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"Bubbling" and Topological Degeneration in the Calculus of Variations

Michael Struwe

Abstract. After recalling first instances of "topological degeneration" and "bubbling" in geometric analysis we present current challenges in applications of variational methods to problems in this field.

1. Introduction

First encounters with the phenomena of topological degeneration and "bubbling" in applications of variational methods to problems in geometric analysis date back to the late 1930's and early 1980's, respectively. We briefly recall these classical results and proceed to describe some current challenges in the field.

2. Minimal surfaces

2.1. Douglas condition

After the break-through solution by Douglas and Rado in 1930/31 to Plateau's problem for disc-type minimal surfaces, in 1939 Jesse Douglas [12] also considered the Plateau problem for minimal surfaces of general topological type. He may have been the first geometric analyst to discover topological degeneration in a variational problem.

For simplicity, consider the problem of finding minimal surfaces of annulus-type spanning two disjoint, closed Jordan curves $\Gamma_{1,2}$ in $\mathbb{R}^3$. Generalizing his approach to Plateau's problem for minimal surfaces of the type of the disc, in modern terminology Douglas sought to characterize annulus-type minimal surfaces as critical points of Dirichlet's integral

$$E(u,\rho) = \frac{1}{2}\int_{A_\rho} |\nabla u|^2 dz$$

among maps $u\colon A_\rho \to \mathbb{R}^3$ defined on an annular region

$$A_\rho = \{z \in \mathbb{R}^2;\ \rho < |z| < 1\},$$

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satisfying the Plateau boundary condition (PBC). That is, each admissible u maps the inner boundary circle $S_\rho = \{z;\ |z| = \rho\}$ monotonically onto Γ_1 and the outer boundary circle $S_1 = \{z;\ |z| = 1\}$ monotonically onto Γ_2, in a manner compatible with the given orientation of the curves $\Gamma_{1,2}$. Moreover, also the parameter $0 < \rho < 1$ is allowed to vary, which determines the conformal type of the domain.

It turns out that also pairs of disc-type surfaces $u_{1,2}\colon B \to \mathbb{R}^3$ play a role in this case, where $B = \{z;\ |z| < 1\}$ is the unit disc in $\mathbb{R}^2$ and where $u_{1,2}$ satisfies PBC in the sense that u_i maps $S_1 = \partial B$ continuously and (weakly) monotonically onto Γ_i, with energy

$$E(u_i) = \frac{1}{2}\int_B |\nabla u_i|^2 dz,\ \ i = 1, 2.$$

In fact, Douglas was able to assert the existence of an annulus-type minimal surface spanning the given contours whenever the "Douglas condition"

$$\inf_{u \in C^1(A_\rho, \mathbb{R}^3) \text{ with PBC}, 0<\rho<1} E(u, \rho) < \inf_{u_{1,2} \in C^1(B, \mathbb{R}^3) \text{ with PBC}} E(u_1) + E(u_2) \qquad (2.1)$$

was satisfied, the idea being that condition (2.1) would prevent a minimizing sequence of annulus-type surfaces from degenerating to a pair of disc-type surfaces.

Today, we understand better what role topologically degenerate solution pairs $u_{1,2}$ of disc-type minimal surfaces play in this context.

2.2. Example

In fact, the simple example of two co-axial parallel planar curves at distance $d > 0$ from each other shows that Douglas' condition is not optimal; moreover, the result of Douglas does not give any information about the set of *all* solutions of the Plateau problem in this case.

Indeed, for sufficiently small $d > 0$ there exist both a stable catenoid minimal surface of energy (or area) $< 2\pi$ spanning the given configuration of curves, approximately given by the thin cylindrical strip between the two circles with area $2\pi d$, and, in addition, an unstable catenoid of energy $> 2\pi$, which as $\rho \downarrow 0$ degenerates to the pair of disc-type surfaces each spanning one of the boundary circles and having a combined energy of size 2π.

Moreover, there exists a critical distance $d^* > 1$ such that as $d \uparrow d^*$ the stable and unstable catenoids merge in a critically unstable one having energy $> 2\pi$, which disappears as we further increase the separation distance, leaving for $d > d^*$ only the pair of disc-type minimal surfaces of energy 2π.

2.3. Critical Point Theory

After the rigorous derivation of all Morse inequalities for disc-type minimal surfaces in [24], and generalizing the above example, for curves $\Gamma_{1,2}$ with disjoint convex hulls the paper [25] establishes the full Morse relations for the energy functional E on the space

$$M = \{(u, \rho);\ u \in H^1 \cap C^0(\bar{A}_\rho, \mathbb{R}^3) \text{ satisfies PBC}, 0 \le \rho < 1\},$$

of functions satisfying PBC with finite Dirichlet integral, where for $\rho = 0$ we let $A_0 = B$ and consider pairs $u = (u_1, u_2)$ of functions $u_i \in H^1 \cap C^0(\bar{B}, \mathbb{R}^3)$ each satisfying PBC for Γ_i, $i = 1, 2$, with $E(u, 0) = E(u_1) + E(u_2)$, thus including topologically degenerate surfaces.

In the above example, the topologically degenerate surface corresponding to the pair $u = (u_1, u_2)$ of minimal discs acts as a critical point of minimum type for the extended variational problem for the energy E on the space M. With M being contractible, the Morse relations then hold in all cases where $0 < d \neq d^*$.

For suitable boundary curves, jointly with Jürgen Jost in [16] we extend the critical point theory to the Plateau problem for arbitrary genus and connectivity, including the contributions to the Morse inequalities of surfaces of lower topological type, with a well-defined pseudo-gradient flow also acting on the Teichmüller space of conformal structures on the domain and possibly deforming the domain to a domain of degenerate type.

3. Harmonic Maps of Closed Surfaces

3.1. Harmonic Maps

Let Σ be a closed surface with Riemannian metric g. For a closed "target" manifold $N \subset \mathbb{R}^n$ weakly harmonic maps $u\colon \Sigma \to N$ are critical points of Dirichlet's integral

$$E(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 d\mu_g$$

in the class

$$H^1(\Sigma, N) = \{u \in H^1(\Sigma, \mathbb{R}^n);\ u(x) \in N \text{ for almost every } x \in \Sigma\}$$

of Sobolev maps $u\colon \Sigma \to N$ with finite energy. By a result of Hélein [14] weakly harmonic maps $u\colon \Sigma \to N$, in fact, are smooth and are classical solutions of the elliptic system of equations

$$-\Delta_g u = A(u, \nabla u, \nabla u) \perp T_u N \ \text{ on } \Sigma,$$

where Δ_g is the Laplacean on Σ in the metric g, and where $A\colon N \times T_p N \times T_p N \to T_p^\perp N$ is the second fundamenal form on N.

3.2. Homotopy Problem

A classical problem, posed by Eells-Sampson [13] in 1964, is the question whether every homotopy class of smooth maps $u\colon \Sigma \to N$ can be represented by a harmonic map. For targets N with non-positive sectional curvature, Eells–Sampson were able to answer the question in the affirmative.

3.3. The Sacks–Uhlenbeck Result

Without any restriction on the curvature of the target, but assuming instead that $\pi_2(N) = 0$, the following existence result of similar generality was obtained by Sacks-Uhlenbeck [21] in 1981.

Theorem 3.1. *Let Σ and N be as above, and suppose that $\pi_2(N) = 0$. Then for any smooth $u_0\colon \Sigma \to N$ there is a harmonic map $u\colon \Sigma \to N$ homotopic to u_0.*

The proof introduces the first (simultaneous with Wente [29]) instance of "bubbling" in geometric analysis.

3.3.1. Approximation. For $\alpha > 1$ Sacks-Uhlenbeck define the α-energy

$$E_\alpha(u) = \frac{1}{2}\int_\Sigma \big((1+|\nabla u|^2)^\alpha - 1\big)d\mu_g$$

for $u \in W^{1,2\alpha}(\Sigma, N) := H^1(\Sigma, N) \cap W^{1,2\alpha}(\Sigma, \mathbb{R}^n)$. Note that by Sobolev's theorem, for any $\alpha > 1$ we have a compact embedding $W^{1,2\alpha}(\Sigma, N) \hookrightarrow C^0(\Sigma, N)$. In particular, for any smooth $u_0 \colon \Sigma \to N$ the homotopy class

$$[u_0]_\alpha = \{u \in W^{1,2\alpha}(\Sigma, N);\ u \sim u_0\}$$

is well-defined and closed with respect to weak convergence in $W^{1,2\alpha}$. Sacks-Uhlenbeck then conclude that for any $\alpha > 1$ there exists $u_\alpha \in [u_0]_\alpha$ that minimizes the α-energy in this class, and from elliptic regularity theory they deduce smoothness of u_α.

Next observe that the space of smooth maps $u \colon \Sigma \to N$ is dense in $H^1(\Sigma, N)$; moreover, the homotopy class

$$[u_0] := [u_0]_1 = \{u \in H^1(\Sigma, N);\ u \sim u_0\}$$

is well-defined. Since clearly for any smooth $u \colon \Sigma \to N$ as $\alpha \downarrow 1$ there holds $E_\alpha(u) \to E(u)$, we then conclude that

$$E_\alpha(u_\alpha) = \inf_{u \in [u_0]_\alpha} E_\alpha(u) \to \inf_{u \in [u_0]} E(u) =: \beta_0 \text{ as } \alpha \downarrow 1.$$

3.3.2. "Bubbling". In a second key step Sacks-Uhlenbeck analyze the behavior of the approximate harmonic maps u_α as $\alpha \downarrow 1$, and they show that either the family $(u_\alpha)_{\alpha > 1}$ is uniformly smoothly bounded, or for some sequence $\alpha_k \downarrow 1$ the corresponding sequence $u_k = u_{\alpha_k}$ exhibits "bubbling" in the following sense: There exist points $p_k \in \Sigma$ and radii $r_k \downarrow 0$ such that in geodesic normal coordinates x around $p_k = 0$ there holds

$$v_k := u_k(r_k x) \to v_\infty \ \text{ in } C^2_{loc}(\mathbb{R}^2),$$

where $v_\infty \colon \mathbb{R}^2 \to N$ is a smooth, non-constant harmonic map with finite Dirichlet integral

$$E(v_\infty, \mathbb{R}^2) \le \lim_{R\to\infty} \liminf_{k\to\infty} E(v_k, B_R(0)) \le \liminf_{k\to\infty} E_{\alpha_k}(u_k) = \beta_0.$$

By conformal invariance of the Dirichlet integral in 2 space dimensions, via stereographic projection $\Phi \colon S^2 \setminus \{p_0\} \to \mathbb{R}^2$ from the south pole $p_0 \in S^2$ the map v_∞ may be lifted to a smooth, non-constant harmonic map $w_\infty = v_\infty \circ \Phi \colon S^2 \setminus \{p_0\} \to N$ with finite Dirichlet integral.

Sacks–Uhlenbeck then proceed to show that in 2 space dimensions any smooth harmonic map $S^2 \setminus \{p_0\} \to N$ with finite Dirichlet integral may be smoothly extended to all of S^2, thus removing the point singularity at p_0.

A simple estimate next shows that the Dirichlet energy of any smooth, non-constant harmonic map $u \colon S^2 \to N$ is larger than a uniform quanta $\varepsilon_0(N) > 0$ of energy. Indeed, compute

$$\|\Delta_{g_{S^2}} u\|^2_{L^2(S^2)} = \int_{S^2} |\Delta_{g_{S^2}} u|^2 d\mu_{g_{S^2}} = -\int_{S^2} \Delta_{g_{S^2}} u \cdot A(u, \nabla u, \nabla u) d\mu_{g_{S^2}}$$
$$\le C \int_{S^2} |\nabla u|^2 |\Delta_{g_{S^2}} u| d\mu_{g_{S^2}} \le C \|\nabla u\|^2_{L^4(S^2)} \|\Delta_{g_{S^2}} u\|_{L^2(S^2)}$$

by Hölder's inequality. The Gagliardo-Nirenberg inequality and elliptic regularity give

$$\|\nabla u\|^2_{L^4(S^2)} \le C\|\nabla u\|_{H^1(S^2)} \|\nabla u\|_{L^2(S^2)} \le C\|\Delta_{g_{S^2}} u\|_{L^2(S^2)} \|\nabla u\|_{L^2(S^2)}.$$

Thus, with a constant $C_0 = C_0(N) > 0$ independent of u we obtain the bound

$$\|\Delta_{g_{S^2}} u\|^2_{L^2(S^2)} \le C_0 \|\Delta_{g_{S^2}} u\|^2_{L^2(S^2)} \|\nabla u\|_{L^2(S^2)}.$$

Hence we conclude

$$E(u, S^2) = \frac{1}{2}\|\nabla u\|^2_{L^2(S^2)} \ge \frac{1}{2C_0^2} =: \varepsilon_0(N).$$

3.3.3. Surgery. Finally, by replacing the "large spherical cap" of u_k inside a suitable ball $B_{Lr_k}(p_k)$ (that is, the part of the rescaled map v_k inside the ball $B_L(0)$) with a "small spherical cap" approximately given by $v_\infty(L^2 z/|z|^2)$ for $|z| < L$, and using that $\pi_2(N) = 0$, for suitably chosen $L = L_k \to \infty$ as $k \to \infty$ Sacks-Uhlenbeck construct a map $\tilde{u}_k$ homotopic to u_k (and hence to u_0) with

$$E(\tilde{u}_k) \le E(u_k) - \varepsilon_0(N) + o(1),$$

where $o(1) \to 0$ as $k \to \infty$. Thus, $\tilde{u}_k \in [u_0]$ with $E(\tilde{u}_k) < \beta_0$ for sufficiently large $k \in \mathbb{N}$, which contradicts the definition of β_0. This shows that, in fact, $u_k \to u_\infty$ smoothly as $k \to \infty$, and $u_\infty \sim u_0$ is harmonic with $E(u_\infty, \Sigma) = \beta_0$.

Remark 3.2. Note that the "bubble map" w_∞ solves the harmonic map equation on S^2, regardless of the topological type of Σ. Thus, also the process of "bubbling" in general involves a change of topology.

4. Nirenberg's Problem

In the classical examples that we looked at so far, solutions arise as relative minimizers: The Sacks-Uhlenbeck map is a minimizer of energy in its homotopy class, while the Douglas condition is a criterion for the existence of a minimizer of Dirichlet's energy of the given topological type. However, also non-minimizing solutions can sometimes be obtained inspite of "bubbling", even though we cannot yet make assertions about *all* such solutions. We illustrate this with Nirenberg's problem.

After the work of Berger [5] and Kazdan-Warner [17] on Riemannian metrics of prescribed curvature, the particular case of finding conformal metrics $g = e^{2u} g_0$ on the sphere S^2 with its standard round metric g_0 having a given function f as Gauss curvature $K_g = f$ has attracted the attention of geometric analysts. This problem, posed by Nirenberg, has given rise to sophisticated analytic approaches and deep insights into the interplay of analysis and geometry and remains a challenge, even though partial answers have been obtained.

In view of the equation

$$K_g = e^{-2u}(-\Delta_0 u + 1)$$

relating K_g and u, where $\Delta_0 = \Delta_{g_0}$ for short, for given $f\colon S^2 \to \mathbb{R}$ we need to solve the nonlinear partial differential equation

$$-\Delta_0 u + 1 = fe^{2u} \quad \text{on } S^2. \tag{4.1}$$

Integrating, for a solution to (4.1) we have the Gauss-Bonnet identity

$$\int_{S^2} fe^{2u} d\mu_0 = 4\pi. \tag{4.2}$$

Thus, (4.1) can only be solved if f is positive somewhere. In the following we therefore suppose that f is smooth and strictly positive.

The problem is variational. Indeed, introducing the Liouville energy

$$S(u) = \fint_{S^2} (|\nabla u|^2 + 2u) d\mu_0,$$

where $\fint_{S^2} = \frac{1}{4\pi}\int_{S^2}$ denotes the average and with $d\mu_0 = d\mu_{g_0}$, and setting

$$E(u) = E_f(u) = S(u) - \log\left(\fint_{S^2} fe^{2u} d\mu_0\right) \tag{4.3}$$

for $u \in H^1(S^2)$, solutions of (4.1) may be characterized as critical points of E among functions u satisfying (4.2).

Via the Möbius group of conformal diffeomorphism $\Phi\colon S^2 \to S^2$, for any $p \in S^2$ the functional E may be compared with the functional

$$E_{f(p)}(u) = S(u) - \log\left(\fint_{S^2} f(p) e^{2u} d\mu_0\right),$$

where f is replaced by the constant $f(p)$. Indeed, given any $p \in S^2$, any $t \geq 1$, letting $\Phi_p\colon S^2 \setminus \{-p\} \to \mathbb{R}^2$ be stereographic projection from the point $-p \in S^2$ and letting $\delta_t\colon \mathbb{R}^2 \ni z \to tz \in \mathbb{R}^2$ be the standard dilation, we obtain the Möbius map $\Phi_{p,t} = \Phi_p^{-1} \circ \delta_t \circ \Phi_p \in M$. Setting $u_{p,t} = u \circ \Phi_{p,t} + \log|\Phi'_{p,t}|$, where we write $|\Phi'| = \sqrt{\det d\Phi}$ for brevity, we then have

$$S(u_{p,t}) = S(u)$$

(see for instance [10], Proposition 2.1) and thus

$$\begin{aligned} E_f(u_{p,t}) &= S(u_{p,t})) - \log\left(\fint_{S^2} fe^{2u_{p,t}} d\mu_0\right) \\ &= S(u) - \log\left(\fint_{S^2} (f \circ \Phi_{p,t}^{-1}) e^{2u} d\mu_0\right) \to E_{f(p)}(u) \end{aligned}$$

as $t \to \infty$. For large $t > 1$, the first and second variation of E at $u_{p,t}$ then may be related to $\nabla f(p)$ and $\nabla^2 f(p)$, respectively. From this observation, Chang-Yang [10] deduce the following existence result.

Theorem 4.1. [Chang-Yang [10], Theorem II’] *Suppose that $f > 0$ is a smooth function satisfying the non-degeneracy condition*

$$\Delta_0 f(p) \neq 0 \text{ at any } p \in S^2 \text{ with } \nabla f(p) = 0 \tag{4.4}$$

and the index count condition

$$\sum_{\nabla f(p)=0, \Delta_0 f(p)<0} (-1)^{ind(p)} \neq 1. \tag{4.5}$$

Then there is a smooth solution u to (4.1).

Remark 4.2. (i) Note that Chang-Yang [10], p. 217, showed that when $f \not\equiv 1$ solutions of (4.1) cannot be relative minimizers of E.

(ii) Representing $S^2 = \partial B_1(0) \subset \mathbb{R}^3$ and denoting as x also the restriction of the coordinate function $x = (x^i)_{1 \leq i \leq 3}$ in $\mathbb{R}^3$ to S^2, Kazdan-Warner [17] observed that the action of the Möbius group on the problem gives rise to the condition

$$\int_{S^2} \langle \nabla x^i, \nabla f \rangle e^{2u} d\mu_0 = 0, \; 1 \leq i \leq 3,$$

for (4.1) to be solvable. Here, $\langle \cdot, \cdot \rangle$ at any point $x \in S^2$ denotes the inner product on $T_x S^2$ inherited from the ambient $\mathbb{R}^3$; see for instance [8]. Thus, equation (4.1) cannot be solved, for instance, for the function $f(x) = 1 + \varepsilon x^1$ for any $0 < \varepsilon < 1$, which, indeed, does not satisfy (4.5).

(iii) In [28] an example was given showing that also the restriction to critical points p of f with $\Delta_0 f(p) < 0$ in condition (4.5) in Theorem 4.1 in general cannot be removed; hence, with the non-degeneracy condition (4.4), condition (4.5) is not only sufficient but in general also necessary for the existence of a solution to (4.1).

4.1. Interpretation

Condition (4.5) in Theorem 4.1 may be interpreted in terms of the “last Morse inequality” related to the variational integral (4.3), that is, in terms of an equation identifying the “topological degree” $d = 1$ of the (contractible) set of admissible functions with the sum of the topological degrees of all critical points of E, including the contributions of the “critical points at infinity” (in a terminology introduced by Bahri [3]). With our above motivation, the latter is given by the left hand side of (4.5). Thus, if that term is different from 1, there has to be a further contribution to the total topological degree of all critical points, then necessarily coming from a solution u to (4.1).

4.2. Open Problem

With this interpretation of condition (4.5) in Theorem 4.1 we might expect to obtain not only an existence result but also a characterization of *all* solutions of (4.1) in case that the problem admits multiple solutions. One particular instance of such a case might be a function f as described in [28], however, modified in such a way that condition (4.5) holds true.

5. A Problem in Gauge Theory

The landmark paper [22] by Sibner–Sibner–Uhlenbeck solved a question in gauge theory that had been open for more than a decade, concerning the existence of non-self-dual solutions to the Yang-Mills equations in the trivial $SU(2)$-bundle over S^4. By analogy with the problem of harmonic maps from $S^2 = \mathbb{C} \cup \{\infty\}$ to itself, where one can show that all non-constant harmonic maps either are conformal or anticonformal, from work of Atiyah-Jones [2] and Bourguignon–Lawson–Simons [6] it had been conjectured that any Yang–Mills connection in the trivial $SU(2)$-bundle over S^4 would either be self-dual or anti-self-dual, as described for all $SU(2)$-bundles on the four-sphere by Atiyah–Hitchin–Drinfeld–Manin [1].

However, in the paper [22] Sibner et al. showed that there exist an infinite number of non-self-dual Yang-Mills connections in this setting by exploiting symmetry.

5.1. m-Equivariant Connections

For a connection D in the trivial $SU(2)$-bundle over S^4 with curvature $F = D \circ D$ let

$$YM(D) = \frac{1}{2}\int_{S^4} |F|^2 d\mu_{g_{S^4}}$$

be the Yang-Mills energy. Note that the Yang-Mills energy is conformally invariant, and one may identify S^4 with $\mathbb{R}^4 \cup \{\infty\}$ via stereographic projection.

Following Atiyah and Braam, Sibner et al. in [22] choose coordinates

$$(z, \theta, x, y) \mapsto (z\cos\theta, z\sin\theta, x, y) \in \mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2.$$

This allows them to define a $U(1)$-action on S^4 by letting

$$q(\theta')(z, \theta, x, y) := (z, \theta + \theta'(mod\, 2\pi), x, y).$$

For $m \in \mathbb{N}$, a connection D then is called m-equivariant if

$$q(\theta)^* D = s(\theta)^{-1} \circ D \circ s(\theta) \ \text{ for all } \theta \in \mathbb{R}/2\pi\mathbb{Z} = U(1),$$

where $s(\theta) = e^{\hat{i}m\theta}$ with a standard basis $(\hat{i}, \hat{j}, \hat{k})$ for the Lie algebra $su(2)$ of $SU(2)$. The m-equivariant, self-dual or anti-self-dual minimizers of the Yang-Mills energy then are called m-instantons or m-anti-instanton, respectively.

5.2. Non-minimal connections

For $m \geq 2$, Sibner et al. obtain the following result.

Theorem 5.1. (Sibner-Sibner-Uhlenbeck [22], Theorem 1) *For every integer $m \geq 2$, there exists a non-minimal m-equivariant solution to the Yang-Mills equations in the trivial $SU(2)$-bundle over S^4.*

We sketch the proof. Let $m \geq 2$. With the help of a construction of Taubes, Sibner et al. construct a loop of connections gluing an m-instanton to an m-anti-instanton with a maximal energy strictly less than $8\pi m$, and then a Palais-Smale sequence of connections D_k at that min–max level. In $n = 4$ dimensions, the Yang–Mills functional does not satisfy the Palais-Smale condition. However, a compactness-"bubbling" alternative as in the case of harmonic maps holds, and we have convergence of a suitable subsequence $D_k \to D_\infty$ on $S^4 \setminus \{x_1, \ldots, x_{i_0}\}$, with finitely many concentration points x_i, $1 \leq i \leq i_0$, where non-trivial m-equivariant Yang–Mills connections over an $SU(2)$-bundle over S^4 split off.

If $i_0 = 0$, the connection D_∞ is the sought-after non-minimizing m-equivariant Yang-Mills connection in the trivial $SU(2)$-bundle over S^4.

Similarly, if $i_0 > 0$ and if there is a "bubble" corresponding to a non-trivial m-equivariant Yang–Mills connection in the trivial bundle, we can take that connection to obtain the assertion of the theorem.

On the other hand, if $i_0 > 0$ and if there is a "bubble" with an m-equivariant Yang–Mills connection in a non-trivial $SU(2)$-bundle over S^4, its topological charge $c_2 = km$ for some $k \in \mathbb{Z} \setminus \{0\}$ has to be compensated by the other "bubbles". Thus, there is a second "bubble" with a non-trivial m-equivariant Yang–Mills connection on some non-trivial $SU(2)$-bundle over S^4 with second Chern class $c_2' = k'm$ for some $k' \in \mathbb{Z} \setminus \{0\}$, and the energies of the two bubbles will add up to a number $4\pi(|c_2| + |c_2'|) \geq 8\pi m$, which is impossible and rules out this case.

Remark 5.2. The proof does not yield the explicit form of the non-self-dual solutions. This contrasts with the self-dual solutions, whose existence can be established by explicit construction.

5.3. Open Problem

For $m = 1$ the construction of Taubes is not available to prove the strict upper bound $\beta_0 < 8\pi$ for the min-max level β_0, and the argument that we sketched above fails. Thus we are missing the most elementary non-self-dual solution related to an instanton-anti-instanton balanced pairing with a $U(1)$-symmetry.

6. Min–Max Willmore Spheres

6.1. Willmore Surfaces

Let Σ be a closed surface. For a smooth immersion $u \colon \Sigma \to \mathbb{R}^3$ let

$$W(u) = \int_\Sigma |H|^2 d\mu_g$$

denote the Willmore energy of u, introduced by Blaschke (and even earlier by Sophie Germain), where $g = u^* g_{\mathbb{R}^3}$ is the pull-back metric and where H is the mean curvature of $S = u(\Sigma)$ induced by the immersion. Critical points of the Willmore energy with respect to variations of the map u are called Willmore surfaces.

Note that, as shown by Blaschke, the Willmore energy is invariant under compositions of the immersion u with conformal transformations of the ambient $\mathbb{R}^3$.

Interest in Willmore surfaces was revived in 1965 when Willmore stated the seminal conjecture that the torus obtained by rotating the circle of unit radius centered at $(\sqrt{2}, 0, 0)$ in the (x, z)-plane around the z-axis was the unique minimizer (modulo conformal transformations) of the Willmore energy of tori. This conjecture was finally confirmed in 2014 by Fernando Codá Marques and André Neves [19].

On the other hand, the beautiful observation of Smale [23] that it is possible to "turn a sphere inside out" by means of a smooth path of immersions also invites a study of corresponding "min-max" Willmore spheres, as proposed by Robert Kusner and recently analyzed by Rivière [20].

6.2. The Cost of Sphere Eversion

Indeed, let Ω be the set of continuous paths $p = (p(t))_{0\le t\le 1}$ of C^2-immersions into $\mathbb{R}^3$ inducing an eversion of the standard sphere and set

$$\beta_0 = \inf_{p\in\Omega, 0\le t\le 1} W(p(t)).$$

It is then tempting to speculate about the existence of a "min–max" Willmore sphere which achieves this least maximal Willmore energy β_0 among paths everting the sphere and a corresponding path $p \in \Omega$. In fact, as observed by Rivière, there are two results that point to a unique candidate.

First, recall that Bryant [7] in 1984 was able to describe all immersed Willmore spheres in $\mathbb{R}^3$ as being given by the images by inversions of simply connected, complete, non-compact minimal surfaces with planar ends, with Willmore energy given by $4\pi k$ and index equal to $k-3$, where k is the number of ends.

On the other hand, by a topological result of Banchoff-Max [4] any path $p \in \Omega$ has to contain at least one immersion with a point of self-intersection of order 4 (a quadruple point). Hence, by a result of Li-Yau [18] we have $\beta_0 \ge 16\pi$.

Inspired by these results, Rivière conjectured that the desired "min-max" Willmore sphere $u = p(t)$ achieving the "min-max" value β_0 along a suitable path $p \in \Omega$, should be given by an inversion of a simply connected, complete minimal surface with $k = 4$ planar ends, thus having index $m = 1$ and energy $W(u) = 16\pi$; moreover, in consequence we should have $\beta_0 = 16\pi$.

6.3. Strategies and Open Questions

Two strategies are proposed by Rivière for the proof of this conjecture. First, one might attempt to construct the conjectured optimal path by solving the Willmore flow (the heat flow for the Willmore energy) with initial data obtained by perturbing the candidate "min-max" Willmore sphere from Bryant's construction in the unstable direction. However, the Willmore flow might blow up, and it is not sure if one can complete the path to reach either the standard embedding of S^2 or its inversion.

As an alternative construction, Rivière employs a standard mountain-pass argument for a suitably penalized version of the Willmore energy. Cleverly adapting the "monotonicity trick" from [26] or [27] to his setting, he obtains a Palais-Smale sequence u_k for W from which he is able to extract finitely many Willmore immersions $\xi_1, \ldots, \xi_{i_0}$ of S^2 minus finitely many "bubble" points such that

$$\beta_0 = \sum_{i=1}^{i_0} W(\xi_i) - 4\pi N$$

for some $N \in \mathbb{N}$; but in order to answer the question posed by Kusner one would need to show that $i_0 = 1$ and $N = 0$, and it not clear how this can be done.

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References

[1] Atiyah, M.F., Hitchin, N.J., Drinfeld, V.G., Manin, Yu.I.: Construction of instantons. Phys. Lett. A **65**(3), 185–187 (1978)

[2] Atiyah, M.F., Jones, J.D.S.: Topological aspects of Yang-Mills theory. Comm. Math. Phys. **61**(2), 97–118 (1978)

[3] Bahri, A.: Critical points at infinity in some variational problems, Pitman Research Notes in Mathematics Series, 182. Longman Scientific & Technical, Harlow; copublished in the United States with John Wiley & Sons, Inc., New York, (1989)

[4] Max, Nelson, Banchoff, Tom: Every sphere eversion has a quadruple point, Contributions to analysis and geometry (Baltimore, Md., 1980), pp. 191-209

[5] Berger, Melvyn S.: Riemannian structures of prescribed Gaussian curvature for compact 2-manifolds. J. Differential Geometry **5**, 325–332 (1971)

[6] Bourguignon, Jean-Pierre, Lawson, H. Blaine, Simons, James: Stability and gap phenomena for Yang-Mills fields, Proc. Nat. Acad. Sci. U.S.A. 76 ,(4), 1550-1553 (1979)

[7] Bryant, Robert L.: A duality theorem for Willmore surfaces. J. Differential Geom. **20**(1), 23–53 (1984)

[8] Chang, Sun-Yung Alice: Non-linear elliptic equations in conformal geometry, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, (2004)

[9] Chang, Sun-Yung A., Gursky, Matthew J., Yang, Paul C.: The scalar curvature equation on 2- and 3-spheres. Calc. Var. Partial Differential Equations **1**(2), 205–229 (1993)

[10] Chang, Sun-Yung Alice, Yang, Paul C.: Prescribing Gaussian curvature on S^2. Acta Math. **159**(3–4), 215–259 (1987)

[11] Donaldson, Simon: Karen Uhlenbeck and the calculus of variations. Notices Amer. Math. Soc. **66**(3), 303–313 (2019)

[12] Douglas, Jesse: Minimal surfaces of higher topological structure. Ann. of Math. (2) **40**(1), 205–298 (1939)

[13] Eells, James, Jr., Sampson, J. H.: Harmonic mappings of Riemannian manifolds. Amer. J. Math. **86**, 109–160 (1964)

[14] Hélein, Frédéric: Régularité des applications faiblement harmoniques entre une surface et une variété riemannienne, C. R. Acad. Sci. Paris Sér. I Math. 312, no. 8, 591-596 (1991)

[15] Hong, Min-Chun., Tian, Gang, Yin, Hao: The Yang-Mills α-flow in vector bundles over four manifolds and its applications. Comment. Math. Helv. **90**(1), 75–120 (2015)

[16] Jost, J., Struwe, M.: Morse-Conley theory for minimal surfaces of varying topological type. Invent. Math. **102**(3), 465–499 (1990)

[17] Kazdan, Jerry L., Warner, F.W.: Curvature functions for compact 2-manifolds. Ann. of Math. (2) **99**, 14–47 (1974)

[18] Li, Peter, Yau, Shing Tung: A new conformal invariant and its applications to the Willmore conjecture and the first eigenvalue of compact surfaces. Invent. Math. **69**(2), 269–291 (1982)

[19] Marques, Fernando C., Neves, André: Min-max theory and the Willmore conjecture. Ann. of Math. (2) **179**(2), 683–782 (2014)

[20] Rivière, Tristan: Willmore minmax surfaces and the cost of the sphere eversion. J. Eur. Math. Soc. (JEMS) **23**(2), 349–423 (2021)

[21] Sacks, J., Uhlenbeck, K.: The existence of minimal immersions of 2-spheres. Ann. of Math.(2) **113**(1), 1–24 (1981)

[22] Sibner, L.M., Sibner, R.J., Uhlenbeck, K.: Solutions to Yang-Mills equations that are not self-dual. Proc. Nat. Acad. Sci. U.S.A. **86**(22), 8610–8613 (1989)

[23] Smale, Stephen: A classification of immersions of the two-sphere. Trans. Amer. Math. Soc. **90**, 281–290 (1958)

[24] Struwe, Michael: On a critical point theory for minimal surfaces spanning a wire in $\mathbb{R}^n$. J. Reine Angew. Math. **349**, 1–23 (1984)

[25] Struwe, Michael: A Morse theory for annulus-type minimal surfaces. J. Reine Angew. Math. **368**, 1–27 (1986)

[26] Struwe, Michael: The existence of surfaces of constant mean curvature with free boundaries. Acta Math. **160**(1–2), 19–64 (1988)

[27] Struwe, Michael: Positive solutions of critical semilinear elliptic equations on non-contractible planar domains. J. Eur. Math. Soc. (JEMS) **2**(4), 329–388 (2000)

[28] Struwe, Michael: A flow approach to Nirenberg's problem. Duke Math. J. **128**(1), 19–64 (2005)

[29] Wente, Henry C.: Large solutions to the volume constrained Plateau problem, Arch. Rational Mech. Anal. 75, no. 1, 59-77 (1980/81)

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