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Eidgenössische Technische Hochschule Zürich
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On higher cardinal characteristics

Bachelor Thesis

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supervised by
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To Aude

Abstract

This thesis aims to study how cardinal characteristics of the continuum can be extended to larger cardinals. We present two different approaches how this can be done, using first the example of splitting numbers and then we turn to sets of mapping between functions and their bounding and dominating numbers.

In the first chapter, we settle the notation that will be used througout the thesis and present some useful basic results and definitions. We also state more advanced results that will be used in the thesis, but the scope and demonstration of which goes beyond it.

In chapter 2 we see how the classical splitting number may be extended to larger cardinals, extending the notion of splittig family to "spitting on κ" in the obvious way. We discuss some results on successor and singular cardinals before turning to the world of large cardinals and their splitting numbers. Here we restrict us to two main types of large cardinals: inaccessible and compact.

In the last chapter, we look at another way of generalizing cardinal characteristics, this time with the example of bounding and dominating numbers. We first replace the well-known cardinal characteristics in a setting enabling straightforward generalization. We then analize properties of the newly born higher cardinal characteristics that can be shown in ZFC and prove a result on the consistency of their relation with regards to the consistency of ZFC.

Summary

1 Introduction

This bachelor thesis has the goal of investigating how cardinal characteristics of the continuum can be generalized for larger cardinals. The first idea was to inspect classical subsets of $\mathscr{P}(\mathbb{R})$ like the Lebesgue σ-algebra or the standard topology, and see if they deliver interesting characteristics for $\mathfrak{c}$. It turned out to not be the case, but the closest to this idea is a generalisation on the set of functions, which we present in the chapter 3. First we introduce the reader to the set-theoretic notation that will be used throughout this thesis, recall some useful basic notions, and define more specific objects which we will use at some point in our reflections. Then, in chapter 2, we investigate the clasiscal splitting number and its generalization to larger cardinals.

We assume that the reader is familiar with the axioms of ZFC and the basic associated definitions, amongst others ordinal and cardinal numbers, cardinal arithmetics, as well as some associated results. We will denote ordinal numbers by greek letters α, β, γ, δ. The class of all ordinals is Ω. For infinite cardinal numbers, we use the greek letters κ, λ, μ, ν. The definitions, basic results, and some more advanced results presented here mainly come from Halbeisen [7], Jech [8][9] and Kanamori [10].

1.1 Basic notions on sets and cardinals

Let κ be a cardinal. We define $\kappa^+ = \bigcap\{\alpha \in \Omega : \kappa < |\alpha|\}$. A cardinal κ is a **successor cardinal** if there is λ such that $\kappa = \lambda^+$. A cardinal κ is a **limit cardinal**, if $\kappa = \bigcup\{\mu : \mu < \kappa\}$. If for all cardinals $\lambda < \kappa$, we have $2^\lambda < \kappa$, we say that κ is a **strong limit cardinal**. ω_α denotes the α-th infinite cardinal number. More precisely, by transfinite induction, $\omega_0 = \omega$, $\omega_{\alpha+1} = \omega_\alpha^+$ for all $\alpha \in \Omega$, and $\omega_\alpha = \bigcup_{\gamma\in\alpha} \omega_\gamma$ for α a non-empty limit ordinal. 2^κ denotes the cardinality of the power set of κ. In particular, the **continuum** is $\mathfrak{c} = 2^\omega$.

For two ordinal numbers α and β we will write $[\alpha, \beta) = \{\alpha\} \cup \{\gamma \in \beta : \alpha \in \gamma\}$. For a set X and a cardinal number κ, $[X]^\kappa$ denotes the set of all subsets of X having cardinality κ.

$$[X]^\kappa = \{y \in \mathscr{P}(X) : |y| = \kappa\}.$$

The set of finite subsets of X is denoted by $\mathrm{fin}(X) = \{y \subseteq X : |y| < \omega\}$. We can extend this notion to arbitrary sizes $< \kappa$ by defining

$$[X]^{<\kappa} = \{y \in \mathscr{P}(X) : |y| < \kappa\}.$$

With this, $\mathrm{fin}(X) = [X]^{<\omega}$.

For two sets X and Y, ${}^X Y$ is the set of all functions from X to Y. For cardinals κ and λ, we denote by ${}^{<\lambda}\kappa$ the set of all functions from a part of λ, whose cardinality is less than λ, to κ.

$$
\begin{aligned}
{}^{<\lambda}\kappa &= \bigcup\{{}^X\kappa : X \subset \lambda \wedge |X| < \lambda\} \\
\kappa^{<\lambda} &= \bigcup\{\kappa^\mu : \mu < \lambda\} \\
\mathrm{Fn}(X, Y) &= \{p \in {}^X Y : |\mathrm{dom}(p)| < \omega\}
\end{aligned}
$$

The **cofinality** of an infinite cardinal κ is

$$\mathrm{cf}(\kappa) = \min\{|\mathscr{C}| : \mathscr{C} \subseteq \kappa \wedge \bigcup \mathscr{C} = \kappa\}.$$

An infinite cardinal κ is called **regular** if $\mathrm{cf}(\kappa) = \kappa$; otherwise, κ is called **singular**.

Lemma 1.1 (Halbeisen [7]).

a) For every infinite cardinal κ, $\mathrm{cf}(\kappa)$ is regular.

b) Every infinite successor cardinal is regular.

c) Every infinite singular cardinal is a limit cardinal.

For a cardinal κ and a set X we define a **tree** $\mathcal{T}$ as any subset of $\bigcup_{\alpha\in\kappa} {}^{\alpha}X$ such that for all $\alpha \in \kappa$ and $t \in {}^{\alpha}X$, if $t \in \mathcal{T}$ then $\forall \beta \in \alpha(t|_{\beta} \in \mathcal{T})$. A **branch** of the tree $\mathcal{T}$ is a subset $B \subseteq \mathcal{T}$ which is totally ordered by $\subseteq$ and $\subseteq$-maximal, i.e. B is not inculded in any other branch. The **height of a branch** B is the least ordinal α such that $t|_{\alpha} = \emptyset$ for all $t \in B$; and the **height of a tree** $\mathcal{T}$ is the least ordinal α such that $t|_{\alpha} = \emptyset$ for all $t \in \mathcal{T}$. The **level** $\alpha \in \kappa$ of $\mathcal{T}$ is the set $\mathcal{T} \cap {}^{\alpha}X$. Note that the most simple non-empty tree $\mathcal{T} = \{\emptyset\}$ is of height 0 and has only an element at level 0.

A tree $\mathcal{T} \subseteq \bigcup_{\alpha\in\kappa} {}^{\alpha}X$ is called **κ-Aronszajn** if its height is κ but it has neither level of cardinality κ nor branch of height κ.

1.2 Independence and consistency

Considering a mathematical theory (i.e. a set of axioms), we can ask ourselves if a particular sentence φ is provable or if its negation $\neg\varphi$ is provable using the axioms of the theory. In fact, in theories strong enough like ZFC, there will always be some sentence ψ which is not provable, neither is its negation. Such sentences are called independent of the initial therory, since the theory does not make any statement over the sentence. There are therefore models of the theory in which the sentence is true, and models in which it is false, and the theory remains consistent[1] by adding ψ or $\neg\psi$ (but not both) to its list of axioms.

For example, the **continuum hypothesis** (CH) is the statement that there is no intermediate cardinal between ω_0 and $\mathfrak{c}$, so $\mathfrak{c} = \omega_1$. The **generalised continuum hypothesis** (GCH) extends this fact for larger cardinals, stating that for every infinite cardinal κ, $2^{\kappa} = \kappa^{+}$. The **singular cardinal hypothesis** (SCH) is the statement that for every singular cardinal κ, $2^{\mathrm{cf}(\kappa)} < \kappa \to \kappa^{\mathrm{cf}(\kappa)} = \kappa^{+}$. Those three statements have been proven to be independent of ZFC.

In order to prove independence of a certain statement from ZFC, we use the forcing technique. To a forcing partial order $\mathbb{P} = (P, \preceq)$, we denote its $\preceq$-smallest element by $\mathbf{0}$ or $\mathbf{0}_{\mathbb{P}}$, and write $\underset{\sim}{x}$ for a $\mathbb{P}$-name of an element x of a generic extension, or $\check{x}$ for the canonical name of an x in the ground model. If a condition $p \in P$ forces a statement ψ of the forcing language, we write $p \Vdash_{\mathbb{P}} \psi$, or $p \Vdash \psi$ if the underlying forcing notion is clear. We will denote the classic **Cohen's forcing** partial order by $\mathbb{C}_{\kappa} = (\mathrm{Fn}(\kappa \times \omega, 2), \subseteq)$.

A **cardinal characteristic** of the continuum is a cardinal number that can consistently lie strictly between ω_0 and $\mathfrak{c}$. Of course, this happens only in models of $\neg$CH. Under $\neg$GCH we can study the behaviour of generalisations of these characteristics to larger cardinals, therefore the name of this thesis: "higher cardinal characteristics".

Example: for an ideal $\mathcal{I}$ over a set X, where $\mathcal{I}$ contains the Fréchet ideal, we have the following cardinal characteristics of $|X|$:

$$\begin{aligned}
\mathrm{add}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} \notin \mathcal{I}\} \\
\mathrm{cov}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup \mathcal{A} = X\} \\
\mathrm{non}(\mathcal{I}) &= \min\{|Y| : Y \subseteq X \wedge Y \notin \mathcal{I}\} \\
\mathrm{cof}(\mathcal{I}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \forall B \in \mathcal{I}(\exists A \in \mathcal{A}(B \subseteq A))\}.
\end{aligned}$$

[1] Note that the consistency of a theory strong enough cannot be proven in the theory itself (ZFC $\nvdash$ Con(ZFC)), it has to be assumed.

1.3 Some large cardinals

We have seen in lemma 1.1 that all infinite successor cardinals are regular and all infinite singular cardinals are limits. The question left open is whether some infinite regular limit cardinals exist or not. Of course, ω is an example for such a cardinal. But are there other (larger) ones? It has been proven (see Kanamori [10]) that this statement is independent of ZFC. The assumptions of the existence of various types of such cardinals, called **large cardinals**, are named "large cardinal hypotheses". Some of these hypotheses are known to be relatively consistent with ZFC, but for others (for the biggest ones), the relative consistency has not been proven yet, or is even questionned. For the study of higher splitting numbers in chapter 2, we will also have a look at the cases of some basic large cardinals, whose existence is known to be independent but consistent with ZFC. Most definitions and basic results about large cardinals presented here are from Jech [9] and Kanamori [10].

An infinite regular limit cardinal is called **weakly inaccessible** if it is uncountable. The condition for a weakly inaccessible cardinal to be uncountable is added only in the aim of making their existence independent of ZFC. Apart from this distinction, ω fulfills the conditions for beeing a weakly inaccessible cardinal. A weakly inaccessible cardinal κ is also **strongly inaccessible** (or just inaccessible), if κ is a strong limit cardinal, i.e. if for all cardinals $\lambda < \kappa$, $2^\lambda < \kappa$. There are models of ZFC (see Garti and Shelah [6]) containing a cardinal which is weakly but not strongly inaccessible. These models are not models of GCH, since GCH implies that the notions of weakly and strongly inaccessible cardinals coincide. Note that for a weakly inaccessible cardinal ω_α, we have $\omega_\alpha = \mathrm{cf}(\omega_\alpha) = \mathrm{cf}(\alpha) \leq \alpha$, so obviously $\omega_\alpha = \alpha$.

To introduce the next large cardinal, recall first that ordinary first-order languages allow for finite formulae written in terms of, amongst other symbols, $\wedge$, $\vee$, $\exists$ and $\forall$. We now consider the language[2] $\mathscr{L}_{\lambda,\mu}$ with λ, μ infinite cardinals, where we add the new logical operators

$$\bigwedge_{\xi\in\alpha} \quad \bigvee_{\xi\in\alpha} \qquad \text{whith } \alpha \in \lambda,$$

and logical quantifiers

$$\exists_{\xi\in\beta} \quad \forall_{\xi\in\beta} \qquad \text{whith } \beta \in \mu.$$

These shall design conjunction, disjunction, existence and universality over a potentially infinite domain of discourse. $\mathscr{L}_{\lambda,\mu}$-formulae shall additionally have less than μ free variables. We recognise then the ordinary first-order logic behind $\mathscr{L}_{\omega,\omega}$.

A $\mathscr{L}_{\lambda,\mu}$-theory shall be said **satisfiable** if it has a model, and **κ-satisfiable** if every subtheory of cardinality less than κ has a model. Note that the compactness theorem states that a $\mathscr{L}_{\omega,\omega}$-theory is satisfiable if and only if it is ω-satisfiable. We can now define a **strongly compact cardinal** as an uncountable cardinal κ for which any κ-satisfiable $\mathscr{L}_{\kappa,\kappa}$-theory is satisfiable; κ shall be called **weakly compact** if any κ-satisfiable $\mathscr{L}_{\kappa,\kappa}$-theory consisting only of sentences using at most κ non-logical symbols is satisfiable. As previously, except for the condition of being uncountable, ω satisfies the definition of both strongly and weakly inaccessible.

It can be shown (see Kanamori [10]) that every strongly compact cardinal is also weakly compact, and that every weakly compact cardinal is also strongly inaccessible. Cox and Lücke [3] showed that the exsitence of strongly inaccessible cardinals that are not weakly compact is consistent with ZFC.

Lemma 1.2 (Jech [8])**.** *A strongly inaccessible cardinal κ is weakly compact if and only if there is no κ-Aronszajn tree.*

[2]For this purpose, a language $\mathscr{L}$ shall encompass all the logical and non-logical symbols.

2 Splitting numbers

The splitting number is a classical cardinal characteristic of the continuum. In this chapter, we define it in a more general way in order to have directly higher cardinal variants of it. We then try to evaluate the possible cardinals that these higher splitting numbers can be, depending on the properties of their defining cardinal. We will distinguish between successor and singular cardinals, and also have a look at some regular limit cardinals. The results presented here are mainly the work of Suzuki [13] and Zapletal [15].

Let κ be an infinite cardinal. A family $\mathscr{S} \subseteq [\kappa]^\kappa$ is a **splitting family** on κ if for all $x \in [\kappa]^\kappa$ there is an s in $\mathscr{S}$ whith $|x \cap s| = |x \setminus s| = \kappa$. In this case we say that s **splits** x. The **splitting number** on κ, denoted by $\mathfrak{s}(\kappa)$, is the least cardinality of a splitting family on κ.

$$\mathfrak{s}(\kappa) = \min\{|\mathscr{S}| : \mathscr{S} \subseteq [\kappa]^\kappa \text{ is splitting on } \kappa\}$$

By the following lemma 2.1, for every infinite cardinal κ, $\mathfrak{s}(\kappa)$ is well-defined and $\mathfrak{s}(\kappa) \leq 2^\kappa$.

Lemma 2.1. *$[\kappa]^\kappa \subseteq \mathscr{P}(\kappa)$ is splitting on κ.*

Proof. Take any $x \in [\kappa]^\kappa$ and find the unique ordinal α and the unique order preserving bijection $h : x \to \alpha$. Define $s = \{h^{-1}(\beta) : \beta \in \alpha \wedge$ "β is a limit ordinal plus an even natural"$\}$. Then $s \in [\kappa]^\kappa$ and $|x \cap s| = |x \setminus s|$. ⊣

The question we ask ourselves is how large and how small the splitting numbers can really be. Can $\mathfrak{s}(\kappa)$ always (i.e. for every cardinal κ) reach 2^κ? Or is there a better upper bound? Or can $\mathfrak{s}(\kappa)$ even be smaller than κ? What about a lower bound in this case?

2.1 The splitting number of the continuum

Let's first have a look at the splitting number $\mathfrak{s} = \mathfrak{s}(\omega_0)$. As already mentioned, $\mathfrak{s} \leq \mathfrak{c}$, but $\mathfrak{s}$ must also be uncountable.

Fact 2.2. $\omega_1 \leq \mathfrak{s}$

Proof. Assume we have a countable splitting family $\mathscr{S} = \{S_0, S_1, S_2, ...\} \subseteq [\omega]^\omega$. Notice that if $s \subset \omega$ splits $x \in [\omega]^\omega$, then so does $\omega \setminus s$. Therefore the family $\mathscr{S}$ remains splitting if we replace some (possibly infinitly many) S_i by their complement $S_i' = \omega \setminus S_i$.

Per induction, we now construct a set $X \in [\omega]^\omega$ which is not split by any of the elements of a slight adaptation of $\mathscr{S}$. Define $y_0 = S_0' = S_0$ and $x_0 = \bigcap y_0$. Since $\mathscr{S}$ is splitting, $|y_0| = \omega$. For $n \in \omega$ and $|y_n| = \omega$, let

$$S_{n+1}' = \begin{cases} S_{n+1} & \text{if } |S_{n+1} \cap y_n| = \omega \\ \omega \setminus S_{n+1} & \text{otherwise.} \end{cases}$$

Also define $y_{n+1} = S_{n+1} \cap y_n$ and

$$x_{n+1} = \bigcap (y_{n+1} \setminus \{x_0, x_1, ..., x_n\}).$$

Note that per definition of S_{n+1}', $|y_{n+1}| = \omega$, so $x_{n+1} \notin \{x_0, x_1, ..., x_n\}$ is well-defined.

With this construction, we have $y_0 \supseteq y_1 \supseteq ... \supseteq y_n \supseteq y_{n+1} \supseteq ...$ and $S_n' \supseteq y_n$, for all $n \in \omega$. Furthermore, fixing an $n \in \omega$ and since we have chosen $x_n \in y_n$, S_n' contains x_n but also contains all subsequent $x_i \in y_i \subseteq S_n'$ for $i \geq n$.

Finally, let $X = \bigcup_{i\in\omega}\{x_i\}$ and consider $\mathscr{S}' = \{S'_0, S'_1, ...\}$, the countable splitting family obtained from $\mathscr{S}$ by the replacements during the construction of the y_n. From $x_{n+1} \notin \{x_0, x_1, ..., x_n\}$ we conclude $X \in [\omega]^\omega$, and from the choice of the $x_n \in y_n$ follows $|X \setminus S'_n| \leq n$ for all $n \in \omega$. This means that X is not split by any element of $\mathscr{S}'$, a contradiction. We conclude $|\mathscr{S}| > \omega$, so $\mathfrak{s} \geq \omega_1$. $\dashv$

So if we assume the continuum hypothesis (CH), we have $\mathfrak{s}(\omega_0) = \omega_0^+ = \mathfrak{c}$. As we shall see later, this fact does not extend to higher splitting numbers. In fact, it is consistent with ZFC for $\mathfrak{s}$ to take many values betwwen ω_1 and $\mathfrak{c}$. This choice of values will restrict itself a little in the following paragraphs, as we study the splitting numbers of larger cardinals.

2.2 Higher splitting numbers

Can we extend the results found for $\mathfrak{s}(\omega_0)$ to any splitting number $\mathfrak{s}(\kappa)$? Notice that the first inequality of fact 2.2 does *not* extend in general to $\kappa < \mathfrak{s}(\kappa)$ as the counterexample in the next fact shows.

Fact 2.3. $\mathfrak{s}(\omega_1) = \omega_0$

Suzuki [13] states this fact and refers for the proof to Motoyoshi [11], a student of Kamo. As the work containing the proof is availible in japanese only, the following proof is an adaptation from that of a more general result in Zapletal [15].

Proof. Since $\omega_1 \leq 2^\omega = \mathfrak{c}$, we can choose an injective function $f : \omega_1 \hookrightarrow {}^\omega 2$. For every $h \in {}^{<\omega}2$, define $S_h = \{\alpha \in \omega_1 : h \subseteq f(\alpha)\}$ and set

$$\mathscr{S} = \{S_h : h \in {}^{<\omega}2 \wedge |S_h| = \omega_1\} \subseteq [\omega_1]^{\omega_1}.$$

We show that $\mathscr{S}$ is splitting on ω_1.

If $\mathscr{S}$ was not splitting on ω_1, choose an $x \in [\omega_1]^{\omega_1}$ which is not split by any $S_h \in \mathscr{S}$. Consider the set $A = \{h \in {}^{<\omega}2 : |S_h \cap x| = \omega_1\}$. Two partial functions h and g in A must be extension-compatible, since otherwise $S_g \cap S_h = \emptyset$ and $\omega_1 = |x \cap S_g| \leq |x \setminus S_h| \leq \omega_1$, so x would be split by S_h. Therefore, A is directed by $\subseteq$, and there is at most one ordinal $\alpha \in \omega_1$ such that each initial segment of $f(\alpha)$ belongs to A. This means $x \subseteq \{\alpha\} \cup \bigcup\{S_h \cap x : h \in {}^{<\omega}2 \setminus A\}$, so x is the countable union of countable sets, a contradiction to the choice $x \in [\omega_1]^{\omega_1}$.

So $\mathscr{S} \subseteq [\omega_1]^{\omega_1}$ is splitting on ω_1, and $|{}^{<\omega}2| = \omega_0$ implies that $|\mathscr{S}| = \omega_0$. Since $\mathfrak{s}(\omega_1)$ must obviously be infinite, we conclude that $\mathfrak{s}(\omega_1) = \omega_0$. $\dashv$

As a consequence, CH implies $\mathfrak{s}(\omega) = \mathfrak{c}$ and $\mathfrak{s}(\mathfrak{c}) = \omega$. This fact may seem counterintuitive at first glance, because we need more sets to split ω than we need to split $\mathfrak{c}$. Recall however that the size of the splitting subsets are different in both cases.

After these preliminary results showing that higher splitting numbers $\mathfrak{s}(\kappa)$ do not necessarily behave like $\mathfrak{s}$, we now have a look more generally at the higher cardinals, distinguishing between different cases. Recall that every cardinal is either a successor cardinal or a limit cardinal. For infinite cardinals, we also make a distinction between cardinals that are singular or regular. First we have a look at the case of successor cardinals (which are always regular), and then we will review the singular cardinal case (which are always limit cardinals). Finally, we will approach the regular limit cardinal case. Notice that ω_0 is such a cardinal, but is the only one of its kind whose existence is provable in ZFC. It was therfore handled separately in chapter 2.1. The existence of larger, hence uncountable, regular limit cardinals is independent of ZFC and relies on large cardinal hypotheses.

2.2.1 The successor case

As successor cardinal, ω_1 showed us that the splitting number of successor cardinals does not behave the same way as $\mathfrak{s}(\omega)$. We could hope for the splitting number of a successor cardinal to be for example his predecessor, or the preceeding limit cardinal.

Proposition 2.4. *Let κ be an infinite successor cardinal. Then $\mathfrak{s}(\kappa) < \kappa$.*

The proof is analogous to that of fact 2.3 and is adapted from Zapletal [15].

Proof. Let λ be the least cardinal such that $2^\lambda \geq \kappa$. λ is well defined since, as κ is a successor, there is a cardinal μ with $\mu^+ = \kappa$, for which $2^\mu \geq \kappa$. So $\{\nu : \kappa \leq 2^\nu\}$ is a non empty set of cardinals.

Since $\kappa \leq 2^\lambda$, we can choose an injective function $f : \kappa \hookrightarrow {}^\lambda 2$. For every $h \in {}^{<\lambda}2$, define $S_h = \{\alpha \in \kappa : h \subseteq f(\alpha)\}$ and set

$$\mathscr{S} = \{S_h : h \in {}^{<\lambda}2 \wedge |S_h| = \kappa\} \subseteq [\kappa]^\kappa.$$

We show that $\mathscr{S}$ is splitting on κ.

If $\mathscr{S}$ was not splitting on κ, choose an $x \in [\kappa]^\kappa$ which is not split by any $S_h \in \mathscr{S}$. Consider the set $A = \{h \in {}^{<\lambda}2 : |S_h \cap x| = \kappa\}$. Two partial functions h and g in A must be extension-compatible, since otherwise $S_g \cap S_h = \emptyset$ and $\kappa = |x \cap S_g| \leq |x \setminus S_h| \leq \kappa$, so x would be split by S_h. Therefore, A is directed by $\subseteq$, and there is at most one ordinal $\alpha \in \kappa$ such that each initial segment of $f(\alpha)$ belongs to A. This means $x \subseteq \{\alpha\} \cup \bigcup\{S_h \cap x : h \in {}^{<\lambda}2 \setminus A\}$, so x is the union of at most $2^{<\lambda} < \kappa$ sets of cardinality less than κ, a contradiction to the choice $x \in [\kappa]^\kappa$.

So $\mathscr{S} \subseteq [\kappa]^\kappa$ is splitting on κ and $\mathfrak{s}(\kappa) \leq |\mathscr{S}| < \kappa$. ⊣

The proof of proposition 2.4 suggests that the splitting number of a successor cardinal must be such that there can be an injective function $\kappa \hookrightarrow {}^{\mathfrak{s}(\kappa)}2$, advocating towards the following conjectures.

Conjecture 2.5 (Lower bound for the splitting number of successor cardinals)**.** *Let κ be an infinite successor cardinal or a non-strong regular limit cardinal. Then $2^{\mathfrak{s}(\kappa)} \geq \kappa$.*

Conjecture 2.6 (The splitting number of a successor can reach its lower bound)**.** *Let κ be a successor cardinal or a non-strong regular limit cardinal and $\lambda = \bigcap\{\nu : \kappa \leq 2^\nu\}$. Then*

$$\mathrm{Con}(\mathsf{ZFC}) \Longrightarrow \mathrm{Con}(\mathsf{ZFC} + \lambda = \mathfrak{s}(\kappa)).$$

2.2.2 The singular case

We will look briefly at the singular cardinal case, taking the example of ω_ω.

Proposition 2.7 (Zapletal [15])**.** $\mathfrak{s}(\omega_\omega) \leq \mathfrak{s}(\omega)$

Proof. Let $\mathscr{S} = \{S_\alpha : \alpha \in \mathfrak{s}\} \subseteq [\omega]^\omega$ be a splitting family. Define $S'_\alpha = \bigcup_{n \in S_\alpha}[\omega_n, \omega_{n+1})$ and $\mathscr{S}' = \{S'_\alpha : \alpha \in \mathfrak{s}\}$. For any $X \in [\omega_\omega]^{\omega_\omega}$ define $n_0 = \bigcap\{m \in \omega : |X \cap [\omega_m, \omega_{m+1})| \geq \omega_0\}$ and

$$n_{j+1} = \bigcup\{\bigcap\{m \in \omega : |X \cap [\omega_m, \omega_{m+1})| \geq \omega_{j+1}\}, n_j + 1\}.$$

Set $Y = \{n_j : j \in \omega\}$. Then for all $\alpha \in \mathfrak{s}$, if S_α splits Y we have also that S'_α splits X. ⊣

The following lemma applies to the example of ω_ω, but also to ω itself. The proof is taken from Dow and Shelah [4].

Lemma 2.8. *Let $\kappa = \omega_\omega$ or $\kappa = \omega$. Then $\mathrm{cf}(\mathfrak{s}(\kappa)) > \omega_0$.*

Proof. Let $\langle \lambda_n : n \in \omega \rangle$ be a sequence cofinal in κ. If $\omega_0 < \mathfrak{s}(\kappa) < \kappa$, then we are done. Otherwise, let $\mathscr{S} = \{S_\alpha : \alpha \in \kappa\} \subseteq [\kappa]^\kappa$. Define $X_0 = \kappa$ and for all $\alpha \in \kappa \setminus \{\emptyset\}$ choose $X_\alpha \in [\kappa]^\kappa$ with $X_\alpha \subseteq \bigcup_{\beta \in \alpha} X_\beta$ and X_α not split by $\{S_\alpha : \alpha \in \lambda_n\}$. A κ-pseudo-intersection X of $\{X_\alpha : \alpha \in \kappa\}$, i.e. $|X \setminus X_\alpha| < \kappa$ for all $\alpha \in \kappa$, is then not split by any S_α with $\alpha \in \kappa$, so $\mathscr{S}$ is not splitting. Such an X exists by construction of the X_α. ⊣

2.2.3 The regular limit case

As mentionned in chapter 1.3, the existence of such cardinals is independent of ZFC. Under the assumption of their existence, we can study the possibilities for their splitting number. We here have a look at the case of three large cardinals. First we have a look at the case of (weakly and strongly) inaccessible cardinals, then we look at a result about weakly compact cardinals and mention even further (stronger) axioms.

In ZFC + inaccessible cardinal axiom

Theorem 2.9 (Motoyoshi [11])**.** *Let κ be an uncountable regular cardinal. Then κ is strongly inaccessible if and only if $\kappa \leq \mathfrak{s}(\kappa)$.*

As Motoyoshi's thesis is availible in japanese only, the following proof is adapted from Zapletal [15].

Proof. Let κ be an uncountable regular cardinal.

⇒) Let κ be strongly inaccessible and $\mathscr{S} \subseteq [\kappa]^\kappa$ of cardinality $|\mathscr{S}| = \lambda < \kappa$. We denote by V_α for an ordinal α the sets in the construction of the cumulative hierarchy of $\mathbf{V} \vDash \mathsf{ZFC}$. By the reflection principle (see for example Halbeisen [7]), there is a suitably large ordinal μ such that V_μ reflects ZFC* (where ZFC* denotes a finite fragment of ZFC containing all the axioms we need to construct: ordinals, cardinals, state the theorems, etc.) and a set model M in V_μ also reflecting ZFC* such that $\kappa, \mathscr{S} \in M$, ${}^\lambda 2 \subset M$ and $|M| < \kappa$. The latter is possible because κ is a strong limit cardinal. By Mostowski's collapse we may also assume that M is transitive.

Since κ is regular, we can choose an $\delta \in \kappa$ such that $\delta \ni \bigcup(M \cap \kappa)$. Define

$$A_0 = \{S \in \mathscr{S} : \alpha \in S\} \text{ and } A_1 = \{S \in \mathscr{S} : \alpha \notin S\}$$

and notice that they both are in M, as well as the set $X = \bigcap A_0 \setminus \bigcup A_1$. Now for any $S \in \mathscr{S}$ we have either $X \setminus S = \emptyset$ or $X \cap S = \emptyset$ depending on if $\alpha \in S$ or not. It remains to show that $|X| = \kappa$ to conclude that X is not split by S and so $\mathscr{S}$ cannot be splitting. Because $\alpha \in X \setminus M$ we have that $X \cap (M \cap \kappa)$ is cofinal in $M \cap \kappa$ and therefore $|X| = \kappa$ since M is a set model of V_μ containing κ.

⇐) By contraposition, let's assume there is a cardinal $\lambda < \kappa$ with $2^\lambda \geq \kappa$. Then, following the proof of proposition 2.4, we can show that $\mathfrak{s}(\kappa) < \kappa$. ⊣

We can now deduce that if a cardinal κ is weakly but not strongly inaccessible, then its splitting number will be small, i.e. $\mathfrak{s}(\kappa) < \kappa$.

Question 2.10 (Can $\mathfrak{s}$(weakly inaccessible) be ω_0?). Is there a model of ZFC containing a weakly (but not strongly) inaccessible cardinal κ, such that $\mathfrak{s}(\kappa) = \omega_0$? If not, how small can it be?

Partial answer. First note that $\mathfrak{c}$ can not be strongly compact since $2^{\omega_0} = \mathfrak{c}$, but if there exists a weakly compact cardinal κ in $\mathbf{V}$, a model of GCH, then $\mathbb{C}_\kappa$ forces that $\mathfrak{c} = \kappa$ is weakly compact in the generic extension. If conjecture 2.6 turned to be true, then the splitting number of a weakly compact cardinal could be as small as ω_0. (?)

In ZFC + weakly compact cardinal axiom

Theorem 2.11 (Suzuki [13])**.** *Let κ be an uncountable regular cardinal. Then κ is weakly compact if and only if $\kappa < \mathfrak{s}(\kappa)$.*

Proof. Let κ be an uncountable regular cardinal.

$\Rightarrow$) If κ is weakly compact, let $\mathscr{S} = \{S_\alpha : \alpha \in \kappa\} \subseteq [\kappa]^\kappa$ be a family of cardinality κ and for all $\alpha \in \kappa$ and $t \in {}^\alpha 2$ define the set

$$X_t = \bigcap\{S_\beta : \beta \in \alpha \wedge t(\beta) = 1\} \setminus \bigcup\{S_\beta : \beta \in \alpha \wedge t(\beta) = 0\}.$$

Consider $\mathcal{T} = \{t \in {}^\alpha 2 : |X_t| = \kappa\}$; obviously $\mathcal{T}$ is a tree, since for any $t \in \mathcal{T}$ we have $s \subseteq t \to X_s \subseteq X_t$. Because κ is strongly inaccessible, there are more sets in $\mathscr{S}$ than functions in ${}^\alpha 2$ for any $\alpha \in \kappa$ and we can find a $t \in {}^\alpha 2 \cap \mathcal{T}$ (reordering at most 2^α elements of $\mathscr{S}$, which does not change the splitting properties of $\mathscr{S}$; note the similarites with fact 2.2), so $\mathcal{T}$ has height κ. Weak compactness of κ implies that there is a branch $B \in \mathcal{T}$ of height κ, since there is no level of size κ in $\mathcal{T}$.

Let $f = \bigcup B : \kappa \to 2$; since for all $\alpha \in \kappa$ we have $|X_{f|_\alpha}| = \kappa$ we can choose an increasing sequence of κ ordinals $\langle \delta_\alpha : \alpha \in \kappa \rangle$ such that $\delta_\alpha \in X_{f|_\alpha}$ for all $\alpha \in \kappa$. This means also $\delta_\alpha \in X_{f|_\gamma}$ for $\alpha \in \gamma \in \kappa$. Consider the set $X = \{\delta_\alpha : \alpha \in \kappa\}$ and note that for any $\gamma \in \kappa$

$$X \cap S_\gamma = \begin{cases} \{\delta_\alpha : \alpha \in \gamma\} & \text{if } f(\gamma) = 0 \\ \{\delta_\alpha : \gamma \in \alpha \in \kappa\} & \text{if } f(\gamma) = 1. \end{cases}$$

So either $|X \cap S_\gamma| < \kappa$ or $|X \setminus S_\gamma| < \kappa$. In both cases X is not split by any S_γ, therefore $\mathscr{S}$ is not a splitting family. Since the choice of $\mathscr{S} \subseteq [\kappa]^\kappa$ was arbitrary and by theorem 2.9, we conclude $\kappa < \mathfrak{s}(\kappa)$.

$\Leftarrow$) If $\kappa < \mathfrak{s}(\kappa)$ we know from theorem 2.9 that κ is strongly inaccessible. If κ was not weakly compact, then there would be a κ-Aronszajn tree $\mathcal{T} \subseteq \bigcup_{\alpha\in\kappa} {}^\alpha 2$ with $|\mathcal{T}| = \kappa$ and we may assume without loss of generality that $\mathcal{T}$ has no terminal nodes. For every $t \in \mathcal{T}$ we define $S_t = \{u \in \mathcal{T} : t \subset u\}$ and then $\mathscr{S} = \{S_t : t \in \mathcal{T} \wedge |S_t| = \kappa\}$.

Now for any $X \subseteq [\mathcal{T}]^\kappa$ there is a level $\alpha \in \kappa$ for which two distinct (i.e. incompatible) $u, v \in {}^\alpha 2$ exist with $|X \cap S_u| = |X \cap S_v| = \kappa$. If it were not the case, then we could build a branch of height κ by induction. Since u and v are incompatible, $S_u \cap S_v = \emptyset$, so $X \cap S_u \subseteq X \setminus S_v$ and we can find a bijection $\phi : \mathcal{T} \to \kappa$ such that $\mathscr{S}' = \{\phi[S_t] : S_t \in \mathscr{S}\}$ is splitting on κ. Since $|\mathscr{S}'| \leq \kappa$ we have a contradiction to $\kappa < \mathfrak{s}(\kappa)$, so κ must be weakly compact. ⊣

For different variations of this proof, see Suzuki [13][12] or Zapletal [15].

Corrolary 2.12. *Let κ be a regular cardinal. Then $\mathfrak{s}(\kappa) = \kappa$ if and only if κ is strongly inaccessible but not weakly compact.*

Proof. The claim follows directly from theorems 2.9 and 2.11, together with the fact that every weakly compact cardinal is strongly inaccessible. Cox and Lücke [3] showed that the existence of such cardinals is consistent with ZFC. ⊣

We could continue so with ever larger cardinal hypotheses and find that the splitting number of such large cardinal κ is at least κ^{++} or κ^{+++}. The problem is that in such cases, we have to rely on the consistency of the existence of such large cardinals with ZFC, because it (partly) has not been demonstrated yet. For example, Zapletal [15] shows a similar result for $\mathfrak{s}(\kappa) > \kappa^+$ but has to admit the consistency of the existence of supercompact cardinals.

	κ regular					κ singular
		κ limit				
	κ successor	κ weakly inaccessible	κ strongly inaccessible	κ weakly compact	$\kappa = \omega_0$	for example: $\kappa = \omega_\omega$
ω_0	$\mathfrak{s}(\kappa) < \kappa$ (conj 2.5: $\kappa \leq 2^{\mathfrak{s}(\kappa)}$)	If κ is not strongly inaccessible (conj 2.5: $\kappa \leq 2^{\mathfrak{s}(\kappa)}$)				
…						$\mathfrak{s}(\kappa) \leq \mathfrak{s}$ and $\mathrm{cf}(\mathfrak{s}(\kappa)) > \omega_0$
κ		Only if κ is also strongly inaccessible. Refer to →	$\mathfrak{s}(\kappa) = \kappa$ *iff* κ is not weakly compact			
κ^+ … 2^κ			Only if κ is also weakly compact. Refer to →	$\kappa < \mathfrak{s}(\kappa)$	$\omega_1 \leq \mathfrak{s} \leq \mathfrak{c}$ and $\mathrm{cf}(\mathfrak{s}) > \omega_0$	$\mathfrak{s}(\kappa) \leq \mathfrak{s}$ and $\mathrm{cf}(\mathfrak{s}(\kappa)) > \omega_0$

Table 2.1: Tabular summary of the values for $\mathfrak{s}(\kappa)$ that are *a priori* consistent with ZFC, including under some large cardinal hypotheses. A grey shaded area means that the splitting number cannot take the corresponding values.

2.3 Overview of the different cases

We can now summarize the different cases we analyzed in table 2.1. The case of singular cardinal was handeled only partly with the example of ω_ω, and some more investigation would be necessary there. We could also add some further large cardinals, but the inclusion of these in the table would not be of great interest.

2.4 Further questions

Question 2.13 (How does iterated splitting behave?). Let κ be a cardinal number and define for any $n \in \omega$ per induction on n: $\mathfrak{s}^0(\kappa) = \kappa$ and $\mathfrak{s}^{n+1}(\kappa) = \mathfrak{s}(\mathfrak{s}^n(\kappa))$. How does $\mathfrak{s}^n(\kappa)$ evolves with $n \to \omega$, depending on κ? Can there be a final fixpoint? An oscillation in a closed loop? A special property that $\mathfrak{s}^n(\kappa)$ will finally keep, for n big enough?

Initial answer. At least for some cases: strongly inaccessible cardinals are fixpoints and maybe some singular cardinals can be too; in models of GCH there is the final loop $\omega_0 \rightleftharpoons \omega_1$ which is reached in finitely many steps from ω_n. (?)

3 Sets of functions

In this chapter, we will generalize the well known bounding and dominating numbers. However, we do this with a different approach than we did for the splitting numbers. This approach is due to Switzer [14], from whose work most of the following comes. First we set the required conditions for the generalization of the cardinal characteristics to larger cardinals. We then define them and study basic relations amongst them, comparing the results the “classical” ones of Cichoń’s diagram. Then we look at further relations and prove a consistency result for models of CH.

On ω we define the following relations between $u, v \in {}^{\omega}\omega$ and $s \in \mathcal{S} = \{s \in {}^{\omega}([\omega]^{<\omega}) : \forall n \in \omega(|s(n)| \leq n)\}$:

$$u \neq^* v \Leftrightarrow |\{n \in \omega : u(n) = v(n)\}| < \omega$$
$$u \leq^* v \Leftrightarrow |\{n \in \omega : u(n) > v(n)\}| < \omega$$
$$u \in^* s \Leftrightarrow |\{n \in \omega : u(n) \notin s(n)\}| < \omega$$

We know two basic cardinal characteristics of the continuum related to the relation $\leq^*$: the **bounding number** $\mathfrak{b}$ is the least cardinality of an unbounded family $B \subseteq {}^{\omega}\omega$, i.e. for which there is no $v \in {}^{\omega}\omega$ such that for all $u \in B$ we have $u \leq^* v$; and the **dominating number** $\mathfrak{d}$ is the least cardinality of a dominating family $D \subseteq {}^{\omega}\omega$, i.e. for which to any $u \in {}^{\omega}\omega$, there is a $v \in D$ such that $u \leq^* v$.

In order to extend these concepts, we need some more definitions. To any relation $R \subseteq Y \times Z$ and $B \subseteq Y$ we say that $z \in Z$ **is an** R**-bound for** B if for all $b \in B$ we have bRz. Moreover, B is said to be R**-bounded** if it has an R-bound, and R**-unbounded** otherwise. A set $D \subseteq Z$ is R**-dominating** if for every $y \in Y$ there is a $d \in D$ such that yRd.
We denote by $\mathfrak{b}(R)$ the least cardinality of an R-unbounded set, and by $\mathfrak{d}(R)$ the least cardinality of an R-dominating set.

More generally, let $\mathcal{I}$ be an ideal over a set X, and Y, Z be sets. From a relation $R \subseteq Y \times Z$ we can define a new relation $R_{\mathcal{I}} \subseteq {}^{X}Y \times {}^{X}Z$ through

$$fR_{\mathcal{I}}g \Leftrightarrow \{x \in X : \neg f(x)Rg(x)\} \in \mathcal{I}.$$

For example, we can recover the previous relations by letting $R \in \{\neq, \leq, \in\}$, $X = Y = Z = \omega$ (or $Z = [\omega]^{<\omega}$ with some constraints in the case $R = \in$) and $\mathcal{I}$ being the Fréchet ideal (i.e. $\mathcal{I} = \mathcal{F} = [\omega]^{<\omega}$).

Proposition 3.1. *The following equalities between cardinal characteristics of the continuum are provable in ZFC:*

a) $\mathfrak{b}(\leq_{\mathcal{F}}) = \mathfrak{b}(\leq^*) = \mathfrak{b} \wedge \mathfrak{d}(\leq_{\mathcal{F}}) = \mathfrak{d}(\leq^*) = \mathfrak{d}$

b) $\mathfrak{b}(\neq_{\mathcal{F}}) = \mathfrak{b}(\neq^*) = \mathrm{non}(\mathcal{M}) \wedge \mathfrak{d}(\neq_{\mathcal{F}}) = \mathfrak{d}(\neq^*) = \mathrm{cov}(\mathcal{M})$

c) $\mathfrak{b}(\in_{\mathcal{F}}) = \mathfrak{b}(\in^*) = \mathrm{add}(\mathcal{N}) \wedge \mathfrak{d}(\in_{\mathcal{F}}) = \mathfrak{d}(\in^*) = \mathrm{cof}(\mathcal{N})$

Proof. The equalities in 1 and the first equality of every group in 2 and 3 yield by definition. For the others, see Bartoszyński and Judah [1]. ⊣

Fremlin [5] proved a number of inequalities between these cardinals together with the least uncountable cardinal ω_1 and the continuum $\mathfrak{c}$. His results are summarized in Cichoń’s diagram, called after the polish mathematician and shown in figure 3.1.

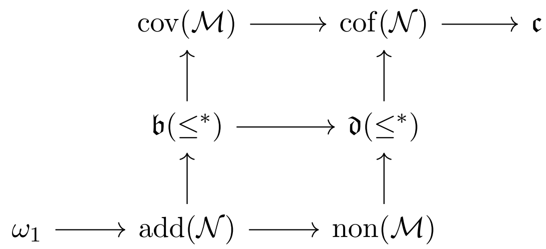

Figure 3.1: A part of Cichoń's diagram [14]. An arrow indicates an increase in cardinality (non-strict).

3.1 Discovering higher bounding and dominating cardinals

One way to extend the now defined characteristics of the continuum to higher cardinals would be to consider functions $u, v \in {}^{\kappa}\kappa$ for some $\kappa > \omega$. This is analog to the approach taken in chapter 2. Switzer [14] however takes an other interesting approach, iterating the process of creating a new relation modulo an ideal, which we are going to describe in this chapter, considering now functions $f : {}^{\omega}\omega \to {}^{\omega}\omega$ or $\mathcal{S}$.

To be able to apply our construction to a relation $R \in {}^{\omega}\omega \times {}^{\omega}\omega$ or ${}^{\omega}\omega \times \mathcal{S}$ (for example $\neq^*$, $\leq^*$ or $\in^*$), we need one or more ideals over ${}^{\omega}\omega$ (note that $|{}^{\omega}\omega| = \mathfrak{c}$). These will mainly be the null-sets or the meager-sets ideals $\mathcal{N}$ and $\mathcal{M}$, but Switzer [14] also presents it for the σ-compact-sets ideal $\mathcal{K}$ and it could be applied to other ideals over ${}^{\omega}\omega$. The reader knowing the ideals $\mathcal{K}$, $\mathcal{M}$ and $\mathcal{N}$ over $\mathbb{R}$ can construct a bijection between $\mathbb{R}$ and ${}^{\omega}\omega$ and map the sets of the ideals back and forth. Alternatively for $\mathcal{N}$, use the Lebesgue measure on ${}^{\omega}\omega$. It is defined for finite sequences as

$$\mu(N_s) = \prod_{i<length(s)} 2^{1-s(i)},$$
$$\text{with } s \in \bigcup_{n\in\omega} {}^{n}\omega \text{ and } N_s = \{u \in {}^{\omega}\omega : s \subseteq u\},$$

and then completed in the standard way.

Using the definitions of $R_{\mathcal{I}}$ to the three mentionned ideals, we get in total 9 new relations and 18 cardinals. For example, with $f, g : {}^{\omega}\omega \to {}^{\omega}\omega$ and $h : {}^{\omega}\omega \to \mathcal{S}$:

- $f \in^*_{\mathcal{M}} h \iff$ for all but a meagre set of $u \in {}^{\omega}\omega$ we have $f(u) \in^* h(u)$ $\iff \{u \in {}^{\omega}\omega : f(u) \not\in^* h(u)\} \in \mathcal{M}$;
- $f \leq^*_{\mathcal{N}} g \iff$ for all but a null set of $u \in {}^{\omega}\omega$ we have $f(u) \leq^* g(u)$ $\iff \{u \in {}^{\omega}\omega : f(u) \not\leq^* g(u)\} \in \mathcal{N}$;
- $f \neq^*_{\mathcal{K}} g \iff$ for all but a σ-compact set of $u \in {}^{\omega}\omega$ we have $f(u) \neq^* g(u)$ $\iff \{u \in {}^{\omega}\omega : f(u) \not\neq^* g(u)\} \in \mathcal{K}$;
- $\mathfrak{b}(\neq^*_{\mathcal{N}})$ is the smallest cardinality of a set $B \subseteq {}^{({}^{\omega}\omega)}({}^{\omega}\omega)$ such that for each g there is an $f \in B$ for which the set $\{u \in {}^{\omega}\omega : \ |\{n \in \omega : f(u)(n) = g(u)(n)\}| = \omega\} \notin \mathcal{N}$, i.e. has non-zero measure;
- $\mathfrak{d}(\neq^*_{\mathcal{N}})$ is the smallest cardinality of a set $D \subseteq {}^{({}^{\omega}\omega)}({}^{\omega}\omega)$ such that for each f there is a $g \in D$ for which the set $\{u \in {}^{\omega}\omega : \ |\{n \in \omega : f(u)(n) = g(u)(n)\}| = \omega\} \in \mathcal{N}$, i.e. is a null set;
- $\mathfrak{b}(\in^*_{\mathcal{M}})$ is the smallest cardinality of a set $B \subseteq {}^{({}^{\omega}\omega)}({}^{\omega}\omega)$ such that for each h there is an $f \in B$ for which the set $\{u \in {}^{\omega}\omega : \ |\{n \in \omega : f(u)(n) \notin h(u)(n)\}| = \omega\} \notin \mathcal{M}$, i.e. is nonmeagre.

Theorem 3.2 (Switzer [14])**.** *Let $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}, \mathcal{K}\}$ be an ideal over ${}^{\omega}\omega$, then the following inequalities hold in* ZFC*:*

$$\begin{array}{ccccccc} & & \mathfrak{b}(\neq^*_{\mathcal{I}}) & \longrightarrow & \mathfrak{d}(\in^*_{\mathcal{I}}) & \longrightarrow & 2^{\mathfrak{c}} \\ & & \uparrow & & \uparrow & & \\ & & \mathfrak{b}(\leq^*_{\mathcal{I}}) & \longrightarrow & \mathfrak{d}(\leq^*_{\mathcal{I}}) & & \\ & & \uparrow & & \uparrow & & \\ \mathfrak{c}^+ & & \mathfrak{b}(\in^*_{\mathcal{I}}) & \longrightarrow & \mathfrak{d}(\neq^*_{\mathcal{I}}) & & \end{array}$$

Figure 3.2: Relations between higher bounding and dominating numbers [14]. An arrow indicates an increase in cardinality (non-strict). Note the resemblance with Cichoń's diagram (figure 3.1) and proposition 3.1, replacing R^* by $R^*_{\mathcal{I}}$. It is intentional that no arrow is drawn from $\mathfrak{c}^+$.

Before we tackle the proof of the seven inequalities of the theorem, we shall prepare ourselves with some tools and a lemma that will help us for the harder two. Let $\mathcal{J} = \{J_{n,k} : k < n \in \omega\}$ be a partition of ω in finite subsets and $J_n = \bigcup_{k<n} J_{n,k}$. We will call the elements of the previously defined set $\mathcal{S} = \{s \in {}^{\omega}([\omega]^{<\omega}) : \forall n \in \omega(|s(n)| \leq n)\}$ **slaloms**. Now let's call an $x : \omega \to {}^{<\omega}\omega$ **$\mathcal{J}$-function** if $\mathrm{dom}(x(n)) = J_n$ for every $n \in \omega$. Similarly we will call $t : \omega \to [{}^{<\omega}\omega]^{<\omega}$ a **$\mathcal{J}$-slalom** if $|t(n)| \leq n$ and $w \in t(n) \to \mathrm{dom}(w) = J_n$ for all $n \in \omega$. For a $\mathcal{J}$-function x and a $\mathcal{J}$-slalom t we will write $x \in^* t$ if $x(n) \in t(n)$ for all but finitely many $n \in \omega$.

Furthermore, to an $u : \omega \to \omega$ we denote by u' the $\mathcal{J}$-function defined by $u'(n) = u|_{J_n}$, and to $f : {}^{\omega}\omega \to {}^{\omega}\omega$ by f' the function defined by $f'(u) = f(u)'$. We write $\mathcal{F}_{\mathcal{J}}$ and $\mathcal{S}_{\mathcal{J}}$ for the set of all $\mathcal{J}$-functions, respectively $\mathcal{J}$-slaloms. In fact, $u \mapsto u'$ is a bijection between ${}^{\omega}\omega$ and $\mathcal{F}_{\mathcal{J}}$. We can construct bijections between ${}^{\omega}\omega$ and $\mathcal{F}_{\mathcal{J}}$, respectively $\mathcal{S}$ and $\mathcal{S}_{\mathcal{J}}$, that keep trace of the ideal, i.e. a homeomorphism / measure isomorphism, for which the bounding and dominating numbers on $\in^*_{\mathcal{I}}$ remain the same. We will use this equivalence for the proof of the two harder cases of theorem 3.2.

Lemma 3.3. *Let t be a $\mathcal{J}$-slalom. Then there is a function $u_t : \omega \to \omega$ such that for all $v : \omega \to \omega$, if $v' \in^* t$ then there are infinitely many $j \in \omega$ for which $u_t(j) = v(j)$.*

Proof. Let t be a $\mathcal{J}$-slalom, with $t(n) = \{w^n_0, ..., w^n_{n-1}\}$. We can define $u_t(j) = w^n_k(j)$ whenever $k < n \in \omega$ and $j \in J_{n,k}$, and will see that this u_t is the one we look for.

Now let $v : \omega \to \omega$ with $v' \in^* t$. If for a particular n we have $v'(n) \in t(n)$, then there is $k < n$ such that $v'(n) = w^n_k$. For all $j \in J_{n,k}$ we then have $v(j) = v'(n)(j) = w^n_k(j) = u_t(j)$. Since there are only finitely many n that do *not* fulfill $v'(n) \in t(n)$, there are infinitely many j for which $u_t(j) = v(j)$. $\dashv$

We can now turn to the proof of the theorem 3.2.

Proof. The first five inequalities are quite straightforward to prove, and the last two make use of the previous lemma.

$\mathfrak{b}(\in^*_{\mathcal{I}}) \leq \mathfrak{b}(\leq^*_{\mathcal{I}})$:
Let $B \subseteq {}^{({}^{\omega}\omega)}({}^{\omega}\omega)$ be $\in^*_{\mathcal{I}}$-bounded by $b : {}^{\omega}\omega \to \mathcal{S}$, i.e. for all $f \in B$ we have $f \in^*_{\mathcal{I}} b$. Define $\tilde{b} : {}^{\omega}\omega \to {}^{\omega}\omega$ through $\tilde{b}(u)(n) = max(b(u)(n)) + 1$. Then $\tilde{b}$ is an $\leq^*_{\mathcal{I}}$-bound for B, so the inequality follows.

$\mathfrak{b}(\leq^*_{\mathcal{I}}) \leq \mathfrak{b}(\neq^*_{\mathcal{I}})$:
Let $B \subseteq {}^{({}^{\omega}\omega)}({}^{\omega}\omega)$ be $\leq^*_{\mathcal{I}}$-bounded by $b : {}^{\omega}\omega \to {}^{\omega}\omega$. Then b is also an $\neq^*_{\mathcal{I}}$-bound to B, so the claim follows.

$\mathfrak{d}(\neq^*_{\mathcal{I}}) \leq \mathfrak{d}(\leq^*_{\mathcal{I}})$:
Let $D \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$ be $\leq^*_{\mathcal{I}}$-dominating, i.e. for every $f : {}^\omega\omega \to {}^\omega\omega$ there is a $d_f \in D$ such that $f \leq^*_{\mathcal{I}} d_f$. This implies directly $f \neq^*_{\mathcal{I}} d_f$, so D is also $\neq^*_{\mathcal{I}}$-dominating, hence the claim.

$\mathfrak{d}(\leq^*_{\mathcal{I}}) \leq \mathfrak{d}(\in^*_{\mathcal{I}})$:
Let $D \subseteq {}^{({}^\omega\omega)}\mathcal{S}$ be $\in^*_{\mathcal{I}}$-dominating, i.e. for every $f : {}^\omega\omega \to {}^\omega\omega$ there is a $d_f \in D$ such that $f \in^*_{\mathcal{I}} d_f$. Define $\tilde{d_f} : {}^\omega\omega \to {}^\omega\omega$ through $\tilde{d_f}(u)(n) = max(d_f(u)(n)) + 1$. Then $\tilde{d_f}$ satisfies $f \leq^*_{\mathcal{I}} \tilde{d_f}$, so the inequality follows.

$\mathfrak{b}(\leq^*_{\mathcal{I}}) \leq \mathfrak{d}(\leq^*_{\mathcal{I}})$:
Let B be $\leq^*_{\mathcal{I}}$-bounded. Then B cannot be $\leq^*_{\mathcal{I}}$-dominating, for it contains no element that will dominate its own $\leq^*_{\mathcal{I}}$-bound, hence the claim.

$\mathfrak{b}(\in^*_{\mathcal{I}}) \leq \mathfrak{d}(\neq^*_{\mathcal{I}})$:
Let $B \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$ be of cardinality $|B| < \mathfrak{b}(\in^*_{\mathcal{I}})$. We will show that B is not $\neq^*_{\mathcal{I}}$-dominating, i.e. that there is a function $f : {}^\omega\omega \to {}^\omega\omega$ for which $\forall g \in B(\{v \in {}^\omega\omega : |\{n \in \omega : f(v)(n) = g(v)(n)\}| = \omega\} \notin \mathcal{I})$, from which the claim follows.

Set $B' = \{g' : g \in B\} \subseteq {}^{({}^\omega\omega)}\mathcal{F}_{\mathcal{J}}$, for which $|B'| = |B|$, so B' is $\in^*_{\mathcal{I}}$-bounded, using here the equivalence between ${}^\omega\omega$ and $\mathcal{F}_{\mathcal{J}}$. Let $\bar{f} : {}^\omega\omega \to \mathcal{S}_{\mathcal{J}}$ be a $\in^*_{\mathcal{I}}$-bound for B', which means that $\{v \in {}^\omega\omega : g'(v) \not\in^* \bar{f}(v)\} \in \mathcal{I}$ for all $g \in B$. Define $f : {}^\omega\omega \to {}^\omega\omega : v \mapsto u_{\bar{f}(v)}$ as per in lemma 3.3. Now by the same lemma, for all $g : {}^\omega\omega \to {}^\omega\omega$ and all $v \in {}^\omega\omega$, if $g'(v) \in^* \bar{f}(v)$ then there are infinitely many $j \in \omega$ for which $f(v)(j) = g(v)(j)$. This means that

$$\{v \in {}^\omega\omega : g'(v) \in^* \bar{f}(v)\} \subseteq \{v \in {}^\omega\omega : |\{n \in \omega : f(v)(n) = g(v)(n)\}| = \omega\},$$

so since for $g \in B$ the first set is not in $\mathcal{I}$ (its complement is), the second one is not either in $\mathcal{I}$, because $\mathcal{I}$ is an ideal. This is the property of f we were looking for.

$\mathfrak{b}(\neq^*_{\mathcal{I}}) \leq \mathfrak{d}(\in^*_{\mathcal{I}})$:
The proof in this case is analogous to the previous one. Let $B \subseteq {}^{({}^\omega\omega)}\mathcal{S}_{\mathcal{J}}$ be of cardinality $|B| < \mathfrak{b}(\neq^*_{\mathcal{I}})$. We will show that B is not $\in^*_{\mathcal{I}}$-dominating, i.e. that there is a function $f : {}^\omega\omega \to {}^\omega\omega$ for which $\forall g \in B(\{v \in {}^\omega\omega : f'(v) \not\in^* g(v)\} \notin \mathcal{I})$, from which the claim follows.

To each $g \in B$ set $\bar{g} : {}^\omega\omega \to {}^\omega\omega : v \mapsto u_{g(v)}$ as per in lemma 3.3. Set $B' = \{\bar{g} : g \in B\} \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$, using here the equivalence between $\mathcal{S}$ and $\mathcal{S}_{\mathcal{J}}$, for which $|B'| = |B|$, so B' is $\neq^*_{\mathcal{I}}$-bounded. Let $f : {}^\omega\omega \to {}^\omega\omega$ be a $\neq^*_{\mathcal{I}}$-bound for B', which means that $\{v \in {}^\omega\omega : |\{n \in \omega : f(v)(n) = \bar{g}(v)(n)\}| = \omega\} \in \mathcal{I}$ for all $g \in B$. Now again by lemma 3.3, for all $g : {}^\omega\omega \to \mathcal{S}$ and all $v \in {}^\omega\omega$, if $f'(v) \in^* g(v)$ then there are infinitely many $j \in \omega$ for which $f(v)(j) = \bar{g}(v)(j)$. This means that

$$\{v \in {}^\omega\omega : f'(v) \in^* g(v)\} \subseteq \{v \in {}^\omega\omega : |\{n \in \omega : f(v)(n) = \bar{g}(v)(n)\}| = \omega\},$$

so since for $g \in B$ the second set is in $\mathcal{I}$, the first one is in $\mathcal{I}$ as well. This is the property of f we were looking for. ⊣

3.2 Further relations between the bounding and dominating cardinals

We continue investigating the higher bounding and dominating numbers. First we mention a few results that are provable in ZFC, then we turn to a consistency proof, other results being only mentionned at the end.

3.2.1 Relations provable in ZFC

Proposition 3.4. *Let $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}, \mathcal{K}\}$ and $R \in \{\neq^*, \leq^*, \in^*\}$. Then $\mathfrak{b}(R) \leq \mathfrak{b}(R_\mathcal{I})$.*
If moreover $\mathfrak{b}(R) = \mathfrak{c} = \mathrm{non}(\mathcal{I})$, then $\mathfrak{c}^+ \leq \mathfrak{b}(R_\mathcal{I})$.

Proof. Let $B = \{f_\alpha : \alpha \in \kappa\} \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$ be a set of cardinality $|B| = \kappa < \mathfrak{b}(R)$. Since for all $u \in {}^\omega\omega$ the set $\{f_\alpha(u) : \alpha \in \kappa\}$ is R-bounded, we can define $g : {}^\omega\omega \to {}^\omega\omega : u \mapsto g(u)$, where $g(u)$ is an R-bound for the mentionned set (consider $g : {}^\omega\omega \to \mathcal{S}$ in the case $R = \in^*$). Now, for all $\alpha \in \kappa$ we have $f_\alpha(u) R g(u)$, so $f_\alpha R_\mathcal{I} g$ since $\{u \in {}^\omega\omega : \neg f_\alpha(u) R g(u)\} = \emptyset \in \mathcal{I}$, and g is an $R_\mathcal{I}$-bound for B. We infer $\mathfrak{b}(R) \leq \mathfrak{b}(R_\mathcal{I})$.

Assume $\mathfrak{b}(R) = \mathfrak{c} = \mathrm{non}(\mathcal{I})$ and let $B' = \{f_\alpha : \alpha \in \mathfrak{c}\} \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$ be a set of cardinality $|B'| = \mathfrak{c}$, as well as $\{u_\alpha : \alpha \in \mathfrak{c}\} = {}^\omega\omega$ be a list of all functions of ${}^\omega\omega$. For any $\beta, \delta \in \mathfrak{c}$, consider the set $X = \{f_\alpha(u_\delta) : \alpha \in \beta\}$; as before, because $\mathfrak{b}(R) = \mathfrak{c}$, the set X is R-bounded by a $v^\delta_\beta \in {}^\omega\omega$. Defining $g : {}^\omega\omega \to {}^\omega\omega : u_\alpha \mapsto g(u) = v^\alpha_\alpha$ (or $g : {}^\omega\omega \to \mathcal{S}$ in the case $R = \in^*$) we have that for every $\delta \ni \alpha$, $f_\alpha(u_\delta) R g(u_\delta)$, therefore $|\{u_\delta \in {}^\omega\omega : \neg f_\alpha(u_\delta) R g(u_\delta)\}| \leq |\alpha| < \mathfrak{c}$. Since $\mathfrak{c} = \mathrm{non}(\mathcal{I})$, we conclude $f_\alpha R_\mathcal{I} g$. Finally, $\mathfrak{c}^+ \leq \mathfrak{b}(R_\mathcal{I})$. ⊣

Furthermore, sometimes under a certain assumption over the characteristics of the continuum, the bouding, respectively the dominating numbers modulo the meager and null ideals are the same. We refer to Switzer [14] and Brendle [2] for the proof.

Theorem 3.5. *Let $R \in \{\neq^*, \leq^*, \in^*\}$. Then*

a) $\mathfrak{d}(R_\mathcal{N}) = \mathfrak{d}(R_\mathcal{M})$

b) If $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$*, then* $\mathfrak{b}(R_\mathcal{N}) = \mathfrak{b}(R_\mathcal{M})$

c) $\mathfrak{c}^+ \leq \mathfrak{d}(R_\mathcal{I})$ *for* $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}, \mathcal{K}\}$

3.2.2 Consistency results

We shall now turn to some inequalities between the newly introduced cardinal characteristics that are consistent with ZFC + CH, but before we do so, we shall mention a few results by Brendle and Switzer [2] in models of ZFC + ¬CH. They showed that $\mathfrak{b}(R_\mathcal{I}) < \mathfrak{c}$ and $\mathfrak{b}(R_\mathcal{I}) = \mathfrak{c}$ are consistent with ZFC, as well as $\mathfrak{b}(R_\mathcal{N}) < \mathfrak{b}(R_\mathcal{M})$ and $\mathfrak{b}(R_\mathcal{N}) > \mathfrak{b}(R_\mathcal{M})$. In what follows, $\mathbf{V}$ shall always be a model of ZFC + CH when not otherwise stipulated.

To begin with, let's introduce the **$\mathcal{N}$-Cohen forcing** defined by $\mathbb{C}_\mathcal{N} = (C_\mathcal{N}, \subseteq)$ where

$$C_\mathcal{N} = \{p : \mathrm{dom}(p) \to {}^\omega\omega : \mathrm{dom}(p) \subseteq {}^\omega\omega \\ \wedge \mathrm{graph}(p) \text{ is Borel} \wedge \mu(\mathrm{dom}(p)) = 0\}.$$

In the upcoming paragraphs, we shall study the properties of this forcing notion and prove a consistency result on the separation of bounding and dominating cardinals. We concentrate ourselves on the ideal of null sets $\mathcal{N}$, but what follows could in a similar way be applied to $\mathcal{I} = \mathcal{M}$ or $\mathcal{K}$ as well. First we see some basic properties of $\mathbb{C}_\mathcal{N}$.

Lemma 3.6. *$\mathbb{C}_\mathcal{N}$ is σ-closed and satisfies $\mathfrak{c}^+$-cc.*

Proof. To see that $\mathbb{C}_\mathcal{N}$ is σ-closed, choose an increasing sequence of $\mathbb{C}_\mathcal{N}$-conditions $p_0 \subseteq p_1 \subseteq p_2 \subseteq \ldots$ of length ω and set $p = \bigcup_{n\in\omega} p_n$. Now obviously p is a function from $\mathrm{dom}(p) \subseteq {}^\omega\omega$ to ${}^\omega\omega$. Since $\mathrm{graph}(p) = \bigcup_{n\in\omega} \mathrm{graph}(p_n)$ and the countable union of Borel sets is Borel, $\mathrm{graph}(p)$ is Borel. In a similar way, since the countable union of null sets is a null set, $\mu(\mathrm{dom}(p)) = \mu(\bigcup_{n\in\omega} \mathrm{dom}(p_n)) = 0$. It follows that $p \in C_\mathcal{N}$.

The requirement for the elements of $p \in C_\mathcal{N}$ that $\mathrm{graph}(p)$ is to be Borel means that they all are Borel subset of ${}^2({}^\omega\omega)$, of which there are $\mathfrak{c}$. It follows directly that $\mathbb{C}_\mathcal{N}$ has the $\mathfrak{c}^+$-chain condition. ⊣

As a consequence, under the assumption of CH, $\mathbb{C}_\mathcal{N}$ preserves all cardinalities. How does $\mathbb{C}_\mathcal{N}$ behave in regards to Borel null sets? Obviously $\mathbb{C}_\mathcal{N}$ does not add reals, and therefore it cannot add any new Borel null sets either. Let $\underset{\sim}{A}$ be a $\mathbb{C}_\mathcal{N}$-name for a subset of ${}^\omega\omega$. If $p \in \mathbb{C}_\mathcal{N}$ forces the sentence "$\underset{\sim}{A}$ is a null set", then we can find a Borel set N and a stronger condition $q \supseteq p$ that forces $\underset{\sim}{A} \subseteq \underset{\cdot}{N}$.

Lemma 3.7. *Let G be $\mathbb{C}_\mathcal{N}$-generic over $\mathbf{V}$. Then ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ is not $\neq^*_\mathcal{N}$-dominating in $\mathbf{V}[G]$.*

Proof. First notice that $\bigcup G$ is a function $\bigcup G : {}^\omega\omega \to {}^\omega\omega$ since G is directed, dense in $\mathbb{C}_\mathcal{N}$, and to a condition $p \in \mathbb{C}_\mathcal{N}$ we can extend any Borel set to the domain of a $q \supseteq p$. So let $g = \bigcup G$, $\underset{\sim}{g}$ its $\mathbb{C}_\mathcal{N}$-name, $p \in P$ and $f \in {}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ and suppose that $p \Vdash \mu(\{u \in {}^\omega\omega : \underset{\cdot}{f}(u) = \underset{\sim}{g}(u)\}) = \underset{\cdot}{0}$. This null set is contained in a Borel null set N, with $N \in \mathbf{V}$ since per lemma 3.6 the forcing notion $\mathbb{C}_\mathcal{N}$ is σ-closed. So we can find a condition $q \supseteq p$ such that $q \Vdash \{u \in {}^\omega\omega : \underset{\cdot}{f}(u) = \underset{\sim}{g}(u)\} \subseteq \underset{\cdot}{N}$ and a $v \in {}^\omega\omega \setminus (N \cup \mathrm{dom}(q))$, since $\mu(N \cup \mathrm{dom}(q)) = 0$.

Set $q' = q \cup \{\langle v, f(v)\rangle\}$ for which obviously $q \subseteq q'$. Now we also have $q' \nVdash \{u \in {}^\omega\omega : \underset{\cdot}{f}(u) = \underset{\sim}{g}(u)\} \subseteq \underset{\cdot}{N}$ which is a contradiction, so our assumption that there is $p \in P$ with the mentionned property fails. This means that for any $f \in {}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$, $\mathbf{V}[G]$ is a model for $\mu(\{u \in {}^\omega\omega : f(u) = g(u)\}) \neq 0$, in other words $f \not\neq^*_\mathcal{N} g$ and so ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ is not $\neq^*_\mathcal{N}$-dominating. ⊣

Lemma 3.8. *Let $\langle u_\alpha : \alpha \in \omega_1\rangle$ be an enumeration of ${}^\omega\omega$ and let $N_0 = \langle N_{0,\alpha} : \alpha \in \omega_1\rangle$ and $N_1 = \langle N_{1,\beta} : \beta \in \omega_1\rangle$ be two (non-necessarily injective) enumerations of the Borel null sets of ${}^\omega\omega$ such that for all $u \in {}^\omega\omega$ there are uncountably many $\langle\alpha, \beta\rangle$ for which $u \notin N_{0,\alpha} \cup N_{1,\beta}$. Then there is an ω_1-sequence $\langle\langle N'_{0,\alpha}, N'_{1,\alpha}\rangle : \alpha \in \omega_1\rangle$ enumerating all of $N_0 \times N_1$ such that $\forall\alpha \in \omega_1(u_\alpha \notin N'_{0,\alpha} \cup N'_{1,\alpha})$.*

Proof. Let $\langle\langle M_{0,\alpha}, M_{1,\alpha}\rangle : \alpha \in \omega_1\rangle$ be any enumeration of $N_0 \times N_1$ and define by transfinite induction $Y_0 = \emptyset$ and

$$\begin{aligned} X_\alpha &= \{\beta \in \omega_1 : \langle M_{0,\beta}, M_{1,\beta}\rangle \notin Y_\alpha \wedge u_\alpha \notin M_{0,\beta} \cup M_{1,\beta}\} \\ \iota_\alpha &= \bigcap X_\alpha \\ \langle N'_{0,\alpha}, N'_{1,\alpha}\rangle &= \langle M_{0,\iota_\alpha}, M_{1,\iota_\alpha}\rangle \\ Y_{\alpha+1} &= Y_\alpha \cup \{\langle N'_{0,\alpha}, N'_{1,\alpha}\rangle\} \\ Y_\alpha &= \bigcup_{\beta\in\alpha} Y_\beta \text{ for } \alpha \text{ a limit ordinal} \end{aligned}$$

for all $\alpha \in \omega_1$. At each step, X_α is not empty because Y_α is always countable and we assumed the existence of uncountably many β for which $u_\alpha \notin M_{0,\beta} \cup M_{1,\beta}$, so ι_α is well-defined. By construction, we ensure that for all $\alpha \in \omega_1$ we have $u_\alpha \notin N'_{0,\alpha} \cup N'_{1,\alpha}$.

We still need to see that $\langle\langle N'_{0,\alpha}, N'_{1,\alpha}\rangle : \alpha \in \omega_1\rangle$ enumerates all of $N_0 \times N_1$. Assume it were not the case and choose $\delta \in \omega_1$ minimal such that $\forall\alpha \in \omega_1(\delta \neq \iota_\alpha)$, i.e. $\langle M_{0,\delta}, M_{1,\delta}\rangle$ has not been enumerated. Let β be the stage in the construction at which we reached that for all $\gamma \in \delta$, $\langle M_{0,\gamma}, M_{1,\gamma}\rangle \in Y_\beta$. Then for all $\alpha \in \omega_1$ with $\alpha \ni \beta$ we have $u_\alpha \in M_{0,\delta} \cup M_{1,\delta}$. Since β is at most countable, there are at least cocountably many such u_α, but this would mean that $M_{0,\delta} \cup M_{1,\delta}$ is not a Borel null set, a contradiction. ⊣

In the following lemma and theorem, we will use the countable support product and iteration of copies of $\mathbb{C}_\mathcal{N}$. We will refer to them interchangably as, since $\mathbb{C}_\mathcal{N}$ does not add new reals nor new Borel null sets, the countable support product and the countable support iterations are equivalent forcing notions.

Lemma 3.9. *Let G be $\mathbb{P}_\alpha$-generic over $\mathbf{V}$ where $\mathbb{P}_\alpha$ is the countable support product of $\alpha \in \omega_2$ copies of $\mathbb{C}_\mathcal{N}$. Then ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ is $\neq^*_\mathcal{N}$-unbounded in $\mathbf{V}[G]$.*

Proof. Assume that there is a $\mathbb{P}_\alpha$-condition p and a $\mathbb{P}_\alpha$-name $\underset{\sim}{g}$ for which $p \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{g} : {}^\omega\dot{\omega} \to {}^\omega\dot{\omega}$ and g is an $\neq^*_\mathcal{N}$-bound for ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ in $\mathbf{V}[G]$. This means that for all $f \in {}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ the set $\{u \in {}^\omega\omega : f(u) \not\neq^* g(u)\} = \{u \in {}^\omega\omega : \ |\{n \in \omega : f(u)(n) = g(u)(n)\}| = \omega\}$ has measure zero.

Let $\langle u_\beta : \beta \in \omega_1\rangle$ be an enumeration of ${}^\omega\omega$. Note that such an enumeration exists because we assume $\mathbf{V} \vDash \mathsf{CH}$. Now for every $\mathbb{P}_\alpha$-condition $q = \langle q_\delta : \delta \in \alpha\rangle$ define $N^q = \bigcup_{\delta\in\alpha} \mathrm{dom}(q_\delta)$ and note that N^q is a Borel null set as we work in the countable support product. Since we consider $\mathbf{V} \vDash \mathsf{CH}$, there are ω_1 Borel null sets and as many $\mathbb{P}_\alpha$-conditions, so by lemma 3.8 we can choose a sequence $\langle\langle N_\beta, q^\beta\rangle : \beta \in \omega_1 \wedge N_\beta$ ranges over all Borel null sets $\wedge\, q^\beta \in \mathbb{P}_\alpha\rangle$ such that for all $\beta \in \omega_1$ we have $u_\beta \notin N_\beta \cup N^{q^\beta}$.

For each $\beta \in \omega_1$ we can find an $r^\beta \geq q^\beta$ that decides $\underset{\sim}{g}(u_\beta)$, i.e. $r^\beta \Vdash_{\mathbb{P}_\alpha} \underset{\sim}{g}(u_\beta) = \dot{v}_\beta$ where $\dot{v}_\beta$ is the canonical $\mathbb{P}_\alpha$-name of a $v_\beta : \omega \to \omega$ in $\mathbf{V}$. The function given by $f : {}^\omega\dot{\omega} \to {}^\omega\dot{\omega} : u_\beta \mapsto \dot{v}_\beta$ is therefore in $\mathbf{V}$. By our assumption, there is a $\mathbb{P}_\alpha$-condition p for which $p \Vdash_{\mathbb{P}_\alpha}$ "$N = \{u \in {}^\omega\omega : \dot{f}(u) \not\neq^* \underset{\sim}{g}(u)\}$ is a null set" and as well $q \geq p$ with $q \Vdash N \subseteq \dot{B}$ where B is a Borel null set. For β such that $B = N_\beta$ we now have $q^\beta \leq r^\beta \Vdash_{\mathbb{P}_\alpha} N \subseteq \dot{N}_\beta$ and $r^\beta \Vdash_{\mathbb{P}_\alpha} \dot{f}(\dot{u}_\beta) = \underset{\sim}{g}(\dot{u}_\beta)$, so $r^\beta \Vdash_{\mathbb{P}_\alpha} \dot{u}_\beta \in N$, a contradiction to the choice of β with $u_\beta \notin N_\beta$. ⊣

With all this work done, we can now finally turn to the consistensy result.

Theorem 3.10 (Switzer [14])**.** *Let $\kappa > \omega_2$ be a regular cardinal, $\mathbb{P}_\alpha$ be the countable support product of $\alpha \leq \kappa$ copies of $\mathbb{C}_\mathcal{N}$ and G be $\mathbb{P}_\kappa$-generic over $\mathbf{V}$, where $\mathbf{V} \vDash \mathsf{GCH}$. Then $\mathbf{V}[G] \vDash (\mathfrak{c}^+ = \omega_2 = \mathfrak{b}(\neq^*_\mathcal{N}) < \mathfrak{d}(\neq^*_\mathcal{N}) = 2^\mathfrak{c} = \kappa)$.*

Proof. First note that as in lemma 3.6, $\mathbb{P}_\kappa$ is σ-closed, and together with the Δ-system lemma (see Halbeisen [7, lemma 14.3]) we also have that $\mathbb{P}_\kappa$ satisfies the ω_2-chain condition. Therefore this forcing notion preserves all cardinalities.

Since we assume $\mathbf{V} \vDash \mathsf{GCH}$, from chapter 3.2.1 we already know that $\mathfrak{c}^+ = \omega_2 \leq \mathfrak{b}(\neq^*_\mathcal{N})$. Also, by an argument analogous to that of Halbeisen [7, theorem 15.19], we have $2^\mathfrak{c} = \kappa$.

Let $\underset{\sim}{f}$ be a $\mathbb{P}_\kappa$-name for a function in ${}^{({}^\omega\omega)}({}^\omega\omega)$ and for all $u \in {}^\omega\omega$ let $\mathcal{A}_u$ be an antichain of $\mathbb{P}_\kappa$-conditions p_v such that $p_v \Vdash_{\mathbb{P}_\kappa} \underset{\sim}{f}(\dot{u}) = \dot{v}$ for all $v \in {}^\omega\omega$. Since $\mathbb{P}_\kappa$ satisfies ω_2-cc and we assumed CH, the set

$$\{\mathrm{supp}(p) : p \in \bigcup_{u\in{}^\omega\omega} \mathcal{A}_u\}$$

is of cardinality ω_1, so there is an $\alpha \in \omega_2$ for which $\underset{\sim}{f}$ is a $\mathbb{P}_\alpha$-name. Therefore, any $\neq^*_\mathcal{N}$-bound of ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ would be in a $\mathbb{P}_\alpha$-generic extension of $\mathbf{V}$ but, by lemma 3.9, ${}^{({}^\omega\omega)}({}^\omega\omega) \cap \mathbf{V}$ is $\neq^*_\mathcal{N}$-unbounded, so it is in $\mathbf{V}[G]$ too.

Finally, let $D \subseteq {}^{({}^\omega\omega)}({}^\omega\omega)$ be of cardinality $|D| = \lambda < \kappa$. Then D has been added to $\mathbf{V}[G]$ at an early stage of the iteration (latest by $\mathbb{P}_\lambda$) and therefore lemma 3.7 implies that D is not $\neq^*_\mathcal{N}$-dominating. ⊣

Adapting the proof of theorem 3.10 to the other ideals $\mathcal{M}$ and $\mathcal{K}$ and using further forcing notions like Hechler and localization forcings, Switzer shows in total four consistency results for the separation of bounding and dominating numbers. We shall summarize these in the following theorem, the proof of which is to be found in Switzer [14, Sections 4.1 and 4.3].

Theorem 3.11 (Switzer [14])**.** *Let $\lambda \geq \kappa \geq \omega_2$ be regular cardinals and $\mathcal{I} \in \{\mathcal{N}, \mathcal{M}, \mathcal{K}\}$. Then the following propositions are equiconsistent with* ZFC*:*

a) $\mathfrak{c}^+ = \mathfrak{b}(\neq^*_\mathcal{I}) < \mathfrak{d}(\neq^*_\mathcal{I}) = 2^\mathfrak{c} = \kappa$

b) $\mathfrak{c}^+ < \mathfrak{b}(\leq^*_\mathcal{I}) = \mathfrak{d}(\neq^*_\mathcal{I}) = 2^\mathfrak{c} = \kappa$

c) $\mathfrak{c}^+ < \mathfrak{b}(\leq^*_{\mathcal{I}}) = \kappa < \mathfrak{d}(\leq^*_{\mathcal{I}}) = 2^{\mathfrak{c}} = \lambda$

d) $\mathfrak{c}^+ < \mathfrak{b}(\in^*_{\mathcal{I}}) = 2^{\mathfrak{c}} = \kappa$

3.3 Further questions

Question 3.12 (Is there an analogous to Cichoń's diagram for even higher iterations?). What about further steps in the construction method? For example with CH on ${}^{({}^{\mathfrak{c}}\mathfrak{c})}({}^{\mathfrak{c}}\mathfrak{c})$ to make it not too heavy in the notation. Has this construction an analogous to Cichoń's diagram?

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