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The lively siblings of the Pentagon theorem

Norbert Hungerbühler

Abstract. The five circles in the classical Pentagon theorem of Miquel are given as circumcircles of five certain triangles in the pentagon. If one chooses instead the circumcircles of five other triangles, one gets a different configuration of circles. This resulting configuration of circles carries three families of five concyclic quadruples of points. Together with the five circumcircles this gives a total of 20 circles. The radical axes of each two of these twenty circles are all concurrent.

Mathematics Subject Classification. 51B10, 05B25, 51E30.

Keywords. Pentagon theorem, Möbius plane.

1. Introduction

The classical version of Miquel's Pentagon theorem on the Riemann sphere can be formulated as follows:

Theorem 1. *Let $h_1, \ldots, h_5$ be five different Möbius circles which intersect each other at a point I and such that any three of them only meet in I. Then, for $i \in \{1, \ldots, 5\}$, h_{i-1} and h_{i+1} meet in I and a second point q_i, and h_{i-2} and h_{i+2} meet in I and a second point s_i (indices read cyclically). Let k_i be the Möbius circle through s_i, q_{i-1}, q_{i+1}. Then, for $i \in \{1, \ldots, 5\}$, k_{i-1} and k_{i+1} meet in q_i and a second point p_i, and the points $p_1, \ldots, p_5$ all lie on one common Möbius circle c.*

The situation is shown in Fig. 1. Miquel's original proof can be found in [4, Théorème III]. It is based on classical angle theorems. A computer assisted algebraic proof which uses null bracket algebra has been published in [3]. A simple algebraic proof based on the cross ratio has been discussed in [1].

The assumption that the Möbius circles h_i intersect (not touch) each other in I implies that the points q_i and s_i are different from I. In addition, since we

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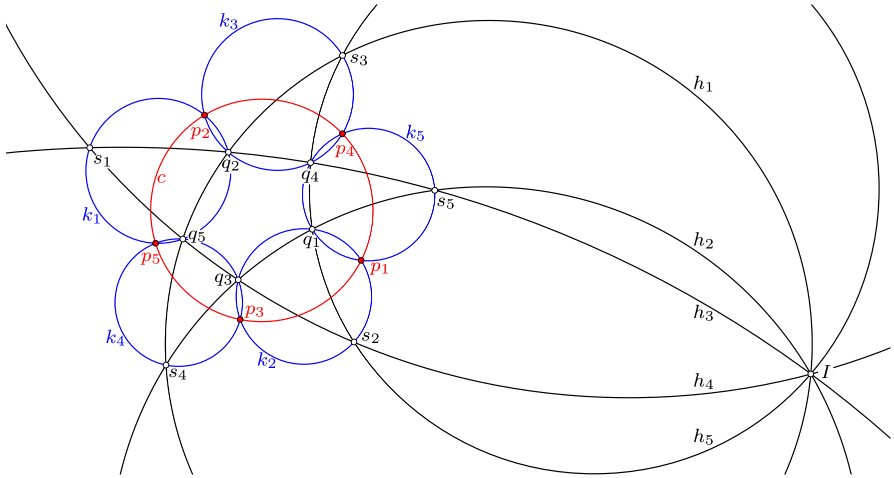

Figure 1 The classical Pentagon theorem

assume that any three of the circles h_i only meet in I, we have that the 10 points q_i, s_i are pairwise distinct. These assumptions can be relaxed if one is interested in degenerate cases of the configuration. Using the Möbius transformation $z \mapsto 1/(z - I)$ we may always assume that $I = \infty$. In this case, the Möbius circles h_i are lines in the complex plane.

The idea is now to replace the circles k_i through s_i, q_{i-1}, q_{i+1} by circles k_i through s_i, q_{i-2}, q_{i+2}. This variant has apparently not yet been treated in the literature. Surprisingly, this configuration shows numerous incidences, even significantly more than the classical Pentagon Theorem 1.

Theorem 2. *Let $h_1, \ldots, h_5$ be five different Möbius circles which intersect each other at a point I and such that any three of them only meet in I. Then, for $i \in \{1, \ldots, 5\}$, h_{i-1} and h_{i+1} meet in I and a second point q_i, and h_{i-2} and h_{i+2} meet in I and a second point s_i (indices read cyclically). Let k_i be the Möbius circle through s_i, q_{i-2}, q_{i+2}, and let p_{ij}, r_{ij} be the points of intersection of k_i and k_j. Then, for $i \in \{1, \ldots, 5\}$, the following quadruples of points are concyclic:*

- $p_{i+1,i-2}, r_{i+1,i-2}, p_{i+2,i-1}, r_{i+2,i-1}$ *lie on a circle c_i (see Fig. 2).*
- $q_i, p_{i-1,i+1}, r_{i-1,i+1}, p_{i+2,i-2}$ *lie on a circle d_i (see Fig. 3).*
- $q_{i-1}, q_{i+1}, p_{i+1,i+2}, p_{i-2,i-1}$ *lie on a circle e_i (see Fig. 4).*

It does not matter which of the two points of intersection of the circles k_i and k_j is denoted by p_{ij} or r_{ij}. However, let us agree that the intersection of k_{i-2} and k_{i+2} is $q_i = r_{i+2,i-2}$ while the second point of intersection of k_{i-2} and k_{i+2} will be denoted $p_{i+2,i-2}$. Note that k_{i-2} and k_{i+2} always have two points of intersection, $q_i = r_{i+2,i-2}$ and $p_{i+2,i-2}$. However, whether k_i and k_{i+2} have common points depends on the position of the points q_i.

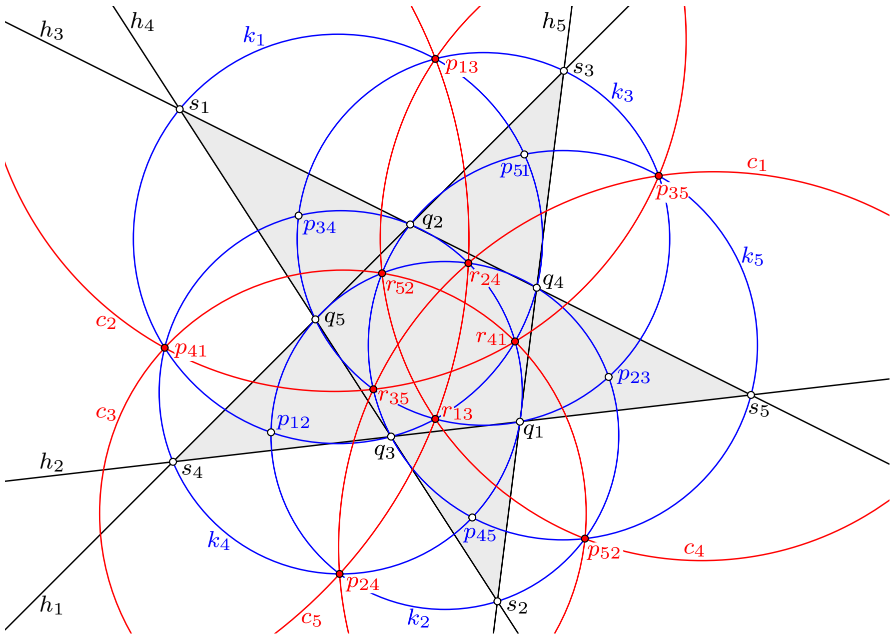

FIGURE 2 First sibling of the Pentagon theorem

A geometric proof of Theorem 2 would certainly be possible. On the other hand, a direct approach using coordinates in order to compute the various points leads to horrible and really long expressions. Here, we propose a proof based on calculations with complex numbers and the theory of the power of a point with respect to a circle.

Interestingly, not only the Miquel Pentagon Theorem has this new relative in Theorem 2, but recently also a new variant of Morley's Five Circles Theorem was discovered (see [2]).

It turns out that Theorem 2 follows easily from another incidence result, the mother of the siblings in Fig. 5, which we formulate now.

Theorem 3. *In the configuration of Theorem 2 the 10 Möbius circles though* I, p_{ij}, r_{ij}, $i \neq j \in \{1, \ldots, 5\}$, *are either concurrent in a point* $J \neq I$, *or they touch each other in the point* I.

2. Siblings of the Pentagon Theorem: The proof

In this section, we carry out the computations in the complex plane $\mathbb{C}$. In particular, z denotes a complex variable, and $\bar{z}$ is its complex conjugate. The equation of a line through two different points p, q is given by

$$(p - z)(\bar{q} - \bar{z}) = (\bar{p} - \bar{z})(q - z).$$

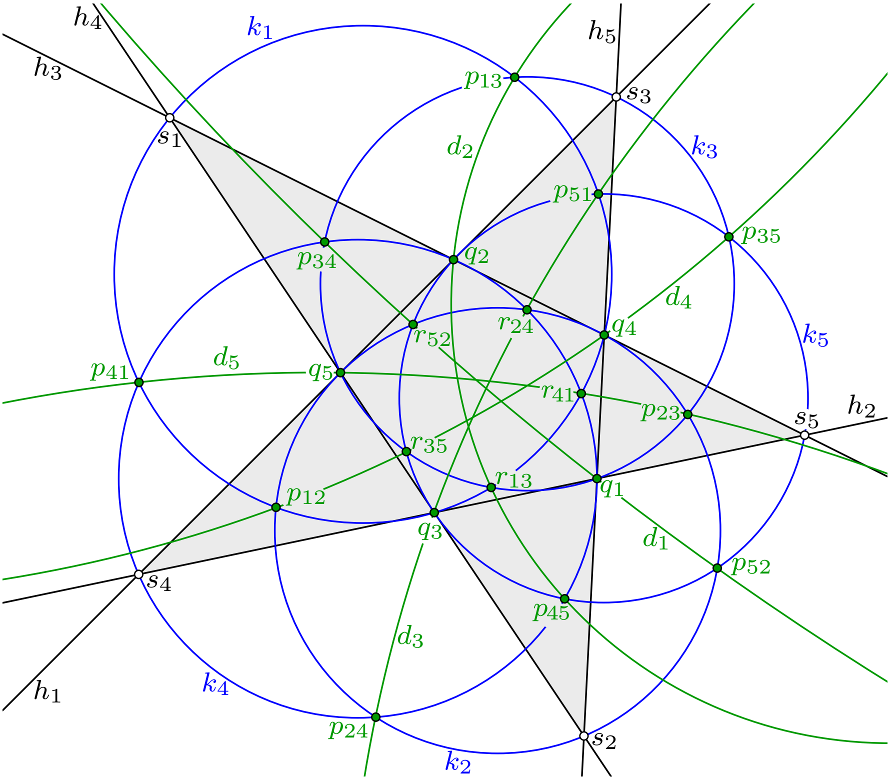

FIGURE 3 Second sibling of the Pentagon theorem

Indeed $z = p$ and $z = q$ are solutions of this equation, and expanded it has the form

$$\bar{a}z + a\bar{z} = c, \quad a \in \mathbb{C} \setminus \{0\}, c \in \mathbb{R},$$

of a line.

Similarly, the equation of a circle through three different points p, q, r (which do not lie on a line) is given by

$$(p - q)(r - z)(\bar{p} - \bar{z})(\bar{r} - \bar{q}) = (\bar{p} - \bar{q})(\bar{r} - \bar{z})(p - z)(r - q)$$

since $z = p, z = q, z = r$ are solutions of this equation, and expanded it has the form

$$(z - c)(\bar{z} - \bar{c}) = r, \quad c \in \mathbb{C}, r \in \mathbb{R}_+,$$

of a circle.

Observe that the group of Möbius transformations

$$z \mapsto \frac{az + b}{cz + d}, \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, \quad a, b, c, d \in \mathbb{C},$$

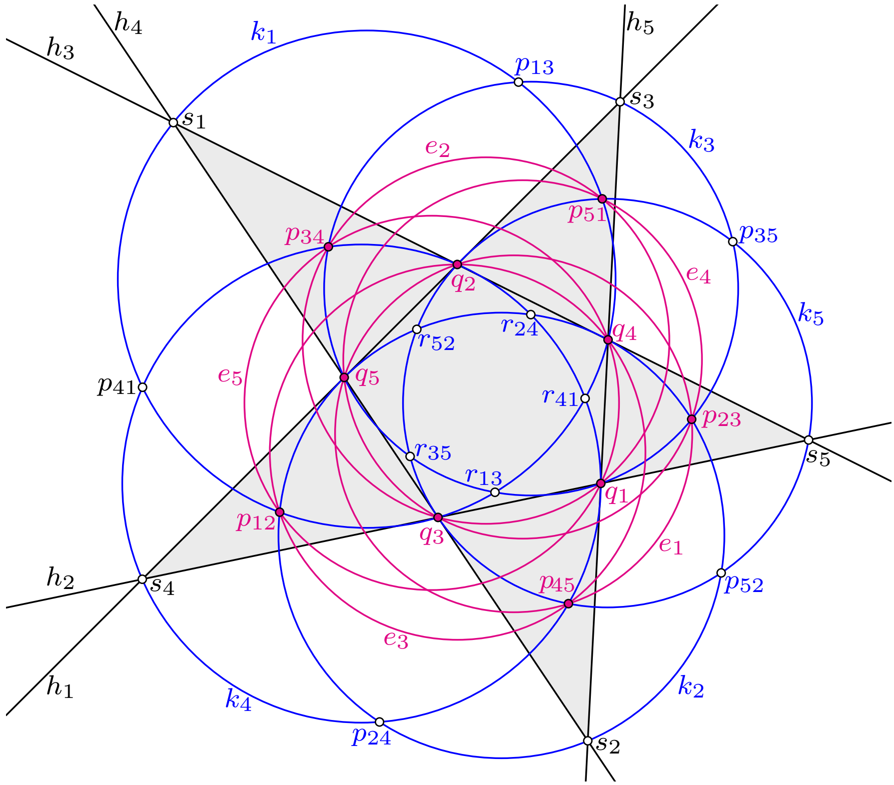

Figure 4 Third sibling of the Pentagon theorem

(with the usual convention $1/0 = \infty, 1/\infty = 0$) is sharply 3-transitive on the set of points and maps blocks (*i.e.* circles or lines) to blocks. The Möbius transformation $z \mapsto 1/(z - I)$ maps the point I to the point ∞. Hence we may assume without loss of generality that I is the point ∞.

We first compute the points s_i. The blocks $h_1, \ldots, h_5$ are lines of the form

$$h_i : (q_{i-1} - z)(\bar{q}_{i+1} - \bar{z}) = (\bar{q}_{i-1} - \bar{z})(q_{i+1} - z).$$

The point $s_i \neq \infty$ is the intersection of the lines h_{i-2} and h_{i+2}. Solving the corresponding linear system of the two equations yields

$$s_i = \frac{(q_{i-2} - q_{i+1})(q_{i+2}\bar{q}_{i-1} - q_{i-1}\bar{q}_{i+2}) - (q_{i-1} - q_{i+2})(q_{i+1}\bar{q}_{i-2} - q_{i-2}\bar{q}_{i+1})}{(q_{i+2} - q_{i-1})(\bar{q}_{i-2} - \bar{q}_{i+1}) - (q_{i-2} - q_{i+1})(\bar{q}_{i+2} - \bar{q}_{i-1})}. \tag{1}$$

The equation of the block k_i through the points s_i, q_{i-2}, q_{i+2} is then given by

$$\begin{aligned}(s_i - q_{i+2})(q_{i-2} - z)(\bar{s}_i - \bar{z})(\bar{q}_{i-2} - \bar{q}_{i+2})\\ = (\bar{s}_i - \bar{q}_{i+2})(\bar{q}_{i-2} - \bar{z})(s_i - z)(q_{i-2} - q_{i+2}).\end{aligned}$$

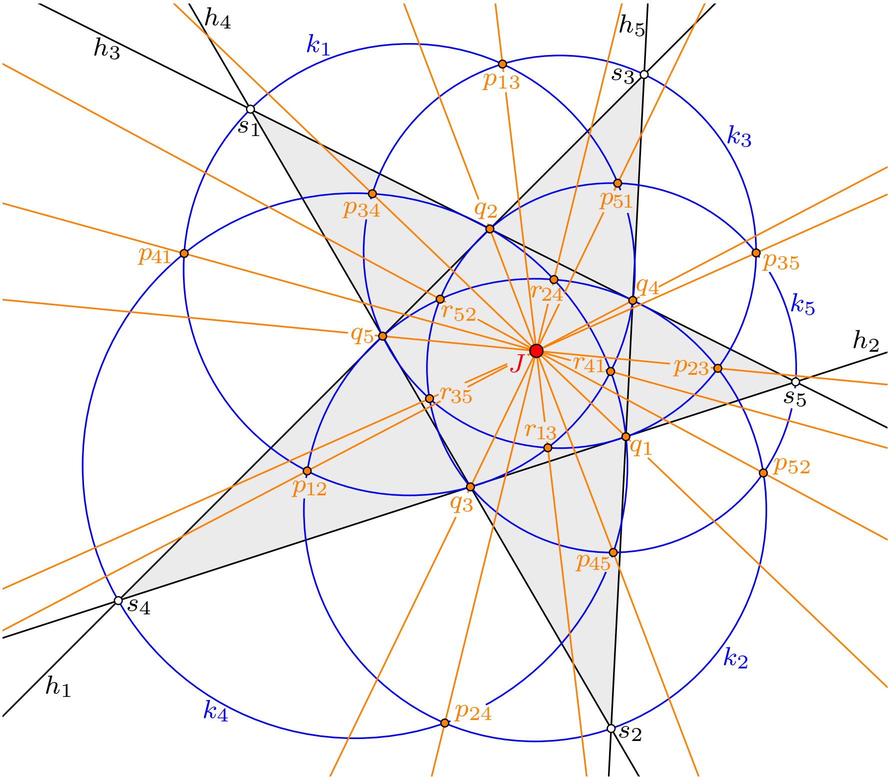

Figure 5 The mother of the siblings

By expanding the products we can bring this equation into the standard form

$$k_i \,:\, (z-c_i)(\bar z-\bar c_i)=r_i \tag{2}$$

with

$$c_i=\frac{q_{i-2}q_{i+2}(\bar q_{i-2}-\bar q_{i+2})+q_{i+2}s_i(\bar q_{i+2}-\bar s_i)+s_iq_{i-2}(\bar s_i-\bar q_{i-2})}{\bar q_{i-2}(q_{i+2}-s_i)+\bar q_{i+2}(s_i-q_{i-2})+\bar s_i(q_{i-2}-q_{i+2})},$$

$$r_i=-\frac{(q_{i-2}-q_{i+2})(\bar q_{i-2}-\bar q_{i+2})(q_{i+2}-s_i)(\bar q_{i+2}-\bar s_i)(s_i-q_{i-2})(\bar s_i-\bar q_{i-2})}{\big(q_{i-2}(\bar q_{i+2}-\bar s_i)+q_{i+2}(\bar s_i-\bar q_{i-2})+\big(s_i(\bar q_{i-2}-\bar q_{i+2})\big)^2}.$$

Using (1) in these formulas we obtain

$$c_i=\frac{q_{i-2}q_{i-1}(\bar q_{i+1}-\bar q_{i-2})+q_{i+2}(q_{i+1}(\bar q_{i+2}-\bar q_{i-1})+q_{i-2}(\bar q_{i-2}+\bar q_{i-1}-\bar q_{i+1}-\bar q_{i+2}))}{q_{i-1}(\bar q_{i+1}-\bar q_{i-2})+q_{i+2}(\bar q_{i-2}-\bar q_{i+1})+(q_{i+1}-q_{i-2})(\bar q_{i+2}-\bar q_{i-1})},$$

$$r_i=-\frac{(q_{i+2}-q_{i-2})(\bar q_{i+2}-\bar q_{i-2})(q_{i-2}-q_{i+1})(\bar q_{i-2}-\bar q_{i+1})(q_{i+2}-q_{i-1})(\bar q_{i+2}-\bar q_{i-1})}{\big(q_{i+2}(\bar q_{i+1}-\bar q_{i-2})+q_{i-1}(\bar q_{i-2}-\bar q_{i+1})-(q_{i+1}-q_{i-2})(\bar q_{i+2}-\bar q_{i-1})\big)^2}.$$

The equation of the radical axis a_{ij} of the circles k_i and k_j results by eliminating $z\bar z$ from their respective equations (2). We obtain

$$a_{ij} \,:\, z(\bar c_j-\bar c_i)+\bar z(c_j-c_i)=c_j\bar c_j-c_i\bar c_i+r_i-r_j. \tag{3}$$

One finds that the intersection of any two of these radical axes is the point

$$J = \frac{\sum_{i=1}^{5} \bar{q}_i(q_{i-1}q_{i-2} - q_{i+1}q_{i+2})}{\sum_{i=1}^{5} \bar{q}_i(q_{i-1} + q_{i-2} - q_{i+1} - q_{i+2})}. \tag{4}$$

This can also be checked by substituting for z the given expression (4) for J in the equation (3) of the radical axis a_{ij}. If the denominator in (4) is different from 0, the radical axes meet in $J \neq I$, otherwise they are parallel. This completes the proof of Theorem 3.

An immediate consequence of Theorem 3 is the following.

Proposition 4. *The point J has the same power with respect to all circles k_i.*

Indeed this follows directly from the fact that the radical axis of any two circles k_i, k_j passes through J.

The proof of Theorem 2 follows now directly from Proposition 4. Namely, consider four arbitrary points p, q, u, v among the points p_{ij}, r_{ij} such that p, q belong to one of the radical axes, and u, v belong to another radical axis. Then p, q, u, v are concyclic. This gives, apart from the circles k_i, exactly the circles c_i, d_i and e_i in Theorem 2. And as a final conclusion we get the following.

Proposition 5. *The radical axes of each two of the circles k_i, c_i, d_i, e_i either all meet in J or are all parallel.*

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Declarations

Conflict of interest The author states that there is no conflict of interest.

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