DISS. ETH NO. 28790

Distances between linear and integer optimal solutions in terms of subdeterminants

A thesis submitted to attain the degree of
DOCTOR OF SCIENCES of ETH ZURICH

(Dr. sc. ETH Zurich)

presented by

INGO STALLKNECHT

MSc Mathematics, Ruprecht-Karls-Universität Heidelberg

born on April 28, 1993
citizen of Germany

accepted on the recommendation of

Prof. Dr. Robert Weismantel
Prof. Dr. Joseph Paat

2023

„Das Schöne, das Wahre:
Es ist nicht draußen, da sucht es der Tor, es ist in dir, du bringst es ewig hervor.“

Friedrich Schiller (1759–1805)

To my parents

Acknowledgement

This thesis would not have been the same without the help of my supervisor Robert Weismantel. I want to thank him for introducing me to such intriguing research questions, for being patient when progress was slow, and for allowing me to extensively pursue my own ideas. Thanks for always being available for discussions and for sharing your expertise and experience with me. It helped me immensely to grow as a person and as a mathematician.

Thanks to Joe, Luze, Jon and Zach for all the fun and enriching discussions on Zoom. I really enjoyed doing research together and it undoubtedly broadened my horizons.

In particular, I want to thank Joe for encouraging me when situations were difficult, and for all the weekdays and weekends we were working on our project in the office together. Your support and mentoring were a godsend for which I will always remain deeply grateful.

I share many good memories with my fellow PhD students Jörg, Stephan, Christoph and Sabrina. Thanks Jörg for explaining to me how everything works here. Christoph, I really enjoyed working together on our paper on $\{a, b, c\}$-modularity. Sabrina, thank you for carefully proofreading this thesis. But above all, I deeply appreciate discussing the small and big questions in life with all of you.

I am grateful to every member of the institute for facilitating such an open-minded, productive and positive atmosphere. Particularly, I want to thank Annette for always being available to help and for making everyone feel at home at the institute.

Finally, I want to thank my family for all the support and love they have given me up till now. This thesis is for you.

Abstract

Let $A \in \mathbb{Z}^{m \times n}$ with $\operatorname{rank}(A) = m$, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. Set $\Delta_m := \max\{|\det(B)|\colon B$ is an $m \times m$ submatrix of $A\}$. We consider integer linear programming problems in standard form $\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{Z}^n_{\geqslant 0}\}$ (IP) and its linear relaxation $\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{R}^n_{\geqslant 0}\}$ (LP). In this thesis, we study the concept of *proximity*, i.e., we describe the ℓ_1-distance of an (LP)-optimal vertex solution to its closest (IP)-optimal solution. Firstly, we establish the first proximity bound which is polynomial in m and Δ_m. Secondly, we determine exactly how many different columns A can have in terms of m and Δ_m if $\Delta_m \leqslant 2$ or $m \leqslant 2$. Moreover, we establish the first upper bound on the number of different columns of A which is polynomial in m and Δ_m. This implies the first proximity bound for integer programming problems in standard form with upper bounds which is polynomial in m and Δ_m. Thirdly, for $c \equiv 0$ and a given vertex solution x^* to (LP), we introduce a new parameter $f(x^*) \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ in terms of which we derive a proximity bound of $(f(x^*) + 1)\Delta_m$, which is essentially tight for $f(x^*) \leqslant m$. Our analysis also yields an efficient feasibility test for (IP) if $f(x^*)$ is constant. Fourthly, in the case $m = 2$, we provide an essentially tight proximity bound of $3\Delta_m$ for all but finitely many b if the vertex solution to (LP) corresponds to a prime determinant. Finally, we provide an algorithm that solves (IP) efficiently if A possesses at most three different $m \times m$ subdeterminants in absolute value.

Zusammenfassung

Sei $A \in \mathbb{Z}^{m\times n}$ mit $\operatorname{rank}(A) = m$, $b \in \mathbb{Z}^m$ und $c \in \mathbb{Z}^n$. Definiere $\Delta_m := \max\{|\det(B)|\colon\ B$ ist eine $m \times m$ Untermatrix von $A\}$. Wir betrachten lineare ganzzahlige Optimierungsprobleme in Standardform $\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{Z}^n_{\geqslant 0}\}$ (IP) sowie deren lineare Relaxierung $\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{R}^n_{\geqslant 0}\}$ (LP). In dieser Arbeit studieren wir das Konzept der *Proximität*, das heißt, wir untersuchen den ℓ_1-Abstand einer (LP)-optimalen Ecklösung zur nächsten (IP)-optimalen Lösung. Zunächst leiten wir die erste in m and Δ_m polynomielle Proximitätsschranke her. Für $\Delta_m \leqslant 2$ oder $m \leqslant 2$ bestimmen wir in Abhängigkeit von m und Δ_m exakt, wie viele verschiedene Spalten A haben kann. Darüber hinaus leiten wir die erste obere Schranke für die Anzahl verschiedener Spalten von A her welche polynomiell in m und Δ_m ist. Dies impliziert die erste in m und Δ_m polynomielle Proximitätsschranke für ganzzahlige Optimierungsprobleme in Standardform mit oberen Schranken. Im darauffolgenden Kapitel führen wir für $c \equiv 0$ und einer gegebenen Ecklösung x^* von (LP) den neuen Parameter $f(x^*) \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$ ein, in dessen Abhängigkeit wir eine Proximitätsschranke von $(f(x^*) + 1)\Delta_m$ herleiten. Dies ist im Wesentlichen bestmöglich für $f(x^*) \leqslant m$. Unsere Analyse impliziert für konstantes $f(x^*)$ auch einen effizienten Test dafür, ob eine zulässige Lösung für (IP) existiert. Außerdem zeigen wir im Falle $m = 2$ eine im Wesentlichen bestmögliche Proximitätsschranke von $3\Delta_m$ für alle bis auf endliche viele b unter der Voraussetzung, dass die zu (LP) gehörige Ecklösung zu einer primen Determinante korrespondiert. Zum Schluss geben wir einen effizienten Algorithmus an, welcher (IP) löst unter der Voraussetzung, dass A höchstens drei verschiedene $m \times m$ Unterdeterminanten im Betrag besitzt.

This thesis includes material from the following publications:

1. J. Lee, J. Paat, I. Stallknecht, and L. Xu. Improving proximity bounds using sparsity. In M. Baïou, B. Gendron, O. Günlük, and A.R. Mahjoub, editors, **Proceedings of the 6th International Symposium on Combinatorial Optimization**, pages 115–127, 2020. Springer.[1]
https://doi.org/10.1007/978-3-030-53262-8_10

2. J. Lee, J. Paat, I. Stallknecht, and L. Xu. Polynomial upper bounds on the number of differing columns of Δ-modular integer programs. **Mathematics of Operations Research**, pages 1-20, 2022. INFORMS.[2]
https://doi.org/10.1287/moor.2022.1339

3. C. Glanzer, I. Stallknecht, and R. Weismantel. On the Recognition of $\{a, b, c\}$-Modular Matrices. In M. Singh and D.P. Williamson, editors, **Proceedings of the 22nd International Integer Programming and Combinatorial Optimization Conference**, pages 238–251, Cham, 2021. Springer.[3]
https://doi.org/10.1007/978-3-030-73879-2_17

4. C. Glanzer, I. Stallknecht, and R. Weismantel. Notes on $\{a, b, c\}$-Modular Matrices. **Vietnam Journal of Mathematics**, 50(2): 469-485, 2022. Springer.[4]
https://doi.org/10.1007/s10013-021-00520-9

Contents

Chapter 1

Introduction

The world we live in today is filled with numerous optimization problems. How does the manager of the municipal transportation network schedule buses efficiently such that waiting times are minimized? How does a hardware manufacturer for microchips find a valid chip architecture that maximizes the number of transistors on the chip? How does a factory with a given set of resources decide which products in which quantities should be produced in order to maximize profits?

Such problems and many others can be modeled in terms of **integer programming problems** (IPs)

$$\max\{c^\intercal x \colon Ax \leqslant b,\ x \in \mathbb{Z}^n\}, \tag{IP}$$

where $A \in \mathbb{Z}^{m\times n}, b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$. While the modeling power of IPs is tremendous, the class of integer programming problems is $\mathcal{NP}$-hard, as already the special case of 0-1 integer linear programming is $\mathcal{NP}$-complete, see [44]. This makes it rather unlikely that an efficient algorithm for general integer programming problems exists.

However, the corresponding linear relaxation

$$\max\{c^\intercal x \colon Ax \leqslant b,\ x \in \mathbb{R}^n\} \tag{LP}$$

can be solved efficiently in general, see [45, 46] and [63, Chapter 13+14]. This naturally gives rise to the idea of local search heuristics:

1. Find an optimal solution $x^* \in \mathbb{R}^n$ to (LP) efficiently.
2. Search in a "small" neighborhood of x^* for an optimal solution to (IP).

But how "far away" from x^* do we need to search in order to find an (IP)-optimal solution? Assume for this chapter that (IP) is feasible and bounded. Set $\Delta := \max\{|\det(B)| \colon B$ is a submatrix of $A\}$. A classical result in this direction is the following by Cook, Gerards, Schrijver and Tardos:

Theorem 1.1 ([21]). *For every* (LP)*-optimal solution $x^* \in \mathbb{R}^n$, there exists an* (IP)*-optimal solution $z^* \in \mathbb{Z}^n$ such that*

$$\|z^* - x^*\|_\infty < n\Delta.$$

Let us now consider IPs in standard form with upper bounds, i.e.,

$$\max\{c^\intercal x\colon Ax = b,\ 0 \leqslant x \leqslant u,\ x \in \mathbb{Z}^n\}, \tag{UIP}$$

$$\max\{c^\intercal x\colon Ax = b,\ 0 \leqslant x \leqslant u,\ x \in \mathbb{R}^n\}, \tag{ULP}$$

where additionally $u \in \mathbb{Z}^n_{\geqslant 0}$. Assume for this chapter that (UIP) is feasible and bounded. In this setting, Eisenbrand and Weismantel established a proximity bound that solely depends on the number of rows m and the largest entry in absolute value $\|A\|_\infty$ of the constraint matrix A.

Theorem 1.2 ([29]). *For every* (ULP)*-optimal vertex solution $x^* \in \mathbb{R}^n$, there exists an* (UIP)*-optimal solution $z^* \in \mathbb{Z}^n$ such that*

$$\|z^* - x^*\|_1 < m(2m\|A\|_\infty + 1)^m.$$

In [29] they also present optimization algorithms which leverage this proximity bound in order to achieve superior running times. Their theorem yields a local search algorithm to solve (UIP) in running time

$$n \cdot O(m)^{(m+1)^2} \cdot O(\|A\|_\infty)^{m\cdot(m+1)} \cdot \log_2^2(m\|A\|_\infty).$$

Furthermore, with not too much additional analysis, this implies a running time of $O(n\|A\|_\infty^2)$ and $O(n^2\|A\|_\infty^2)$ for the unbounded and bounded knapsack problem, respectively. This is an improvement by a factor of n upon the previously best bounds by Tamir [70].

It is thus worthwhile to study proximity. But can we do better than these results? In this thesis we will study many settings in which we can obtain stronger proximity bounds.

In **Chapter 2**, we combine the concept of sparsity with a refinement of the analysis by Cook et al. [21]. For problems in standard form without upper bounds, this leads to a proximity bound of

$$\|z^* - x^*\|_1 < (m+1) \cdot S \cdot \Delta_m,$$

where S is the smallest possible size of the support of an integer optimal solution and $\Delta_m := \max\{|\det(B)|\colon B$ is an $m \times m$ submatrix of $A\}$. Further analysis results in a proximity bound of

$$\|z^* - x^*\|_1 < 3m^2 \log_2(\sqrt{2m} \cdot \Delta_m^{1/m})\Delta_m.$$

Note that for problems in standard form without upper bounds this is the first proximity bound which is polynomial in m and Δ_m.

In **Chapter 3**, we study the maximal number of different columns that can appear in an integral matrix $A \in \mathbb{Z}^{m\times n}$ with full row rank and largest $m \times m$ subdeterminant in absolute value Δ_m. More precisely, we study for $m, k \in \mathbb{Z}_{\geqslant 1}$ the parameter

$$\mathfrak{c}(k, m) := \max\left\{n\colon A \in \mathbb{Z}^{m\times n} \text{ has differing columns, } \operatorname{rank} A = m, \text{ and } \Delta_m(A) \leqslant k\right\},$$

where A has differing columns if and only if A does not contain the vector consisting only of zeros and if $A_i = \pm A_j,\ i,j \in \{1,\ldots,n\}$ implies $i = j$. We show that

$$\mathfrak{c}(k,m) = \frac{1}{2}(m^2+m) + m(k-1)$$

if $m \leqslant 2$ or $k \leqslant 2$. Moreover, we establish that

$$\mathfrak{c}(k,m) \ \leqslant\ \begin{cases} 1/2 \cdot (m^2+m) + m(k-1), & \text{if } k \leqslant 2,\\ 1/2 \cdot (m^2+m)k^2, & \text{if } k \geqslant 3, \end{cases}$$

which yields the first polynomial upper bound on $\mathfrak{c}(k,m)$ in terms of the largest $m \times m$ subdeterminant in absolute value and the number of rows. For IPs in standard form with arbitrary upper bounds u, these findings imply a proximity bound of

$$\|z^* - x^*\|_1 \leqslant (m+1)\Delta_m \cdot (2\mathfrak{c}(\Delta_m, m) + 1) \leqslant m(m+1)^2\Delta_m^3 + (m+1)\Delta_m.$$

This is the first proximity bound for problems of this type which is polynomial in m and Δ_m.

In **Chapter 4**, we provide proximity bounds in two different settings. First of all, for $c \equiv 0$ and a given vertex solution x^* with $\operatorname{supp}(x^*) = \{1,\ldots,m\}$ to (LP), we introduce a new parameter $f(x^*) \in \mathbb{Z}_{\geqslant 1} \cup \{\infty\}$, which describes the number of columns of A that are needed such that its sum lies in the interior of $\operatorname{cone}(-A_{\cdot,1},\ldots,-A_{\cdot,m})$. In this context, we derive a proximity bound of $(f(x^*) + 1)\Delta_m$, which is best possible for $f(x^*) \leqslant m$ in the sense, that there exist examples which have proximity of $\mathcal{O}(f(x^*)\Delta_m)$. Furthermore, we obtain an efficient feasibility test for (IP) if $f(x^*)$ is constant.

Secondly, we provide structural results for problems in standard form in the context of proximity in the case $m = 2$. These results imply a proximity bound of

$$\|z^* - x^*\|_1 \leqslant 3\Delta_2$$

for all but finitely many $b \in \mathbb{Z}^2$ if an LP-basis corresponding to x^* has a prime determinant in absolute value. This can be seen as a refinement of a result by Oertel, Paat and Weismantel [58] for such constraint matrices. Their result implies in this particular case a proximity bound of $3\Delta_2$ with a probability of 1 for a randomly chosen b. Furthermore, for arbitrary b, our analysis reveals a proximity bound of $4\Delta_2$ in the special case of cardinality constrained knapsack problems.

In **Chapter 5**, we provide an algorithm that efficiently optimizes over an IP in standard form if the constraint matrix possesses at most three different $m \times m$ subdeterminants in absolute value.

Chapter 2

Improving Proximity Bounds Using Sparsity

The following chapter appeared in [51].

2.1 Introduction

Let $A \in \mathbb{Z}^{m\times n}$ with $\operatorname{rank}(A) = m$, $c \in \mathbb{Z}^n$, and $b \in \mathbb{Z}^m$. Denote the largest absolute value of a minor of A of order $k \in \{1,\dots,m\}$ by

$$\Delta_k := \Delta_k(A) := \max\{|\det(B)| : B \text{ is a } k\times k \text{ submatrix of } A\}.$$

Note that $\Delta_1 = \|A\|_\infty$ is the largest absolute entry of A. For simplicity, we set $\Delta := \Delta_m$. We consider the standard form integer program

$$\max\{c^\intercal z : Az = b,\ z \in \mathbb{Z}^n_{\geqslant 0}\} \tag{IP}$$

and its linear relaxation

$$\max\{c^\intercal x : Ax = b,\ x \in \mathbb{R}^n_{\geqslant 0}\}. \tag{LP}$$

We assume that (IP) is both feasible and bounded.

Given an optimal vertex solution x^* to (LP), we investigate the question of (LP) to (IP) proximity: can we bound the distance from x^* to some optimal solution z^* of (IP)? We refer to any bound τ on $\|x^* - z^*\|_1$ as a **proximity bound**. Proximity bounds have a variety of implications in the theory of integer programming. For example, a proximity bound of τ translates into a bound of $\tau \cdot \|c\|_\infty$ on the so-called integrality gap [21, 36, 41]. Furthermore, strong proximity bounds reduce the time needed for a local search algorithm to find an optimal (IP) solution starting from an optimal (LP) solution, see, e.g. [29].

One of the first seminal results on proximity is by Cook et al. [21], who established that there exists an optimal solution z^* to (IP) satisfying

$$\|z^* - x^*\|_\infty \leqslant n \cdot \max\{\Delta_k : k \in \{1, \dots, m\}\}. \tag{2.1}$$

Cook et al. actually consider problems in inequality form, i.e., with constraints $Ax \leqslant b$ rather than $Ax = b$ and $x \geqslant 0$, but their results easily translate to the standard form setting. A closer analysis reveals that Δ suffices for the standard form problem rather than $\max\{\Delta_k : k \in \{1, \dots, m\}\}$ stated in (2.1). Furthermore, if we naively extend (2.1) to a bound on proximity in terms of the ℓ_1-norm, then we obtain $\|z^* - x^*\|_1 \leqslant n^2\Delta$. Another closer analysis gives us the bound

$$\|z^* - x^*\|_1 \leqslant (m+1)n\Delta. \tag{2.2}$$

See the proof of Lemma 2.3 for the two 'closer analyses' referred to above. Cook et al.'s bound has been generalized to various problems including those with separable convex objective functions [36, 41, 75] or with mixed integer constraints [62], and extended to alternative data parameters such as k-regularity [50, 76] and the magnitude of Graver basis elements [25].

The proximity bound in (2.2) depends on the dimension n. In 2018 Eisenbrand and Weismantel [29] proved that proximity is independent of the dimension by establishing the bound

$$\|z^* - x^*\|_1 \leqslant m(2m \cdot \|A\|_\infty + 1)^m. \tag{2.3}$$

Eisenbrand and Weismantel use the so-called Steinitz Lemma with the ℓ_∞-norm [67] in their proof of (2.3). Their proof can be modified using the norm $\|x\|_* = \|B^{-1}x\|_\infty$, where B is an $m \times m$ submatrix of A with $|\det(B)| = \delta$, to obtain the bound

$$\|z^* - x^*\|_1 \leqslant m(2m+1)^m \cdot \Delta. \tag{2.4}$$

The proximity bounds (2.3) and (2.4) also hold for standard form problems with additional upper bound constraints on the variables. Oertel et al. established that the upper bound $\|z^* - x^*\|_1 \leqslant (m+1) \cdot (\Delta - 1)$ holds for most problems, where 'most' is defined parametrically with b treated as input [58].

As for lower bounds on proximity, it is not difficult to come up with examples demonstrating $\|z^* - x^*\|_1 \geqslant m \cdot (\Delta - 1)$ and $\|z^* - x^*\|_1 \geqslant \|A\|_\infty^m$. Aliev et al. [2] give a tight lower bound $\Delta - 1$ on proximity in terms of the ℓ_∞-norm when $m = 1$. However, it remains an open question if (2.2), (2.3), or (2.4) is tight in general.

2.1.1 Statement of Results and Overview of Proof Techniques

The focus of this paper is to create stronger proximity bounds. Recall that $\Delta := \Delta_m$. Our first main result is an improvement over (2.4) for fixed Δ. We always consider the logarithm $\log(\cdot)$ to have base two.

Theorem 2.1. *For every optimal* (LP) *vertex solution x^*, there exists an optimal* (IP) *solution z^* such that*

$$\|z^* - x^*\|_1 < 3m^2 \log(\sqrt{2m} \cdot \Delta^{1/m}) \cdot \Delta.$$

Theorem 2.1 demonstrates that proximity in the ℓ_1-norm between (IP) solutions and (LP) **vertex** solutions is bounded by a polynomial in m and Δ. We focus on vertex solutions because proximity may depend on n for general non-vertex (LP) solutions. For example, suppose that $c = 0$ and take any feasible solution to (LP), which is optimal in this case, such that each of the n components is in $1/2 + \mathbb{Z}$.

Our second main proximity result is in terms of $\|A\|_\infty$ **after** A **is transformed by a suitable unimodular matrix**. Recall that a unimodular matrix $U \in \mathbb{Z}^{m\times m}$ satisfies $|\det(U)| = 1$, so the $m \times m$ minors of UA have the same magnitudes as those of A. Moreover, the optimal solutions of (IP) are the same as the optimal solutions to

$$\max\{c^\intercal z : UAz = Ub,\ z \in \mathbb{Z}^n_{\geqslant 0}\}. \qquad (U\text{ - IP})$$

Given an $m \times m$ submatrix B of A, we can find a unimodular matrix U in polynomial time such that UB is upper triangular. The Hermite normal form provides one method for computing such U, see [63]. If B satisfies $|\det(B)| = \Delta$, then $\Delta \leqslant \|UB\|^m_\infty$, and we can apply Theorem 2.1 to obtain the bound

$$\|z^* - x^*\|_1 < 3m^2 \log(\sqrt{2m} \cdot \|UB\|_\infty) \cdot \|UB\|^m_\infty. \qquad (2.5)$$

The previous bound is predicated on the knowledge of an $m \times m$ submatrix of maximum absolute determinant, which is NP-hard to find [47]. However, Di Summa et al. [22] established that this submatrix can be approximated in polynomial time. In particular, they demonstrated that there exists an $m \times m$ submatrix B of A satisfying

$$\Delta \leqslant |\det(B)| \cdot (2\log(m + 1))^{m/2} \qquad (2.6)$$

that can be found in time polynomial in m and n.[1] We can use this approximate largest absolute determinant to derive our second main result. We denote the linear relaxation of (U - IP) by (U - LP).

Theorem 2.2. *Let B be an $m \times m$ submatrix B of A satisfying* (2.6) *and $U \in \mathbb{Z}^{m\times m}$ a unimodular matrix such that UB is upper triangular. Then for every optimal* (U - LP) *vertex solution x^*, there exists an optimal* (U - IP) *solution z^* satisfying*

$$\|z^* - x^*\|_1 < 3m^2 \log(2\sqrt{m \log(m + 1)} \cdot \|UB\|_\infty) \cdot (2 \log(m + 1))^{m/2} \cdot \|UB\|^m_\infty.$$

It is worth reemphasizing that the proximity bound in Theorem 2.2 can be determined in polynomial time, which is in contrast to the bound in (2.5), and the dependence on m is significantly less than the bound in (2.3).

The proofs of Theorems 2.1 and 2.2 are based on combining proof techniques of Cook et al. [21] with results on the sparsity of optimal solutions to (IP). The **support** of a vector $x \in \mathbb{R}^n$ is defined as

$$\mathrm{supp}(x) := \{i \in \{1, \ldots, n\} : x_i \neq 0\}.$$

[1] The approximation result of Di Summa et al. involves an ε factor of precision and the running time is polynomial in $m, n, 1/\varepsilon$. For the sake of presentation, we have fixed this ε to $1/m$ and obtain a polynomial time algorithm in m, n.

A classic theorem of Carathéodory states that $|\operatorname{supp}(x^*)| \leqslant m$ for every vertex solution of (LP). It turns out that the minimum support of an optimal solution to (IP) is not much larger; denote this value by

$$S := \min\{|\operatorname{supp}(z^*)| : z^* \text{ is an optimal solution for (IP)}\}.$$

Aliev et al. [3, 4] established that

$$S \leqslant m + \log\left(\sqrt{\det(AA^\intercal)}\right) \leqslant 2m\log(2\sqrt{m}\cdot\|A\|_\infty). \tag{2.7}$$

For other results regarding sparsity, see [27, 57] for general A and [12, 13, 20, 64] for matrices that form a Hilbert basis. See also the manuscript of Aliev et al. [1], who give improved sparsity bounds for **feasible solutions** to special classes of integer programs and provide efficient algorithms for finding such solutions. Using sparsity we derive the following proximity bound, which forms the basis for Theorems 2.1 and 2.2.

Lemma 2.3. *For every optimal* (LP) *vertex solution x^*, there exists an optimal* (IP) *solution z^* such that*

$$\|z^* - x^*\|_1 < (m+1)\cdot S\cdot\Delta.$$

Lemma 2.3 improves (2.2) by replacing the dependence on n to S. Lemma 2.3 is stated for a generic sparsity bound, so one could use it together with (2.7) to achieve a proximity bound in terms of Δ and $\|A\|_\infty$. In order to provide a bound for proximity that is uniform in the data parameter, we prove a new sparsity result in terms of Δ. A bound in terms of Δ is also of interest because it is invariant under unimodular transformations of A.

Theorem 2.4. *There exists an optimal* (IP) *solution z^* such that*

$$|\operatorname{supp}(z^*)| < 2m\log\left(\sqrt{2m}\cdot\Delta^{1/m}\right).$$

Our proximity bounds can be generalized to mixed integer programs. Given an index set $\mathcal{I} \subseteq \{1,\dots,n\}$, the mixed integer program with integrality constraints indexed by $\mathcal{I}$ is

$$\max\{c^\intercal z : Az = b,\ z \geqslant 0,\ z_i \in \mathbb{Z}\ \forall\, i \in \mathcal{I}\}. \tag{MIP}$$

Similarly to [3, Corollary 4], we establish the extension of Theorem 2.4 to (MIP).

Corollary 2.5. *There exists an optimal* (MIP) *solution z^* satisfying*

$$|\operatorname{supp}(z^*)| < m + 2m\log\left(\sqrt{2m}\cdot\Delta^{1/m}\right) = 2m\log\left(2\sqrt{m}\cdot\Delta^{1/m}\right).$$

We obtain the following proximity result by applying Corollary 2.5.

Corollary 2.6. *For every optimal* (LP) *vertex solution x^*, there exists an optimal* (MIP) *solution z^* such that*

$$\|z^* - x^*\|_1 < 3m^2\log(2\sqrt{m}\cdot\Delta^{1/m})\cdot\Delta.$$

Our results also extend to integer programs in general form. Let $A \in \mathbb{Z}^{m\times n}$ and $B \in \mathbb{Z}^{m\times d}$ be matrices satisfying $\operatorname{rank}([A, B]) = m$. Note that it is not necessary to assume that $\operatorname{rank}(A) = m$ in our general form results. Let $C \in \mathbb{Z}^{t\times d}$, $c \in \mathbb{Z}^{n+d}$, $b_1 \in \mathbb{Z}^m$, and $b_2 \in \mathbb{Z}^t$. The **general form integer program** is

$$\max\left\{c^\intercal z : \begin{array}{rcl}[A,\ B]\, z & = & b_1\\ [0,\ C\ ]\, z & \leqslant & b_2\end{array},\ \ z \in \mathbb{Z}^{n+d},\ \ z_i \geqslant 0\ \forall\ i \in \{1,\dots,n\}\right\}. \qquad \text{(GIP)}$$

We define the general form linear program (GLP) similarly. Previously cited bounds on proximity hold for (GIP). However, our analysis reveals that proximity for (GIP) depends on the potentially smaller data parameter

$$\delta := \max\left\{|\det(E)| : \begin{array}{c} E \text{ is any submatrix of } \begin{pmatrix} A & B\\ 0 & C\end{pmatrix}\\ \text{defined using the first } m \text{ rows}\end{array}\right\}.$$

If $t = 0$ and $d = 0$, then (GIP) is a standard form problem and $\delta = \Delta_m(A)$. If $m = 0$ and $n = 0$, then (GIP) is an inequality form problem and $\delta = \max\{\Delta_k(C) : k \in \{1,\dots,d\}\}$.

Corollary 2.7. *For every optimal* (GLP) *vertex solution x^*, there exists an optimal* (GIP) *solution z^* such that*

$$\|z^* - x^*\|_1 < \min\{m+t+1, n+d\} \cdot \Big(\min\big\{n-m, 2m \cdot \log(\sqrt{2m}\cdot \delta^{1/m})\big\} + d\Big) \cdot \delta.$$

The proximity bound in Corollary 2.7 matches the best known bounds in both the standard form setting and the inequality form setting.

Going beyond integer linear optimization problems, it would be ideal for proximity bounds in terms of sparsity to extend to integer programs with separable convex objective functions (see [36, 41] for similarities between the linear and separable convex setting). However, for separable convex **maximization problems**, strong proximity bounds do not exist for exact solutions, in general, even though sparsity results apply. In contrast, for separable convex **minimization problems**, strong sparsity bounds do not exist for exact solutions, in general, even though the classic proximity techniques apply.

The paper is structured as follows. In Section 2.2, we present our proofs of the proximity bound derived from a generic sparsity bound (Lemma 2.3) and the sparsity bounds (Theorem 2.4 and Corollary 2.5). Then in Section 2.3, we provide the proofs of the proximity results for the standard form integer programs (Theorem 2.1 and Theorem 2.2). The proof of the mixed integer case (Corollary 2.6) is omitted because it is the same as the proof of the pure integer case except that a different sparsity bound is applied. Additionally, we provide a proof of the proximity result in the general form setting (Corollary 2.7).

2.2 Proofs regarding sparsity

Given $A \in \mathbb{R}^{m\times n}$ and $I \subseteq \{1,\dots,n\}$, we let $A_I \in \mathbb{R}^{m\times|I|}$ denote the columns of A indexed by I. If $I = \{i\}$ for some $i \in \{1,\dots,n\}$, then $A_i := A_I$. Similarly, given $u \in \mathbb{R}^n$, we let $u_I \in \mathbb{R}^{|I|}$ denote the components of u indexed by I.

Proof of Lemma 2.3. We prove the result by projecting the optimization problems onto the union of the supports of x^* and an optimal (IP) solution with minimal support. Let $\bar{z} \in \mathbb{Z}^n_{\geqslant 0}$ be an optimal (IP) solution with minimum support. By the definition of S we have $|\operatorname{supp}(\bar{z})| = S$. As x^* is an optimal vertex solution of (LP) we also have $|\operatorname{supp}(x^*)| \leqslant m$. Define

$$H := \operatorname{supp}(x^*) \cup \operatorname{supp}(\bar{z}),$$

and note that

$$|H| = |\operatorname{supp}(x^*) \cup \operatorname{supp}(\bar{z})| \leqslant |\operatorname{supp}(x^*)| + |\operatorname{supp}(\bar{z})| \leqslant |\operatorname{supp}(x^*)| + S. \tag{2.8}$$

If $n = m$, then A is invertible and there exists a unique solution $A^{-1}b$ to the system $Ax = b$. In this case $x^* = \bar{z} = A^{-1}b$. Therefore, $\|x^* - \bar{z}\|_1 = 0$. For the rest of the proof, we assume that $n > m$ and $H = \{1,\dots,|H|\}$.

Consider the optimization problems

$$\max\left\{c_H^\intercal z : A_H z = b,\ z \in \mathbb{Z}^{|H|}_{\geqslant 0}\right\} \tag{IP2}$$

and

$$\max\left\{c_H^\intercal x : A_H x = b,\ x \in \mathbb{R}^{|H|}_{\geqslant 0}\right\}. \tag{LP2}$$

Observe that x^*_H is an optimal vertex solution for (LP2), and $\bar{z}_H$ is an optimal solution for (IP2).

Rewrite (LP2) in inequality form:

$$\max\left\{c_H^\intercal x : A_H x = b, -I_H x \leqslant 0,\ x \in \mathbb{R}^{|H|}\right\},$$

where I_H is the $|H| \times |H|$ identity matrix. Partition the rows of $-I_H$ into D_1 and D_2 such that $D_1\bar{z}_H < D_1 x^*_H$ and $D_2\bar{z}_H \geqslant D_2 x^*_H$. Define the pointed polyhedral cone

$$K := \left\{u \in \mathbb{R}^{|H|} : A_H u = 0,\ D_1 u \leqslant 0,\ D_2 u \geqslant 0\right\}. \tag{2.9}$$

Observe that $\bar{z}_H - x^*_H \in K$. By (2.8) we see that the rank of A_H is at least $|\operatorname{supp}(x^*)|$ because the columns of A corresponding to the support x^* are linearly independent. Thus, the dimension of K, which we denote by $\dim(K)$, is at most $|\operatorname{supp}(x^*)| + S - \operatorname{rank}(A_H) \leqslant S$.

Let $U := \{u^1,\dots,u^t\} \subseteq \mathbb{R}^{|H|}\backslash\{0\}$ be a set of vectors that generate the extreme rays of K, i.e.,

$$K = \left\{\sum_{i=1}^t \lambda_i u^i : \lambda_i \geqslant 0\ \forall\ i \in \{1,\dots,t\}\right\}$$

and u^i satisfies $|H| - 1$ linearly independent constraints in (2.9) at equality for each $i \in \{1,\dots,t\}$.

Claim 1. *For each $\tilde{u} \in U$ we have*

$$|\operatorname{supp}(\tilde{u})| \leqslant m+1. \tag{2.10}$$

Also, each $\tilde{u} \in U$ can be scaled to have integer components and satisfy $\|\tilde{u}\|_\infty \leqslant \Delta$.

Proof. Set $T := \operatorname{supp}(\tilde{u})$ and without loss of generality assume $T = \{1, \ldots, |T|\}$. Recall that $\tilde{u}$ satisfies a set of $|H| - 1$ linearly independent constraints in (2.9) at equality. One such set is composed of $|H| - |T|$ constraints from the system $D_1\tilde{u} \leqslant 0,\ D_2\tilde{u} \geqslant 0$ and $|T| - 1$ constraints from $0 = A_H\tilde{u} = A_T\tilde{u}_T$. By this choice of constraints it follows that

$$|\operatorname{supp}(\tilde{u})| = |T| \leqslant m+1$$

and $|T| - 1 \leqslant \operatorname{rank}(A_T) \leqslant \min\{|T|, m\}$. Recalling $n > m$ and $\operatorname{rank}(A) = m$, the latter inequalities imply that there exists an index set $\overline{T}$ satisfying $T \subseteq \overline{T} \subseteq \{1, \ldots, n\}$, $A_{\overline{T}} \in \mathbb{Z}^{m \times (m+1)}$ and $\operatorname{rank}(A_{\overline{T}}) = m$.

Let $\bar{u} \in \mathbb{R}^{m+1}$ denote the vector obtained by appending $m + 1 - |T|$ zeros to $\tilde{u}_T$. There exists an index set $I \subseteq \overline{T}$ with $|I| = m$ and A_I invertible. Let i denote the singleton in $\overline{T} \backslash I$. Because $A_{\overline{T}}\bar{u} = 0$, we have

$$A_I\bar{u}_I = -A_i\bar{u}_i.$$

If $\bar{u}_i = 0$, then $\bar{u} = 0$ and so $\tilde{u} = 0$. However, this contradicts that $\tilde{u} \in U$. Hence, $\bar{u}_i \neq 0$. Scale $\bar{u}$ such that $|\bar{u}_i| = |\det(A_I)|$. Applying Cramer's rule demonstrates that

$$|\bar{u}_j| = |\det(A_{I \cup \{i\} \backslash \{j\}})| \qquad \forall\ j \in I.$$

Hence, $\bar{u}$, and consequently $\tilde{u}$, can be scaled to have integer components with $\|\tilde{u}\|_\infty \leqslant \Delta$.

□

For the rest of the proof we assume that each $\tilde{u} \in U$ is scaled such that the conclusions of Claim 1 hold.

Recall $\bar{z}_H - x^*_H \in K$. By Carathéodory's theorem, there exists an index set $I \subseteq \{1, \ldots, t\}$ with $|I| \leqslant \dim(K) \leqslant S$ and $\lambda_i \in \mathbb{R}_{\geqslant 0}$ for each $i \in I$ such that

$$\bar{z}_H - x^*_H = \sum_{i \in I} \lambda_i u^i.$$

Set $w := \sum_{i \in I} \lfloor \lambda_i \rfloor u^i$. Using standard techniques in proximity proofs (see, e.g., [21, Theorem 1]), it can be verified that $\tilde{z} := \bar{z}_H - w$ is a feasible solution to (IP2), $\tilde{x} := x^*_H + w$ is a feasible solution to (LP2), and

$$c_H^\intercal \bar{z}_H + c_H^\intercal x^*_H = c_H^\intercal \tilde{z} + c_H^\intercal \tilde{x}. \tag{2.11}$$

Because x^*_H is optimal for (LP2), we have $c_H^\intercal \tilde{x} \leqslant c_H^\intercal x^*_H$. Combining this with (2.11) proves that $c_H^\intercal \tilde{z} \geqslant c_H^\intercal \bar{z}_H$. Because $\bar{z}_H$ is optimal for (IP2), we have $c_H^\intercal \tilde{z} = c_H^\intercal \bar{z}_H$ and $\tilde{z}$ is also an optimal solution to (IP2).

Define $z^* \in \mathbb{Z}^n_{\geqslant 0}$ component-wise to be

$$z_i^* := \begin{cases} \tilde{z}_i & \text{if } i \in \{1, \dots, |H|\} \\ 0 & \text{otherwise.} \end{cases}$$

By construction z^* is an optimal solution to (IP) because $Az^* = A_H \tilde{z} = A_H \bar{z}_H = b$ and $c^\intercal z^* = c_H^\intercal \tilde{z} = c_H^\intercal \bar{z}_H = c^\intercal \bar{z}$. By (2.8) and (2.10), we arrive at the final result:

$$\begin{aligned} \|z^* - x^*\|_1 = \|\tilde{z} - x_H^*\|_1 &\leqslant \sum_{i \in I} (\lambda_i - \lfloor \lambda_i \rfloor) \|u^i\|_1 \\ &< (m+1) \cdot S \cdot \|u^i\|_\infty \leqslant (m+1) \cdot S \cdot \Delta. \end{aligned}$$

□

Proof of Theorem 2.4. Let z^* be an optimal solution of (IP) with minimum support. The definition of S states that $S := |\operatorname{supp}(z^*)|$. Define $\tilde{A} \in \mathbb{Z}^{m \times S}$ as the submatrix of A corresponding to the support of z^*. If $S \leqslant 2m$, then the result holds. Thus, assume that $S > 2m$, which implies $\log({}^S\!/\!_m) < ({}^S\!/\!_m) - 1$. Theorem 1 in Aliev et al. [3] states that

$$S < m + \log\left(\sqrt{\det(\tilde{A}\tilde{A}^\intercal)}\right).$$

The Cauchy-Binet formula for $\det(\tilde{A}\tilde{A}^\intercal)$ states that

$$\det(\tilde{A}\tilde{A}^\intercal) = \sum_{\substack{B \text{ is an } m \times m \\ \text{submatrix of } \tilde{A}}} \det(B)^2.$$

See, e.g., [43]. Combining the previous inequalities yields

$$\begin{aligned} S &< m + \log\left(\sqrt{\det(\tilde{A}\tilde{A}^\intercal)}\right) \leqslant m + \log\left(\sqrt{\binom{S}{m}\Delta^2}\right) \\ &\leqslant m + \log\left(S^{m/2}\Delta\right) = m + \frac{m}{2}\log\left(\frac{S}{m}\right) + \frac{m}{2}\log(m) + \log(\Delta) \\ &< m + \frac{m}{2}\left(\frac{S}{m} - 1\right) + \frac{m}{2}\log(m) + \log(\Delta) = \frac{S}{2} + \frac{m}{2} + \frac{m}{2}\log(m) + \log(\Delta). \end{aligned}$$

Therefore,

$$|\operatorname{supp}(z^*)| = S < m + m\log(m) + 2\log(\Delta) \leqslant 2m\log\left(\sqrt{2m} \cdot \Delta^{1/m}\right).$$

□

Proof of Corollary 2.5. Let z^* be an optimal (MIP) solution with minimal support. By applying Theorem 2.4 to the standard form integer program with constraint matrix $A_\mathcal{I}$ and right hand side $b - A_\mathcal{J} z_\mathcal{J}^* \in \mathbb{Z}^m$, where $\mathcal{J} := \{1, \dots, n\} \backslash \mathcal{I}$, we see that

$$|\operatorname{supp}(z_\mathcal{I}^*)| < 2m\log\left(\sqrt{2m} \cdot \Delta^{1/m}\right).$$

Similarly, by considering the standard form linear program with constraint matrix $A_{\mathcal{J}}$ and right hand side $b - A_{\mathcal{I}} z^*_{\mathcal{I}}$, we see that $|\operatorname{supp}(z^*_{\mathcal{J}})| \leqslant m$. Hence,

$$|\operatorname{supp}(z^*)| \leqslant |\operatorname{supp}(z^*_{\mathcal{J}})| + |\operatorname{supp}(z^*_{\mathcal{I}})| \leqslant m + 2m \log\big(\sqrt{2m} \cdot \Delta^{1/m}\big).$$

□

2.3 Results on proximity

Proof of Theorem 2.1. For now assume that $m \geqslant 2$. Combining Lemma 2.3 with Theorem 2.4 demonstrates that there exists an optimal (IP) solution z^* satisfying

$$\begin{aligned}\|z^* - x^*\|_1 &< (m+1) \cdot S \cdot \Delta \\ &\leqslant (m+1) \cdot 2m \log(\sqrt{2m} \cdot \Delta^{1/m}) \cdot \Delta \\ &\leqslant 3m^2 \log(\sqrt{2m} \cdot \Delta^{1/m}) \cdot \Delta.\end{aligned}$$

This completes the proof when $m \geqslant 2$.

It is left to consider the case $m = 1$. Here we have $\Delta = \|A\|_\infty$. If $\Delta = 1$, then x^* is integral and there is nothing to prove. Thus, assume $\Delta \geqslant 2$. The proximity bound (2.3) states that there exists an optimal solution z^* to (IP) satisfying

$$\|z^* - x^*\|_1 \leqslant 2 \cdot \|A\|_\infty + 1 = 2\Delta + 1 < 3\Delta \leqslant 3m^2 \log(\sqrt{2m} \cdot \Delta^{1/m}) \cdot \Delta.$$

This completes the proof.

□

We use the following result to prove Theorem 2.2.

Lemma 2.8 (Theorem 1 in [22]). *For every $\varepsilon > 0$, there exists an $m \times m$ submatrix B of A satisfying*

$$\Delta \leqslant |\det(B)| \cdot (e \cdot \ln((1+\varepsilon) \cdot m))^{m/2}.$$

The matrix B can be found in time that is polynomial in m, n, and $1/\varepsilon$.

Setting $\varepsilon = 1/m$ in Lemma 2.8 yields the approximation factor

$$\begin{aligned}\Delta &\leqslant |\det(B)| \cdot (e \cdot \ln(m+1))^{m/2} \\ &= |\det(B)| \cdot \left(\frac{e}{\log(e)} \cdot \log(m+1)\right)^{m/2} \\ &\leqslant |\det(B)| \cdot (2 \cdot \log(m+1))^{m/2},\end{aligned}$$

which is precisely (2.6).

Proof of Theorem 2.2. Let B be an $m \times m$ submatrix of A satisfying (2.6). There exists a unimodular matrix $U \in \mathbb{Z}^{m \times m}$ such that UB is an upper triangular matrix with non-negative diagonal entries d_i.

Unimodular matrices preserve the absolute value of $m \times m$ determinants, so $|\det(B)| = |\det(UB)|$. By (2.6) we see that

$$\begin{aligned} \Delta &\leqslant |\det(B)| \cdot (2 \cdot \log(m+1))^{m/2} = |\det(UB)| \cdot (2 \cdot \log(m+1))^{m/2} \\ &= \left(\prod_{i=1}^{m} d_i\right) \cdot (2 \cdot \log(m+1))^{m/2} \leqslant \|UB\|_\infty^m \cdot (2 \cdot \log(m+1))^{m/2}. \end{aligned}$$

By applying Theorem 2.1 we obtain the bound

$$\begin{aligned} \|z^* - x^*\|_1 &< 3m^2 \log(\sqrt{2m} \cdot \Delta^{1/m}) \cdot \Delta \\ &< 3m^2 \log(2\sqrt{m \log(m+1)} \|UB\|_\infty) \cdot (2 \log(m+1))^{m/2} \cdot \|UB\|_\infty^m. \end{aligned}$$

This completes the proof.

□

Next, we present a proof of Corollary 2.7. We advise the reader that the proof is similar to the proof of Lemma 2.3.

Proof of Corollary 2.7. Let z^* be an optimal solution to (GIP) with minimal support on the first n components. Consider the n-dimensional integer program obtained by fixing the last d variables of (GIP) to the last d components of z^*. Similarly, consider the n-dimensional linear program obtained by fixing the last d variables of (GLP) to the last d components of x^*. These lower dimensional problems are in standard form. Therefore, by applying Theorem 2.4 to these lower dimensional problems and recalling $\Delta_m(A) \leqslant \delta$ we see that

$$\bar{S} := |\operatorname{supp}(z^*) \cap \{1, \ldots, n\}| \leqslant 2m \cdot \log(\sqrt{2m} \cdot \delta^{1/m})$$

and

$$|\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)| \leqslant \min\{n, \bar{S} + m\} + d.$$

We project the original optimization problems onto the variables corresponding to $|\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)|$ to bound proximity. We complete the proof of the corollary by showing

$$\|z^* - x^*\|_1 < \min\{m + t + 1, n + d\} \cdot \min\{n + d - m, \bar{S} + d\} \cdot \delta.$$

As in the proof of Lemma 2.3, we create a pointed cone from the constraints defining (GIP). In order to assure a pointed cone, we introduce redundant constraints. Let $b_3 \in \mathbb{Z}^d$ be the vector where every component is $\lceil \|x^*\|_\infty \rceil + \|z^*\|_\infty$, and define

$$D := \begin{pmatrix} 0 & C \\ -I_n & 0 \\ 0 & I_d \end{pmatrix} \in \mathbb{Z}^{(t+n+d)\times(n+d)} \quad \text{and} \quad f = \begin{pmatrix} b_2 \\ 0 \\ b_3 \end{pmatrix} \in \mathbb{Z}^{t+n+d}.$$

By construction, z^* is an optimal solution to the integer program

$$\max \left\{ c^\intercal z : \begin{array}{l} [A, B]z = b_1,\ Dz \leqslant f,\ z \in \mathbb{Z}^{n+d}, \\ z_i = 0\ \forall\ i \in \{1, \ldots, d+n\} \backslash (\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)) \end{array} \right\},$$

and x^* is an optimal vertex solution to the corresponding linear relaxation.

Subdivide the rows of D such that $D_1 z^* < D_1 x^*$ and $D_2 z^* \geqslant D_2 x^*$. Define the polyhedral cone

$$K := \left\{ u \in \mathbb{R}^{n+d} : \begin{array}{l} [A,B]u = 0, \begin{pmatrix} D_1 \\ -D_2 \end{pmatrix} u \leqslant 0, \\ u_i = 0 \;\forall\; i \in \{1,\dots,d+n\} \backslash (\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)) \end{array} \right\}.$$

Observe that $z^* - x^* \in K$. Moreover, the introduction of b_3 and I_d ensures that $\operatorname{rank}(D) = n + d$ and that K is pointed. We bound $\dim(K)$ in two ways by counting the number of linearly independent equations in the definition of K. First, there are m linearly independent constraints of the form $[A,B]u = 0$ because $\operatorname{rank}([A,\ B]) = m$. Second, there are $n + d - |\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)|$ many linearly independent constraints of the form $u_i = 0$. Additional linearly independent equations that are independent from this second set are $|\operatorname{supp}(x^*) \cap \{1,\dots,n\}|$ many rows from $[A,B]u = 0$ corresponding to the independent columns of $\operatorname{supp}(x^*) \cap \{1,\dots,n\}$. Hence, $\dim(K)$ can be upper bounded as follows:

$$\begin{aligned} \dim(K) &\leqslant \min\{n + d - m, |[\operatorname{supp}(x^*) \cup \operatorname{supp}(z^*)] \cap \{n+1,\dots,n+d\}| + \bar{S}\} \\ &\leqslant \min\{n + d - m, d + \bar{S}\}. \end{aligned}$$

Let U be a finite set generating the extreme rays of K. The proof of Claim 1 can be used to demonstrate that for each $\tilde{u} \in U$ we have

$$|\operatorname{supp}(\tilde{u})| \leqslant \min\{m + t + 1, n + d\},$$

and each $\tilde{u} \in U$ can be scaled such that $\tilde{u} \in \mathbb{Z}^{n+d}$ and $\|\tilde{u}\|_\infty \leqslant \delta$. The main difference that arises when repeating this proof is that the matrices D_1 and D_2 contain rows of $[0, C]$ rather than simply rows from the identity matrix as was the case in the proof of Lemma 2.3. It is this difference that necessitates the choice of the data parameter δ and dictates its definition.

By Carathéodory's theorem there exist $k \leqslant \dim(K)$ vectors $u^1,\dots,u^k \in U$ and coefficients $\lambda_1,\dots,\lambda_k \in \mathbb{R}_{\geqslant 0}$ such that $z^* - x^* = \sum_{i=1}^k \lambda_i u^i$. Because $z^* - \sum_{i=1}^k \lfloor \lambda_i \rfloor u^i$ is also an optimal solution to (GIP), we can assume without loss of generality that $\lambda_1,\dots,\lambda_k < 1$ (the reasoning is similar to that in the proof of Lemma 2.3). This implies that

$$\begin{aligned} \|z^* - x^*\|_1 &\leqslant \sum_{i=1}^k \lambda_i \|u^i\|_1 < \min\{m + t + 1, n + d\} \cdot \dim(K) \cdot \delta \\ &\leqslant \min\{m + t + 1, n + d\} \cdot \min\{n + d - m, \bar{S} + d\} \cdot \delta. \end{aligned}$$

This completes the proof. □

Chapter 3

Polynomial Upper Bounds on the Number of Differing Columns of Δ-Modular Integer Programs

The following chapter appeared in [52]. Thanks to Jim Geelen for bringing awareness to the results in [30], Stefan Kuhlmann for his comments on our lower bound construction, and Rico Zenklusen for his comments regarding [15].

3.1 Introduction.

The feasible region of an integer linear program with box constraints can be written as

$$\mathrm{IP} := \{\mathbf{x} \in \mathbb{Z}^n \colon A\mathbf{x} = \mathbf{b},\ \boldsymbol{\ell} \leqslant \mathbf{x} \leqslant \mathbf{u}\},$$

for a constraint matrix $A \in \mathbb{Z}^{m\times n}$ with $\operatorname{rank} A = m$ and vectors $\mathbf{b} \in \mathbb{Z}^m$ and $\boldsymbol{\ell}, \mathbf{u} \in (\mathbb{Z} \cup \{\pm\infty\})^n$ with $\boldsymbol{\ell} < \mathbf{u}$. Integer programs have been used for many decades to model problems in operations research, computer science, and mathematics; see [16, 56, 63] and the references therein. One parameter that impacts the structure of IP is the largest absolute $m \times m$ minor of A, which we denote by

$$\Delta(A) := \max\{|\det B| \colon B \text{ is an } m \times m \text{ submatrix of } A\}.$$

We say that A is **Δ-modular** if $\Delta(A) \leqslant \Delta$.

To illustrate the impact that $\Delta(A)$ has on IP, consider the distance between IP and its linear relaxation

$$\mathrm{LP} := \{\mathbf{x} \in \mathbb{R}^n \colon A\mathbf{x} = \mathbf{b},\ \boldsymbol{\ell} \leqslant \mathbf{x} \leqslant \mathbf{u}\}.$$

This distance, which is referred to as **proximity** in the literature, is defined as the maximum distance from any vertex of LP to the closest feasible IP solution:

$$\pi := \max_{\substack{\mathbf{x}^* \text{ is a vertex} \\ \text{of LP}}} \ \min_{\mathbf{z}^* \in \mathrm{IP}} \|\mathbf{x}^* - \mathbf{z}^*\|_1.$$

We assume $\mathrm{IP} \neq \emptyset$ whenever discussing π. Proximity is often used in the analysis of integer programming algorithms. For instance, proximity can also be used to bound the state space of a dynamic program [29]. Proximity also translates into an upper bound on the integrality gap: for an objective vector $\mathbf{c} \in \mathbb{R}^n$ and vectors $\mathbf{x}^* \in \mathrm{LP}$ and $\mathbf{z}^* \in \mathrm{IP}$ that maximize $\mathbf{x} \to \mathbf{c}^\intercal \mathbf{x}$ over LP and IP, respectively, the integrality gap $|\mathbf{c}^\intercal \mathbf{x}^* - \mathbf{c}^\intercal \mathbf{z}^*|$ is at most $\pi \cdot \|\mathbf{c}\|_\infty$. In a seminal paper by Cook et al. [21], they showed that $\pi \leqslant n^2 \Delta(A)$.[1] Eisenbrand and Weismantel [29] proved $\pi \leqslant m(2m\|A\|_\infty + 1)^m$, where $\|A\|_\infty$ is the largest absolute entry of A; this was the first upper bound on π that was independent of n. Their proof approach extends[2] to show

$$\pi \leqslant m(2m+1)^m \Delta(A).$$

In the special case when $\boldsymbol{\ell} = \mathbf{0}$ and $\mathbf{u} \equiv \infty$, Lee et al. [51] demonstrated that $\pi \leqslant 3m^2 \log_2(\sqrt{2m}\Delta(A)^{1/m})\Delta(A)$; their proof crucially relied on sparsity results that are not applicable when $\boldsymbol{\ell}$ and $\mathbf{u}$ take general values [4]. No upper bounds on π have been provided that are polynomial in $\Delta(A)$ and m for general values of $\boldsymbol{\ell}$ and $\mathbf{u}$.

Testing if $\mathrm{IP} \neq \emptyset$ is NP-hard in general [19], although it can be tested in polynomial time if n is fixed [53]. The parameter $\Delta(A)$ is also known to influence how efficiently we can test if $\mathrm{IP} \neq \emptyset$, at least when $\Delta(A)$ is small. For example, every vertex of LP is integer valued when $\Delta(A) = 1$. Therefore, testing if $\mathrm{IP} \neq \emptyset$ simplifies to testing if $\mathrm{LP} \neq \emptyset$. Matrices with $\Delta(A) = 1$ are called **unimodular**, and after elementary row operations they are equivalent to totally unimodular (TU) matrices. The study of TU matrices dates back to Hoffman and Kruskal [42] with one prominent example being the vertex-edge incidence matrix of a bipartite graph. It remains an open question if $\mathrm{IP} \neq \emptyset$ can be tested efficiently when $\Delta(A)$ is fixed.

If $\Delta(A) = 2$, then A is called **bimodular**. One prominent example of a bimodular matrix is the vertex-edge incidence matrix of a graph whose so-called odd cycle packing number is one; see [17, 18] for combinatorial optimization algorithms over such graphs. When the constraint matrix is bimodular, the vertices of LP may not be integer valued. However, such matrices do impose the nice property that if $\mathrm{IP} \neq \emptyset$, then every vertex of LP lies on an edge containing a vector in IP [72]; Veselov and Chirkov used this property in a polynomial time algorithm to test if $\mathrm{IP} \neq \emptyset$ when A contains no $m \times m$ minors equal to zero. Artmann et al. [8] used a more combinatorial approach to design an optimization algorithm that runs in strongly polynomial time for general bimodular matrices. The algorithm in [8] heavily relies on Seymour's combinatorial characterization of TU matrices [65]. Cevallos et al. [15, Theorem 5.4] argue that compact linear extended formulations (LEFs) do not always exist for bimodular integer programs; in their paper, they write *"A natural approach to solve bimodular integer programs would have been to try to find a compact LEF of the feasible solutions to (conic) bimodular in-*

[1]Cook et al.'s original result considers inequality-form polyhedra, the ℓ_∞ rather than ℓ_1-distance, and **totally Δ-modular matrices**, which have all absolute minors bounded by Δ. A closer analysis revealed that $\Delta(A)$ suffices; see [51, Lemma 3].

[2]See the footnote on Page 3 of [58] for a discussion on this extension.

teger programs, thus avoiding the partially involved combinatorial techniques used in [8], which is so far the only method to efficiently solve bimodular integer programs. Theorem 5.4 shows that this approach cannot succeed." Glanzer et al. used combinatorial properties of the constraint matrix to optimize over IP efficiently in the setting when A has at most three distinct absolute determinants [32]. These examples illustrate the importance of combinatorial properties of the constraint matrix; this leads to a third, more combinatorial, property of IP that is influenced by $\Delta(A)$: the number of differing columns that A can have.

We say that two vectors $\mathbf{a}, \overline{\mathbf{a}} \in \mathbb{R}^d$ **differ** if $\mathbf{a} \neq \pm\overline{\mathbf{a}}$.[3] We say that a multiset $X \subseteq \mathbb{R}^d$ has **differing columns** if every pair of vectors in X differs and $\mathbf{0} \notin X$. We also treat the matrix A as a multiset of its columns, and we denote the number of differing columns by $|A|$. For $m, \Delta \in \mathbb{Z}_{\geqslant 1}$, we denote the maximum number of differing columns in a Δ-modular matrix by

$$\mathfrak{c}(\Delta, m) := \max\left\{|A| \colon A \subseteq \mathbb{Z}^m,\ \mathrm{rank}A = m,\ \text{and } \Delta(A) \leqslant \Delta\right\}.$$

The $\mathrm{rank}A = m$ condition is necessary. Otherwise, one can add a row of all zeros to any integer-valued matrix with $m-1$ rows; the resulting matrix A will have $\Delta(A) = 0$.

One of the first bounds on $\mathfrak{c}(\Delta, m)$ is due to Heller [40], who proved $\mathfrak{c}(1, m) = {}^1\!/\!_2 \cdot (m^2 + m)$. Early generalizations of Heller's result focused on $\mathfrak{c}(\Delta, m)$ for fixed values of Δ. Lee showed $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{f}_L(\Delta) \cdot m^{2\Delta}$ [50, Proposition 10.1] for some function $\mathfrak{f}_L$, and Anstee showed $\mathfrak{c}(\Delta, m) \in O(m^{2\Delta(1+\log_2 \Delta)})$ for totally Δ-modular matrices [6, Theorem 3.2]. In the case when A only contains primitive columns, Kung showed $|A| \leqslant \mathfrak{f}_K(\Delta) \cdot m^2$ for a super-polynomial function $\mathfrak{f}_K$ [49, Theorem 1.1], and $|A| \leqslant m^2$ when no nonzero minor is divisible by three [48, Theorem 1.1]. Oxley and Walsh recently showed $|A| \leqslant {}^1\!/\!_2 \cdot (m^2 + m) + m - 1$ when A is bimodular, but only when A contains primitive columns, and only when m is sufficiently large [60, Theorem 1.1]. The best known upper bound on $\mathfrak{c}(\Delta, m)$ for fixed Δ is given by Geelen et al. [30, Theorem 2.2.4]. They demonstrated that $\mathfrak{c}(\Delta, m) \leqslant {}^1\!/\!_2 \cdot m^2 + \mathfrak{f}_G(\Delta)m$, where $\mathfrak{f}_G(\Delta)$ is a number that can be lower bounded by the "power tower" with base Δ iterated three times.

For fixed values of m, the best known upper bound on $\mathfrak{c}(\Delta, m)$ was due to Heller [40] and Glanzer et al. [34]:

$$\mathfrak{c}(\Delta, m) \leqslant \begin{cases} {}^1\!/\!_2 \cdot (m^2 + m), & \text{if } \Delta = 1; \\ m^2\Delta, & \text{if } \Delta = 2, 3; \\ {}^1\!/\!_2 \cdot m^2 \Delta^{2+\log_2 \log_2(\Delta)}, & \text{if } \Delta \geqslant 4. \end{cases} \tag{3.1}$$

The inequality $\mathfrak{c}(3, m) \leqslant 3m^2$ is present in the analysis in [34, Subsection 3.3] but not stated. In summary, neither Geelen et al. nor Glanzer et al. provided an upper bound on $\mathfrak{c}(\Delta, m)$ that is polynomial in Δ.

An interesting variation of $\mathfrak{c}(\Delta, m)$ is considered by Oxley and Walsh [60] and Kung [48, 49], who considered the maximum number of differing primitive columns

[3] Glanzer et al. [34] used the term **distinctive** rather than differing.

in a Δ-modular matrix, which we denote by $\mathfrak{c}^{\mathrm{p}}(\Delta, m)$; a **primitive** vector $\mathbf{v} = (v_1, \ldots, v_t)$ is an integer valued vector with $\gcd\{v_1, \ldots, v_t\} = 1$. It is easy to see that $\mathfrak{c}^{\mathrm{p}}(1, m) = \mathfrak{c}(1, m)$, and only when $\Delta \geqslant 2$ is there a distinction between $\mathfrak{c}^{\mathrm{p}}(\Delta, m)$ and $\mathfrak{c}(\Delta, m)$. By identifying excluded minors in matroids representable by bimodular matrices, Oxley and Walsh gave a bound of $\mathfrak{c}^{\mathrm{p}}(2, m) = \mathfrak{c}(1, m) + m - 1$ for sufficiently large values of m. Our analysis shows $\mathfrak{c}^{\mathrm{p}}(2, m) = \mathfrak{c}(1, m) + m$ for $m \in \{3, 5\}$ and $\mathfrak{c}^{\mathrm{p}}(2, m) = \mathfrak{c}(1, m) + m - 1$ otherwise; for a lower bound, see our tight example analysis in Section 3.4, and for a matching upper bound, see (3.20) (3.21), (3.22) in Section 3.4. Of course, the big open question in this line of work is the determination of $\mathfrak{c}(\Delta, m)$ and $\mathfrak{c}^{\mathrm{p}}(\Delta, m)$ for general values of Δ.

3.1.1 Statement of results.

Our first main result is the exact value of $\mathfrak{c}(2, m)$. This is the first tight column number bound since Heller's result.

Theorem 3.1 (An exact bound when $\Delta = 2$)**.** *For every $m \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\mathfrak{c}(2, m) = \frac{1}{2}(m^2 + m) + m.$$

Our proof of Theorem 3.1 reveals new combinatorial properties about bimodular matrices. We show that submatrices contain at most one disjoint **circuit**, which is an inclusion-wise minimal set of linearly dependent columns; this generalizes a result of Heller that certain submatrices of TU matrices have no circuits. See Lemma 3.12 for a precise statement and Section 3.3 for more discussion. As previously quoted, combinatorial properties of the constraint matrix are critical in algorithms designed for Δ-modular IPs. For this reason, we believe our combinatorial analysis may be of independent interest.

Our proof of Theorem 3.1 requires a lower bound on $\mathfrak{c}(2, m)$. We give a bound for general Δ.

Proposition 3.2 (A lower bound on $\mathfrak{c}(\Delta, m)$)**.** *For every $\Delta, m \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\frac{1}{2}(m^2 + m) + m(\Delta - 1) \leqslant \mathfrak{c}(\Delta, m).$$

Geelen et al.'s result implies that $\mathfrak{c}(\Delta, m) = \mathfrak{c}(1, m) + \mathfrak{h}(\Delta)m$ for some function $\mathfrak{h}$. Heller's result and Theorem 3.1 support our conjecture that $\mathfrak{h}(\Delta) = \Delta - 1$; we prove this when $m \leqslant 2$.

Proposition 3.3 (An exact bound when $m \leqslant 2$)**.** *Suppose $m \leqslant 2$. For every $\Delta \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\mathfrak{c}(\Delta, m) = \frac{1}{2}(m^2 + m) + m(\Delta - 1).$$

Our second main result is the first upper bound on $\mathfrak{c}(\Delta, m)$ that is polynomial in Δ and m.

Theorem 3.4 (An upper bound on $\mathfrak{c}(\Delta, m)$). *For every $\Delta, m \in \mathbb{Z}_{\geqslant 1}$, we have*

$$\mathfrak{c}(\Delta, m) \leqslant \begin{cases} 1/2 \cdot (m^2 + m) + m(\Delta - 1), & \text{if } \Delta \leqslant 2; \\ 1/2 \cdot (m^2 + m)\Delta^2, & \text{if } \Delta \geqslant 3. \end{cases}$$

Our third main results connects $\mathfrak{c}(\Delta, m)$ with the proximity value π. We apply Theorem 3.4 to establish the first upper bound on π that is polynomial in m and $\Delta(A)$. Unlike in [51], our new bound applies when the variable bounds $\boldsymbol{\ell}$ and $\mathbf{u}$ are arbitrary.

Theorem 3.5 (LP to IP proximity). *Set $\Delta := \Delta(A)$, where $A \in \mathbb{Z}^{m \times n}$ is the constraint matrix in* IP. *The proximity value π satisfies*

$$\pi \leqslant (m+1)\Delta(2\mathfrak{c}(\Delta, m) + 1) \leqslant m(m+1)^2\Delta^3 + (m+1)\Delta.$$

Column number bounds can also be applied to bound so-called Graver basis elements in test sets for integer programs; see, e.g., [54, §3.7]. By directly substituting Theorem 3.4 into the results in [54, §3.7], one derives a bound of $O(m^3\Delta^3)$ on the ℓ_1-norm of Graver basis vectors; the previously known bound of $O(m^m\Delta)$ can be found in Diaconis et al. [23, Theorem 1] or by modifying a proof of Eisenbrand et al. [26, Lemma 2].

The paper proceeds as follows. We begin with a proof of Proposition 3.2 because it is used to establish the equations in Theorem 3.1 and Proposition 3.3; see Section 3.2. Our new combinatorial results for bimodular matrices are given in Section 3.3. Sections 3.4, 3.5, 3.6, and 3.7 contain the proofs of Theorem 3.1, Proposition 3.3, Theorem 3.4, and Theorem 3.5, respectively.

Notation and preliminaries. We use bold font to denote vectors in dimension two or higher. $\mathbf{0}$, $\mathbf{1}$, and $\mathbb{I}_k$ denote the all-zero matrix, the all-one matrix, and the $k \times k$ identity matrix, respectively. We denote the ith standard unit vector in $\mathbb{R}^t$ by $\mathbf{e}_t^i$. For the ith standard unit vector in $\mathbb{R}^m$, we drop the subscript and write $\mathbf{e}^i$. We use $\mathbf{a}^\intercal$ to denote a row vector. We write $[\mathbf{b}^1|\cdots|\mathbf{b}^t]$ to denote a matrix with columns $\mathbf{b}^1, \ldots, \mathbf{b}^{t-1}$, and $\mathbf{b}^t$. We often partition the rows of a matrix, and it is convenient to refer to these inline; for matrices $B \in \mathbb{Z}^{r \times t}$ and $C \in \mathbb{Z}^{s \times t}$, we adopt the notation

$$(B, C) := \begin{bmatrix} B \\ C \end{bmatrix} \in \mathbb{Z}^{(r+s) \times t}.$$

A **basis** is an invertible (square) matrix. We let $\operatorname{conv} B$ and $\operatorname{span} B$ denote the convex hull of and the linear space spanned by the columns of $B \subseteq \mathbb{R}^d$, respectively. For $\mathbf{v} = (v_1, \ldots, v_t) \in \mathbb{R}^t$, we denote the support of $\mathbf{v}$ by $\operatorname{supp} \mathbf{v} := \{i = 1, \ldots, t \colon v_i \neq 0\}$. A Δ-modular matrix B with differing columns is **maximal** if there does not exist a Δ-modular matrix $B' \supsetneq B$ with differing columns.

We use **elementary operations** to refer to elementary row operations that preserve integrality. Elementary operations do not affect differing columns or Δ-modularity of a matrix. We write $B \sim B'$ if B and B' are equivalent up to elementary operations. We also freely swap columns and multiply them by -1 because these operations do not affect differing columns or Δ-modularity.

We often analyze the determinant structure of matrices with linearly dependent rows. To do this, we note that every matrix $B \in \mathbb{Z}^{m \times n}$ can be transformed via elementary operations into a matrix $(\overline{B}, \mathbf{0})$, where $\overline{B} \in \mathbb{Z}^{\mathrm{rank} B \times n}$ has full row rank. We always use $\overline{B}$ to denote a full row rank projection of B obtained via elementary operations. Elementary operations preserve linear relationships, so the following holds:

$$\text{Let } [B|\mathbf{b}] \sim \left[\begin{array}{c|c} \overline{B} & \overline{\mathbf{b}} \\ \mathbf{0} & \mathbf{0} \end{array}\right] \text{ and } \mathbf{v} \in \mathbb{R}^{|B|}. \text{ We have } B\mathbf{v} = \mathbf{b} \text{ if and only if } \overline{B}\mathbf{v} = \overline{\mathbf{b}}. \tag{3.2}$$

Consequently,

$$\text{if } A \text{ is } \Delta\text{-modular with } \mathrm{rank} A = m \text{ and } B \subseteq A, \text{ then } \overline{B} \text{ is } \Delta\text{-modular.} \tag{3.3}$$

3.2 A proof of Proposition 3.2.

We use a generalization of the lower bound construction given by Heller. Let $m, \Delta \in \mathbb{Z}_{\geqslant 1}$, and let $A \in \mathbb{Z}^{m \times n}$ consist of the following columns:

(i) $\mathbf{e}^i$ for every $i = 1, \ldots, m$.

(ii) $k\mathbf{e}^1$ for every $k = 2, \ldots, \Delta$.

(iii) $k\mathbf{e}^1 - \mathbf{e}^i$ for every $k = 1, \ldots, \Delta$ and $i = 2, \ldots, m$.

(iv) $\mathbf{e}^i - \mathbf{e}^j$ for every $2 \leqslant i < j \leqslant m$.

The following example illustrates A for $m = 4$ and $\Delta = 3$:

$$\left[\begin{array}{c|c|c|c||c|c||c|c|c|c|c|c|c|c|c||c|c|c}
1 & 0 & 0 & 0 & 2 & 3 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & -1
\end{array}\right].$$

From the definition, we see that A has differing columns, $\mathrm{rank} A = m$, and

$$|A| = m + (\Delta - 1) + (m-1)\Delta + \binom{m-1}{2} = \frac{1}{2}\left(m^2 + m\right) + m(\Delta - 1).$$

If $m = 1$, then it is easy to verify that A is Δ-modular. Assume that $m \geqslant 2$ and the proposition is true for $m - 1$. Consider a matrix $B \subseteq A$. We prove $\Delta(A) \leqslant \Delta$ by proving $|\det B| \leqslant \Delta$.

Let $\hat{A}$ be the matrix formed by the last $m - 1$ rows of A. The matrix $C \subseteq \hat{A}$ corresponding to *(iv)* form the incidence matrix of a directed graph on $m - 1$ vertices, and $\hat{A} \backslash C$ is a multiset of standard unit vectors or negatives thereof. It is well known that $\Delta(C) = 1$ and $\Delta(\hat{A}) = 1$; see [63, (4) on Page 268]. Therefore, if B contains a column of the form *(ii)*, then by expanding $\det B$ along this column and using $\Delta(\hat{A}) = 1$, we conclude $|\det B| \leqslant \Delta$.

For any column of A, if we project out one of the last $m-1$ components, then the resulting column is either $\mathbf{0}$ or it is of the form of one of *(i)*–*(iv)*, albeit in dimension $m-1$, and possibly negated. Thus, if any of the last $m-1$ rows of B contains exactly one non-zero entry, which necessarily equals ± 1, then we expand $\det B$ along this row and induct on m to conclude $|\det B| \leqslant \Delta$. In particular, if B contains a column of the form *(i)*, then $|\det B| \leqslant \Delta$.

Assume that B only contains columns of the form *(iii)* and *(iv)*, and each of the last $m-1$ rows of B contain at least two non-zero entries. The invertible matrix B must contain at least one column of the form *(iii)* otherwise the first row would be all-zero. Consider a column of the form *(iii)* and suppose B also contains a column of the form *(iv)* with overlapping support, say B contains $\mathbf{a} = k\mathbf{e}^1 - \mathbf{e}^i$ and $\mathbf{a}' = \mathbf{e}^i - \mathbf{e}^j$. The matrix $[B|\mathbf{a} + \mathbf{a}']\backslash\{\mathbf{a}'\} = [B|k\mathbf{e}^1 - \mathbf{e}^j]\backslash\{\mathbf{a}'\}$ has the same absolute determinant as B and contains one more column satisfying *(iii)* than B does. After performing this replacement at most $m-2$ more times, we can assume that B does not contain columns of the form *(i)* or *(ii)*, each of the last $m-1$ rows of B contains at least two non-zero entries, and B does not contain a column of the form *(iii)* and a column of the form *(iv)* with overlapping supports. Given that B contains a column $\mathbf{a} = k\mathbf{e}^1 - \mathbf{e}^i$ of the form *(iii)* and the ith row of B contains at least two non-zero entries, there exists another $\mathbf{a}' \in B$ whose support contains the index i. After the previous replacement steps, we know that $\mathbf{a}'$ must also be of the form *(iii)*, that is, $\mathbf{a}' = k'\mathbf{e}^1 - \mathbf{e}^i$ for some $k' \neq k$. Note that $\mathbf{a} - \mathbf{a}' = (k - k')\mathbf{e}^1$ and $[B|\mathbf{a} - \mathbf{a}']\backslash\{\mathbf{a}'\}$ contains the column $(k-k')\mathbf{e}^1$, which is of the form *(i)* or *(ii)*. Hence, $|\det B| = |\det([B|\mathbf{a} - \mathbf{a}']\backslash\{\mathbf{a}'\})| \leqslant \Delta$. □

3.3 Structural properties of bimodular matrices.

In order to motivate the results in this section, we turn to a result of Heller. Consider a TU matrix with differing columns of the form

$$A = \left[\begin{array}{c|c} 1 & \boldsymbol{\beta}^\intercal \\ \mathbf{0} & \hat{A} \end{array}\right], \tag{3.4}$$

where $\boldsymbol{\beta} \in \mathbb{Z}^{n-1}$ and $\hat{A} \in \mathbb{Z}^{(m-1)\times(n-1)}$. Although A has differing columns, the matrix $\hat{A}$ may not. After possibly multiplying columns of A by -1, suppose two non-differing columns of $\hat{A}$ are actually equal. Heller showed that the set of columns in $\hat{A}$ with multiplicity at least two is linearly independent; see [40, (ii) on page 1358]. This linear independence is crucial in his determination of $\mathfrak{c}(1, m)$. The results in this section can be viewed as a generalization of Heller's result to bimodular matrices. It is not hard to find examples where this linear independence fails to hold for bimodular matrices. Rather than linear independence, we show that the set of columns in $\hat{A}$ with multiplicity at least two can have at most one circuit after appropriate elementary operations; see Lemma 3.12.

This section is outlined as follows. First, we formally define the set M of columns with multiplicity at least two; see Equation (3.6). Next, we provide general results of bimodular matrices in Subsection 3.3.1. We argue that a bimodular matrix can have at most one non-primitive column (Lemma 3.6 **(i)**), and we analyze circuits in the absence of non-primitive columns in Subsection 3.3.2. Finally, we provide more precise structural statements about A when it contains two or three linearly independent columns whose sum is divisible by two; see Subsections 3.3.3 and 3.3.4.

Let $A \subseteq \mathbb{Z}^m$ be a maximal bimodular matrix with $\text{rank}A = m$ and differing columns. For any primitive column $\mathbf{a}^0 \in A$, we can transform $\mathbf{a}^0$ to $\mathbf{e}^1$ via elementary operations to relabel A as (3.4). The columns in (3.4) depend on the primitive column $\mathbf{a}^0$ mapped to $\mathbf{e}^1$, and we make specific choices of $\mathbf{a}^0$ in later subsections. By multiplying columns of A by -1, we assume that

$$\text{if two columns } \mathbf{b}, \mathbf{c} \in \hat{A} \text{ do not differ, then } \mathbf{b} = \mathbf{c}. \tag{3.5}$$

Assumption (3.5) implies that $\hat{A}$ contains a unique maximal set of differing columns, which we denote by $A/\mathbf{e}^1$. We note that if A is a representation of a matroid $\mathcal{M}$, then $A/\mathbf{e}^1$ is a representation of the matroid obtained from $\mathcal{M}$ by first contracting the element $\mathbf{e}^1$, then removing "loops", and finally removing "parallel" columns that are negations of each other. Note that the matrix $A/\mathbf{e}^1$ may contain a column and a dilation $\alpha\mathbf{b}$ for $|\alpha| \geqslant 2$ because we consider differing columns; this distinguishes our use of "/" from the regular "simplification" of $A/\mathbf{e}^1$ in matroid theory. Nevertheless, our use of the notation "/" is meant to evoke the common notation for matroid contraction.

A matrix $B = [\mathbf{e}^1|\mathbf{a}^1|\cdots|\mathbf{a}^{m-1}] = [(1,\mathbf{0})|(\beta_1,\mathbf{b}^1)|\cdots|(\beta_{m-1},\mathbf{b}^{m-1})] \subseteq A$ is a basis if and only if $B/\mathbf{e}^1 = [\mathbf{b}^1|\cdots|\mathbf{b}^{m-1}] \subseteq A/\mathbf{e}^1$ is a basis because $|\det B| = |\det B/\mathbf{e}^1|$. Therefore, $A/\mathbf{e}^1$ is bimodular. For each $\mathbf{b} \in \mathbb{Z}^{m-1}$, we define the **original set** of columns in A corresponding to $\mathbf{b}$ to be

$$O(\mathbf{b}) := \left\{ \begin{bmatrix} \beta \\ \mathbf{b} \end{bmatrix} \in A \right\}.$$

A column $\mathbf{a} \in A$ is said to be an **original of** $\mathbf{b} \in A/\mathbf{e}^1$ if $\mathbf{a} = (\beta, \mathbf{b})$ for some $\beta \in \mathbb{Z}$. Denote the set of columns of $A/\mathbf{e}^1$ with multiple originals by

$$M := \{\mathbf{b} \in A/\mathbf{e}^1 : |O(\mathbf{b})| \geqslant 2\}. \tag{3.6}$$

As a reminder, throughout this section we assume bimodularity and maximality of A, as well as (3.4) and (3.5).

3.3.1 General properties of A and M.

Lemma 3.6. *The matrix A satisfies the following properties:*

(i) A contains at most one non-primitive column, which needs to be of the form $2\mathbf{a}$ for some $\mathbf{a} \in \mathbb{Z}^m$. Moreover, if A only contains primitive columns, then $|O(\mathbf{0})| = 1$.

(ii) *If* $\mathbf{a} \in \mathbb{Z}^m \cap \operatorname{conv}[\mathbf{0}|A| - A]$*, then* $[\mathbf{a}|A]$ *is bimodular. In particular, if* $\mathbf{b} \in A/\mathbf{e}^1$ *and* $k := |O(\mathbf{b})|$*, then* $O(\mathbf{b}) = \{(\beta, \mathbf{b}), \ldots, (\beta + k - 1, \mathbf{b})\} \subseteq A$*.*

(iii) *For each* $\mathbf{b} \in A/\mathbf{e}^1$*, it follows that* $|O(\mathbf{b})| \leqslant 3$*.*

Proof. **(i)** Let $\alpha\mathbf{a} \in A$ be non-primitive with $\alpha \geqslant 2$ and $\mathbf{a} \in \mathbb{Z}$. Let $[\alpha\mathbf{a}|B] \subseteq A$ be a basis. We have $2 \geqslant |\det[\alpha\mathbf{a}|B]| = \alpha|\det[\mathbf{a}|B]| \geqslant \alpha$ because A is bimodular and $[\mathbf{a}|B]$ is integer valued.

If $2\mathbf{a}, 2\mathbf{c} \in A$ are distinct non-primitive columns, then they must be linearly independent because A has differing columns. Let $[2\mathbf{a}|2\mathbf{c}|B] \subseteq A$ be a basis. We have $2 \geqslant |\det[2\mathbf{a}|2\mathbf{c}|B]| \geqslant 4$, which is a contradiction.

(ii) We can write $\mathbf{a} = \sum_{i=1}^t \mathbf{c}^i v_i$, where $\mathbf{c}^1, \ldots, \mathbf{c}^t \in [\mathbf{0}|A| - A]$, $v_1, \ldots, v_t \geqslant 0$, and $\sum_{i=1}^t v_i = 1$. Fix $[\mathbf{a}^1|\cdots|\mathbf{a}^{m-1}] \subseteq A$. By multi-linearity of the determinant and the fact that $[\mathbf{0}|A| - A]$ is bimodular, it follows that

$$\left|\det\left[\ \mathbf{a}\,\middle|\,\mathbf{a}^1\,\middle|\cdots\middle|\,\mathbf{a}^{m-1}\ \right]\right| = \left|\sum_{i=1}^t v_i \det\left[\ \mathbf{c}^i\,\middle|\,\mathbf{a}^1\,\middle|\cdots\middle|\,\mathbf{a}^{m-1}\ \right]\right| \leqslant \left|\sum_{i=1}^t v_i\right| 2 = 2.$$

Hence, $[\mathbf{a}|A]$ is bimodular.

Let $\mathbf{b} \in A/\mathbf{e}^1$ with $k = |O(\mathbf{b})|$. We have $O(\mathbf{b}) = \{(\beta_1, \mathbf{b}), \ldots, (\beta_k, \mathbf{b})\} \subseteq A$ with $\beta_1 < \beta_2 < \cdots < \beta_k$ by Assumption (3.5). By the maximality of A and the previous paragraph, we have $(\hat{\beta}, \mathbf{b}) \in A$ for every $\beta_1 \leqslant \hat{\beta} \leqslant \beta_k$. Hence, $\{\beta_1, \beta_2, \ldots, \beta_k\} = \{\beta_1, \beta_1 + 2, \ldots, \beta_1 + k - 1\}$.

(iii) Assume $k := |O(\mathbf{b})| \geqslant 4$ for some $\mathbf{b} \in A/\mathbf{e}^1$. By **(ii)**, we know that $(\beta, \mathbf{b}), (\beta + 3, \mathbf{b}) \in A$ for some $\beta \in \mathbb{Z}$. Let $[B|(\beta, \mathbf{b})|(\beta + 3, \mathbf{b})] \subseteq A$ be a basis. We have

$$2 \geqslant \left|\det\left[\ B\ \middle|\ \begin{matrix}\beta\\ \mathbf{b}\end{matrix}\ \middle|\ \begin{matrix}\beta+3\\ \mathbf{b}\end{matrix}\ \right]\right| = \left|\det\left[\ B\ \middle|\ \begin{matrix}\beta\\ \mathbf{b}\end{matrix}\ \middle|\ \begin{matrix}3\\ \mathbf{0}\end{matrix}\ \right]\right| \geqslant 3,$$

which is a contradiction.

□

Recall the 'bar' notation defined in (3.2): We write $C \sim (\overline{C}, \mathbf{0}) \in \mathbb{Z}^{m \times d}$, where $\overline{C} \in \mathbb{Z}^{\operatorname{rank}C \times d}$.

Lemma 3.7. *Assume that* A *only contains primitive columns. Let* $C \subseteq M$ *be a circuit. Then*

(i) $3 \leqslant |C| \leqslant 4$*.*

(ii) *There exists a vector* $\boldsymbol{\gamma} \in \mathbb{Z}^{|C|}$ *such that* $\left|\det\left(\boldsymbol{\gamma}^\intercal, \overline{C}\right)\right| = 2$*, where* $\left(\boldsymbol{\gamma}^\intercal, \overline{C}\right) \in \mathbb{Z}^{|C| \times |C|}$*.*

(iii) *If* $\overline{C}$ *is unimodular, then* $\boldsymbol{\gamma}$ *in (ii) satisfies* $1/2 \cdot \sum_{\mathbf{a} \in (\boldsymbol{\gamma}^\intercal, C)} \mathbf{a} \in \mathbb{Z}^m$*.*

(iv) *If* $\overline{C}$ *is not unimodular, then* $|C| = 3$*. Consequently,* $\boldsymbol{\gamma}$ *in (ii) satisfies* $1/2 \cdot (\mathbf{a} + \mathbf{a}')$ *for two columns* $\mathbf{a}, \mathbf{a}' \in (\boldsymbol{\gamma}^\intercal, C)$*.*

Proof. Set $t := |C|$ and $C := [\mathbf{b}^1|\cdots|\mathbf{b}^t]$. We have $t \geqslant 3$ otherwise A contains a non-primitive column. By Cramer's rule, $\mathbf{b}^t = \sum_{i=1}^{t-1} \mathbf{b}^i v_i$ for some $(v_1, \ldots, v_{t-1}) =: \mathbf{v} \in \{\pm 2, \pm 1, \pm 1/2\}^{t-1}$. If $v_i \in \{\pm 2\}$ for some i, then swap the roles of i and t so that $\mathbf{v} \in \{\pm 1, \pm 1/2\}^{t-1}$. By (3.3), we can assume that $C = \overline{C}$ to simplify the remaining proof.

For each $i = 1, \ldots, t$, the inclusion $\mathbf{b}^i \in M$ implies $(\beta, \mathbf{b}^i) \in A$ for at least two choices of $\beta \in \mathbb{Z}$. By Lemma 3.6 **(ii)-(iii)**, we have $(\beta, \mathbf{b}^i) \in A$ for every $\beta \in \{\beta_i, \ldots, \beta_i + k_i\}$, where $k_i \in \{1, 2\}$. At least one value $e_i \in \{\beta_i, \ldots, \beta_i + k_i\}$ is even.

Define the sets

$$\begin{aligned} \Omega &:= \{\boldsymbol{\omega} = (\omega_1, \ldots, \omega_{t-1}) \colon \omega_i \in \{\beta_i, \beta_i + 1\} \ \forall\, i = 1, \ldots, t-1\}, \text{ and} \\ \Sigma &:= \{\boldsymbol{\omega}^\intercal \mathbf{v} \colon \boldsymbol{\omega} \in \Omega\}. \end{aligned}$$

Each component of $\mathbf{v}$ is non-zero, so $|\Sigma| \geqslant t \geqslant 3$. For each $\boldsymbol{\omega} \in \Omega$ and $\omega_t \in \{\beta_t, \beta_t + 1\}$,

$$\begin{aligned} \left|\det\left[\begin{array}{c|c|c|c} \omega_1 & \cdots & \omega_{t-1} & \omega_t \\ \mathbf{b}^1 & & \mathbf{b}^{t-1} & \mathbf{b}^t \end{array}\right]\right| &= \left|\det\left[\begin{array}{c|c|c|c} \omega_1 & \cdots & \omega_{t-1} & \omega_t - \boldsymbol{\omega}^\intercal \mathbf{v} \\ \mathbf{b}^1 & & \mathbf{b}^{t-1} & \mathbf{0} \end{array}\right]\right| \\ &= \left|\det\left[\ \mathbf{b}^1 \middle| \cdots \middle| \mathbf{b}^{t-1}\ \right]\right| \cdot |\omega_t - \boldsymbol{\omega}^\intercal \mathbf{v}| \leqslant 2. \end{aligned}$$

Suppose $|\det[\mathbf{b}^1|\cdots|\mathbf{b}^{t-1}]| = 1$, i.e., $\overline{C}$ is unimodular. By Cramer's rule, $\mathbf{v} \in \{\pm 1\}^{t-1}$ and $\Sigma \subseteq \mathbb{Z}$. Hence, $\omega_t - \boldsymbol{\omega}^\intercal \mathbf{v} \in \{\pm 2, \pm 1, 0\}$ for each $\omega_t \in \{\beta_t, \beta_t + 1\}$ and $\boldsymbol{\omega}^\intercal \mathbf{v} \in \Sigma$. This implies that $4 \geqslant |\Sigma| \geqslant t \geqslant 3$ and that there exists at least one choice $\hat{\boldsymbol{\omega}}^\intercal \mathbf{v}$ and $\hat{\omega}_t$ such that $|\hat{\omega}_t - \hat{\boldsymbol{\omega}}^\intercal \mathbf{v}| = 2$. From the previous equation, we see that Property **(ii)** holds with $\boldsymbol{\gamma} := (\hat{\boldsymbol{\omega}}, \hat{\omega}_t)$. We also have that

$$\sum_{\mathbf{a} \in (\boldsymbol{\gamma}^\intercal, C)} \mathbf{a} = \begin{bmatrix} \hat{\omega}_t \\ \mathbf{b}^t \end{bmatrix} - \sum_{i=1}^{t-1} v_i \begin{bmatrix} \hat{\omega}_i \\ \mathbf{b}^i \end{bmatrix} + \sum_{i=1}^{t-1} (1+v_i) \begin{bmatrix} \hat{\omega}_i \\ \mathbf{b}^i \end{bmatrix} = \begin{bmatrix} \hat{\omega}_t - \hat{\boldsymbol{\omega}}^\intercal \mathbf{v} \\ \mathbf{0} \end{bmatrix} + \sum_{i=1}^{t-1} (1+v_i) \begin{bmatrix} \hat{\omega}_i \\ \mathbf{b}^i \end{bmatrix}$$

has all even components because $\mathbf{v} \in \{\pm 1\}^{t-1}$; this implies that **(iii)** holds.

Suppose $|\det[\mathbf{b}^1|\cdots|\mathbf{b}^{t-1}]| = 2$, i.e., $\overline{C}$ is not unimodular. We have $\omega_t - \boldsymbol{\omega}^\intercal \mathbf{v} \in \{\pm 1, \pm 1/2, 0\}$ for each $\omega_t \in \{\beta_t, \beta_t + 1\}$ and $\boldsymbol{\omega}^\intercal \mathbf{v} \in \Sigma$. If $t \geqslant 4$, then $\sigma_{\max} - \sigma_{\min} \geqslant 3/2$, where $\sigma_{\min}$ and $\sigma_{\max}$ are the minimum and maximum values in Σ, respectively. Therefore, $(\beta_t + 1) - \sigma_{\min} > 1$ or $\beta_t - \sigma_{\max} < -1$, which is a contradiction. Thus, $|\Sigma| = t = 3$, and $\mathbf{v} \in \{\pm 1/2\}^{t-1}$. Furthermore, there is at least one choice $\hat{\boldsymbol{\omega}}^\intercal \mathbf{v}$ and $\hat{\omega}_t$ such that $|\hat{\omega}_t - \hat{\boldsymbol{\omega}}^\intercal \mathbf{v}| = 1$. Property **(ii)** holds with $\boldsymbol{\gamma} := (\hat{\boldsymbol{\omega}}, \hat{\omega}_t)$. We have $\mathbf{b}^3 = \mathbf{b}^1 v_1 + \mathbf{b}^2 v_2 \in \mathbb{Z}^{m-1}$, which implies $1/2 \cdot (\mathbf{b}^1 + \mathbf{b}^2) \in \mathbb{Z}^{m-1}$ because $\mathbf{v} \in \{\pm 1/2\}^2$. Similarly, $1/2 \cdot (\hat{\omega}_1 + \hat{\omega}_2) \in \mathbb{Z}$ because $\mathbf{v} \in \{\pm 1/2\}^2$, $|\hat{\omega}_3 - \hat{\boldsymbol{\omega}}^\intercal \mathbf{v}| = 1$, and $\hat{\omega}_3 \in \mathbb{Z}$. Hence, $1/2 \cdot ((\hat{\omega}_1, \mathbf{b}^1) + (\hat{\omega}_2, \mathbf{b}^2)) \in \mathbb{Z}^m$, which implies that **(iv)** holds. □

3.3.2 Circuits in M when A contains only primitive columns.

In this subsection, we assume that A only contains primitive columns and M contains a circuit; these assumptions allow us to apply Lemma 3.7. Choose $B^* \subseteq A$ satisfying

$$B^* \text{ is linearly independent and } \frac{1}{2} \cdot \sum_{\mathbf{a} \in B^*} \mathbf{a} \in \mathbb{Z}^m, \tag{3.7}$$

and minimizing $|B^*|$. The set B^* exists and $|B^*| \leqslant 4$ by Lemma 3.7 **(iii)-(iv)**. Furthermore, $2 \leqslant |B^*|$ because we assumed that A only contains primitive columns. After applying elementary operations, we assume that

$$B^* = \left[\ \mathbf{e}^1 \middle| \cdots \middle| \mathbf{e}^{|B^*|-1} \middle| \mathbf{e}^1 + \cdots + \mathbf{e}^{|B^*|-1} + 2\mathbf{e}^{|B^*|} \ \right]. \tag{3.8}$$

Lemma 3.8. *After possibly multiplying columns of A by -1, we can assume that*

$$A \cap \operatorname{span} B^* \subseteq B^* \cup \left\{ \mathbf{e}^{|B^*|} + \sum_{i=1}^{|B^*|-1} \alpha_i \mathbf{e}^i \colon \alpha_1, \ldots, \alpha_{|B^*|-1} \in \{0,1\} \right\}. \tag{3.9}$$

Furthermore, $|O(\mathbf{b})| = 1$ for each $\mathbf{b} \in B^*/\mathbf{e}^1$ and

$$M = \{\mathbf{b} \in A/\mathbf{e}^1 \colon |O(\mathbf{b})| = 2\}. \tag{3.10}$$

Proof. Set $s := |B^*| \geqslant 2$. Let $\mathbf{a} \in A \cap \operatorname{span} B^* \backslash B^*$. By Cramer's rule and the assumption that A is bimodular, we can write $\mathbf{a} = B^*\mathbf{v}$, where $\mathbf{v} = (v_1, \ldots, v_s) \in \{\pm 1, \pm 1/2, 0\}^s$. For proving (3.9), it suffices to show $v_s = 1/2$ because $\mathbf{a} = B^*\mathbf{v} \in \mathbb{Z}^m$ implies $v_i \in \{\pm 1/2\}$ for all $i = 1, \ldots, s-1$.

Set $I := \operatorname{supp}(v_1, \ldots, v_{s-1})$. Suppose $v_s = 0$; then $\mathbf{v} \in \{0, \pm 1\}^s$ and $|I| \leqslant 1$ otherwise $1/2 \cdot (\mathbf{a} + (\mathbf{e}^1 + \cdots + \mathbf{e}^{s-1} + 2\mathbf{e}^s) + \sum_{i \in \{1,\ldots,s-1\} \backslash I} \mathbf{e}^i) \in \mathbb{Z}^m$, which contradicts the minimality of B^*. However, $|I| \leqslant 1$ implies $\mathbf{a} \in B^*$, which is a contradiction. Suppose $v_s \in \{\pm 1\}$; then $\mathbf{v} \in \{0, \pm 1\}^s$ and $|I| = 0$ otherwise $1/2 \cdot (\mathbf{a} + \sum_{i \in \{1,\ldots,s-1\} \backslash I} \mathbf{e}^i) \in \mathbb{Z}^m$, which contradicts the minimality of B^*. However, $|I| = 0$ implies $\mathbf{a} = \mathbf{0}$, which contradicts that A has differing columns. Thus, $v_s = \pm 1/2$ and by possibly replacing $\mathbf{a}$ by $-\mathbf{a}$, we assume that $v_s = 1/2$. This proves (3.9).

It follows directly from (3.9) that $|O(\mathbf{b})| = 1$ for each $\mathbf{b} \in B^*/\mathbf{e}^1$. Assume to the contrary that $|O(\mathbf{b})| \geqslant 3$ for some $\mathbf{b} \in A/\mathbf{e}^1$. It follows from Lemma 3.6 **(ii)-(iii)** that $O(\mathbf{b}) = \{\mathbf{a}, \mathbf{a} + \mathbf{e}^1, \mathbf{a} + 2\mathbf{e}^1\}$ for some $\mathbf{a} \in A$. Inclusion (3.9) implies that if $\mathbf{c} \in A \cap \operatorname{span} B^*$, then $\mathbf{c} + 2\mathbf{e}^1 \notin A$. Therefore $\mathbf{a} \notin \operatorname{span} B^*$, and $C = [\mathbf{e}^2 | \cdots | \mathbf{e}^{s-1} | \mathbf{e}^1 + \cdots + \mathbf{e}^{s-1} + 2\mathbf{e}^s | \mathbf{a} | \mathbf{a} + 2\mathbf{e}^1] \subseteq A$ has linearly independent columns. Recall (3.2): we have $C \sim (\overline{C}, \mathbf{0})$, where

$$\overline{C} = \left[\ \mathbf{e}^2_{s+1} \middle| \cdots \middle| \mathbf{e}^{s-1}_{s+1} \middle| \mathbf{e}^1_{s+1} + \cdots + \mathbf{e}^{s-1}_{s+1} + 2\mathbf{e}^s_{s+1} \middle| \overline{\mathbf{a}} \middle| \overline{\mathbf{a}} + 2\mathbf{e}^1_{s+1} \ \right] \in \mathbb{Z}^{(s+1)\times(s+1)}$$

is invertible and $\left|\det \overline{C}\right| \geqslant 4$, which contradicts (3.3). □

Define the matrices

$$C^* := \left\{ \mathbf{e}^{|B^*|-1}_{m-1} + \sum_{i=1}^{|B^*|-2} \alpha_i \mathbf{e}^i_{m-1} \ : \ \alpha_1, \ldots, \alpha_{|B^*|-2} \in \{0,1\} \right\} \tag{3.11}$$

and

$$D^* := B^*/\mathbf{e}^1 = \left[\ \mathbf{e}^1_{m-1} \middle| \mathbf{e}^2_{m-1} \middle| \cdots \middle| \mathbf{e}^{|B^*|-2}_{m-1} \middle| \mathbf{e}^1_{m-1} + \cdots + \mathbf{e}^{|B^*|-2}_{m-1} + 2\mathbf{e}^{|B^*|-1}_{m-1} \ \right]. \tag{3.12}$$

By (3.9), we can assume that C^* contains all columns in $(A \cap \operatorname{span} B^*)/\mathbf{e}^1$ that have multiple originals, i.e., $M \cap \operatorname{span} D^* \subseteq C^*$.

Lemma 3.9. *If* $[\mathbf{b}|\mathbf{b}+\mathbf{d}] \subseteq M$ *for some* $\mathbf{b} \in M \backslash C^*$ *and* $\mathbf{d} \in \operatorname{span} D^*$*, then* $\mathbf{d} \in [D^*| - D^*]$.

Proof. Set $s := |B^*| - 1 \geqslant 1$. By Cramer's rule and the bimodularity of $A/\mathbf{e}^1$, we have $\mathbf{d} = D^*\mathbf{v}$, where $\mathbf{v} = (v_1, \ldots, v_s) \in \{\pm 1, \pm 1/2, 0\}^s$. Set $D^* = [\mathbf{d}^1|\cdots|\mathbf{d}^s]$. Recall (3.2):

$$[D^*|\mathbf{b}|\mathbf{b}+\mathbf{d}] \sim \left[\begin{array}{c|c|c} \overline{D^*} & \mathbf{0} & \overline{\mathbf{d}} \\ 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]$$

where $\overline{D^*} = [\overline{\mathbf{d}^1}|\cdots|\overline{\mathbf{d}^s}] = [\mathbf{e}^1_s|\cdots|\mathbf{e}^{s-1}_s|\mathbf{e}^1_s + \cdots + \mathbf{e}^{s-1}_s + 2\mathbf{e}^s_s]$ and $\overline{\mathbf{d}} = \overline{D^*}\mathbf{v}$. For every choices of $(\gamma_1, \mathbf{b})$ and $(\gamma_2, \mathbf{b}+\mathbf{d})$ in A, we have

$$\left|\det \left[\begin{array}{c|c|c|c|c|c} 0 & & 0 & 1 & \gamma_1 & \gamma_2 \\ \mathbf{d}^1 & \cdots & \mathbf{d}^{s-1} & \mathbf{d}^s & \mathbf{0} & \overline{\mathbf{d}} \\ 0 & & 0 & 0 & 1 & 1 \end{array}\right]\right| = 2|\gamma_2 - \gamma_1 - v_s| \leqslant 2.$$

If $v_s \notin \mathbb{Z}$, then $2(\gamma_2 - \gamma_1 - v_s)$ is odd and contained in $\{\pm 2, \pm 1, 0\}$. There are two choices for both γ_1 and γ_2 because $[\mathbf{b}|\mathbf{b}+\mathbf{d}] \subseteq M$. However, this means that there are at least three distinct odd values of $2(\gamma_2 - \gamma_1 - v_s)$ in $\{\pm 2, \pm 1, 0\}$, which is a contradiction. Hence, $v_s \in \mathbb{Z}$. This implies that $\mathbf{v} \in \{\pm 1, 0\}^s$ because $D^*\mathbf{v} = \mathbf{d} \in \mathbb{Z}^{m-1}$.

Set $I := \operatorname{supp}(v_1, \ldots, v_s)$. If $|I| \geqslant 2$, then $1/2 \cdot (\mathbf{d} + \sum_{i \in \{1,\ldots,s\} \backslash I} \mathbf{d}^i) \in \mathbb{Z}^{m-1}$. This implies there exist originals of $\mathbf{b}, \mathbf{b}+\mathbf{d}$, and $\mathbf{d}^i$ for each $i \in \{1, \ldots, s\} \backslash I$ that satisfy (3.7). However, there are only $s + 2 - |I| < s + 1 = |B^*|$ columns here, which contradicts the minimality of B^*. Hence, $|I| = 1$ and $\mathbf{d} \in [D^*| - D^*]$. □

Lemma 3.10. *If* $C = [\mathbf{b}^1|\cdots|\mathbf{b}^t] \subseteq M$ *is a circuit, then* $[D^*|\mathbf{b}^{j_1}|\cdots|\mathbf{b}^{j_{t-1}}]$ *contains a circuit for every choice of indices* $j_1, \ldots, j_{t-1} \in \{1, \ldots, t\}$.

Proof. Set $s := |B^*| - 1 \geqslant 1$, and set $D^* = [\mathbf{d}^1|\cdots|\mathbf{d}^s]$. Assume to the contrary that $[D^*|\mathbf{b}^1|\cdots|\mathbf{b}^{t-1}]$ has linearly independent columns. Using linear independence and (3.2), we have

$$[D^*|C] \sim \left[\begin{array}{c|c} \overline{D} & \mathbf{0} \\ \mathbf{0} & \overline{C} \\ \mathbf{0} & \mathbf{0} \end{array}\right],$$

where $\overline{D^*} = [\overline{\mathbf{d}^1}|\cdots|\overline{\mathbf{d}^s}] = [\mathbf{e}^1_s|\cdots|\mathbf{e}^{s-1}_s|\mathbf{e}^1_s + \cdots + \mathbf{e}^{s-1}_s + 2\mathbf{e}^s_s]$ and $\overline{C} \in \mathbb{Z}^{(t-1)\times t}$. By Lemma 3.7 **(ii)**, there exists some $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_t) \in \mathbb{Z}^t$ such that $(\boldsymbol{\gamma}^\intercal, C) \subseteq A$ and $|\det(\boldsymbol{\gamma}^\intercal, \overline{C})| = 2$. Therefore,

$$\left|\det \left[\begin{array}{c|c|c|c|c} 0 & & 0 & 1 & \boldsymbol{\gamma}^\intercal \\ \overline{\mathbf{d}^1} & \cdots & \overline{\mathbf{d}^{s-1}} & \overline{\mathbf{d}^s} & \mathbf{0} \\ 0 & & 0 & 0 & \overline{C} \end{array}\right]\right| = \left|\det \overline{D^*}\right| \cdot \left|\det(\boldsymbol{\gamma}^\intercal, \overline{C})\right| = 4,$$

which contradicts (3.3). □

Lemma 3.11. *If $C \subseteq M$ is a circuit and $|C\backslash C^*| \geqslant 2$, then $C\backslash C^* = [\mathbf{b}|\mathbf{b}+\mathbf{d}]$ for some $\mathbf{b} \in M\backslash C^*$ and $\mathbf{d} \in D^*$. Given that $|C| \in \{3,4\}$ from Lemma 3.7 (i), it follows that $C \cap C^* \neq \varnothing$.*

Proof. Set $s := |B^*| - 1 \geqslant 1$ and $t := |C|$. By Lemma 3.10, we know that $\operatorname{rank}[D^*|C] \leqslant \operatorname{rank} D^* + \operatorname{rank} C - 1 = s + t - 2$. Also, by $|C\backslash C^*| \geqslant 2$ and $M \cap \operatorname{span} D^* \subseteq C^*$, we know that $\operatorname{rank}[D^*|C] \geqslant \operatorname{rank} D^* + 1 = s + 1$. By Lemma 3.7 **(i)**, we have $t \in \{3,4\}$. In both cases, we argue that $|C\backslash C^*| = 2$ and $\operatorname{rank}[D^*|C] = s+1$. It will then follow from Lemma 3.9 and after possibly multiplying the column by -1, that $C\backslash C^* = [\mathbf{b}|\mathbf{b}+\mathbf{d}]$ for $\mathbf{b} \in M\backslash C^*$ and $\mathbf{d} \in D^*$.

Assume that $t = 3$; then $\operatorname{rank}[D^*|C] = s+1$. If $|C\backslash C^*| = 3$, then $[D^*|C] \sim [D^*|\mathbf{e}^{s+1}_{m-1}|\mathbf{e}^{s+1}_{m-1} + \mathbf{d}^1|\mathbf{e}^{s+1}_{m-1} + \mathbf{d}^2]$ for distinct $\mathbf{d}^1, \mathbf{d}^2 \in \operatorname{span} D^*$. By Lemma 3.9, we have $\mathbf{d}^1, \mathbf{d}^2 \in [D^*| - D^*]$. The matrix $[\mathbf{e}^{s+1}_{m-1} + \mathbf{d}^1|\mathbf{e}^{s+1}_{m-1} + \mathbf{d}^2]$ is contained in M but $\mathbf{d}^2 - \mathbf{d}^1 \notin [D^*| - D^*]$ for any two distinct columns in D^*; this contradicts Lemma 3.9. Therefore, $|C\backslash C^*| = 2$ when $t = 3$.

Assume that $t = 4$. If $\operatorname{rank}[D^*|C] = s+1$, then $|C\backslash C^*| = 2$ as in the case $t = 3$. Assume to the contrary that $\operatorname{rank}[D^*|C] = s+2$. By (3.2), we can assume that

$$[D^*|C] \sim \left[\begin{array}{c|c|c|c} \overline{D^*} & \mathbf{0} & \overline{D^*}\mathbf{u}^3 & \overline{D^*}\mathbf{u}^4 \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{v}^3 & \mathbf{v}^4 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right],$$

where $\overline{D^*} = [\mathbf{e}^1_s|\cdots|\mathbf{e}^{s-1}_s|\mathbf{e}^1_s + \cdots + \mathbf{e}^{s-1}_s + 2\mathbf{e}^s_s]$, $\mathbf{u}^3 = (u^3_1, \ldots, u^3_s)$ and $\mathbf{u}^4$ are contained in $\{0, \pm 1/2, \pm 1\}^s$, and $\mathbf{v}^3 = (v^3_1, v^3_2)$ and $\mathbf{v}^4 = (v^4_1, v^4_2)$ are contained in $\{0,\pm 1\}^2$. Lemma 3.7 **(iv)** implies C is unimodular because $t = 4$, and Lemma 3.7 **(iii)** implies $1/2 \cdot (\mathbf{1} + \mathbf{v}^3 + \mathbf{v}^4) \in \mathbb{Z}^2$. We derive a contradiction in two cases.

First, assume that $\mathbf{v}^3 \in \{\pm 1\}^2$ or $\mathbf{v}^4 \in \{\pm 1\}^2$. Say $\mathbf{v}^3 \in \{\pm 1\}^2$; then $\mathbf{v}^4 = \mathbf{0}$ and $|C\backslash C^*| = 3$. For $\delta \in \mathbb{Z}$ and $\boldsymbol{\gamma} = (\gamma_1, \gamma_2) \in \mathbb{Z}^2$, the matrix

$$E(\boldsymbol{\gamma}, \delta) := \left[\begin{array}{c|c|c|c|c|c} 0 & & 0 & 1 & \boldsymbol{\gamma}^\intercal & \delta \\ \overline{\mathbf{d}^1} & \cdots & \overline{\mathbf{d}^{s-1}} & \overline{\mathbf{d}^s} & \mathbf{0} & \overline{D^*}\mathbf{u}^3 \\ \mathbf{0} & & \mathbf{0} & \mathbf{0} & \mathbb{I}_2 & \mathbf{v}^3 \end{array}\right] \in \mathbb{Z}^{(s+3)\times(s+3)}$$

has an absolute determinant of $2|\delta - \boldsymbol{\gamma}^\intercal \mathbf{v}^3 - u^3_s| = |2(\delta - \boldsymbol{\gamma}^\intercal\mathbf{v}^3) - 2u^3_s| \in \{0,1,2\}$. Given that $C \subseteq M$, there are two choices for each of γ_1, γ_2, and δ such that $(E(\boldsymbol{\gamma},\delta), \mathbf{0}) \subseteq A$. Thus, there are at least four distinct values of $2(\delta - \boldsymbol{\gamma}^\intercal\mathbf{v}^3) - 2u^3_s$ in $\{\pm 2, \pm 1, 0\}$ that have the same parity, namely, the same parity as $2u^3_s$. However, this is a contradiction.

Second, assume that $\mathbf{v}^3 \in \{\pm \mathbf{e}^1_2\}$ and $\mathbf{v}^4 \in \{\pm \mathbf{e}^2_2\}$ or that $\mathbf{v}^3 \in \{\pm \mathbf{e}^2_2\}$ and $\mathbf{v}^4 \in \{\pm \mathbf{e}^1_2\}$; then $|C\backslash C^*| = 4$. By Lemma 3.9, we have $\overline{D^*}\mathbf{u}^3, \overline{D^*}\mathbf{u}^4 \in [\overline{D^*}| - \overline{D^*}]$. By possibly multiplying the column by -1, we assume $\overline{D^*}\mathbf{u}^3 \in \overline{D^*}$. Set $F := D^*\backslash\{D^*\mathbf{u}^3\}$ and $\overline{F} := \overline{D^*}\backslash\{\overline{D^*}\mathbf{u}^3\}$. By Lemma 3.7 **(ii)**, there exists a vector $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_4) \in \mathbb{Z}^4$ such that $(\boldsymbol{\gamma}^\intercal, C) \subseteq A$ and $2 = |\det(\boldsymbol{\gamma}^\intercal, \overline{C})|$. Let $\boldsymbol{\delta} \in \mathbb{Z}^{s-1}$ be such that $(\boldsymbol{\delta}^\intercal, F) \subseteq A$.

We have

$$\left|\det\begin{bmatrix} \boldsymbol{\delta}^\intercal & \gamma_1 & \gamma_2 & \gamma_3 & \gamma_4 \\ \overline{F} & \mathbf{0} & \mathbf{0} & \overline{D^*\mathbf{u}^3} & \overline{D^*\mathbf{u}^4} \\ \mathbf{0} & \mathbf{e}_2^1 & \mathbf{e}_2^2 & \mathbf{v}^3 & \mathbf{v}^4 \end{bmatrix}\right| = \left|\det\begin{bmatrix} \overline{F} & \mathbf{0} & \overline{D^*\mathbf{u}^3} \\ \mathbf{0} & \mathbb{I}_2 & \mathbf{v}^3 \end{bmatrix}\right| \cdot \left|\det(\boldsymbol{\gamma}^\intercal, \overline{C})\right|$$
$$= 2\left|\det \overline{D^*}\right| = 4,$$

which is a contradiction.

Therefore, $|C\backslash C^*| = 2$ and $\operatorname{rank}[D^*|C] = s+1$ when $t = 4$. □

We arrive at our main result in this section. We repeat assumptions for the reader.

Lemma 3.12. *Assume that A only contains primitive columns and let $C^1 \subseteq M$ be a circuit. Choose $B^* \subseteq A$ that satisfies* (3.7) *and minimizes $|B^*|$, and assume that B^* has the form* (3.8). *Recall C^* from* (3.11) *and D^* from* (3.12).

(i) If $|B^*| = 2$, then $C^1 = [C^*|\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_{m-1}]$ for some $\mathbf{b} \in A/\mathbf{e}^1$. Furthermore, C^1 is the unique circuit in M, so $|M| \leqslant m$.

(ii) If $|B^*| = 3$, then $C^1 = [C^*|\mathbf{b}|\mathbf{b} + \mathbf{d}]$ for some $\mathbf{b} \in A/\mathbf{e}^1$ and $\mathbf{d} \in D^*$. Furthermore, C^1 is the unique circuit in M, so $|M| \leqslant m$.

(iii) If $|B^*| = 4$, then either $C^1 = C^*$, or $|C^1 \cap C^*| = 2$ and $|C^1| = 4$. Moreover, if M contains multiple circuits, say $C^1 \neq C^*$, then M contains precisely three circuits: C^1, C^*, and the symmetric difference $C^1 \triangle C^*$. Regardless of the number of circuits, $|M| \leqslant m + 1$.

Proof. We have $|C^1| \in \{3,4\}$ by Lemma 3.7 **(i)**.

(i) Note that $C^* = [\mathbf{e}^1_{m-1}]$, so $|C^1\backslash C^*| \geqslant 2$. It follows from Lemma 3.11 that $|C^1\backslash C^*| = 2$ and $C\backslash C^* = [\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_{m-1}]$ for some $\mathbf{b} \in M\backslash C^*$. Therefore, $C^1 = [C^*|\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_{m-1}]$.

If M contains another circuit C^2, then $C^2 = [C^*|\mathbf{b}'|\mathbf{b}' + 2\mathbf{e}^1_{m-1}]$ with $\mathbf{b} \neq \mathbf{b}'$. The column $\mathbf{e}^1_{m-1}$ is linearly dependent on $[\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_{m-1}]$ and on $[\mathbf{b}'|\mathbf{b}' + 2\mathbf{e}^1_{m-1}]$. Hence, $[\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_{m-1}|\mathbf{b}'|\mathbf{b}' + 2\mathbf{e}^1_{m-1}] \subseteq M\backslash C^*$ contains a circuit C^3. However, $|C^3\backslash C^*| = |C^3| \geqslant 3$ because $C^3 \subseteq M\backslash C^*$; this contradicts Lemma 3.11.

(ii) Note that $C^* = [\mathbf{e}^2_{m-1}|\mathbf{e}^1_{m-1} + \mathbf{e}^2_{m-1}]$, so $|C^1\backslash C^*| \geqslant 1$. If $|C^1\backslash C^*| = 1$, then $C^1 \subseteq \operatorname{span} C^* = \operatorname{span} D^*$, which contradicts $M \cap \operatorname{span} D^* \subseteq C^*$. Thus, $|C^1\backslash C^*| \geqslant 2$. It follows from Lemma 3.11 that $|C^1\backslash C^*| = 2$ and $C^1\backslash C^* = [\mathbf{b}|\mathbf{b}+\mathbf{d}]$ for some $\mathbf{b} \in M\backslash C^*$ and $\mathbf{d} \in D^*$. Hence, $C^1 = [C^*|\mathbf{b}|\mathbf{b}+\mathbf{d}]$ because $\operatorname{rank}[\mathbf{e}^2_{m-1}|\mathbf{b}|\mathbf{b} + \mathbf{d}] = \operatorname{rank}[\mathbf{e}^1_{m-1} + \mathbf{e}^2_{m-1}|\mathbf{b}|\mathbf{b} + \mathbf{d}] = 3$.

If M contains another circuit C^2, then $C^2 = [C^*|\mathbf{b}'|\mathbf{b}' + \mathbf{d}']$ for some $\mathbf{b}' \in M\backslash C^*$ and $\mathbf{d}' \in D^*$. We know that $[\mathbf{b}|\mathbf{b}+\mathbf{d}|\mathbf{b}'|\mathbf{b}'+\mathbf{d}'] \subseteq M\backslash C^*$ does not contain a circuit because such a circuit would not satisfy Lemma 3.11. Therefore, $\operatorname{rank}[\mathbf{b}|\mathbf{b}+\mathbf{d}|\mathbf{b}'|\mathbf{b}'+\mathbf{d}'] = 4$. However, $\mathbf{e}^1_{m-1}+\mathbf{e}^2_{m-1}$ is linearly dependent on $[\mathbf{e}^2_{m-1}|\mathbf{b}|\mathbf{b} + \mathbf{d}]$ and on $[\mathbf{e}^2_{m-1}|\mathbf{b}'|\mathbf{b}' + \mathbf{d}']$, which implies that $[\mathbf{e}^2_{m-1}|\mathbf{b}|\mathbf{b} + \mathbf{d}|\mathbf{b}'|\mathbf{b}' + \mathbf{d}']$ is a circuit with five columns; this contradicts Lemma 3.7 **(i)**.

(iii) If $|C^1| = 3$ and C^1 is unimodular, then $|B^*| \leqslant 3$ by Lemma 3.7 **(iii)**, which contradicts the minimality of $|B^*|$ in Case **(iii)**. If $|C^1| = 3$ and C^1 is not unimodular, then $|B^*| \leqslant 2$ by Lemma 3.7 **(iv)**, which again contradicts the minimality of $|B^*|$ in this case. Hence, $|C^1| = 4$.

Suppose $C^1 \neq C^*$; then $|C^1 \backslash C^*| \geqslant 1$ because both matrices are circuits. Recall $M \cap \operatorname{span} D^* \subseteq C^*$; see the sentence after (3.12). If $|C^1 \backslash C^*| = 1$, then $C^1 \subseteq M \cap \operatorname{span} C^* = M \cap \operatorname{span} D^* \subseteq C^*$ because C^1 is a circuit; this contradicts $|C^1 \backslash C^*| = 1$. It then follows from Lemma 3.11 that $|C^1 \backslash C^*| = 2$. Hence, $C^1 = [\mathbf{c}^1|\mathbf{c}^2|\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1]$ for some $\mathbf{c}^1, \mathbf{c}^2 \in C^*$, $\mathbf{b}^1 \in M \backslash C^*$, and $\mathbf{d}^1 \in D^*$. We have

$$[C^*|\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1] \sim \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 1 & 0 & d_1^1 \\ 0 & 0 & 1 & 1 & 0 & d_2^1 \\ 1 & 1 & 1 & 1 & 0 & d_3^1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \tag{3.13}$$

where $\overline{\mathbf{d}} := (d_1^1, d_2^1, d_3^1) \in \overline{D^*} = [(1,0,0)|(0,1,0)|(1,1,2)]$.

Given that C^1 is a circuit, $\mathbf{d}^1$ linearly depends on $\mathbf{c}^1$ and $\mathbf{c}^2$. From this and (3.13), we can determine $[\mathbf{c}^1|\mathbf{c}^2]$ (the index sets refer to the matrix on the right hand side of (3.13)):

$$\begin{array}{l} \text{If } \overline{\mathbf{d}} = (1,0,0), \text{ then } [\mathbf{c}^1|\mathbf{c}^2] \text{ is indexed by } \{1,2\} \text{ or } \{3,4\}. \\ \text{If } \overline{\mathbf{d}} = (0,1,0), \text{ then } [\mathbf{c}^1|\mathbf{c}^2] \text{ is indexed by } \{1,3\} \text{ or } \{2,4\}. \\ \text{If } \overline{\mathbf{d}} = (1,1,2), \text{ then } [\mathbf{c}^1|\mathbf{c}^2] \text{ is indexed by } \{1,4\} \text{ or } \{2,3\}. \end{array} \tag{3.14}$$

Claim 3.13. *If $C^2 \subseteq M$ is a circuit and $C^2 \neq C^*$, then $C^1 \backslash C^* = C^2 \backslash C^*$.*

Proof of Claim. Assume to the contrary that there exists a circuit $C^2 \subseteq M$ with $C^2 \neq C^*$ and $C^1 \backslash C^* \neq C^2 \backslash C^*$. As was the case with C^1, we can write $C^2 = [\mathbf{c}^3|\mathbf{c}^4|\mathbf{b}^2|\mathbf{b}^2 + \mathbf{d}^2]$ for some $\mathbf{c}^3, \mathbf{c}^4 \in C^*, \mathbf{b}^2 \in M \backslash C^*$ and $\mathbf{d}^2 \in D^*$. We know that $[\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1] \neq [\mathbf{b}^2|\mathbf{b}^2 + \mathbf{d}^2]$ because $C^1 \backslash C^* \neq C^2 \backslash C^*$. Also, $[\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1|\mathbf{b}^2|\mathbf{b}^2 + \mathbf{d}^2] \subseteq M \backslash C^*$ does not contain a circuit because such a circuit would not satisfy Lemma 3.11. Thus, $\operatorname{rank}[\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1|\mathbf{b}^2|\mathbf{b}^2 + \mathbf{d}^2] = 4$ and $\mathbf{d}^1 \neq \mathbf{d}^2$. It follows from (3.14) that $\{\mathbf{c}^1, \mathbf{c}^2\} \backslash \{\mathbf{c}^3, \mathbf{c}^4\} \neq \varnothing$; say $\mathbf{c}^1 \notin \{\mathbf{c}^3, \mathbf{c}^4\}$. Similarly, we can assume that $\mathbf{c}^3 \notin \{\mathbf{c}^1, \mathbf{c}^2\}$. Consequently, any three columns of $[\mathbf{d}^1|\mathbf{d}^2|\mathbf{c}^1|\mathbf{c}^3]$ are linearly independent. However, this implies $[\mathbf{b}^1|\mathbf{b}^1 + \mathbf{d}^1|\mathbf{b}^2|\mathbf{b}^2 + \mathbf{d}^2|\mathbf{c}^1|\mathbf{c}^3]$ is a circuit of cardinality six, which contradicts Lemma 3.7 **(i)**. ◇

Recall $|M \cap C^*| \geqslant 2$. By Claim 3.13, every circuit in M is contained in (3.13). By Lemma 3.11, any circuit in M besides C^* uses columns 5 and 6 alongside two of the first four columns. This shows that either M only contains one circuit, namely C^1, or M contains the three circuits C^1, C^*, and $C^1 \triangle C^*$.

Suppose $|M \cap C^*| \leqslant 3$. Hence, C^* is not contained in M. For any two columns in $M \cap C^*$ and any choice of (d_1^1, d_2^1, d_3^1), there exists at most one pair of columns in $M \cap C^*$ that form a circuit with columns 5 and 6; see (3.14). Hence, M contains at most one circuit, so $|M| \leqslant m$.

Suppose $|M \cap C^*| = 4$; then $M \cap C^* = C^*$. Let $\mathbf{c} \in M \cap C^*$. The matrix $M \backslash \{\mathbf{c}\}$ satisfies $|M \cap C^* \backslash \{\mathbf{c}\}| = 3$. Therefore, $|M \backslash \{\mathbf{c}\}| \leqslant m$ from the previous paragraph. Hence, $|M| \leqslant m + 1$.

□

3.3.3 Additional structural properties when $|B^*| = 2$.

Lemma 3.14. *Assume that A only contains primitive columns. Choose $B^* \subseteq A$ that satisfies* (3.7) *and minimizes $|B^*|$, and assume that B^* has the form* (3.8)*. If $|B^*| = 2$, $m \geqslant 3$ and $|M| = m$, then*

$$|A| \leqslant \frac{1}{2}\left(m^2 + m\right) + 3. \tag{3.15}$$

Proof. The assumption $|M| = m$ implies that M contains a circuit, so we can apply Lemma 3.12. Suppose $m = 3$. The following claim shows that (3.15) holds.

Claim 3.15.

$$A \sim \left[\mathbf{e}^1|\mathbf{e}^1 + 2\mathbf{e}^2|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2|\mathbf{e}^3|\mathbf{e}^1 + \mathbf{e}^3|\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3|2\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3|\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3\right]$$

Proof of Claim. By Lemma 3.12 **(i)** and (3.11), the circuit has the form $[\mathbf{e}^1_2|\mathbf{b}|\mathbf{b} + 2\mathbf{e}^1_2]$ for some $\mathbf{b} \in A/\mathbf{e}^1$. After elementary operations, we have that A is a superset of

$$\left[\begin{array}{c|c|c|c|c|c|c} B^* & \beta_0 & \beta_0+1 & \beta_1 & \beta_1+1 & \beta_2 & \beta_2+1 \\ & \mathbf{e}^1_2 & \mathbf{e}^1_2 & \mathbf{b} & \mathbf{b} & \mathbf{b}+2\mathbf{e}^1_2 & \mathbf{b}+2\mathbf{e}^1_2 \end{array}\right]$$
$$\sim \left[\begin{array}{c|c|c|c|c|c|c|c} 1 & 1 & \beta_0 & \beta_0+1 & \beta_1 & \beta_1+1 & \beta_2 & \beta_2+1 \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right].$$

We assume, without loss of generality, that the equivalence in the latter displayed equation is an equation. We must have $\beta_0 = 0$ because it is the midpoint of the columns of B^*; see Lemma 3.6 **(ii)**. By subtracting the third row from the first row β_1 many times, we assume $\beta_1 = 0$; we then see that $2|\beta_2 - 2| \leqslant 2$ using bimodularity with columns 2, 6, and 7 and that $2|\beta_2| \leqslant 2$ using bimodularity with columns 2, 5, and 8. Hence, $\beta_2 = 1$. By Lemma 3.6 **(ii)** and the maximality of A, it follows that $1/2 \cdot ((\beta_2 + 1, 2, 1) + (\beta_1, 0, 1)) = (1, 1, 1) \in A$. Thus, A has at least $\mathfrak{c}(2, 3) = 9$ columns.

The following equation follows from the definition of $A/\mathbf{e}^1$:

$$|A| = |O(\mathbf{0})| + |A/\mathbf{e}^1| + \sum_{\mathbf{b} \in A/\mathbf{e}^1} \big(|O(\mathbf{b})| - 1\big).$$

By using (3.10) and Lemma 3.6 **(i)**, we see that

$$|A| = |O(\mathbf{0})| + |A/\mathbf{e}^1| + \sum_{\mathbf{b} \in A/\mathbf{e}^1} \big(|O(\mathbf{b})| - 1\big) = 1 + |A/\mathbf{e}^1| + |M|.$$

The matrix $A/\mathbf{e}^1 \subseteq \mathbb{Z}^2$ is bimodular and contains $2\mathbf{e}^1_2$ and $\mathbf{e}^1_2$. It is quickly verified that (after multiplying columns by -1), we have $|A/\mathbf{e}^1| \leqslant 5$; see also the proof of Proposition 3.3 for an argument of this. Hence,

$$|A| = 1 + |A/\mathbf{e}^1| + |M| \leqslant 1 + 5 + 3 = 9.$$

Thus, $|A| = 9$ and A can be transformed via elementary operations to the form in Claim 3.15. ◇

Suppose $m \geqslant 4$. It follows from Lemma 3.12 **(i)** that $M = [\mathbf{e}^1_{m-1}|\mathbf{b}^1|\cdots|\mathbf{b}^{m-1}]$ and M contains exactly one circuit $[\mathbf{e}^1_{m-1}|\mathbf{b}^1|\mathbf{b}^2] = [\mathbf{e}^1_{m-1}|\mathbf{b}^1|\mathbf{b}^1 + 2\mathbf{e}^1_{m-1}]$. Hence,

$$E := \left[\ \mathbf{b}^1 \,\middle|\, \mathbf{b}^2 \,\middle|\, \cdots \,\middle|\, \mathbf{b}^{m-1}\ \right]$$

is a basis and $|\det E| = |\det[\mathbf{b}^1|\mathbf{b}^2|\cdots|\mathbf{b}^{m-1}]| = |\det[\mathbf{b}^1|2\mathbf{e}^1_{m-1}|\cdots|\mathbf{b}^{m-1}]| = 2$. Observe that

$$E \sim \left[\ \mathbf{e}^2_{m-1} \,\middle|\, \mathbf{e}^2_{m-1} + 2\mathbf{e}^1_{m-1} \,\middle|\, \mathbf{e}^3_{m-1} \,\middle|\, \cdots \,\middle|\, \mathbf{e}^{m-1}_{m-1}\ \right],$$

so if $E\mathbf{w} \in A/\mathbf{e}^1$ for some $\mathbf{w} = (w_1, \ldots, w_{m-1}) \in \mathbb{R}^{m-1}$, then $w_3, \ldots, w_{m-1} \in \{-1, 0, 1\}$.

For $i = 1, \ldots, m-1$, let $\beta_i \in \mathbb{Z}$ be such that $(\beta_i, \mathbf{b}^i), (\beta_i + 1, \mathbf{b}^i) \in A$. Define

$$\Gamma := \{\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-1})\colon \gamma_i \in \{\beta_i, \beta_i + 1\}\ \forall\ i = 1, \ldots, m-1\}.$$

For each $(\beta, \mathbf{b}) \in A$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_{m-1}) \in \Gamma$, Cramer's rule and bimodularity of $A/\mathbf{e}^1$ imply $E^{-1}\mathbf{b} =: \mathbf{v} = (v_1, \ldots, v_{m-1}) \in \{\pm 1, \pm 1/2, 0\}^{m-1}$. Furthermore,

$$\left|\det\left[\begin{array}{c|c} \boldsymbol{\gamma}^\intercal & \beta \\ E & \mathbf{b} \end{array}\right]\right| = \left|\det\left[\begin{array}{c|c} \boldsymbol{\gamma}^\intercal & \beta - \boldsymbol{\gamma}^\intercal\mathbf{v} \\ E & \mathbf{0} \end{array}\right]\right| = 2|\beta - \boldsymbol{\gamma}^\intercal\mathbf{v}| \leqslant 2,$$

so $|\beta - \boldsymbol{\gamma}^\intercal\mathbf{v}| \leqslant 1$. As $v_3, \ldots, v_{m-1} \in \{-1, 0, 1\}$, the maximum $\sigma_{\max}$ and minimum $\sigma_{\min}$ of $\{\beta - \boldsymbol{\gamma}^\intercal\mathbf{v}\colon \boldsymbol{\gamma} \in \Gamma\}$ satisfy $2 \geqslant \sigma_{\max} - \sigma_{\min} \geqslant 1/2 \cdot |\operatorname{supp}(v_1, v_2)| + |\operatorname{supp}(v_3, \ldots, v_{m-1})|$. Thus, we have $|\operatorname{supp}(v_3, \ldots, v_{m-1})| \leqslant 2$ because $|\beta - \boldsymbol{\gamma}^\intercal\mathbf{v}| \leqslant 1$ holds for all $\boldsymbol{\gamma} \in \Gamma$. This leads to a natural partition of A. For $j = 0, 1, 2$, define

$$A^j := \left\{(\beta, \mathbf{b}) \in A\colon E^{-1}\mathbf{b} = (v_1, \ldots, v_{m-1}) \text{ with } |\operatorname{supp}(v_3, \ldots, v_{m-1})| = j\right\}.$$

We have $M \subseteq [A^0|A^1]$ because $M = [\mathbf{e}^1_{m-1}|\mathbf{b}^1|\cdots|\mathbf{b}^{m-1}]$ and $[\mathbf{e}^1_{m-1}|\mathbf{b}^1|\mathbf{b}^2]$ is a circuit.

Claim 3.16.

$$A^0 \sim [\mathbf{e}^1|\mathbf{e}^1 + 2\mathbf{e}^2|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2|\mathbf{e}^3|\mathbf{e}^1 + \mathbf{e}^3|\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3|2\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3|\mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3]$$

Proof of Claim. By definition, $A^0 = A \cap \operatorname{span}\{\mathbf{e}^1, (\beta_1, \mathbf{b}^1), (\beta_2, \mathbf{b}^1 + 2\mathbf{e}^1_{m-1})\}$ and $\operatorname{rank} A^0 = 3$. Recall (3.2): $A^0 \sim (\overline{A^0}, \mathbf{0})$, where $\overline{A^0}$ is a full row rank bimodular matrix. One such sequence of elementary operations maps $\mathbf{e}^1$ to $\mathbf{e}^1_3$, $\mathbf{e}^2$ to $\mathbf{e}^2_3$, and $\mathbf{b}^1$ to $\mathbf{e}^3_3$. By Claim 3.15, we know that $|A^0| = 9$ and A^0 can be transformed via elementary operations to the form described in Claim 3.16. ◇

Claim 3.17. *For each* $i = 3, \dots, m-1$, *there are at most four columns* $(\beta, \mathbf{b}) \in A$ *such that* $E^{-1}\mathbf{b} = (v_1, \dots, v_{m-1})$ *satisfies* $\operatorname{supp}(v_3, \dots, v_{m-1}) = \{i\}$. *Consequently,* $|A^1| \leqslant 4(m-3)$.

Proof of Claim. Following Claim 3.16, we assume

$$A^0 = [\mathbf{e}^1|\mathbf{e}^1+2\mathbf{e}^2|\mathbf{e}^2|\mathbf{e}^1+\mathbf{e}^2|\mathbf{e}^3|\mathbf{e}^1+\mathbf{e}^3|\mathbf{e}^1+2\mathbf{e}^2+\mathbf{e}^3|2\mathbf{e}^1+2\mathbf{e}^2+\mathbf{e}^3|2\mathbf{e}^1+\mathbf{e}^2+\mathbf{e}^3]. \quad (3.16)$$

Set

$$F^i := \left\{(\beta, \mathbf{b}) \in A \colon E^{-1}\mathbf{b} = (v_1, \dots, v_{m-1}) \text{ satisfies } \operatorname{supp}(v_3, \dots, v_{m-1}) = \{i\}\right\}.$$

We claim that

$$\mathbf{a} - \mathbf{a}' \in [A^0| - A^0] \text{ for every pair of distinct columns } \mathbf{a}, \mathbf{a}' \in F^i. \quad (3.17)$$

Assume to the contrary that (3.17) is violated by some $\mathbf{a}, \mathbf{a}' \in F^i$. Recall (3.2): $[A^0|\mathbf{a} - \mathbf{a}'] \sim (\overline{F}, \mathbf{0})$, where $\overline{F}$ has full row rank and differing columns. We have $\operatorname{rank}\overline{F} = 3$ because $\mathbf{a}, \mathbf{a}' \in F^i$. Claim 3.15 established that a rank-3 bimodular matrix A containing A^0 has at most nine differing columns. It follows that $\overline{F}$ is not bimodular. In particular, there exists a basis $\left[\overline{\mathbf{a} - \mathbf{a}'}|\overline{\mathbf{c}}|\overline{\mathbf{d}}\right] \subseteq \overline{F}$ such that $|\det\left[\overline{\mathbf{a} - \mathbf{a}'}|\overline{\mathbf{c}}|\overline{\mathbf{d}}\right]| \geqslant 3$. By (3.3), any basis in A containing $[\mathbf{a}|\mathbf{a}'|\mathbf{c}|\mathbf{d}]$ has an absolute determinant of at least three, which contradicts that A is bimodular. This shows that (3.17) is true.

Note that F^i contains $(\beta_i, \mathbf{b}^i)$ and $(\beta_i + 1, \mathbf{b}^i)$. Let $\mathbf{a}, \mathbf{a}' \in F^i \backslash \{(\beta_i, \mathbf{b}^i), (\beta_i + 1, \mathbf{b}^i)\}$. The following are columns of $[A^0| - A^0]$ according to (3.17): $\mathbf{a} - (\beta_i, \mathbf{b}^i)$, $\mathbf{a} - (\beta_i + 1, \mathbf{b}^i) = \mathbf{a} - (\beta_i, \mathbf{b}^i) - \mathbf{e}^1$, $\mathbf{a}' - (\beta_i, \mathbf{b}^i)$, and $\mathbf{a}' - (\beta_i + 1, \mathbf{b}^i) = \mathbf{a}' - (\beta_i, \mathbf{b}^i) - \mathbf{e}^1$. From (3.16) and the assumption that A only contains primitive columns, it follows that

$$\mathbf{a} - \begin{bmatrix} \beta_i \\ \mathbf{b}^i \end{bmatrix}, \ \mathbf{a}' - \begin{bmatrix} \beta_i \\ \mathbf{b}^i \end{bmatrix} \in \left\{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^3, -(\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3), 2\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3, -\mathbf{e}^2, -\mathbf{e}^3\right\}.$$

Furthermore, $\mathbf{a} - \mathbf{a}' = (\mathbf{a} - (\beta_i, \mathbf{b}^i)) - (\mathbf{a}' - (\beta_i, \mathbf{b}^i))$ is a column of $[A^0| - A^0]$. Define $S^1 := \{\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^3, -(\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3)\}$ and $S^2 := \{2\mathbf{e}^1 + 2\mathbf{e}^2 + \mathbf{e}^3, -\mathbf{e}^2, -\mathbf{e}^3\}$. It is quickly checked that if both $(\mathbf{a} - (\beta_i, \mathbf{b}^i))$ and $(\mathbf{a}' - (\beta_i, \mathbf{b}^i))$ are in S^1 or both are in S^2, then $\mathbf{a} - \mathbf{a}' \notin [A^0| - A^0]$. Hence, there are at most two columns $F^i \backslash \{(\beta_i, \mathbf{b}^i), (\beta_i + 1, \mathbf{b}^i)\}$. Equivalently, $|F^i| \leqslant 4$. ◇

Claim 3.18. $|A^2| \leqslant \binom{m-3}{2}$.

Proof of Claim. Let $(\beta, \mathbf{b}) \in A^2$. Set $E^{-1}\mathbf{b} =: \mathbf{v} = (v_1, \dots, v_{m-1})$, where $\operatorname{supp}(v_3, \dots, v_{m-1}) = \{i, j\}$ for some $i, j \in \{3, \dots, m-1\}$. To prove the claim, it suffices to show that there is no other column $(\beta', \mathbf{b}') \in A^2$ such that $E^{-1}\mathbf{b}' =: \mathbf{v}' = (v_1', \dots, v_{m-1}')$ satisfies $\operatorname{supp}\mathbf{v}' = \{i, j\}$. Indeed, this will show that a column of A^2 is uniquely determined by two indices in $\{3, \dots, m-1\}$.

For simplicity, assume $i = 3$ and $j = 4$. Recall $2 \geqslant \sigma_{\max} - \sigma_{\min} \geqslant 1/2 \cdot |\operatorname{supp}(v_1, v_2)| + |\operatorname{supp}(v_3, \dots, v_{m-1})| \geqslant |\operatorname{supp}(v_3, v_4)| = 2$. Therefore, it must hold that $\operatorname{supp}\mathbf{v} =$

$\{3,4\}$; in particular, $v_1 = v_2 = 0$. Assume to the contrary that A contains another column $(\beta', \mathbf{b}')$ such that $E^{-1}\mathbf{b}' =: \mathbf{v}' = (v'_1, \ldots, v'_{m-1})$ satisfies $\operatorname{supp} \mathbf{v}' = \{3,4\}$. As was the case with $\mathbf{v}$, we have $\operatorname{supp} \mathbf{v}' = \{3,4\}$. Recall that $M \subseteq [A^0|A^1]$. This implies $\mathbf{b} \notin M$ and $|O(\mathbf{b})| = 1$. Therefore, $\mathbf{b}$ and $\mathbf{b}'$ are distinct because $|O(\mathbf{b}')| = 1$ and $(\beta, \mathbf{b}) \neq (\beta', \mathbf{b}')$. In fact, the two columns differ according to assumption (3.5). Given that $v_3, v_4, v'_3, v'_4 \in \{-1, 1\}$, the columns (v_3, v_4) and (v'_3, v'_4) must have different sign patterns, say $v_3 = v'_3 = 1$ and $v_4 = -v'_4 = 1$. By multilinearity of the determinant,

$$\left|\det\left[\ \mathbf{b}^1 \middle| \mathbf{b}^2 \middle| \mathbf{b} \middle| \mathbf{b}' \middle| \mathbf{b}^5 \middle| \cdots \middle| \mathbf{b}^{m-1}\ \right]\right| = \left|\det\begin{bmatrix} v_3 & v'_3 \\ v_4 & v'_4 \end{bmatrix}\right| \cdot \left|\det\left[\ \mathbf{b}^1 \middle| \cdots \middle| \mathbf{b}^{m-1}\ \right]\right|$$
$$= 2|\det E| = 4.$$

This contradicts that $A/\mathbf{e}^1$ is bimodular. $\diamond$

Finally, we prove (3.15) by combining Claims 3.16, 3.17, and 3.18:

$$|A| = |A^0| + |A^1| + |A^2| \leqslant 9 + 4(m-3) + \binom{m-3}{2} = \frac{1}{2}(m^2 + m) + 3.$$

$\square$

3.3.4 Additional structural properties when $|B^*| = 3$.

Throughout Section 3.3, we have assumed that A has the form (3.4). In order to transform A into this form, we identify a primitive column to transform into $\mathbf{e}^1$. Our choice of a primitive column thus far has been somewhat arbitrary when in reality there are multiple choices. For example, if $[\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1+\mathbf{e}^2+2\mathbf{e}^3] \subseteq A$, then any of these columns can be chosen. Moreover, these columns are interchangeable: if we label the columns as $\mathbf{a}^1, \mathbf{a}^2$, and $\mathbf{a}^3$, then for any permutation $\sigma \in \mathcal{S}^3$ there are elementary operations such that $[\mathbf{a}^{\sigma(1)}|\mathbf{a}^{\sigma(2)}|\mathbf{a}^{\sigma(3)}] = [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$. The discussion surrounding Equation (3.18) in the proof of Lemma 3.19 illustrates this symmetry in more detail. In Lemma 3.19, we consider swapping the roles of $\mathbf{e}^1$ and another primitive column in $[\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$. In order to formalize our argument, we define $A/\mathbf{a}$ for a primitive column $\mathbf{a} \in A$ to be the matrix $A/\mathbf{e}^1$ after identifying $\mathbf{a}$ with $\mathbf{e}^1$. We use $M_\mathbf{a}$ to denote the set of columns of $A/\mathbf{a}$ with at least two originals in A, and we use $O(C)$ to denote the set of original columns in A corresponding to a subset $C \subseteq A/\mathbf{a}$.

Lemma 3.19. *Assume that A only contains primitive columns and M contains a circuit. Choose $B^* \subseteq A$ that satisfies* (3.7) *and minimizes $|B^*|$, and assume that B^* has the form* (3.8)*. If $|B^*| = 3$, then there exists at least one column $\mathbf{a} \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$ such that $M_\mathbf{a}$ does not contain a circuit. It follows that $|M_\mathbf{a}| \leqslant m - 1$.*

Proof. Let $\mathbf{a} \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$. As stated in the previous paragraph, we can assume $\mathbf{a} = \mathbf{e}^1$ by applying elementary operations to A. Lemma 3.12 **(ii)** implies $|M_\mathbf{a}| \leqslant m$ for each $\mathbf{a} \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$.

Claim 3.20. *Suppose there exists a column* $\mathbf{a} \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$ *such that* $M_{\mathbf{a}}$ *contains a circuit* C*; then there exists a column* $\mathbf{a}' \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]\backslash\{\mathbf{a}\}$ *such that* $M_{\mathbf{a}'}$ *contains a circuit* C'*. Furthermore,* $O(C) = O(C')$*.*

Proof of Claim. Without loss of generality, $\mathbf{a} = \mathbf{e}^1$. It follows from Lemma 3.12 **(ii)** that $C = [\mathbf{e}^1_{m-1} + \mathbf{e}^2_{m-1}|\mathbf{e}^2_{m-1}|\mathbf{e}^3_{m-1}|\mathbf{d} + \mathbf{e}^3_{m-1}]$ for some $\mathbf{d} = (d_1, d_2, \mathbf{0}) \in [\mathbf{e}^1_{m-1}|\mathbf{e}^1_{m-1} + 2\mathbf{e}^2_{m-1}]$. Given that $C \subseteq M$, we have $(\alpha, \mathbf{e}^2_{m-1} + \mathbf{e}^3_{m-1})$ and $(\alpha + 1, \mathbf{e}^2_{m-1} + \mathbf{e}^3_{m-1})$ are in A for some $\alpha \in \mathbb{Z}$. By (3.9), we can assume $\alpha = 0$, so $\mathbf{e}^2 + \mathbf{e}^3, \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 \in A$. Similarly, we can assume $\mathbf{e}^3, \mathbf{e}^1 + \mathbf{e}^3 \in A$. Suppose $(\beta, \mathbf{d} + \mathbf{e}^3_{m-1}), (\beta + 1, \mathbf{d} + \mathbf{e}^3_{m-1}) \in A$. Hence,

$$A = [B^* \mid O(C) \mid A'] = \left[\begin{array}{ccccccccccc|c} 1&0&1&0&1&0&1&0&1&\beta&\beta+1 & \\ 0&1&1&1&1&0&0&0&0&d_1&d_1 & \\ 0&0&2&1&1&1&1&0&0&d_2&d_2 & A' \\ 0&0&0&0&0&0&0&1&1&1&1 & \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} & \end{array}\right]. \tag{3.18}$$

Columns 4-11 in (3.18) correspond to $O(C)$.

Suppose $\mathbf{d} = \mathbf{e}^1_{m-1}$, or equivalently suppose $(d_1, d_2) = (1, 0)$. It follows that $\beta = 0$, and Columns 4-11 correspond to $O(C')$, where $C' \subseteq M_{\mathbf{e}^2}$ is a circuit. This proves the claim.

Suppose $\mathbf{d} = \mathbf{e}^1_{m-1} + 2\mathbf{e}^2_{m-1}$ or equivalently suppose $(d_1, d_2) = (1, 2)$. From (3.3), the top left 4×11 submatrix of (3.18) is bimodular. From this, we see that $\beta = 1$: if $\beta \leqslant 0$, then the first four rows of columns 2,7,9, and 10 form a basis with absolute determinant greater than two, and if $\beta \geqslant 2$, then the first four rows of columns 2,6,8, and 11 form a basis with absolute determinant greater than two. Conduct the following three elementary row operations to A followed by multiplying columns by -1: (1) subtract the second row from the first row and subtract twice the second row from the third row; (2) multiply the third row by -1; (3) add the third row to the first row; (4) negate columns 6 and 7:

$$A \sim \left[\begin{array}{ccccccccccc|c} 1&1&0&0&1&1&0&0&1&0&1 & \\ 0&1&1&1&1&0&0&0&0&1&1 & \\ 0&2&0&1&1&1&1&0&0&0&0 & A'' \\ 0&0&0&0&0&0&0&1&1&1&1 & \\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0} & \end{array}\right]. \tag{3.19}$$

Columns 4-11 in (3.18) correspond to $O(C)$, where $C \subseteq A/\mathbf{e}^1$ is a circuit. Similarly, Columns 4-11 in (3.19) correspond to $O(C')$, where $C' \subseteq A/\mathbf{e}^2$ is a circuit. Furthermore, Columns 4-11 in (3.18) are equivalent up to row and column operations to Columns 4-11 in (3.19). Now, $\mathbf{e}^2$ on the right hand side of (3.19) is equivalent to $\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3$ in (3.18). Therefore, in the original representation of A in (3.18), we conclude that there is a circuit $C' \subseteq M_{\mathbf{e}^1+\mathbf{e}^2+2\mathbf{e}^3}$ such that $O(C') = O(C)$. $\diamond$

Assume to the contrary that $M_{\mathbf{e}^1}$, $M_{\mathbf{e}^2}$, and $M_{\mathbf{e}^1+\mathbf{e}^2+2\mathbf{e}^3}$ each contain a circuit. Call these circuits C^1, C^2, and C^3, respectively. By Claim 3.20, for each $i \in \{1, 2, 3\}$, there exists some $j_i \in \{1, 2, 3\}\backslash\{i\}$ such that $O(C^i) = O(C^{j_i})$. Since there is an odd number of circuits here, we conclude that $O(C^1) = O(C^2) = O(C^3)$. This means that columns 4-11 in (3.18) equal $O(C^1), O(C^2)$, and $O(C^3)$. Suppose $(d_1, d_2) = (1, 0)$. Denote column 4 by $\mathbf{c}$. There is no column $\mathbf{c}'$ among 5-11 such that $\mathbf{c} - \mathbf{c}' \in \{\pm(\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3)\}$. Therefore, $\mathbf{c}$ is not an original column of C^3, which is a contradiction. Similarly, if $(d_1, d_2) = (1, 2)$, then we see that column 10 is not an original column of $O(C^2)$. This proves Lemma 3.19. □

This ends our discussion on new combinatorial properties of bimodular constraint matrices. We reiterate to the reader that we believe Lemmas 3.12 and 3.19 may be of independent interest in future research. Next, we apply these properties to prove Theorem 3.1.

3.4 A proof of Theorem 3.1.

Proposition 3.2 proves $\mathfrak{c}(2, m) \geqslant 1/2 \cdot (m^2 + m) + m$. We prove $\mathfrak{c}(2, m) = 1/2 \cdot (m^2 + m) + m$ using induction on m. It is quickly verified that when $m = 1$ the unique maximal bimodular matrix is $[1|2]$ (up to multiplying columns by -1). This proves $\mathfrak{c}(2, 1) = 2$. Assume that $m \geqslant 2$ and

$$\mathfrak{c}(2, k) = \frac{1}{2}\left(k^2 + k\right) + k \qquad \forall\ k = 1, \ldots, m - 1.$$

Let $A \in \mathbb{Z}^{m \times n}$ be a maximal bimodular matrix with $\operatorname{rank} A = m$ and differing columns. After elementary operations, we can assume that $\mathbf{e}^1 \in A$. For the inductive step, we use the following relationship between $|A|$ and $|A/\mathbf{e}^1|$:

$$|A| = |O(\mathbf{0})| + |A/\mathbf{e}^1| + \sum_{\mathbf{b} \in A/\mathbf{e}^1} \big(|O(\mathbf{b})| - 1\big).$$

Recall $M = \{\mathbf{b} \in A/\mathbf{e}^1 \colon |O(\mathbf{b})| \geqslant 2\}$. We consider two cases: when A only contains primitive columns and when A contains a non-primitive column.

Case 1. Assume that A only contains primitive columns. We have $|O(\mathbf{0})| = 1$ by Lemma 3.6 **(i)**, and $|M| = \sum_{\mathbf{b} \in A/\mathbf{e}^1} (|O(\mathbf{b})| - 1)$ by (3.10).

Subcase 1.1. Assume that M contains a circuit. Choose $B^* \subseteq A$ that satisfies (3.7) and minimizes $|B^*|$, and assume that B^* has the form (3.8). We have $2 \leqslant |B^*| \leqslant 4$ according to Lemma 3.7.

Suppose $|B^*| = 2$. If $m = 2$, then $A \subseteq \operatorname{span} B^*$ by Lemma 3.8. It follows from (3.9) that $A \subseteq A \cap \operatorname{span} B^* \subseteq [B^*|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2]$. Hence, $|A| \leqslant 4 = 1/2 \cdot (m^2 + m) + m - 1$. Suppose $m \geqslant 3$. Lemma 3.12 **(i)** implies $|M| \leqslant m$. If $|M| = m$, then $|A| \leqslant 1/2 \cdot (m^2 + m) + 3$ by Lemma 3.14. If $|M| \leqslant m - 1$, then

$$|A| = 1 + |A/\mathbf{e}^1| + |M| \leqslant 1 + \mathfrak{c}(2, m-1) + (m - 1) \leqslant 1/2 \cdot \big(m^2 + m\big) + m - 1.$$

Therefore,

$$|A| \leqslant \begin{cases} 1/2 \cdot (m^2 + m) + m - 1, & \text{if } m \geqslant 4 \text{ or } m = 2; \\ 1/2 \cdot (m^2 + m) + m, & \text{if } m = 3. \end{cases} \tag{3.20}$$

Suppose $|B^*| = 3$. We have $[\mathbf{e}^1_{m-1}|\mathbf{e}^1_{m-1} + 2\mathbf{e}^2_{m-1}] \subseteq A/\mathbf{e}^1$ and every column of $A/\mathbf{e}^1$ is primitive because $2 < |B^*| = 3$. Therefore, $A/\mathbf{e}^1$ satisfies Inequality (3.20). By Lemma 3.19, there exists a column $\mathbf{a} \in [\mathbf{e}^1|\mathbf{e}^2|\mathbf{e}^1 + \mathbf{e}^2 + 2\mathbf{e}^3]$ such that $|M_{\mathbf{a}}| = \sum_{\mathbf{b}\in A/\mathbf{a}}(|O(\mathbf{b})| - 1) \leqslant m - 1$. Therefore,

$$|A| \leqslant \begin{cases} 1/2 \cdot (m^2 + m) + m - 2, & \text{if } m \geqslant 5 \text{ or } m = 3; \\ 1/2 \cdot (m^2 + m) + m - 1, & \text{if } m = 4. \end{cases} \tag{3.21}$$

Suppose $|B^*| = 4$. If $m = 4$, then $|A| \leqslant 12 = 1/2 \cdot (m^2 + m) + m - 2$ by (3.9). Suppose $m \geqslant 5$; then $|M| \leqslant m + 1$ by Lemma 3.12 **(iii)**. The matrix $B^*/\mathbf{e}^1$ satisfies $|B^*/\mathbf{e}^1| = 3$. Furthermore, $B^*/\mathbf{e}^1$ has minimal cardinality among all subsets of $A/\mathbf{e}^1$ satisfying (3.7) otherwise we would contradict the minimality of B^*. Hence, $A/\mathbf{e}^1$ satisfies Inequality (3.21) for $m - 1$. Therefore,

$$|A| \leqslant \begin{cases} 1/2 \cdot (m^2 + m) + m - 1, & \text{if } m \geqslant 6; \\ 1/2 \cdot (m^2 + m) + m, & \text{if } m = 5; \\ 1/2 \cdot (m^2 + m) + m - 2, & \text{if } m = 4. \end{cases} \tag{3.22}$$

Subcase 1.2. Assume that M does not contain a circuit. This implies $|M| \leqslant m - 1$ and $|A| \leqslant 1 + \mathfrak{c}(2, m - 1) + (m - 1) \leqslant 1/2 \cdot (m^2 + m) + m - 1$.

Case 2. Assume that A contains a non-primitive column $\mathbf{a}$. The column $1/2 \cdot \mathbf{a}$ is contained in A because A is maximal, and the column is primitive by Lemma 3.6. By transforming $1/2 \cdot \mathbf{a}$ to $\mathbf{e}^1$ using elementary operations, $\mathbf{a}$ transforms to $2\mathbf{e}^1$, and we can write

$$A = \left[\ 2\mathbf{e}^1 \mid A'\ \right] = \left[\begin{array}{c|c} 2 & \boldsymbol{\beta}^\intercal \\ \mathbf{0} & \hat{A} \end{array}\right],$$

where $A' \in \mathbb{Z}^{m\times(n-1)}$ and $\mathbf{e}^1 \in A'$. From this identity we see that $\hat{A} \supseteq A/\mathbf{e}^1$ is unimodular, so $|A/\mathbf{e}^1| \leqslant \mathfrak{c}(1, m - 1) = 1/2 \cdot (m^2 - m)$.

We refer to known results in matroid theory to complete the case; see [59] for a thorough introduction on matroids. From Lemma 3.6 **(i)** and **(ii)**, it follows that $\mathbf{a} - \mathbf{a}' \notin p \cdot \mathbb{Z}^m$ for any distinct columns $\mathbf{a}, \mathbf{a}' \in A'$ and any prime number $p \geqslant 3$. This, along with the assumption that A' is bimodular, demonstrates that the matrix A' is a representation of a matroid $\mathcal{M}$ over the field GF(p) for any prime number $p \geqslant 3$. Similarly, the matrix $A/\mathbf{e}^1$ is a representation of the simplification $\mathcal{M}/\mathbf{e}^1$ of the minor of $\mathcal{M}$ obtained by contracting $\mathbf{e}^1$; here we use the fact that $A/\mathbf{e}^1$ is defined to have differing columns. According to Kung [48, Lemma 2.2.1], $\mathcal{M}$ does not contain the Reid geometry. By [48, Theorem 3.1], it follows that $|A'| - |A/\mathbf{e}^1| = |\mathcal{M}| - |\mathcal{M}/\mathbf{e}^1| \leqslant 2m - 1$. Therefore,

$$|A| = 1 + |A'| \leqslant 1 + (2m - 1) + |A/\mathbf{e}^1| \leqslant 2m + \frac{1}{2}(m^2 - m) = \frac{1}{2}(m^2 + m) + m.$$

We remark that this matroid argument does not apply to **Case 1**, where $A/\mathbf{e}^1$ may not be unimodular so $|A/\mathbf{e}^1| \leqslant \mathfrak{c}(1, m - 1)$ cannot be used in the last inequality. □

Tight examples. Proposition 3.2 provides an example of a bimodular matrix A with a non-primitive column that satisfies $|A| = \mathfrak{c}(2, m)$; therefore, the upper bound in **Case 2** is tight. In the following paragraphs, we discuss the tightness of the bounds (3.20), (3.21), and (3.22), which consider when A only contains primitive columns. We highlight two special cases: when $m = 3$ and $|B^*| = 2$ and when $m = 5$ and $|B^*| = 4$. These cases are special because according to (3.20), (3.21), (3.22) and **Subcase 1.2**, they are the only cases when $|A|$ may equal $\mathfrak{c}(2, m)$.

When $|B^*| = 2$, the bound (3.20) is attainable. If $m \geqslant 4$ or $m = 2$, then a tight example comes from deleting the non-primitive column from the example for $\mathfrak{c}(2, m)$ in Section 3.2. This example is the vertex-edge incidence matrix of the directed complete graph on m vertices together with the identity matrix and $m - 1$ extra columns $\mathbf{e}^1 + \mathbf{e}^i$ for $i = 2, \ldots, m$. In this example, B^* corresponds to $[\mathbf{e}^1 - \mathbf{e}^2 | \mathbf{e}^1 + \mathbf{e}^2]$. For the special case when $m = 3$, a tight example with $\mathfrak{c}(2, 3) = 9$ columns is the matrix in Claim 3.15:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 & 0 & 0 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{r|c|c|c|r|r|c|c|c} 1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

This example is the vertex-edge incidence matrix of the directed complete graph on three vertices appended to an identity matrix and three extra columns $\mathbf{e}^1 + \mathbf{e}^2, \mathbf{e}^1 + \mathbf{e}^3, \mathbf{e}^2 + \mathbf{e}^3$.

When $|B^*| = 3$, the upper bound (3.21) for $m \geqslant 5$ or $m = 3$ can be achieved by the example of the vertex-edge incidence matrix of the directed complete graph with the identity matrix and $m - 2$ extra columns $\mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^i$ for $i = 3, \ldots, m$. For $m = 4$, the upper bound (3.21) can be achieved by the previous example with one extra column $\mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^3 - \mathbf{e}^4$. In these examples, one choice of B^* is $[\mathbf{e}^3 | \mathbf{e}^1 - \mathbf{e}^2 | \mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^3]$.

When $|B^*| = 4$, the upper bound (3.22) is tight for $m = 4, 5$. By setting A to the right hand side (3.9), we obtain a tight example when $m = 4$; here, $B^* = [\mathbf{e}^1 | \mathbf{e}^2 | \mathbf{e}^3 | \mathbf{e}^1 + \mathbf{e}^2 + \mathbf{e}^3 + 2\mathbf{e}^4]$. For the special case when $m = 5$, a tight example with $20 = \mathfrak{c}(2, 5)$ many columns consists of the twelve columns in the previous example for $m = 4$ and eight extra columns:

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|r|r|c|c} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right].$$

This example is equivalent to the vertex-edge incidence matrix of the directed complete graph minus the column $\mathbf{e}^1 - \mathbf{e}^2$, along with the identity matrix and six extra columns: $\mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^i$ for $i = 3, 4, 5$ and $\mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^3 - \mathbf{e}^4 - \mathbf{e}^5 + \mathbf{e}^i$ for $i = 3, 4, 5$. Here, B^* corresponds to $[\mathbf{e}^3 | \mathbf{e}^1 - \mathbf{e}^4 | \mathbf{e}^2 - \mathbf{e}^4 | \mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^3]$.

The bound (3.22) for $m \geqslant 6$ already shows that the maximal example in this case has at most $\mathfrak{c}(2, m) - 1$ differing columns. However, we believe that this bound can be improved. Our current maximal example found for $m = 6$ has $1/2 \cdot (m^2 + m) + m - 2 = \mathfrak{c}(2, m) - 2 = 25$ columns. The example is the vertex-edge incidence matrix of the directed complete graph with identity matrix except $\mathbf{e}^1, \mathbf{e}^2$, along with $-\mathbf{e}^1 + \mathbf{e}^i + \mathbf{e}^j$ for $i \neq j \in \{3, 4, 5, 6\}$. Here, one choice of B^* is $[\mathbf{e}^3 - \mathbf{e}^4 | \mathbf{e}^5 - \mathbf{e}^6 | -\mathbf{e}^1 + \mathbf{e}^3 + \mathbf{e}^4 | -\mathbf{e}^1 + \mathbf{e}^5 + \mathbf{e}^6]$. Our current maximal example found for $m \geqslant 7$ has $1/2 \cdot (m^2 + m) + m - 3 = \mathfrak{c}(2, m) - 3$ columns. The example is the vertex-edge incidence matrix of the directed complete graph without the column $\mathbf{e}^1 - \mathbf{e}^2$, along with the identity matrix and the columns $\mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^i$ for $i = 3, \ldots, m$. Here, B^* corresponds to $[\mathbf{e}^3 | \mathbf{e}^1 - \mathbf{e}^4 | \mathbf{e}^2 - \mathbf{e}^4 | \mathbf{e}^1 + \mathbf{e}^2 - \mathbf{e}^3]$.

3.5 A proof of Proposition 3.3.

If $m = 1$, then $[1 | \cdots | \Delta]$ is the unique maximal Δ-modular matrix with differing columns (up to multiplication by -1). Thus, the result holds for $m = 1$.

Suppose $m = 2$. Let $A \in \mathbb{Z}^{2 \times n}$ be a Δ-modular matrix with differing columns that satisfies $\operatorname{rank} A = 2$ and $|A| = \mathfrak{c}(\Delta, 2)$. Let $[\mathbf{b}^1 | \mathbf{b}^2] \subseteq A$ satisfy $|\det[\mathbf{b}^1 | \mathbf{b}^2]| = \Delta$. Each column $\mathbf{a} \in A$ can be written as $\mathbf{a} = v_1 \mathbf{b}^1 + v_2 \mathbf{b}^2$ for $v_1, v_2 \in [-1, 1]$. Otherwise, say if $|v_1| > 1$, then we derive the contradiction $|\det[\mathbf{a} | \mathbf{b}^2]| = |v_1| \cdot |\det[\mathbf{b}^1 | \mathbf{b}^2]| > \Delta$. After possibly multiplying columns of A by -1, we assume that $v_2 \in [0, 1]$ for each column $v_1 \mathbf{b}^1 + v_2 \mathbf{b}^2 \in A$.

Set $\Pi := \{v_1 \mathbf{b}^1 + v_2 \mathbf{b}^2 \in \mathbb{Z}^2 \colon v_1, v_2 \in [0, 1)\}$. It is well known that $|\Pi| = |\det[\mathbf{b}^1 | \mathbf{b}^2]| = \Delta$; see [10, §VII]. Partition Π as $\Pi = \{\mathbf{0}\} \cup \Pi^1 \cup \Pi^2 \cup \Pi^{\text{int}}$, where

$$\begin{array}{lcl} \Pi^1 & := & \{v_1 \mathbf{b}^1 \in \Pi \colon v_1 \in (0, 1)\}, \\ \Pi^2 & := & \{v_2 \mathbf{b}^2 \in \Pi \colon v_2 \in (0, 1)\}, \\ \Pi^{\text{int}} & := & \{v_1 \mathbf{b}^1 + v_2 \mathbf{b}^2 \in \Pi \colon (v_1, v_2) \in (0, 1)^2\} \quad . \end{array}$$

After multiplying columns by -1, we assume that if $v_1 \mathbf{b}^1 \in A$ for some $v_1 \in [-1, 1]$, then $v_1 \geqslant 0$. Hence, we assume $A \cap (\Pi^1 + \{\mathbf{0}, -\mathbf{b}^1\}) \subseteq \Pi^1$.[4] We partition $A \backslash \Pi$ as follows:

$$\begin{aligned} A \backslash \Pi = \; & \big(A \cap \big(\Pi^{\text{int}} - \mathbf{b}^1\big)\big) \\ & \cup \; \big(A \cap \big(\Pi^1 + \{\mathbf{b}^2, -\mathbf{b}^1 + \mathbf{b}^2\}\big)\big) \cup \big(A \cap \{\mathbf{b}^2, \mathbf{b}^1 + \mathbf{b}^2, -\mathbf{b}^1 + \mathbf{b}^2\}\big) && (3.23) \\ & \cup \; \big(A \cap \big(\Pi^2 + \{\pm \mathbf{b}^1\}\big)\big) \cup \{\mathbf{b}^1\}. && (3.24) \end{aligned}$$

Suppose $v_1 \mathbf{b}^1 + \mathbf{b}^2 \in A$ for some $v_1 \in [-1, 1]$; it follows that if $w_1 \mathbf{b}^1 + \mathbf{b}^2 \in A$ for some $w_1 \in [-1, 1]$, then $|v_1 - w_1| \leqslant 1$. Indeed, otherwise we obtain the contradiction $|\det[v_1 \mathbf{b}^1 + \mathbf{b}^2 | w_1 \mathbf{b}^1 + \mathbf{b}^2]| = |v_1 - w_1| \cdot |\det[\mathbf{b}^1 | \mathbf{b}^2]| > \Delta$. This implies that the set in (3.23) has cardinality at most $|\Pi^1| + 2$; furthermore, the cardinality is equal to $|\Pi^1| + 2$ if and only if A contains two columns $v_1 \mathbf{b}^1 + \mathbf{b}^2$ and $(v_1 + 1) \mathbf{b}^1 + \mathbf{b}^2$ for

[4]For sets $X, Y \subseteq \mathbb{R}^d$, the *Minkowski sum* is $X + Y := \{\mathbf{x} + \mathbf{y} \colon \mathbf{x} \in X, \ \mathbf{y} \in Y\}$. For $\mathbf{y} \in \mathbb{R}^d$, we write $X + \mathbf{y}$ instead of $X + \{\mathbf{y}\}$.

some $v_1 \in [-1, 1]$. Similarly, it can be shown that the set in (3.24) has cardinality at most $|\Pi^2| + 2$.

We claim that either the set in (3.23) has cardinality at most $|\Pi^1| + 1$ or the set in (3.24) has cardinality at most $|\Pi^2| + 1$. Assume that the set in (3.23) has cardinality $|\Pi^1| + 2$. Thus, A contains two columns of the form $v_1\mathbf{b}^1 + \mathbf{b}^2$ and $(v_1 + 1)\mathbf{b}^1 + \mathbf{b}^2$. Replace $\mathbf{b}^2$ with $v_1\mathbf{b}^1 + \mathbf{b}^2$; the result is another basis in A of absolute determinant Δ. After this replacement, we can assume that A contains the columns $\mathbf{b}^2$ and $\mathbf{b}^1 + \mathbf{b}^2$. The matrix A cannot contain a column of the form $-\mathbf{b}^1+v_2\mathbf{b}^2$ for $v_2 > 0$ otherwise $|\det[\mathbf{b}^1+\mathbf{b}^2|-\mathbf{b}^1+v_2\mathbf{b}^2]| = |1+v_2|\cdot|\det[\mathbf{b}^1|\mathbf{b}^2]| > \Delta$. Hence, the set in (3.24) is contained in $(A \cap (\Pi^2 + \mathbf{b}^1)) \cup \{\mathbf{b}^1\}$, which contains at most $|\Pi^2| + 1$ many elements. In other words, the union of the sets in (3.23) and (3.24) has cardinality at most $|\Pi^1| + |\Pi^2| + 3$.

The column $\mathbf{0}$ is in $\Pi \backslash A$, and $|\Pi| = \Delta$. Therefore, $|A \cap \Pi| = |A \cap (\Pi^{\text{int}} \cup \Pi^1 \cup \Pi^2)| \leqslant |\Pi| - 1$. The set $A \cap \left(\Pi^{\text{int}} - \mathbf{b}^1\right)$ is contained in a translation of Π^{int}, so $|A \cap \left(\Pi^{\text{int}} - \mathbf{b}^1\right)| \leqslant |\Pi^{\text{int}}|$. By combining our upper bounds and applying the claim in the previous paragraph, we obtain

$$|A| = |A \cap \Pi| + |A \backslash \Pi| \leqslant (|\Pi| - 1) + (|\Pi^{\text{int}}| + |\Pi^1| + |\Pi^2| + 3) = 2(|\Pi| - 1) + 3 = 2\Delta + 1.$$

The equation $\mathfrak{c}(\Delta, 2) = 2\Delta + 1$ then follows from Proposition 3.2.

3.6 A proof of Theorem 3.4.

Let $A \in \mathbb{Z}^{m \times n}$ be a Δ-modular matrix with $\operatorname{rank} A = m$ and differing columns that satisfies $\mathfrak{c}(\Delta, m) = |A|$. Let $B \subseteq A$ satisfy $|\det B| = \Delta$. If $\Delta \leqslant 2$, then Theorem 3.4 follows from Theorem 3.1 and Heller's result. Therefore, assume $\Delta \geqslant 3$. We can assume that B is in Hermite Normal Form (See [63, §4.1]):[5]

$$B = \left[\ \mathbf{b}^1 | \cdots | \mathbf{b}^m\ \right] \sim \left[\begin{array}{c|c|c|c} \mathbb{I}_{m-k} & * & \cdots & * \\ & \delta_1 & \ddots & \vdots \\ & & \ddots & * \\ & & & \delta_k \end{array}\right],$$

where $\delta_1, \ldots, \delta_k \geqslant 2$, $\prod_{i=1}^k \delta_i = \Delta$, and for each $\mathbf{b}^i = (b^i_1, \ldots, b^i_m)$, we have $0 \leqslant b^i_j < b^i_i$ for all $j = 1, \ldots, i-1$ and $b^i_j = 0$ for all $j = i+1, \ldots, m$.

Each column $\mathbf{a} = (a_1, \ldots, a_m) \in A \backslash B$ satisfies $|a_m| \leqslant \delta_k$ because $|a_m| \cdot \prod_{i=1}^{k-1} \delta_i = |\det[\mathbf{b}^1 | \cdots | \mathbf{b}^{m-1} | \mathbf{a}]| \leqslant \Delta$. After possibly multiplying columns by -1, we can assume that $a_m \in \{0, \ldots, \delta_k\}$ for all columns $\mathbf{a}$. For $r \in \mathbb{Z}$, define

$$A[r] := \left\{(a_1, \ldots, a_m) \in A \colon a_m = r\right\}.$$

For each prime number p, define

$$\hat{A}[p] := \bigcup_{i=1,\ p|i}^{\delta_k} A[i].$$

[5]We use the convention that blank entries in a matrix are zero.

By applying a union bound, we see that

$$\mathfrak{c}(\Delta, m) = |A| \leqslant |A[1]| + \sum_{\substack{p=2,\\ p \text{ prime}}}^{\delta_k} \left|\hat{A}[p]\right|. \tag{3.25}$$

We use (3.25) to upper bound $|A|$ in terms of $\mathfrak{c}(1,m),\ldots,$ and $\mathfrak{c}(\Delta-1,m)$. Our analysis distinguishes between the cases $k=1$ and $k \geqslant 2$.

Case 1. Assume that $k=1$. Recall $\Delta \geqslant 3$. For this range of Δ, Glanzer et al. [34, Subsection 3.2] showed that

$$\mathfrak{c}(\Delta, m) \leqslant \sum_{\substack{p=2,\\ p \text{ prime}}}^{\Delta} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) + 2\mathfrak{c}\left(\left\lfloor \frac{\Delta}{2} \right\rfloor, m\right). \tag{3.26}$$

Case 2. Assume that $k \geqslant 2$. For a prime $p \in \{2,\ldots,\delta_k\}$ and an integer i divisible by p, we can divide the mth row of $A[i]$ by p. We have

$$\left|\hat{A}[p]\right| \;\leqslant\; \left|\left[\ \mathbf{b}^1 \middle| \cdots \middle| \mathbf{b}^{m-1} \middle| \hat{A}[p]\ \right]\right| \;\leqslant\; \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) \tag{3.27}$$

because the columns in the middle expression form a $\lfloor \Delta/p \rfloor$-modular matrix with rank-m.

Consider $A[1]$. For each $\mathbf{a} = (a_1,\ldots,a_m) \in A[1]$, we have

$$\begin{aligned}
\left|\det[\mathbf{b}^1 \mid \cdots \mid \mathbf{b}^{m-2} \mid \mathbf{a} \mid \mathbf{b}^m]\right| &= \left|\det\begin{bmatrix} \mathbb{I}_{m-k} & * & \cdots & * & * & * \\ & \delta_1 & \cdots & * & * & * \\ & & \ddots & \vdots & \vdots & \vdots \\ & & & \delta_{k-2} & * & * \\ & & & & a_{m-1} & b^m_{m-1} \\ & & & & 1 & \delta_k \end{bmatrix}\right| \\
&= \left|a_{m-1}\delta_k - b^m_{m-1}\right| \prod_{i=1}^{k-2} \delta_i \\
&\leqslant \Delta.
\end{aligned}$$

Thus, $|a_{m-1}\delta_k - b^m_{m-1}| \leqslant \delta_{k-1}\delta_k$. The Hermite Normal Form assumption implies $0 \leqslant b^m_{m-1} < \delta_k$, so we have $|a_{m-1}| \leqslant \delta_{k-1}$. For each $r \in \mathbb{Z}$, define the column set

$$A[1,r] := \left\{(a_1,\ldots,a_m) \in A[1] \colon |a_{m-1}| = r\right\}.$$

With these sets, we can upper bound $|A[1]|$:

$$|A[1]| \leqslant |A[1,\delta_{k-1}]| + \sum_{i=0}^{\lfloor \log_2(\delta_{k-1}-1) \rfloor} \left| \bigcup_{s \in \mathbb{Z},\ s \text{ odd}} A[1, s2^i] \right|.$$

The matrix $[\mathbf{b}^1|\cdots|\mathbf{b}^{m-1}|A[1,\delta_{k-1}]]$ is Δ-modular of full row rank, and the $(m-1)$st row is divisible by δ_{k-1}; dividing the $(m-1)$st row by δ_{k-1} shows $|A[1,\delta_{k-1}]| \leqslant \mathfrak{c}(\Delta/\delta_{k-1}, m)$.

For each $i = 0, \ldots, \lfloor \log_2(\delta_{k-1} - 1) \rfloor$, define

$$\overline{A}[i] := \bigcup_{s \in \mathbb{Z},\ s \text{ odd}} A[1, s2^i].$$

If $\overline{A}[i] = A[1, s2^i]$ for a single odd integer s, then perform the following elementary operation to the full row rank matrix $\left[\mathbf{b}^1|\cdots|\mathbf{b}^{m-1}|\overline{A}[i]\right]$: subtract $s2^i$ times the mth row from the $(m-1)$st row. The $(m-1)$st row of the resulting matrix is divisible by δ_{k-1}, so $|\overline{A}[i]| \leqslant \mathfrak{c}(\Delta/\delta_{k-1}, m)$.

Suppose $\overline{A}[i] \neq A[1, s2^i]$ for a single odd integer s. For each column $\mathbf{a} = (a_1, \ldots, a_{m-2}, s2^i, 1) \in \overline{A}[i]$, if we add 2^i times the mth row to the $(m-1)$st row, then the resulting column is $(a_1, \ldots, a_{m-2}, (s+1)2^i, 1)$; in particular, the $(m-1)$st entry is divisible by 2^{i+1} because s is odd. Perform this elementary operation to the full rank matrix $\left[\mathbf{b}^1|\cdots|\mathbf{b}^{m-2}|\overline{A}[i]\right]$: add 2^i times the mth row to the $(m-1)$st row. The $(m-1)$st row of the resulting matrix is divisible by 2^{i+1}. Hence, $|\overline{A}[i]| \leqslant \mathfrak{c}(\lfloor \Delta/2^{i+1} \rfloor, m)$.

Substituting (3.27) and our bounds for $|A[1,\delta_{k-1}]|$ and $|\overline{A}[i]|$ into (3.25), we obtain the following upper bound for $\mathfrak{c}(\Delta, m)$:

$$\sum_{\substack{p=2,\\ p \text{ prime}}}^{\delta_k} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) + \mathfrak{c}\left(\frac{\Delta}{\delta_{k-1}}, m\right) + \sum_{\ell=0}^{\lfloor \log_2(\delta_{k-1}-1) \rfloor} \max\left\{\mathfrak{c}\left(\frac{\Delta}{\delta_{k-1}}, m\right), \mathfrak{c}\left(\left\lfloor \frac{\Delta}{2^{\ell+1}} \right\rfloor, m\right)\right\}$$

If $\delta_{k-1} = 2$, then $\lfloor \log_2(\delta_{k-1} - 1) \rfloor = 0$ and

$$\mathfrak{c}(\Delta, m) \leqslant \sum_{\substack{p=2,\\ p \text{ prime}}}^{\delta_k} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) + 2\mathfrak{c}\left(\frac{\Delta}{2}, m\right). \tag{3.28}$$

If $\delta_{k-1} = 3$, then $\lfloor \log_2(\delta_{k-1} - 1) \rfloor = 1$ and

$$\mathfrak{c}(\Delta, m) \leqslant \sum_{\substack{p=2,\\ p \text{ prime}}}^{\delta_k} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) + 2\mathfrak{c}\left(\frac{\Delta}{3}, m\right) + \mathfrak{c}\left(\left\lfloor \frac{\Delta}{2} \right\rfloor, m\right). \tag{3.29}$$

If $\delta_{k-1} \geqslant 4$, then

$$\mathfrak{c}(\Delta, m) \leqslant \sum_{\substack{p=2,\\ p \text{ prime}}}^{\delta_k} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) + 2\mathfrak{c}\left(\frac{\Delta}{\delta_{k-1}}, m\right) + \sum_{\ell=0}^{\lfloor \log_2(\delta_{k-1}-1) \rfloor - 1} \mathfrak{c}\left(\left\lfloor \frac{\Delta}{2^{\ell+1}} \right\rfloor, m\right). \tag{3.30}$$

This completes **Case 2.**

We now use the two cases to bound $\mathfrak{c}(\Delta, m)$. Define $\mathfrak{g}(\Delta, m) := 1/2 \cdot (m^2 + m)\Delta^2$. We use induction on Δ to show $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{g}(\Delta, m)$. Glanzer et al. demonstrated that $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{g}(\Delta, m)$ for $\Delta \leqslant 3$; see (3.1). Thus, we assume that $\Delta \geqslant 4$ and $\mathfrak{c}(\delta, m) \leqslant \mathfrak{g}(\delta, m)$ for each $\delta < \Delta$. To prove $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{g}(\Delta, m)$, it suffices to upper bound the right hand sides of (3.26), (3.29), and (3.30) by $\mathfrak{g}(\Delta, m)$. Given that $\Delta \geqslant 4$, we do not need to bound (3.28) because it is less than (3.26).

Using the definition of $\mathfrak{g}(\delta, m)$ and the prime zeta function $\mathfrak{p}(s) := \sum_{p \text{ prime}} 1/p^s$, we arrive at the bound

$$\sum_{\substack{p=2,\\ p \text{ prime}}}^{\Delta} \mathfrak{g}\left(\left\lfloor \frac{\Delta}{p} \right\rfloor, m\right) \leqslant \sum_{\substack{p=2,\\ p \text{ prime}}}^{\Delta} \frac{1}{p^2} \mathfrak{g}(\Delta, m) \leqslant \mathfrak{p}(2)\mathfrak{g}(\Delta, m). \tag{3.31}$$

We know $\mathfrak{p}(2) < 1/2$ [66, A085548]. We extend (3.26) using the induction hypothesis, (3.31), and the definition of $\mathfrak{g}(\lfloor \Delta/2 \rfloor, m)$:

$$\mathfrak{c}(\Delta, m) \leqslant \left(\mathfrak{p}(2) + \frac{1}{2}\right) \mathfrak{g}(\Delta, m) < \mathfrak{g}(\Delta, m).$$

Similarly, we extend (3.29):

$$\mathfrak{c}(\Delta, m) \leqslant \left(\mathfrak{p}(2) + \frac{2}{9} + \frac{1}{4}\right) \mathfrak{g}(\Delta, m) < \mathfrak{g}(\Delta, m).$$

We extend (3.30) using $\sum_{i=0}^{t-1} 1/4^{i+1} = 1/3 \cdot (1 - 1/4^t) < 1/3$, the induction hypothesis and (3.31):

$$\mathfrak{c}(\Delta, m) \leqslant \left(\mathfrak{p}(2) + \frac{2}{\delta_{k-1}^2} + \frac{1}{3}\right) \mathfrak{g}(\Delta, m) \leqslant \left(\mathfrak{p}(2) + \frac{1}{8} + \frac{1}{3}\right) \mathfrak{g}(\Delta, m) < \mathfrak{g}(\Delta, m).$$

□

As a final remark, the term Δ^2 in Theorem 3.4 comes from our ability to bound $\mathfrak{g}(\Delta, m)$ via induction. If we apply our proof analysis to some other upper bound $\overline{\mathfrak{g}}(\Delta, m) = m^2\Delta^q$, where $1 \leqslant q \leqslant 2$, then it suffices for q to belong to the following set:

$$\left\{ q \in \mathbb{R}_{\geqslant 0} \colon \max\left\{ \frac{2}{3^q} + \frac{1}{2^q}, \frac{2}{4^q} + \frac{1}{2^q - 1}, \frac{2}{2^q} \right\} \leqslant 1 - \mathfrak{p}(q) \right\}.$$

From numerical computation, $q = 1.95$ is in this set. Thus, $\mathfrak{c}(\Delta, m) \leqslant m^2\Delta^{1.95}$. We chose to present $m^2\Delta^2$ for simplicity.

An open question is whether $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{h}(m)\Delta$ for a polynomial $\mathfrak{h}$. Since posting the first version of this manuscript, we became aware of geometric arguments by Gennadiy Averkov and Matthias Schymura that $\mathfrak{c}(\Delta, m) \leqslant \mathfrak{f}(m)\Delta$, where $\mathfrak{f}$ is superpolynomial [9]. In fact, we can give a quick argument showing that $\mathfrak{c}(\Delta, m) \leqslant 3^m\Delta$. If $B \subseteq A$ satisfies $|\det B| = \Delta$, then every $\mathbf{a} \in A$ satisfies $\|B^{-1}\mathbf{a}\|_\infty \leqslant 1$. Hence, $A \subseteq \Pi + B\{-1, 0, 1\}^m$, where $\Pi := \{B\mathbf{v} \in \mathbb{Z}^m \colon \mathbf{v} \in [0, 1)^m\}$. From this, we see that $|A| \leqslant 3^m \cdot |\Pi| = 3^m\Delta$.

3.7 A proof of Theorem 3.5.

Recall that when it comes to bounding π the matrix A does not necessarily have differing columns as is the case with the other results in this manuscript.

Let $\mathbf{x}^*$ be a vertex of LP satisfying $\pi = \min_{\mathbf{z}\in\mathrm{IP}} \|\mathbf{x}^* - \mathbf{z}\|_1$. By standard LP results, there exists a vector $\mathbf{c} \in \mathbb{R}^n$ such that $\mathbf{x}^*$ is the unique maximizer of $\mathbf{x} \to \mathbf{c}^\intercal\mathbf{x}$ over LP; see [16, Chapter 3]. By possibly perturbing $\mathbf{c}$, we can assume that there is a unique maximizer $\mathbf{z}^*$ of $\mathbf{z} \to \mathbf{c}^\intercal\mathbf{z}$ over IP. Note that $\mathbf{z}^*$ is a vertex of $\operatorname{conv}\mathrm{IP}$. Let $k \leqslant 2\mathfrak{c}(\Delta, m) + 1$ denote the number of distinct columns of A. By applying [61, Theorem 2] with $T = \varnothing$ and $B = A$, there exists a matrix $W \in \mathbb{Z}^{k\times n}$ such that

$$\operatorname{conv}\mathrm{IP} = \operatorname{conv}\{\mathbf{x} \in \mathrm{LP} \colon W\mathbf{x} \in \mathbb{Z}^k\}.$$

The previous equation implies $\mathbf{z}^*$ is also the unique maximizer of $\mathbf{z} \to \mathbf{c}^\intercal\mathbf{z}$ over the **mixed integer** linear set $\{\mathbf{x} \in \mathrm{LP} \colon W\mathbf{x} \in \mathbb{Z}^k\}$, which has k many integer constraints.

Consider the difference vector $\mathbf{x}^* - \mathbf{z}^*$. From the equation $\pi = \min_{\mathbf{z}\in\mathrm{IP}} \|\mathbf{x}^* - \mathbf{z}\|_1$, it follows that $\pi \leqslant \|\mathbf{x}^* - \mathbf{z}^*\|_1$. The difference vector was first analyzed by Cook et al. [21] and later in [36, 41, 51, 62, 75]. The proof of mixed integer proximity in [62, Theorem 2] established that

$$\mathbf{x}^* - \mathbf{z}^* = \sum_{i=1}^{n} \lambda_i \mathbf{u}^i,$$

where $\mathbf{u}^1, \ldots, \mathbf{u}^n \in \mathbb{Z}^n$ and $\lambda_1, \ldots, \lambda_n \geqslant 0$ satisfy $\sum_{i=1}^n \lambda_i \leqslant k$. The result [51, Claim 8] demonstrated that $\|\mathbf{u}^i\|_1 \leqslant (m+1)\Delta$ for each $i = 1, \ldots, n$. Therefore, $\pi \leqslant \|\mathbf{x}^* - \mathbf{z}^*\|_1 \leqslant (m+1)\Delta k$. The result now follows from Theorem 3.4. □

Chapter 4

Further Proximity Bounds

4.1 A Proximity Bound with Respect to the Vertex Parameter $f(x^*)$

The results in this section are joint work with Robert Weismantel.

4.1.1 Introduction

Set $[l] := \{1, \ldots, l\}$. Let $A \in \mathbb{Z}^{m \times n}$, $\operatorname{rank}(A) = m, b \in \mathbb{Z}^m$ and

$$P := \{x \in \mathbb{R}^n : Ax = b, \ x \geqslant 0\}.$$

Let $x^* \in P$ be a vertex and assume $\operatorname{supp}(x^*) \subseteq \{1, \ldots, m\}$. Set $\delta := |\det(A_1, \ldots, A_m)|$ and $\Delta_m := \max\{|\det(B)| : B$ is an $m \times m$ submatrix of $A\}$.

If $\Delta_m = 1$, then every vertex of P is integral. If $\Delta_m = 2$ and P is full dimensional, then $P \cap \mathbb{Z}^n \neq \emptyset$, see [72]. Deciding feasibility for $P \cap \mathbb{Z}^n$, i.e., one either finds $z \in P \cap \mathbb{Z}^n$ or outputs that $P \cap \mathbb{Z}^n = \emptyset$, is $\mathcal{NP}$-complete [44]. However, the work of Lenstra [53] implies that for constant n one can efficiently decide feasibility for $P \cap \mathbb{Z}^n$.

In the following, we want to analyze the distance from x^* to the "closest" point in $P \cap \mathbb{Z}^n$. Note that this can be seen as a special case of the previously studied proximity question for $c \equiv 0$.

Statement of Results

Among all bases $A_{\cdot,I}$ with $\operatorname{supp}(x^*) \subseteq I =: \{i_1, \ldots, i_m\}$ and $\tilde{z} \in \mathbb{Z}^n_{\geqslant 0}$ satisfying

$$A\tilde{z} \in \operatorname{int}(\operatorname{cone}(-A_{i_1}, \ldots, -A_{i_m})),$$

i.e., $A\tilde{z}$ lies in the interior of $\operatorname{cone}(-A_{i_1}, \ldots, -A_{i_m})$, the smallest possible value of $\|\tilde{z}\|_1$ is called the *vertex number* $f(x^*)$ of the vertex x^*. Without loss of generality, assume $I = \{1, \ldots, m\}$ and $\delta > 0$. If such $\tilde{z}$ does not exist, then we set $f(x^*) = \infty$. One natural reason to consider the parameter $f(x^*)$ is the fact that if $f(x^*)$ is finite, then there exists an integral point in P with small support.

Lemma 4.1 (Support Bound). *If $f(x^*) < \infty$ and $P \cap \mathbb{Z}^n \neq \emptyset$, then there exists $z^* \in P \cap \mathbb{Z}^n$ such that*

$$|\operatorname{supp}(z^*)| \leqslant m + \min\{m, f(x^*)\} + \log_2(\delta).$$

Consider the polyhedron

$$\begin{aligned} P' &:= \{x \in \mathbb{R}^{n-m} : A^{-1}_{\cdot,[m]} A_{\cdot,[n]\setminus[m]} x \leqslant A^{-1}_{\cdot,[m]} b,\ x \geqslant 0\} \\ &=: \{x \in \mathbb{R}^{n-m} : \overline{A}x \leqslant \overline{b},\ x \geqslant 0\} \end{aligned}$$

and note that finding an integral point in P can be efficiently reduced to finding an integral point in P'. The strong bound on the support in Lemma 4.1 facilitates the proof of the following proximity bound.

Theorem 4.2 (Proximity Upper Bound). *If $P \cap \mathbb{Z}^n \neq \emptyset$, then there exists $z^* \in P' \cap \mathbb{Z}^{n-m}$ such that*

$$\|z^*\|_1 < (f(x^*) + 1)\Delta_m.$$

This proximity bound is tight for $f(x^*) \in \{1, \ldots, m\}$ in the sense that there exist problems in standard form having proximity of $\mathcal{O}(f(x^*)\Delta_m)$.

Lemma 4.3 (Proximity Lower Bound). *For every $m \in \mathbb{Z}_{\geqslant 1}$, $k \in \{1, \ldots, m\}$ and $l \in \mathbb{Z}_{\geqslant 2}$, there exists $A \in \mathbb{Z}^{m \times n}, b \in \mathbb{Z}^m$ and a vertex $x^* \in P$, $\operatorname{supp}(x^*) \subseteq \{1, \ldots, m\}$ such that $k = f(x^*)$, $l = \Delta_m$ and there exists $z^* \in P' \cap \mathbb{Z}^{n-m}$ such that*

$$\|z^*\|_1 = (f(x^*) + 1)(\Delta_m - 1) - 1.$$

Another motivation to study the parameter $f(x^*)$ is that for constant $f(x^*)$ it gives rise to a polynomial time feasibility test for $P \cap \mathbb{Z}^n$. Denote by $L(A)$ the running time to find some $\tilde{z} \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $A\tilde{z} \in \operatorname{int}(\operatorname{cone}(-A_1, \ldots, -A_m))$ and denote by $\operatorname{HNF}(A)$ the running time needed to find the Hermite Normal Form of A and denote by $\operatorname{LP}(A)$ an upper bound on the running time to decide feasibility for the linear problem

$$A_{\cdot,[m]}x + A_{\cdot,I}y = 0,\ x \in \mathbb{R}^m_{>0},\ y \in \mathbb{R}^{|I|}_{\geqslant 0} \tag{4.1}$$

for any $I \subseteq [n]\setminus[m], |I| \leqslant m$. Note that the Hermite Normal Form can be efficiently calculated [69] and that (4.1) can be efficiently solved [45, 46]. Denote by $\theta \in \mathbb{R}$ the smallest exponent such that any two integral $m \times m$ matrices can be multiplied in time $\mathcal{O}(m^\theta)$. It is known that $\theta < 2.373$, see [5].

Theorem 4.4 (Feasibility Test). *Let $f(x^*) < \infty$. Then:*

1. *There exists an algorithm that decides feasibility for $P \cap \mathbb{Z}^n$ in time*

$$\mathcal{O}\big(L(A) + \operatorname{HNF}(A) + m^\theta\big).$$

2. *It holds*

$$L(A) \in \mathcal{O}\Big(n + f(x^*) \cdot m^{\theta + f(x^*)}\Big).$$

3. *It holds*
$$L(A) \in \mathcal{O}\Big(n^m \cdot \mathrm{LP}(A)\Big).$$

In particular, the feasibility test is efficient if either $f(x^*)$ or m is constant.

Note that $L(A)$ can often be bounded by a polynomial even if $f(x^*)$ and m are not constant, which can be seen in some of the following examples in which we discuss the parameter $f(x^*)$.

Examples

- $\mathrm{cone}(A) = \mathbb{R}^m$**:** In this case, we have $f(x^*) < \infty$ for any $b \in \mathbb{Z}^m$ and any vertex $x^* \in P$.

- $f(x^*) = 1$**:** The existence of a column vector A_i of A satisfying $A_i = A_{\cdot,[m]}\lambda,\ \lambda \in \mathbb{R}^m_{<0}$ implies $f(x^*) = 1$.

- **Vector in lattice?:** Whether $b \in \mathbb{Z}^m$ lies in the lattice spanned by $A \in \mathbb{Z}^{m\times n}$ or not can be formulated as the feasibility problem $Ax - Ay = b,\ x, y \in \mathbb{Z}^n_{\geqslant 0}$. In this case, we have $f(\begin{bmatrix} x^* & \mathbf{0}\end{bmatrix}) \leqslant m$ and $L(\begin{bmatrix} A & -A\end{bmatrix}) \in \mathcal{O}(n)$ which implies that the feasibility test in Theorem 4.4 is efficient.

- **Corner Polyhedra:** The corner polyhedron of P with respect to the vertex $x^*,\ \mathrm{supp}(x^*) = [m]$ is obtained by relaxing the non-negativity constraints on the first m variables. The corresponding feasibility problem can be equivalently written as $Ax - A_{\cdot,[m]}y = b,\ x \in \mathbb{Z}^n_{\geqslant 0}, y \in \mathbb{Z}^m_{\geqslant 0}$. It holds $f(\begin{bmatrix} x^* & \mathbf{0}\end{bmatrix}) \leqslant m$ and $L(\begin{bmatrix} A & -A_{\cdot,[m]}\end{bmatrix}) \in \mathcal{O}(n)$ which implies that the feasibility test in Theorem 4.4 is efficient.

This section is structured as follows: In subsection 4.1.2, we establish basic properties of the parameter $f(x^*)$ (Lemma 4.1 and Lemma 4.3). In subsection 4.1.3, we prove Theorem 4.2. In subsection 4.1.4, we prove Theorem 4.4.

4.1.2 Properties of $f(x^*)$

Denote by $\mathbf{e}_i$ the i-th unit vector. Define the parallelepipeds

$$\Pi := \Big\{x \in \mathbb{R}^m \colon x = \sum_{i=1}^m \lambda_i A_i,\ 0 \leqslant \lambda_i < 1\Big\},$$
$$\Pi_0 := \Big\{x \in \mathbb{R}^m \colon x = \sum_{i=1}^m \lambda_i A_i,\ -1 < \lambda_i < 1\Big\},$$

and consider the residue classes $\mathbb{Z}^m/\Pi$. For $x, r \in \mathbb{Z}^m$ we write $x \equiv_\Pi r$ if $x - r \in \mathcal{L}(A_1, \ldots, A_m)$, i.e., $x - r$ lies in the lattice spanned by $A_1, \ldots, A_m$. Define a norm $\|\cdot\|_{\Pi_0}$ on $\mathbb{R}^m$ using the gauge function

$$\|x\|_{\Pi_0} := \inf\Big\{\lambda > 0\colon \frac{x}{\lambda} \in \Pi_0\Big\}.$$

Let $x = \sum_{i=1}^m \lambda_i A_i$, $\lambda_i \in \mathbb{R}$. Then, $\|x\|_{\Pi_0} = \max_{i=1,\ldots,m}\{|\lambda_i|\}$. Set $\|x\|_\pi := \min_{i=1,\ldots,m}\{|\lambda_i|\colon \lambda_i \neq 0\}$ for $x \neq 0$.

Proof of Lemma 4.1. It is known that $|\mathbb{Z}^m/\Pi| = \delta$, see [10, p.289 Corollary (2.6)]. The fundamental theorem of finite abelian groups tells us that the group $\mathbb{Z}^m/\Pi$ (with respect to $+$) is isomorphic to $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$ satisfying $m_1 \cdot \ldots \cdot m_l = \delta$.

Claim 4.5. *A minimal generating set of $\mathbb{Z}^m/\Pi$ has at most $\log_2(\delta)$ many elements.*

Proof of Claim. Let $S = \{s_1, \ldots, s_{|S|}\}$ be a minimal generating set of $\mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$ and for the sake of contradiction assume that $|S| > \log_2(\delta)$. This implies $2^{|S|} > \delta$ which yields that there exist distinct $x, y \in \{0,1\}^{|S|}$ (wlog $x_1 = 1$ and $y_1 = 0$) such that $\sum_{i=1}^{|S|} x_i s_i = \sum_{i=1}^{|S|} y_i s_i$. Thus, $s_1 = \sum_{i=2}^{|S|}(y_i - x_i)s_i$ which contradicts the minimality of S and thus proves the claim. $\diamond$

Claim 4.5 and $P \cap \mathbb{Z}^n \neq \emptyset$ imply that there exist $I \subseteq \{m+1, \ldots, n\}$, $|I| \leqslant \log_2(\delta)$ and $\hat{z} \in \mathbb{Z}^{|I|}_{\geqslant 0}$, $\|\hat{z}\|_1 \leqslant \delta - 1$ such that $b \equiv_\Pi A_{\cdot,I}\hat{z}$.

The definition of $f(x^*)$ and Carathéodory's theorem together with our assumptions imply that there exist $J \subseteq \{m+1, \ldots, n\}$, $|J| \leqslant \min\{m, f(x^*)\}$ and $\overline{z} \in \mathbb{Z}^{|J|}_{\geqslant 0}$ such that $A_{\cdot,J}\overline{z} \in \operatorname{int}(\operatorname{cone}(-A_1, \ldots, -A_m))$. Thus, there exists $k \in \mathbb{Z}_{\geqslant 0}$ such that $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(k\overline{z}) \in \operatorname{cone}(A_1, \ldots A_m)$. This implies $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(\delta k\overline{z}) \in \operatorname{cone}(A_1, \ldots A_m)$ and $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(\delta k\overline{z}) \equiv_\Pi 0$. Furthermore, there exists $z^* \in P \cap \mathbb{Z}^n$ satisfying $\operatorname{supp}(z^*) \subseteq \{1, \ldots, m\} \cup I \cup J$. This yields

$$|\operatorname{supp}(z^*)| \leqslant m + \min\{m, f(x^*)\} + \log_2(\delta).$$

$\square$

Proof of Lemma 4.3. Define $A \in \mathbb{Z}^{m\times(m+k+1)}$ by

$$[\mathbf{e}_1|\ldots|\mathbf{e}_{m-1}|l\mathbf{e}_m|\mathbf{e}_1+\cdots+\mathbf{e}_{m-1}+(l-1)\mathbf{e}_m|-\mathbf{e}_1|\ldots|-\mathbf{e}_{k-2}|-\mathbf{e}_{k-1}-\cdots-\mathbf{e}_{m-1}|-l\mathbf{e}_m]$$

and $b := \mathbf{e}_1 + \cdots + \mathbf{e}_m \in \mathbb{Z}^m$ and consider the corresponding polyhedron P. Note that for this choice of A it holds that $l = \Delta_m$ and the vertex $x^* := \mathbf{e}_1 + \cdots + \mathbf{e}_{m-1} + \frac{1}{\Delta_m}\mathbf{e}_m \in P$ satisfies $k = f(x^*)$. Moreover, the integral point

$$z^* := \mathbf{e}_1 + \cdots + \mathbf{e}_{m-1} + (\Delta_m - 1)(\mathbf{e}_{m+1} + \cdots + \mathbf{e}_{m+f(x^*)}) + (\Delta_m - 2)\mathbf{e}_{m+f(x^*)+1} \in P \cap \mathbb{Z}^n$$

satisfies

$$\|z^*_{[n]\setminus[m]}\|_1 = (f(x^*)+1)(\Delta_m - 1) - 1,$$

which finishes the proof. $\square$

4.1.3 Proof of Theorem 4.2

In order to prove Theorem 4.2, it suffices to show that there exists $z^* \in P \cap \mathbb{Z}^n$ such that

$$\|z^*_{[n]\setminus[m]}\|_1 < (f(x^*)+1)\Delta_m.$$

This statement is obvious for $f(x^*) = \infty$. Thus, assume $f(x^*) < \infty$. Choose I and J as in the proof of Lemma 4.1. Due to Lemma 4.1, wlog $\{1,\dots,n\} = \{1,\dots,m\} \cup I \cup J$. There exists $\overline{z} \in \mathbb{Z}^{|J|}_{\geqslant 0}$ such that $A_{\cdot,J}\overline{z} \in \operatorname{int}(\operatorname{cone}(-A_1,\dots,-A_m))$ and $\|\overline{z}\|_1 = f(x^*)$. Choose $\hat{z} \in \mathbb{Z}^{|I|}_{\geqslant 0}$ such that $A_{\cdot,[m]}x + (A_{\cdot,J}\overline{z})y = b - A_{\cdot,I}\hat{z},\ x \in \mathbb{Z}^m_{\geqslant 0}, y \in \mathbb{Z}_{\geqslant 0}$ is feasible and $k := \|\hat{z}\|_1$ is minimal. Note that $k \leqslant \delta - 1$. Consider the subgroup $\langle A_{\cdot,J}\overline{z}\rangle$ of $\mathbb{Z}^m/\Pi$.

Claim 4.6. *It holds* $|\langle A_{\cdot,J}\overline{z}\rangle| \leqslant \frac{\delta}{k+1}$.

Proof of Claim. For the sake of contradiction, assume that $|\langle A_{\cdot,J}\overline{z}\rangle| > \frac{\delta}{k+1}$. First of all, note that $A_{\cdot,I}z \notin \langle A_{\cdot,J}\overline{z}\rangle$ for all $z \in \mathbb{Z}^{|I|}, 0 \leqslant z \leqslant \hat{z}, z \neq 0$ as otherwise we obtain a contradiction to the minimality of k. Secondly, the sets $A_{\cdot,I}z^{(i)} + \langle A_{\cdot,J}\overline{z}\rangle,\ z^{(i)} \in \mathbb{Z}^{|I|}, 0 = z^{(0)} \leqslant \dots \leqslant z^{(k)} = \hat{z}, \|z^{(i)}\|_1 = i$ are pairwise disjoint. This implies that the cardinality of the union of these sets is larger than $(k+1)\cdot\frac{\delta}{k+1} = \delta$, which is a contradiction to $|\mathbb{Z}^m/\Pi| = \delta$ and thus proves the claim. ◇

Claim 4.7. *It holds* $\|A_i\|_{\Pi_0} \leqslant \frac{\Delta_m}{\delta}$ *for all* $i \in [n]$.

Proof of Claim. The case $i \leqslant m$ is trivial. Thus, assume $i > m$. Write $A_i = \lambda_1 A_i + \dots + \lambda_m A_m$. Choose $j \in \{1,\dots,m\}$ such that $|\lambda_j|$ is maximal. Consider

$$|\det(A_1,\dots,A_m)| = |\det(A_1,\dots,A_{j-1},\lambda_j A_j, A_{j+1},\dots,A_m)| = |\lambda_j|\delta \leqslant \Delta_m.$$

This yields $\|A_i\|_{\Pi_0} = |\lambda_j| \leqslant \frac{\Delta_m}{\delta}$. ◇

By subadditivity of the norm, this implies $\|A_{\cdot,I}\hat{z}\|_{\Pi_0} \leqslant k\frac{\Delta_m}{\delta}$.

Claim 4.8. *It holds* $\|A_{\cdot,J}\overline{z}\|_\pi \geqslant \frac{k+1}{\delta}$.

Proof of Claim. For the sake of contradiction, assume that $\|A_{\cdot,J}\overline{z}\|_\pi < \frac{k+1}{\delta}$. Then,

$$\Big\| |\langle A_{\cdot,J}\overline{z}\rangle| \cdot A_{\cdot,J}\overline{z} \Big\|_\pi < \frac{\delta}{k+1}\cdot\frac{k+1}{\delta} = 1,$$

which is a contradiction to $|\langle A_{\cdot,J}\overline{z}\rangle|\cdot A_{\cdot,J}\overline{z} \equiv_{\Pi_0} 0$. Thus, $\|A_{\cdot,J}\overline{z}\|_\pi \geqslant \frac{k+1}{\delta}$. ◇

Set $r := \lfloor \frac{k}{k+1}\Delta_m \rfloor$. There exists $s \in \mathbb{Z}$ satisfying $0 \leqslant s \leqslant |\langle A_{\cdot,J}\overline{z}\rangle| \leqslant \frac{\delta}{k+1}$ such that $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r+s)\overline{z} \equiv_\Pi 0$.

Claim 4.9. *It holds that*

$$A_{\cdot,[m]}x = b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r+s)\overline{z},\ x \geqslant 0$$

is feasible.

Proof of Claim. First of all, note that

$$\|A_{\cdot,J}(r\overline{z})\|_\pi \geqslant r \cdot \frac{k+1}{\delta} \geqslant k\frac{\Delta_m}{\delta} - \frac{k}{k+1}\frac{k+1}{\delta} > k\frac{\Delta_m}{\delta} - 1 \geqslant \|A_{\cdot,I}\hat{z}\|_{\Pi_0} - 1.$$

Due to $A_{\cdot,J}\overline{z} \in \operatorname{int}(\operatorname{cone}(-A_1, \ldots, -A_m))$ and $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r+s)\overline{z} \equiv_\Pi 0$, we obtain $A_{\cdot,I}\hat{z} + A_{\cdot,J}(r+s)\overline{z} \in b + \operatorname{cone}(-A_1, \ldots, -A_m)$. This shows the feasibility of $A_{\cdot,[m]}x = b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r\overline{z}),\ x \geqslant 0$. ◇

Define $z^* \in \mathbb{Z}^n_{\geqslant 0}$ in the following way.

$$z^*_i := \begin{cases} \hat{z}_i \text{ for } i \in I\backslash J, \\ (r+s)\overline{z}_i \text{ for } i \in J\backslash I, \\ \hat{z}_i + (r+s)\overline{z}_i \text{ for } i \in I \cap J, \\ \left(A^{-1}_{\cdot,[m]}\big(b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r+s)\overline{z}\big)\right)_i \text{ for } i \in \{1, \ldots, m\}, \\ 0 \text{ otherwise.} \end{cases}$$

Note that $z^*_{[m]}$ is integral because $b - A_{\cdot,I}\hat{z} - A_{\cdot,J}(r+s)\overline{z} \equiv_\Pi 0$. By construction, it holds that $z^* \in P \cap \mathbb{Z}^n$. We have

$$\begin{aligned} \|(r+s)\overline{z}\|_1 = (r+s)f(x^*) &\leqslant \left(\left\lfloor \frac{k}{k+1}\Delta_m \right\rfloor + \frac{\delta}{k+1}\right) f(x^*) \\ &\leqslant (k+1)\frac{\Delta_m}{k+1} f(x^*) \leqslant f(x^*)\Delta_m. \end{aligned}$$

Due to $\|\hat{z}\|_1 = k \leqslant \Delta_m - 1$, we have

$$\|z^*_{[n]\backslash[m]}\|_1 < (f(x^*) + 1)\Delta_m.$$

□

4.1.4 Proof of Theorem 4.4

1. First of all, we want to decide whether there exists $x \in \mathbb{Z}^n$ such that $Ax = b$, which is a necessary condition for $P \cap \mathbb{Z}^n = \varnothing$. Note that we can find a unimodular matrix $U \in \mathbb{Z}^{n \times n}$ such that $\begin{bmatrix} H_{\cdot,[m]} & \mathbf{0} \end{bmatrix} = H := AU \in \mathbb{Z}^{m \times n}$ is in Hermite Normal Form in time $\operatorname{HNF}(A)$. Thus, we have

$$\begin{aligned} &Ax = b,\ x \in \mathbb{Z}^n \\ \Leftrightarrow\ &(AU)(U^{-1}x) = b,\ x \in \mathbb{Z}^n \\ \Leftrightarrow\ &Hy = b,\ y \in \mathbb{Z}^n \\ \Leftrightarrow\ &H_{\cdot,[m]}y_{[m]} = b,\ y_{[m]} \in \mathbb{Z}^m. \end{aligned}$$

   Feasibility of the last equation can be efficiently tested by forward substitution. If the equation is not feasible, then we can output $P \cap \mathbb{Z}^n = \varnothing$. Thus, we can assume that the latter equation is feasible, which gives rise to an

$\overline{x} \in \mathbb{Z}^n$ such that $A\overline{x} = b$. Define $v \in \mathbb{Z}^n_{\geqslant 0}$ by setting $v_i, i \in [n]$ as the smallest non-negative integer such that $\overline{x}_i + \delta v_i \geqslant 0$ and define $\overline{z} := \overline{x} + \delta v \in \mathbb{Z}^n_{\geqslant 0}$. Cramer's Rule and $\delta = |\det(A_1, \ldots, A_m)|$ imply that there exists $\lambda \in \mathbb{Z}^m$ such that

$$Av = \frac{\lambda_1}{\delta} A_1 + \cdots + \frac{\lambda_m}{\delta} A_m.$$

In particular, we obtain $A(\delta v) = A_{\cdot,[m]}\lambda$. We can find $\tilde{z} \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $A\tilde{z} \in \text{int}(\text{cone}(-A_1, \ldots, -A_m))$ in time $L(A)$. By Cramer's Rule, there exists $\tilde{\lambda} \in \frac{1}{\delta}\mathbb{Z}^m_{<0}$ such that $A\tilde{z} = A_{\cdot,[m]}\tilde{\lambda}$. Set $k := 0$ if $\lambda \in \mathbb{Z}^m_{\leqslant 0}$ and

$$k := \max_{i \in [n]} \left\{ \left\lceil \frac{\lambda_i}{|\tilde{\lambda}_i|} \right\rceil : \lambda_i > 0 \right\} \in \mathbb{Z}_{\geqslant 1}$$

otherwise. This yields

$$\begin{aligned} A(\overline{z} + k\delta\tilde{z}) &= A\overline{x} + A(\delta v) + k\delta A\tilde{z} \\ &= b + A_{\cdot,[m]}\lambda + k\delta A_{\cdot,[m]}\tilde{\lambda} = b + A_{\cdot,[m]}(\lambda + k\delta\tilde{\lambda}) \end{aligned}$$

with $\lambda + k\delta\tilde{\lambda} \in \mathbb{Z}^m_{\leqslant 0}$. Define $\hat{z} \in \mathbb{Z}_{\geqslant 0}$ by $\hat{z}_i = -\lambda_i - k\delta\tilde{\lambda}_i$ for $i \in [m]$ and $\hat{z}_i = 0$ otherwise. Then, $z^* := \overline{z} + k\delta\tilde{z} + \hat{z} \in \mathbb{Z}^n_{\geqslant 0}$ satisfies $Az^* = b$ and thus $z^* \in P \cap \mathbb{Z}^n$. For the running time, note that a linear equation system given by an $m \times m$ constraint matrix can be solved in time $\mathcal{O}(m^\theta)$.

2. For a given $z \in \mathbb{Z}^n_{\geqslant 0}$, the inclusion $Az \in \text{int}(\text{cone}(-A_1, \ldots, -A_m))$ is equivalent to the feasibility of $-A_{\cdot,[m]}\lambda = Az,\ \lambda \in \mathbb{R}^m_{>0}$. The latter can be checked in time $\mathcal{O}(m^\theta)$. We construct an algorithm that successively tests for all $z \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $\|z\|_1 = 1, \ldots, f(x^*)$ whether feasibility is given. Due to the definition of $f(x^*)$, our algorithm will find $\tilde{z} \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $A\tilde{z} \in \text{int}(\text{cone}(-A_1, \ldots, -A_m))$. Note that there exist $m + m^2 + \cdots + m^{f(x^*)} \leqslant f(x^*)m^{f(x^*)}$ many vectors $z \in \mathbb{Z}^n_{\geqslant 0}$ satisfying $1 \leqslant \|z\|_1 \leqslant f(x^*)$. Since we need to perform a feasibility test for any such z, we obtain the stated running time.

3. Note that the number of inclusion-wise maximal sets $I \subseteq [n]\backslash[m], |I| \leqslant m$ is bounded by $\binom{n}{m} \leqslant n^m$. Note that there exists such I satisfying $\text{cone}(A_{\cdot,I}) \cap \text{int}(\text{cone}(-A_1, \ldots, -A_m)) \neq \varnothing$ because $f(x^*) < \infty$ and Carathéodory's theorem. We construct an algorithm that successively tests for all such I until it finds one satisfying this condition. This can be modeled in terms of the feasibility problem (4.1), which by definition can be solved in time $\text{LP}(A)$. Scaling the resulting y-vector (which can be assumed to be rational) to an integral vector and appending zeros yields a desired $\tilde{z}$-vector.

□

4.2 Proximity Bounds in the Case $m = 2$

The results in this section are joint work with Joseph Paat and Robert Weismantel. Let $A \in \mathbb{Z}^{2\times n},\ b \in \mathbb{Z}^2$ and $c \in \mathbb{Z}^n$. For $i \in \{1, \ldots, n\}$ we denote the i-th column of A by A_i. Furthermore, let Δ_2 be the maximal 2×2 subdeterminant in absolute value of A, i.e., $\Delta_2 := \max\{|\det(B)|\colon B$ is a 2×2 submatrix of $A\}$. Set $\delta := |\det(A_1, A_2)|$. We are interested in feasible optimization problems of the form

$$\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{R}^n_{\geqslant 0}\}, \tag{LP}$$

$$\max\{c^\intercal x\colon Ax = b,\ x \in \mathbb{Z}^n_{\geqslant 0}\}. \tag{IP}$$

Without loss of generality, we assume throughout this section that there exists an (LP)-optimal solution x^* satisfying $\operatorname{supp}(x^*) \subseteq \{1,2\}$. Our goal is to show that for each (LP)-optimal solution x^* there exists an (IP)-optimal solution z^* which is close. More precisely, we want to bound $\|z^* - x^*\|_1$ solely in terms of Δ_2 in contrast to the proximity bound by Cook et al. [21] which is in terms of the number of variables and $\Delta := \max\{|\det(B)|\colon B$ is a submatrix of $A\}$.

To the best of our knowledge, the strongest proximity bound in this setting which is linear in Δ_2 is an immediate consequence of the analysis by Eisenbrand and Weismantel [29]:

$$\|x^* - z^*\|_1 < m \cdot 4^m \Delta_2 = 32\Delta_2.$$

A proximity bound of $3\Delta_2$ would be best possible in the sense that for each $\Delta_2 \in \mathbb{Z}_{\geqslant 1}$ there exists a problem instance with an (LP)-optimal solution x^* such that for each (IP)-optimal solution z^* it holds that $\|x^* - z^*\|_1 > 3\Delta_2 - 5$.

4.2.1 Structural Results

Define the parallelepipeds

$$\Pi := \Big\{x \in \mathbb{R}^2\colon x = \lambda_1 A_1 + \lambda_2 A_2,\ 0 \leqslant \lambda_1, \lambda_2 < 1\Big\},$$
$$\Pi_0 := \Big\{x \in \mathbb{R}^2\colon x = \lambda_1 A_1 + \lambda_2 A_2,\ -1 < \lambda_1, \lambda_2 < 1\Big\},$$

and consider the residue classes $\mathbb{Z}^2/\Pi$. For $x, r \in \mathbb{Z}^2$ we write $x \equiv_\Pi r$ if $x = r + m_1 A_1 + m_2 A_2$ for $m_1, m_2 \in \mathbb{Z}$. Define a norm $\|\cdot\|_{\Pi_0}$ on $\mathbb{R}^2$ using the gauge function

$$\|x\|_{\Pi_0} := \inf\Big\{\lambda > 0\colon \frac{x}{\lambda} \in \Pi_0\Big\}.$$

The following lemma is a natural extension of cycle-type arguments.

Lemma 4.10. *Let x^* be an* (LP)*-optimal vertex solution satisfying* $\operatorname{supp}(x^*) \subseteq \{1,2\}$ *and y be an* (IP)*-optimal solution. If there exists $\overline{y} \in \mathbb{Z}^n$ with $0 \leqslant \overline{y} \leqslant y$ and $A(y - \overline{y}) = (\lfloor x_1^* \rfloor - \tilde{y}_1)A_1 + (\lfloor x_2^* \rfloor - \tilde{y}_2)A_2$ for some $\tilde{y}_1,\ \tilde{y}_2 \in \mathbb{Z}$ with $\lfloor x_i^* \rfloor - \tilde{y}_i \geqslant 0$ for $i \in \{1,2\}$, then it holds for $\tilde{y} := (\tilde{y}_1, \tilde{y}_2, 0, \ldots, 0)^\intercal \in \mathbb{Z}^n$ that $\overline{y} + \lfloor x^* \rfloor - \tilde{y} \in \mathbb{Z}^n$ is an optimal solution of* (IP).

Proof. Note that $\overline{y} + \lfloor x^* \rfloor - \tilde{y} \in \mathbb{Z}^n$ is a feasible solution to (IP). For the sake of contradiction let us assume $c^\intercal(\overline{y} + \lfloor x^* \rfloor - \tilde{y}) < c^\intercal y$.

Assume that $\lfloor x_i^* \rfloor - \tilde{y}_i > 0$ for both $i \in \{1,2\}$. Without loss of generality, assume $\frac{x_1^*}{x_2^*} \leqslant \frac{\lfloor x_1^* \rfloor - \tilde{y}_1}{\lfloor x_2^* \rfloor - \tilde{y}_2}$. Set $\gamma := \frac{x_1^*}{\lfloor x_1^* \rfloor - \tilde{y}_1} > 0$.

Due to our assumptions, $\hat{x} := \left(0, x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}, 0, \ldots, 0\right)^\intercal + \gamma(y - \overline{y}) \in \mathbb{R}^n_{\geqslant 0}$. Furthermore, it holds that

$$A\hat{x} = \gamma \sum_{i=1}^{2} (\lfloor x_i^* \rfloor - \tilde{y}_i) A_i + \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) A_2 = x_1^* A_1 + x_2^* A_2 = b$$

and

$$\begin{aligned} c^\intercal \hat{x} &= \gamma c^\intercal (y - \overline{y}) + c_2 \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) \\ &> \gamma \sum_{i=1}^{2} c_i (\lfloor x_i^* \rfloor - \tilde{y}_i) + c_2 \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) = c_1 x_1^* + c_2 x_2^* = c^\intercal x^* \end{aligned}$$

which is a contradiction and hence shows the statement in this case.

Assume now that there exists an $i \in \{1, 2\}$ such that $\lfloor x_i^* \rfloor - \tilde{y}_i = 0$. If both terms are zero, then $A(y - \overline{y}) = 0$ which implies $c^\intercal (y - \overline{y}) = 0$ and thus $c^\intercal y = c^\intercal \overline{y} = c^\intercal (\overline{y} + \lfloor x^* \rfloor - \tilde{y})$. Due to symmetry, we only consider $\lfloor x_2^* \rfloor - \tilde{y}_2 = 0 < \lfloor x_1^* \rfloor - \tilde{y}_1$.

Then we have for $\hat{x} := (0, x_2^*, 0, \ldots, 0)^\intercal + \gamma(y - \overline{y}) = (0, x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}, 0, \ldots, 0)^\intercal + \gamma(y - \overline{y}) \in \mathbb{R}^n_{\geqslant 0}$ that

$$A\hat{x} = \gamma \sum_{i=1}^{2} (\lfloor x_i^* \rfloor - \tilde{y}_i) A_i + \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) A_2 = x_1^* A_1 + x_2^* A_2 = b$$

and because $x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1} = 0 \leqslant x_2^*$ that

$$\begin{aligned} c^\intercal \hat{x} &= \gamma c^\intercal (y - \overline{y}) + c_2 \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) \\ &> \gamma \sum_{i=1}^{2} c_i (\lfloor x_i^* \rfloor - \tilde{y}_i) + c_2 \left(x_2^* - x_1^* \frac{\lfloor x_2^* \rfloor - \tilde{y}_2}{\lfloor x_1^* \rfloor - \tilde{y}_1}\right) = c_1 x_1^* + c_2 x_2^* = c^\intercal x^* \end{aligned}$$

which is a contradiction and finishes the proof. □

For an optimal solution z^* to (IP) we have $Az^* =: u_1 + \cdots + u_M$, where the u_i are copies of the columns of A, i.e., $u_i = A_j$ for a $j \in \{1, \ldots, n\}$. We write $u_i = \alpha_1^{(i)} A_1 + \alpha_2^{(i)} A_2$ for all $i \in \{1, \ldots, M\}$ and introduce the index sets

$$\begin{aligned} I_b^{\nwarrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_1^{(i)} \leqslant 0,\ \alpha_2^{(i)} \geqslant 0 \text{ and } u_i \not\equiv_\Pi 0\}, \\ I_b^{\nearrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_1^{(i)} > 0,\ \alpha_2^{(i)} \geqslant 0 \text{ and } u_i \not\equiv_\Pi 0\}, \\ I_b^{\downarrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_2^{(i)} < 0 \text{ and } u_i \not\equiv_\Pi 0\}, \\ I^{\swarrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_1^{(i)} \leqslant 0,\ \alpha_2^{(i)} < 0 \text{ and } u_i \equiv_\Pi 0\}, \\ I^{\searrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_1^{(i)} > 0,\ \alpha_2^{(i)} < 0 \text{ and } u_i \equiv_\Pi 0\}, \\ I^{\uparrow} &:= \{i \in \{1, \ldots, M\} \colon \alpha_2^{(i)} \geqslant 0 \text{ and } u_i \equiv_\Pi 0\}. \end{aligned}$$

Keep in mind that $I_b^{\nwarrow} \dot{\cup} I_b^{\nearrow} \dot{\cup} I_b^{\downarrow} \dot{\cup} I^{\swarrow} \dot{\cup} I^{\searrow} \dot{\cup} I^{\uparrow} = \{1, \ldots, M\}$. Moreover, we define $I_b := I_b^{\nwarrow} \dot{\cup} I_b^{\nearrow} \dot{\cup} I_b^{\downarrow}$ and

$$
\begin{aligned}
I_{A_1} &:= \{i \in \{1, \ldots, M\} : u_i = A_1\}, \\
I_{A_2} &:= \{i \in \{1, \ldots, M\} : u_i = A_2\},
\end{aligned}
$$

which implies $I_{A_1} \dot{\cup} I_{A_2} \subseteq I^{\uparrow}$. In this setting, we also define

$$
\begin{aligned}
\alpha_1 &:= \sum_{i \in I_b \dot{\cup} I^{\swarrow} \dot{\cup} I^{\searrow}} \alpha_1^{(i)}, \\
\alpha_2 &:= \sum_{i \in I_b \dot{\cup} I^{\swarrow} \dot{\cup} I^{\searrow}} \alpha_2^{(i)}.
\end{aligned}
$$

Lemma 4.11. *It holds* $\|u_i\|_{\Pi_0} \leqslant \frac{\Delta_2}{\delta}$ *for* $i \in \{1, \ldots, M\}$.

Proof. Assume for the sake of contradiction that $|\alpha_j^{(i)}| > \frac{\Delta_2}{\delta}$ for $i \in \{1, \ldots, M\}$ and $j \in \{1, 2\}$. Let $\sigma : \{1, 2\} \to \{1, 2\}$ with $\sigma(1) = 2$ and $\sigma(2) = 1$ and set $\beta_j := 0$ and $\beta_{\sigma(j)} := 1$. Then

$$
\Delta_2 \geqslant |\det(u_i, A_{\sigma(j)})| = \left|\det\left((A_1, A_2)\begin{pmatrix} \alpha_1^{(i)} & \beta_1 \\ \alpha_2^{(i)} & \beta_2 \end{pmatrix}\right)\right| = \delta \left|\det \begin{pmatrix} \alpha_1^{(i)} & \beta_1 \\ \alpha_2^{(i)} & \beta_2 \end{pmatrix}\right|
$$

which implies $\frac{\Delta_2}{\delta} \geqslant |\alpha_j^{(i)} \beta_{\sigma(j)}| = |\alpha_j^{(i)}| > \frac{\Delta_2}{\delta}$ which is a contradiction. Hence, $\|u_i\|_{\Pi_0} \leqslant \frac{\Delta_2}{\delta}$, i.e. $|\alpha_1^{(i)}|, |\alpha_2^{(i)}| \leqslant \frac{\Delta_2}{\delta}$ for all $i \in \{1, \ldots, M\}$. □

The following lemma gives us some information about the previously introduced index sets.

Lemma 4.12. *Let* x^* *be an* (LP)-*optimal vertex solution satisfying* $\operatorname{supp}(x^*) \subseteq \{1, 2\}$ *and* $x_1^* \geqslant \Delta_2$. *Assume that* z^* *is an* (IP)-*optimal solution satisfying* $|I_b| \leqslant \delta - 1$. *Then:*

1. $|\alpha_1| \leqslant \Delta_2$
2. $I^{\uparrow} = I_{A_1} \dot{\cup} I_{A_2}$

Proof. Without loss of generality, we can assume that Lemma 4.10 can not be applied anymore to z^*. Lemma 4.11 states that $\|u_i\|_{\Pi_0} \leqslant \frac{\Delta_2}{\delta}$ for all $i \in \{1, \ldots, M\}$. This implies that

$$
\Big\|\sum_{i \in I_b} u_i\Big\|_{\Pi_0} \leqslant \sum_{i \in I_b} \|u_i\|_{\Pi_0} \leqslant (\delta - 1)\frac{\Delta_2}{\delta}.
$$

If $\sum_{i \in I_b} \alpha_2^{(i)} \leqslant x_2^*$, we can use Lemma 4.10 in order to show that $\sum_{i \in I_b \dot{\cup} I_{A_1} \dot{\cup} I_{A_2}} u_i$ is corresponding to our (IP)-optimal solution z^* because of

$$
\sum_{i \in I_b} \alpha_1^{(i)} \leqslant (\delta - 1)\frac{\Delta_2}{\delta} < \Delta_2 \leqslant x_1^*
$$

and $\sum_{i \in I_b} u_i \equiv_\Pi b$. This implies $I^{\uparrow} = I_{A_1} \dot\cup I_{A_2}$ and $|\alpha_1| \leqslant \Delta_2$.

Otherwise, $\frac{\delta-1}{\delta}\Delta_2 \geqslant \sum_{i \in I_b} \alpha_2^{(i)} > x_2^*$. Our goal is to show that $|\alpha_1| \leqslant \Delta_2$ in this case. Due to symmetry, it suffices to establish that $\alpha_1 \geqslant -\Delta_2$. For this purpose, we analyze the terms $\sum_{i \in I_b^{\nwarrow}} \alpha_2^{(i)}$ and $\sum_{i \in I_b^{\nearrow}} \alpha_2^{(i)}$ separately. By definition of Δ_2 and the inequality

$$\left|\det\left(\begin{pmatrix} A_1 & A_2 \end{pmatrix}\begin{pmatrix} \alpha_1^{(i)} & \alpha_1^{(s)} \\ \alpha_2^{(i)} & \alpha_2^{(s)} \end{pmatrix}\right)\right| \leqslant \Delta_2,$$

we have $\alpha_1^{(s)}\alpha_2^{(i)} - \alpha_1^{(i)}\alpha_2^{(s)} \leqslant \frac{\Delta_2}{\delta}$ and $\alpha_1^{(i)} \geqslant \frac{\alpha_1^{(s)}\alpha_2^{(i)} - \frac{\Delta_2}{\delta}}{\alpha_2^{(s)}}$ for $i \in I^{\swarrow} \dot\cup I^{\searrow}$ and $s \in I_b^{\nwarrow} \dot\cup I_b^{\nearrow}$.

$\sum_{s \in I_b^{\nwarrow}} \alpha_2^{(s)}$: Let $I_1^{\swarrow} \subseteq I^{\swarrow}$. We have

$$\begin{aligned}
\sum_{s \in I_b^{\nwarrow}} \alpha_1^{(s)} + \sum_{i \in I_1^{\swarrow}} \alpha_1^{(i)} &\geqslant \sum_{s \in I_b^{\nwarrow}} \alpha_1^{(s)} + \sum_{i \in I_1^{\swarrow}} \sum_{s \in I_b^{\nwarrow}} \frac{\alpha_2^{(s)}}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \left(\frac{\alpha_2^{(i)}\alpha_1^{(s)} - \frac{\Delta_2}{\delta}}{\alpha_2^{(s)}} \right) \\
&= \sum_{s \in I_b^{\nwarrow}} \alpha_1^{(s)} + \frac{1}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \left(\sum_{i \in I_1^{\swarrow}} \sum_{s \in I_b^{\nwarrow}} \alpha_2^{(i)}\alpha_1^{(s)} - \sum_{i \in I_1^{\swarrow}} \sum_{s \in I_b^{\nwarrow}} \frac{\Delta_2}{\delta} \right) \\
&\geqslant \sum_{s \in I_b^{\nwarrow}} \alpha_1^{(s)} \left(1 - \frac{|I_1^{\swarrow}|}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \right) - \frac{|I_1^{\swarrow}|}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \frac{|I_b^{\nwarrow}|\Delta_2}{\delta} \\
&\geqslant -\frac{|I_b^{\nwarrow}|\Delta_2}{\delta} \left(1 - \frac{|I_1^{\swarrow}|}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \right) - \frac{|I_1^{\swarrow}|}{\sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}} \frac{|I_b^{\nwarrow}|\Delta_2}{\delta} \\
&= -\frac{|I_b^{\nwarrow}|\Delta_2}{\delta}
\end{aligned}$$

because $|I_1^{\swarrow}| \leqslant \sum_{l \in I_b^{\nwarrow}} \alpha_2^{(l)}$.

$\sum_{s \in I_b^{\nearrow}} \alpha_2^{(s)}$: Set $I_2^{\swarrow} := I^{\swarrow} \setminus I_1^{\swarrow}$ and choose $i_0 \in I_2^{\swarrow}$. Define $\tilde{I}_2^{\swarrow} := I_2^{\swarrow} \setminus \{i_0\}$. We have

$$\begin{aligned}
\sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} + \sum_{j \in \tilde{I}_2^{\swarrow}} \alpha_1^{(j)} &\geqslant \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} - \sum_{j \in \tilde{I}_2^{\swarrow}} \sum_{s \in I_b^{\nearrow}} \frac{\alpha_2^{(s)}}{\sum_{s \in I_b^{\nearrow}} \alpha_2^{(l)}} \left(\frac{\frac{\Delta_2}{\delta} - \alpha_1^{(s)}\alpha_2^{(j)}}{\alpha_2^{(s)}} \right) \\
&= \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} + \frac{1}{\sum_{l \in I_b^{\nearrow}} \alpha_2^{(l)}} \left(\sum_{j \in \tilde{I}_2^{\swarrow}} \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)}\alpha_2^{(j)} - \sum_{j \in \tilde{I}_2^{\swarrow}} \sum_{s \in I_b^{\nearrow}} \frac{\Delta_2}{\delta} \right) \\
&= \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} + \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} \frac{\sum_{j \in \tilde{I}_2^{\swarrow}} \alpha_2^{(j)}}{\sum_{l \in I_b^{\nearrow}} \alpha_2^{(l)}} - \frac{|\tilde{I}_2^{\swarrow}|}{\sum_{l \in I_b^{\nearrow}} \alpha_2^{(l)}} \frac{|I_b^{\nearrow}|}{\delta}\Delta_2 \\
&\geqslant \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} - \sum_{s \in I_b^{\nearrow}} \alpha_1^{(s)} - \frac{|I_b^{\nearrow}|}{\delta}\Delta_2 = -\frac{|I_b^{\nearrow}|}{\delta}\Delta_2
\end{aligned}$$

because $|\tilde{I}_2^{\swarrow}| \leqslant \sum_{l\in I_b^{\nearrow}} \alpha_2^{(l)}$ and $\frac{\sum_{j\in\tilde{I}_2^{\swarrow}} \alpha_2^{(j)}}{\sum_{l\in I_b^{\nearrow}} \alpha_2^{(l)}} \geqslant -1$. This yields

$$\sum_{s\in I_b^{\nearrow}} \alpha_1^{(s)} + \sum_{j\in\tilde{I}_2^{\swarrow}} \alpha_1^{(j)} + \alpha_1^{(i_0)} \geqslant -\frac{|I_b^{\nearrow}|+1}{\delta}\Delta_2.$$

All in all, we obtain

$$\begin{aligned}\alpha_1 := &\sum_{i\in I_b^{\nwarrow}\dot\cup I_1^{\swarrow}} \alpha_1^{(i)} + \sum_{i\in I_b^{\nearrow}\dot\cup I_2^{\swarrow}} \alpha_1^{(i)} + \sum_{i\in I_b^{\downarrow}\dot\cup I^{\searrow}} \alpha_1^{(i)} \\ \geqslant & -\frac{|I_b^{\nwarrow}|}{\delta}\Delta_2 - \frac{|I_b^{\nearrow}|+1}{\delta}\Delta_2 - \frac{|I_b^{\downarrow}|}{\delta}\Delta_2 \geqslant -\Delta_2\end{aligned}$$

because $|I_b| \leqslant \delta - 1$. Due to symmetry, one also obtains $\alpha_1 \leqslant \Delta_2$.

Now we can use Lemma 4.10 to show that $\sum_{i\in I_b\dot\cup I^{\swarrow}\dot\cup I^{\searrow}\dot\cup I_{A_1}\dot\cup I_{A_2}} u_i$ corresponds to z^* because

$$\sum_{i\in I_b\dot\cup I^{\swarrow}\dot\cup I^{\searrow}} \alpha_1^{(i)} \leqslant \Delta_2 \leqslant x_1^*.$$

In particular, we obtain $I^{\uparrow} = I_{A_1}\dot\cup I_{A_2}$.

□

Lemma 4.13. *Let x^* be an* (LP)*-optimal vertex solution satisfying* $\mathrm{supp}(x^*) \subseteq \{1,2\}$ *and $x_1^* \geqslant \Delta_2$. Assume that z^* is an* (IP)*-optimal solution satisfying* $|I_b| \leqslant \delta - 1$.

If $\sum_{i\in I_b} \alpha_2^{(i)} > x_2^*$, then

1. $\left||I_{A_1}| - x_1^*\right| \leqslant \Delta_2$,
2. $\left||I_{A_2}| - x_2^*\right| \leqslant \frac{\Delta_2}{\delta}$,
3. $|I^{\swarrow}\dot\cup I^{\searrow}| \leqslant \frac{\delta-1}{\delta}\Delta_2$.

If $\sum_{i\in I_b} \alpha_2^{(i)} \leqslant x_2^*$, then

1. $\left||I_{A_1}| - x_1^*\right| \leqslant \Delta_2$,
2. $\left||I_{A_2}| - x_2^*\right| \leqslant \Delta_2$,
3. $I^{\swarrow}\dot\cup I^{\searrow} = \emptyset$.

Proof. Lemma 4.12 implies $|\alpha_1| \leqslant \Delta_2$ and $I^{\uparrow} = I_{A_1}\dot\cup I_{A_2}$. Without loss of generality, we assume that Lemma 4.10 can not be applied anymore. We know that $\left||I_{A_1}| - x_1^*\right| \leqslant \lceil|\alpha_1|\rceil \leqslant \Delta_2$ because $|\alpha_1| \leqslant \Delta_2$.

1. Assume $\sum_{i\in I_b} \alpha_2^{(i)} > x_2^*$. For the sake of contradiction assume $|I^{\swarrow}\dot\cup I^{\searrow}| > \frac{\delta-1}{\delta}\Delta_2$. We can find $S \subseteq I^{\swarrow}\dot\cup I^{\searrow}$ with $|S| \leqslant \frac{\delta-1}{\delta}\Delta_2$ such that $\sum_{i\in I_b\dot\cup S} \alpha_2^{(i)} \leqslant x_2^*$ because $\sum_{i\in I_b} \alpha_2^{(i)} \leqslant (\delta-1)\frac{\Delta_2}{\delta}$ and $\alpha_2^{(i)} \in \mathbb{Z}$ for all $i \in I^{\swarrow}\dot\cup I^{\searrow}$. By Lemma 4.10 this would lead to a new (IP)-optimal solution induced by $\sum_{i\in I_b\dot\cup S\dot\cup I^{\uparrow}} u_i$

contradicting our assumption, i.e. $|I^{\swarrow} \dot{\cup} I^{\searrow}| \leqslant \frac{\delta-1}{\delta}\Delta_2$. Due to $|\alpha_1^{(i)}|, |\alpha_2^{(i)}| \leqslant \frac{\Delta_2}{\delta}$ we have $x_2^* - \sum_{i \in I_b \dot{\cup} I^{\swarrow} \dot{\cup} I^{\searrow}} \alpha_2^{(i)} < \frac{\Delta_2}{\delta}$, otherwise we could apply Lemma 4.10 again. This yields $\big||I_{A_2}| - x_2^*\big| \leqslant \frac{\Delta_2}{\delta}$ because $I^{\uparrow} = I_{A_1} \dot{\cup} I_{A_2}$.

2. Assume $\sum_{i \in I_b} \alpha_2^{(i)} \leqslant x_2^*$. The proof of Lemma 4.12 implies that $\sum_{i \in I_b \dot{\cup} I_{A_1} \dot{\cup} I_{A_2}} u_i$ equals z^*. This implies $I^{\swarrow} \dot{\cup} I^{\searrow} = \emptyset$. Because $\| \sum_{i \in \{1,\ldots,M\} \setminus I^{\uparrow}} u_i \|_{\Pi_0} \leqslant \Delta_2$ and $\alpha_2^{(i)} = 1$ for $i \in I_{A_2}$ we know that $|\alpha_2| \leqslant \Delta_2$ and $\big||I_{A_2}| - x_2^*\big| \leqslant \lceil |\alpha_2| \rceil \leqslant \Delta_2$.

□

The same type of results hold for $x_2^* \geqslant \Delta_2$. But due to symmetry it suffices to consider $x_1^* \geqslant \Delta_2$.

4.2.2 Proximity Results

The following corollary can be interpreted as a refinement of a result by Oertel, Paat and Weismantel [58] in the special case $m = 2$ and δ prime. In this setting, the result by Oertel et al. implies a proximity bound of $3\Delta_2$ with a probability of 1 for a randomly chosen b. The following result extends this proximity bound to all but finitely many b.

Corollary 4.14. *Let δ be prime and let $b \in \mathbb{Z}^2$ such that $\|b\|_{\Pi_0} \geqslant \Delta_2$. Then for every* (LP)*-optimal vertex solution x^* satisfying* $\operatorname{supp}(x^*) \subseteq \{1, 2\}$*, there exists an* (IP)*-optimal solution z^* such that*

$$\|z^* - x^*\|_1 < 3\Delta_2.$$

Proof. Let z^* be an (IP)-optimal solution such that Lemma 4.10 can not be applied anymore. Then it holds that

$$u_1 + \cdots + u_M := Az^* = b = Ax^* = x_1^* A_1 + x_2^* A_2,$$

where the u_i are copies of the columns of A, i.e., for all $i \in \{1, \ldots, M\}$ there exists $j \in \{1, \ldots, n\}$ such that $u_i = A_j$. Without loss of generality, we assume $x_1^* \geqslant \Delta_2$ because $\|x_1^* A_1 + x_2^* A_2\|_{\Pi_0} = \|b\|_{\Pi_0} \geqslant \Delta_2$.

Claim 4.15. *It holds $|I_b| \leqslant \delta - 1$.*

Proof of Claim. Choose $U \subseteq I_b$ such that $\sum_{i \in U} u_i \equiv_{\Pi} b$. We can assume that $|U| \leqslant \delta - 1$ because otherwise there exist disjunct $U_1, U_2 \subseteq U$ such that $\sum_{i \in U_1} u_i \equiv_{\Pi} -\sum_{i \in U_2} u_i$ because $|\mathbb{Z}^2/\Pi| = \delta - 1$. This implies that we can replace U by $U \setminus \{U_1 \dot{\cup} U_2\}$. By applying this argument successively, we obtain $|U| \leqslant \delta - 1$. For all $U \subseteq I_b$, $|U| \leqslant \delta - 1$ satisfying $\sum_{i \in U} u_i \equiv_{\Pi} b$ and $\sum_{i \in \tilde{U}} u_i \not\equiv_{\Pi} 0$ for all $\tilde{U} \subsetneq U$, we choose U such that $\sum_{i \in U} \alpha_2^{(i)}$ is minimal.

If $\sum_{i \in U} \alpha_2^{(i)} \leqslant x_2^*$, we can use Lemma 4.10 to show that $\sum_{i \in U \dot{\cup} I_{A_1} \dot{\cup} I_{A_2}} u_i$ is corresponding to our (IP)-optimal solution z^* which yields $U = I_b$ and $|I_b| \leqslant \delta - 1$. Thus, we can assume $\frac{\delta-1}{\delta}\Delta_2 \geqslant \sum_{i \in U} \alpha_2^{(i)} > x_2^*$. Due to the minimality assumption on U and because $\mathbb{Z}^2/\Pi$ is a cyclic group, we obtain $I_b^{\downarrow} \subseteq U$. Following the proof of Lemma 4.12 and applying Lemma 4.10 yields that $\sum_{i \in U \dot\cup I^{\swarrow} \dot\cup I^{\searrow} \dot\cup I_{A_1} \dot\cup I_{A_2}} u_i$ corresponds to z^*. Once again, this implies $U = I_b$ and $|I_b| \leqslant \delta - 1$. ◇

Thus, we can apply Lemma 4.12 and obtain $|\alpha_1| \leqslant \Delta_2$ and $I^{\uparrow} = I_{A_1} \dot\cup I_{A_2}$. We have

$$\|z^* - x^*\|_1 = |I_b| + |I^{\swarrow} \dot\cup I^{\searrow}| + \big||I_{A_1}| - x_1^*\big| + \big||I_{A_2}| - x_2^*\big|.$$

If $\sum_{i \in I_b} \alpha_2^{(i)} \leqslant x_2^*$, then by Lemma 4.13

$$\begin{aligned} \|z^* - x^*\|_1 &= |I_b| + |I^{\swarrow} \dot\cup I^{\searrow}| + \big||I_{A_1}| - x_1^*\big| + \big||I_{A_2}| - x_2^*\big| \\ &\leqslant \delta - 1 + \Delta_2 + \Delta_2 < 3\Delta_2. \end{aligned}$$

If $\frac{\delta-1}{\delta}\Delta_2 \geqslant \sum_{i \in I_b} \alpha_2^{(i)} > x_2^*$, then Lemma 4.13 yields

$$\begin{aligned} \|z^* - x^*\|_1 &= |I_b| + |I^{\swarrow} \dot\cup I^{\searrow}| + \big||I_{A_1}| - x_1^*\big| + \big||I_{A_2}| - x_2^*\big| \\ &\leqslant \delta - 1 + \frac{\delta - 1}{\delta}\Delta_2 + \Delta_2 + \frac{\Delta_2}{\delta} = \delta + 2\Delta_2 - 1 < 3\Delta_2. \end{aligned}$$

□

In some settings, Corollary 4.14 can be used to obtain strong proximity bounds for all $b \in \mathbb{Z}^2$. To exemplify this, we consider feasible **cardinality constrained knapsack problems (CCK)**:

$$\begin{aligned} \max\ & c^\intercal x \\ \sum_{i=1}^{n} w_i x_i &\leqslant b_1 \\ \sum_{i=1}^{n} x_i &\leqslant b_2 \\ x &\in \mathbb{Z}^n_{\geqslant 0} \end{aligned}$$

where $c, w \in \mathbb{Z}^n$. Intuitively, the additional constraint limits the number of items that can be put into the knapsack. We assume that (CCK) is feasible and define $\overline{\Delta}$ as the largest absolute subdeterminant of the contraint matrix of (CCK). The corresponding problem in standard form ($\overline{\text{CCK}}$)

$$\begin{aligned} \max\ & \overline{c}^\intercal x \\ \overline{A}x &= b \\ x &\in \mathbb{Z}^{n+2}_{\geqslant 0} \end{aligned}$$

is obtained by adding slack variables. Note that $\overline{A}_{2,\cdot} = (1, \dots, 1, 0, 1)$ and $\overline{\Delta}$ is the largest 2×2 subdeterminant of $\overline{A}$ in absolute value.

Corollary 4.16. *For each optimal vertex solution x^* to the linear relaxation of ($\overline{\text{CCK}}$) with an LP-basis having a prime determinant in absolute value, there exists a ($\overline{\text{CCK}}$)-optimal solution z^* such that*

$$\|x^* - z^*\|_1 < 4\overline{\Delta}.$$

Proof. Let x^* be such an optimal vertex solution to the linear relaxation of ($\overline{\text{CCK}}$). Consider the following three cases.

$b_2 \geqslant 2\overline{\Delta}$: Then there exists $i \in [n+2]$ such that $x^*_i \geqslant \overline{\Delta}$. By Corollary 4.14, there exists a ($\overline{\text{CCK}}$)-optimal solution z^* such that $\|x^* - z^*\|_1 < 3\overline{\Delta}$.

$0 \leqslant b_2 < 2\overline{\Delta}$: Let z^* be any ($\overline{\text{CCK}}$)-optimal solution. Then

$$\|x^* - z^*\|_1 \leqslant \|x^*\|_1 + \|z^*\|_1 < 2\overline{\Delta} + 2\overline{\Delta} = 4\overline{\Delta}.$$

$b_2 < 0$: In this case ($\overline{\text{CCK}}$) is infeasible.

□

Chapter 5

On the Optimization over $\{a, b, c\}$-Modular Matrices

The following chapter is a subset of [32] and its extended version [33]. Note that there is some overlap in the introduction and preliminaries with the thesis of Christoph Glanzer [31]. While in his thesis the recognition problem is being covered (Theorem 5.1 and Theorem 5.2), in this chapter the optimization problem will be tackled (Theorem 5.3).

5.1 Introduction

A matrix is called **totally unimodular** (TU) if all of its subdeterminants are equal to 0, 1 or -1. Within the past 60 years the community has established a deep and beautiful theory about TU matrices. A landmark result in the understanding of such matrices is Seymour's decomposition theorem [65]. It shows that TU matrices arise from network matrices and two special matrices using row, column, transposition, pivoting, and so-called k-sum operations. As a consequence of this theorem it is possible to recognize in polynomial time whether a given matrix is TU [63, 71]. An implementation of the algorithm in [71] by Walter and Truemper [74] returns a certificate if A is not TU: For an input matrix with entries in $\{0, \pm 1\}$, the algorithm finds a submatrix $\tilde{A}$ which is minimal in the sense that $|\det(\tilde{A})| = 2$ and every proper submatrix of $\tilde{A}$ is TU. We refer to Schrijver [63] for a textbook exposition of Seymour's decomposition theorem, a recognition algorithm arising therefrom and further material on TU matrices.

There is a well-established relationship between totally unimodular and **unimodular** matrices, i.e., matrices whose $n \times n$ subdeterminants are equal to 0, 1 or -1. In analogy to this we define for $A \in \mathbb{Z}^{m \times n}$ and $m \geqslant n$,

$$D(A) := \{|\det(A_{I,\cdot})| : I \subseteq [m],\ |I| = n\},$$

the set of all $n \times n$ subdeterminants of A in absolute value, where $A_{I,\cdot}$ is the submatrix formed by selecting all rows with indices in I. It follows straightforwardly from the recognition algorithm for TU matrices that one can efficiently decide whether $D(A) \subseteq \{1,0\}$. A technique in [8, Section 3] allows us to recognize in polynomial time whether $D(A) \subseteq \{2,0\}$. If all $n \times n$ subdeterminants of A are nonzero, the results in [7] can be applied to calculate $D(A)$ given that $\max\{k\colon k \in D(A)\}$ is constant. Nonetheless, with the exception of these results we are not aware of other instances for which it is known how to determine $D(A)$ in polynomial time.

The main motivation for the study of matrices with bounded subdeterminants comes from integer optimization problems (IPs). It is a well-known fact that IPs of the form $\max\{c^\mathsf{T} x\colon Ax \leqslant b,\ x \in \mathbb{Z}^n\}$ for $A \in \mathbb{Z}^{m\times n}$ of full column rank, $b \in \mathbb{Z}^m$ and $c \in \mathbb{Z}^n$ can be solved efficiently if $D(A) \subseteq \{1,0\}$, i.e., if A is unimodular. This naturally leads to the question whether these problems remain efficiently solvable when the assumptions on $D(A)$ are further relaxed. Quite recently in [8] it was shown that when $D(A) \subseteq \{2,1,0\}$ and $\mathrm{rank}(A) = n$, integer optimization problems can be solved in strongly polynomial time. Recent results have also led to understand IPs when A is **nondegenerate**, i.e., if $0 \notin D(A)$. The foundation to study the nondegenerate case was laid by Veselov and Chirkov [72]. They describe a polynomial time algorithm to solve IPs if $D(A) \subseteq \{2,1\}$. In [7] the authors showed that IPs over nondegenerate constraint matrices are solvable in polynomial time if the largest $n \times n$ subdeterminant of the constraint matrix is bounded by a constant.

The role of bounded subdeterminants in complexity questions and in the structure of IPs and LPs has also been studied in [28, 35, 38, 61], as well as in the context of combinatorial problems in [17, 18, 55]. The sizes of subdeterminants also play an important role when it comes to the investigation of the **diameter of polyhedra**, see [24] and [11].

5.1.1 Statement of Results

A matrix $A \in \mathbb{Z}^{m\times n}$, $m \geqslant n$, is called **$\{a,b,c\}$-modular** if $D(A) = \{a,b,c\}$, where $a \geqslant b \geqslant c \geqslant 0$.[1] The paper presents three main results. First, we prove the following structural result for a subclass of $\{a,b,0\}$-modular matrices.

Theorem 5.1 (Decomposition Property). *Let $a \geqslant b > 0$, $\gcd(\{a,b\}) = 1$ and assume that $\{a,b\} \neq \{2,1\}$. Using row permutations, multiplications of rows by -1 and elementary column operations, any $\{a,b,0\}$-modular matrix $A \in \mathbb{Z}^{m\times n}$ can be brought into a block structure of the form*

$$\begin{bmatrix} L & 0 & 0/a \\ 0 & R & 0/b \end{bmatrix}, \tag{5.0}$$

in time polynomial in n, m and $\log ||A||_\infty$, where $L \in \mathbb{Z}^{m_1\times n_1}$ and $R \in \mathbb{Z}^{m_2\times n_2}$ are TU, $n_1 + n_2 = n - 1$, $m_1 + m_2 = m$. In the representation above the rightmost

[1]For reasons of readability, we will stick to the notation that $a \geqslant b \geqslant c$ although the order of these elements is irrelevant.

column has entries in $\{0, a\}$ *and* $\{0, b\}$*, respectively. The matrix* $\begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}$ *contains the* $(n-1)$*-dimensional unit matrix as a submatrix.*

The first $n-1$ columns of (5.1) are TU since they form a 1-sum of two TU matrices (see [63, Chapter 19.4]). This structural property lies at the core of the following recognition algorithm. We say that a matrix A possesses a **duplicative relation** if it has nonzero $n \times n$ subdeterminants k_1 and k_2 satisfying $2 \cdot |k_1| = |k_2|$.

Theorem 5.2 (Recognition Algorithm). *There exists an algorithm that solves the following recognition problem in time polynomial in* n*,* m *and* $\log ||A||_\infty$*: Either, calculate* $D(A)$*, or give a certificate that* $|D(A)| \geqslant 4$*, or return a duplicative relation.*

For instance, Theorem 5.2 cannot be applied to check whether a matrix is $\{4, 2, 0\}$-modular, but it can be applied to check whether a matrix is $\{3, 1, 0\}$-, or $\{6, 4, 0\}$-modular. More specifically, Theorem 5.2 recognizes $\{a, b, c\}$-modular matrices unless $(a, b, c) = (2 \cdot k, k, 0)$, $k \in \mathbb{Z}_{\geqslant 1}$. In particular, this paper does not give a contribution as to whether so-called **bimodular matrices** (the case $k = 1$) can be recognized efficiently.

The decomposition property established in Theorem 5.1 is a major ingredient for the following optimization algorithm to solve standard form IPs over $\{a, b, c\}$-modular constraint matrices for any constant $a \geqslant b \geqslant c \geqslant 0$.

Theorem 5.3 (Optimization Algorithm). *Consider a standard form integer program of the form*

$$\max\{c^\mathsf{T} x \colon Bx = b,\ x \in \mathbb{Z}^n_{\geqslant 0}\}, \tag{5.1}$$

for $b \in \mathbb{Z}^m$*,* $c \in \mathbb{Z}^n$ *and* $B \in \mathbb{Z}^{m \times n}$ *of full row rank, where* $D(B^\mathsf{T})$ *is constant, i.e.,* $\max\{k \colon k \in D(B^\mathsf{T})\}$ *is constant.*[2] *Then, in time polynomial in* n*,* m *and the encoding size of the input data, one can solve* (5.1) *or output that* $|D(B^\mathsf{T})| \geqslant 4$*.*

Notably, in Theorem 5.3, the assumption that $D(B^\mathsf{T})$ is constant can be dropped if B is **degenerate**, i.e., if $0 \in D(B^\mathsf{T})$.

5.1.2 An Example for $\{a, b, c\}$-Modular Matrices

- *Generalized network flow.* Let $G = (V, E)$ be a directed graph whose vertices can be partitioned as $V = S \cup \{v\} \cup T$ such that no arc runs between S and T, and no arc runs from v to S. Consider a generalized network flow problem in G, where $s \in S$ and $t \in T$, with capacities $u \colon E \to \mathbb{Z}_{>0}$ and gains

$$\gamma(e) := \begin{cases} \frac{a}{b} & \text{if } e \text{ runs from } S \text{ to } v, \\ 1 & \text{otherwise,} \end{cases}$$

[2]Note that we use $D(B^\mathsf{T})$ instead of $D(B)$ since B has full row rank and we refer to its $m \times m$ subdeterminants.

where $a, b \in \mathbb{Z}_{>0}$. Then, the natural formulation of finding a maximal s-t flow in G with respect to u and γ is an integer optimization problem whose constraint matrix A satisfies $D(A^\mathsf{T}) \subseteq \{a, b, 0\}$ after multiplying the constraint corresponding to v by b.

5.2 Notation and Preliminaries

For $k \in \mathbb{Z}_{\geqslant 1}$, $[k] := \{1, \ldots, k\}$. $\mathcal{I}_n$ is the n-dimensional unit matrix, where we leave out the subscript if the dimension is clear from the context. For a matrix $A \in \mathbb{Z}^{m \times n}$, we denote by $A_{i,\cdot}$ the i-th row of A. For a subset I of $[m]$, $A_{I,\cdot}$ is the submatrix formed by selecting all rows with indices in I, in increasing order. An analogous notation is used for the columns of A. For $k \in \mathbb{Z}$, we write $I_k := \{i \in [m] \colon A_{i,n} = k\}$, the indices of the rows whose n-th entry is equal to k. Set $||A||_\infty := \max_{i \in [m], j \in [n]} |A_{ij}|$. For simplicity, we assume throughout the document that the input matrix A to any recognition algorithm satisfies $m \geqslant n$ and $\operatorname{rank}(A) = n$ as $\operatorname{rank}(A) < n$ implies that $D(A) = \{0\}$. Left-out entries in figures and illustrations are equal to zero.

$D(A)$ is preserved under elementary column operations, permutations of rows and multiplications of rows by -1. By a series of elementary column operations on A, any nonsingular $n \times n$ submatrix B of A can be transformed to its Hermite normal form (HNF), in which B becomes a lower-triangular, nonnegative submatrix with the property that each of its rows has a unique maximum entry on its main diagonal [63]. This can be done in time polynomial in n, m and $\log ||A||_\infty$ [69].

At various occasions we will make use of a simple adaptation of the HNF described in [7, Section 3]: Choose a nonsingular $n \times n$ submatrix B of A. When applying this transformation, either, the matrix B will be clear from the context, otherwise, choose B to be any nonsingular $n \times n$ submatrix of A. Permute the rows of A such that $A_{[n],\cdot} = B$. Apply elementary column operations to A such that B is in HNF. After additional row and column permutations and multiplications of rows by -1,

$$A = \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & & & & \\ * & \cdots & * & \delta_1 & & & \\ \vdots & \vdots & \vdots & \ddots & \ddots & & \\ * & \cdots & \cdots & \cdots & * & \delta_l & \\ A_{n,1} & \cdots & \cdots & \cdots & \cdots & A_{n,n-1} & A_{n,n} \\ A_{n+1,1} & \cdots & \cdots & \cdots & \cdots & A_{n+1,n-1} & A_{n+1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m,1} & \cdots & \cdots & \cdots & \cdots & A_{m,n-1} & A_{m,n} \end{bmatrix},$$

where $A_{\cdot,n} \geqslant 0$, $|\det(B)| = (\prod_{i=1}^{l} \delta_i) \cdot A_{n,n}$, all entries marked by $*$ are numbers between 0 and the corresponding entry on the main diagonal minus one, and $\delta_i \geqslant 2$

for all $i \in [l]$. In particular, the rows of $A_{[n],\cdot}$ whose entries on the main diagonal are strictly larger than one are at positions $n-l,\dots,n$.

We note that it is not difficult to efficiently recognize nondegenerate matrices given a constant upper bound on $|D(A)|$. This will allow us to exclude the nondegenerate case in all subsequent algorithms. We wish to emphasize that the results in [7] can be applied to solve this task given that $\max\{k\colon k \in D(A)\}$ is constant.

Lemma 5.4. *Given a constant $d \in \mathbb{Z}$, there exists an algorithm that solves the following recognition problem in time polynomial in n, m and $\log ||A||_\infty$: Either, calculate $D(A)$, or give a certificate that $|D(A)| \geqslant d+1$, or that $0 \in D(A)$.*

5.3 Proof of Theorem 5.3

One ingredient to the proof of Theorem 5.3 is the following result by Gribanov, Malyshev and Pardalos [37] which reduces the standard form IP (5.1) to an IP in inequality form in dimension $n-m$ such that the subdeterminants of the constraint matrices are in relation.

Lemma 5.5 ([37, Corollary 1.1, Remark 5, Theorem 3][3])**.** *In time polynomial in n, m and $\log ||B||_\infty$,* (5.1) *can be reduced to the inequality form IP*

$$\max\{h^\top y\colon Cy \leqslant g,\ y \in \mathbb{Z}^{n-m}\}, \tag{5.2}$$

where $h \in \mathbb{Z}^{n-m}$, $g \in \mathbb{Z}^n$ and $C \in \mathbb{Z}^{n\times(n-m)}$, with $D(C) = \frac{1}{\gcd(D(B^\top))} \cdot D(B^\top)$.

To prove this reduction, the authors apply a theorem by Shevchenko and Veselov [73] which was originally published in Russian. For completeness of presentation, we will provide an alternative but similar proof of Lemma 5.5 which uses the following well-known determinant identity instead of the aforementioned result.

Lemma 5.6 (Jacobi's complementary minor formula, see [14, Lemma A.1e])**.** *Let $A \in \mathbb{Z}^{n\times n}$ be invertible and $I, J \subseteq [n]$, $|I| = |J| = k$ for $k \in [n]$. Then,*

$$\det(A_{I,J}) = \det(A) \cdot (-1)^{\sum_{i\in I} i + \sum_{j\in J} j} \cdot \det\left(A^{-1}_{\overline{J},\overline{I}}\right),$$

where $\overline{I} := [n]\backslash I$ and $\overline{J} := [n]\backslash J$.

Alternative proof of Lemma 5.5. We closely follow the proof of [37]. First, we reformulate (5.1) in such a way that the $\gcd$ of the full rank subdeterminants of the constraint matrix becomes 1. To this end, calculate the Smith normal form of B, i.e., find $P \in \mathbb{Z}^{m\times m}$ and $Q \in \mathbb{Z}^{n\times n}$ unimodular and nonsingular such that $B = P[S \mid 0]Q$, where $S \in \mathbb{Z}^{m\times m}$ is a diagonal matrix satisfying $\prod_{i=1}^m S_{i,i} = \gcd(D(B^\top))$. This can be done in time polynomial in m, n and $\log ||B||_\infty$, see [68].

[3]Note that in [37], the one-to-one correspondence between $D(C)$ and $D(B^\top)$ is not explicitly stated in Corollary 1.1 but follows from Theorem 3.

Thus, $Bx = b \Leftrightarrow [\mathcal{I}_m \mid 0]Qx = S^{-1}P^{-1}b$. For simplicity, set $b' := S^{-1}P^{-1}b$. If $b' \notin \mathbb{Z}^m$, then (5.1) is infeasible. Thus, assume that $b' \in \mathbb{Z}^m$. Summarizing, solving (5.1) is equivalent to solving

$$\max\{c^\mathsf{T}x\colon [\mathcal{I}_m \mid 0]Qx = b',\ x \in \mathbb{Z}^n_{\geqslant 0}\}, \tag{5.3}$$

where due to multiplying by S^{-1}, $D(([\mathcal{I}_m \mid 0]Q)^\mathsf{T}) = \frac{1}{\gcd(D(B^\mathsf{T}))} \cdot D(B^\mathsf{T})$.

Secondly, we reduce (5.3) to an IP in inequality form. Since Q is unimodular, substituting $z := Qx$ yields

$$\begin{aligned} &\{x \in \mathbb{Z}^n\colon [\mathcal{I}_m \mid 0]Qx = b'\} \\ =\ & Q^{-1}\{z \in \mathbb{Z}^n\colon z_{[m]} = b'\} \\ =\ & \{Q^{-1}_{\cdot,[m]}b' + Q^{-1}_{\cdot,[n]\setminus[m]}y\colon y \in \mathbb{Z}^{n-m}\}. \end{aligned}$$

Plugging this identity into (5.3) yields

$$\begin{aligned} & \max\{c^\mathsf{T}(Q^{-1}_{\cdot,[m]}b' + Q^{-1}_{\cdot,[n]\setminus[m]}y)\colon Q^{-1}_{\cdot,[m]}b' + Q^{-1}_{\cdot,[n]\setminus[m]}y \geqslant 0,\ y \in \mathbb{Z}^{n-m}\} \\ = \quad & c^\mathsf{T}Q^{-1}_{\cdot,[m]}b' + \max\{h^\mathsf{T}y\colon Cy \leqslant g,\ y \in \mathbb{Z}^{n-m}\}, \end{aligned}$$

where $h^\mathsf{T} := c^\mathsf{T}Q^{-1}_{\cdot,[n]\setminus[m]}$, $g := Q^{-1}_{\cdot,[m]}b'$ and $C := -Q^{-1}_{\cdot,[n]\setminus[m]}$.

Recall that $D(([\mathcal{I}_m \mid 0]Q)^\mathsf{T}) = \frac{1}{\gcd(D(B^\mathsf{T}))} \cdot D(B^\mathsf{T})$. As $[\mathcal{I}_m \mid 0]Q = Q_{[m],\cdot}$, it remains to show that $D((Q_{[m],\cdot})^\mathsf{T}) = D(Q^{-1}_{\cdot,[n]\setminus[m]})$. Lemma 5.6 applied to $A := Q$ for $I := [m]$ and $J \subseteq [n]$, $|J| = m$, yields

$$|\det(Q_{[m],J})| = |\det(Q^{-1}_{J^c,[n]\setminus[m]})|,$$

i.e., the claim follows. □

As a second ingredient to the proof of Theorem 5.3, we will make use of some results for **bimodular integer programs** (BIPs). BIPs are IPs of the form

$$\max\{c^\mathsf{T}x\colon Ax \leqslant b,\ x \in \mathbb{Z}^n\},$$

where $c \in \mathbb{Z}^n$, $b \in \mathbb{Z}^m$ and $A \in \mathbb{Z}^{m\times n}$ is **bimodular**, i.e., $\operatorname{rank}(A) = n$ and $D(A) \subseteq \{2, 1, 0\}$. As mentioned earlier, [8] proved that BIPs can be solved in strongly polynomial time. Their algorithm uses the following structural result for BIPs by [72] which will also be useful to us.

Theorem 5.7 ([72, Theorem 2], as formulated in [8, Theorem 2.1]). *Assume that the linear relaxation* $\max\{c^\mathsf{T}x\colon Ax \leqslant b,\ x \in \mathbb{R}^n\}$ *of a BIP is feasible, bounded and has a unique optimal vertex solution* v. *Denote by* $I \subseteq [m]$ *the indices of the constraints which are tight at* v, *i.e.,* $A_{I,\cdot}v = b_I$. *Then, an optimal solution* x^* *of* $\max\{c^\mathsf{T}x\colon A_Ix \leqslant b_I,\ x \in \mathbb{Z}^n\}$ *is also optimal for the BIP.*

Proof of Theorem 5.3. Using Lemma 5.5, we reduce the standard form IP (5.1) to (5.2). Note that $\gcd(D(C)) = 1$. Let us denote (5.2) with objective vector $h \in \mathbb{Z}^{n-m}$ by $\mathtt{IP}_{\leqslant}(h)$ and its natural linear relaxation by $\mathtt{LP}_{\leqslant}(h)$. We apply Theorem 5.2 to C and perform a case-by-case analysis depending on the output.

1. The algorithm calculates and returns $D(C)$. If $0 \notin D(C)$, C is nondegenerate and $\mathtt{IP}_{\leqslant}(h)$ can be solved using the algorithm in [7]. Thus, assume that $0 \in D(C)$. C has no duplicative relations. As $\gcd(D(C)) = 1$, this implies that C is $\{a, b, 0\}$-modular for $a \geqslant b > 0$, where $\gcd(\{a, b\}) = 1$ and $(a, b) \neq (2, 1)$. Thus, C satisfies the assumptions of Theorem 5.1. As a consequence of Theorem 5.1, there exist elementary column operations which transform C such that its first $n - m - 1$ columns are TU, i.e., there is $U \in \mathbb{Z}^{(n-m)\times(n-m)}$ unimodular such that $CU = [T \mid d]$, where T is TU and $d \in \mathbb{Z}^n$. Substituting $z := U^{-1}y$ yields the equivalent problem
$$\max\{h^{\mathsf{T}}Uz\colon [T \mid d]z \leqslant g,\ z \in \mathbb{Z}^{n-m}\}, \tag{5.4}$$
where we have used that $y = Uz \in \mathbb{Z}^{n-m} \Leftrightarrow z \in \mathbb{Z}^{n-m}$ as U preserves integrality. Let z^* be an optimal solution to the mixed-integer linear program
$$\max\{h^{\mathsf{T}}Uz\colon [T \mid d]z \leqslant g,\ z \in \mathbb{R}^{n-m},\ z_{n-m} \in \mathbb{Z}\}, \tag{5.5}$$
which can be found in polynomial time [63, Chapter 18.4]. If no such solution exists, (5.4) is infeasible. Fixing $z_{n-m} := z^*_{n-m}$ in (5.5) induces an LP in dimension $n - m - 1$. Let $\bar{z}$ be a vertex solution to this LP, which can be found efficiently (see, for example, [39]).[4] Since T is TU, $\bar{z} \in \mathbb{Z}^{n-m-1}$. The solution $[\bar{z} \mid z^*_n]$ has the same objective value as z^* and is optimal for (5.4) since it is integral and (5.5) is a relaxation of (5.4).

2. The algorithm returns that $|D(C)| \geqslant 4$. Then, $|D(B^{\mathsf{T}})| = |D(C)| \geqslant 4$.

3. The algorithm returns a duplicative relation, i.e., $\{2 \cdot k, k\} \subseteq D(C)$, $k > 0$. This case is more involved because we do not have any information as to which other elements might be contained in $D(C)$.

   Assume w.l.o.g. that $\mathtt{LP}_{\leqslant}(h)$ is feasible and that $\mathtt{LP}_{\leqslant}(h)$ is bounded. We postpone the unbounded case to the end of the proof. Calculate an optimal vertex solution v to $\mathtt{LP}_{\leqslant}(h)$. If $v \in \mathbb{Z}^{n-m}$, then v is also optimal for $\mathtt{IP}_{\leqslant}(h)$. Thus, assume that $v \notin \mathbb{Z}^{n-m}$ and let $I \subseteq [n]$ be the indices of tight constraints at v, i.e., $C_{I,\cdot}v = g_I$. In what follows, we prove that we may assume w.l.o.g. that (a) $0 \in D(C)$, (b) $k = 1$, and (c) C_I is bimodular.

   a) From Lemma 5.4 applied to C for $d = 3$ we obtain three possible results: $|D(C)| \geqslant 4$, $0 \notin D(C)$ or $0 \in D(C)$. In the first case we are done and in the second case, C is nondegenerate and $\mathtt{IP}_{\leqslant}(h)$ can be solved using the algorithm in [7]. Therefore, w.l.o.g., $0 \in D(C)$.

   b) If $\{2 \cdot k, k, 0\} \subseteq D(C)$ for $k > 1$, it follows from $\gcd(D(C)) = 1$ that $|D(C)| \geqslant 4$. Therefore, w.l.o.g., $k = 1$.

[4] As T has full column rank, the feasible region is pointed, i.e., such a vertex exists.

c) Since $v \notin \mathbb{Z}^{n-m}$, it holds that $1 \notin D(C_I)$ as otherwise, $v \in \mathbb{Z}^{n-m}$ due to Cramer's rule. Apply Theorem 5.2 once more, but this time to C_I. If the algorithm returns that $|D(C_I)| \geqslant 4$, then $|D(C)| \geqslant 4$. If the algorithm returns a duplicative relation, i.e., $\{2 \cdot s, s\} \subseteq D(C_I)$, then $s \neq 1$ as $1 \notin D(C_I)$. Since by (a) and (b), $\{2, 1, 0\} \subseteq D(C)$, it follows that $\{2 \cdot s, 2, 1, 0\} \subseteq D(C)$. Thus, $|D(C)| \geqslant 4$. If the algorithm calculates and returns $D(C_I)$, then it has either found that $D(C_I) \subseteq \{2, 0\}$ or it finds an element $t \in D(C_I) \backslash \{2, 0\}$. Then, $t \neq 1$ as $1 \notin D(C_I)$, implying that $\{t, 2, 1, 0\} \subseteq D(C)$ and $|D(C)| \geqslant 4$. Thus, w.l.o.g., C_I is bimodular.

Let $\texttt{IP}^{\text{cone}}_{\leqslant}(h) := \max\{h^{\mathsf{T}} y \colon C_I y \leqslant g_I,\ y \in \mathbb{Z}^{n-m}\}$. As C_I is bimodular, this is a BIP. By possibly perturbing the vector h (e.g. by adding $\frac{1}{M} \cdot \sum_{i \in I} C_{i,\cdot}$ for a sufficiently large $M > 0$), we can assume that v is the unique optimal solution to $\texttt{LP}_{\leqslant}(h)$, which will allow us to apply Theorem 5.7. Solve $\texttt{IP}^{\text{cone}}_{\leqslant}(h)$ using the algorithm by [8]. If $\texttt{IP}^{\text{cone}}_{\leqslant}(h)$ is infeasible, so is $\texttt{IP}_{\leqslant}(h)$. Let $y \in \mathbb{Z}^{n-m}$ be an optimal solution for $\texttt{IP}^{\text{cone}}_{\leqslant}(h)$. We claim that either, y is also optimal for $\texttt{IP}_{\leqslant}(h)$ or that $|D(C)| \geqslant 4$: If y is feasible for $\texttt{IP}_{\leqslant}(h)$, it is also optimal since $\texttt{IP}^{\text{cone}}_{\leqslant}(h)$ is a relaxation of $\texttt{IP}_{\leqslant}(h)$. If C is bimodular, Theorem 5.7 states that y is feasible for $\texttt{IP}_{\leqslant}(h)$, i.e., it is optimal for $\texttt{IP}_{\leqslant}(h)$. Thus, if y is not feasible for $\texttt{IP}_{\leqslant}(h)$, $D(C)$ contains an element which is neither 0 nor 1 nor 2. As $\{2, 1, 0\} \subseteq D(C)$ by (a) and (b), this implies that $|D(C)| \geqslant 4$.

It remains to explain why we may assume that $\texttt{LP}_{\leqslant}(h)$ is bounded. If not, $\texttt{IP}_{\leqslant}(h)$ is either infeasible or unbounded. More precisely, $\texttt{IP}_{\leqslant}(h)$ is unbounded if and only if $\{y \in \mathbb{Z}^{n-m} \colon Cy \leqslant g\}$ is feasible. We reduce the feasibility test of this set to a bounded IP of the same form as above: Set $s := C_{1,\cdot}$. By construction, $\texttt{LP}_{\leqslant}(s)$ is bounded. Solve $\texttt{IP}_{\leqslant}(s)$ using our algorithm above. Either, we determine a feasible point of $\{y \in \mathbb{Z}^{n-m} \colon Cy \leqslant g\}$ in which case $\texttt{IP}_{\leqslant}(h)$ is unbounded, we find that this set is infeasible, or we find that $|D(C)| \geqslant 4$.

□

Bibliography

[1] I. Aliev, G. Averkov, J. De Loera, and T. Oertel. Optimizing sparsity over lattices and semigroups. In D. Bienstock and G. Zambelli, editors, **Proceedings of the 21st International Integer Programming and Combinatorial Optimization Conference**, pages 40–51, Cham, 2020. Springer.

[2] I. Aliev, M. Henk, and T. Oertel. Distances to lattice points in knapsack polyhedra. **Mathematical Programming**, 182(3):808–816, 2019.

[3] I. Aliev, J. De Loera, F. Eisenbrand, T. Oertel, and R. Weismantel. The support of integer optimal solutions. **SIAM Journal on Optimization**, 28(3):2152–2157, 2018.

[4] I. Aliev, J. De Loera, T. Oertel, and C. O'Neil. Sparse solutions of linear diophantine equations. **SIAM Journal on Applied Algebra and Geometry**, 1(1):239–253, 2017.

[5] J. Alman and V.V. Williams. A refined laser method and faster matrix multiplication. In D. Marx, editor, **Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)**, pages 522–539, 2021.

[6] R.P. Anstee. Forbidden configurations, discrepancy and determinants. **European Journal of Combinatorics**, 11(1):15–19, 1990.

[7] S. Artmann, F. Eisenbrand, C. Glanzer, T. Oertel, S. Vempala, and R. Weismantel. A note on non-degenerate integer programs with small subdeterminants. **Operations Research Letters**, 44(5):635–639, 2016.

[8] S. Artmann, R. Weismantel, and R. Zenklusen. A strongly polynomial algorithm for bimodular integer linear programming. In **Proceedings of the 49th Annual ACM SIGACT Symposium on Theory of Computing**, STOC 2017, pages 1206–1219, New York, 2017. Association for Computing Machinery.

[9] G. Averkov and M. Schymura. Personal Communication. 2021.

[10] A. Barvinok. **A Course in Convexity**, volume 54. Graduate Studies in Mathematics, American Mathematical Society, Providence, Rhode Island, 2002.

[11] N. Bonifas, M. Di Summa, F. Eisenbrand, N. Hähnle, and M. Niemeier. On sub-determinants and the diameter of polyhedra. **Discrete & Computational Geometry**, 52(1):102–115, 2014.

[12] W. Bruns and J. Gubeladze. Normality and covering properties of affine semigroups. **Journal für die reine und angewandte Mathematik**, 510:151 – 178, 2004.

[13] W. Bruns, J. Gubeladze, M. Henk, A. Martin, and R. Weismantel. A counterexample to an integer analogue of Carathéodory's theorem. **Journal für die reine und angewandte Mathematik**, 510:179–185, 1999.

[14] S. Caracciolo, A.D. Sokal, and A. Sportiello. Algebraic/combinatorial proofs of cayley-type identities for derivatives of determinants and pfaffians. **Advances in Applied Mathematics**, 50(4):474–594, 2013.

[15] A. Cevallos, S. Weltge, and R. Zenklusen. Lifting linear extension complexity bounds to the mixed-integer setting. **Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms**, pages 788–807, 2018.

[16] M. Conforti, G. Cornuéjols, and G. Zambelli. **Integer Programming**. Graduate Texts Mathematics. Springer, Cham, 2014.

[17] M. Conforti, S. Fiorini, T. Huynh, G. Joret, and S. Weltge. The stable set problem in graphs with bounded genus and bounded odd cycle packing number. In **Proceedings of the 31st ACM-SIAM Symposium on Discrete Algorithms**, pages 2896–2915, 2020.

[18] M. Conforti, S. Fiorini, T. Huynh, and S. Weltge. Extended formulations for stable set polytopes of graphs without two disjoint odd cycles. In D. Bienstock and G. Zambelli, editors, **Proceedings of the 21st International Integer Programming and Combinatorial Optimization Conference**, pages 104–116, Cham, 2020. Springer.

[19] S.A. Cook. The complexity of theorem-proving procedures. In **Proceedings of the 3rd ACM Symposium on Theory of Computing**, pages 151–158, 1971.

[20] W. Cook, J. Fonlupt, and A. Schrijver. An integer analogue of Carathéodory's theorem. **Journal of Combinatorial Theory, Series B**, 40(1):63–70, 1986.

[21] W. Cook, A.M.H. Gerards, A. Schrijver, and É. Tardos. Sensitivity theorems in integer linear programming. **Mathematical Programming**, 34:251–264, 1986.

[22] M. Di Summa, F. Eisenbrand, Y. Faenza, and C. Moldenhauer. On largest volume simplices and sub-determinants. In **Proceedings of the 26th Annual ACM-SIAM Symposium on Discrete Algorithms**, pages 315–323, 2015.

[23] P. Diaconis, R.L. Graham, and B. Sturmfels. Primitive Partition Identities. **Combinatorics, Paul Erdös is Eighty**, 2:173–192, 1993.

[24] M. Dyer and A. Frieze. Random walks, totally unimodular matrices, and a randomised dual simplex algorithm. **Mathematical Programming**, 64(1-3):1–16, 1994.

[25] F. Eisenbrand, C. Hunkenschröder, K. Klein, M. Koutecký, A. Levin, and S. Onn. An algorithmic theory of integer programming. **Online Preprint: arXiv:1904.01361**, 2019.

[26] F. Eisenbrand, C. Hunkenschröder, and K-M. Klein. Faster algorithms for integer programs with block structure. In I. Chatzigiannakis, C. Kaklamanis, D. Marx, and D. Sannella, editors, **45th International Colloquium on Automata, Languages, and Programming**, volume 107, pages 49:1–49:13. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2018.

[27] F. Eisenbrand and G. Shmonin. Carathéodory bounds for integer cones. **Operations Research Letters**, 34(5):564–568, 2006.

[28] F. Eisenbrand and S. Vempala. Geometric random edge. **Mathematical Programming**, 164(1-2):325–339, 2017.

[29] F. Eisenbrand and R. Weismantel. Proximity results and faster algorithms for integer programming using the Steinitz lemma. In **Proceedings of the 29th Annual ACM-SIAM Symposium on Discrete Algorithms**, pages 808–816, 2018.

[30] J. Geelen, P. Nelson, and Z. Walsh. Excluding a line from $\mathbb{C}$-representable matroids. **Online Preprint: arXiv:2101.12000**, 2021.

[31] C. Glanzer. **Properties of Integer Programs with Bounded Subdeterminants**. PhD thesis, ETH Zürich, Zürich, 2021. Diss.-Nr. 27808.

[32] C. Glanzer, I. Stallknecht, and R. Weismantel. On the Recognition of $\{a, b, c\}$-Modular Matrices. In M. Singh and D.P. Williamson, editors, **Proceedings of the 22nd International Integer Programming and Combinatorial Optimization Conference**, pages 238–251, Cham, 2021. Springer.

[33] C. Glanzer, I. Stallknecht, and R. Weismantel. Notes on $\{a, b, c\}$-Modular Matrices. **Vietnam Journal of Mathematics**, 50(2):469–485, 2022.

[34] C. Glanzer, R. Weismantel, and R. Zenklusen. On the number of distinct rows of a matrix with bounded subdeterminants. **SIAM Journal of Discrete Mathematics**, 32:1706–1720, 2018.

[35] C. Glanzer, R. Weismantel, and R. Zenklusen. On the number of distinct rows of a matrix with bounded subdeterminants. **SIAM Journal on Discrete Mathematics**, 32(3):1706–1720, 2018.

[36] F. Granot and J. Skorin-Kapov. Some proximity and sensitivity results in quadratic integer programming. **Mathematical Programming**, 47(2):259–268, 1990.

[37] D.V. Gribanov, D.S. Malyshev, and P.M. Pardalos. A note on the parametric integer programming in the average case: sparsity, proximity, and FPT-algorithms. **Online Preprint: arXiv:2002.01307**, 2020.

[38] D.V. Gribanov and S.I. Veselov. On integer programming with bounded determinants. **Optimization Letters**, 10(6):1169–1177, 2016.

[39] M. Grötschel, L. Lovász, and A. Schrijver. **Geometric Algorithms and Combinatorial Optimization**, volume 2. Springer Science & Business Media, 2012.

[40] I. Heller. On linear systems with integral valued solutions. **Pacific Journal of Mathematics**, 7(3):1351–1364, 1957.

[41] D.S. Hochbaum and J.G. Shanthikumar. Convex separable optimization is not much harder than linear optimization. **Journal of the ACM**, 37(4):843–862, 1990.

[42] A.J. Hoffman and J.B. Kruskal. Integral boundary points of convex polyhedra. **Linear Inequalities and Related Systems**, 24:223–246, 1956.

[43] R.A. Horn and C.R. Johnson. **Matrix Analysis 2nd Edition**. Cambridge University Press New York, NY, USA, 2012.

[44] R.M. Karp. Reducibility among combinatorial problems. In **Complexity of Computer Computations**, pages 85–103, New York, 1972. Plenum Press.

[45] L.G. Khachiyan. A polynomial algorithm in linear programming (in Russian). **Doklady Akademii Nauk SSSR**, 244:1093–1096, 1979.

[46] L.G. Khachiyan. Polynomial algorithms in linear programming (in Russian). **Zhurnal Vychislitel'noi Matematiki i Matematicheskoi Fiziki**, 20:51–68, 1980.

[47] L.G. Khachiyan. On the complexity of approximating extremal determinants in matrices. **Journal of Complexity**, 11(1):138–153, 1995.

[48] J. Kung. Combinatorial geometries representable over $\mathrm{GF}(3)$ and $\mathrm{GF}(q)$. I. The number of points. **Discrete Computational Geometry**, 5:83–95, 1990.

[49] J. Kung. The long-line graph of a combinatorial geometry. II geometries representable over two fields of different characteristics. **Journal of Combinatorial Theory Series B**, 50:41–53, 1990.

[50] J. Lee. Subspaces with well-scaled frames. **Linear Algebra and its Applications**, 115:21–56, 1989.

[51] J. Lee, J. Paat, I. Stallknecht, and L. Xu. Improving proximity bounds using sparsity. In M. Baïou, B. Gendron, O. Günlük, and A.R. Mahjoub, editors, **Proceedings of the 6th International Symposium on Combinatorial Optimization**, pages 115–127. Springer, 2020.

[52] J. Lee, J. Paat, I. Stallknecht, and L. Xu. Polynomial upper bounds on the number of differing columns of Δ-modular integer programs. **Mathematics of Operations Research**, pages 1–20, 2022.

[53] H.W. Lenstra. Integer programming with a fixed number of variables. **Mathematics of Operations Research**, 8(4):538–548, 1983.

[54] J.A. De Loera, R. Hemmecke, and M. Köppe. **Algebraic and Geometric Ideas in the Theory of Discrete Optimization**. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2012.

[55] M. Nǎgele, B. Sudakov, and R. Zenklusen. Submodular minimization under congruency constraints. **Combinatorica**, 39(6):1351–1386, 2019.

[56] G.L. Nemhauser and L.A. Wolsey. **Integer and Combinatorial Optimization**. Wiley interscience series in discrete mathematics and optimization. Wiley, 1988.

[57] T. Oertel, J. Paat, and R. Weismantel. Sparsity of integer solutions in the average case. In A. Lodi and V. Nagarajan, editors, **Proceedings of the 20th International Integer Programming and Combinatorial Optimization Conference**, pages 341–353, 2019.

[58] T. Oertel, J. Paat, and R. Weismantel. The distributions of functions related to parametric integer optimization. **SIAM Journal on Applied Algebra and Geometry**, 4(3):422–440, 2020.

[59] J.G. Oxley. **Matroid theory**. Oxford University Press, 1992.

[60] J.G. Oxley and Z. Walsh. 2-modular matrices. **Online Preprint: arXiv:2105.04525**, 2021.

[61] J. Paat, M. Schlöter, and R. Weismantel. The integrality number of an integer program. In D. Bienstock and G. Zambelli, editors, **Proceedings of the 21st International Integer Programming and Combinatorial Optimization Conference**, pages 338–350, Cham, 2020. Springer.

[62] J. Paat, R. Weismantel, and S. Weltge. Distances between optimal solutions of mixed-integer programs. **Mathematical Programming**, 179(1):455–467, 2018.

[63] A. Schrijver. **Theory of linear and integer programming**. John Wiley & Sons, Inc. New York, NY, 1986.

[64] A. Sebő. Hilbert bases, Carathéodory's theorem and combinatorial optimization. In **Proceedings of the 1st International Integer Programming and Combinatorial Optimization Conference**, pages 431–455, 1990.

[65] P.D. Seymour. Decomposition of regular matroids. **Journal of Combinatorial Theory, Series B**, 28(3):305–359, 1980.

[66] N.J.A. Sloane et al. The on-line encyclopedia of integer sequences. **Published electronically at http://oeis.org**, 1, 2018.

[67] E. Steinitz. Bedingt konvergente reihen und konvexe systeme. **Journal für die reine und angewandte Mathematik**, 143:128–176, 1913.

[68] A. Storjohann. **Algorithms for matrix canonical forms**. PhD thesis, ETH Zürich, Zürich, 2000. Diss.-Nr. 13922.

[69] A. Storjohann and G. Labahn. Asymptotically fast computation of hermite normal forms of integer matrices. In **Proceedings of the 1996 International Symposium on Symbolic and Algebraic Computation**, ISSAC '96, pages 259–266, New York, 1996. Association for Computing Machinery.

[70] A. Tamir. New pseudopolynomial complexity bounds for the bounded and other integer knapsack related problems. **Operations Research Letters**, 37(5):303–306, 2009.

[71] K. Truemper. A decomposition theory for matroids. v. testing of matrix total unimodularity. **Journal of Combinatorial Theory, Series B**, 49(2):241–281, 1990.

[72] S.I. Veselov and A.J. Chirkov. Integer program with bimodular matrix. **Discrete Optimization**, 6(2):220–222, February 2009.

[73] S.I. Veselov and V.N. Shevchenko. Bounds for the maximal distance between the points of certain integer lattices (in russian). **Combinatorial-Algebraic Methods in Applied Mathematics**, pages 26–33, 1980.

[74] M. Walter and K. Truemper. Implementation of a unimodularity test. **Mathematical Programming Computation**, 5(1):57–73, 2013.

[75] M. Werman and D. Magagnosc. The relationship between integer and real solutions of constrained convex programming. **Mathematical Programming**, 51(1–3):133–135, 1991.

[76] L. Xu and J. Lee. On proximity for k-regular mixed-integer linear optimization. In H. Le Thi, H. Le, and T. Pham Dinh, editors, **Optimization of Complex Systems: Theory, Models, Algorithms and Applications**, pages 438–447, Cham, 2019. Springer.