Defining what is a probability of failure for systems modelled by stochastic simulators

Other Conference Item

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Risk, Safety & Uncertainty Quantification

Defining what is a probability of failure for systems modelled by stochastic simulators

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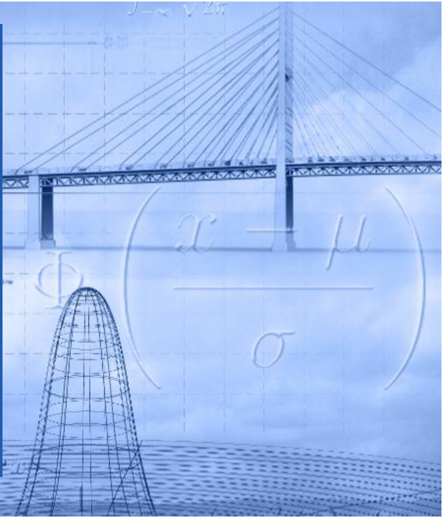


Overview on reliability analysis

Goal: estimate the probability of failure

Melchers and Beck (2015)

- Failure is defined through a limit state function $g : \boldsymbol{x} \in \mathcal{D}_{\boldsymbol{X}} \subset \mathbb{R}^M \mapsto \mathbb{R}$, such that the failure domain $\mathcal{D}_f$ is given by $\{\boldsymbol{x} : g(\boldsymbol{x}) \leqslant 0\}$

$$\boxed{P_f = \mathbb{P}\left(g\left(\boldsymbol{X}\right) \leqslant 0\right) = \int_{\{\boldsymbol{x}:g(\boldsymbol{x})\leqslant 0\}} f_{\boldsymbol{X}}(\boldsymbol{x})\mathrm{d}\boldsymbol{x}}$$

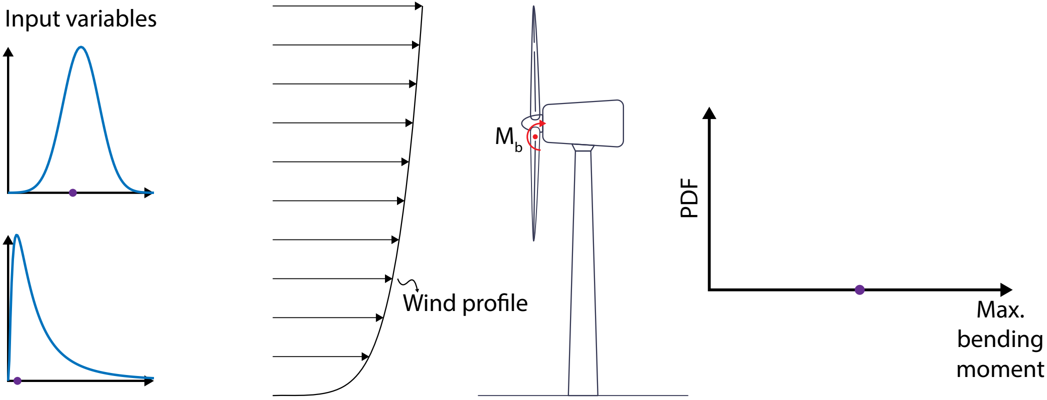
Features

- Multidimensional integration
- Implicit domain
- Failure is rare, $P_f \in [10^{-8}, 10^{-2}]$

- g is often based on a computational model $\mathcal{M}$
- g is a deterministic simulator

Deterministic simulators

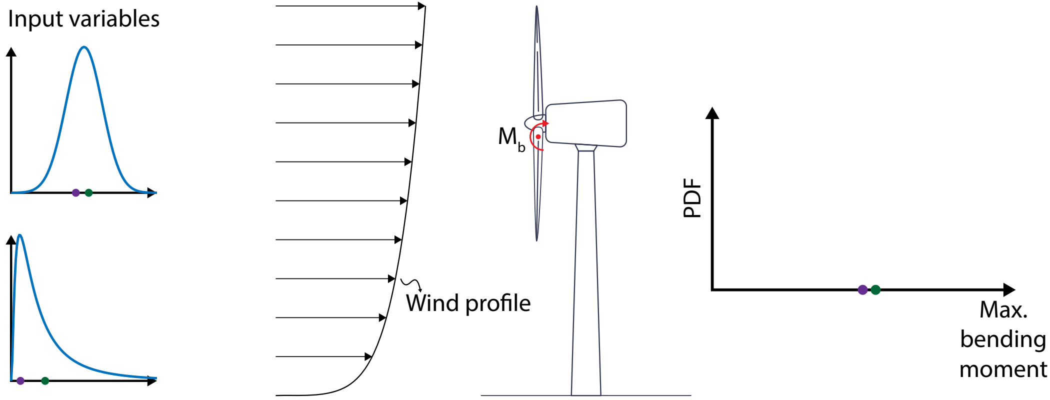
Zhu and Sudret (2020)



Each run of the model leads to a single output

Deterministic simulators

Zhu and Sudret (2020)



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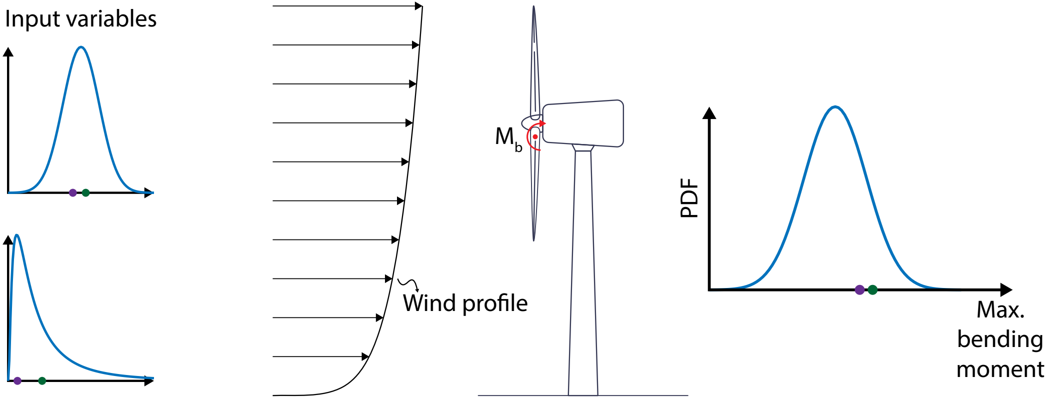
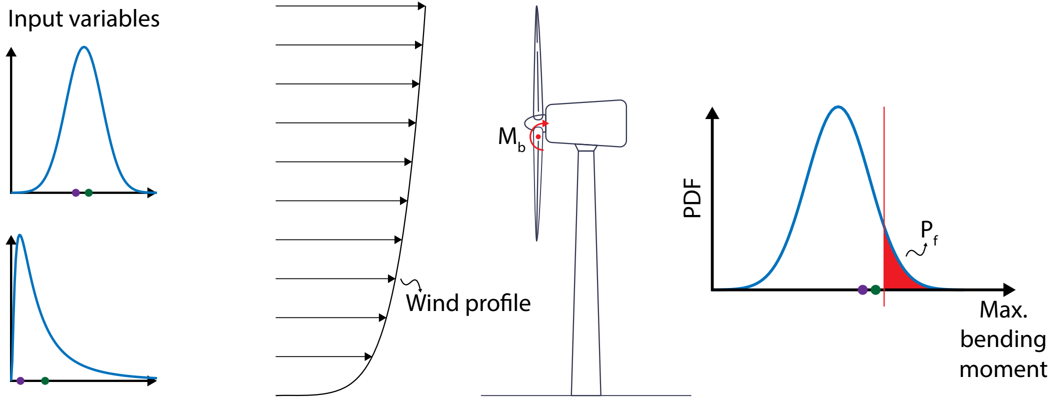


Running the model multiple times allows characterizing the output random variable

Deterministic simulators

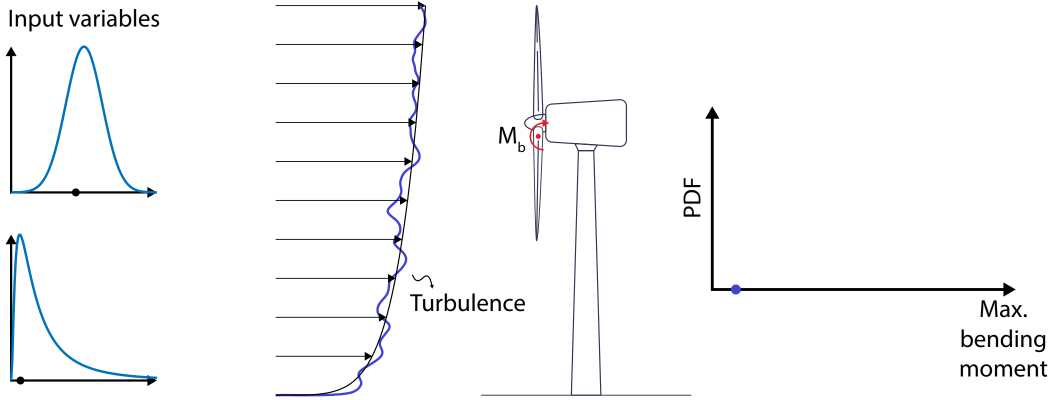
Zhu and Sudret (2020)



Finally, we can compute any quantity of interest (QoI)

Stochastic simulators

Zhu and Sudret (2020)

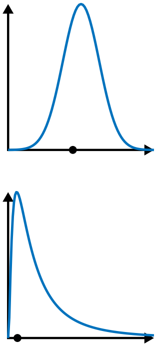


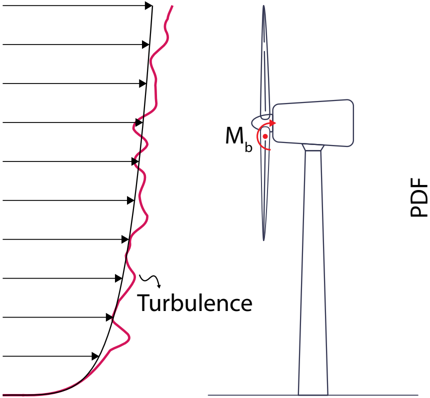
Wind fields depend on *macroscopic parameters* and incorporate wind stochasticity

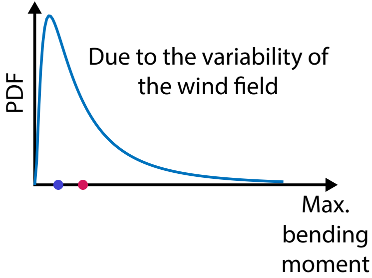
Stochastic simulators

Zhu and Sudret (2020)

Input variables



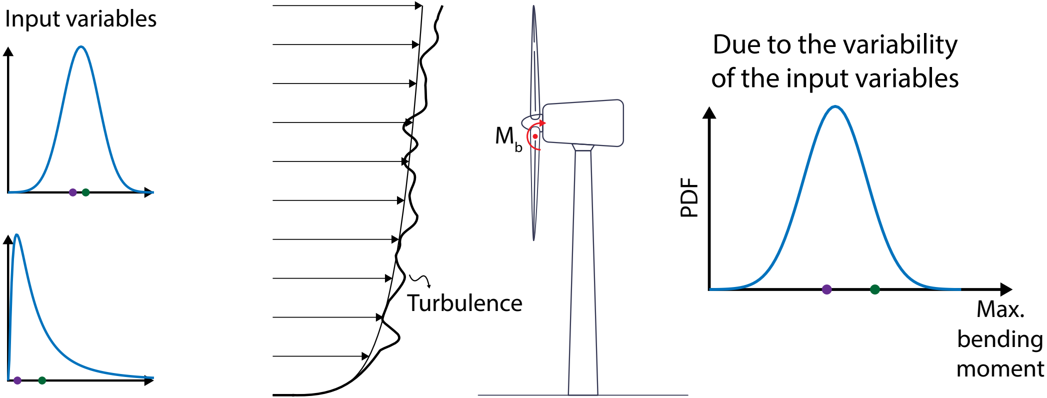




Repeated runs of the simulator lead to different outputs

Stochastic simulators

Zhu and Sudret (2020)



If the stochasticity is fixed, the model reduces to a deterministic simulator

Formal definition

Lüthen, Marelli and Sudret (2023)

Deterministic simulators

- A deterministic simulator $\mathcal{M}_d$ is a map:

$$\begin{aligned} \mathcal{M}_d : \mathcal{D}_X &\to \mathbb{R} \\ \boldsymbol{x} &\mapsto \mathcal{M}_d(\boldsymbol{x}) \end{aligned}$$

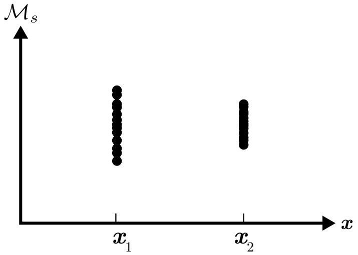
- $\mathcal{D}_{\boldsymbol{X}}$ is the input parameter space with joint PDF $f_{\boldsymbol{X}}$

Stochastic simulators

- A real-valued stochastic simulator $\mathcal{M}_s$ is a map:

$$\begin{aligned} \mathcal{M}_s : \mathcal{D}_X \times \Omega &\to \mathbb{R} \\ (\boldsymbol{x}, \omega) &\mapsto \mathcal{M}_s(\boldsymbol{x}, \omega) \end{aligned}$$

- Stochasticity in the model is represented by an abstract random event $\omega \in \Omega$



- Randomness in computational models is achieved by introducing latent variables $\boldsymbol{Z}(\omega)$:

$$\mathcal{M}_s(\boldsymbol{x}, \omega) \equiv \mathcal{M}_d(\boldsymbol{x}, \boldsymbol{Z}(\omega))$$

Stochastic simulators

Lüthen, Marelli and Sudret (2023)

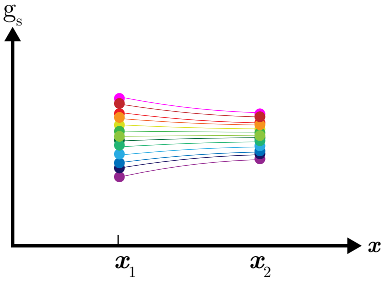
Two possible points of view

- Fixing $\boldsymbol{x}$ or ω leads to different behaviours

Random function view

- For a given ω_0, the output behaves as a deterministic function:

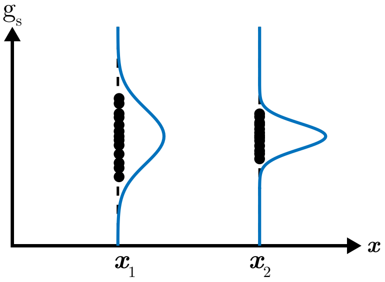
$$g_s\left(\cdot, \omega_0\right) \equiv g\left(\boldsymbol{x}, \omega_0\right)$$



Random variable view

- For a given $\boldsymbol{x}_0$, the output behaves as a random variable:

$$Y_{\boldsymbol{x}_0} = g_s\left(\boldsymbol{x}_0, \cdot\right) \equiv Y_{\boldsymbol{x}_0} \mid \boldsymbol{X} = \boldsymbol{x}_0$$



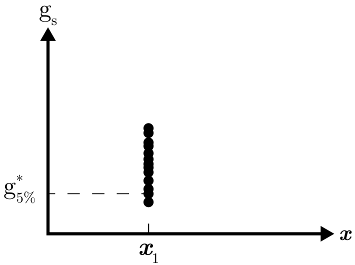
Reliability analysis on stochastic models

- The goal is to compute the probability that $g_s\left(\boldsymbol{X}, \omega\right) \leqslant 0$
- No straightforward definition due to stochasticity
- Current approaches aim to bypass the latter

Possible approaches

- Computing characteristic values

$$g^*_\alpha\left(\boldsymbol{x}\right) = \inf\left\{x \in \mathbb{R} : \mathbb{P}\left(Y_{\boldsymbol{x}_0} \leqslant x\right) > \alpha\right\}$$



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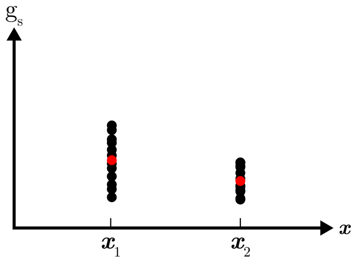
- Denoising

$$\hat{g}(\boldsymbol{x}) = \mathbb{E}_\omega[g_s(\boldsymbol{x}, \omega)]$$



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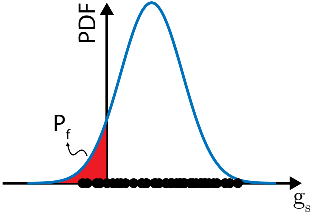
- Denoising

$$\hat{g}(\boldsymbol{x}) = \mathbb{E}_\omega\left[g_s(\boldsymbol{x}, \omega)\right]$$

- MCS

$$P_{f,\text{MCS}} = \mathbb{E}_\omega\left[\mathbb{P}\left(g_s(\cdot, \boldsymbol{\omega}_0) \leqslant 0\right)\right] \equiv \mathbb{E}_{\boldsymbol{X}}\left[\mathbb{P}\left(g_s(\boldsymbol{x}_0, \cdot) \leqslant 0\right)\right]$$



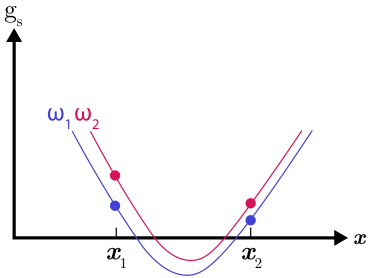
Loss of information occurs because the stochasticity is not explicitly considered

Reliability analysis on stochastic models

Random function view

- The conditional P_f reads:

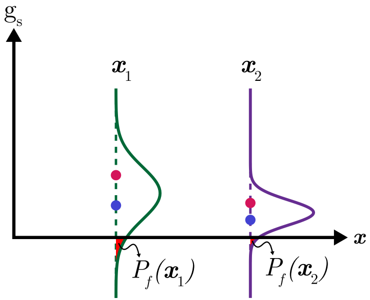
$$P_f\left(\omega_0\right) = \mathbb{P}\left(g_s\left(\boldsymbol{X}, \omega_0\right) \leqslant 0\right)$$



Random variable view

- The conditional P_f reads:

$$P_f\left(\boldsymbol{x}_0\right) = \mathbb{P}_\omega\left(g_s\left(\boldsymbol{x}_0, \omega\right) \leqslant 0\right)$$



Due to conditioning, P_f is a random variable

Reliability analysis on stochastic models

Practical implications

- A single realization of P_f is not of interest. Instead, we look for the statistics and quantiles associated with this random variable
- Reliability analysis becomes ambiguous and more complex due to the additional latent variables in the simulator
- The points of view are not interchangeable. They are different random variables that have the same expected value, $P_{f,\text{MCS}}$
- There is no unequivocal reliability measure nor a "best" point of view

Best approach will depend on the problem at hand

Toy example – $R - S$ problem

- Consider the deterministic R-S problem:

$$\boldsymbol{X} = \{R, S\}$$
$$g(\boldsymbol{X}) = R - S$$

- Conversion into stochastic simulator by introducing two latent variables A and B:

$$g_s(\boldsymbol{X}; A, B) = A \cdot R - B \cdot S$$

- where:
  - Input space

$$R \sim \mathcal{N}\left(5, 0.8^2\right); S \sim \mathcal{N}\left(2, 0.6^2\right)$$

  - Latent space

$$A \sim \mathcal{N}\left(1, 0.1^2\right); B \sim \mathcal{N}\left(1, 0.1^2\right)$$

Monte Carlo approach

- Two-level approach for obtaining the conditional distributions

Random function view

- Draw samples from the latent variables
- For each draw, the stochastic model reads:

$$g_s\left(\boldsymbol{X}; A_0, B_0\right) = A_0 \cdot R - B_0 \cdot S$$

- Compute $P_f(\omega_0)$

Random variable view

- Draw samples from the input variables
- For each draw, the stochastic model reads:

$$g_s\left(\boldsymbol{x}_0; A, B\right) = A \cdot R_0 - B \cdot S_0$$

- Compute $P_f(\boldsymbol{x}_0)$

- Distributions were chosen such that the conditional probability of failure was analytically obtained

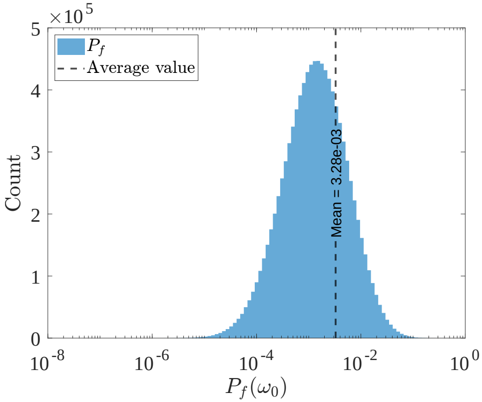
Toy example – $R - S$ problem

Results – Random function view

- Many possible reliability measures:

Mean	$3.28 \cdot 10^{-3}$
Median	$1.34 \cdot 10^{-3}$
$P_{f_{95\%}}$	$1.26 \cdot 10^{-2}$
95% CI	$\left[7.29 \cdot 10^{-5}, 1.86 \cdot 10^{-2}\right]$

- Probability of failure for particular ω_0, caused only due the uncertainty in the input variables
- Practical application: design



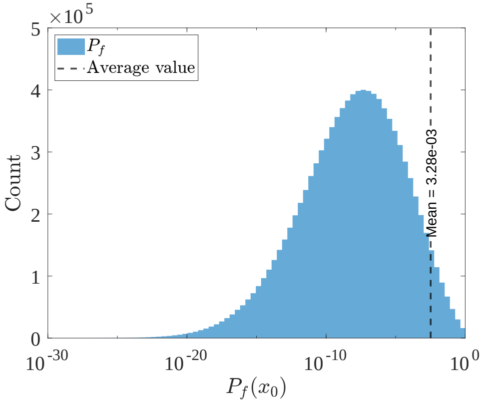
Toy example – $R - S$ problem

Results – Random variable view

- Many possible reliability measures:

Mean	$3.28 \cdot 10^{-3}$
Median	$1.26 \cdot 10^{-8}$
$P_{f_{95\%}}$	$2.54 \cdot 10^{-3}$
95% CI	$\left[5.15 \cdot 10^{-17}, 1.98 \cdot 10^{-2}\right]$

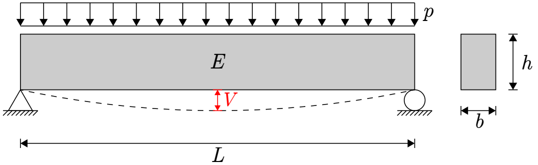
- Probability of failure at a particular design, caused only due to the stochasticity in the model
- Practical application: robust design



Simply supported beam

- Limit state function

$$g_s\left(\boldsymbol{x};E\right) = 0.015 - \frac{5pL^4}{32Ebh^3}$$



- Input space

Variable	Description	Distribution	Mean	Std. deviation
b	Beam width	Lognormal	0.15 m	7.5 mm
h	Beam height	Lognormal	0.3 m	15 mm
L	Length	Lognormal	5 m	50 mm
p	Uniform load	Lognormal	10 kN/m	2 kN/m

- Latent space

Variable	Description	Distribution	Mean	Std. deviation
E	Young's modulus	Lognormal	30,000 MPa	4,500 MPa

Simply supported beam

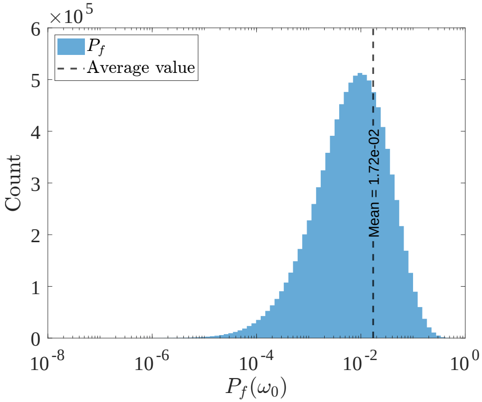
Results – Random function view

- Possible realiability measures:

Mean	$1.72 \cdot 10^{-2}$
Median	$7.20 \cdot 10^{-3}$
$P_{f_{95\%}}$	$6.80 \cdot 10^{-2}$
95% CI	$\left[1.67 \cdot 10^{-4}, 9.56 \cdot 10^{-2}\right]$

- Due to two-level Monte-Carlo, many calls to the limit state function

Need for stochastic emulators



Simply supported beam

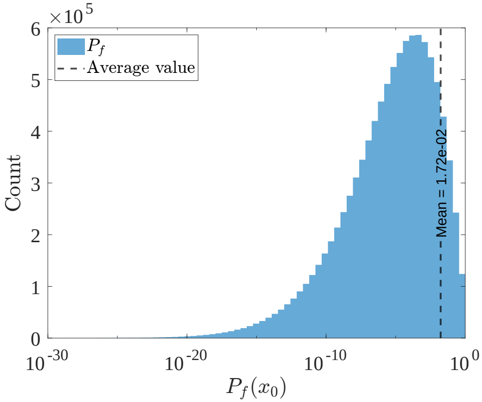
Results – Random variable view

- Possible realiability measures:

Mean	$1.72 \cdot 10^{-2}$
Median	$1.29 \cdot 10^{-5}$
$P_{f_{95\%}}$	$8.32 \cdot 10^{-2}$
95% CI	$\left[1.75 \cdot 10^{-14}, 2.01 \cdot 10^{-1}\right]$

- Due to two-level Monte-Carlo, many calls to the limit state function

Need for stochastic emulators



Less relevant in practice due to its huge variability

Concluding remarks

Conclusions

- Aleatory uncertainty is inherent to the problem and cannot be disregarded
- Due to inherent stochasticity of the models, performing reliability analysis becomes more complex
- No unequivocal reliability measure exists and most suitable reliability measure will depend on the problem
- Despite that, the random variable view leads to uninteresting pointwise P_f

Upcoming steps

- Use of stochastic emulators to overcome the computational burden
- Definition of suitable reliability measures w.r.t. case studies

Chair of Risk, Safety & Uncertainty Quantification

www.rsuq.ethz.ch

The Uncertainty Quantification Software

UQLab

www.uqlab.com

The Uncertainty Quantification Community

UQWorld

www.uqworld.org

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References and further reading

- Zhu, X., and Sudret, B. (2020). Replication-based emulation of the response distribution of stochastic simulators using generalized lambda distributions. International Journal for Uncertainty Quantification, 10(3), 249-275.

- Zhu, X., and Sudret, B. (2021). Emulation of stochastic simulators using generalized lambda models. SIAM/ASA Journal on Uncertainty Quantification, 9(4), 1345-1380.

- Zhu, X., and Sudret, B. (2023). Stochastic polynomial chaos expansions to emulate stochastic simulators. International Journal for Uncertainty Quantification, 13(2), 31-52.

- Lüthen, N., Marelli, S., and Sudret, B. (2023). A spectral surrogate model for stochastic simulators computed from trajectory samples. Computer Methods in Applied Mechanics and Engineering, 406, 115875.