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ARTICLE

On the choosability of H -minor-free graphs

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Abstract

Given a graph H , let us denote by $f_\chi(H)$ and $f_\ell(H)$, respectively, the maximum chromatic number and the maximum list chromatic number of H -minor-free graphs. Hadwiger's famous colouring conjecture from 1943 states that $f_\chi(K_t) = t - 1$ for every $t \geq 2$. A closely related problem that has received significant attention in the past concerns $f_\ell(K_t)$, for which it is known that $2t - o(t) \leq f_\ell(K_t) \leq O(t(\log \log t)^6)$. Thus, $f_\ell(K_t)$ is bounded away from the conjectured value $t - 1$ for $f_\chi(K_t)$ by at least a constant factor. The so-called H -Hadwiger's conjecture, proposed by Seymour, asks to prove that $f_\chi(H) = v(H) - 1$ for a given graph H (which would be implied by Hadwiger's conjecture).

In this paper, we prove several new lower bounds on $f_\ell(H)$, thus exploring the limits of a list colouring extension of H -Hadwiger's conjecture. Our main results are:

- For every $\varepsilon > 0$ and all sufficiently large graphs H we have $f_\ell(H) \geq (1 - \varepsilon)(v(H) + \kappa(H))$, where $\kappa(H)$ denotes the vertex-connectivity of H .
- For every $\varepsilon > 0$ there exists $C = C(\varepsilon) > 0$ such that asymptotically almost every n -vertex graph H with $\lceil Cn \log n \rceil$ edges satisfies $f_\ell(H) \geq (2 - \varepsilon)n$.

The first result generalizes recent results on complete and complete bipartite graphs and shows that the list chromatic number of H -minor-free graphs is separated from the desired value of $(v(H) - 1)$ by a constant factor for all large graphs H of linear connectivity. The second result tells us that for almost all graphs H with superlogarithmic average degree $f_\ell(H)$ is separated from $(v(H) - 1)$ by a constant factor arbitrarily close to 2. Conceptually these results indicate that the graphs H for which $f_\ell(H)$ is close to the conjectured value $(v(H) - 1)$ for $f_\chi(H)$ are typically rather sparse.

Keywords: chromatic number; graph minors; list coloring; Hadwiger's conjecture

2020 MSC Codes: Primary: 05C15, 05C80, 05C83

1. Introduction

All graphs considered in this paper are finite, have no loops and no parallel edges. Given graphs G and H , we say that G contains H as a *minor*, in symbols, $G \geq H$, if a graph isomorphic to H can be obtained from a subgraph of G by contracting edges.

Hadwiger's colouring conjecture, first stated in 1943 by Hugo Hadwiger [8], is among the most famous and important open problems in graph theory. It claims a deep relationship between the chromatic number of graphs and their containment of graph minors, as follows.

Conjecture 1 (Hadwiger [8]). *Let $t \in \mathbb{N}$. If a graph G is K_t -minor-free, then $\chi(G) \leq t - 1$.*

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Hadwiger's conjecture has been proved for all values $t \leq 6$, see [26] for the most recent result in this sequence, resolving the K_6 -minor-free case. For $t = 5$, the conjecture states that K_5 -minor-free graphs are 4-colorable. Since planar graphs are K_5 -minor-free, this special case already generalises the famous four colour theorem that was proved in 1976 by Appel, Haken and Koch [1, 2].

Given that during 80 years of study little progress has been made towards resolving Hadwiger's conjecture for $t \geq 7$, it seems natural to approach the conjecture via meaningful relaxations. For instance, much of recent work has focused on its asymptotic version. The so-called *linear Hadwiger conjecture* states that for some absolute constant $C \geq 1$, every K_t -minor-free graph is $\lfloor Ct \rfloor$ -colorable. Starting with a breakthrough result by Norin, Postle and Song [20] in 2019, there has been a set of papers providing some exciting progress towards this conjecture [21, 23–25]. This culminated in the currently best known upper bound of $O(t \log \log t)$ for the chromatic number of K_t -minor-free graphs by Delcourt and Postle [6] in 2021.

Another natural relaxation, proposed by Seymour [27, 28], suggests replacing the condition that the considered graphs exclude K_t as a minor by the stronger condition that they exclude a particular, possibly non-complete graph H on t vertices as a minor.

Conjecture 2 (*H*-Hadwiger's conjecture [27, 28]). *H*-minor-free graphs are $(\nu(H) - 1)$ -colorable.

Note that Hadwiger's conjecture would imply the truth of this statement for every H . Also note that this upper bound on the chromatic number would be best possible for every H , as the complete graph $K_{\nu(H)-1}$ has chromatic number $\nu(H) - 1$ but is too small to host an H -minor.

H-Hadwiger's conjecture can easily be verified using a degeneracy-colouring approach if H is a forest, and it is also known to be true for spanning subgraphs of the Petersen graph [9]. A particular case of *H*-Hadwiger's conjecture which has received special attention in the past is when $H = K_{s,t}$ is a complete bipartite graph. Woodall [37] conjectured in 2001 that every $K_{s,t}$ -minor-free graph is $(s + t - 1)$ -colorable. Also this problem remains open, but if true it would resolve *H*-Hadwiger's conjecture for all bipartite H . Several special cases of this conjecture have been solved by now. Most notably, Kostochka [14, 15] proved that for some function $t_0(s) = O(s^3 \log^3 s)$, *H*-Hadwiger's conjecture holds whenever $H = K_{s,t}$ and $t \geq t_0(s)$. The conjecture is also true for $H = K_{3,3}$, which can be seen using the structure theorem for $K_{3,3}$ -minor-free graphs by Wagner [35] and the fact that planar graphs are 5-colorable. In addition, the statement has been proved for $H = K_{2,t}$ when $t \geq 1$ [5, 18, 37, 38], for $H = K_{3,t}$ when $t \geq 6300$ [16] and for $H = K_{3,4}$ [10]. In a different direction, Norin and Turcotte [22] recently proved *H*-Hadwiger's conjecture for all sufficiently large bipartite graphs of bounded maximum degree that belong to a class of graphs with strongly sublinear separators.

1.1. List colouring *H*-minor-free graphs

In this paper, we shall be concerned with the *list chromatic number* of graphs that exclude a fixed graph H as a minor. List colouring is a well-known and popular subject in the area of graph colouring, whose introduction dates back to the seminal paper of Erdős, Rubin and Taylor [7]. A *list assignment* for a graph G is a mapping $L: V(G) \rightarrow 2^{\mathbb{N}}$ assigning to every vertex $v \in V(G)$ a finite set $L(v)$ of colours, also called the *list* of v . An *L-colouring* of G is a proper colouring $c: V(G) \rightarrow \mathbb{N}$ for which every vertex must choose a colour from its list, that is, $c(v) \in L(v)$ for every $v \in V(G)$. Finally, we say that G is *k-choosable* for some integer $k \geq 1$ if there exists a proper *L*-colouring for every list assignment L satisfying $|L(v)| \geq k$ for all $v \in V(G)$. The *list chromatic number* of G , denoted $\chi_\ell(G)$, is the smallest integer k such that G is *k-choosable*. Note that trivially $\chi(G) \leq \chi_\ell(G)$ for every graph G , but conversely no relationship holds, as $\chi_\ell(G)$ is unbounded even on bipartite graphs G , see [7].

The first open problem regarding list colouring of minor-closed graph classes was raised already in 1979 in the seminal paper by Erdős, Rubin and Taylor [7], who asked to determine

the maximum list chromatic number of planar graphs. This question was answered in the 1990s in work of Thomassen [33] and Voigt [34]. Thomassen proved that every planar graph is 5-choosable, and Voigt gave the first examples of planar graphs G with list chromatic number $\chi_\ell(G) = 5$.

The latter result also answered a question by Borowiecki [4] in the negative, who had asked whether one could potentially strengthen Hadwiger's conjecture to the list colouring setting by asserting that every K_t -minor-free graph G satisfies $\chi_\ell(G) \leq t - 1$.

Given the previous discussion, it is natural to study the maximum list chromatic number of K_t -minor-free graphs, see also [39] for an open problem garden entry about this problem. To make the following presentation more convenient, for every graph H we denote by $f_\chi(H)$ and $f_\ell(H)$, respectively, the maximum (list) chromatic number of H -minor-free graphs. Note that with this notation, the H -Hadwiger's conjecture amounts to saying that $f_\chi(H) = \nu(H) - 1$.

Let us briefly summarise previous work regarding bounds on $f_\ell(K_t)$. The construction of Voigt mentioned above shows that $f_\ell(K_5) \geq 5$. Thomassen's result regarding the 5-choosability of planar graphs was later extended by Škrekovski [29] to K_5 -minor-free graphs, thus proving that $f_\ell(K_5) = 5$. Until today none of the values $f_\ell(K_t)$ with $t \geq 6$ have been determined precisely, a list of the currently best known lower and upper bounds for $f_\ell(K_t)$ for small values of t can be found in [3]. In 2007, Kawarabayashi and Mohar [12] made two conjectures regarding the asymptotic behaviour of $f_\ell(K_t)$, namely that (A) $f_\ell(K_t) = O(t)$, this is known as the *list linear Hadwiger conjecture*, and that (B) $f_\ell(K_t) \leq \frac{3}{2}t$ for every t . In 2010, Wood [36], inspired by the fact that $f_\ell(K_5) = 5$, proposed an even stronger conjecture stating that $f_\ell(K_t) = t$ for every $t \geq 5$. This strong conjecture was refuted in 2011 by Barát, Joret and Wood, who gave a construction showing that $f_\ell(K_t) \geq \frac{4}{3}t - O(1)$. However, the weaker conjecture (B) by Kawarabayashi and Mohar still remained open. Recently, a new lower bound of $f_\ell(K_t) \geq 2t - o(t)$ was established by the second author [30], thus refuting conjecture (B). As for upper bounds, the best currently known bound is $f_\ell(K_t) \leq Ct(\log \log t)^6$, which was established in 2020 by Postle [25]. Some previous work also addressed bounds on $f_\ell(H)$ when H is non-complete. In particular, Woodall [27] conjectured in 2001 that $f_\ell(K_{s,t}) = s + t - 1$ for all integers $s, t \geq 1$, and proved this in the case when $s = t = 3$. From the previously mentioned works [5, 18, 37, 38] it was also known that $f_\ell(K_{2,t}) = t + 1$ for $t \geq 1$. Additionally, a result by Jørgensen [10] implied the truth of the conjecture for $K_{3,4}$, and Kawarabayashi [11] proved that $f_\ell(K_{4,t}) \leq 4t$ for every t . Despite this positive evidence, Woodall's conjecture was recently disproved by the second author [31] showing that $f_\ell(K_{s,t}) \geq (1 - o(1))(2s + t)$ for all large values of $s \leq t$. A positive result comes from the aforementioned result of Norin and Turcotte [22], which also works for list colourings and shows that $f_\ell(H) = \nu(H) - 1$ for all large bipartite graphs H of bounded maximum degree in a graph class with strongly sublinear separators.

1.2. Our contribution

The above discussion shows that when excluding a sufficiently large complete or a sufficiently large balanced complete bipartite graph H , the value of $f_\ell(H)$ exceeds the trivial lower bound $f_\ell(H) \geq \nu(H) - 1$ by at least a constant factor. This means that, in a strong sense, one cannot hope for extending Hadwiger's conjecture to list colouring with the same quantitative bounds. However, note that if H is a complete or a balanced complete bipartite graph, then H is quite dense in the sense that it has a quadratic number of edges. On the other extreme of the spectrum, the previously mentioned result by Norin and Turcotte [22] shows that $f_\ell(H) = \nu(H) - 1$ *does* hold for large classes of graphs H with a constant maximum degree (and thus, with a linear number of edges). This naturally opens up a new question, as follows: How sparse must the desired minor H be, such that one *can* hope for a list colouring extension of H -Hadwiger's conjecture? Concretely, which structural and density properties of graphs H guarantee that $f_\ell(H) = \nu(H) - 1$? While one might

be tempted to hope for a nice description of the class of all graphs H satisfying $f_\ell(H) = v(H) - 1$, Theorem 3 below speaks a word of caution: Any given graph F can be augmented, by the addition of sufficiently many isolated vertices, to a graph H in this class.

Theorem 3. *For every graph F there exists $k_0 = k_0(F)$ such that for every $k \geq k_0$ the graph H obtained from F by the addition of k isolated vertices satisfies $f_\ell(H) = v(H) - 1$. In fact, every H -minor-free graph is $(v(H) - 2)$ -degenerate.*

This shows that arbitrary graphs F can show up as induced subgraphs of graphs H with $f_\ell(H) = v(H) - 1$. To avoid such artificial constructions and to make a nice structural description of the graph class at hand more likely, it seems natural to ask for the largest class that is closed under taking subgraphs such that all members H of this class satisfy $f_\ell(H) = v(H) - 1$.¹

Problem 4. *Characterise the class \mathcal{H} of graphs H such that $f_\ell(H') = v(H') - 1$ for all $H' \subseteq H$.*

The main contributions of this paper are Theorems 5 and 7 below, which establish new lower bounds on $f_\ell(H)$ and strongly limit the horizon for positive instances of Problem 4. The first result proves a lower bound on $f_\ell(H)$ in terms of $v(H)$ and the vertex-connectivity $\kappa(H)$, implying that $f_\ell(H)$ exceeds $v(H)$ by a constant factor for all large graphs of linear connectivity.²

Theorem 5. *For every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that every graph H on at least n_0 vertices satisfies $f_\ell(H) \geq (1 - \varepsilon)(v(H) + \kappa(H))$.*

In particular, this result immediately generalises both of the lower bounds of $f_\ell(K_t) \geq 2t - o(t)$ and $f_\ell(K_{s,t}) \geq (1 - o(1))(2s + t)$ previously established by the second author in [30, 31] by noting that $\kappa(K_t) = t - 1$ and $\kappa(K_{s,t}) = s$ for $s \leq t$. It also has the following simple consequence, showing that the graphs in \mathcal{H} have a subquadratic number of edges.

Corollary 6. *For every $n \in \mathbb{N}$, let $h(n)$ denote the maximum possible number of edges of an n -vertex graph in \mathcal{H} . Then $\lim_{n \rightarrow \infty} \frac{h(n)}{n^2} = 0$.*

Proof. Towards a contradiction, suppose the statement is not true. Then there is some constant $\delta > 0$ such that there exist arbitrarily large graphs $H \in \mathcal{H}$ with average degree at least $\delta v(H)$. By a classical result of Mader [17], every graph of average degree at least $4(k - 1)$ for some integer $k \geq 2$ contains a k -connected subgraph. As \mathcal{H} is closed under subgraphs, this implies that there are arbitrarily large graphs $H \in \mathcal{H}$ with connectivity at least $\frac{\delta}{4}v(H)$. Then, using $\varepsilon := \frac{\delta}{8}$ and Theorem 5, for sufficiently large $H \in \mathcal{H}$ with average degree at least $\delta v(H)$, we have $f_\ell(H) \geq (1 - \varepsilon)(1 + \frac{\delta}{4})v(H) = (1 + \frac{\delta}{8} - \frac{\delta^2}{32})v(H) > v(H)$. However, we have $f_\ell(H) = v(H) - 1$ by the definition of \mathcal{H} , which yields the desired contradiction and concludes the proof. \square

Our second result addresses to what extent sparsity of H can push $f_\ell(H)$ closer to the trivial lower bound $v(H) - 1$, by showing that for any fixed $\varepsilon > 0$, asymptotically almost all n -vertex graphs H with average degree of order $C \log n$ for a sufficiently large constant C are far from being in \mathcal{H} , in the sense that $f_\ell(H)$ is separated from $v(H) - 1$ by a factor of at least $2 - \varepsilon$.

Theorem 7. *For every $\varepsilon > 0$ there exists a constant $C = C(\varepsilon) > 0$ such that asymptotically almost every graph H on n vertices with $\lceil Cn \log n \rceil$ edges satisfies $f_\ell(H) \geq (2 - \varepsilon)n$.*

¹This is done in the spirit of the definition of perfect graphs,¹ where a nice characterisation of graphs with $\chi(G) = \omega(G)$ seems elusive, but the largest class of graphs with this property that is closed under taking induced subgraphs admits a beautiful structural description by the strong perfect graph theorem.

²Here, by graphs of linear connectivity we mean n -vertex graphs H that are αn -connected for some small but absolute constant $\alpha > 0$.

Together with Corollary 6, this hints at the graphs in \mathcal{H} typically being quite sparse. It also shows that the lower bound $f_\ell(K_t) \geq 2t - o(t)$ for complete graphs from [30] applies in equal strength to almost all t -vertex graphs H with $\omega(t \log t)$ edges, despite them being (much) sparser than K_t .

Our proofs of Theorems 5 and 7 are based on several extensions and refinements of the probabilistic approach for lower bounding $f_\ell(K_t)$ and $f_\ell(K_{s,t})$ introduced by the second author in [30, 31]. However, several new ideas are required to overcome obstacles arising from the largely increased generality of the setup. For instance, to prove Theorem 7 one has to construct graphs avoiding rather sparse graphs H as a minor. While the constructions in [30, 31] were based on the fact that clique sums of graphs under mild assumptions preserve K_t - and $K_{s,t}$ -minor-freeness, a corresponding statement is no longer true for sparse graphs H of much lower connectivity.

1.3. Organisation of the paper

In Section 2 we prove two probabilistic results on random bipartite graphs that exhibit properties of these graphs that are crucial for our constructions in the proofs of Theorems 5 and 7. We then present the proofs of our main results Theorem 5 and Theorem 7 in, respectively, Section 3 and Section 4. Finally, in Section 5 we separately prove Theorem 3. The latter proof is self-contained and independent of the results in the other three sections.

1.4. Notation and terminology

By $\kappa(G)$ we denote the *vertex-connectivity* of a graph G , i.e., the minimum k such that G is k -connected. Given integers $m, n \geq 1$ and an edge-probability $p \in [0, 1]$, we use $G(m, n; p)$ to denote the bipartite Erdős-Rényi random graph with bipartition classes A and B of sizes m and n , respectively, and in which a pair ab with $a \in A$ and $b \in B$ is chosen as an edge of $G(m, n; p)$ independently with probability p . For integers $m, n \geq 1$ we denote by $G(n; m)$ a random graph drawn uniformly from all graphs on vertex set $[n] = \{1, \dots, n\}$ with exactly m edges.

While the original definition of the graph minor-containment relation \geq is via edge contractions and deletions, for proving the results in this paper it will be more convenient to think about *graph minor models*. Given a graph G and a graph H , an H -minor model is a collection $(Z_h)_{h \in V(H)}$ of pairwise disjoint and non-empty subsets of $V(G)$ with the property that $G[Z_h]$ is a connected graph for every $h \in V(H)$ and such that for every edge $h_1 h_2 \in E(H)$, there exists at least one edge in G with endpoints in Z_{h_1} and Z_{h_2} . The sets $Z_h, h \in V(H)$ are also called the *branch sets* of the minor model. It is well-known and easy to see that for every pair of graphs G and H we have $G \geq H$ if and only if there exists an H -minor model in G .

2. Probabilistic lemmas

In this short preparatory section we prove two simple auxiliary results (Lemmas 9 and 11) that will be used in the proofs of both our main results in Section 3 and 4. The lemmas capture two simple but important properties exhibited by bipartite Erdős-Rényi random graphs. These properties will later be used to lower bound the list chromatic number of the graphs in our constructions for Theorems 5 and 7 and to argue that they exclude a given graph as a minor.

Two basic tools from probability theory that we will use in the following are the classical Chernoff concentration bounds, stated below. A standard application of the Chernoff bounds yields an upper bound on the maximum degree of bipartite graphs with linear expected degree, stated below without proof.

Lemma 8 (Chernoff). *Let X be a binomially distributed random variable. Then the following bounds hold for every $\delta \in (0, 1]$:*

$$\mathbb{P}(X \geq (1 + \delta)\mathbb{E}(X)) \leq \exp\left(-\frac{\delta^2}{3}\mathbb{E}(X)\right), \mathbb{P}(X \leq (1 - \delta)\mathbb{E}(X)) \leq \exp\left(-\frac{\delta^2}{2}\mathbb{E}(X)\right).$$

Lemma 9. *Let $p \in (0, 1]$ be a constant. Then w.h.p. the random bipartite graph $G = G(n, n; p)$ has maximum degree at most $2pn$.*

In order to compactly state and refer to our next lemma below, it is convenient for us to introduce a technical definition for the following relationship between a graph H and a bipartite graph G with vertex bipartition A, B . Let G^{\complement} denote the graph complement of G . We are interested in the existence of \tilde{H} -minor models in G^{\complement} , where \tilde{H} is a subgraph of H . For fixed integers k, l , consider the situation where $X_1, \dots, X_k \subseteq A, Y_1, \dots, Y_l \subseteq B$ are pairwise disjoint subsets of A and B and $x_1, \dots, x_k, y_1, \dots, y_l \in V(H)$ are distinct vertices of H . Let \tilde{H} be the induced subgraph by the vertices $\{x_1, \dots, x_k, y_1, \dots, y_l\}$ and let $Z_{x_i} := X_1, \dots, Z_{x_k} := X_k, Z_{y_1} := Y_1, \dots, Z_{y_l} := Y_l$. Then the branch sets $(Z_v)_{v \in V(\tilde{H})}$ form an \tilde{H} -minor model in G^{\complement} if and only if each branch set is connected and for each edge of the form $x_i y_j \in E(H)$ there is an edge between Z_{x_i} and Z_{y_j} in G^{\complement} . Therefore, $(Z_v)_{v \in V(\tilde{H})}$ is not an \tilde{H} -minor model in G^{\complement} if there is an edge $x_i y_j \in E(H)$ such that G contains all edges between X_i and Y_j . This relationship, with some additional constraints on branch set size and subgraph size, is captured by the following property.

Definition 10 (Property P). *Let $0 < \delta < 1, s \in \mathbb{N}$ and let H be a graph on n vertices. We say that a bipartite graph G with bipartition $\{A, B\}$ satisfies property $P(H, \delta, s)$ if for all integers $k, l \geq \delta n$ the following holds:*

If $x_1, \dots, x_k, y_1, \dots, y_l \in V(H)$ are distinct vertices satisfying $e_H(\{x_1, \dots, x_k\}, \{y_1, \dots, y_l\}) \geq s$ and $X_1, \dots, X_k \subseteq A, Y_1, \dots, Y_l \subseteq B$ are pairwise disjoint sets of size at most $\frac{1}{\delta}$ each, then there exists an index pair $(i, j) \in [k] \times [l]$ such that $x_i y_j \in E(H)$ and $xy \in E(G)$ for every $(x, y) \in X_i \times Y_j$.

Lemma 11. *Let $\delta, p \in (0, 1)$ be constants. Then there exists a constant $D = D(\delta, p) > 1$ and a sequence $q_n = 1 - o(1)$ such that with $s = s(n) := \lceil Dn \log n \rceil$ for every n -vertex graph H the random bipartite graph $G = G(n, n; p)$ satisfies $P(H, \delta, s)$ with probability at least q_n .*

Proof. Choose any constant $D > \max\{1, 3p^{-(1/\delta^2)}\}$. Let A, B be the vertex bipartition of G with $|A| = |B| = n$, let H be an n -vertex graph and let $k, l \geq \delta n$. There are at most n^n choices of distinct vertices $x_1, \dots, x_k, y_1, \dots, y_l \in V(H)$ and at most n^{2n} choices of disjoint vertex sets $X_1, \dots, X_k \subseteq A, Y_1, \dots, Y_l \subseteq B$. Consider a fixed such choice satisfying the premises in Definition 10 and the random event that for every pair $(i, j) \in [k] \times [l]$ such that $x_i y_j \in E(H)$, not all of the potential edges between X_i and Y_j are included in G . The probability that this holds is $\prod_{x_i y_j \in E(H)} (1 - p^{|X_i||Y_j|}) \leq (1 - p^{(1/\delta^2)})^{Dn \log n}$, where we used the premises that the sets X_i, Y_j are of size at most $\frac{1}{\delta}$ and that there are at least $s \geq Dn \log n$ edges of the form $x_i y_j \in E(H)$. Using a union bound over the choices described above, we have

$$\begin{aligned} \mathbb{P}(G \text{ does not satisfy property } P(H, \delta, s)) &\leq n^{3n} (1 - p^{(1/\delta^2)})^{Dn \log n} \\ &\leq \exp(3n \log n - p^{(1/\delta^2)} Dn \log n). \end{aligned}$$

We have $3 - p^{(1/\delta^2)} D < 0$ and thus the above expression tends to 0 as $n \rightarrow \infty$. Setting $q_n := 1 - \exp((3 - p^{(1/\delta^2)} D)n \log n)$ then concludes the proof of the lemma. □

3. Proof of Theorem 5

In this section, we present the proof of Theorem 5. We start off by making use of Lemmas 9 and 11 from the previous section to establish the existence of small H -minor-free graphs that are in a sense “almost complete”, as follows.

Lemma 12. *For every $\varepsilon \in (0, \frac{1}{2})$ there exists an integer $N = N(\varepsilon)$ such that for every $n \geq N$ and every n -vertex graph H with $\kappa(H) \geq \varepsilon n$ there exists a graph F with the following properties:*

- *The vertex set of F can be partitioned into two disjoint sets A and B such that both A and B form cliques in F and $|A| = \lfloor (1 - 2\varepsilon)\kappa(H) \rfloor$, $|B| = \lfloor (1 - 2\varepsilon)n \rfloor$.*
- *Every vertex in B has at most εn non-neighbors in F .*
- *F is H -minor-free.*

Proof. Define $p := \frac{\varepsilon}{2}$ and $\delta := \varepsilon^2$. By Lemma 9 there is a sequence $p_n = 1 - o(1)$ such that $G(n, n; p)$ has maximum degree at most $2pn = \varepsilon n$ with probability at least p_n , and by Lemma 11 there exists an absolute constant $D > 0$ and a sequence $q_n = 1 - o(1)$ such that for every n -vertex graph H the probability that $G(n, n; p)$ satisfies property $\mathcal{P}(H, \delta, \lceil Dn \log n \rceil)$ is at least q_n . Let n_1 be such that $p_n, q_n > \frac{1}{2}$ for every $n \geq n_1$. Moreover, let $n_2 \in \mathbb{N}$ be chosen large enough such that the inequality $\delta^2 n^2 \geq Dn \log n$ holds for every $n \geq n_2$. Finally, we put $N := \max\{n_1, n_2\}$ and let $n \geq N$ be arbitrary. By our choice of N , there then exists at least one bipartite graph G with bipartition $\{A', B'\}$ such that $|A'| = |B'| = n$, G has maximum degree at most εn , and G satisfies property $\mathcal{P}(H, \delta, \lceil Dn \log n \rceil)$. Let $A \subseteq A', B \subseteq B'$ be chosen (arbitrarily) such that $|A| = \lfloor (1 - 2\varepsilon)\kappa(H) \rfloor$, $|B| = \lfloor (1 - 2\varepsilon)n \rfloor$. Note that this is possible as $\kappa(H) < v(H) = n$. We now define F as the graph complement of the induced subgraph $G[A \cup B]$ of G . Since A and B are independent sets in G , they form cliques in F . Thus the first item of the lemma is satisfied. To verify the second item, it suffices to note that since G has maximum degree at most εn , the same is true for $G[A \cup B]$, and thus every vertex in F can have at most εn non-neighbors in F .

It thus remains to prove that F is indeed H -minor-free. Towards a contradiction, suppose that there exists an H -minor model $(Z_h)_{h \in V(H)}$ in F . Let X_A, X_B, X_{AB} be the partition of $V(H)$ defined as follows: $X_A := \{h \in V(H) | Z_h \subseteq A\}$ and $X_B := \{h \in V(H) | Z_h \subseteq B\}$ contain those branch sets which are subsets of A or of B , respectively, and $X_{AB} := \{h \in V(H) | Z_h \cap A \neq \emptyset \neq Z_h \cap B\}$ contains the branch sets which overlap with both A and B .

Our goal now is to find at least δn vertices in X_A and in X_B whose corresponding vertex subsets of A or of B have size at most $\frac{1}{\delta}$ and which have at least $Dn \log n$ edges between them. We will then be able to use property $\mathcal{P}(H, \delta, \lceil Dn \log n \rceil)$ to complete the proof.

Note that we have $|X_B| + |X_{AB}| \leq |B| \leq (1 - 2\varepsilon)n$ as the sets in $(Z_h)_{h \in V(H)}$ are pairwise disjoint. Given that $|X_A| + |X_B| + |X_{AB}| = v(H) = n$, this implies that $|X_A| \geq 2\varepsilon n$. Since the sets $(Z_h)_{h \in X_A}$ are disjoint and since $|A| \leq (1 - 2\varepsilon)\kappa(H) < (1 - 2\varepsilon)n < n$, there cannot be more than δn sets of size greater than $\frac{1}{\delta}$ in the collection $(Z_h)_{h \in X_A}$. Hence, there exists $k \geq 2\varepsilon n - \delta n \geq \delta n$ and k distinct vertices $x_1, \dots, x_k \in X_A$ such that $|Z_{x_i}| \leq \frac{1}{\delta}$ for $i = 1, \dots, k$. Note that H has minimum degree at least $\kappa(H)$, for otherwise one could separate a vertex in H from the rest of the graph by deleting fewer than $\kappa(H)$ vertices. Using this, we have

$$\begin{aligned} |N_H(x_i) \cap X_B| &\geq \deg_H(x_i) - |X_A \cup X_{AB}| \geq \delta(H) - |A| \\ &\geq \kappa(H) - (1 - 2\varepsilon)\kappa(H) = 2\varepsilon\kappa(H) \geq 2\varepsilon^2 n = 2\delta n \end{aligned}$$

for every $i = 1, \dots, k$, where in the last step we used that $\kappa(H) \geq \varepsilon n$ by assumption. Consider for any fixed index $i \in [k]$ the set collection $(Z_h)_{h \in N_H(x_i) \cap X_B}$. Since the sets are pairwise disjoint and contained in the set B of size at most n , as above it follows that at most δn sets in this collection can be of size greater than $\frac{1}{\delta}$. Consequently, for each $i \in [k]$ there exists a subset $N_i \subseteq N_H(x_i) \cap X_B$ of size at least $2\delta n - \delta n = \delta n$ such that $|Z_h| \leq \frac{1}{\delta}$ for every $h \in N_i$ and $i \in [k]$. Let $y_1, \dots, y_l \in X_B$

be distinct vertices such that $\{y_1, \dots, y_l\} = \bigcup_{i=1}^k N_i$. Then clearly, $l \geq |N_1| \geq \delta n$. Furthermore, we have

$$e_H(\{x_1, \dots, x_k\}, \{y_1, \dots, y_l\}) \geq \sum_{i=1}^k |N_i| \geq k \cdot \delta n \geq \delta^2 n^2 \geq Dn \log n,$$

where in the last step we used our assumption that $n \geq N \geq n_2$.

We can now use that G satisfies property $P(H, \delta, \lceil Dn \log n \rceil)$, which directly implies that there exists a pair $(i, j) \in [k]^2$ such that $x_i y_j \in E(H)$ and G contains all edges of the form xy where $(x, y) \in Z_{x_i} \times Z_{y_j}$. However, by definition of F this means that there exists no edge in F which has endpoints in both Z_{x_i} and Z_{y_j} . This is a contradiction to our initial assumption that $(Z_h)_{h \in V(H)}$ form an H -minor model in F . Thus, F does not contain H as a minor, which establishes the third item of the lemma and concludes the proof. \square

Our next lemma below guarantees that for sufficiently well-connected graphs H , the property of being H -minor-free is preserved when pasting together two graphs along a sufficiently small clique. This statement will then be used in the proof of Theorem 5 to glue several copies of the H -minor-free graph from Lemma 12 along a common clique, thus eventually creating a graph that is still H -minor-free but has an increased list chromatic number. The lemma is folklore in the graph minors community, see also Section 3.1 in [19].

Lemma 13. *Let G_1, G_2 be H -minor-free graphs and let $C := V(G_1) \cap V(G_2)$. If C forms a clique in both G_1 and G_2 and if $|C| < \kappa(H)$, then the graph union $G_1 \cup G_2$ is also H -minor-free.*

The last ingredient required to complete the proof of Theorem 5 is a simple but important idea on how to lower-bound the list chromatic number of a graph that is obtained from a fixed graph F by repeated pasting along the same clique. Since the statement will also be reused for the proof of Theorem 7 in the next section, we decided to isolate it here. We use the following terminology:

Definition 14 (Pasting). *Let F be a graph, let $S \subseteq V(F)$ and $K \in \mathbb{N}$. A K -fold pasting of F at S is any graph that can be expressed as the union of K isomorphic copies F_1, \dots, F_K of F with the property that $V(F_i) \cap V(F_j) = S$ for all $1 \leq i < j \leq K$.*

Lemma 15. *Let $m, n, d \in \mathbb{N}$ with $d \leq m$ and let F be a graph whose vertex set is partitioned into two cliques A, B such that every vertex in B has at least $|A| - d$ neighbours in A . Let $K = (|A| + |B| - 1)^{|A|}$ and let $F^{(K)}$ be a K -fold pasting of F at A . Then $\chi_\ell(F^{(K)}) \geq |A| + |B| - d$.*

Proof. Let F_1, \dots, F_K be an ordering of the copies of F in the pasting graph $F^{(K)}$, and let B_1, \dots, B_K be the corresponding copies of B . Let $f: [|A| + |B| - 1]^A \rightarrow [K]$ be an arbitrary bijection and let $c_1, \dots, c_K: A \rightarrow [|A| + |B| - 1]$ be the ordering of colour assignments to A that satisfies $f(c_i) = i$ for all $i \in [K]$. Consider the list assignment $L: V(F^{(K)}) \rightarrow 2^{|A|+|B|-1}$ defined as follows:

- $L(a) := [|A| + |B| - 1]$ for all $a \in A$
- $L(b) := [|A| + |B| - 1] \setminus \{c_i(a) \mid a \in A \setminus N_{F_i}(b)\}$ for all $b \in B_i$ for all $i \in [K]$

Given that every vertex in B by assumption has at most d non-neighbors in F , we have $|L(v)| \geq |A| + |B| - 1 - d$ for all $v \in V(F^{(K)})$. Now assume towards a contradiction that $F^{(K)}$ admits a proper L -colouring c and let $i \in [K]$ be the unique index satisfying $c_i(a) = c(a)$ for all $a \in A$. Then let $c_{F_i}: A \cup B_i \rightarrow [|A| + |B| - 1]$ be the colouring c restricted to the graph F_i . Since $v(F_i) = |A| + |B|$, there exist by the pigeonhole principle vertices $u, v \in V(F_i)$ with $c(u) = c(v)$. Since c is proper and A, B are cliques, we have $uv \notin E(F_i)$ and $u \in A, v \in B$ without loss of generality. However, $c(u) \notin L(v)$ by the construction of L , a contradiction. \square

By assembling the previously established pieces, we can now easily deduce Theorem 5.

Proof of Theorem 5. Let a constant $\varepsilon > 0$ be given choose $\tilde{\varepsilon} \in (0, \frac{\varepsilon}{4})$. Let $N = N(\tilde{\varepsilon})$ be as in Lemma 12. We now set $n_0 := \max\{N, \lceil \frac{4}{\varepsilon^2} \rceil\}$ and claim that Theorem 5 holds for this choice of n_0 .

Let H be a graph on $n \geq n_0$ vertices. We have to prove that $f_\ell(H) \geq (1 - \varepsilon)(n + \kappa(H))$. If $\kappa(H) < \varepsilon n$, then this follows directly from the trivial lower bound via

$$f_\ell(H) \geq v(H) - 1 = n - 1 \geq (1 - \varepsilon^2)n = (1 - \varepsilon)(n + \varepsilon n) > (1 - \varepsilon)(n + \kappa(H)).$$

Thus, we may now assume $\kappa(H) \geq \varepsilon n$, in particular, $\kappa(H) \geq \tilde{\varepsilon} n$. Using $n \geq N$ and Lemma 12 we now find that there exists an H -minor-free graph F whose vertex set is partitioned into two cliques A, B such that $|A| = \lfloor (1 - 2\tilde{\varepsilon})\kappa(H) \rfloor < \kappa(H)$ and $|B| = \lfloor (1 - 2\tilde{\varepsilon})n \rfloor$, and such that every vertex in B has at most $\tilde{\varepsilon} n$ non-neighbors in F . Let $d := \lfloor \tilde{\varepsilon} n \rfloor$ and $K := (|A| + |B| - 1)^{|A|}$. Let $F^{(K)}$ denote a K -fold pasting of F at the clique A . Since every vertex in B has at least $|A| - d$ neighbours in A , we can apply Lemma 15 to find that

$$\begin{aligned} \chi_\ell(F^{(K)}) &\geq |A| + |B| - d \geq (1 - 2\tilde{\varepsilon})(\kappa(H) + n) - 2 - \tilde{\varepsilon} n \\ &\geq (1 - 3\tilde{\varepsilon})(n + \kappa(H)) - 2 \geq (1 - \varepsilon)(n + \kappa(H)), \end{aligned}$$

using $n \geq n_0 \geq \frac{4}{\varepsilon^2}$ in the last step. In addition, since $|A| < \kappa(H)$, the graph $F^{(K)}$ is H -minor-free by repeated application of Lemma 13. We conclude that $f_\ell(H) \geq (1 - \varepsilon)(v(H) + \kappa(H))$, as desired. \square

4. Proof of Theorem 7

In this section, we present the proof of Theorem 7. The theorem claims a lower bound on $f_\ell(H)$ for almost all graphs H on n vertices and $\lceil Cn \log n \rceil$ edges for some large constant $C > 0$. However, in fact the only condition on the graph H our lower bound proof relies upon is the following pseudo-random graph property, guaranteeing the existence of many edges between every pair of disjoint linear-size vertex subsets in H .

Definition 16 (Property Q, graph family \mathcal{Q}_n). Let $\delta > 0$ and $D > 1$ be arbitrary. We say that a graph H with n vertices satisfies property $Q(\delta, D)$ if for every two disjoint vertex sets $A, B \subseteq V(H)$ with $|A|, |B| \geq \delta n$, we have $e_H(A, B) \geq Dn \log n$. Let $\mathcal{Q}_n(\delta, D)$ denote the family of n -vertex graphs H that satisfy property $Q(\delta, D)$.

Crucially, property $Q(\delta, D)$ is satisfied for almost all graphs on n vertices with an average degree of $C \log n$ for a large enough constant C . The proof uses a standard probabilistic argument and is therefore omitted.

Lemma 17. Let $\delta > 0, D > 1$ be arbitrary and let $m: \mathbb{N} \rightarrow \mathbb{N}$ be defined as $m(n) = \lceil \frac{D^2}{\delta^2} n \log n \rceil$. Then with high probability as $n \rightarrow \infty$, a random graph $H = G(n; m(n))$ drawn uniformly from all n -vertex graphs with $m(n)$ edges satisfies property $Q(\delta, D)$.

In our next step towards proving Theorem 7, we establish the following statement somewhat analogous to Lemma 12, showing how to build small and close-to-complete H -minor-free graphs for a given graph $H \in \mathcal{Q}_n(\delta, D)$.

Lemma 18. Let $\delta \in (0, 1), D > 1, n \in \mathbb{N}$, and $H \in \mathcal{Q}_n(\delta, D)$ be arbitrary. Moreover, let G be a bipartite graph with bipartition $\{A, B\}$, $|A| = |B| = \lfloor (1 - 3\delta)n \rfloor$ satisfying property $P(H, \delta, s)$ for $s = \lceil Dn \log n \rceil$. Then its complement graph G^c does not contain $H[U]$ as a minor for any $U \subseteq V(H)$ with $|U| \geq (1 - \delta)n$.

Proof. Assume G^c contains $H[U]$ as a minor for some $U \subseteq V(H)$ with $|U| \geq (1 - \delta)n$. Let $(Z_h)_{h \in U}$ be an $H[U]$ -minor model in G^c and define $X_A := \{h \in U \mid Z_h \subseteq A\}$, $X_B := \{h \in U \mid Z_h \subseteq B\}$, and $X_{AB} := \{h \in U \mid Z_h \cap A \neq \emptyset \neq Z_h \cap B\}$. We have $|X_A| + |X_{AB}| \leq |A|, |X_B| + |X_{AB}| \leq$

$|B|$, and $|X_A| + |X_B| + |X_{AB}| = |U| \geq (1 - \delta)n$, which implies $|X_A|, |X_B| \geq (1 - \delta)n - (1 - 3\delta)n = 2\delta n$.

Since the branch sets $(Z_h)_{h \in X_A}$ in A and the branch-sets $(Z_h)_{h \in X_B}$ in B are pairwise disjoint, at most $\delta(1 - 3\delta)n < \delta n$ branch sets in each of $(Z_h)_{h \in X_A}$ and $(Z_h)_{h \in X_B}$ can be larger than $\frac{1}{\delta}$. Thus, there are at least $2\delta n - \delta n = \delta n$ branch sets of size at most $\frac{1}{\delta}$ in $(Z_h)_{h \in X_A}$ as well as in $(Z_h)_{h \in X_B}$. Thus for $k := l := \lceil \delta n \rceil$, there exist distinct vertices $x_1, \dots, x_k \in X_A, y_1, \dots, y_l \in X_B$ such that $|Z_{x_i}|, |Z_{y_j}| \leq \frac{1}{\delta}$ for all $1 \leq i, j \leq k = l$. Since $H \in \mathcal{Q}_n(\delta, D)$, we have $e_H(\{x_1, \dots, x_k\}, \{y_1, \dots, y_l\}) \geq \lceil Dn \log n \rceil = s$. Next we use our assumption that G satisfies property $P(H, \delta, s)$. It implies that there exists an edge $x_i y_j \in E(H)$ with $(i, j) \in [k] \times [l]$ such that G contains all the edges xy with $(x, y) \in Z_{x_i} \times Z_{y_j}$. Then, however, there is an edge between vertices x_i and y_j in H , but no edge between the corresponding branch sets Z_{x_i} and Z_{y_j} in G^G , a contradiction. \square

The next auxiliary statement we need is Lemma 19 below, which establishes a weak analogue of Lemma 13 for graphs $H \in \mathcal{Q}_n(\delta, D)$. Note that as these graphs may have sublinear minimum degree and connectivity, Lemma 13 cannot be used to obtain the same statement.

Lemma 19. *Let $\delta > 0, D > 1, H \in \mathcal{Q}_n(\delta, D)$ and let F be a graph with a clique $W \subseteq V(F)$ of size $\lfloor (1 - 3\delta)n \rfloor$. Let $K \in \mathbb{N}$ and let $F^{(K)}$ be a K -fold pasting of F at W . If $F^{(K)}$ contains H as a minor, then there exists $U \subseteq V(H)$ with $|U| \geq (1 - \delta)n$ such that F contains $H[U]$ as a minor.*

Proof. In the following, let F_1, \dots, F_K denote the copies of F such that $F^{(K)} = \bigcup_{i=1}^K F_i$.

Suppose $F^{(K)}$ has an H -minor and fix an H -minor model $(Z_h)_{h \in V(H)}$ in H . Let us denote $X_W := \{h \in V(H) \mid Z_h \cap W \neq \emptyset\}$ and $\xi_W := |X_W|$, and $X_i := \{h \in V(H) \mid Z_h \subseteq V(F_i) \setminus W\}$ and $\xi_i := |X_i|$ for every $i \in [K]$. Note that since every branch-set Z_h induces a connected subgraph of F , every vertex $h \in V(H)$ appears in exactly one of the sets X_W, X_1, \dots, X_K , i.e., they form a partition of $V(H)$. In particular, we have $\xi_W + \sum_{i=1}^K \xi_i = v(H) = n$.

We have $\xi_W \leq |W| \leq n - 3\delta n$ and thus $\sum_{i=1}^K \xi_i = n - \xi_W \geq 3\delta n$. In the following, let us w.l.o.g. assume $[K]$ is ordered such that $\xi_1 \geq \xi_2 \geq \dots \geq \xi_K$. We claim that $\xi_1 \geq (1 - \delta)n - \xi_W$. Towards a contradiction, suppose in the following that $\xi_1 < (1 - \delta)n - \xi_W$. We first note that using this assumption, we have that $\sum_{i=2}^K \xi_i = n - (\xi_W + \xi_1) > n - (1 - \delta)n = \delta n$.

Now suppose for a first case that $\xi_1 \geq \delta n$. Then the two disjoint sets of vertices X_1 and $\bigcup_{i=2}^K X_i$ in H are both of size at least δn . By property $Q(\delta, D)$ this implies that $e_H(X_1, \bigcup_{i=2}^K X_i) \geq Dn \log n > 0$. In particular there exists $2 \leq i \leq K$ and an edge $uv \in E(H)$ for some $u \in X_1$ and $v \in X_i$. This implies that there must exist an edge in $F^{(K)}$ connecting a vertex in $Z_u \subseteq V(F_1) \setminus W$ to a vertex in $Z_v \subseteq V(F_i) \setminus W$. However, by construction of $F^{(K)}$ no such edges exist, and so we arrive at the desired contradiction in this first case.

For the second case, suppose that $\xi_1 < \delta n$ (and thus in particular $\xi_i < \delta n$ for all $i \in [K]$). Let $j \in [K]$ be the smallest index such that $\sum_{i=1}^j \xi_i > \delta n$ (this is well-defined, since $\sum_{i=1}^K \xi_i \geq 3\delta n$, see above). By the minimality of j , we have $\sum_{i=1}^j \xi_i = \xi_j + \sum_{i=1}^{j-1} \xi_i \leq \delta n + \delta n = 2\delta n$. This implies that $\sum_{i=j+1}^K \xi_i = \sum_{i=1}^K \xi_i - \sum_{i=1}^j \xi_i \geq 3\delta n - 2\delta n = \delta n$. In consequence, we find that the two disjoint vertex sets $\bigcup_{i=1}^j X_i, \bigcup_{i=j+1}^K X_i$ in H are both of size at least δn . Hence, using property $Q(\delta, D)$ we have $e_H(\bigcup_{i=1}^j X_i, \bigcup_{i=j+1}^K X_i) \geq Dn \log n > 0$. Similar as above, this implies the existence of two indices i, i' with $1 \leq i \leq j < i' \leq K$ such that there exists an edge between $V(F_i) \setminus W$ and $V(F_{i'}) \setminus W$ in $F^{(K)}$. As this is impossible by construction of $F^{(K)}$, a contradiction follows also in the second case. Thus our initial assumption $\xi_1 < (1 - \delta)n - \xi_W$ was false.

We therefore have $|X_1 \cup X_W| = \xi_1 + \xi_W \geq (1 - \delta)n$. Let $U := X_1 \cup X_W$. For every $h \in U$, let $Z'_h := Z_h$ if $h \in X_1$ and $Z'_h := Z_h \cap V(F_1)$ if $h \in X_W$. We now show that $(Z'_h)_{h \in U}$ is an $H[U]$ -minor model in F_1 , which will then conclude the proof of the lemma.

First of all, note that $F_1[Z_h']$ is a connected graph for every $h \in U$. If $h \in X_1$, then $F_1[Z_h'] = F^{(K)}[Z_h]$ is connected since $(Z_h)_{h \in V(H)}$ is an H -minor model. And if $h \in X_W$, then the connectivity of $F_1[Z_h'] = F^{(k)}[Z_h \cap V(F_1)]$ follows since (1) $F^{(K)}[Z_h]$ is connected and (2) every path connecting two vertices in Z_h' that is contained in $F^{(K)}[Z_h]$ can be shortened to a path whose vertex set is completely contained in $V(F_1)$ by short-cutting every segment of the path that starts and ends in the clique W by the direct connection between its endpoints.

Let us now consider any edge $uv \in E(H[U])$. Then there must exist an edge $xy \in E(F^{(K)})$ with $x \in Z_u, y \in Z_v$. If we have $x, y \in V(F_1)$, then this witnesses the existence of an edge between Z_u' and Z_v' in F_1 , as desired. If on the other hand at least one of x, y lies outside of $V(F_1)$, then we necessarily must have $Z_u \cap W \neq \emptyset \neq Z_v \cap W$, and thus there exists an edge in the clique induced by W (and thus also in F_1) that connects a vertex in Z_u' to a vertex in Z_v' . All in all, this shows that F_1 contains $H[U]$ as a minor. Since $|U| \geq (1 - \delta)n$, this concludes the proof. \square

With the previous auxiliary results at hand, we can now deduce Theorem 7.

Proof of Theorem 7. Let a constant $\varepsilon \in (0, 1)$ be given. Let $\delta > 0$ be chosen small enough such that $7\delta < \varepsilon$, set $p := \frac{\delta}{2}$, let $D = D(\delta, p) > 1$ be the constant given by Lemma 11, and let $C := \frac{D^2}{\delta^2}$.

For every $n \in \mathbb{N}$, put $s = s(n) = \lceil Dn \log n \rceil$. By Lemma 17, a random graph $H = G(n; \lceil Cn \log n \rceil)$ chosen uniformly from all n -vertex graphs with $\lceil Cn \log n \rceil$ edges satisfies property $Q(\delta, D)$ w.h.p. as $n \rightarrow \infty$. Now assume the graph H satisfies property $Q(\delta, D)$. By Lemmas 9 and 11, w.h.p. as $n \rightarrow \infty$, the random bipartite graph $G = G(\lfloor (1 - 3\delta)n \rfloor, \lfloor (1 - 3\delta)n \rfloor; p)$ has maximum degree at most $2p\lfloor (1 - 3\delta)n \rfloor \leq \delta n$ and satisfies property $P(H, \delta, s)$. Now fix n large enough and consider a graph G with bipartition $\{A, B\}$, $|A| = |B| = \lfloor (1 - 3\delta)n \rfloor$ satisfying these two properties. By Lemma 18, G^{\square} does not contain any induced subgraph $H[U]$ as a minor for any $U \subseteq V(H)$ with $|U| \geq (1 - \delta)n$. Let $K := (|A| + |B| - 1)^{|A|}$ and let $(G^{\square})^{(K)}$ be a K -fold pasting of G^{\square} at A . Then by Lemma 19, $(G^{\square})^{(K)}$ does not contain H as a minor. Moreover, by Lemma 15, applied with $d = \lfloor \delta n \rfloor$, we find that $(G^{\square})^{(K)}$ has list chromatic number at least $|A| + |B| - d > 2(1 - 3\delta)n - \delta n - 2 > (2 - \varepsilon)n$ for n large enough. This shows that w.h.p. the random graph $H = G(n; \lceil Cn \log n \rceil)$ satisfies $f_{\ell}(H) \geq (2 - \varepsilon)n$, which concludes the proof. \square

5. Proof of Theorem 3

In this section we give the proof of Theorem 3, which is self-contained and independent of the results in the previous sections. A basic tool from extremal graph theory used in the proof is Turán’s theorem, in the following form:

Theorem 20 (Turán). Let $k \in \mathbb{N}$, $k \geq 2$ and let G be a graph. If $e(G) > (1 - \frac{1}{k-1})\frac{v(G)^2}{2}$ then G contains a clique on k vertices.

We also use the following classical result regarding the minimum degree of K_t -minor-free graphs, as independently proved by Kostochka [13] and Thomason [32].

Theorem 21 ([13, 32]). For every integer $t \geq 1$ there exists an integer $d = d(t) = O(t\sqrt{\log t})$ such that every graph of minimum degree at least d contains K_t as a minor. In particular, for every graph F there exists $d = d(F) \in \mathbb{N}$ such that all graphs of minimum degree at least d contain F as a minor.

Proof of Theorem 3. We start by fixing an integer $d \in \mathbb{N}$ as guaranteed by Theorem 21, i.e. such that every graph of minimum degree at least d contains F as a minor. We now define $k_0(F) := \min\{d + 1, 9 \cdot v(F)^3\}$. Let $k \geq k_0(F)$ be any given integer. Let H denote the graph obtained from F by adding k isolated vertices. We will now show that every H -minor-free graph is $(v(H) - 2)$ -degenerate, which then easily implies $f_{\ell}(H) = v(H) - 1$.

Towards a contradiction, suppose that there exists an H -minor-free graph G which is not $(v(H) - 2)$ -degenerate, and let G be chosen such that $v(G)$ is minimised. Note that the minimality assumption on G immediately implies that $\delta(G) \geq v(H) - 1 = v(F) + k - 1$. Observe that since $\delta(G) \geq k > d$, the graph $G - x$ for some $x \in V(G)$ has minimum degree at least d and thus must contain F as a minor. Let $X \subseteq V(G)$ be chosen of minimum size subject to $G[X]$ containing F as a minor. Note that from the above it follows that $|X| \leq v(G) - 1$ and hence that $V(G) \setminus X \neq \emptyset$. Let $(Z_f)_{f \in V(F)}$ be an F -minor model in $G[X]$. By minimality of X , we have that $(Z_f)_{f \in V(F)}$ forms a partition of X . With the goal of bounding the number of edges in $G - X$, we present our next argument as a separate claim. We will later use this bound and Turán’s theorem to show the existence of an F -subgraph in $G - X$.

Claim 22. For every $v \in V(G) \setminus X$ and every $f \in V(F)$, we have $|N(v) \cap Z_f| < 9v(F)$.

Proof. Let $v \in V(G) \setminus X$ and $f \in V(F)$ be arbitrary. For $|Z_f| = 1$ the inequality $|N(v) \cap Z_f| \leq 1 < 9v(F)$ trivially holds for every $v \in V(G) \setminus X$. We may therefore assume $|Z_f| \geq 2$. Let T_f denote a spanning tree of the connected graph $G[Z_f]$, and let $L_f \subseteq Z_f$ be the set of leaves in T_f .

We first show that T_f has at most $v(F) - 1$ leaves. Note that for every $l \in L_f$ the graph $G[Z_f \setminus \{l\}]$ is still connected. However, by minimality of X , $G[X \setminus \{l\}]$ does not contain F as a minor, and thus in particular the set system consisting of $Z_f \setminus \{l\}$ together with the remaining branch-sets $(Z_{f'})_{f' \in V(F), f' \neq f}$ cannot be an F -minor model in G . In consequence, there has to exist some $f' \in V(F) \setminus \{f\}$ such that among all vertices in Z_f , the vertex l is the only one that has a neighbour in $Z_{f'}$. Since the above argument applies to any choice of $l \in L_f$, and since the respective elements f' have to be distinct for different choices of l , it follows that $|L_f| \leq |V(F) \setminus \{f\}| = v(F) - 1$.

We next describe a decomposition of T_f into strictly less than $2v(F)$ edge-disjoint and internal-vertex-disjoint paths. Let $T_{f'}$ be a tree without degree 2-vertices such that T_f is a subdivision of $T_{f'}$, i.e., every edge in $T_{f'}$ corresponds to one maximal path of T_f all whose internal vertices are of degree 2. Then, since $T_{f'}$ is a tree and thus has average degree strictly less than 2, it has more leaves than vertices of degree 3 or more. As the number of leaves in $T_{f'}$ is exactly $|L_f|$, we have $v(T_{f'}) \leq |L_f| + (|L_f| - 1) \leq 2v(F) - 3$ and therefore $e(T_{f'}) = v(T_{f'}) - 1 \leq 2v(F) - 4 < 2v(F)$. This means that T_f can be expressed as the edge-disjoint union of a collection of paths $(P_i)_{i=1}^r$ where $r < 2v(F)$ and the internal vertices of each path P_i are of degree 2 in T_f .

Now choose a vertex set $Y \subseteq Z_f$ of size at most $v(F) - 1$ as follows: For each edge $ff' \in E(F)$, pick some vertex $y_{f'} \in Z_{f'}$ that has at least one neighbour in $Z_{f'}$ and add it to Y . Let \mathcal{R} denote the collection of internally disjoint paths in T_f obtained from $(P_i)_{i=1}^r$ by splitting each path P_i into its maximal subpaths that do not contain internal vertices in Y . It is easy to see that $|\mathcal{R}| \leq r + |Y| < 2v(F) + v(F) = 3v(F)$, and that T_f equals the union of the paths in \mathcal{R} .

We next claim that for every vertex $v \in V(G) \setminus X$ and every $R \in \mathcal{R}$, we have $|N(v) \cap V(R)| \leq 3$. Indeed, suppose that v has at least 4 distinct neighbours on R . Let x and y be the two neighbours of v on R that are closest to the endpoints of R . Define R' as the path obtained from R by replacing its subpath between x and y (which has to contain at least two internal vertices) by the path $x - v - y$ of length two. Let A be the set of vertices on R strictly between x and y and observe that $|A| \geq 2$ and $A \cap Y = \emptyset$. For $X' := (X \setminus A) \cup \{v\}$ we have $|X'| < |X|$ and we can find an F -minor model in $G[X']$, namely the branch-sets $(Z_f \setminus A) \cup \{x\}$ together with $(Z_{f'})_{f' \in V(F), f' \neq f}$. Notice that $G[(Z_f \setminus A) \cup \{x\}]$ is indeed connected as all the internal vertices of R are of degree 2 in T_f . Also, since $Y \subseteq Z_f \setminus A$, there still exists a connection from a vertex in $(Z_f \setminus A) \cup \{x\}$ (namely, $y_{f'}$) to a vertex in $Z_{f'}$ for every edge $ff' \in E(F)$. This contradicts our initial choice of X and proves that our assumption was wrong, so indeed every vertex $v \in V(G) \setminus X$ satisfies $|N(v) \cap V(R)| \leq 3$ for every $R \in \mathcal{R}$.

Therefore, we have $|N(v) \cap Z_f| \leq \sum_{R \in \mathcal{R}} |N(v) \cap V(R)| \leq 3|\mathcal{R}| < 9v(F)$ for every $v \in V(G) \setminus X$, which concludes the proof of the claim. \square

It follows immediately from Claim 22 that $|N(v) \cap X| \leq \sum_{f \in V(F)} |N(v) \cap Z_f| < 9v(F)^2$ for every $v \in V(G) \setminus X$. Additionally recalling that $\delta(G) \geq v(F) + k - 1 \geq k$, we find that for every $v \in V(G) \setminus X$, we have $\deg_{G-X}(v) = |N(v) \setminus X| = \deg(v) - |N(v) \cap X| > k - 9v(F)^2$. Having established $V(G) \setminus X \neq \emptyset$ at the beginning of the proof, it now follows that $G - X$ is a graph of minimum degree greater than $k - 9v(F)^2$. Also note that since $G[X]$ contains F as a minor, we are not able to find k distinct vertices in $V(G) \setminus X$ as these could be used to augment the F -minor in $G[X]$ to an H -minor in G , contradicting our assumptions. We thus have $v(G - X) < k$. Using our choice of k_0 and $k \geq k_0$, it now follows that

$$\delta(G - X) > k - 9v(F)^2 > \left(1 - \frac{1}{v(F) - 1}\right) k > \left(1 - \frac{1}{v(F) - 1}\right) v(G - X).$$

Therefore, $G - X$ has more than $\left(1 - \frac{1}{v(F) - 1}\right) \frac{v(G - X)^2}{2}$ edges and thus Theorem 20 implies the existence of a clique on $v(F)$ vertices in $G - X$. In particular, $G - X$ and thus G contain a subgraph isomorphic to F . Let $K \subseteq V(G)$ be the vertex set of such a copy of F . Then, since $v(G) \geq \delta(G) + 1 \geq v(F) + k$, there are at least k vertices outside of K in G , which can be added to the copy of F on vertex set K to create a subgraph of G that is isomorphic to H . In particular, this means that G contains H as a minor, a contradiction. All in all, we find that our initial assumption, namely regarding the existence of a smallest counterexample G to our claim, was wrong. This concludes the proof that all H -minor-free graphs are $(v(H) - 2)$ -degenerate.

It is a well-known fact and easy to prove by induction that for every $a \in \mathbb{N}$ all a -degenerate graphs are $(a + 1)$ -choosable. Thus what we have proved also implies that every H -minor-free graph is $(v(H) - 1)$ -choosable, as desired. All in all, it follows that $f_\ell(H) = v(H) - 1$, concluding the proof of the theorem. \square

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