrm{U}(\widehat{\mathfrak{g}})}(w) \cap \mathrm{U}\big(\widehat{\mathfrak{g}}_{\mathrm{alg}}\big) = \mathrm{Ann}_{\mathrm{U}(\widehat{\mathfrak{g}}_{\mathrm{alg}})}(w_{\mathrm{alg}})\,.$$

□

Hence the action of meromorphic $\mathfrak{g}$-valued functions on the singular modules is given by Laurent polynomials only. We will drop the subscript "alg" from all notations.

4 Relation with (irregular) meromorphic connections

There are canonical vector space isomorphisms $\big(\mathfrak{g} \otimes z^i\big)^\vee \simeq \mathfrak{g} \otimes z^{-(i+1)}dz$, for $i \in \mathbb{Z}$. They are induced from the nondegenerate $\mathcal{L}G$-invariant residue-pairing

$$\mathcal{L}\mathfrak{g}dz \times \mathcal{L}\mathfrak{g} \longrightarrow \mathbb{C}, \qquad (X \otimes \omega, Y \otimes g) \longmapsto (X \mid Y) \cdot \mathrm{Res}_{z=0}(g\omega)\,, \tag{15}$$

where $\mathcal{L}\mathfrak{g}dz := \mathfrak{g} \otimes \mathbb{C}(\!(z)\!)dz$, G is a connected simply-connected (simple) Lie group with Lie algebra $\mathfrak{g}$, and $\mathcal{L}G$ the associated loop group.

Thus after fixing a level $\kappa \in \mathbb{C}$ the families of singular modules (10) and (13) are both naturally parametrised by elements

$$\mathcal{A} = dQ + \Lambda\frac{dz}{z} \in z^{-1}\mathfrak{h}\big[z^{-1}\big]dz\,. \tag{16}$$

Namely the residue term $\Lambda z^{-1}dz \in \mathfrak{h} \otimes z^{-1}dz$ corresponds to the tame part $\lambda \in \mathfrak{h}^\vee$ of a singular character, and the irregular type

$$Q = \sum_{i=1}^{p-1} \frac{A_i}{z^i} \in \mathfrak{h}(\!(z)\!)\big/\mathfrak{h}[\![z]\!]\ , \qquad \text{with } A_i \in \mathfrak{h} \text{ for all } i\,,$$

is such that $d\left(A_i z^{-i}\right) = -iA_i z^{-i-1}dz \in \mathfrak{h} \otimes z^{-i-1}dz$ corresponds to the wild part $a_i \in \big(\mathfrak{h} \otimes z^i\big)^\vee$. The meromorphic 1-forms (16) should be thought of as principal parts of germs of meromorphic connections at a point on a Riemann surface (with semisimple formal residue and untwisted irregular type; here we are considering "very good" orbits in the terminology of [14]).

As mentioned in the introduction, the crucial facts are:

(1) $\mathfrak{g}_p = \mathrm{Lie}(G_p)$, where $G_p := G\big(\mathbb{C}[\![z]\!]/z^p\mathbb{C}[\![z]\!]\big)$ is the group of $(p-1)$-jets of bundle automorphisms for the trivial principal G-bundle on a (formal) disc;

(2) The level-zero complex symplectic reduction for the diagonal G-action—on products of coadjoint G_p-orbits—yields a description of an open de Rham space $\mathcal{M}^*_{dR}$, viz. a moduli spaces of isomorphism classes of *irregular* meromorphic connections on a holomorphically trivial principal bundle over the Riemann sphere (with prescribed positions of poles and irregular types [10, § 5]; see [8] for $G = GL_m(\mathbb{C})$).

Moreover the diagonal G-action will correspond to taking $\mathfrak{g}$-coinvariants for the tensor product of finite singular modules, generalising the tame case (see Sects. 9 and 10).

Remark 4.1 **(Birkhoff groups/Lie algebras)** Consider the subgroup $B_p \subseteq G_p$ of elements with constant term 1. Then G acts on B_p by conjugation, and there are natural identification $G_p \simeq G \ltimes B_p$ and $\mathfrak{g}_p \simeq \mathfrak{g} \ltimes \mathfrak{b}_p$, where $\mathfrak{b}_p = \mathrm{Lie}(B_p)$.[8] This yields a vector space decomposition $\mathfrak{g}_p^\vee \simeq \mathfrak{g}^\vee \oplus \mathfrak{b}_p^\vee$: in the duality (15) the former summand corresponds to formal residues with zero irregular types, and the latter to irregular types with zero residue (so in particular $q \in \mathfrak{b}_p^\vee$). △

5 Bases, gradings and filtrations

Denote $\Pi = \{\theta_i\}_i \subseteq \mathcal{R}^+$ the set of simple roots, and choose an order $\mathcal{R}_+ = (\alpha_1, \ldots, \alpha_s)$ for the set of positive roots. If $r := \mathrm{rk}(\mathfrak{g})$ we may assume $\theta_i = \alpha_i$ for $i \in \{1, \ldots, r\}$. Let then $(F_\alpha)_{\alpha \in \mathcal{R}^+}$ and $(E_\alpha)_{\alpha \in \mathcal{R}^+}$ be bases of $\mathfrak{n}^-$ and $\mathfrak{n}^+$ with $(F_\alpha, E_\alpha) \in \mathfrak{g}_{-\alpha} \oplus \mathfrak{g}_\alpha$, and such that $(F_\alpha, H_\alpha := [E_\alpha, F_\alpha], E_\alpha)$ is an $\mathfrak{sl}_2$-triple. (We may at times write $E_{-\alpha} := F_\alpha$ for the sake of a uniform notation.)

In particular $(H_\theta)_{\theta \in \Pi}$ is a basis of $\mathfrak{h}$, and we get a (ordered) Cartan–Weyl basis of $\mathfrak{g}$:

$$(X_1, \ldots, X_{2s+r}) := (F_{\alpha_1}, \ldots, F_{\alpha_s}, H_{\theta_1}, \ldots, H_{\theta_r}, E_{\alpha_1}, \ldots, E_{\alpha_s}) . \tag{17}$$

For a multi-index $n \in \mathbb{Z}_{\geqslant 0}^{2s+r}$ define

$$X^n := X_1^{n_1} \cdots X_{2s+r}^{n_{2s+r}} \in U(\mathfrak{g}) .$$

By the PBW theorem these monomials provide a $\mathbb{C}$-basis of $U(\mathfrak{g})$.

5.1 PBW-bases of singular modules

Let $\boldsymbol{\beta} = (\beta_i)_{i \geqslant 0}$ be a sequence of non-negative integers with finite support, and consider another sequence with values in the index set of the Cartan–Weyl basis (17), i.e. $\boldsymbol{k} = (k_i)_{i \geqslant 0} \in \{1, \ldots, r + 2s\}^{\mathbb{Z}_{\geqslant 0}}$. Then define

$$X_{\boldsymbol{k}} z^{\boldsymbol{\beta}} := \prod_{i \in \boldsymbol{\beta}^{-1}(\mathbb{Z}_{>0})} X_{k_i} z^{\beta_i} \in U(\mathcal{L}\mathfrak{g}_{\mathrm{alg}}) .$$

[8] Beware to distinguish the positive/negative deeper Borel subalgebra $\mathfrak{b}_p^{\pm}$ from the Birkhoff subalgebra $\mathfrak{b}_p$.

Lemma 5.1 (PBW-basis of algebraic affine enveloping algebras) *A* $\mathbb{C}$*-basis of* $U(\mathcal{L}\mathfrak{g}_{\mathrm{alg}})$ *is given by*

$$\mathcal{B} := \left\{ X_{k'} z^{-\beta'} \cdot X^n \cdot X_k z^\beta \right\}_{k',\beta',n,k,\beta}, \tag{18}$$

where β' *is nonincreasing,* β *is nondecreasing, and* $k'_j \leqslant k'_{j+1}$ *(resp.* $k_j \leqslant k_{j+1}$*) if* $\beta_j = \beta_{j+1}$ *(resp.* $\beta'_j = \beta'_{j+1}$*).*

This is one statement of the PBW theorem for the countable-dimensional Lie algebra $\mathcal{L}\mathfrak{g}_{\mathrm{alg}} = \mathfrak{g} \otimes \mathbb{C}[z^{\pm 1}]$—we have monomials over a totally ordered basis.

Corollary 5.1 (PBW-basis of affine singular modules) *A* $\mathbb{C}$*-basis of the affine singular module* $\widehat{W}$ *can be extracted from*

$$\mathcal{B}_{\widehat{W}} := \left\{ X_{k'} z^{-\beta'} \cdot X^n \cdot X_k z^\beta w \right\}_{k',\beta',n,k,\beta}, \tag{19}$$

where β'*,* k'*,* n*,* k*, and* β *are as above.*

Proof The family generates over $\mathbb{C}$ since $U(\mathcal{L}\mathfrak{g}_{\mathrm{alg}})w = \widehat{W}$, and using Lem. 5.1. □

Remark In (19) one may take β bounded above by $p - 1$, as $z^p \mathfrak{g}[\![z]\!] w = (0)$. △

Using this set of generators we can prove smoothness.

Lemma 5.2 *The singular modules are smooth.*

Proof This is clear in the finite case, as $z^p \mathfrak{g}[\![z]\!] W = (0)$.

In the affine case choose $X \in \mathfrak{g}$ and an element $\widehat{w} = X_{k'} z^{-\beta'} X^n X_k z^\beta w$ of (19). Then the vanishing $X z^N \widehat{w} = 0$ holds for

$$N \geqslant p + \sum_{i \geqslant 0} \beta'_i \in \mathbb{Z}_{>0},$$

and the conclusion follows since (19) is a set of generators. □

Lemma 5.3 (PBW-basis of depth-p finite enveloping algebras) *A* $\mathbb{C}$*-basis of* $U(\mathfrak{g}_p)$ *is given by*

$$\mathcal{B} := \left\{ X^n \cdot X_k z^\beta \right\}_{n,k,\beta}, \tag{20}$$

where n*,* k *and* β *are as above, with the condition of Rem. 5.1. Moreover restricting to* $X_i, X_{k_j} \in \mathfrak{n}^-$ *for* $i \in \{1, \ldots, 2s + r\}$ *and* $j \geqslant 0$ *yields a* $\mathbb{C}$*-basis of* $U(\mathfrak{n}_p^-)$*.*

This is one statement of the PBW theorem for the finite-dimensional Lie algebras $\mathfrak{g}_p$ and $\mathfrak{n}_p^-$.

Corollary 5.2 (PBW-basis of finite singular modules) *A $\mathbb{C}$-basis of the finite singular module $W \subseteq \widehat{W}$ is given by*

$$\mathcal{B}_W := \left\{ X^{\mathfrak{n}} \cdot X_k z^{\beta} w \right\}_{\mathfrak{n},k,\beta} , \tag{21}$$

where all conditions of Lem. 5.3 apply.

Proof The family generates since $W = U(\mathfrak{n}_p^-)w$, and using Lem. 5.3 (the generating part). But $U(\mathfrak{n}_p^-)$ has trivial intersection with the annihilator of w, hence the family is free by Lem. 5.3 (the linear independence part). □

In particular W is a free rank-1 left $U(\mathfrak{n}_p^-)$-module.

5.2 Gradings for z-degree

We first define two positive $\mathbb{Z}$-gradings on $\widehat{W}$.

Definition 5.1 Choose $k \in \mathbb{Z}$. Then:

- the subspace $\widehat{\mathcal{F}}_k^- = \widehat{\mathcal{F}}_k^-(\widehat{W}) \subseteq \widehat{W}$ is the $\mathbb{C}$-span of the vectors of (19) with $\sum_i \beta_i' = k$;
- the subspace $\widehat{\mathcal{F}}_k^+ = \widehat{\mathcal{F}}_k^+(\widehat{W}) \subseteq \widehat{W}$ is the $\mathbb{C}$-span of the vectors of (19) with $\sum_i \beta_i = k$.

By definition $\widehat{\mathcal{F}}_0^- = W$, $\widehat{\mathcal{F}}_0^+ = \widehat{W}^-$, and

$$\mathfrak{g} \otimes z^{-i}\left(\widehat{\mathcal{F}}_k^-\right) = \widehat{\mathcal{F}}_{k+i}^- , \qquad \text{for } i \geqslant 0 . \tag{22}$$

In particular $\left(\widehat{W}, \widehat{\mathcal{F}}_\bullet^-\right)$ is a $\mathbb{Z}$-graded $\mathfrak{g}\left[z^{-1}\right]$-module, where $\mathfrak{g}\left[z^{-1}\right]$ is a $\mathbb{Z}$-graded Lie algebra with grading defined by $\deg(\mathfrak{g} \otimes z^{-i}) = i$.

The other grading instead does not yield a graded module; but we can obtain one inducing a (positive) grading on $W \subseteq \widehat{W}$.

Definition 5.2 For $k \in \mathbb{Z}$ set $\mathcal{F}_k^+ := \widehat{\mathcal{F}}_k^+ \cap W$.

It follows that $\mathcal{F}_0^+ = U(\mathfrak{g})w \subseteq W$, and

$$\mathfrak{n}^- \otimes z^i\left(\mathcal{F}_k^+\right) \subseteq \mathcal{F}_{k+i}^+ , \qquad \text{for } k, i \geqslant 0 , \tag{23}$$

so the space $\left(W, \mathcal{F}_\bullet^+\right)$ is a $\mathbb{Z}$-graded $\mathfrak{n}^-[\![z]\!]$-module, where $\mathfrak{n}^-[\![z]\!]$ is a $\mathbb{Z}$-graded Lie algebra with grading defined by $\deg(\mathfrak{n}^- \otimes z^i) = i$.

5.3 Filtrations

We consider the filtration $\widehat{\mathcal{F}}^-_{\leqslant\bullet}$ on $\widehat{W}$ associated with the grading of Def. 5.1 for the negative z-degree. It follows from (22) that

$$\widehat{\mathcal{F}}^-_{\leqslant k+1} = \sum_{m+l=k} \mathfrak{g} \otimes z^{-m-1}\big(\widehat{\mathcal{F}}^-_l\big)\,, \qquad \mathfrak{g} \otimes z^i\big(\widehat{\mathcal{F}}^-_{\leqslant k}\big) \subseteq \widehat{\mathcal{F}}^-_{\leqslant k}\,, \tag{24}$$

for $k, i \geqslant 0$.

Finally we consider on $U(\mathfrak{g})w = U(\mathfrak{n}^-)w \subseteq W$ the natural filtration $\mathcal{E}_{\leqslant\bullet}$ induced from that of $U(\mathfrak{n}^-)$, so that $\mathcal{E}_{\leqslant 0} = \mathbb{C}w$. Note

$$\mathfrak{n}^-\big(\mathcal{E}_{\leqslant k}\big) + \mathcal{E}_{\leqslant k} = \mathcal{E}_{\leqslant k+1}\,, \tag{25}$$

and further $\mathfrak{n}^-$ acts nontrivially on the associated graded of $\big(U(\mathfrak{g})w, \mathcal{E}_{\leqslant\bullet}\big)$:

$$\mathfrak{n}^-\big(\mathrm{gr}(\mathcal{E})_k\big) \subseteq \mathrm{gr}(\mathcal{E})_{k+1}\,, \tag{26}$$

where as customary $\mathrm{gr}(\mathcal{E})_k := \mathcal{E}_{\leqslant k}/\mathcal{E}_{\leqslant k-1}$ for $k \in \mathbb{Z}_{\geqslant 0}$—and $\mathcal{E}_{\leqslant -1} := (0)$.

5.4 Weight gradings

For $\mu \in \mathfrak{h}^\vee$ define

$$\widehat{\mathcal{F}}_\mu\big(\widehat{W}\big) = \widehat{\mathcal{F}}_\mu := \Big\{\, \widehat{w} \in \widehat{W} \;\Big|\; H\widehat{w} = \mu(H)\widehat{w} \text{ for } H \in \mathfrak{h} \,\Big\} \subseteq \widehat{W},$$

and analogously $\mathcal{F}_\mu(W) = \mathcal{F}_\mu := W \cap \widehat{\mathcal{F}}_\mu \subseteq W$.

Proposition 5.1 *The singular modules are $\mathfrak{h}$-semisimple, i.e.*

$$\widehat{W} = \bigoplus_{\mu \in \mathfrak{h}^\vee} \widehat{\mathcal{F}}_\mu\,, \qquad W = \bigoplus_{\mu \in \mathfrak{h}^\vee} \mathcal{F}_\mu\,.$$

Proof This follows from the fact that all elements of (19) and (21) are $\mathfrak{h}$-weight vectors, which in turn is proven recursively using the identities

$$H \cdot X_\alpha z^i \widehat{w} = \langle \mu + \alpha, H\rangle X_\alpha z^i \widehat{w}\,, \qquad H \cdot H' z^i \widehat{w} = \langle \mu, H\rangle \cdot H' z^i \widehat{w}\,,$$

for $\alpha \in \mathcal{R}$, $H, H' \in \mathfrak{h}$, $i \in \mathbb{Z}$ and $\widehat{w} \in \widehat{\mathcal{F}}_\mu$. □

Remark In the finite case one may define the $\mathfrak{h}_p$-weight spaces, i.e. the subspaces of vectors $\widehat{w} \in W$ such that $Hz^i\widehat{w} = \langle \mu_i, Hz_i\rangle \widehat{w}$ for $\boldsymbol{\mu} = (\mu_0, \dots, \mu_{p-1}) \in \mathfrak{h}_p^\vee$. However the very first recursion fails for $p \geqslant 2$: if $H \in \mathfrak{h}$ is such that $\langle \alpha, H\rangle \neq 0$ then

$$Hz \cdot X_{-\alpha} w = \langle a_1, Hz\rangle X_{-\alpha} w - \langle \alpha, H\rangle X_{-\alpha} z \cdot w \notin \mathbb{C}(X_{-\alpha} w)\,,$$

where w is the cyclic vector, so the finite singular modules are not $\mathfrak{h}_p$-semisimple. △

The proof of Proposition 5.1 implies all weights are contained inside $\lambda + Q \subseteq \mathfrak{h}^\vee$, where $Q := \mathbb{Z}\mathcal{R}$ is the root lattice.

Remark Consider the z-linear extension of the adjoint action $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ on $\mathcal{L}\mathfrak{g}$. Decomposing $\mathcal{L}\mathfrak{g} = \bigoplus_{\alpha\in\mathcal{R}} \mathcal{L}\mathfrak{g}_\alpha \oplus \mathcal{L}\mathfrak{h}$ we see $\mathcal{L}\mathfrak{g}$ is naturally a $\mathfrak{h}^\vee$-graded Lie algebra (with nontrivial weights still given by $\mathcal{R} \cup \{0\}$), and the proof of Proposition 5.1 shows the singular modules are $\mathfrak{h}^\vee$-graded. △

In the finite case one can go further recovering the standard notion of positivity. Namely $(\mathfrak{h}^\vee, \preceq)$ is a poset by defining $\mu' \preceq \mu$ by $\mu - \mu' \in Q_+$, where $Q_+ := \mathbb{Z}_{\geqslant 0}\mathcal{R}^+ \subseteq Q$ is the *positive* root lattice.

Lemma 5.4 *One has $\mathcal{F}_\lambda = \mathbb{C}w$ and $W = \bigoplus_{\mu\preceq\lambda} \mathcal{F}_\mu$.*

Proof It follows from the fact that W is generated over $U(\mathfrak{n}_p^-)$ by a $\mathfrak{h}_p$-weight vector annihilated by $\mathfrak{n}_p^+$: it is a highest-weight $\mathfrak{g}_p$-module. □

In particular (21) consists of weight vectors, and the line $\mathbb{C}w \subseteq W$ has the highest weight.

In view of Lem. 5.4 the weight spaces are naturally parametrised by elements $\nu \in Q_+$, via $\mathcal{F}_\nu := \mathcal{F}_{\lambda-\nu}$. Now for an element $\nu \in \mathfrak{h}^\vee$ denote

$$\mathrm{Mult}_{\mathcal{R}^+}(\nu) := \left\{ \boldsymbol{m} = (m_\alpha)_\alpha \in \mathbb{Z}_{\geqslant 0}^{\mathcal{R}^+} \;\middle|\; \sum_{\alpha\in\mathcal{R}^+} m_\alpha \cdot \alpha = \nu \right\} \subseteq \mathbb{Z}_{\geqslant 0}^{\mathcal{R}^+},$$

so that the cardinality of $\mathrm{Mult}_{\mathcal{R}^+}(\nu)$ is the *finite* number of ways of expressing ν as a $\mathbb{Z}_{\geqslant 0}$-linear combination of positive roots. In particular $\mathrm{Mult}_{\mathcal{R}^+}(0) = \{\mathbf{0}\}$, and $\mathrm{Mult}_{\mathcal{R}^+}(\nu) = \varnothing$ for $\nu \notin Q_+$.

Finally for $\boldsymbol{m} \in \mathbb{Z}_{\geqslant 0}^{\mathcal{R}^+}$ denote

$$\mathrm{WComp}_p(\boldsymbol{m}) := \left\{ \boldsymbol{\varphi} = (\varphi_\alpha)_\alpha \;\middle|\; \varphi_\alpha \colon \{0, \ldots, p-1\} \to \mathbb{Z}_{\geqslant 0}, \sum_{i=0}^{p-1} \varphi_\alpha(i) = m_\alpha \right\},$$

which is the *finite* set of weak p-compositions of the integers $m_\alpha \geqslant 0$.[9] In particular $\mathrm{WComp}_1(\boldsymbol{m})$ is a singleton containing the element $\boldsymbol{\varphi}$ with $\varphi_\alpha(0) = m_\alpha$ for all $\alpha \in \mathcal{R}^+$.

Proposition 5.2 *For $\nu \in \mathfrak{h}^\vee$ one has*

$$\dim(\mathcal{F}_\nu) = \sum_{\boldsymbol{m}\in\mathrm{Mult}_{\mathcal{R}^+}(\nu)} \binom{\boldsymbol{m}+p-1}{\boldsymbol{m}} < \infty, \tag{27}$$

where $\binom{\boldsymbol{m}+p-1}{\boldsymbol{m}} := \prod_{\alpha\in\mathcal{R}^+} \binom{m_\alpha+p-1}{m_\alpha}$.

[9] A *composition* of m_α is a sequence of positive integers summing to m_α; it is a p-composition if the sequence has finite length $p \geqslant 1$; and it is *weak* if zero is allowed.

Proof Choose $\mu \in \mathfrak{h}^\vee$ and set $\nu = \lambda - \mu$. Then for $\boldsymbol{m} \in \mathrm{Mult}_{\mathcal{R}^+}(\nu)$ and $\boldsymbol{\varphi} \in \mathrm{WComp}_p(\boldsymbol{m})$ consider the vector

$$w^{\boldsymbol{\varphi}} := \prod_{i=0}^{p-1} \left(\prod_{\alpha \in \mathcal{R}^+} \left(X_{-\alpha} z^i\right)^{\varphi_\alpha(i)} w \right) \in \mathcal{B}_W . \tag{28}$$

The family $\{ w^{\boldsymbol{\varphi}} \}_{\boldsymbol{\varphi}} \subseteq W$ is free since it consists of distinct elements extracted from (21) (beware of the ordering in the product), and by construction $w^{\boldsymbol{\varphi}} \in \mathcal{F}_\nu$.

Conversely the vectors (28) exhaust (21), from which one can extract a basis of $\mathcal{F}_\nu$, so the conclusion follows from standard combinatorial identities. □

Thus Proposition 5.2 strengthen Lem. 5.4: the given sum is empty for $\nu \notin Q_+$, and $\mathrm{WComp}_p(\boldsymbol{0})$ is a singleton containing the element $\boldsymbol{\varphi}$ with $\varphi_\alpha(i) = 0$ for $i \in \{ 0, \ldots, p-1 \}$.

As expected (27) generalises the standard fact that $\dim\left(\mathcal{F}_\nu\right) = |\mathrm{Mult}_{\mathcal{R}^+}(\nu)|$ for Verma modules, i.e. it generalises the character of Verma modules. The difference in the general case is that one must also specify a z-degree for each occurrence of a positive root.

Remark This notion of positivity is lost with the (finite) modules of Sect. 14: in particular they have *infinite*-dimensional weight spaces and are less suited to yield irregular versions of conformal blocks. △

For example consider the case where $\nu = \theta \in \Pi$ is a simple root. One has $\mathrm{Mult}_{\mathcal{R}^+}(\theta) = \left\{ \boldsymbol{m}^\theta \right\}$, with $\boldsymbol{m}^\theta_\alpha := \delta_{\theta,\alpha}$. Also $\mathrm{WComp}_p(\boldsymbol{m}^\theta) = \left\{ \boldsymbol{\varphi}^{\theta,i} \right\}_i$, where

$$\varphi^{\theta,i}_\alpha(j) = \delta_{\alpha,\theta}\delta_{ij} , \qquad i,j \in \{ 0, \ldots, p-1 \} .$$

Hence $X_{\boldsymbol{\varphi}^{\theta,i}} = X_{-\theta} z^i$, so we recover

$$\dim\left(\mathcal{F}_\theta\right) = p , \qquad \mathcal{F}_\theta = \mathrm{span}_{\mathbb{C}} \left\{ X_{-\theta} w, \ldots, X_{-\theta} z^{p-1} w \right\} .$$

Remark It follows from the above that

$$U\left(\mathfrak{n}^+[\![z]\!]\right)\mathcal{F}_\nu = \bigoplus_{0 \preceq \nu' \preceq \nu} \mathcal{F}_{\nu'} , \qquad \text{for } \nu \in Q_+ .$$

Hence the module W is *locally* $\mathfrak{n}^+[\![z]\!]$-finite, i.e. the vector spaces $U\left(\mathfrak{n}^+[\![z]\!]\right)\widehat{w} \subseteq W$ are finite-dimensional for all $\widehat{w} \in W$.

One is tempted to say that W lies in a "Bernstein–Gelfand–Gelfand category $\mathcal{O}[\![z]\!]$" [32]—of $\mathfrak{h}$-semisimple finitely generated left $U\left(\mathfrak{g}[\![z]\!]\right)$-modules which are locally $\mathfrak{n}^+[\![z]\!]$-finite. △

5.4.1 Archetypal case

One may get to the end of this story when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with the standard basis (F, H, E) and the standard A_1-root system $\mathcal{R} = \{ \pm\alpha \}$, where α is positive and $\langle \alpha, H \rangle = 2$. Then $Q_+ = \mathbb{Z}_{\geqslant 0}\alpha$, so simply $\mathrm{Mult}_{\mathcal{R}^+}(\nu) = \{ m \}$ for elements $\nu = m\alpha$ with $m \in \mathbb{Z}_{\geqslant 0}$.

Thus (27) reduces to

$$\dim\left(\mathcal{F}_{m\alpha}\right) = \left|\mathrm{WComp}_p(m)\right| = \binom{m+p-1}{m}. \tag{29}$$

In the tame case one recovers the line generated by $F^m v$, whereas in the general case a basis is given by

$$w^{\varphi} = \prod_{i=0}^{p-1} \left(Fz^i\right)^{\varphi(i)} \cdot v, \qquad \text{for } \varphi \in \mathrm{WComp}_p(m). \tag{30}$$

6 Dual modules

In view of Proposition 5.1 we consider the restricted duals of the $\mathfrak{h}^\vee$-graded singular modules, i.e. the $\mathfrak{h}^\vee$-graded vector spaces

$$\widehat{W}^* := \bigoplus_{\mu \in \mathfrak{h}^\vee} \widehat{\mathcal{F}}_\mu^\vee \subseteq \widehat{W}^\vee, \qquad W^* := \bigoplus_{\mu \in \mathfrak{h}^\vee} \mathcal{F}_\mu^\vee \subseteq W^\vee. \tag{31}$$

They are naturally equipped with a *right* $U(\widehat{\mathfrak{g}})$- and $U(\mathfrak{g}_p)$-module structure (respectively), namely

$$\left\langle \widehat{\psi} X z^i, \widehat{w} \right\rangle = \left\langle \widehat{\psi}, X z^i \widehat{w} \right\rangle, \quad \widehat{\psi} K = \kappa \widehat{\psi}, \qquad \text{for } i \in \mathbb{Z},\ X \in \mathfrak{g},\ \widehat{\psi} \in \widehat{W}^*,\ \widehat{w} \in \widehat{W},$$

and analogously in the finite case.

To get a left action compose with a Lie algebra morphism $\widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}^{\mathrm{op}}$ (resp. $\mathfrak{g}_p \to \mathfrak{g}_p^{\mathrm{op}}$), or rather with the induced ring morphism $U(\widehat{\mathfrak{g}}) \to U\left(\widehat{\mathfrak{g}}^{\mathrm{op}}\right) = U\left(\widehat{\mathfrak{g}}\right)^{\mathrm{op}}$ (resp. $U(\mathfrak{g}_p) \to U(\mathfrak{g}_p)^{\mathrm{op}}$). In particular a Lie algebra morphism $\theta \colon \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$ has a unique $\mathbb{Z}$-graded extension $\widehat{\theta} \colon \mathcal{L}\mathfrak{g} \to \mathcal{L}\left(\mathfrak{g}^{\mathrm{op}}\right) = \left(\mathcal{L}\mathfrak{g}\right)^{\mathrm{op}}$: in the finite case one can then consider the restriction $\widehat{\theta} \colon \mathfrak{g}[\![z]\!] \to \mathfrak{g}[\![z]\!]^{\mathrm{op}}$, which is compatible with the projections $\mathfrak{g}[\![z]\!] \twoheadrightarrow \mathfrak{g}_p$ and $\mathfrak{g}[\![z]\!]^{\mathrm{op}} \twoheadrightarrow \mathfrak{g}_p^{\mathrm{op}}$; in the affine case one may further ask that θ is $(\cdot \mid \cdot)$-orthogonal, and extend the definition by $\widehat{\theta}(K) := -K$.

In what follows we only consider morphisms of this type.

Definition 6.1 *(Dual singular modules)* The affine (resp. finite) θ-dual singular module $\widehat{W}_\theta^*$ (resp. W_θ^*) is the left $U(\widehat{\mathfrak{g}})$-module (resp. $U(\mathfrak{g}_p)$-module) defined by the morphism $\theta \colon \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$.

The $\mathrm{U}\big(\mathfrak{g}[\![z]\!]\big)$-linear inclusion map $W \hookrightarrow \widehat{W}$ then dually corresponds to a $\mathrm{U}\big(\mathfrak{g}[\![z]\!]\big)$-linear restriction map $\widehat{W}^*_\theta \twoheadrightarrow W^*_\theta$.

Remark 6.1 **(Dual/contragredient modules)** Basic examples of morphisms $\theta\colon \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$ preserving $(\cdot \mid \cdot)$ are the tautological $\theta_0 = -\mathrm{Id}_{\mathfrak{g}}$, and the *transposition* θ_1, defined by

$$\theta_1(E_\alpha) = E_{-\alpha}\,, \quad \theta_1\big|_{\mathfrak{h}} = \mathrm{Id}_{\mathfrak{h}}\,, \qquad \text{for } \alpha \in \mathcal{R}\,.$$

We refer to θ_0-duals simply as *dual* modules, and to θ_1-duals as *contragredient* modules. △

Consider then the element $\psi \in W^*$ dual to the cyclic vector in the basis (21), i.e. $\langle \psi, w\rangle = 1$ and ψ vanishes on all other vectors of (21)—whence $\mathcal{F}^\vee_\lambda = \mathbb{C}\psi$.

Assume hereafter that $\theta(\mathfrak{h}) = \mathfrak{h}^{\mathrm{op}}$ (up to conjugating θ by an inner automorphism of $\mathfrak{g}$), and canonically identify $\mathfrak{h} \simeq \mathfrak{h}^{\mathrm{op}}$ and their duals. Then we have a well defined pull-back map $\theta^* \in \mathrm{GL}(\mathfrak{h}^\vee)$, which we extend z-linearly to $\big(\mathfrak{h} \otimes z^i\big)^\vee \simeq \mathfrak{h}^\vee \otimes z^i$. Moreover by orthogonality the subspace $\mathfrak{n}^+ \oplus \mathfrak{n}^- \subseteq \mathfrak{g}$ is θ-stable.

Lemma 6.1 *The vector $\psi \in W^*$ satisfies the relations*

$$\begin{aligned} z^p\mathfrak{g}[\![z]\!]\psi &= (0) = \theta^{-1}(\mathfrak{n}^-)[\![z]\!]\psi\,, \\ Hz^i\psi &= \langle \theta^* a_i, Hz^i\rangle\psi\,, \qquad \textit{for } H \in \mathfrak{h},\ i \in \{0,\dots,p-1\}\,. \end{aligned} \tag{32}$$

Proof Use (11), (23), $z^p\mathfrak{g}[\![z]\!]W = (0)$, and the fact that $\widehat{\theta}\colon \mathfrak{g}[\![z]\!] \to \mathfrak{g}[\![z]\!]^{\mathrm{op}}$ preserves the z-grading of Def. 5.2. □

In particular $\mathfrak{n}^-[\![z]\!]\psi = (0)$ in the dual case, and $\mathfrak{n}^+[\![z]\!]\psi = (0)$ in the contragredient case.

6.1 Dual weight grading

Denote $\theta_* := \big(\theta^*\big)^{-1} = \big(\theta^{-1}\big)^*$, and introduce the notation $\widehat{\mathcal{F}}^*_\mu \subseteq \widehat{W}^*_\theta$ and $\mathcal{F}^*_\mu \subseteq W^*_\theta$ for the $\mathfrak{h}$-weight spaces.

Lemma 6.2 *One has $\widehat{\mathcal{F}}^\vee_\mu = \widehat{\mathcal{F}}^*_{\theta^*\mu}$ and $E_\alpha z^i \widehat{\mathcal{F}}^\vee_\mu \subseteq \widehat{\mathcal{F}}^\vee_{\mu+\theta_*\alpha}$, for $\mu \in \mathfrak{h}^\vee$, $\alpha \in \mathcal{R}$, $i \in \mathbb{Z}$, and analogously in the finite case—restricting to $i \in \mathbb{Z}_{\geqslant 0}$.*

Proof Let $\widehat{I}_\nu\colon \widehat{W} \to \widehat{W}$ be the idempotent for the direct summand $\widehat{\mathcal{F}}_\mu \subseteq \widehat{W}$, viz. the endomorphism such that $\widehat{I}_\mu\big|_{\widehat{W}(\mu')} = \delta_{\mu,\mu'}\,\mathrm{Id}_{\widehat{W}(\mu')}$. Then by definition $\widehat{\psi} \in \widehat{\mathcal{F}}^\vee_\mu$ means $\widehat{\psi} = \widehat{\psi} \circ \widehat{I}_\mu$, and by construction

$$\theta(H)\widehat{I}_\mu = \widehat{I}_\mu\theta(H) = \langle \theta^*\mu, H\rangle \widehat{I}_\mu \in \mathrm{End}\big(\widehat{W}\big)\,, \qquad \text{for } \mu \in \mathfrak{h}^\vee,\ H \in \mathfrak{h}\,.$$

Hence for $\widehat{w} \in \widehat{W}$ one has

$$\left\langle H\widehat{\psi}, \widehat{w} \right\rangle = \left\langle \widehat{\psi}, \widehat{I}_\mu\big(\theta(H)\widehat{w}\big) \right\rangle = \langle \theta^*\mu, H\rangle \left\langle \widehat{\psi}, \widehat{w} \right\rangle ,$$

whence the inclusion $\widehat{\mathcal{F}}^\vee_\mu \subseteq \widehat{\mathcal{F}}^*_{\theta^*\mu}$, and the equality follows from (31).

The latter inclusion follows from $\theta(E_\alpha)z^i\widehat{\mathcal{F}}_\mu \subseteq \widehat{\mathcal{F}}_{\mu-\theta_*\alpha}$ for $\alpha \in \mathcal{R}$, which is a straightforward computation using (2).

The same pair of arguments applies verbatim to the finite case. □

Hence (31) is the $\mathfrak{h}$-weight decomposition of θ-dual singular modules, and the weights are contained inside $\theta^*(\lambda + Q) \subseteq \mathfrak{h}^\vee$ (resp. $\theta^*(\lambda + Q^+)$) in the affine (resp. finite) case. By Lem. 5.4 we conclude that $\psi \in W^*_{\theta_0}$ is a *lowest*-weight vector of lowest weight $\theta_0^*\lambda = -\lambda$, whereas $\psi \in W^*_{\theta_1}$ is a highest-weight vector of highest weight $\theta_1^*\lambda = \lambda$.

In particular in the contragredient case matching the cyclic vector with its dual yields a canonical morphism $\Phi\colon W \to W^*_{\theta_1}$, hence a generalisation of the Shapovalov form

$$S\colon W \otimes W \longrightarrow \mathbb{C}, \qquad \widehat{w} \otimes \widehat{w}' \longmapsto \langle \Phi\big(\widehat{w}\big), \widehat{w}'\rangle .$$

This may be degenerate, particularly since the image of the canonical morphism is the submodule

$$W'_\theta := U\big(\mathfrak{g}[\![z]\!]\big)\psi \subseteq W^*_\theta ,$$

which in general is a proper submodule. Nonetheless we can recursively find the obstruction for ψ to generate the θ-dual module. To give a necessary condition consider the vector

$$\widehat{w} = E_{-\alpha}z^{p-1}w \in \mathcal{F}_{\lambda-\alpha} , \qquad \alpha \in \mathcal{R}^+ .$$

By Lem. 6.2 a linear form $\widehat{\psi} \in W'_\theta$ that vanishes on $\mathcal{B}_W \setminus \{ \widehat{w} \}$ must lie in the span of $\left\{ \theta^{-1}(E_\alpha)\psi, \ldots, \theta^{-1}(E_\alpha)z^{p-1}\psi \right\} \subseteq \mathcal{F}^\vee_{\lambda-\alpha}$, so consider a generic element

$$\widehat{\psi} = \widehat{\psi}(b_0, \ldots, b_{p-1}) = \sum_{j=0}^{p-1} b_j\theta^{-1}(E_\alpha)z^j\psi , \qquad b_j \in \mathbb{C} .$$

Using $z^p\mathfrak{g}[\![z]\!]W = (0) = \mathfrak{n}^+[\![z]\!]w$ and $\langle \psi, w\rangle = 1$ yields

$$\left\langle \widehat{\psi}, \widehat{w} \right\rangle = b_{p-1}\langle a_{p-1}, H_\alpha z^{p-1}\rangle,$$

so we need the highest irregular part to be *regular* (cf. Sect. 8).

Conversely we have the following.

Proposition 6.1 *One has $W'_\theta = W^*_\theta$ for parameters (λ, q) in a dense subspace of the affine space $\mathfrak{h}^\vee_p$—with respect to the strong/classical topology.*

Proof Clearly $\mathcal{F}^\vee_\lambda \subseteq W'_\theta$, and then we reason recursively on the $\mathfrak{h}^\vee$-weight space decomposition of W.

Choose $\widehat{w} \in \mathcal{B}_W \cap \mathcal{F}_\mu$, and consider the vectors $\widehat{w}_\alpha(k) := E_{-\alpha} z^k \widehat{w} \in \mathcal{F}_{\mu-\alpha}$, for $\alpha \in \mathcal{R}^+$ and $k \in \{0, \ldots, p-1\}$. As $\widehat{w}$, α and k vary, the vectors $\widehat{w}_\alpha(k)$ exhaust $\mathcal{B}_W \cap \mathcal{F}_{\mu-\alpha}$, so we must find coefficients $b_{ij} \in \mathbb{C}$ such that $\left\langle \widehat{\psi}_\alpha(i), \widehat{w}_\alpha(k) \right\rangle = \delta_{ik}$, where

$$\widehat{\psi}_\alpha(i) = \sum_{j=0}^{p-1} b_{ij} \theta^{-1}(E_\alpha) z^j \widehat{\psi} \in \mathcal{F}^\vee_{\mu-\alpha}, \qquad \text{for } i \in \{0, \ldots, p-1\},$$

and where $\widehat{\psi} \in \mathcal{F}^\vee_\mu$ is the dual of $\widehat{w}$—lying in W'_θ by the recursive hypothesis.

Now one has

$$\left\langle \widehat{\psi}_\alpha(i), \widehat{w}_\alpha(k) \right\rangle = \sum_{j=0}^{p-1} b_{ij} \left\langle \widehat{\psi}, E_\alpha z^j E_{-\alpha} z^k \widehat{w} \right\rangle,$$

hence the given condition means $BM = \mathrm{Id}_{\mathbb{C}^p}$, where B and M are the p-by-p matrices with coefficients $B_{ij} = b_{ij}$ and $M_{jk} = \left\langle \widehat{\psi}, E_\alpha z^j E_{-\alpha} z^k \widehat{w} \right\rangle$, respectively (the latter selects the component of $E_\alpha z^j E_{-\alpha} z^k \widehat{w} \in \mathcal{F}_\mu$ along the line $\mathbb{C}\widehat{w}$, in the basis (21)). A solution exists if and only if $\det(M) \neq 0$.

Now the determinant of $M = M(\widehat{w}, \alpha)$ is a degree-p polynomial whose coefficients depend polynomially on (λ, q), hence it amounts to a polynomial function $\mathfrak{h}^\vee_p \to \mathbb{C}$. Thus $W'_\theta = W^*_\theta$ by taking (λ, q) in a countable intersection of open dense subsets. □

Finally we can choose a complementary subspace to W inside $\widehat{W}$, and extend $\psi \colon W \to \mathbb{C}$ by zero to the whole of $\widehat{W}$—e.g. extract a PBW-basis from (19). Then one can consider the module $\widehat{W}'_\theta \subseteq \widehat{W}^*_\theta$ generated by this extension over $\mathcal{L}\mathfrak{g}$, and define gradings/filtrations on $\widehat{W}'_\theta \twoheadrightarrow W'_\theta$ analogously to §§ 3.2 and 3.3, using the generating set (18), the basis (20), and the standard filtration of $U\big(\theta^{-1}(\mathfrak{n}^+)\big)$. These satisfy the analogous identities of (22)–(25).

7 Segal–Sugawara operators

For $n \in \mathbb{Z}$ define

$$L_n := \frac{1}{2(\kappa + h^\vee)} \sum_{j \in \mathbb{Z}} \left(\sum_k : X_k z^{-j} \cdot X^k z^{n+j} : \right), \tag{33}$$

where $(X_k)_k$ and $(X^k)_k$ are $(\cdot \mid \cdot)$-dual bases of $\mathfrak{g}$, $\kappa \neq -h^\vee$ is a *noncritical* level, and in the normal-ordered product one puts elements of $\mathfrak{g}[\![z]\!] \subseteq \mathcal{L}\mathfrak{g}$ to the right.

The Sugawara operators (33) (due to Segal in this particular form) are well-defined elements of the level-κ completion of $U(\widehat{\mathfrak{g}})$ with respect to the system of left ideals

$U(\widehat{\mathfrak{g}})z^{\bullet}\mathfrak{g}[\![z]\!] \subseteq U(\widehat{\mathfrak{g}})$. If follows from Lem. 5.2 that there are well-defined actions of (33) on the modules $W \subseteq \widehat{W}$.

7.1 Cyclic vector as Sugawara eigenvector

The cyclic vector $w \in \widehat{W}$ is a common eigenvector for the Sugawara operators when $n \gg 0$. To get explicit formulæ for the eigenvalues we recall further euclidean properties of the Cartan–Weyl basis (17).

Remark 7.1 **(On bases and dualities)** Recall that $(H_\alpha \mid H_\alpha)E^{\pm\alpha} = 2E_{\mp\alpha}$. Using the pairing $(\cdot \mid \cdot)\colon \mathfrak{h}^\vee \otimes \mathfrak{h}^\vee \to \mathbb{C}$ induced by the minimal-form duality $\mathfrak{h} \simeq \mathfrak{h}^\vee$ this can be written $2E^{\pm\alpha} = (\alpha \mid \alpha)E_{\mp\alpha}$.

Then we replace the simple-root basis of $\mathfrak{h}$ with a $(\cdot \mid \cdot)$-orthonormal basis, denoted $(H_k)_k$—i.e. we "divide" by the Cartan matrix—, and for $i \in \mathbb{Z}$ we transfer the basis and the pairings to $\mathfrak{g} \otimes z^i$ and $\big(\mathfrak{g} \otimes z^i\big)^\vee \simeq \mathfrak{g}^\vee \otimes z^i$ using the canonical vector space isomorphism $\mathfrak{g} \simeq \mathfrak{g} \otimes z^i$. Then one has the tautological basis-independent identity

$$(\mu \mid \mu') = \sum_{k=1}^{r} \langle \mu, H_k z^i \rangle \langle \mu', H_k z^j \rangle \qquad \text{for } \mu \in \mathfrak{h}^\vee \otimes z^i, \mu' \in \mathfrak{h}^\vee \otimes z^j \,.$$

△

Denote as customary $\rho := \frac{1}{2}\sum_{\alpha \in \mathcal{R}^+} \alpha \in \mathfrak{h}^\vee$ the half-sum of positive roots.

Proposition 7.1 *The cyclic vector* w *is a common eigenvector for the operators* (33) *with* $n \geqslant p-1$. *If* $n > 2(p-1)$ *then* $L_n w = 0$, *else* $L_n w = l_n w$ *with*

$$l_n := \frac{1}{2\big(\kappa + h^\vee\big)} \sum_{j=1-p+n}^{p-1} (a_j \mid a_{n-j}) \,, \qquad \textit{for } n \in \{\, p, \ldots, 2(p-1) \,\} \,, \tag{34}$$

and

$$l_{p-1} := \frac{1}{2\big(\kappa + h^\vee\big)} \left(\sum_{j=0}^{p-1} (a_j \mid a_{p-1-j}) + 2p(\rho \mid a_{p-1}) \right) . \tag{35}$$

Proof Postponed to § B.1. □

Hence the cyclic vector is a Gaiotto–Teschner irregular state of order $p-1$ [28] arising from affine Lie algebras.

Remark 7.2 This generalises the standard fact that $L_n v = 0$ for $n > 0$, and that v is an L_0-eigenvector, with nonzero eigenvalue for generic values of $\lambda \in \mathfrak{h}^\vee$. Namely if $p = 1$ then (35) reduces to

$$L_0 v = \Delta_\lambda v \,, \qquad \Delta_\lambda = \frac{(\lambda \mid \lambda + 2\rho)}{2(\kappa + h^\vee)} \,,$$

reverting to the notation $\lambda = a_0$, which recovers the *conformal weight* corresponding to the action of the quadratic Casimir (4). △

7.2 Action on finite modules

Later we will use the action of the operator L_{-1} on the finite module $W \subseteq \widehat{W}$.

Using $z^p \mathfrak{g}[\![z]\!] W = (0)$ we see nonvanishing terms arise for $1 - p \leqslant j \leqslant p$ in (33), and resolving the ordered product yields

$$L_{-1}\widehat{w} = \frac{1}{\kappa + h^\vee} \sum_{j=1}^{p} \left(\sum_k X_k z^{-j} X^k z^{j-1} \right) \widehat{w} \,, \qquad \widehat{w} \in W \,. \tag{36}$$

As expected $L_{-1}\widehat{w} \notin W$, but it can be put back into the finite module via the loop-algebra action (see Sect. 9).

Remark We see (36) generalises the usual formula from the tame case:

$$L_{-1}\widehat{v} = \frac{1}{\kappa + h^\vee} \sum_k X_k z^{-1} X^k \widehat{v} \,, \qquad \widehat{v} \in V \,. \tag{37}$$

△

8 Irregular conformal blocks: first version

Consider the Riemann sphere $\Sigma := \mathbb{C}P^1$, choose an integer $n \geqslant 1$ and mark points $p_1, \ldots, p_n \in \Sigma$. Denote $J = \{ 1, \ldots, n \}$ the ordered set of labels for the points and $\mathbf{p} = (p_1, \ldots, p_n)$ the ordered set of points.

Let $\mathscr{O}_\Sigma$ be the structure sheaf of regular functions on Σ, seen as a (smooth) complex projective curve. Then consider the stalks $\mathscr{O}_j = \mathscr{O}_{\Sigma, p_j}$ at the marked points, their (unique) maximal ideals $\mathfrak{M}_j = \mathfrak{M}_{p_j} \subseteq \mathscr{O}_j$ of germs of functions vanishing at p_j, the completions $\widehat{\mathscr{O}}_j := \varprojlim_n \mathscr{O}_j / \mathfrak{M}_j^n$, and their field of fractions $\widehat{\mathscr{O}}_j \hookrightarrow \widehat{\mathscr{K}_j}$.

Remark If z_j is a local coordinate on Σ vanishing at p_j then

$$\mathscr{O}_j \simeq \mathbb{C}[z_j] \,, \qquad \mathfrak{M}_j = z_j \mathbb{C}[z_j] \,, \qquad \widehat{\mathscr{O}}_j \simeq \mathbb{C}[\![z_j]\!] \,, \qquad \widehat{\mathscr{K}_j} \simeq \mathbb{C}(\!(z_j)\!) \,.$$

More generally this follows by choosing a uniformiser, i.e. a generator of the maximal ideal(s). △

Then consider the loop algebras $(\mathcal{L}\mathfrak{g})_j := \mathfrak{g} \otimes \widehat{\mathscr{K}_j}$ and the associated affine Lie algebras $\widehat{\mathfrak{g}}_j \twoheadrightarrow (\mathcal{L}\mathfrak{g})_j$. There are canonical isomorphisms $\widehat{\mathfrak{g}}_i \simeq \widehat{\mathfrak{g}}_j$ for $i, j \in J$, and the subscripts distinguish the local picture at the marked points.

Now for $j \in J$ further choose an integer $r_j \geqslant 1$, and set up singular modules as in Sect. 3. Hence consider the Lie subalgebras $\mathfrak{S}^{(r_j)} \subseteq \widehat{\mathfrak{g}}_j$, a common level $\kappa \in \mathbb{C}$ for the central elements, and singular characters $\chi_j = \chi(\lambda_j, q_j, \kappa)$, where $\lambda_j \in \mathfrak{h}^\vee$ and

$$q_j = \big((a_j)_1, \dots, (a_j)_{r_j - 1}\big), \qquad (a_j)_i \in \big(\mathfrak{h} \otimes z^i\big)^\vee .$$

This yields singular modules $W^{(r_j)}_{\chi_j} =: W_j \subseteq \widehat{W}_j := \widehat{W}^{(r_j)}_{\chi_j}$, and we consider the vector spaces

$$\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{\boldsymbol{p},\boldsymbol{\chi}} := \bigotimes_{j \in J} \widehat{W}_j , \qquad \mathcal{H} = \mathcal{H}_{\boldsymbol{p},\boldsymbol{\chi}} := \bigotimes_{j \in J} W_j , \tag{38}$$

where $\boldsymbol{\chi} = (\chi_j)_{j \in J}$. Clearly $\mathcal{H} \subseteq \widehat{\mathcal{H}}$, and the dependence on the choice of marked points is void (it becomes relevant after considering the action of $\mathfrak{g}$-valued meromorphic functions in Sect. 6.2).

The spaces (38) are endowed with natural structures of left modules for the associative algebras $U\big(\widehat{\mathfrak{g}}\big)^{\otimes n} \simeq \bigotimes_{j \in J} U\big(\widehat{\mathfrak{g}}_j\big)$ and $\bigotimes_{j \in J} U\big(\mathfrak{g}_{r_j}\big)$, respectively.

Moreover for indices $i \neq j \in J$ denote $\iota^{(ij)} \colon U(\mathcal{L}\mathfrak{g})^{\otimes 2} \longrightarrow U(\mathcal{L}\mathfrak{g})^{\otimes n}$ the natural inclusion on the i-th and j-th slot, defined on pure tensors by

$$X \otimes Y \longmapsto 1^{\otimes i-1} \otimes X \otimes 1^{\otimes j-i-1} \otimes Y \otimes 1^{\otimes n-j} , \tag{39}$$

for $i < j$, and analogously for $i > j$. Finally define $\iota^{(ii)} \colon U(\mathcal{L}\mathfrak{g})^{\otimes 2} \longrightarrow U(\mathcal{L}\mathfrak{g})^{\otimes n}$ by $X \otimes Y \mapsto 1^{\otimes i-1} \otimes XY \otimes 1^{\otimes n-i}$. This yields an action of quadratic loop-algebra tensors on (38).

8.1 Tame isomonodromy times

We now vary part of the parameters defining the spaces (38), namely the marked points. An *admissible* deformation is one where they do not coalesce, so marked points vary inside the configuration space

$$C_n := \mathrm{Conf}_n(\Sigma) \subseteq \Sigma^n ,$$

of ordered n-tuples of (labeled) points on Σ.

The space C_n is the space of *tame* isomonodromy times. It is a complex manifold of dimension n.

Remark The terminology points again to meromorphic connections on the sphere, cf. introduction.

Let us repeat that the positions of the poles and the irregular types together control Stokes data of irregular meromorphic G-connections over the sphere. In turn Stokes data generalise the conjugacy class of the monodromy representation $\nu \colon \pi_1\big(\Sigma^\circ, b\big) \longrightarrow G$, where $\Sigma^\circ := \Sigma \setminus \big\{ p_j \big\}_{j \in J}$ is the punctured sphere with the poles removed and $b \in \Sigma^\circ$ a base point [9].

Then one may consider admissible deformations of the connections along which Stokes data (locally) do not vary, which yields by definition *isomonodromic* deformations. This can be set up as a system of nonlinear differential equations where the positions of the poles and the irregular types are precisely the independent variables, hence they become the "times" of isomonodromic deformations: the positions of the poles are the tame/regular times, and the rest are the wild/irregular ones.

Geometrically these differential equations constitute a *nonlinear* flat/integrable symplectic connection in the local system of moduli spaces $\mathcal{M}^*_{\mathrm{dR}}$ of meromorphic connections, as the marked points and the irregular types vary (i.e. as the *wild* Riemann surface structure on the sphere varies [13]). △

Remark 8.1 Let $\Sigma \supseteq U \xrightarrow{z} \mathbb{C}$ be a local affine chart on Σ—so $\Sigma \simeq \mathbb{C} \cup \{\infty\}$. Then coordinates on the open subset $C_n(U) := \mathrm{Conf}_n(U) \subseteq C_n$ are given by $\mathbf{t}\colon C_n(U) \to \mathbb{C}^n$, where $\mathbf{t} = (t_j)_{j\in J}$ and $t_j(\mathbf{p}) := z(p_j)$—so $C_n(U) \simeq \mathrm{Conf}_n(\mathbb{C})$, and $C_n \simeq \mathrm{Conf}_n(\mathbb{C}) \cup \mathrm{Conf}_{n-1}(\mathbb{C})$. This yields an atlas on the configuration space. △

Now for a J-tuple χ of singular characters we consider the vector bundles $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_{\bullet,\chi} \to C_n$ and $\mathcal{H} = \mathcal{H}_{\bullet,\chi} \to C_n$, whose fibres over $\mathbf{p} \in C_n$ are the spaces (38), respectively. We have an inclusion $\mathcal{H} \subseteq \widehat{\mathcal{H}}$, and global vector bundle trivialisations:

$$\widehat{\mathcal{H}} \simeq \bigotimes_{J\in J} U\big(\widehat{\mathfrak{n}}^-\big) \otimes_{U(\mathfrak{n}^-)} U\big(\mathfrak{n}^-_{r_j}\big) \times C_n \longrightarrow C_n\,,$$

by (14), and the simpler

$$\mathcal{H} \simeq \bigotimes_{j\in J} U\big(\mathfrak{n}^-_{r_j}\big) \times C_n \longrightarrow C_n\,,$$

by $W_j \simeq U\big(\mathfrak{n}^-_{r_j}\big)$. The point here is that both vector space isomorphisms do not depend on the choice of marked points (nor on the character, cf. 6.4).

8.2 Action of meromorphic functions: punctual version

Given marked points $p_j \in \Sigma$ consider the effective divisor $D := \sum_{j\in J}\big[p_j\big]$ on Σ, and denote as customary $\mathscr{O}_{*D}(\Sigma) = \mathscr{O}_{\Sigma,*D}(\Sigma)$ the vector space of meromorphic functions along Σ with poles at most on (the support of) D. Then let $\mathfrak{g}_{*D}(\Sigma) := \mathfrak{g} \otimes \mathscr{O}_{*D}(\Sigma)$ be the Lie algebra of $\mathfrak{g}$-valued such meromorphic functions, with bracket coming from $\mathfrak{g}$:

$$[f, g](p) := \big[f(p), g(p)\big] \in \mathfrak{g}\,, \qquad \text{for } f, g \in \mathfrak{g}_{*D}(\Sigma)\,, p \in \Sigma\,.$$

Taking Laurent expansions at p_j yields a linear map $\tau_j\colon \mathscr{O}_{*D}(\Sigma) \to \widehat{\mathscr{K}_j}$, and tensoring with $\mathfrak{g}$ a linear map $\mathfrak{g}_{*D}(\Sigma) \to \mathcal{L}\mathfrak{g}_j \subseteq \widehat{\mathfrak{g}}_j$.

Remark If z_j is a local coordinate on Σ vanishing at p_j, and $f \in \mathscr{O}_{*D}(\Sigma)$, then there are coefficients $f_i \in \mathbb{C}$ such that

$$\tau_j(f) = f(z_j) = \sum_{i \geqslant -\operatorname{ord}_{p_j}(f)} f_i z_j^i \in \mathbb{C}(\!(z_j)\!)\,,$$

where $\operatorname{ord}_p(f) \geqslant 0$ is the order of $p \in \Sigma$ as a pole of f. △

Thus there is an arrow

$$\tau\colon \mathfrak{g}_{*D}(\Sigma) \longrightarrow \operatorname{End}\big(\widehat{\mathcal{H}}\big)\,, \qquad \tau(X \otimes f) := \sum_{j \in J} \big(X \otimes \tau_j(f)\big)^{(j)}\,. \tag{40}$$

Using (2), and the fact that the sum of the residues of a meromorphic 1-form on Σ vanishes, shows that (40) is a morphism of Lie algebras.

Then the action $\tau\colon \mathfrak{g}_{*D}(\Sigma) \to \mathfrak{gl}\big(\widehat{\mathcal{H}}\big)$ endows $\widehat{\mathcal{H}}$ with a left $\mathfrak{g}_{*D}(\Sigma)$-module structure.

Definition 8.1 *(Irregular covacua, first version)* The space of irregular covacua at the pair $(\mathbf{p}, \boldsymbol{\chi})$ is the space of coinvariants of the $\mathfrak{g}_{*D}(\Sigma)$-module $\widehat{\mathcal{H}}$:

$$\mathscr{H} := \widehat{\mathcal{H}}_{\mathfrak{g}_{*D}} = \widehat{\mathcal{H}}_{\mathbf{p},\boldsymbol{\chi}} \big/ \mathfrak{g}_{*D}(\Sigma)\widehat{\mathcal{H}}_{\mathbf{p},\boldsymbol{\chi}}\,. \tag{41}$$

Remark In our terminology (41) would be better called the space of *singular* covacua, and be irregular/wild when $r_j \geqslant 2$ for some $j \in J$. △

The space of *vacua* is the dual of (41), and provides a mathematical construction of an irregular version of conformal blocks:

$$\begin{aligned} \mathscr{H}^\dagger &:= \operatorname{Hom}_{\mathfrak{g}_{*D}(\Sigma)}\big(\widehat{\mathcal{H}}_{\mathbf{p},\boldsymbol{\chi}}, \mathbb{C}\big) \\ &= \Big\{ \langle \psi \,\Big|\, \in \widehat{\mathcal{H}}^\vee_{\mathbf{p},\boldsymbol{\chi}} \,|\, \langle \psi, \tau(X \otimes f)\widehat{\boldsymbol{w}} \rangle = 0 \text{ for } X \in \mathfrak{g}\,,\ f \in \mathscr{O}_{*D}(\Sigma)\,,\ \widehat{\boldsymbol{w}} \in \widehat{\mathcal{H}}_{\mathbf{p},\boldsymbol{\chi}} \Big\}\,. \end{aligned} \tag{42}$$

Note we invert the usual usage of the dagger, since in this paper we focus on the space of coinvariants of $\widehat{\mathcal{H}}$—rather than the space of invariants of its dual (cf. [6]).

By (40), the fundamental identity inside the space of covacua is

$$\Big[\big(X \otimes \tau_i(f)\big)^{(i)} \widehat{\boldsymbol{w}}\Big] = -\sum_{j \in J \setminus \{i\}} \Big[\big(X \otimes \tau_j(f)\big)^{(j)} \widehat{\boldsymbol{w}}\Big]\,, \tag{43}$$

for $i \in J$, where square brackets denote equivalence classes modulo $\mathfrak{g}_{*D}(\Sigma)\widehat{\mathcal{H}}_{\mathbf{p},\boldsymbol{\chi}}$.

8.3 Action of meromorphic functions: global version

Now we want to globalise the action (40) over the space of configurations of n-tuples of points on the sphere, i.e. we want a map of sheaves of Lie algebras on C_n.

To define the domain sheaf consider the projection

$$\pi_\Sigma \colon \Sigma^{n+1} \longrightarrow \Sigma^n, \qquad (p, p_1, \dots, p_n) \longmapsto (p_1, \dots, p_n)\,.$$

Then set

$$Y := \pi_\Sigma^{-1}(C_n) = \left\{ (p, p_1, \dots, p_n) \;\middle|\; p_i \neq p_j \text{ for } i \neq j \right\} \subseteq \Sigma^{n+1}\,,$$

so that $\pi_\Sigma \colon Y \twoheadrightarrow C_n$ is the universal family of n-pointed spheres.

Now for $j \in J$ define the hyperplane $P_j := \left\{ p = p_j \right\} \subseteq \Sigma^{n+1}$, consider the effective divisor $\mathcal{D} := \sum_{j \in J} \left[Y \cap P_j \right]$ on Y, and let $\mathscr{O}_{*\mathcal{D}} = \mathscr{O}_{Y,*\mathcal{D}}$ be the sheaf of meromorphic functions on Y with poles at most along (the support of) $\mathcal{D}$. Then we have the push-forward sheaf $(\pi_\Sigma)_* \mathscr{O}_{*\mathcal{D}}$ on C_n, and by tensoring we obtain the sheaf of Lie algebras $\mathfrak{g}_{*\mathcal{D}} := \mathfrak{g} \otimes (\pi_\Sigma)_* \mathscr{O}_{*\mathcal{D}}$.

Remark If $U' \subseteq C_n$ is open then $\mathfrak{g}_{*\mathcal{D}}(U')$ is then the Lie algebra of $\mathfrak{g}$-valued meromorphic functions on $\Sigma \times U'$, such that the restriction to $\Sigma \times \{ \mathbf{p} \} \simeq \Sigma$ has poles at most at the set $\left\{ p_j \right\}_{j \in J}$ for all $\mathbf{p} \in U'$, as wanted. △

Now for $U' \subseteq C_n$ open we consider the Laurent expansion $\tau_j(U')(f)$ of functions $f \in \mathcal{O}_{*\mathcal{D}}\left(\pi_\Sigma^{-1}(U')\right)$ along the divisor $Y \cap P_j$. Then tensoring with $\mathfrak{g}$ yields a map of sheaves

$$\tau_j \colon \mathfrak{g}_{*\mathcal{D}} \longrightarrow \mathscr{O}_{C_n} \otimes \mathcal{L}\mathfrak{g}_j \subseteq \mathscr{O}_{C_n} \otimes \widehat{\mathfrak{g}}_j\,,$$

where $\mathscr{O}_{C_n}$ is the structure sheaf on the configuration space.

Remark If z_j is a local coordinate on Σ vanishing at p_j, $U' = \mathrm{Conf}_n(U)$ for $U \subseteq \Sigma$ an open affine subset, and $f \in (\pi_\Sigma)_* \mathscr{O}_{*\mathcal{D}}(U')$, then there are suitable functions $f_i \colon U' \to \mathbb{C}$ such that

$$\tau_j(U')(f) = f(z_j, t_1, \dots, t_n) = \sum_i f_i(t_1, \dots, t_n) z_j^i \in \mathscr{O}_{C_n}(U') \otimes \mathbb{C}(\!(z_j)\!)\,,$$

using the local coordinates $(t_j)_{j \in J}$ on $U' \subseteq C_n$ of Rem. 8.1. By definition the functions f_i may have poles on the hyperplanes $\left\{ t_i = t_j \right\} \subseteq \mathbb{C}^n$. △

Finally summing the action over each slot of the tensor product we have a sheaf-theoretic analogue of (40), acting on sections of $\widehat{\mathcal{H}}$.

8.4 Irregular isomonodromy times

One may add the other possible deformations, e.g. with the following setup.

Recall the *regular* parts of the Cartan subalgebra and its dual are the complements of (co)root hyperplanes:

$$\mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{\alpha \in \mathcal{R}} \mathrm{Ker}(\alpha), \qquad \mathfrak{h}^{\vee}_{\mathrm{reg}} := \mathfrak{h}^{\vee} \setminus \bigcup_{\alpha \in \mathcal{R}} \mathrm{Ker}\big(\mathrm{ev}_{H_\alpha}\big),$$

and analogously for $\mathfrak{h} \otimes z^i$ and its dual.

Then consider irregular parts $q_j \in \mathfrak{b}^{\vee}_{r_j}$ such that the most irregular coefficient $(a_j)_{r_j-1}$ is regular, and define an *admissible* deformation of as one in which the most irregular coefficient does not cross coroot hyperplanes.

Remark This is the analogous condition as for the marked points: the open charts $C_n(\mathbb{C}) \subseteq \mathbb{C}^n$ are regular parts for Cartan subalgebras of rank-n type-A simple Lie algebras. △

Doing so we get to the space of isomonodromy times

$$\mathbf{B} = C_n \times \prod_{j \in J} \big(\mathfrak{h}^{\vee}_{r_j}\big)_{\mathrm{reg}}, \tag{44}$$

where

$$\big(\mathfrak{h}^{\vee}_{r_j}\big)_{\mathrm{reg}} = \prod_{i=1}^{r_j-2} \big(\mathfrak{h} \otimes z^i\big)^{\vee} \times \big(\mathfrak{h} \otimes z^{r_j-1}\big)^{\vee}_{\mathrm{reg}},$$

and in turn $(\mathfrak{h} \otimes z^i)_{\mathrm{reg}} := \big\{ Hz^i \bigm| H \in \mathfrak{h}_{\mathrm{reg}} \big\} \subseteq \mathfrak{h} \otimes z^i$ for $i \in \mathbb{Z}$.

The space (44) is a complex manifold of dimension $d = n + r\sum_{j \in J}(r_j - 1)$, where $r = \mathrm{rk}(\mathfrak{g})$. As expected it coincides with the space of tame isomonodromy times if $r_j = 1$ for $j \in J$.

Remark If there is just one irregular module W_j with $r_j = 2$ then

$$\mathbf{B} = C_n \times \big(\mathfrak{h} \otimes z\big)^{\vee}_{\mathrm{reg}},$$

and one recovers the base space for the FMTV connection [23]—up to the canonical vector space isomorphism $\mathfrak{h} \otimes z \simeq \mathfrak{h}$. If further the variations of marked points are neglected then (44) becomes the base space for the "Casimir" connection of De Concini and Millson–Toledano Laredo (DMT) [40, 52]. △

Then in (38) one can let both $\boldsymbol{p} \in C_n$ and $\boldsymbol{\chi} \in \prod_{j \in J}\big(\mathfrak{h}^{\vee}_{r_j}\big)_{\mathrm{reg}}$ vary, getting a vector bundle over the base space (44). This also comes with a canonical vector bundle trivialisation, reasoning in the same way as for $\mathcal{H} \subseteq \widehat{\mathcal{H}}$ (namely (14) is also independent of χ).

Finally one may extend the sheaf $\mathfrak{g}_{*\mathcal{D}}$ trivially along the Cartan directions. Namely the projection $\pi_{C_n} : \mathbf{B} \twoheadrightarrow C_n$ is open, so one may take the naïf pullback sheaf:

$$\pi^*_{C_n}\mathfrak{g}_{*\mathcal{D}}(U) = \mathfrak{g}_{*\mathcal{D}}\big(\pi_{C_n}(U)\big)\,, \qquad \text{for } U \subseteq \mathbf{B} \text{ open}\,.$$

9 Conformal blocks in terms of finite modules: first version

Throughout this section fix a pair $(\mathbf{p}, \chi)$ to define the spaces $\mathcal{H} \subseteq \widehat{\mathcal{H}}$ as in (38). Compose the inclusion $\mathcal{H} \hookrightarrow \widehat{\mathcal{H}}$ with the canonical projection $\pi_{\mathscr{H}} \colon \widehat{\mathcal{H}} \twoheadrightarrow \mathscr{H}$ to obtain a map $\iota \colon \mathcal{H} \to \mathscr{H}$.

To study the image of ι consider the tensor product filtration

$$\widehat{\mathfrak{F}}^-_{\leqslant \bullet} := \bigotimes_{j \in J} \big(\widehat{\mathcal{F}}^-_j\big)_{\leqslant \bullet}\,, \tag{45}$$

where $\big(\widehat{\mathcal{F}}^-_j\big)_{\leqslant \bullet}$ is the filtration defined in § 3.3 on $\widehat{W}_j$. By definition $\widehat{\mathfrak{F}}^-_{\leqslant 0} = \mathcal{H}$, and we push (45) forward to a filtration $\widehat{\mathscr{F}}^-_{\leqslant \bullet}$ on $\mathscr{H}$, along the surjection $\pi_{\mathscr{H}}$. Note $\widehat{\mathscr{F}}^-_{\leqslant \bullet}$ is exhaustive, since $\widehat{\mathfrak{F}}^-_{\leqslant \bullet}$ is.

Proposition 9.1 *The map ι is surjective,*

Proof We will show that $\widehat{\mathscr{F}}^-_{\leqslant k}$ lies in the image of ι by induction on $k \geqslant 0$. The base is given by $\widehat{\mathscr{F}}^-_{\leqslant 0} = \pi_{\mathscr{H}}(\mathcal{H})$.

Now we use (43) for a function $f_i \in \mathscr{O}_{*D}(\Sigma)$ with a pole at p_i, and only there. Such a function is e.g. defined by $f_i(z) = (z - t_i)^{-m}$, with the notations of Rem. 8.1, working in a local chart containing $\mathbf{p}$.

Hence $\tau_j\big(f_i\big) \in \widehat{\mathscr{O}}_j$ for $j \neq i$, and if $\widehat{w} \in \widehat{\mathfrak{F}}^-_{\leqslant k}$ the rightmost identity of (24) shows that the right-hand side of (43) lies in $\widehat{\mathscr{F}}^-_{\leqslant k}$. Then by induction the image of ι contains $\widehat{\mathscr{F}}^-_{\leqslant k}$ and all the vectors on the left-hand side of (43), and the conclusion follows from the leftmost identity of (24). $\square$

Proposition 9.2 *One has* $\operatorname{Ker}(\iota) = \mathfrak{g}\mathcal{H} \subseteq \mathcal{H}$.

Proof Consider an element $\widehat{w} = \tau(X \otimes f)\widehat{u}$ with $\widehat{u} = \bigotimes_{j \in J} \widehat{u}_j \in \widehat{\mathcal{H}}$, $f \in \mathfrak{g}_{*D}(\Sigma)$ and $X \in \mathfrak{g}$.

If the function f is noncostant then it has a pole, say at $p_j \in \Sigma$. It follows that $\tau_j(f) \notin \widehat{\mathscr{O}}_j$, whence $X \otimes \tau_j(f)^{(j)}\widehat{u}_j \notin \big(\widehat{\mathcal{F}}^+_j\big)_{\leqslant 0}$ by (22), and $\widehat{w} \notin \mathcal{H} = \widehat{\mathfrak{F}}^+_{\leqslant 0}$.

Thus to have element of the kernel we must restrict to $f \in \mathbb{C}$. Then using (22) again we see that $X \otimes f = X \otimes \tau_j(f) \in \mathfrak{g}$ preserves the grading $\big(\widehat{\mathcal{F}}^-_j\big)_\bullet$ on $\widehat{W}_j$, so $(X \otimes f)\widehat{u}_j \in W_j = \big(\widehat{\mathcal{F}}^-_j\big)_0$ implies $\widehat{u}_j \in W_j$.

Conversely $\mathfrak{g}\mathcal{H} \subseteq \mathfrak{g}_{*D}(\Sigma)\widehat{\mathcal{H}} \cap \mathcal{H}$ lies in the kernel. $\square$

Hence there is an identification $\mathscr{H} \simeq \mathcal{H}_{\mathfrak{g}} = \mathcal{H}/\mathfrak{g}\mathcal{H}$, generalising the analogous standard fact for the tame case.

To go further one may appeal to the tensor product of the weight grading of § 3.4, which is a $\big(\mathfrak{h}^\vee\big)^J$-grading on $\mathcal{H}$. Namely we consider the subspaces

$$\mathcal{F}_{\boldsymbol{\mu}} = \mathcal{F}_{\boldsymbol{\mu}}(\mathcal{H}) := \bigotimes_{j \in J} \mathcal{F}_{\mu_j}(W_j) \subseteq \mathcal{H}\,, \qquad \text{for } \boldsymbol{\mu} = (\mu_j)_{j\in J} \in \big(\mathfrak{h}^\vee\big)^J\,. \tag{46}$$

By (40) the subspace $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}}$ lies inside the weight space of weight $|\boldsymbol{\mu}| := \sum_j \mu_j \in \mathfrak{h}^\vee$ for the tensor product $\mathfrak{h}$-action.

If $|\boldsymbol{\mu}| \neq 0$ then $\boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}} \subseteq \mathfrak{h}\boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}}$ is annihilated by $\pi_{\mathscr{H}}$, so we still have a surjective map

$$\mathcal{H} \supseteq \bigoplus_{|\boldsymbol{\mu}|=0} \boldsymbol{\mathcal{F}}_{\boldsymbol{\mu}} \xrightarrow{\pi_{\mathscr{H}}} \mathscr{H}\,, \tag{47}$$

and by construction the $\mathfrak{h}$-action is trivialised on this subspace.

Remark The condition $|\boldsymbol{\mu}| = 0$ is reminiscent of meromorphic connections: it is equivalent to the vanishing of the sum of the residues over Σ—in the duality (15). △

9.1 Auxiliary tame module

Suppose one of the modules is tame, e.g. the last one: $r_n = 1$ and $W_n = V_n$. Then we split the tensor product as

$$\mathcal{H} = \mathcal{H}' \otimes V_n\,, \qquad \mathcal{H}' := \bigotimes_{j\in J'} W_j\,,$$

where $J' := J \setminus \{\, n \,\}$, and we embed

$$\mathcal{H}' \longrightarrow \mathcal{H}\,, \qquad \bigotimes_{j\in J'} \widehat{w}_j \longmapsto \bigotimes_{j\in J'} \widehat{w}_j \otimes v_n\,,$$

where $v_n \in V_n$ is the cyclic/highest-weight vector.

Proposition 9.3 *One has* $\iota(\mathcal{H}') = \mathscr{H}$.

Proof Denote $\mathcal{E}^{(n)}_{\leqslant \bullet}$ the filtration on $U(\mathfrak{g})v_n \subseteq V_n$ defined in § 3.3, which is exhaustive in this (tame) case. We will prove by induction on $k \geqslant 0$ that $\iota\big(\mathcal{H}'\big)$ contains the classes of all vectors inside $\mathcal{H}' \otimes \mathcal{E}^{(n)}_{\leqslant k}$, noting the base follows from the identity $\mathcal{E}^{(n)}_{\leqslant 0} = \mathbb{C}v_n$.

For the inductive step we use the constant version of (43). For $X \in \mathfrak{g}$ this shows that the class of $X^{(n)}\widehat{\boldsymbol{w}}$ lies in $\iota\big(\mathcal{H}'\big)$ as soon as that of $\widehat{\boldsymbol{w}} \in \mathcal{H}' \otimes \mathcal{E}^{(n)}_{\leqslant k}$ does, which is precisely the inductive hypothesis. Hence the conclusion follows from (25). □

Now $\mathbb{C}v_n = \mathcal{F}_{\lambda_n}(V_n)$, so (47) yields a surjection:

$$\mathcal{H}' \supseteq \bigoplus_{|\boldsymbol{\mu}| = -\lambda_n} \boldsymbol{\mathcal{F}}'_{\boldsymbol{\mu}} \xrightarrow{\pi_{\mathscr{H}}} \mathscr{H}\,, \qquad \text{where } \boldsymbol{\mu} = (\mu_j)_{j \in J'} \in \left(\mathfrak{h}^\vee\right)^{J'}, \tag{48}$$

writing $|\boldsymbol{\mu}| = \sum_{j \in J'} \mu_j \in \mathfrak{h}^\vee$ analogously to the above, and where $\boldsymbol{\mathcal{F}}'_{\boldsymbol{\mu}} \subseteq \mathcal{H}'$ is the tensor product of the weight-gradings over $J' \subseteq J$ (analogously to (46)). Note the direct sum is just the weight space of weight $-\lambda_n \in \mathfrak{h}^\vee$ for the (tensor) action of $\mathfrak{h}$ on $\mathcal{H}'$; let us temporarily denote this space by $\mathcal{H}'(-\lambda_n)$.

Lemma 9.1 *The kernel of (48) equals* $\mathfrak{n}^+\mathcal{H}' \cap \mathcal{H}'(-\lambda_n) \subseteq \mathcal{H}'(-\lambda_n)$.

Proof We must show that no coinvariants can arise from the residual $\mathfrak{n}^-$-action.

To this end recall $\mathfrak{n}^-$ has nontrivial action on the associated graded of the filtration $\mathcal{E}_{\leqslant \bullet}$ of § 3.3: more precisely (26) yields

$$\mathfrak{n}^-\left(\mathcal{H}' \otimes \mathrm{gr}\left(\mathcal{E}^{(n)}\right)_k\right) \subseteq \left(\mathcal{H}' \otimes \mathrm{gr}\left(\mathcal{E}^{(n)}\right)_k\right) \oplus \left(\mathcal{H}' \otimes \mathrm{gr}\left(\mathcal{E}^{(n)}\right)_{k+1}\right) \subseteq \mathcal{H}\,,$$

for $k \in \mathbb{Z}_{\geqslant 0}$; but there can be no vanishing of components in the latter direct summand since V_n is freely generated over $U(\mathfrak{n}^-)$, and this applies in particular to $v_n \in \mathcal{E}^{(n)}_{\leqslant 0} \simeq \mathrm{gr}\left(\mathcal{E}^{(n)}_0\right)$. □

The punchline is the final identification

$$\mathscr{H} \simeq \mathcal{H}'(-\lambda_n) \big/ \left(\mathfrak{n}^+\mathcal{H}' \cap \mathcal{H}'(-\lambda_n)\right). \tag{49}$$

7.1.1. *On dimensions.* To go further we use the results of § 3.4; in particular we employ the notation $\mathcal{F}_\nu(W_j) := \mathcal{F}_{\lambda_j - \nu} \subseteq W_j$ for $\nu \in Q^+$—i.e. we parametrise the weights $\mu_j = \lambda_j - \nu \preceq \lambda_j$ by $\nu \in Q^+$.

By definition the weight space of weight $-\lambda_n \in \mathfrak{h}^\vee$ for the $\mathfrak{h}$-action on $\mathcal{H}'$, denoted $\mathcal{H}'(-\lambda_n)$ above, is the direct sum of the spaces $\boldsymbol{\mathcal{F}}'_{\boldsymbol{\nu}} \subseteq \mathcal{H}'$ such that $0 = \lambda_n + \sum_{j \in J'}(\lambda_j - \nu_j)$, so only elements such that $|\boldsymbol{\nu}| = |\boldsymbol{\lambda}| \in \mathfrak{h}^\vee$ will contribute to coinvariants. This actually depends on the sum of the tame parts of the singular characters, hence we ought to change notation:

$$\mathcal{H}'_{|\boldsymbol{\lambda}|} := \bigoplus_{|\boldsymbol{\nu}| = |\boldsymbol{\lambda}|} \boldsymbol{\mathcal{F}}'_{\boldsymbol{\nu}} \subseteq \mathcal{H}'\,. \tag{50}$$

Proposition 9.4 *The $\mathfrak{h}$-weight space $\mathcal{H}'_{|\boldsymbol{\lambda}|} \subseteq \mathcal{H}'$ has dimension*

$$\dim\left(\mathcal{H}'_{|\boldsymbol{\lambda}|}\right) = \sum_{|\boldsymbol{\nu}| = |\boldsymbol{\lambda}|} \left(\prod_{j \in J'} \sum_{\boldsymbol{m} \in \mathrm{Mult}_{\mathcal{R}^+}(\nu_j)} \binom{\boldsymbol{m} + r_j - 1}{\boldsymbol{m}}\right) < \infty\,. \tag{51}$$

Proof It follows from (27), taking the products of the dimensions of the weight spaces $\mathcal{F}_{\nu_j} \subseteq W_j$.

The dimension is finite since for $\nu \in Q_+$ there are finitely many J'-tuples $\boldsymbol{\nu} \in \left(Q^+\right)^{J'}$ such that $|\boldsymbol{\nu}| = \nu$—analogously to $|\mathrm{Mult}_{\mathcal{R}^+}(\nu)| < \infty$. □

We deduce the following.

Corollary 9.1 *If one module is tame then the space* (41) *is finite-dimensional for all choices of marked points and singular characters.*

In particular the weight space is trivial if $|\boldsymbol{\lambda}| \notin Q^+$, and the simplest nontrivial case is when $|\boldsymbol{\lambda}| = 0$. Then $|\boldsymbol{\nu}| = 0$ implies $\nu_j = 0$ for $j \in J'$, so $\mathcal{H}'_0$ is the line generated by the tensor product $\bigotimes_{i \in J'} w_i$ of the cyclic vectors $w_i \in W_i$.

The next nontrivial example is when $|\boldsymbol{\lambda}| = \theta \in \Pi$ is a simple root. Now $|\boldsymbol{\nu}| = \theta$ implies $\boldsymbol{\nu} \in \left\{ \boldsymbol{\nu}^{\theta,i} \right\}_i$, with $\nu_j^{\theta,i} = \delta_{ij}\theta$ for $i, j \in J'$. Then one finds the singleton $\mathrm{Mult}_{\mathcal{R}^+}\left(\nu_j^{\theta,i}\right) = \left\{ \delta_{ij} \boldsymbol{m}^\theta \right\}$, with $m_\alpha^\theta = \delta_{\alpha\theta}$. On the whole (51) reduces to

$$\dim\left(\mathcal{H}'_\theta\right) = \sum_{i \in J'} \left(\prod_{j \in j'} \binom{\delta_{ij}\boldsymbol{m}^\theta + r_j - 1}{\delta_{ij}\boldsymbol{m}^\theta} \right) = \sum_{i \in J'} \binom{\boldsymbol{m}^\theta + r_i - 1}{\boldsymbol{m}^\theta} = \sum_{i \in J'} r_i \,,$$

independently of the choice of simple root.

A basis is given by the pure tensors

$$\widehat{\boldsymbol{w}}_i^j := \bigotimes_{k=1}^{i-1} w_k \otimes F_\theta z^j w_i \otimes \bigotimes_{k=i+1}^{n-1} w_k \,,$$

for $i \in J'$ and $j \in \{ 0, \ldots, r_i - 1 \}$.

Remark 9.1 One way to ensure coinvariants are nontrivial is the following: for a given configurations of points $\boldsymbol{p} = (p_j)_{j \in J}$ consider the Lie subalgebra of $\mathfrak{g}$-valued meromorphic functions with poles at p_j, and further with a *zero* elsewhere, say at $p' \in \Sigma \setminus \{ p_j \}_j$. Then the proof of Proposition 9.1 can easily be adapted working in the chart where $p' = \infty$—as the function $f_i(z) = (z - t_i)^{-m}$ vanishes at infinity.

Thus there is still a surjection of $\mathcal{H}$ on the space of coinvariants, and similarly to Proposition 9.2 only constant functions lie in the kernel. Hence in this setup the kernel is trivial and $\mathcal{H} \neq (0)$ itself is the space of coinvariants.

Another way to ensure nontriviality is to put a θ-dual module in the tensor product (introduced in Sect. 6). Further when it is tame then one still has a finite-dimensional space, see Sect. 10. △

9.1.1 Archetypal case

Consider the same setup of Sect. 3.4.1 for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$. In this case $|\boldsymbol{\lambda}| = m\alpha$ for an integer $m \geqslant 0$.

Proposition 9.5 *One has*

$$\dim\left(\mathcal{H}'_{m\alpha}\right) = \binom{m+R-1}{m}, \qquad \textit{where } R := \sum_{j\in J'} r_j\,. \tag{52}$$

A basis is provided by the pure tensors

$$\widehat{w}^{\Phi} = \bigotimes_{j\in J'} \left(\prod_{i=0}^{r_j-1} \left(Fz^i\right)^{\Phi(i,j)} w_j \right),$$

where $\Phi \in \mathrm{WComp}_R(m)$*—identifying* $\{1,\ldots,R\} \simeq \coprod_{j\in J'}\{0,\ldots,r_j-1\}$.

Proof Fix an integer $m \geqslant 0$ and look for $\boldsymbol{\nu} \in (\mathbb{Z}_{\geqslant 0}\alpha)^{J'}$ satisfying $|\boldsymbol{\nu}| = m\alpha$. Such elements are given by weak J'-compositions of m, i.e. functions $\phi\colon J' \to \mathbb{Z}_{\geqslant 0}$ satisfying $\sum_{j\in J'}\phi(j) = m$, with bijection

$$\phi \longmapsto \boldsymbol{\nu}^{\phi}, \qquad \nu_j^{\phi} := \phi(j)\alpha\,.$$

Then by definition $\mathrm{Mult}_{\mathcal{R}^+}\left(\nu_j^{\phi}\right) = \{\phi(j)\}$ for $j \in J'$, so we need only give elements $\varphi_j \in \mathrm{WComp}_{r_j}\left(\phi(j)\right)$ to allocate the z-degrees of the occurrences of $-\alpha$ at each slot of the tensor product.

The data of ϕ and $\boldsymbol{\varphi} = (\varphi_j)_j$ is equivalent to that of the weak R-composition $\Phi\colon R \to \mathbb{Z}_{\geqslant 0}$ defined by $\Phi(i,j) = \varphi_j(i)$, and the result follows. □

Remark In the tame case (52) simplifies to

$$\dim\left(\mathcal{H}'_{m\alpha}\right) = \left|\mathrm{WComp}_{J'}(m)\right| = \binom{m+|J'|-1}{m},$$

and a basis is given by the pure tensors

$$\widehat{\boldsymbol{v}}^{\phi} = \bigotimes_{j\in J'} F^{\phi(j)} v_j \qquad \text{for } \phi \in \mathrm{WComp}_{J'}(m)\,.$$

This is somehow the opposite of (30): there we had an arbitrary singular module, here we have a tensor product of arbitrarily many tame modules. △

10 Irregular conformal blocks: second version

We now vary the setup of Sect. 8 giving a special role to one of the marked points (e.g. the last one): choose a $(\cdot \mid \cdot)$-orthogonal morphism $\theta\colon \mathfrak{g} \to \mathfrak{g}^{\mathrm{op}}$ and at the last marked point put a θ-dual module $\widehat{W}'_\theta \twoheadrightarrow W'_\theta$ as defined in Sect. 6.

In this case the tensor product splits as

$$\widehat{\mathcal{H}} = \widehat{W}'_n \otimes \bigotimes_{j \in J'} \widehat{W}_j\,, \qquad \mathcal{H} = W'_n \otimes \bigotimes_{j \in J'} W_j\,, \tag{53}$$

where $J' = J \setminus \{ n \}$ as in § 7.1—and omitting the subscript θ. These are naturally subspaces of $\operatorname{Hom}(\widehat{W}_n, \widehat{\mathcal{H}}')$ and $\operatorname{Hom}(W_n, \mathcal{H}')$, respectively, where $\mathcal{H}'$ is as in § 7.1 and $\widehat{\mathcal{H}}' := \bigotimes_{j \in J'} \widehat{W}_j$. Moreover they still assemble into trivial vector bundles $\widehat{\boldsymbol{\mathcal{H}}} \twoheadrightarrow \boldsymbol{\mathcal{H}}$ over the space $C_n = \operatorname{Conf}_n(\Sigma)$—but also over the full space (44) of isomonodromy times.

The Lie algebra of $\mathfrak{g}$-valued meromorphic functions on Σ acts on the leftmost tensor product of (53). Thinking in terms of linear maps $\widehat{\psi}\colon \widehat{W}_n \to \widehat{\mathcal{H}}'$, and using (40) and the dual actions of Sect. 6, one has the formula

$$\left\langle \tau(X \otimes f)\widehat{\psi}, \widehat{w} \right\rangle = \sum_{j \in J'} \big(X \otimes \tau_j(f)\big)^{(j)} \left\langle \widehat{\psi}, (\theta(X) \otimes \tau_n(f))\widehat{w} \right\rangle \in \widehat{\mathcal{H}}'\,,$$

where $X \in \mathfrak{g}$, $f \in \mathscr{O}_{*D}(\Sigma)$ and $\widehat{w} \in \widehat{W}_n$. Taking coinvariants of the resulting left module yields a second version of the space of irregular covacua, still denoted $\mathscr{H}$: again the space of vacua $\mathscr{H}^{\dagger}$ is defined as in (42), and provides a second version of irregular conformal blocks. Moreover the material of Sect. 8 goes through, and there is an action of the sheaf of Lie algebras $\mathfrak{g}_{*\mathcal{D}}$ on sections of $\widehat{\boldsymbol{\mathcal{H}}}$ and $\boldsymbol{\mathcal{H}}$.

10.1 On coinvariants

Consider first the natural inclusion $\iota\colon \widehat{W}'_n \otimes \mathcal{H}' \hookrightarrow \widehat{\mathcal{H}}$, which can be composed with the canonical projection $\pi_{\mathscr{H}}\colon \widehat{\mathcal{H}} \to \mathscr{H}$.

Reasoning as in Proposition 9.1 (which may be thought of as the case $\widehat{W}'_n = \mathbb{C}$) shows this composition is surjective. Then reasoning as in Proposition 9.2 shows the kernel is obtained from the action of meromorphic functions with no poles at $\{ p_1, \dots, p_{n-1} \} \subseteq \Sigma$, but only (at most) at the point p_n. Hence there is a vector space isomorphism

$$\mathscr{H} \simeq \widehat{W}'_n \otimes \mathcal{H}' \big/ \mathfrak{g}_{*p_n}(\Sigma)\big(\widehat{W}'_n \otimes \mathcal{H}'\big)\,,$$

thinking of $p_n \in \Sigma$ as a divisor.

Now a function with a pole at most at p_n is either constant, or its Laurent expansion at p_n lies in $z_n^{-1}\mathfrak{g}\big[z_n^{-1}\big] \subseteq (\mathcal{L}\mathfrak{g})_n$, where as usual z_n is a local coordinate on Σ vanishing at p_n. Hence a coinvariant function is uniquely determined by its restriction to $W_n \subseteq \widehat{W}_n$, and since now poles are not allowed we get the following.

Proposition 10.1 *There is a canonical vector space identification*

$$\mathscr{H} \simeq W'_n \otimes \mathcal{H}' \big/ \mathfrak{g}(W'_n \otimes \mathcal{H}')\,.$$

Thus in this case as well we can reduce the discussion to $\mathfrak{g}$-coinvariants for the tensor product of finite modules.

Now suppose the dual module is tame, and adapt the discussion of § 7.1. We see there is a surjective map $\mathcal{H}' \to \mathscr{H}$, where again $\mathcal{H}' = \bigotimes_{j \in J'} W_j$—embedded in $\mathcal{H}$ via $\widehat{\boldsymbol{w}} \mapsto \psi \otimes \widehat{\boldsymbol{w}}$, where $\psi \in V'_n$ is the cyclic vector. Reasoning as in Lem. 9.1 the $\theta^{-1}(\mathfrak{n}^-)$-action cannot give coinvariant elements, so we are left with the action of $\mathfrak{h} \oplus \theta^{-1}(\mathfrak{n}^+)$.

In the dual case where $\theta = \theta_0 = -\mathrm{Id}_{\mathfrak{g}}$ we have $\theta^{-1}(\mathfrak{n}^+) = \mathfrak{n}^+$, so we are essentially back to § 7.1.1. The contragredient case where $\theta = \theta_1$ (the transposition) instead allows to go further. In this case $\theta^{-1}(\mathfrak{n}^+) = \mathfrak{n}^-$, whence a new identification $\mathscr{H} \simeq \mathcal{H}'_{\mathfrak{h}^-}$, and to trivialise the $\mathfrak{h}$-action we consider again the zero-weight subspace inside $\mathcal{H}'$. This is again (50), whose (finite) dimension is given in Proposition 9.4.

Finally in this setup we can recover nontriviality, as follows. Recall we attach weights $\boldsymbol{\lambda} = (\lambda_j)_{j \in J} \in \left(\mathfrak{h}^\vee\right)^J$ to the marked points, and that we consider the sum $|\boldsymbol{\lambda}| = \sum_{j \in J} \lambda_j \in \mathfrak{h}^\vee$. The weight space is $\mathcal{H}'_{|\boldsymbol{\lambda}|} \subseteq \mathcal{H}'$, hence

$$\mathscr{H} \simeq \mathcal{H}'_{|\boldsymbol{\lambda}|} \big/ \left(\mathfrak{n}^- \mathcal{H}' \cap \mathcal{H}'_{|\boldsymbol{\lambda}|}\right).$$

Compare with (49): as expected the roles of the nilpotent subalgebras $\mathfrak{n}^\pm$ are exchanged—by θ.

Proposition 10.2 *Suppose $n \geqslant 3$ and $|\boldsymbol{\lambda}| \in Q^+$: then the space of coinvariants is nontrivial—for any choice of wild parts.*

A fortiori then nontriviality holds if the n-th module is not tame.

Proof For $\widehat{w} \in \mathcal{F}_{|\lambda|}(W_1) \subseteq W_1$ consider the pure tensor

$$\widehat{\boldsymbol{w}} := \widehat{w} \otimes \bigotimes_{2 \leqslant i \leqslant n-1} w_i \in \mathcal{H}'_{|\boldsymbol{\lambda}|}.$$

(In brief put the cyclic in all slots except the first, and put a vector of suitable weight in the first slot.) An argument analogous to the proof of Lem. 9.1 shows that $\widehat{\boldsymbol{w}}_i \notin \mathfrak{n}^- \mathcal{H}'$, which means exactly that $\left[\widehat{\boldsymbol{w}}_i\right] \neq 0$ inside $\mathscr{H}$.

More precisely denote $\mathcal{E}^{(j)}_{\leqslant \bullet}$ the filtration on $U(\mathfrak{g}) w_j = U(\mathfrak{n}^-) w_j \subseteq W_j$ induced from $U(\mathfrak{n}^-)$, as in § 3.3, with associated grading $\mathrm{gr}\left(\mathcal{E}^{(j)}\right)_\bullet$. Then consider the tensor product $\mathbb{Z}^{J''}_{\geqslant 0}$-grading

$$\mathrm{gr}(\boldsymbol{\mathcal{E}})_\bullet := \bigotimes_{j \in J''} \mathrm{gr}\left(\mathcal{E}^{(j)}\right)_\bullet, \qquad \text{where } J'' := J' \setminus \{1\}.$$

Using (26) yields

$$\mathfrak{n}^- \left(W_1 \otimes \mathrm{gr}(\boldsymbol{\mathcal{E}})_{\boldsymbol{k}}\right) \subseteq \left(W_1 \otimes \mathrm{gr}(\boldsymbol{\mathcal{E}})_{\boldsymbol{k}}\right) \oplus \bigoplus_{i=2}^{n-1} \left(W_1 \otimes \mathrm{gr}(\boldsymbol{\mathcal{E}})_{\boldsymbol{k} + \varepsilon_i}\right),$$

for $k \in \mathbb{Z}^{J''}_{\geqslant 0}$, where $\varepsilon_i \in \mathbb{Z}^{J''}$ is the i-th vector of the canonical $\mathbb{Z}$-basis. Again the vanishing of components in the latter direct summands cannot happen, since the $U(\mathfrak{n}^-)$-action is free on singular modules. □

Remark If $n = 2$ instead simply $\mathscr{H} \simeq \mathcal{F}_\nu(W) / \big(\mathfrak{n}^- W \cap \mathcal{F}_\nu(W)\big)$ for $\nu \in Q^+$, and we must further distinguish the tame/wild case.

In the tame case $V = \mathfrak{n}^- V \oplus \mathbb{C}v$, so nontriviality implies $v \in \mathcal{F}_\nu$: this forces $\nu = 0$ and $\mathscr{H} \simeq \mathcal{F}_\lambda(V) = \mathbb{C}v$.

In the wild case instead write $\nu = \sum_{\alpha \in \mathcal{R}^+} m_\alpha \alpha$ for $m_\alpha \in \mathbb{Z}_{\geqslant 0}$, and consider the vector

$$\widehat{w}_\nu := \prod_{\alpha \in \mathcal{R}^+} \big(X_{-\alpha} z\big)^{m_\alpha} w \in \mathcal{F}_\nu(W) \,,$$

ordering again the positive roots along the Cartan–Weyl basis (17) (note this makes sense at all depths $p \geqslant 2$). Clearly $\widehat{w}_\nu \notin \mathfrak{n}^- W$, since all occrrences of root vectors have positive z-degree, hence $\mathscr{H} \neq (0)$ always in this case. △

Remark One may also consider the tensor products of the grading of Def. 5.2, in addition to the $\mathfrak{h}$-weight grading—i.e. use the fact that every finite module is a graded $\mathfrak{n}^-[\![z]\!]$-module. Namely there is a decomposition

$$\mathcal{H}' = \bigoplus_{k \in \mathbb{Z}^{J'}} \boldsymbol{\mathcal{F}}^+_k \,, \qquad \text{where} \qquad \boldsymbol{\mathcal{F}}^+_k = \bigotimes_{j \in J'} \mathcal{F}^+_{k_j}(W_j) \,,$$

which is preserved by the tensor product $\mathfrak{b}^-$-action, so $\mathscr{H} \simeq \bigoplus_{k \in \mathbb{Z}^{J'}} (\boldsymbol{\mathcal{F}}^+_k)_{\mathfrak{b}^-}$. This is a new feature: in the tame case the grading in positive z-degree is trivial. △

11 Flat connections

Consider a particular case of the setup of Sect. 8: mark $n + 1$ (ordered) points on Σ, vary the first $n \geqslant 1$ of them, and *fix* singular characters at those points.

Thus we work on a closed subspace of $\mathrm{Conf}_{n+1}(\Sigma)$, which is naturally identified with the local chart $U' = \mathrm{Conf}_n(U) \subseteq C_n$ of Rem. 8.1 where $p_{n+1} = \infty$—whence $\{ p_1, \dots, p_n \} \subseteq U \simeq \mathbb{C}$. The label set becomes $J = \{ 1, \dots, n, \infty \}$, and we write $J' := J \setminus \{ \infty \}$.

Then we have two versions of spaces of covacua: either we put a singular module at infinity, or a θ-dual. In any case we consider the restrictions of the vector bundles $\boldsymbol{\mathcal{H}} \subseteq \widehat{\boldsymbol{\mathcal{H}}}$ over $U' \simeq C_n(\mathbb{C}) := \mathrm{Conf}_n(\mathbb{C})$, as well as for the sheaves $(\pi_\Sigma)_* \mathcal{O}_{*\mathcal{D}}$ and $\mathfrak{g}_{*\mathcal{D}}$ on U'—and keep the same notation for them.

Then we want to define a connection $\widehat{\nabla}$ on $\widehat{\boldsymbol{\mathcal{H}}} \to C_n(\mathbb{C})$ which is compatible with the action of the sheaf of Lie algebras $\mathfrak{g}_{*\mathcal{D}}$. In the given trivialisation this will be of the form $\widehat{\nabla} = d - \widehat{\varpi}$, where $\widehat{\varpi}$ is a 1-form on $C_n(\mathbb{C})$ with values in endomorphisms of the fibres, and with a view towards (generalisations of) KZ [38] we set

$$\langle \widehat{\varpi}, \partial_{t_i} \rangle := L^{(i)}_{-1} \,, \qquad \text{for } i \in J' \,,$$

where we use the coordinates $\mathbf{t}\colon C_n(\mathbb{C}) \to \mathbb{C}^n$ of Rem. 8.1 and the Sugawara operator (33). This is a translation-invariant 1-form on the parallelisable manifold $C_n(\mathbb{C})$, so in particular $d\widehat{\varpi} = 0$. Further the actions of L_{-1} on different slots commute, so $\left[\widehat{\varpi} \wedge \widehat{\varpi}\right] = 0$, and the connection $\widehat{\nabla}$ is (strongly) flat.

11.1 Compatibility with the action of meromorphic functions

We now consider a natural connection D on the sheaf $\mathfrak{g}_{*\mathcal{D}}$—a linear map $D\colon \mathfrak{g}_{*\mathcal{D}} \to \Omega^1_{C_n(\mathbb{C})} \otimes \mathfrak{g}_{*\mathcal{D}}$ satisfying Leibniz's rule. Namely we set

$$D(X \otimes f) := X \otimes df\,,$$

where $d\colon \Omega^0_{C_n(\mathbb{C})} \to \Omega^1_{C_n(\mathbb{C})}$ is the standard de Rham differential.

Proposition 11.1 *One has*

$$\widehat{\nabla}\big(\tau(X \otimes f)\widehat{\boldsymbol{w}}\big) = \tau\big(D(X \otimes f)\big)\widehat{\boldsymbol{w}} + \tau(X \otimes f)\widehat{\nabla}\widehat{\boldsymbol{w}}\,, \tag{54}$$

where $X \in \mathfrak{g}$, and f and $\widehat{\boldsymbol{w}}$ are local sections of $(\pi_\Sigma)_*\mathscr{O}_{*\mathcal{D}}$ and $\widehat{\mathfrak{H}}$, respectively.

To prove this we use the following well-known fact.

Lemma 11.1 ([35], Lem. 12.8) *One has the identity $\left[L_{-1}, Xz^m\right] = -mXz^{m-1}$ inside the level-κ completion of $U\big(\widehat{\mathfrak{g}}\big)$, for $X \in \mathfrak{g}$ and $m \in \mathbb{Z}$.*

Proof of Proposition 11.1 For $i \in J'$ and for local sections $\widehat{\boldsymbol{w}}$ and $X \otimes f$ of $\widehat{\mathcal{H}}$ and $\mathfrak{g}_{*\mathcal{D}}$—respectively—we must prove that

$$\partial_{t_i}\big(\tau(X \otimes f)\widehat{\boldsymbol{w}}\big) - \left[L^{(i)}_{-1}, \tau(X \otimes f)\right]\widehat{\boldsymbol{w}} = \tau\big(X \otimes \partial_{t_i} f\big)\widehat{\boldsymbol{w}} + \tau(X \otimes f)\partial_{t_i}\widehat{\boldsymbol{w}}\,.$$

Now for $j \in J'$ we have the expansions

$$\tau_j(f) = \sum_k f_k(t_1, \ldots, t_n) z_j^k\,,$$

where f_k is a regular function on an open subset of $C_n(\mathbb{C})$, and we take the local coordinate $z_j = z - t_j$ on Σ—vanishing at p_j. Since $\partial_{t_i}(z_j) + \delta_{ij} = 0$ one has

$$\partial_{t_i}\big(\tau_j(f)\big) = \tau_j(\partial_{t_i} f) + \delta_{ij}\left[L_{-1}, \tau_j(f)\right],$$

using Lem. 11.1. Hence by (40):

$$\partial_{t_i}\big(\tau(X \otimes f)\widehat{\boldsymbol{w}}\big) = \tau\big(X \otimes \partial_{t_i} f\big)\widehat{\boldsymbol{w}} + \left[L_{-1}, X \otimes \tau_i(f)\right]^{(i)}\widehat{\boldsymbol{w}} + \tau(X \otimes f)(\partial_{t_i}\widehat{\boldsymbol{w}})\,,$$

and we conclude with

$$\left[L^{(i)}_{-1}, \tau(X \otimes f)\right] = \left[L^{(i)}_{-1}, \big(X \otimes \tau_i(f)\big)^{(i)}\right] = \left[L_{-1}, X \otimes \tau_i(f)\right]^{(i)}.$$

□

Thus a reduced connection is well defined on $\mathscr{H} \to C_n(\mathbb{C})$, since $\widehat{\nabla}$ preserves the sheaf of sections with values in the subspaces $\mathfrak{g}_{*D}\widehat{\mathcal{H}}_{\boldsymbol{p},\chi} \subseteq \widehat{\mathcal{H}}_{\boldsymbol{p},\chi}$, by (54). We conclude the sheaf of covacua has a natural structure of flat vector bundle over the space of tame isomonodromy times, so in particular the dimension of (41) is constant along variations of the marked points—when finite. After dualising the connection, the same statements follow for the spaces of vacua.

11.2 Description on finite modules: first version

By the results of Sect. 9 it is possible to describe the reduction of $\widehat{\nabla}$ as the $\mathfrak{g}$-reduction of a connection ∇ living on the vector sub-bundle $\mathcal{H} \subseteq \widehat{\mathcal{H}}$, and further as a connections acting on $\mathcal{H}' \subseteq \mathcal{H}$ when the module at infinity is tame.

The goal is to find an explicit expression for ∇. For this we will use the following elementary fact, where we further set $z_\infty := z^{-1}$—a local coordinate vanishing at infinity.

Lemma 11.2 *(Expansions at irregular singularities) For $i \in J'$ and for an integer $m > 0$ one has*

$$\tau_j\left(z_i^{-m}\right) = \begin{cases} \sum_{l \geqslant 0} \binom{m+l-1}{l} \frac{z_j^l}{(t_i - t_j)^l (t_j - t_i)^m}, & j \in J \setminus \{i\}, \\ \sum_{l \geqslant 0} \binom{m+l-1}{l} t_i^l z_\infty^{m+l}, & j = \infty. \end{cases} \tag{55}$$

11.2.1 Tame case

Suppose $r_j = 1$ for $j \in J$. Then using (55) with $m = 1$ yields

$$X \otimes \tau_j\left(z_i^{-1}\right)\widehat{v}_j = \frac{X}{t_j - t_i}\widehat{v}_j, \qquad X \otimes \tau_\infty\left(z_i^{-1}\right)\widehat{v}_\infty = 0,$$

for $X \in \mathfrak{g}$, $i \neq j \in J'$, $\widehat{v}_j \in V_j$ and $\widehat{v}_\infty \in V_\infty$—since $z_j\mathfrak{g}[\![z_j]\!]V_j = (0)$ for $j \in J$. Hence by (43) one has the following identity inside $\mathscr{H}$—with tacit use of $\pi_{\mathscr{H}}$:

$$\left(X \otimes z_i^{-1}\right)^{(i)} \widehat{\boldsymbol{v}} \otimes \widehat{v}_\infty = \sum_{j \in J' \setminus \{i\}} \frac{X^{(j)}}{t_i - t_j} \widehat{\boldsymbol{v}} \otimes \widehat{v}_\infty,$$

where $\widehat{\boldsymbol{v}} = \bigotimes_{j \in J'} \widehat{v}_j \in \mathcal{H}$. In particular the action is trivial at infinity.

Looking at (37) and writing $L_{-1}^{(i)}(\widehat{\boldsymbol{v}} \otimes \widehat{v}_\infty) = \widehat{\boldsymbol{v}}_i \otimes \widehat{v}_\infty$ we find

$$\widehat{\boldsymbol{v}}_i = \frac{1}{\kappa + h^\vee} \sum_{j \in J' \setminus \{i\}} \left(\sum_k \frac{(X^k)^{(i)} X_k^{(j)}}{t_i - t_j} \right) \widehat{\boldsymbol{v}} = \frac{1}{\kappa + h^\vee} \sum_{j \in J' \setminus \{i\}} \frac{\Omega^{(ij)}}{t_i - t_j} \widehat{\boldsymbol{v}}, \tag{56}$$

where $\Omega^{(ij)} := \iota^{(ij)}(\Omega)$ denotes the embedding (39) of the quadratic tensor (5)—with $m = l = 0$.

One recovers the KZ connection [38] on the sub-bundle $\mathcal{H}'_{|\boldsymbol{\lambda}|} \hookrightarrow \mathcal{H}$, taking V_∞ as auxiliary tame module.

11.2.2 Tame modules in the finite part

Now allow $r_\infty \geqslant 1$ to be arbitrary. What changes is

$$X \otimes \tau_\infty(z_i^{-1})\widehat{w}_\infty = \sum_{l=0}^{r_\infty - 2} t_i^l X z_\infty^{l+1} \cdot \widehat{w}_\infty \,,$$

for $X \in \mathfrak{g}$, $\widehat{w}_\infty \in W_\infty$ and $i \in J'$, using the case $m = 1$ of (55). So the action is nontrivial at infinity if $r_\infty \geqslant 2$.

Then by (43) one has the following identity inside $\mathscr{H}$—with tacit use of $\pi_{\mathscr{H}}$:

$$\big(X \otimes z_i^{-1}\big)^{(i)} \widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty = \left(\sum_{j \in J' \setminus \{ i \}} \frac{X^{(j)}}{t_i - t_j} - \sum_{l=0}^{r_\infty - 2} t_i^l \big(X z^{l+1}\big)^{(\infty)} \right) \widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty \,.$$

Thus looking at (37) one finds $L_{-1}^{(i)}(\widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty) = \widehat{\boldsymbol{v}}_i \otimes w_\infty + \mathcal{D}_i(\widehat{\boldsymbol{v}} \otimes \widehat{w})$, where $\widehat{\boldsymbol{v}}_i$ is as in (56), and

$$\mathcal{D}_i(\widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty) = \frac{1}{\kappa + h^\vee} \sum_{l=0}^{r_\infty - 2} t_i^l \Omega_{0,l+1}^{(i\infty)} (\widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty) \,,$$

using again the embedding $\iota^{(i\infty)}(\Omega_{0,l+1})$ of (5) defined by (39).

Remark E.g. if $r_\infty = 2$ then the new operator acts by

$$\mathcal{D}_i(\widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty) = \frac{\Omega_{01}^{(i\infty)} \widehat{\boldsymbol{v}} \otimes \widehat{w}_\infty}{\kappa + h^\vee} \,. \tag{57}$$

In this case the reduced connection is close to the dynamical KZ connection, i.e. [23, Eq. 3]. We will recover the very same "dynamical" Cartan term in Sect. 14. △

11.2.3 Tame module at infinity

Suppose symmetrically $r_\infty = 1$, but r_j is arbitrary for $j \in J'$.

Proposition 11.2 *One has* $L_{-1}^{(i)} \widehat{\boldsymbol{w}} \otimes \widehat{v}_\infty = \widehat{\boldsymbol{w}}_i \otimes \widehat{v}_\infty$, *with*

$$\widehat{\boldsymbol{w}}_i = -\frac{1}{\kappa + h^\vee} \sum_{j \in J' \setminus \{ i \}} \left(\sum_{m=0}^{r_i - 1} \sum_{l=0}^{r_j - 1} \binom{m+l}{l} \frac{\Omega_{ml}^{(ij)} \widehat{\boldsymbol{w}}}{(t_i - t_j)^l (t_j - t_i)^{m+1}} \right) . \tag{58}$$

Proof Postponed to § B.2. □

This is an irregular generalisation of the KZ connection, corresponding to an action of the universal connection of [46].[10]

Remark The flat connection (58) is known to admit an isomonodromy system as semiclassical limit (see op. cit.): precisely the irregular isomonodromy system on $\mathbb{C}P^1$ for variations of the positions of the poles (the tame isomonodromy times, as considered in [37]).

This generalises the same fact from the tame case: the quantisation of the Schlesinger system [48] yields the KZ connection [30, 46]. △

11.2.4 General case

Finally take $r_\infty \geqslant 1$ to be generic as well.

Proposition 11.3 *One has* $L_{-1}^{(i)}\widehat{\boldsymbol{w}} \otimes \widehat{w}_\infty = \widehat{\boldsymbol{w}}_i \otimes \widehat{w}_\infty + \mathcal{D}_i(\widehat{\boldsymbol{w}} \otimes \widehat{w}_\infty)$, *with* $\widehat{\boldsymbol{w}}_i$ *as in* (58) *and*

$$\mathcal{D}_i(\widehat{\boldsymbol{w}} \otimes \widehat{w}_\infty) = \frac{1}{\kappa + h^\vee} \sum_{m=0}^{r_i-1} \sum_{l=0}^{r_\infty-m-1} \binom{m+l}{l} t_i^l \Omega^{(i\infty)}_{m,m+l+1}(\widehat{\boldsymbol{w}} \otimes \widehat{w}_\infty). \tag{59}$$

Proof This is a generalisation of Proposition 11.2 where moreover

$$X \otimes \tau_\infty(z_i^{-m})\widehat{w}_\infty = \sum_{l=0}^{r_\infty-m-1} \binom{m+l-1}{l} t_i^l X z_\infty^{m+l} \cdot \widehat{w}_\infty\,,$$

for $X \in \mathfrak{g}$, $\widehat{w}_\infty \in W_\infty$ and $i \in J'$, using the general case of (55). Now the action is nontrivial at infinity for $r_\infty \geqslant m+1$, and the result still follows from (36). □

11.3 Description on finite modules: second version

Finally one may consider the setup of Sect. 10, i.e. put a θ-dual module W'_θ at infinity. In the analogue of §§ 9.2.1 and 9.2.3—when the module at infinity is tame—the description of the reduced connection does not change, using (32). In the remaining cases one finds the action of the same quadratic tensors on the last slot, acting on the θ-dual.

Hence in the next section we will introduce a *universal* versions of the reduced connection, looking at (58) and (59), to treat the two versions on the same footing.

[10] Compare also (58) with [27, Eqs. B.6 and B.7], where $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$: this should be a formalisation of fn. 6 of op. cit.

12 Universal connections

Fix again a depth $p \geqslant 1$, an integer $n \geqslant 1$, and the finite ordered sets

$$\{ 1, \dots, n \} = J' \subseteq J = \{ 1, \dots, n, \infty \} .$$

Consider then the nonautomous (quantum) Hamiltonian systems

$$\widehat{H}_i = \widehat{H}_i^{(p)} \colon C_n(\mathbb{C}) \to U(\mathfrak{g}_p)^{\otimes |J|} ,$$

with Hamiltonians $\widehat{H}_i = \widehat{H}_i' + \widehat{H}_i''$ for $i \in J'$, where

$$\widehat{H}_i'(\mathbf{t}) := \frac{1}{\kappa + h^\vee} \sum_{j \in J' \setminus \{ i \}} \left(\sum_{m,l=0}^{p-1} \Omega_{ml}^{(ij)} \binom{m+l}{l} (-1)^m (t_i - t_j)^{-1-m-l} \right) , \tag{60}$$

and

$$\widehat{H}_i''(\mathbf{t}) := \frac{1}{\kappa + h^\vee} \sum_{m,l=0}^{p-1} \Omega_{m,m+l+1}^{(i\infty)} \binom{m+l}{l} t_i^l , \tag{61}$$

as suggested by (58) and (59).

These Hamiltonians are equivalent to the *universal* connection (at depth p):

$$\nabla_p = d - \varpi_p , \quad \varpi_p = \varpi_p' + \varpi_p'' , \quad \varpi_p' := \sum_{J'} \widehat{H}_i' dt_i , \quad \varpi_p'' := \sum_{J'} \widehat{H}_i'' dt_i , \tag{62}$$

defined on the trivial vector bundle $U(J, p) := C_n(\mathbb{C}) \times U(\mathfrak{g}_p)^{\otimes |J|} \to C_n(\mathbb{C})$ by means of the $U(\mathfrak{g}_p)^{\otimes |J|}$-valued 1-forms ϖ_p' and ϖ_p'' on the base space. This generalises [46] with a nontrivial action at infinity.

Then for every choice of singular modules labeled by J there is an action of (62) on $\mathcal{H}$ for $p \gg 0$, which reproduces the most general case of § 9.2 (with θ-duals or not), so in particular there are induced *integrable* quantum Hamiltonian systems. Hence one expects (62) to be flat before taking representations, as we will show.

Remark One directly checks that

$$\frac{\partial \widehat{H}_j'}{\partial t_i} - \frac{\partial \widehat{H}_i'}{\partial t_j} = 0 , \qquad \text{and} \qquad \frac{\partial \widehat{H}_j''}{\partial t_i} = \delta_{ij} , \quad \text{for } i, j \in J' ,$$

so (strong) flatness is equivalent to the commutativity of the quantum Hamiltonians. △

12.1 Flatness at finite distance

The 1-form defining the Hamiltonians (60) can be written

$$\varpi'_p = \frac{1}{\kappa + h^\vee} \sum_{i \neq j \in J'} r_p^{(ij)}(t_i - t_j) d(t_i - t_j),$$

where $r_p : \mathbb{C} \setminus \{0\} \to \mathfrak{g}_p^{\otimes 2}$ is the following rational function:

$$r_p(t) := -\sum_{m,l=0}^{p-1} \Omega_{ml} \otimes (-1)^m \binom{m+l}{l} t^{-1-m-l}. \tag{63}$$

Remark It is easy to see that r_p is skew-symmetric, meaning

$$r_p^{(ij)}(t) + r_p^{(ji)}(-t) = 0, \qquad \text{for } t \in \mathbb{C} \setminus \{0\},\ i, j \in J'. \tag{64}$$

△

The study of the connection $\nabla'_p := d - \varpi'_p$ is closely related to the theory of the classical Yang–Baxter equation (CYBE) [4]. In particular flatness (for $|J'| \geqslant 3$) is equivalent to the CYBE for (63) in the Lie algebra $\mathfrak{g}_p$, i.e. to the following identity inside $\mathfrak{g}_p^{\otimes 3}$:

$$\left[r_p^{(12)}(t_{12}), r_p^{(13)}(t_{13})\right] + \left[r_p^{(13)}(t_{13}), r_p^{(23)}(t_{23})\right] + \left[r_p^{(12)}(t_{12}), r_p^{(23)}(t_{23})\right] = 0,$$

where $t_{ij} := t_i - t_j$.

Theorem 12.1 (cf. [46]) *The rational function* (63) *is a solution of the CYBE.*

Proof We will reduce the proof to the well-known case $p = 1$, where $\mathfrak{g}_p = \mathfrak{g}$. In this case we have the classical result that the rational function $r_1(t) = \Omega t^{-1}$ is a skew-symmetric solution of the CYBE [4], which is an easy consequence of the Drinfeld–Kohno relations $\left[\Omega^{(ij)}, \Omega^{(ik)} + \Omega^{(jk)}\right] = 0$, and the Arnold relations [2]:

$$\frac{1}{t_{ij}t_{jk}} + \frac{1}{t_{jk}t_{ki}} + \frac{1}{t_{ki}t_{ij}} = 0. \tag{65}$$

To prove the general case consider the identification $\mathfrak{g}_p^{\otimes 2} \simeq \mathfrak{g}^{\otimes 2} \otimes A(2, p)$, where $A(n, p) := \mathbb{C}[\![w_1, \dots, w_n]\!]/\mathfrak{I}_p$ is the quotient of the power-series ring by the ideal $\mathfrak{I}_p = \left(w_1^p, \dots, w_n^p\right)$ generated by $\left\{ w_1^p, \dots, w_n^p \right\}$. In this identification $\Omega_{ml} = \Omega \otimes w_1^m w_2^l$, and (63) can be written

$$r_p(t) = \Omega \otimes \tau_{(0,0)}^{(p)}(f_t) \in \mathfrak{g}_p^{\otimes 2}, \qquad \text{where} \qquad f_t(w_i, w_j) := \frac{1}{t + w_i - w_j},$$

and where $\tau^{(p)}_{(0,0)}(f_t)$ is the class mod $\mathfrak{I}_p$ of the Taylor expansion of f_t at the origin. Then, up to the identification $\mathfrak{g}_p^{\otimes 3} \simeq \mathfrak{g}^{\otimes 3} \otimes A(3,p)$, the CYBE follows again from (65), with t_i replaced by $t_i - w_i$, for $i \in \{1,2,3\}$. $\square$

Hence we have an inverse system of classical r-matrices, with respect to the canonical projections $\mathfrak{g}[\![z]\!]/z^{\bullet+1}\mathfrak{g}[\![z]\!] \twoheadrightarrow \mathfrak{g}[\![z]\!]/z^{\bullet}\mathfrak{g}[\![z]\!]$, corresponding to an inverse system of flat vector bundles $\big(\mathbb{U}(n,p), \nabla'_p\big)$ over the space of configurations of J'-tuples of points in the complex plane. The inverse limit of the vector bundles is naturally identified with the trivial vector bundle with fibre $\mathbb{U}\big(\mathfrak{g}[\![z]\!]\big)^{\widehat{\otimes}|J|}$, the completion of the n-th tensor power of the positive part of the loop algebra.

Remark The inverse limit $r_\infty(t) = \varprojlim_p r_p(t) \in \mathfrak{g}^{\otimes 2}[t^{-1}][\![z_1,z_2]\!]$ is a solution of the CYBE in a completion of $\mathfrak{g}[\![z]\!]^{\otimes 3} \otimes \mathscr{O}_{C_3(\mathbb{C})}\big(C_3(\mathbb{C})\big)$.

Analogously on the representation-theoretic side one may consider characters of the Lie subalgebra

$$\mathfrak{S}^{(\infty)} := \bigcap_{p \geqslant 1} \mathfrak{S}^{(p)} = \mathfrak{b}^+[\![z]\!] \oplus \mathbb{C}K \subseteq \widehat{\mathfrak{g}},$$

using (7). Then $\mathfrak{S}^{(\infty)}_{\mathrm{ab}} \simeq \mathfrak{h}[\![z]\!] \oplus \mathbb{C}K$, so the induced non-smooth modules $\widehat{W}^{(\infty)}$ depend on infinitely many Cartan parameters (and a level κ), and are generated over $\mathbb{U}(\mathcal{L}\mathfrak{n}^-)$ by a cyclic vector annihilated by $\mathfrak{n}^+[\![z]\!]$. Under (15) the parameters of these modules correspond to principal parts of connections with *essential* singularities. $\triangle$

12.2 Flatness overall

The 1-form defining the Hamiltonians (61) can be written

$$\varpi''_p = \frac{1}{\kappa + h^\vee} \sum_{i \in J'} s_p^{(i\infty)}(t_i)\mathrm{d}t_i\,,$$

where $s_p : \mathbb{C} \setminus \{0\} \to \mathfrak{g}_p^{\otimes 2}$ is the following rational function:

$$s_p(t) := \sum_{m,l=0}^{p-1} \Omega_{m,m+l+1} \otimes \binom{m+l}{l} t^l\,.$$

Theorem 12.2 *The universal connection ∇_p is flat for* $p \geqslant 1$.

Proof Reasoning as in the proof of Theorem 12.1 consider the function

$$g_t(w_i, w_j) := \frac{w_j}{1 - w_j(t + w_i)}\,.$$

Then one directly checks that the Taylor expansion of g_t at the origin satisfies

$$s_p(t) = \Omega \otimes \tau^{(p)}_{(0,0)}(g_t),$$

and we can conclude by proving a version of the CYBE in the Lie algebra $\mathfrak{g}_p$.

Namely by Theorem 12.1 the commutator of two Hamiltonians becomes

$$\big[\widehat{H}_i, \widehat{H}_j\big] = \big[r_p^{(ij)}(t_{ij}), s_p^{(i\infty)}(t_i)\big] + \big[r_p^{(ij)}(t_{ij}), s_p^{(j\infty)}(t_j)\big] + \big[s_p^{(i\infty)}(t_i), s_p^{(j\infty)}(t_j)\big],$$

using the fact that actions on disjoint pairs of slots commute, and the skew-symmetry (64). Now we can use the standard Drinfeld–Kohno relations to reduce flatness (for all $p \geqslant 1$) to a variation of the Arnold relations (65), namely to the following identity:

$$g_{t_i}(w_i, w_\infty) g_{t_j}(w_j, w_\infty) + f_{t_{ij}}(w_i, w_j)\big(g_{t_i}(w_i, w_\infty) - g_{t_j}(w_j, w_\infty)\big) = 0,$$

where $f_t = f_t(w_i, w_j)$ is as in the proof of Theorem 12.1. □

Remark 12.1 One can give a more symmetric expression of (62), with no special role for the marked point at infinity.

To this end consider the generating function

$$\varphi(w_i, w_j) := \frac{1}{w_i - w_j}, \tag{66}$$

which is a meromorphic function on $\mathbb{C}^2$ with poles along $\{ w_i = w_j \} \subseteq \mathbb{C}^2$—and only there. It can be extended (by zero) to a meromorphic function on the complex surface $\Sigma^2 \setminus \{ (\infty, \infty) \}$, so we can take Taylor expansions $\tau_{(p_i,p_j)}(\varphi)$ of φ at any pair of *distinct* points $p_i, p_j \in \Sigma$—using the local coordinates w_i^{-1} and w_j^{-1} at infinity.

Then analogously to the above one checks that

$$\tau^{(p)}_{(p_i,p_j)}(\varphi) = r_p(t_{ij}), \qquad \tau^{(p)}_{(p_i,\infty)}(\varphi) = s_p(t_i),$$

for points $p_i, p_j \in \Sigma$ at finite distance of coordinates $t_i, t_j \in \mathbb{C}$, respectively.

Hence

$$\varpi_p = \frac{1}{\kappa + h^\vee} \sum_{i \neq j \in J} \tau^{(p)}_{(p_i,p_j)}(\varphi) dt_{ij},$$

and all marked points are treated the same.

Then the flatness of (62) for $p \geqslant 1$ is equivalent to *generalised* Arnold relations, relating the Taylor expansions of (66) at pairs extracted from a triple of distinct points on the Riemann sphere. △

Hence we find again an inverse system of flat vector bundles $\big(U(J, p), \nabla_p\big)$, over the space of configurations of J'-tuples of points in the complex plane.

12.3 Connection on coinvariants

The universal connection (62) is well defined for sections with values in the space of $\mathfrak{g}$-coinvariants of $U(\mathfrak{g}_p)^{\otimes n}$.

To prove this consider the canonical embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}_p \simeq \mathfrak{g} \ltimes \mathfrak{b}_p$ and the universal embedding $\mathfrak{g}_p \hookrightarrow U(\mathfrak{g}_p)$. Composing them we let $\mathfrak{g}$ act on $U(\mathfrak{g}_p)$ in the regular representation, and finally the tensor product action (analogous of (40) in the case of constant functions). Then we get a $\mathfrak{g}$-action on differential forms with values in the flat vector bundle $\big(U(J,p), \nabla_p\big)$.

Proposition 12.1 *The* $\mathfrak{g}$*-action is flat for all* $p \geqslant 1$.

Note this is a particular case of a compatibility such as (54), for constant sections of the trivial bundle $C_n(\mathbb{C}) \times \mathfrak{g} \to C_n(\mathbb{C})$, equipped with the trivial connection.

Proof Postponed to § B.3. □

It follows that (62) preserves sections with values in $\mathfrak{g}U(\mathfrak{g}_p)^{\otimes |J|} \subseteq U(\mathfrak{g}_p)^{\otimes |J|}$, so a reduced (flat) connection is well defined on the space of $\mathfrak{g}$-coinvariants of the tensor product. This was to be expected, as it holds for the induced connections above.

13 On conformal transformations

Consider the action of Möbius transformations on $\Sigma = \mathbb{P}(\mathbb{C}^2)$, that is

$$g.\big[t_1 \colon t_2\big] = \big[at_1 + bt_2 \colon ct_1 + dt_2\big] ,$$

for $(t_1, t_2) \in \mathbb{C}^2 \setminus \{ \mathbf{0} \}$, with $g = g(a, b, c, d)$ given by numbers $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$. In the standard affine chart $U = \Sigma \setminus \{[1 \colon 0]\} \xrightarrow{t} \mathbb{C}$ we then have the subgroup of affine transformation of the complex plane, with diagonal action on $C_n(\mathbb{C}) \subseteq \mathbb{C}^n$, and with induced pull-back (right) action on sections of vector bundles over that base.

In particular translations $t \mapsto t + b$ correspond to $a = d = 1$ and $c = 0$. This is the 1-parameter subgroup corresponding to the infinitesimal generator $E \in \mathrm{Lie}\big(\mathrm{PSL}(2, \mathbb{C})\big) = \mathfrak{sl}(2, \mathbb{C})$, and the associated infinitesimal action reads

$$\frac{d\big(\widehat{w} \circ \gamma\big)(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d\widehat{w}(\mathbf{t} + \varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \sum_{i \in J'} \frac{\partial \widehat{w}}{\partial t_i} ,$$

considering the path $\gamma \colon \varepsilon \mapsto g(1, \varepsilon, 0, 1)$.

Analogously dilations correspond to the 1-parameter subgroup generated by $H \in \mathfrak{sl}(2, \mathbb{C})$, and the associated infinitesimal action is given by the Euler vector field

$$\frac{d\big(\widehat{w} \circ \gamma\big)(\varepsilon)}{d\varepsilon}\bigg|_{\varepsilon=0} = \frac{d\widehat{w}\big((1 + \varepsilon)^2 \mathbf{t}\big)}{d\varepsilon}\bigg|_{\varepsilon=0} = 2 \sum_{i \in J'} t_i \frac{\partial \widehat{w}}{\partial t_i} ,$$

considering the path $\gamma \colon \varepsilon \mapsto g\bigl(1+\varepsilon, 0, 0, (1+\varepsilon)^{-1}\bigr)$.

Proposition 13.1 *Suppose the module at infinity is tame. Then the action of affine transformations on horizontal sections of the bundle of covacua reads*

$$\widehat{w}(\mathbf{t}') = \prod_{i \in J'} \exp\bigl(a L_0^{(i)}\bigr) \cdot \widehat{w}(\mathbf{t}), \tag{67}$$

where $\mathbf{t}' = (t'_i)_{i \in J'}$ *with* $t'_i = e^{2a} t_i + b$. *In particular horizontal sections are invariant under translations.*

Proof Postponed to B.4. □

Remark As in the tame case, the $\mathfrak{g}$-coinvariance implies

$$\sum_{i \neq j \in J} \Omega^{(ij)} \widehat{w} + \sum_{k \in J} \Omega^{(kk)} \widehat{w} = 0,$$

in the space $\mathscr{H}$. The action of $\Omega^{(kk)}$ is that of the quadratic Casimir (4) on the k-th slot, so this term acts diagonally and can be exponentiated to find the usual conformal weight (cf. Rem. 7.2). The point is that in general the dilation action has further nonscalar terms. △

14 A different dynamical term at infinity

In this section we generalise the dynamical KZ connection [23], varying the setup of Sect. 3.

Namely note another natural family of Lie algebras $\underline{\mathfrak{S}}^{(p)} \subseteq \mathfrak{S}^{(p)} \subseteq \widehat{\mathfrak{g}}$ is given by

$$\underline{\mathfrak{S}}^{(p)} := \mathfrak{h}[\![z]\!] + z^p \mathfrak{g}[\![z]\!] \oplus \mathbb{C}K.$$

The derived Lie algebra of $\underline{\mathfrak{S}}^{(1)}$ yields the first "level subalgebra" of [21], then the two differ for $p \geqslant 2$. One can then define (smooth) induced modules $\widehat{\underline{W}}$ as in Sect. 3, where $\widehat{\underline{W}} = \widehat{\underline{W}}_{\underline{\chi}}^{(p)}$ depends on a character $\underline{\chi} \colon \underline{\mathfrak{S}}^{(p)} \to \mathbb{C}$. However one does not recover the standard affine Verma module as the starting element of the family, contrary to (10)—which is one motivation behind Def. 3.1.

Moreover one has $\underline{\mathfrak{S}}^{(p)} \simeq \mathfrak{h}_{2p} \oplus \mathbb{C}K$, analogously to Lem. 3.1, so for $p = 1$ a character is defined by elements $\underline{\lambda} \in \mathfrak{h}^\vee$ and by the irregular Cartan term $\mu \in (\mathfrak{h} \otimes z)^\vee$ (plus the choice of a level κ). Hence for $p = 1$ we see (15) matches up the parameters of $\widehat{\underline{W}}$ with principal parts of meromorphic connections at poles of order two, but in general only poles of even order can be obtained with this construction, contrary to (10)—which is another motivation behind Def. 3.1.

So we can put the module $\widehat{\underline{W}} = \widehat{\underline{W}}_{\underline{\chi}}^{(1)}$ at infinity in the tensor product $\widehat{\mathcal{H}}$, and consider the spaces of coinvariants $\mathscr{H}$ as in Sect. 8. The proofs of Props. 9.1, 9.2

and 9.3 can be adjusted introducing suitable filtrations on $\widehat{\underline{W}}$ and $\underline{W} = \mathrm{U}(\mathfrak{g}[\![z]\!])\underline{w}$, where $\underline{w} \in \widehat{\underline{W}}$ is the cyclic vector, as well as the whole of § 9.1. Hence in brief one can use $\underline{W}$ as auxiliary module at infinity, which yields a different "dynamical" Cartan term in the reduced connection—with respect to (57).

Namely (57) simplifies to

$$\mathcal{D}_i(\widehat{\boldsymbol{v}} \otimes \underline{w}) = \frac{1}{\kappa + h^\vee} \sum_k \mu_k H_k^{(i)} \cdot \widehat{\boldsymbol{v}} \otimes \underline{w},$$

where $(H_k)_k$ is a $(\cdot \mid \cdot)$-orthonormal basis of $\mathfrak{h}$, using $(\mathfrak{n}^+ \oplus \mathfrak{n}^-) \otimes z_\infty \cdot \underline{w} = 0$, $H_k z_\infty \cdot \underline{w} = \mu_k \underline{w}$, and writing $\mu_k = \langle \mu, H_k z_\infty \rangle$.

We see the reduced connection generalises the dynamical KZ equations, i.e. [23, Eq. 3], and it coincides with it when the modules over finite points are tame.[11] So we recover the Felder–Markov–Tarasov–Varchenko connection (FMTV) over variations of marked points as a particular case of this construction.

Remark 14.1 Note the whole of the FMTV connection also allows for variations of the irregular part $\mu \in (\mathfrak{h} \otimes z)^\vee$, in addition to the deformations à la Klarès considered here [37]. In particular when there is only one simple pole the resulting flat connection for variations of μ is the DMT connection [40, 52], which is derived from a representation-theoretic setup in [21, § 3.11], and [22, § 3.7] (for the latter see also [53]). △

Remark 14.2 **(On quantisation of isomonodromy connections)** Just as in the case of the KZ connection, a different derivation of these flat connections has been obtained by (filtered) deformation quantisation of isomonodromy systems, this time importantly for *irregular* meromorphic connections.

Namely [9] derived the DMT connection from the quantisation of a dual version of the Schlesinger system. This is related to the usual Schlesinger system by the Harnad duality [29], i.e. the Fourier–Laplace transform (cf. [50] on the quantum side). In the same spirit, the whole of FMTV connection can be obtained by quantising the isomonodromy system of Jimbo–Miwa–Môri–Sato [33] (see [44, § 11]; more generally see op. cit. and [45] for a further extension to connections with poles of order three including all the above cases). △

15 Outlook

As explained in the introduction we also wish to consider flat quantum connections along variations of irregular types (i.e. variations of "wild" Riemann surface structures on the sphere [7]). Two viable viewpoints to introduce them are:

(1) The quantisation of the *full* irregular isomonodromy connections, in the spirit of [9, 44], generalising the simply-laced quantum connections (which quantise the simply-laced isomonodromy systems [12]);

[11] Replace $\kappa + h^\vee \in \mathbb{C}$ with "κ" and $\sum_k \mu_k H_k \in \mathfrak{h}$ with "μ" to retrieve the exact [23, Eq. 3].

(2) Considering quantum symmetries: the quantum/Howe duality [3] was used in [52] to relate KZ and the "Casimir" connection of De Concini and Millson–Toledano Laredo (DMT) [40], and at the level of isomonodromy systems corresponds to the Harnad duality [29]. An analogous quantisation of the Fourier–Laplace transform may be taken here in order to turn the variations of marked points into variations of irregular types, extending the viewpoint of [11, 45].

Another natural direction to pursue is the higher-genus case, noting in that case the moduli spaces of connections on holomorphically trivial bundles have positive codimension inside the full de Rham spaces.

Finally one may try to introduce integrality conditions, and lift this Lie-algebra representation setup to Lie groups, with a view towards the geometric quantisation of coadjoint G_p-orbits (along the lines of the Borel–Weyl–Bott theorem [17, 49, 51], or more generally of the orbit method [36]). Another approach we will try in this direction is that of the quantisation of the nilpotent Birkhoff orbits $\mathcal{O}_B \subseteq \mathfrak{b}_p^\vee$ [8].

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Appendix A. Standard notions/notations

Duals The (algebraic) dual of a vector space W is written $W^\vee = \mathrm{Hom}(W, \mathbb{C})$, and the natural pairing $W^\vee \otimes W \to \mathbb{C}$ is denoted $\alpha \otimes w \mapsto \langle \alpha, w \rangle$. If I is a set and $W = \bigoplus_i \mathcal{F}_i(W)$ an I-graded vector space then the *restricted/graded* dual of $(W, \mathcal{F}_\bullet)$ is the I-graded vector space $W^* := \bigoplus_{i \in I} \mathcal{F}_i(W)^\vee \subseteq \prod_{i \in I} \mathcal{F}_i(W)^\vee \simeq W^\vee$.
Gradings and filtrations If $(I, \leqslant)$ is a totally ordered set and $W = \bigoplus_{i \in I} \mathcal{F}_i(W)$ an I-graded vector space, the associated I-filtration on W is defined by the subspaces $\mathcal{F}_{\leqslant i} := \bigoplus_{j \leqslant i} \mathcal{F}_j(W)$.

If I and J are sets and $W_j = \bigoplus_{i \in I} \mathcal{F}_i^{(j)}(W_j)$ a J-family of I-graded vector spaces, the *tensor product* I^J-grading on $W = \bigotimes_{j \in J} W_j$ is defined by the subspaces

$$\mathcal{F}_{\mathbf{i}} := \Bigg(\bigotimes_{j \in J} \mathcal{F}^{(j)}_{\mathbf{i}(j)} \Bigg), \qquad \text{for } \mathbf{i} \colon J \to I\,.$$

If further $(I, \leqslant)$ is a totally ordered $\mathbb{Z}$-module then the *tensor product* I-filtration on W is defined by the subspaces

$$\mathcal{F}_{\leqslant i} := \bigoplus_{\sum_{j \in J} \mathbf{i}(j) \leqslant i} \mathcal{F}_{\mathbf{i}}, \qquad \text{for } i \in I.$$

Lie-algebraic constructions Let L be a Lie algebra. The *abelianisation* of L is the abelian Lie algebra $L_{ab} := L \big/ \big[L, L\big]$, and the *opposite* of L is the Lie algebra L^{op} on the same vector space, with bracket $\big[X, Y\big]_{L^{op}} := \big[Y, X\big]_L$ for $X, Y \in L$.

If $p \geqslant 1$ is an integer and "z" a variable then the associate Lie algebra of *depth* p is

$$L_p := L[\![z]\!] / z^p L[\![z]\!] \simeq L \otimes \big(\mathbb{C}[\![z]\!] / z^p \mathbb{C}[\![z]\!]\big),$$

coming with a projection $L_p \twoheadrightarrow L_1 = L$. There is then a canonical vector space isomorphism $L_p \simeq \bigoplus_{i=0}^{p-1} L \otimes z^i$, which can be upgraded to an isomorphism of Lie algebras if one defines a Lie bracket on the direct sum by truncating terms of degree greater than $p - 1$.

If W is a left L-module then the space of L-*coinvariants* is $W_L := W / LW$, where $LW := \sum_{X \in L} XW \subseteq W$—in particular L_{ab} is the space of ad_L-coinvariants.

Appendix B. Computations

B.1. **Proof of Prop. 7.1**

Proof Set $a_k^{(j)} := \langle a_j, H_k z^j \rangle$ for $k \in \{0, \ldots, r\}$ and $j \in \{0, \ldots, p-1\}$, and further $a_\alpha^{(j)} := \langle a_j, H_\alpha z^j \rangle$ for $\alpha \in \mathcal{R}$.

By (11) we see that $: X_k z^{-j} X^k z^{n+j} : w \neq 0$ implies $1 - p \leqslant j \leqslant p - 1 - n$, so $n \leqslant 2(p-1)$ is necessary for nonvanishing terms.

Now importantly for $n \in \{p-1, \ldots, 2(p-1)\}$ and $j \in \{1-p, \ldots, p-1-n\}$ one has

$$-j, n+j \in \{1-p+n, \ldots, p-1\} \subseteq \{0, \ldots, p-1\},$$

so the normal ordered products are void in (33). Then for $\alpha \in \mathcal{R}^+$ and $i \in \{1, \ldots, r\}$ one computes

$$H_k z^{-j} H_k z^{j+n} w = a_k^{(-j)} a_k^{(j+n)} w, \qquad E^\alpha z^{-j} E_\alpha z^{n+j} w = 0,$$

and

$$\frac{(H_\alpha \mid H_\alpha)}{2} E_\alpha z^{-j} E^\alpha z^{n+j} w = H_\alpha z^n w = \delta_{n,p-1} a_\alpha^{(n)} w.$$

Hence

$$2\left(\kappa+h^{\vee}\right)L_n w=\sum_{j=1-p}^{p-1-n}\left(\sum_{k=1}^{r}\left(H_k z^{-j}H_k z^{j+n}\right)+\sum_{\alpha\in\mathcal{R}^+}\left(E_\alpha z^{-j}E^{\alpha}z^{j+n}\right)\right)w$$
$$=\left(\sum_{j,k}\left(a_k^{(-j)}a_k^{(j+n)}\right)+\delta_{n,p-1}(2p-n-1)\sum_{\alpha\in\mathcal{R}^+}\left(\frac{(\alpha\mid\alpha)}{2}a_\alpha^{(n)}\right)\right)w,$$

which implies (34) and (35) using $\frac{(\alpha|\alpha)}{2}\langle\mu,H_\alpha z^i\rangle=(\alpha\mid\mu)$, for $\mu\in\mathfrak{h}^{\vee}\otimes z^i$. □

B.2. **Proof of Prop.** 11.2

Proof Using the general case of (55) yields

$$X\otimes\tau_j(z_i^{-m})\widehat{w}_j=\sum_{l=0}^{r_j-1}\binom{m+l-1}{l}\frac{Xz_j^l\widehat{w}_j}{(t_i-t_j)^l(t_j-t_i)^m},\quad X\otimes\tau_\infty(z_i^{-m})\widehat{v}_\infty=0,$$

for $X\in\mathfrak{g}$, $i\neq j\in J'$, $\widehat{w}_j\in W_j$ and $\widehat{v}_\infty\in V_\infty$—since $z_\infty\mathfrak{g}[\![z_\infty]\!]V_\infty=0=z_j^{r_j}\mathfrak{g}[\![z_j]\!]V_j$.

Hence by (43) one has the identity $\left(X\otimes z_i^{-m}\right)^{(i)}\widehat{\boldsymbol{w}}\otimes\widehat{v}_\infty=\widehat{\boldsymbol{w}}_{i,m,X}\otimes\widehat{v}_\infty$ inside $\mathscr{H}$, where

$$\widehat{\boldsymbol{w}}_{i,m,X}=-\sum_{j\in J'\setminus\{i\}}\left(\sum_{l=0}^{r_j-1}\binom{m+l-1}{l}\frac{\left(Xz^l\right)^{(j)}\widehat{\boldsymbol{w}}}{(t_i-t_j)^l(t_j-t_i)^m}\right).$$

The result then follows from (36). □

B.3. **Proof of Prop.** 12.1

Proof We prove the $\mathfrak{g}$-action commutes with $\nabla_p\colon\Omega^0\left(U(n,p)\right)\to\Omega^1\left(U(n,p)\right)$. Since the $\mathfrak{g}$-action is independent of the point on the base space, this is equivalent to $\varpi_p'\otimes X\psi-X\left(\varpi_p'\otimes\psi\right)=0=\varpi_p''\otimes X\psi-X\left(\varpi_p''\otimes\psi\right)$, for $X\in\mathfrak{g}$.

Now by (60) one has

$$\left(\kappa+h^{\vee}\right)\left(\varpi_p'\otimes X\psi-X\left(\varpi_p'\otimes\psi\right)\right)$$
$$=\sum_{i\neq j}\sum_{m,l=0}^{p-1}(-1)^m\binom{m+l}{l}t_{ij}^{-1-m-l}dt_{ij}\otimes\left(\sum_{k\in J'}[\Omega_{ml}^{(ij)}X^{(k)}]\psi\right),$$

and analogously by (61)

$$\left(\kappa+h^{\vee}\right)\left(\varpi_p''\otimes X\psi-X\left(\varpi_p''\otimes\psi\right)\right)$$
$$=\sum_{i\in J'}\sum_{m,l=0}^{p-1}\binom{m+l}{l}t_i^l\otimes\left(\sum_{k\in J'}[\Omega_{ml}^{(i\infty)},X^{(k)}]\psi\right).$$

Hence it is enough to show that

$$\sum_{k \in J'} \left[\Omega^{(ij)}_{ml}, X^{(k)}\right] = 0 \in U(\mathfrak{g}_p)^{\otimes n},$$

for all $i \neq j \in J$ and for all $m, l \in \mathbb{Z}$. Finally by (5) we have

$$\sum_{k \in J'} \left[\Omega^{(ij)}_{ml}, X^{(k)}\right] = \sum_r \Big(\left[X_r, X\right] z^m\Big)^{(i)} \left(X_r z^l\right)^{(j)} + \left(X_r z^m\right)^{(i)} \Big(\left[X_r, X\right] z^l\Big)^{(j)},$$

where we let $(X_r)_r$ be a $(\cdot \mid \cdot)$-orthonormal basis of $\mathfrak{g}$, which vanishes by (6). □

B.4. **Proof of Prop.** 13.1

Proof Indeed if $\widehat{w}$ is a ∇'_p-horizontal section of $U(J, p) \to C_n(\mathbb{C})$ then

$$E\widehat{\boldsymbol{w}} = \sum_{i \in J'} \left(\sum_{j \in J' \setminus \{i\}} r^{(ij)}_p(t_{ij}) \right) \widehat{\boldsymbol{w}},$$

which vanishes by the skew-symmetry (64), and which implies the statement about translations after taking $\mathfrak{g}_p$-modules.

As for dilations, in the universal case of a ∇'_p-horizontal section one finds

$$\frac{H}{2}\widehat{\boldsymbol{w}} = \sum_{i \in J'} \left(\sum_{j \in J' \setminus \{i\}} t_i r^{(ij)}_p(t_{ij}) \right) \widehat{\boldsymbol{w}} = \sum_{i \neq j \in J'} t_{ij} r^{(ij)}_p(t_{ij}) \widehat{\boldsymbol{w}},$$

and we must consider the induced action on finite singular modules. Now one computes

$$L_0 \widehat{w} = \frac{1}{\kappa + h^\vee} \sum_{j=1}^{p} \left(\sum_k X_k z^{-j} X_k z^j \right) \widehat{w}, \quad \text{for } \widehat{w} \in W,$$

analogously to (36), using a $(\cdot \mid \cdot)$-orthonormal basis $(X_k)_k$ of $\mathfrak{g}$. Then reasoning as in § 9.2.3 the induced slot-wise action on coinvariants is

$$L^{(i)}_0 \widehat{\boldsymbol{w}} = -\frac{1}{\kappa + h^\vee} \sum_{j \in J' \setminus \{i\}} \left(\sum_{m=0}^{r_i - 1} \sum_{l=0}^{r_j - 1} \binom{m+l}{l} (-1)^m t_{ij}^{-m-l} \Omega^{(ij)}_{ml} \right) \widehat{\boldsymbol{w}},$$

with tacit use of the projection $\pi_{\mathscr{H}} \colon \mathcal{H}'_{|\boldsymbol{\lambda}|} \to \mathscr{H}$, and on the whole

$$\frac{H}{2}\widehat{\boldsymbol{w}} = \left(L_0 - L^{(\infty)}_0\right)\widehat{\boldsymbol{w}} = \sum_{i \in J'} L^{(i)}_0 \widehat{\boldsymbol{w}},$$

by (63). This is the action of an endomorphism on the *finite-dimensional* vector space $\mathscr{H}$, and the statement follows by integrating the resulting (linear, first-order) differential equation. □

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