A fresh look at the complexity of pivoting in linear complementarity

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A Fresh Look at the Complexity of Pivoting in Linear Complementarity

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Abstract

The linear complementarity problem (LCP) has a broad range of applications in various real-life optimization problems. It has its origin as a framework unifying linear, convex quadratic programs, and bimatrix game problems. Nowadays, many other problems arising in economic theory, geometry, and game theory are known to have formulations as LCPs. All the mentioned problems are of particular importance in mathematical optimization, and many researchers are seeking efficient solving methods.

Deciding whether a general LCP has a solution is NP-complete. Hence, research focuses on LCPs with special characteristics. Our main interest is in LCPs with P-matrices, for which many theoretical results suggest the existence of a polynomial-time algorithm. For instance, Megiddo proved that if the P-matrix LCP is NP-hard, then NP equals coNP. However, despite many promising results, no efficient algorithm is known. We consider simple principal pivoting methods. These are methods that are identified by a pivot rule and make decisions based on the combinatorial structure of the underlying problem instance. It makes perfect sense to employ abstract combinatorial models to study the LCP and principal pivoting methods. We make use of combinatorial settings such as oriented matroids and unique-sink orientations (USOs). This thesis is accordingly split into two parts.

In part I, we combinatorially generalize the LCP by formulating the oriented matroid complementarity problem (OMCP), whose definition is due to Todd. We then investigate P-matrices and their subclasses, such as (hidden) K-matrices, and define the combinatorial counterparts of oriented matroids, e.g., P-matroids.

We present alternative characterizations of K-matroids. This extends a theorem formulated by Fiedler and Pták on algebraic characterizations of K-matrices to the setting of oriented matroids. Our proof is elementary and simplifies the original proof substantially by exploiting oriented matroid duality. As an application, we show that any simple principal pivoting method applied to K-matroid OMCPs terminates very quickly, by a purely combinatorial argument.

Furthermore, we define the class of cyclic-P-matroids. The correspond-
ing algebraic counterparts, called cyclic-P-matrices, have many interesting properties. For instance, they are closed under principal pivot transforms and principal submatrices. The definition is such that it provides a construction scheme for P-matrices of arbitrary order. No other scheme to construct nontrivial P-matrices is currently known.

Part II discusses results related to USOs, which model the behavior of principal pivoting methods. We study several subclasses of P-matroid OMCPs, and investigate peculiarities that distinguish the corresponding USOs.

We reduce the oriented matroid programming (OMP) over combinatorial cubes to the hidden K-matroid OMCP. Through this result, the hidden K-matroid OMCP is at least as hard with respect to principal pivoting as the OMP over combinatorial cubes. This translates to a stronger variant in the algebraic setting. Namely, LP USOs and hidden K-matrix USOs are the same.

In a joint project with Foniok, Gärtner, and Sprecher, we determined the number of USOs arising from different classes of P-matrix LCPs. By the obtained bounds on the sizes of USO classes, we get a rough idea of the algorithmic complexity of principal pivoting methods. The intuition is that small classes are more likely to allow a polynomial-time pivoting strategy. We also compare the obtained class sizes with the sizes of USO classes that have purely combinatorial definitions.

In a joint project with our external collaborator Miyata, we enumerated the P-matroid USOs of the 4-cube by using oriented matroid generation techniques proposed by Finschi and Fukuda. Through the insight gained, we conclude that for an oriented matroid, being a P-matroid is more a matter of correctly labeled and oriented elements than a property of the underlying combinatorial structure.
Zusammenfassung

Das lineare Komplementaritätsproblem (LKP) hat eine grosse Brandbreite an Anwendungen in verschiedenen alltäglichen Optimierungsproblemen. Ursprünglich war es ein Werkzeug um lineare bzw. konvexe quadratische Programme und Bimatrix-Spiele in einer einheitlichen Form zu formulieren. Heutzutage sind viele andere Probleme aus den Wirtschaftswissenschaften und der Spieltheorie bekannt, die als LKPs formuliert werden können. Diese Probleme sind von bedeutender Wichtigkeit in der mathematischen Optimierung, und viele Wissenschaftler sind auf der Suche nach effizienten Lösungsmethoden.


lich kombinatorische Argumente.


In Teil II werden Resultate in Bezug auf USOs besprochen. USOs modellieren das Verhalten von Pivot Methoden auf LKPs mit P-Matroiden. Wir studieren die verschiedenen Klassen an LKPs, und untersuchen die dazugehörenden USOs auf ihre Struktur.


In einem gemeinsamen Projekt mit Foniok, Gärtner und Sprecher haben wir die Anzahl an USOs bestimmt die durch die verschiedenen Unterklassen von LKPs mit P-Matrizen in Erscheinung treten. Dadurch ist es uns möglich abzuschätzen welche Klassen in polynomieller Zeit lösbar sind. Intuitiv sind kleinere Klassen stärker strukturiert. Desweiteren werden die erhaltenen Klassengrössen mit Grössen von USO Klassen verglichen, welche eine rein kombinatorische Definition besitzen.

# Contents

1 Introduction 1
   1.1 Motivation ............................................... 1
   1.2 Goals .................................................. 2
   1.3 Methodology and overview of results ...................... 3
      1.3.1 Part I: Oriented matroids ......................... 3
      1.3.2 Part II: Unique-sink orientations ................. 6

2 P-matrix linear complementarity problem 11
   2.1 Matrix classes and complexity issues ................. 13

3 Basics of oriented matroids 19
   3.1 Introduction ........................................... 19
   3.2 A first example: point configurations ................. 20
   3.3 Circuits ................................................. 29
   3.4 Bases and chirotopes .................................. 34
   3.5 Minors .................................................. 38
   3.6 Reorientations .......................................... 42
   3.7 Duality ............................................... 43
   3.8 Extensions .............................................. 48
I Oriented Matroid Complementarity Problem 53

Motivation 55

4 From LCPs to OMCPs 57

5 Classes of oriented matroids 61
  5.1 P-matroids 61
  5.2 Z-matroids 70
  5.3 K-matroids 72
  5.4 Hidden Z-matroids 76
  5.5 Hidden K-matroids 81
    5.5.1 Alternative hidden K-matroid characterization 81
  5.6 Cyclic-P-matroids 85
    5.6.1 Alternating matroids 87
    5.6.2 Definition and properties of cyclic-P-matroids 91

6 P-matroid OMCPs and simple principal pivoting methods 99
  6.1 K-matroid OMCPs 103
II Unique-Sink Orientations 107

Motivation 109

7 Cubes and unique-sink orientations 113
  7.1 From P-matroid OMCPs to USOs 115

8 Oriented matroid classes and USO classes 117
  8.1 P-matroid USOs 117
  8.2 K-matroid USOs 121
  8.3 Hidden K-matroid OMCP is at least as hard as OMP over cubes 122
  8.4 Cyclic-P-matroid USOs 126

9 Simple principal pivoting for USOs 131
  9.1 Pivot rules and complexity issues 132

10 Counting USOs 139
  10.1 An upper bound for P-matrix USOs 140
  10.2 A lower bound for K-matrix USOs 142
  10.3 The number of USOs arising from a fixed P-matrix 147
  10.4 A lower bound for strongly Holt-Klee and locally uniform USOs 149
  10.5 Conclusions 151

11 Enumeration of P-matroid USOs of the 4-cube 155
Chapter 1

Introduction

1.1 Motivation

For a given matrix $M \in \mathbb{R}^{n \times n}$ and vector $q \in \mathbb{R}^n$, the linear complementarity problem (LCP) is to find vectors $w, z \in \mathbb{R}^n$ so that

$$w - Mz = q, \quad w, z \geq 0, \quad w^Tz = 0.$$  \hspace{1cm} (1.1)

The principal motivation for studying the linear complementarity problem is the broad range of applications in various real-life optimization problems. It has its origin as a framework unifying linear, convex quadratic programs, and bimatrix game problems. Nowadays, many other problems arising in economic theory, geometry, and game theory are known to have formulations as LCPs. All of the mentioned problems are of particular importance in mathematical optimization, and many researchers are seeking efficient solving methods.

In the beginning, Dantzig’s simplex method [16], a type of pivoting algorithm, was the only practical solving method for linear programs (LPs). Up until the early seventies, people had no idea about the algorithmic complexity of Dantzig’s simplex method. Many believed it to be a polynomial-time method, as it runs fast on instances arising from applications in practice. In 1972, Klee and Minty [45] constructed an artificial family of LPs
on which Dantzig’s method is inefficient. It is still an open question as to whether there exists a “smarter” pivot rule that terminates in a polynomial number of pivot steps, and thus also in polynomial time.

These days, Khachiyan’s ellipsoid method [42] and Karmarkar’s interior-point method [41], both originating from nonlinear optimization, are available to solve linear and convex quadratic programs in polynomial time. These methods are polynomial with respect to the input length in bits, where the ellipsoid method is too slow to be of any practical interest. It is an open problem whether there exist strongly polynomial methods. These are methods where the number of arithmetic operations is polynomially bounded by the number of variables and constraints. There is therefore strong interest in the development and analysis of pivoting algorithms.

Linear and convex quadratic problems reduce to LCPs with positive semidefinite (psd) matrices. The reductions are accomplished by writing the complementary slackness conditions and Karush-Kuhn-Tucker conditions respectively in the form of LCPs. The reader is referred to [66] for details on these reductions. Every psd matrix is a so-called sufficient matrix. LCPs with sufficient matrices can have none, exactly one, or infinitely many solutions. Pivoting algorithms exist for LCPs with sufficient matrices. For instance, the criss-cross method [33] is finite and terminates with a solution or a certificate that there is no solution.

1.2 Goals

The ultimate goal is either to find a strongly polynomial algorithm for linear and convex quadratic programs and, more generally, for LCPs with sufficient matrices, or to prove that no such algorithm exists.

This thesis makes new contributions toward this ambitious goal. We are concerned with classes of LCPs that are identified by properties of the associated matrix. We mainly consider P-matrices, a proper subclass of sufficient matrices, and several subclasses of P-matrices such as (hidden) K-matrices. A P-matrix is a square matrix whose principal minors\(^1\) are all positive.

\(^1\)The principal minors of a square matrix are the determinants of the principal submatrices.
Although no efficient algorithm solving LCPs with P-matrices is known, their existence is suggested by several theoretical results. In this thesis, we focus on (simple) principal pivoting methods, which make decisions based on combinatorial properties of the underlying problem instance. We aim to improve the understanding of the combinatorial peculiarities that distinguish LCPs with P-, (hidden) K-matrices, and other types of matrices. Knowledge about such peculiarities helps in the analysis and evaluation of pivoting strategies. It may lead to smarter, more advanced pivot rules, or even to the discovery of classes of LCPs that are identified by a special characteristic that can be exploited by some pivot rule. A lot has been done in the area of deterministic rules, which might not have enough potential. In contrast, randomized rules are promising and encourage further study. At present, however, little is known about their behavior and algorithmic complexity.

1.3 Methodology and overview of results

In our studies, we employ abstract combinatorial settings such as oriented matroids and unique-sink orientations (USOs). These two combinatorial concepts are established tools and used in many other areas of mathematics as well. The thesis is accordingly split into two parts. In Part I, we present the results related to the generalization of matrix classes and the LCP in the setting of oriented matroids. Part II discusses results related to USOs, the combinatorial abstraction of principal pivoting methods.

1.3.1 Part I: Oriented matroids

The concept of oriented matroids was introduced by Bland and Las Vergnas [7] and Lawrence and Folkman [24]. Oriented matroids generalize combinatorial properties of signed linear dependencies in an abstract setting. An introduction to oriented matroids is given in Chapter 3, which can be skipped if the reader is already familiar with the basic theory.

We build on Todd’s [80] approach in order to generalize LCPs. In the following, we explain the basic idea of the generalization, and the reader is referred to Chapter 4 for details.
Note that for an LCP\((M, q)\), every solution \((w^*, z^*)\) is such that \(w^*_i = 0\) or \(z^*_i = 0\) for each \(i \in [n] := \{1, 2, \ldots, n\}\). This property directly follows from the nonnegativity and complementarity constraints in (1.1).

Consider the set
\[
\hat{\mathcal{V}} := \left\{ \text{sign } x : [I_n \quad -M \quad -q] x = 0 \right\},
\] (1.2)
where \(\text{sign } x := (\text{sign } x_1, \ldots, \text{sign } x_n)\). The collection \(\hat{\mathcal{V}}\) is the set of vectors of an oriented matroid.

The oriented matroid complementary problem (OMCP) is to find a sign vector \(X \in \hat{\mathcal{V}}\) so that
\[
X \geq 0, \quad X_{2n+1} = +, \quad X_i = 0 \text{ or } X_{i+n} = 0 \text{ for every } i \in [n].
\]

Once a solution \(X\) is found, a solution \((w^*, z^*)\) to the corresponding LCP is easily obtained. We have \(w^*_i = 0\) whenever \(X_i = 0\) and \(z^*_i = 0\) otherwise.

Every LCP induces an OMCP in this way. On the other hand, the setting of oriented matroids is generalizing. There are oriented matroids that cannot be represented by any matrix \(M\) and vector \(q\) as in (1.2). Consequently, every result obtained in the setting of oriented matroid has a natural translation into the algebraic context, while for the opposite direction, such a translation is sometimes not possible.

Todd [80], Todd and Morris [63], and Fukuda and Terlaky [33] studied oriented matroids given by sets of sign vectors
\[
\mathcal{V} := \left\{ \text{sign } x : [I_n \quad -M] x = 0 \right\},
\] (1.3)
where \(M\) is a P-matrix, symmetric positive definite, and sufficient matrix, respectively. Considering Todd’s result, the class of oriented matroids generalizing P-matrices consists of the oriented matroids on \(2n\) elements, where for every sign vector \(X \in \mathcal{V}\), we have \(X_i = X_{i+n} = +\) or \(X_i = X_{i+n} = -\) for some \(i \in [n]\). An oriented matroid satisfying this property is by definition a P-matroid.

In this thesis we elaborate on the following subjects.
Generalization of matrix classes in the setting of oriented matroids. Similarly to the above mentioned classes, many other matrix classes have combinatorial counterparts in the setting of oriented matroids. In Chapter 5, we consider P-matrices with special characteristics, such as (hidden) K-matrices, and find equivalent classes of oriented matroids. Our approach is first to study algebraic properties of the matrices in question, and then to extract the combinatorial peculiarities.

In Section 5.1, we discuss alternative characterizations for P-matroids. Theorem 5.4 extends a result of Todd by another characterization. Todd’s original proof is quite involved, even for people who are accustomed to oriented matroids. Our proof is much shorter and benefits from oriented matroid duality.

In Section 5.3, we derive a result for the subclass of K-matroids by generalizing algebraic characterizations of K-matrices, which were given by Fiedler and Pták [21]. The combinatorial counterpart for K-matroids is summarized in Theorem 5.16. As a byproduct, our proof yields a simple and purely combinatorial proof of the original Fiedler and Pták Theorem. Rather than on algebraic properties, it heavily relies on duality.

Finally, in Section 5.5, we generalize hidden K-matrices. For all of these classes of oriented matroids, we also present properties that are important and crucial for the study of solving methods for the OMCP.

Solving methods for the OMCP. Many solving methods for the LCP, such as pivoting algorithms, rely on combinatorial, rather than algebraic properties of the underlying problem instance. We focus on simple principal pivoting methods, which are identified by pivot rules. In Chapter 6, we translate these into solving methods for OMCPs, and then investigate the properties satisfied by P-matroid OMCPs.

In Section 6.1, we generalize a result for the K-matrix LCP [25]. We prove that the K-matroid OMCP is solved by simple principal pivoting methods in a linear number of pivot steps, regardless of which pivot rule is applied. In doing so, we also provide an alternative, purely combinatorial proof for the original algebraic result.

Definition and study of cyclic-P-matroids. We define a new subclass of P-matroids by considering $2n$ points on the moment curve in $\mathbb{R}^{n-1}$,
lifted into \( \mathbb{R}^n \). For \( n = 3 \), the vector configuration in \( \mathbb{R}^{n \times 2n} \) of the form

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\
t_1^2 & t_2^2 & t_3^2 & t_4^2 & t_5^2 & t_6^2
\end{pmatrix}
\]

yields P-matroids by appropriately picking values for the \( t_i \)'s and negating some columns. For instance, if we let \( t := (1, 3, 5, 2, 4, 6) \) and negate the first \( n \) columns, then we get

\[
V := \begin{pmatrix}
-1 & -1 & -1 & 1 & 1 & 1 \\
-1 & -3 & -5 & 2 & 4 & 6 \\
-1 & -9 & -25 & 4 & 16 & 36
\end{pmatrix}.
\]

The nullspace of \( V \) defines a P-matroid, in the way of (1.3). For the purpose of illustration, we compute

\[
(V_{[n][n]})^{-1}V = \begin{pmatrix}
1 & 0 & 0 & -3/8 & 1/8 & -3/8 \\
0 & 1 & 0 & -3/4 & -3/4 & 5/4 \\
0 & 0 & 1 & 1/8 & -3/8 & -15/8
\end{pmatrix}, \quad (1.4)
\]

where the last \( n \) columns build the negative of an \( n \times n \) P-matrix.

The class of so-called cyclic-P-matroids is introduced and discussed in Section 5.6. The corresponding realizations are called cyclic-P-matrices accordingly, and have many interesting properties. For instance, they are closed under principal pivot transforms and principal submatrices. Furthermore, the definition is such that it provides a construction scheme for P-matrices of any order. No other such scheme is currently known. Moreover, we are able to give a combinatorial description of extensions by right-hand sides \( q \). This may become useful in the analysis of solving methods.

### 1.3.2 Part II: Unique-sink orientations

Pivoting algorithms for LPs move along edges of the feasible region such that the objective value does not decrease until an optimum is reached. Similarly, Stickney and Watson [76] modeled the behavior of principal
pivoting methods for the P-matrix LCP on digraphs where the underlying undirected graph is a combinatorial cube. Principal pivoting methods are sometimes called Bard-type methods in the literature, and were first studied by Zoutendijk [89] and Bard [4].

Next, we give a short introduction to principal pivoting, and discuss the abstraction of unique-sink orientations (USOs).

Consider the LCP($M, q$) for a matrix $M \in \mathbb{R}^{n \times n}$ and vector $q \in \mathbb{R}^n$. We assume that $M$ is a P-matrix, i.e., the determinant of every principal submatrix is positive. If so, the LCP has a unique solution for every $q$ [73].

For any $B \subseteq [n]$, let $A_B$ be the $n \times n$ matrix whose $j$th column is the $j$th column of $-M$ if $j \in B$, and the $j$th column of the identity matrix $I_n$ otherwise. Since $M$ is a P-matrix, the submatrix $A_B$ is nonsingular, and we call $B$ a basis.

If $A_B^{-1}q \geq 0$, we have discovered the unique solution. Let

$$w_i := \begin{cases} 0 & \text{if } i \in B \\ (A_B^{-1}q)_i & \text{if } i \notin B \end{cases}, \quad z_i := \begin{cases} (A_B^{-1}q)_i & \text{if } i \in B \\ 0 & \text{if } i \notin B \end{cases}.$$ 

On the other hand, if $A_B^{-1}q \not\geq 0$, then $w$ and $z$ as defined above satisfy $w - Mz = q$ and $w^Tz = 0$, but $w, z \geq 0$ fails.

In principal pivoting, one tries to improve the situation by replacing basis $B$ with the symmetric difference\(^2\) $B \oplus C$, where $C$ is some nonempty subset of the “bad indices” $\{i : (A_B^{-1}q)_i < 0\}$. The greedy choice is to let $C$ be the set of all bad indices. For some P-matrix LCPs, however, such strategy cycles and never terminates [66]. In simple principal pivoting, a pivot rule is employed to select a single “bad index”.

To model the behavior of principal pivoting methods in the setting of USOs, we require nondegeneracy. An LCP($M,q$) is nondegenerate if no $A_B^{-1}q$ for $B \subseteq [n]$ has any zero component.

Through a bijective mapping every basis $B \subseteq [n]$ is assigned to a vertex of the $n$-cube so that it is connected to its adjacent bases $B \oplus \{i\}$ for $i \in [n]$. The edges incident to $B$ are directed according to the components of $A_B^{-1}q$. From the perspective of $B$, edge $\{B, B \oplus \{i\}\}$ is outgoing if $(A_B^{-1}q)_i < 0$, and incoming otherwise. Since $M$ is a P-matrix, the direction corresponds

\(^2\)The symmetric difference $B \oplus C$ is $(B \setminus C) \cup (C \setminus B)$. 
to the direction computed from the viewpoint of $B \oplus \{i\}$. Moreover, in the arising digraph, every face of the cube has a unique local sink. Such an orientation is called *unique-sink orientation (USO)*.

Figure 1.1 depicts the USO arising from the cyclic-P-matrix in (1.4) and right-hand side $q = (1/8, -5/4, -3/8)^T$.

![Figure 1.1: A P-matrix USO.](image)

In the model of USOs, a simple principal pivoting method behaves as follows: it starts in a vertex of free choice and follows a directed path according to some pivot rule until the sink is reached. The sink is the unique vertex with no outgoing edge and corresponds to the solution. The length of the traveled path corresponds to the number of *pivot steps*. The realized USO may contain directed cycles, and termination of the algorithm depends on the applied pivot rule.

Not every USO arises from a P-matrix LCP. In other words, the model is generalizing. We call an orientation *P-matrix USO* if it arises from a P-matrix LCP. Researchers are interested in sufficient and necessary conditions for a USO to be a P-matrix USO. Such conditions are of significant importance in the development and analysis of pivoting strategies.

In this thesis, we elaborate on the following topics.

**Study of USOs that arise from OMCPs with P-matroids.** Principal pivoting algorithms can be generalized to solving methods for the P-matroid OMCP. Likewise, their behavior is modeled by USOs. In Chapter 8, we study subclasses of P-matroid OMCPs, and investigate properties
that distinguish the corresponding USOs. Most of the known algebraic results have equivalent counterparts in the setting of oriented matroids.

In Section 8.4, we give a combinatorial description of the extensions of cyclic-P-matroids by right-hand sides $q$. The description rests on observations by Ziegler [88]. Our overall aim is to give an exact characterization of USOs arising from cyclic-P-matroids, but we do not achieve this objective so far.

The hidden K-matrix LCP and LP over combinatorial cubes are equally difficult with respect to principal pivoting. In Section 8.3, we study properties of hidden K-matrix USOs. Theorem 8.13 states that any USO is a hidden K-matrix USO if and only if it is an orientation arising from an LP over a combinatorial cube. Hence, every hidden K-matrix USO is acyclic, i.e., does not contain directed cycles. It follows that the two problems are equally difficult with respect to principal pivoting. The proof is partly done in the setting of oriented matroids. Theorem 8.12 reduces the oriented matroid programming (OMP) over combinatorial cubes to the hidden K-matroid OMCP. The OMP is a generalization of linear programming in the setting of oriented matroids.

History-based pivot rule for LP USOs. Chapter 9 summarizes results on the algorithmic complexity of different pivot rules in simple principal pivoting. The discussion references many known results and contributes a few new ones. Many deterministic pivot rules were shown to be inefficient for LPs over combinatorial cubes. Randomized rules are promising, but difficult to analyze. We motivate the use of a deterministic history-based pivot rule for LP USOs, and prove that it runs in a quadratic number of pivot steps on some other rules’ worst-case orientations.

Sizes of USO classes related to the P-matrix LCP. In Chapter 10, we determine the number of USOs arising from different classes of P-matrix LCPs. In doing so, we get a rough idea about the algorithmic complexity of principal pivoting methods. The intuition is that small classes are more likely to allow a polynomial-time pivoting strategy. We compare the obtained class sizes with the sizes of USO classes that have purely combinatorial definitions. For instance, we compare the number of LP USOs with the number of acyclic USOs. This helps in estimating the amount
of remaining or left unknown structure contained in LP USOs. Note that it is an open question as to whether the USO classes arising from LCPs allow a combinatorial characterization at all.

The obtained results are summarized in Table 1.1. Previously known results are cited.

<table>
<thead>
<tr>
<th>class</th>
<th>lower bound</th>
<th>upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>K-matrix USOs</td>
<td>(2^{\Omega(n^3)})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>LP USOs [19]</td>
<td>(2^{\Omega(n^3)})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>P-matrix USOs</td>
<td>(2^{\Omega(n^3)})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>strongly Holt-Klee USOs</td>
<td>(2^{\Omega(2^n/\sqrt{n})})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>Holt-Klee USOs [19]</td>
<td>(2^{\Omega(2^n/\sqrt{n})})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>locally uniform USOs</td>
<td>(2^{\Omega(2^n/\sqrt{n})})</td>
<td>(2^O(n^3))</td>
</tr>
<tr>
<td>acyclic USOs [53]</td>
<td>(2^{2^{n-1}})</td>
<td>((n + 1)^{2^n})</td>
</tr>
<tr>
<td>all USOs [53]</td>
<td>(n^{\Omega(2^n)})</td>
<td>(n^{O(2^n)})</td>
</tr>
</tbody>
</table>

Table 1.1: Summary of the number of USOs.

**Enumeration of P-matroid USOs of the 4-cube.** Stickney and Watson [76] listed all P-matrix USOs of the 3-cube. In Chapter 11, we enumerate all P-matroid USOs of the 4-cube by using oriented matroid generation techniques proposed by Finschi and Fukuda [23].

Surprisingly, every P-matroid USO of the 4-cube is also a P-matrix USO, even though there exist P-matroids on eight elements that are not realizable; i.e., they cannot be represented in the form of (1.3). It is an open question as to whether this observation is preserved in higher dimensions.

Interestingly, every P-matroid USO of the 3-cube, in addition to most of the ones of the 4-cube, do also arise from cyclic-P-matroid OMCPs.

Finally, we observe that for an oriented matroid, being a P-matroid is more a matter of correctly labeled and oriented elements than a property of the underlying combinatorial structure.
Chapter 2

P-matrix linear complementarity problem

For a given matrix $M \in \mathbb{R}^{n \times n}$ and vector $q \in \mathbb{R}^n$, the linear complementarity problem (LCP) is to find two vectors $w, z \in \mathbb{R}^n$ so that

$$w - Mz = q,$$  
$$w, z \geq 0,$$  
$$w^T z = 0.$$  

A pair $(w, z)$ that satisfies (2.1a) and the nonnegativity condition (2.1b) is feasible. If the pair additionally satisfies the complementarity condition (2.1c), then it is a solution.

A general LCP may not have any solution, and deciding whether there is one is NP-complete [11]. This is even the case for LCPs with sparse matrices, which is shown by a polynomial-time reduction from monotone one-in-three 3SAT to LCPs with matrices containing at most two nonzero elements per row [77].

Our main interest is in the special case where $M$ is a P-matrix.

**Definition 2.1.** A *P-matrix* is a matrix in $\mathbb{R}^{n \times n}$ whose principal minors are all positive.
For many graph games, the problem of finding an optimal strategy can be formulated as a generalized linear complementarity problem (GLCP) with a block P-matrix. For instance, simple stochastic games admit such a formulation [37]. These games in turn admit a polynomial-time reduction from several other stochastic games [2, 9], mean payoff games [90], and parity games [40]. A discussion of the GLCP would go beyond the scope of this thesis, the interested reader is referred to Rüst’s PhD thesis [71].

Binary simple stochastic games reduce to P-matrix LCPs, which have many interesting properties, one of which is as follows.

**Theorem 2.2** (Samelson, Thrall and Wesler [73]). A matrix $M \in \mathbb{R}^{n \times n}$ is a P-matrix if and only if the LCP($M, q$) has a unique solution for every $q \in \mathbb{R}^n$.

The theorem has a beautiful geometric interpretation. Assume that we are given an LCP($M, q$), where $M = (m_i)_{i \in [n]} \in \mathbb{R}^{n \times n}$ and $q \in \mathbb{R}^n$. The following remarks are motivated by the fact that for any solution ($w^*, z^*$), we have $w_i^* = 0$ or $z_i^* = 0$ for each $i \in [n]$.

For any $B \subseteq [n]$, let $A_B$ be the $n \times n$ matrix whose $j$th column is the $j$th column of $-M$ if $j \in B$ and the unit vector $e_j$ otherwise. We consider the collection of cones spanned by the columns of such $A_B$, and call them complementary cones.

![Figure 2.1: Two LCPs of order 2. Problem (b) is a P-matrix LCP.](image-url)
The LCP$(M, q)$ is basically to find a complementary cone containing $q$, since then computing a solution is straightforward.

If $M$ is a P-matrix, then $A_B$ for any $B \subseteq [n]$ is nonsingular, and we call $A_B$ the basis matrix with respect to basis $B$. Consequently, every complementary cone has full dimension. Furthermore, no two complementary cones intersect in their interior, and the union of all complementary cones covers the whole space. Otherwise, there would exist $q$ for which the LCP$(M, q)$ has more than one solution or no solution, contradicting Theorem 2.2.

Figure 2.1 depicts two LCPs of order 2. We verify that the right-hand side $q$ of problem (a) is in the interior of exactly two complementary cones, and thus there are two solutions. Consequently, it is not a P-matrix LCP, unlike problem (b).

### 2.1 Matrix classes and complexity issues

In this section, we discuss matrix classes and complexity results related to the corresponding LCPs.

We start with the definition of sufficient matrices, a class defined by Cottle, Pang, and Venkateswaran [14] in connection with solving convex quadratic programs as LCPs.

**Definition 2.3.** A matrix $M \in \mathbb{R}^{n \times n}$ is column sufficient if for every $x \in \mathbb{R}^n$, we have:

$$x_i(Mx)_i \leq 0 \text{ for all } i \in [n] \text{ implies } x^T M x = 0.$$

A matrix $M \in \mathbb{R}^{n \times n}$ is row sufficient if $M^T$ is column sufficient. A matrix is sufficient if it is both column and row sufficient.

A matrix $M \in \mathbb{R}^{n \times n}$ is positive semidefinite (psd) if $x^T M x \geq 0$ for every $x \in \mathbb{R}^n$. Likewise, matrix $M$ is positive definite (pd) if $x^T M x > 0$ for every nonzero $x \in \mathbb{R}^n$.

**Definition 2.4.** A $P_0$-matrix is a matrix in $\mathbb{R}^{n \times n}$ whose principal minors are all nonnegative.
Every P-matrix is obviously a $P_0$-matrix. Furthermore, every pd matrix is a psd matrix, which in turn is sufficient. Other inclusions are proven in [13], we have

$$\text{pd matrices} \subset \text{psd matrices} \subset \text{sufficient matrices} \subset P_0\text{-matrices}$$

and

$$\text{pd matrices} \subset \text{P-matrices} \subset \text{sufficient matrices}.$$  

Figure 2.2 depicts the relationships between the matrix classes. Classes for which the corresponding LCPs are polynomial-time solvable are depicted in green, those for which the LCPs are NP-hard in red, and the others with unknown computational complexity in black. An extensive discussion of complexity results follows.

As mentioned previously, convex quadratic programming reduces to the psd matrix LCP. On the other hand, every LCP with a psd matrix can be formulated as a convex quadratic program in a straightforward manner. There exist efficient interior-point methods for both convex quadratic programming and the psd matrix LCP [47]. Such methods even solve P-matrix LCPs [87], but are not efficient anymore. No strongly polynomial algorithm is known for the psd matrix LCP.

The decision problems whether a square matrix is a $P$-, $P_0$-, or sufficient matrix are coNP-complete [15, 82]. Deciding whether a $P_0$-matrix LCP has a solution is NP-complete [46].

The computational complexity of LCPs with $P$- or sufficient matri-
ces is unknown, but many theoretical results suggest the existence of a polynomial-time algorithm.

Megiddo [54] observed that if the P-matrix LCP is NP-hard, then NP=coNP, which is considered to be very unlikely. This result carries over to the superclass of LCPs with sufficient matrices. Deciding whether a sufficient LCP has a solution is in \( \text{NP} \cap \text{coNP} \). The NP case is straightforward; any solution is a certificate for the “yes” answer. The coNP case is more involved. Fukuda and Terlaky [33] defined a kind of dual LCP, and formulated a theorem stating that either the LCP with a sufficient matrix or its dual has a solution, but not both. Then a solution of the dual is a certificate for the “no” answer. It is well-know that \( P \subseteq \text{NP} \cap \text{coNP} \). The intersection also contains problems such as integer factorization, i.e., the decision problem: does \( n \) has a prime factor less than \( k \)? If any problem in the intersection is NP-hard, then NP=coNP.

Furthermore, given any LCP(\( M, q \)), an EP-theorem indicates the existence of a polynomial-time algorithm that solves it, proves that no solution exists, or certifies that \( M \) is not sufficient [32].

Papadimitrou [70] introduced the complexity class PPAD in order to clarify the complexity status of several intriguing optimization problems. The class contains problems for which a solution is guaranteed to exist and which are not NP-hard, otherwise NP=coNP [55]. The class is defined through the so-called END OF LINE problem. A problem is in PPAD if it reduces to the END OF LINE problem; it is PPAD-complete if it admits a reduction from the END OF LINE problem. By the large number of known PPAD-complete problems, there is strong evidence that PPAD-complete problems are not polynomial-time solvable. Both the P-matrix LCP and bimatrix games are well-known to be members of PPAD [70], whereas bimatrix games are PPAD-complete [10]. So, bimatrix games, and the corresponding LCP formulations, are quite probably hard problems. It is an open problem whether the P-matrix LCP is PPAD-complete. Actually, there are some indications that it is not [17], and thus not as hard as bimatrix games.

We conclude that LCPs with P- and sufficient matrices are worth further study.

Klafszky and Terlaky [43] extended the criss-cross methods, originally
pivoting methods for linear programs that are identified by pivot rules [79],
to solving methods for OMCPs. Fukuda and Terlaky [33] generalized suf-
cient matrices in the setting of oriented matroids and proved finiteness of
the criss-cross method on OMCPs with sufficient matroids. These results
translate into the algebraic setting, and thus highlight the importance of
combinatorial generalizations.

The P-matrix LCP has been studied extensively. Criss-cross methods
and simple principal pivoting methods behave exactly the same on P-
matrix LCPs. Many deterministic pivot rules were proven to be finite,
but need an exponential number of pivot steps in the worst case. Some
randomized rules are promising, but analysis is a difficult task. Their
algorithmic complexity remains unclear; related results are discussed in
Chapter 9. Another well-studied algorithm is Lemke’s pivoting method,
which is closely related to the Lemke-Howson algorithm for the bimatrix
game problem [49]. Lemke’s algorithm is finite for the P-matrix LCP, but
needs an exponential number of pivot steps in the worst case as well [66].

To come to a conclusion, we note that the existence of a polynomial-
time algorithm for the P-matrix and sufficient matrix LCP is suggested by
many theoretical results. On the other hand, some practical observations
are discouraging, and the problems are very close to NP-hardness. Note
that for any $P_0$-matrix $M$ matrices $M + \epsilon I$ for $\epsilon > 0$ are P-matrices.

In the remainder of this section, we discuss subclasses of P-matrices
that are of theoretical and of practical interest. Figure 2.3 gives an overview
of the subclasses.

Figure 2.3: Matrix classes related to P-matrices.
Definition 2.5. A Z-matrix is a matrix in $\mathbb{R}^{n \times n}$ whose off-diagonal elements are all nonpositive.

Definition 2.6. A K-matrix is a P-matrix that is also a Z-matrix.

In the literature, K-matrices are sometimes called Minkowski- or just M-matrices. Chandrasekaran [8] formulated a pivoting algorithm that decides feasibility and eventually outputs a solution of Z-matrix LCPs in strongly polynomial-time. Any simple principal pivoting method solves K-matrix LCPs in at most $2n$ pivot steps, regardless of which pivot rule is applied [25].

The class of hidden Z-matrices is a natural extension of Z-matrices and was introduced by Mangasarian in connection with solving LCPs as linear programs [51].

Definition 2.7. A hidden Z-matrix is a matrix $M \in \mathbb{R}^{n \times n}$ for which there exist Z-matrices $A, B \in \mathbb{R}^{n \times n}$ such that

$$
MA = B, \text{ and } r^T A + s^T B > 0 \text{ for some } r, s \geq 0,
$$

(2.2)

Every Z-matrix $M$ is a hidden Z-matrix, just let $A := I, B := M$, $s := 0$ and pick any $r > 0$.

Definition 2.8. A hidden K-matrix is a P-matrix that is also a hidden Z-matrix.

Deciding whether a square matrix is a hidden K-matrix can be done in polynomial-time [69]. Furthermore, a polynomial-time algorithm exists for hidden K-matrix LCPs. A solution is obtained by solving linear programs using the ellipsoid method [51, 69]. The procedure is polynomial, however, no practically efficient method is known.

In this thesis, we get to know the new class of cyclic-P-matrices, whose definition is given in the setting of oriented matroids. They are shown to intersect the classes of K-, hidden K-, and P-matrices that are not hidden Z. The computational complexity of the cyclic-P-matrix LCP remains an open question.
Chapter 3

Basics of oriented matroids

3.1 Introduction

In this chapter, we give an overview of the theory of oriented matroids. An oriented matroid is a combinatorial object that is represented by a collection of sign vectors. The motivation to study oriented matroids comes from their broad range of applications in mathematics. Many geometric objects have combinatorial properties, and the setting of oriented matroids perfectly fits for the study. Examples are point configurations, hyperplane arrangements, and convex polytopes in the Euclidean space. We can even do graph theory in the setting of oriented matroids.

Our motivation is to study combinatorial properties of the linear complementarity problem. Along this line, we continue the work of Todd [80], Todd and Morris [63], and Fukuda and Terlaky [33].

Considering oriented matroids has many advantages. They have been extensively studied and provide an axiomatic foundation with valuable tools. They help improve the knowledge of the underlying combinatorics of the problem in question. While the enumeration of certain geometric objects is difficult, oriented matroids in contrast can be enumerated. In many applications, computers are used to verify conjectures.
The origin of oriented matroids dates back to the late 60s. The motivation was to extend ordinary matroids by signs. The first attempt was made by Folkman, and the first axiomatic description of oriented matroids was developed by Folkman and Lawrence [24] and Bland and Las Vergnas [7]. These papers are even today recommended references. For the most comprehensive and complete introduction, we refer the reader to the monograph by Björner, Vergnas, Sturmfels, White, and Ziegler [5]. Another good overview can be found in Finschi’s PhD thesis [22].

In the next section, we access oriented matroids by studying point configurations. The main concepts and terminologies are introduced. The first contact with oriented matroids may be laborious. Point configurations allow for a soft introduction. The basic theory is developed in the subsequent sections. We discuss the equivalent axiomatic systems, minors, duality, and extensions. Readers who are familiar with the theory of oriented matroids may skip this chapter and continue with the main content.

3.2 A first example: point configurations

We give an example of a point configuration and study how its combinatorial type is related to oriented matroids. The combinatorial type, commonly called order type, is the relative position of the points with respect to each other. If the points describe a convex polytope, i.e., the points are in convex position, then the order type is determined by the face lattice. For general point configurations, the order type is determined by the collection of all partitions of the points by hyperplanes.

The enumeration of order types is a difficult problem, and has been done for small dimensions only. The approach of Finschi and Fukuda [22] has been to enumerate order types by generalizing point sets in the setting of oriented matroids. Such a procedure leads to order types that cannot be realized by a concrete point set, namely those that arise from non-realizable oriented matroids. The decision problem whether an oriented matroid is realizable is NP-hard [57, 75], and practical methods exist only for oriented matroids with few elements.
A point configuration is given by a set $P = \{p_1, \ldots, p_n\}$ of $n$ points in $\mathbb{R}^d$. Any hyperplane partitions the set into points in one open half-space, the other open half-space, and points lying on the hyperplane. Such a partition induces a sign vector $X \in \{+, -, 0\}^n$.

Consider the point configuration in the plane in Figure 3.1 with depicted hyperplane $H$, where the open half-space containing $p_4$ is supposed to be the positive half-space. The hyperplane induces the sign vector $X = (-00+)$, where $X_1 = -$ as point $p_1$ is in the negative half-space, $X_2 = 0$ as $p_2$ lies on $H$ and so forth.

![Figure 3.1: A point configuration in $\mathbb{R}^2$.](image)

The collection of all sign vectors obtained by such partitions is the order type of the point configuration.

Assume that any hyperplane is given by a normal vector $a \in \mathbb{R}^d$ and translation value $b \in \mathbb{R}$. We denote such a hyperplane by

$$H(a, b) := \{x \in \mathbb{R}^d : a^T x = b\}.$$

The set of sign vectors obtained by hyperplane partitionings is given by

$$\mathcal{V} := \{(\text{sign } a^T p_1 - b, \text{sign } a^T p_2 - b, \ldots, \text{sign } a^T p_n - b) : a \in \mathbb{R}^d \text{ and } b \in \mathbb{R}\},$$

and completely describes the point configuration. For instance, the points
Basics of oriented matroids are in convex position if for every point \( p_i \) there is \( X \in \mathcal{V} \) with \( X_i = 0 \) and \( X_j = + \) for all \( j \neq i \). If the points represent the vertices of a convex polytope, the points \( p_i \) and \( p_j \) are connected by an edge of the polytope if and only if there is \( X \in \mathcal{V} \) with \( X_i = X_j = 0 \) and \( X_k = + \) for all \( k \neq i, j \).

**Sign vectors.** We introduce basic notations and operations on sign vectors.

For oriented matroids, the *ground set*, in our case the points, is denoted by \( E \). Usually \( E := [n] \). A *sign vector* on \( E \) is a vector \( X \in \{+,0,-\}^E \).

The components of \( X \) contained in \( F \subseteq E \) are denoted by \( X_F \). Let

\[
X^+ := \{ e \in E : X_e = + \}, \\
X^0 := \{ e \in E : X_e = 0 \} \quad \text{and} \\
X^- := \{ e \in E : X_e = - \}.
\]

We write \( X_F \geq 0 \) if \( F \subseteq X^+ \cup X^0 \) and accordingly \( X_F \leq 0 \) if \( F \subseteq X^- \cup X^0 \).

The sign vector obtained from \( X \) by deleting the components in \( F \subseteq E \) is denoted by \( X \setminus F \).

The *support* of \( X \) is the set \( X := X^+ \cup X^- \). The *opposite* of \( X \) is the sign vector denoted by \( -X \) with \( (-X)^+ = X^- \), \( (-X)^- = X^+ \), and \( (-X)^0 = X^0 \).

The *composition* \( X \circ Y \) of two sign vectors \( X \) and \( Y \) is the sign vector given by

\[
(X \circ Y)_e := \begin{cases} 
X_e & \text{if } X_e \neq 0, \\
Y_e & \text{otherwise.}
\end{cases}
\]

The *product* \( X \cdot Y \) is the sign vector given by

\[
(X \cdot Y)_e := \begin{cases} 
0 & \text{if } X_e = 0 \text{ or } Y_e = 0, \\
+ & \text{if } X_e = Y_e \text{ and } X_e \neq 0, \\
- & \text{otherwise.}
\end{cases}
\]

The *separation set* \( D(X,Y) := \{ e \in E : (X \cdot Y)_e = - \} \) contains all elements that disagree in \( X \) and \( Y \).

**Vectors of an oriented matroid.** The collection \( \mathcal{V} \) of sign vectors arising from a point configuration is well-structured and satisfies the axioms of an
oriented matroid.

**Definition 3.1 (vector axioms).** An oriented matroid in vector representation is a pair \( \mathcal{M} = (E, \mathcal{V}) \), where \( \mathcal{V} \) is a set of sign vectors on ground set \( E \) satisfying the following axioms:

(V1) \( 0 \in \mathcal{V} \).

(V2) If \( X \in \mathcal{V} \), then \(-X \in \mathcal{V}\).

(V3) If \( X, Y \in \mathcal{V} \), then \( X \circ Y \in \mathcal{V} \).

(V4) If \( X, Y \in V \) and \( e \in D(X, Y) \), there is \( Z \in \mathcal{V} \) such that

\[
Z_e = 0 \quad \text{and} \quad Z_f = (X \circ Y)_f \quad \text{for all} \quad f \in E \setminus D(X, Y).
\]

The collection \( \mathcal{V} \) is the set of vectors of the oriented matroid \( \mathcal{M} \).

Axiom (V4) has many variants, one of which is the following.

**Proposition 3.2** (Bland and Las Vergnas [7]). Let \( \mathcal{V} \) be a set of sign vectors satisfying (V1), (V2), and (V3). Then (V4) is equivalent to

(V4s) If \( X, Y \in \mathcal{V} \), \( e \in D(X, Y) \) and \( f \in X \setminus D(X, Y) \), then there is \( Z \in \mathcal{V} \) such that

\[
Z_e = 0, \quad Z_f = X_f, \quad Z^+ \subseteq X^+ \cup Y^+ \quad \text{and} \quad Z^- \subseteq X^- \cup Y^-.
\]

We verify that the sign vectors obtained from a point set partitioning satisfy all axioms of an oriented matroid.

The axioms (V1) and (V2) are straightforward to check. The zero vector arises from the hyperplane \( H(0, 0) \). Secondly, if \( H(a, b) \) yields a sign vector \( X \), then \( H(-a, b) \) yields \(-X\).

For axiom (V3), consider any two \( X, Y \in \mathcal{V} \), suppose that they arise from the hyperplanes \( H(a_x, b_x) \) and \( H(a_y, b_y) \), respectively. The composition \( X \circ Y \) arises from \( H(a_x + \epsilon a_y, b_x + \epsilon b_y) \) for sufficiently small \( \epsilon > 0 \).

It remains to show axiom (V4). Let \( X, Y \in V \) be any two sign vectors, where \( D(X, Y) \) is not empty. We construct sign vector \( Z \) with \( Z_e = 0 \) for \( e \in D(X, Y) \) of free choice and \( Z_f = (X \circ Y)_f \) for all \( f \in E \setminus D(X, Y) \).
Vector $Z$ is obtained by a convex combination of $X$ and $Y$. More precisely, it arises from the hyperplane $H(a_x + ta_y, b_x + tb_y)$ for $t = -(a^T_x p_e - b_x)/(a^T_y p_e - b_y)$.

The reader may investigate how (V3) and (V4) fit into the geometric picture.

**Circuits.** Let $X$ and $Y$ be sign vectors on $E$. We say $X$ conforms to $Y$, denoted by $X \preceq Y$, if $X_e = Y_e$ or $X_e = 0$ for every $e \in E$. The vectors of an oriented matroid $M = (E, \mathcal{V})$ form a partially ordered set with respect to conformity, also called *poset*.

The Hasse diagram of the poset $(\mathcal{V}, \preceq)$ of our point configuration is depicted in Figure 3.2. An artificial vector is added at the top.

![Hasse diagram](image)

**Figure 3.2:** The poset $(\mathcal{V}, \preceq)$ of the point configuration.

The vectors of an oriented matroid are not minimal in the sense that they can be reconstructed from a proper subset of vectors. The *atoms* in $(\mathcal{V}, \preceq)$ are the *circuits* of an oriented matroid. The circuits are given by

$$C := \{C \in \mathcal{V}\setminus\{0\} : \text{there is no } X \in \mathcal{V}\setminus\{C, 0\} \text{ such that } X \preceq C\}.$$
They uniquely determine the oriented matroid and contain all information required to recover the vectors. Every vector \( X \in \mathcal{V} \) is a \textit{conformal composition} of circuits. More precisely, we have \( X = C^1 \circ C^2 \circ \cdots \circ C^k \), where \( C^i \in C \) and \( C^i \preceq X \) for each \( i \in [k] \).

The collection of circuits satisfies an axiomatic system as well. These axioms are discussed and shown to be equivalent to the vector axioms in Section 3.3. We use the notation \( \mathcal{M} = (E, \mathcal{C}) \) for an oriented matroid given in terms of its circuits.

**Basis orientation.** There is another way to describe an oriented matroid, which leads to an equivalent axiomatic system dealing with bases. It requires some preparation.

We apply homogenization to the point configuration, which is illustrated in Figure 3.3.

![Figure 3.3: Homogenization of the point configuration.](image)

A general point configuration in \( \mathbb{R}^d \) is lifted into \( \mathbb{R}^{d+1} \) by constructing new points \( v_i := (p_i^T, 1)^T \). Let \( V \) be the \( (d + 1) \times n \) matrix whose \( i \)th
column is $v_i$. Then the oriented matroid $\mathcal{M} = (E, \mathcal{V})$ arising from the configuration is determined by

$$\mathcal{V} = \{\text{sign } V^T x : x \in \mathbb{R}^{d+1}\}.$$ 

The geometric interpretation is as follows. Each $v_i$ is considered to be the normal vector of a central hyperplane $H_i$ oriented such that $v_i$ points into the positive half-space. The arrangement $\mathcal{A} = (H_i)_{i \in [n]}$ of oriented hyperplanes represents the vectors of $\mathcal{M}$. The sign vector $X := \text{sign } V^T x$ is the same for all $x \in \mathbb{R}^{d+1}$ belonging to the same region or face in the arrangement. Hence, each face represents some distinct vector $X \in \mathcal{V}$. For an improved presentation, we usually cut the oriented hyperplanes with the unit sphere $S^d$. We let $S_i := H_i \cap S^d$ and get an arrangement $\mathcal{A} = (S_i)_{i \in [n]}$ of $(d-1)$-dimensional oriented spheres embedded in the unit sphere $S^d$. In principle, it is enough to depict the visible part only, since the vectors on the back are the opposite of the vectors in the front. We then speak of an oriented semisphere arrangement. Note that the vertices in the arrangement correspond to the circuits of the oriented matroid.

A chirotope, sometimes called basis orientation, is a map from ordered $(d+1)$-subsets of the ground set to signs. We define a chirotope as

$$\chi : (E^{d+1}) \rightarrow \{+, -, 0\}, \; \chi(B) \mapsto \text{sign } \det V_B,$$

where $(E^{d+1})$ contains all ordered $(d+1)$-subsets of $E$ and the columns of submatrix $V_B$ of $V$ are in order with $B$.

The pair $\mathcal{M} = (E, \chi)$ denotes an oriented matroid given in terms of a chirotope. The rank of $\mathcal{M}$ is defined to be $d + 1$. The chirotope $\chi$ satisfies an axiomatic system, which can be shown to be equivalent to the vector and circuit axiomatic systems. The reader is referred to Section 3.4 for details.

With respect to point configurations, chirotopes have the following interpretation. In our case $d = 2$. Consider any two points $p_i$ and $p_j$ and let $H$ be the unique hyperplane through these points. Let $p_s$ and $p_t$ be some other points. Then $\chi(i, j, s) = 0$ means that $p_s$ lies on $H$ as well. If both $\chi(i, j, s)$ and $\chi(i, j, t)$ are nonzero and have equal sign, then $p_s$ and $p_t$ lie in the same half-space. Opposite signs means they lie in opposite
3.2 A first example: point configurations

Duality. The concept of duality plays an important role in the theory of oriented matroids. Recall that we constructed an oriented matroid $\mathcal{M} = (E, \mathcal{V})$, where $\mathcal{V} := \{\text{sign } V^T x : x \in \mathbb{R}^{d+1}\}$. In other words, we considered the sign vectors in the rowspace of some vector configuration $V \in \mathbb{R}^{d+1 \times n}$. Both rowspace and nullspace of any matrix are vector subspaces and induce some oriented matroid.

The dual oriented matroid of $\mathcal{M}$ is denoted by $\mathcal{M}^* = (E, \mathcal{V}^*)$, where $\mathcal{V}^* := \{\text{sign } x : Vx = 0\}$. Strictly speaking, the chirotope as defined above describes a basis orientation of the dual matroid $\mathcal{M}^*$. Since the dual of any matroid is unique, the chirotope still uniquely determines the order type of the point configuration.

Realizability. The setting of oriented matroids abstracts combinatorial properties of geometric configurations. In fact, it is a proper generalization. Some oriented matroids are not realized by any vector configuration, or equivalently, by any oriented hyperplane or sphere arrangement. Nevertheless, it is still possible to visualize any oriented matroid in a topological setting. The Topological Representation Theorem [24] states that every oriented matroid is realized by an oriented pseudosphere arrangement. The converse also holds. Every oriented pseudosphere arrangement realizes an oriented matroid.

An oriented pseudosphere arrangement $\mathcal{A} = (S_e)_{e \in E}$ consists of $(d-1)$-dimensional oriented topological spheres $S_e$ embedded in the unit sphere $S^d$, where certain intersection properties, which clearly hold for linear sphere arrangements, are satisfied. For instance, every nonempty intersection $S_F = \bigcap_{e \in F} S_e$ for $F \subseteq E$ is again a topological sphere.

Figure 3.4 depicts some pseudosphere arrangement embedded in the unit sphere $S^2$.

The realizability problem is to decide whether a pseudosphere arrangement is stretchable; that is, whether it can be smoothly deformed into a sphere arrangement while the combinatorial structure is preserved. Here, we define the realizability problem in terms of vector configurations.

Definition 3.3. Let $\mathcal{M}$ be a rank $r$ oriented matroid on $E$ with vectors $\mathcal{V}$ and chirotope $\chi$. The matroid is realizable if there exists a vector
configuration $V = (v_i)_{i \in E} \in \mathbb{R}^{r \times |E|}$ such that

$$V = \{\text{sign } x : Vx = 0\},$$

or equivalently, such that for all $B \in (E^r)$, we have

$$\chi(B) = \text{sign } \det V_B,$$

where the columns of $V_B$ are in order with $B$. The vector configuration $V$ is a realization matrix. If there is no such realization, then $\mathcal{M}$ is non-realizable.

Since the determinant of any square matrix is a polynomial in the entries of the matrix, the realizability problem reduces to determining whether a system of polynomial equations and strict inequalities has a solution. Such a problem is an instance of the existential theory of the reals (ETR) problem, which is for given multivariate polynomials $f_i, i \in [r]$, and $g_j, j \in [s]$, and $h_k, k \in [t]$ to decide whether the polynomial system $f_i > 0$, $g_j \leq 0$, and $h_k = 0$ for $i \in [r], j \in [s], \text{ and } k \in [t]$ has a solution.

The ETR problem is NP-hard because the quadratic programming can be reduced to ETR in polynomial time [72]. In fact, Mnëv [57] proved that deciding realizability for oriented matroids is polynomially equivalent to ETR. Hence, the realizability problem for oriented matroids is NP-hard. This hardness result even holds for matroids of low rank. Shor [75] gave

![Figure 3.4: A pseudosphere arrangement.](image)
a polynomial time reduction of the ETR to *pseudoline arrangements* in
the plane, which are well-known to correspond to pseudosphere arrange-
ments embedded in $S^2$. The construction is such that an instance of the
ETR problem has a solution if and only if the corresponding pseudoline
arrangement is stretchable.

### 3.3 Circuits

We start with any oriented matroid $\mathcal{M} = (E, V)$ in vector representa-
tion. It is a good idea to think of a realizable oriented matroid. Thus,
assume that $V = \{\text{sign } x : Vx = 0\}$ for some vector configuration $V =
(v_i)_{i \in E} \in \mathbb{R}^{r \times n}$ of full row rank. Nevertheless, we have in mind that there
are oriented matroids that are non-realizable.

A circuit represents a linear combination of minimally dependent vec-
tors in the vector configuration.

**Definition 3.4.** Let $\mathcal{M} = (E, V)$ be an oriented matroid in vector repre-
sentation. The set

$$C := \{C \in V \setminus \{0\} : \text{there is no } X \in V \setminus \{C, 0\} \text{ such that } X \preceq C\}.$$ 

is the collection of *circuits* of $\mathcal{M}$.

The circuits provide all information to recover the vectors of the ori-
ented matroid. Every vector can be expressed as a conformal composition
of circuits.

**Definition 3.5.** Let $C$ be a set of sign vectors on $E$. A *conformal com-
position* is a composition $C^1 \circ C^2 \circ \cdots \circ C^k$, where $C^i \in C$ for $i \in [k]$ and
$D(C^i, C^j) = \emptyset$ for $i, j \in [k]$.

**Theorem 3.6.** Let $\mathcal{M} = (E, V)$ be an oriented matroid in vector repre-
sentation with circuits $C$. Then every vector $X \in V \setminus \{0\}$ is a conformal
composition of circuits in $C$.

For a proof of Theorem 3.6, the following elimination operation for
oriented matroids is required.
Proposition 3.7 (Fukuda [29]). Let $\mathcal{V}$ be a set of sign vectors satisfying (V1), (V2), and (V3). Then (V4) is equivalent to

(V4c) For all $X, Y \in \mathcal{V}$ and nonempty $G \subseteq D(X, Y)$, there is $Z \in \mathcal{V}$ such that

1. $Z_e = 0$ for some $e \in G$,
2. $Z_G \preceq X_G$ and
3. $Z_f = (X \circ Y)_f$ for all $f \in E \setminus D(X, Y)$.

The axiom (V4c) constitutes a conformal elimination operation whose strength shows up if applied to vectors $X, Y \in \mathcal{V}$ such that $-Y \preceq X$.

We let $G = D(X, Y) = Y$. According to (V4c), there is $Z \in \mathcal{V}$, which conforms to $X$ and satisfies $Z_e = 0$, while $X_e \neq 0$ for some $e \in G$.

Proof of Theorem 3.6. Let $\mathcal{M} = (E, \mathcal{V})$ be an oriented matroid in vector representation and $\mathcal{C}$ its circuits. We prove that every vector $X \in \mathcal{V} \setminus \{0\}$ can be expressed as a conformal composition of circuits in $\mathcal{C}$. The proof is done by induction. For the base case, observe that if $X \in \mathcal{C}$, we are obviously done. Consider any $X \in \mathcal{V}$ with $|X| = m$. Construct the composition $Y = C_1 \circ C_2 \circ \ldots \circ C_k$ of all $C_i \in \mathcal{C}$ that conform to $X$. Note that $Y$ itself conforms to $X$ and is not the zero vector, otherwise $X$ in $\mathcal{C}$. Either $X = Y$ and we are done or $X \setminus Y$ is not empty. In the latter case, we apply conformal elimination (V4c) to $X$ and $-Y$ with $G = D(X, -Y) = Y$. The resulting $Z$ conforms to $X$ and is such that $Z_e = 0$ for some $e \in G$. Thus, we have $|Z| < m$. By induction hypothesis, we assume that vector $Z$ can be expressed as a conformal composition $D_1 \circ D_2 \circ \ldots \circ D_l$ of circuits $D_j \in \mathcal{C}$. Note that $Z_f = X_f$ for all $f \in X \setminus Y$. Therefore $Y \circ Z$ is a conformal composition of $X$.

The circuits of an oriented matroid satisfy certain axioms, and we are able to characterize an oriented matroid in terms of circuits.

Definition 3.8 (circuit axioms). An oriented matroid in circuit representation is a pair $\mathcal{M} = (E, \mathcal{C})$, where $\mathcal{C}$ is a set of sign vectors on $E$ satisfying the following axioms:

(C1) $0 \notin \mathcal{C}$.

(C2) If $C \in \mathcal{C}$, then $-C \in \mathcal{C}$.
3.3 Circuits

(C3) For all $C, D \in \mathcal{C}$, we have: $C \subseteq D \implies C = D$ or $C = -D$.

(C4) If $C, D \in \mathcal{C}$, $C \neq -D$ and $e \in D(C, D)$, then there is $Z \in \mathcal{C}$ such that

\begin{align*}
Z_e &= 0, \\
Z^+ &\subseteq C^+ \cup D^+ \text{ and} \\
Z^- &\subseteq C^- \cup D^-.
\end{align*}

The collection $\mathcal{C}$ is the set of circuits of the oriented matroid $\mathcal{M}$.

The operation in axiom (C4) is called weak circuit elimination. In the definition of an oriented matroid in terms of circuits, the axiom can be replaced by a stronger variant.

**Proposition 3.9** (Bland and Las Vergnas [7]). Let $\mathcal{C}$ be a set of sign vectors satisfying (C1), (C2), and (C3). Then (C4) is equivalent to

(C4s) If $C, D \in \mathcal{C}$, $e \in D(C, D)$ and $f \in C \setminus D(C, D)$, then there is $Z \in \mathcal{C}$ such that

\begin{align*}
Z_e &= 0, \\
Z_f &= C_f, \\
Z^+ &\subseteq C^+ \cup D^+ \text{ and} \\
Z^- &\subseteq C^- \cup D^-.
\end{align*}

Theorem 3.10 below summarizes the translation from vectors to circuits and vice versa. It proves that the collection of circuits constructed from the vectors satisfies the circuit axioms in Definition 3.8. On the other hand, if we start with an oriented matroid in circuit representation, and construct the vectors by conformal compositions, then they satisfy the vector axioms in Definition 3.1. The theorem also states that it is enough to consider the vectors with minimal support as circuits, general compositions of circuits as vectors, respectively. We do not necessarily need to take conformity into account.

**Theorem 3.10** (vectors $\leftrightarrow$ circuits).

(i) Let $\mathcal{M} = (E, \mathcal{V})$ be an oriented matroid in vector representation.
Then the sets
\[ C := \{ C \in V \setminus \{0\} : \text{there is no } X \in V \setminus \{C, 0\} \text{ such that } X \preceq C \} \]
and
\[ C' := \{ C \in V \setminus \{0\} : \text{there is no } X \in V \setminus \{C, -C, 0\} \text{ such that } X \subseteq C \} \]
are equal and satisfy the circuit axioms in Definition 3.8.

(ii) Let \( M = (E, C) \) be an oriented matroid in circuit representation. Then the sets
\[ V := \{0\} \cup \{ C^1 \circ \cdots \circ C^k : D(C^i, C^j) = \emptyset \text{ for all } C^i, C^j \in C \} \]
and
\[ V' := \{0\} \cup \{ C^1 \circ \cdots \circ C^k : C^i \in C \} \]
are equal and satisfy the vector axioms in Definition 3.1.

Proof. (i). Let \( M = (E, V) \) be an oriented matroid in vector representation and \( C, C' \) as defined in the theorem.

(\( C' \subseteq C \)). For any \( C \in C' \), there is no \( X \in V \setminus \{C, -C, 0\} \) with \( X \subseteq C \). Hence, there is no \( X \in V \setminus \{C, 0\} \) with \( X \preceq C \). Therefore \( C \in C \) as well.

(\( C \subseteq C' \)). For any \( C \in C \), there is no \( X \in V \setminus \{C, 0\} \) with \( X \preceq C \). For the sake of a contradiction, suppose that there is \( X \in V \setminus \{C, -C, 0\} \) with \( X \subseteq C \). The set \( D(C, X) \) is not empty, and some element \( e \in D(C, X) \) can be eliminated from \( C \) and \( X \) by applying conformal elimination (V4c) with \( G = D(C, X) \). The resulting vector \( Z \) conforms to \( C \). Since \( C \in C \), there is no such vector.

(C1). Since \( 0 \notin C \) by definition, the axiom is clearly satisfied.

(C2). For any \( C \in C \), there is no \( X \in V \setminus \{C, 0\} \) with \( X \preceq C \). By axiom (V2), there is no \( X \in V \setminus \{-C, 0\} \) with \( X \preceq -C \). Therefore \( -C \in C \) as well.

(C3). The axiom is clearly satisfied by definition of \( C' \).

(C4). Let \( C, D \in C \) with \( C \neq -D \) and \( e \in D(C, D) \). Eliminate \( e \) from \( C \) and \( D \) by applying vector elimination (V4). The resulting vector \( Z \) is such that \( Z_e = 0 \), \( Z^+ \subseteq C^+ \cup D^+ \) and \( Z^- \subseteq C^- \cup D^- \). It is a candidate for a resulting circuit of (C4). If by any chance \( Z \in C \), we are done. Otherwise, there are \( X \in V \setminus \{Z, 0\} \) with \( X \preceq Z \), where all of them are candidates. At least one such vector is minimal with respect to conformity, and is thus contained in \( C \).
(ii). Let \( \mathcal{M} = (E, \mathcal{C}) \) be an oriented matroid in circuit representation and \( \mathcal{V}, \mathcal{V}' \) as defined in the theorem.

\[ \mathcal{V} \subseteq \mathcal{V}' \]. This directly follows from the definition of \( \mathcal{V} \) and \( \mathcal{V}' \).

\[ \mathcal{V}' \subseteq \mathcal{V} \]. Any \( X \in \mathcal{V}' \) is a composition \( X = C^1 \circ \ldots \circ C^k \) of circuits \( C^i \in \mathcal{C} \). The set \( D(C^i, C^j) \) may not be empty for all pairs of contributing circuits. We prove that \( X \) can also be expressed as a conformal composition of circuits. We proceed by induction on the length of the composition. For the base case, note that \( X^1 := C^1 \) is obviously a conformal composition. By induction hypothesis, assume that \( X^{k-1} := C^1 \circ \ldots \circ C^{k-1} \) equals the conformal composition \( D^1 \circ \ldots \circ D^l \). If by any chance \( X = D^1 \circ \ldots \circ D^l \), we are done. In any case \( X = D^1 \circ \ldots \circ D^l \circ C^k \). If \( D(C^k, D^i) = \emptyset \) for every \( D^i \), we are done. Otherwise, pick any \( D^i \) with \( D(C^k, D^i) \neq \emptyset \). Eliminate an element \( e \) from \( C^k \) and \( D^i \) by applying (C4s) with some distinct element \( f \) for which \( C^k_f \neq 0 \), but \( X^{k-1}_f = 0 \). For the resulting circuit \( Z \), we have \( |D(Z, D^i)| < |D(C^k, D^i)| \) and \( Z_f = C^k_f \). We successively eliminate elements from the most recently obtained \( Z \) and \( D^i \) with distinct element \( f \) until we find \( Z \) with \( D(Z, D^i) = \emptyset \) and \( Z_f = C^k_f \). Repeatedly apply this procedure to \( Z \) and other \( D^j \) for which \( D(Z, D^j) \neq \emptyset \). We finally find \( Z \) with \( D(Z, D^j) = \emptyset \) for all \( D^j \) and \( Z_f = C^k_f \). If \( X = D^1 \circ \ldots \circ D^l \circ Z \), we are done. Otherwise, start all over with some other \( f \) for which \( C^k_f \neq 0 \), but \( (D^1 \circ \ldots \circ D^l \circ Z)_f = 0 \).

(V1). Since \( 0 \in \mathcal{V} \) by definition, the axiom is clearly satisfied.

(V2). Circuit axiom (C2) states that for every \( C \in \mathcal{C} \), we have \( -C \in \mathcal{C} \). Therefore for every \( X \in \mathcal{V}' \), the opposite \( -X \in \mathcal{V}' \) as well.

(V3). Let \( X, Y \) be any two vectors in \( \mathcal{V}' \). Both \( X \) and \( Y \) are compositions of circuits. Consequently, the composition \( X \circ Y \) can be expressed as a composition of circuits, and thus is in \( \mathcal{V}' \).

(V4s). Let \( X, Y \in \mathcal{V} \) with \( e \in D(X, Y) \) and \( f \in X \setminus D(X, Y) \). We prove that there is \( Z \in \mathcal{V} \) with \( Z_e = 0 \), \( Z_f = X_f \), \( Z^+ \subseteq X^+ \cup Y^+ \), and \( Z^- \subseteq X^- \cup Y^- \). We apply case distinction. First, suppose that \( f \in X \setminus Y \). By definition of \( \mathcal{V} \), there is a circuit \( C \) conforming to \( X \) with \( C_f = X_f \). There is also a circuit \( D \) conforming to \( Y \) with \( D_e = Y_e \). If \( C_e = 0 \), we are done with \( Z := C \). Otherwise, we are done by eliminating \( e \) from \( C \) and \( D \) by applying (C4s) with distinct element \( f \). Secondly, suppose that \( f \in D(X, -Y) \). There are circuits \( C \) and \( D \) conforming to \( X \) and \( Y \),
respectively, such that $C_f = D_f = X_f = Y_f$. If $C_e = 0$ or $D_e = 0$, we are done with $Z := C$ or $Z := D$, respectively. Otherwise, $C_e = -D_e \neq 0$. We eliminate $e$ from $C$ and $D$ by applying (C4s) with distinct element $f$, and we are done with the resulting circuit.

\[\square\]

### 3.4 Bases and chirotopes

The third axiomatic system for oriented matroids deals with bases and their orientations. For a vector configuration $V = (v_i)_{i \in E} \in \mathbb{R}^{r \times n}$ of full row rank, a basis is a subset maximal independent columns. The signs of $\det V_B$ for $B \in (E^r)$ translate into basis orientations in the setting of oriented matroids.

Let $\mathcal{M} = (E, C)$ be an oriented matroid in circuit representation. A set $I \subseteq E$ is independent if no circuit is contained in $I$. The set $I$ is maximal independent if additionally for each $e \in E \setminus I$, there is a circuit contained in $I \cup e$.

**Definition 3.11.** Let $\mathcal{M} = (E, C)$ be an oriented matroid in circuit representation. A set $B \subseteq E$ is a basis of $\mathcal{M}$ if it is maximal independent.

For every basis $B$ and $e \in E \setminus B$, there is a unique circuit $C$ with $C_e = +$ and $C \subseteq B \cup e$. If it were not unique, say there are $C$ and $D$, then elimination of $e$ from $C$ and $-D$ by (C4) would result in some circuit that is contained in $B$.

**Definition 3.12.** Let $\mathcal{M} = (E, C)$ be an oriented matroid in circuit representation, $B \subseteq E$ any basis and $e \in E \setminus B$. The unique circuit $C \in \mathcal{C}$ with $C_e = +$ and $C \subseteq B \cup e$ is the fundamental circuit with respect to basis $B$ and element $e$. We denote it by $C(B, e)$.

**Lemma 3.13.** All bases of an oriented matroid have the same cardinality.

**Proof.** Let $\mathcal{M}$ be any oriented matroid on $E$. For the sake of a contradiction, suppose that there are bases $B, B' \subseteq E$ such that $|B'| < |B|$. Basis $B'$ is no subset of $B$, otherwise it would not be maximal independent. Consider the unique circuit $C := C(B, e)$ for any $e \in |B'| \setminus |B|$. We cannot have $C_f = 0$ for all $f \in B \setminus B'$, otherwise $C \subseteq B'$. In other words, set $B'$
would not be a basis. Pick any \( f \in B \backslash B' \) with \( C_f \neq 0 \), and consider the set \((B \cup e) \setminus f\). From the uniqueness of \( C \), it follows that there is no circuit \( D \) with \( D \subseteq (B \cup e) \setminus f\). The set \((B \cup e) \setminus f\) is independent and contains one more element of \( B' \) than \( B \) contains. We repeat these steps until we finally find an independent set for which \( B' \) is a proper subset. Then \( B' \) is not maximal independent, and a contradiction is established. \( \square \)

The proof of Lemma 3.13 implies that every circuit \( C \) is the fundamental circuit with respect to some basis. Without loss of generality, suppose that \( C \) is positive at some element \( e \); otherwise \( -C \) is. The set \( C \setminus e \) is independent because of (C3). If it is not maximal independent, it can be extended to a basis \( B \) by adding elements of any other basis. Since \( C \) is contained in \( B \cup e \), the unique fundamental circuit \( C(B, e) \) equals \( C \).

**Definition 3.14.** The **rank** of an oriented matroid \( M \) is the cardinality of any of its bases. We denote it by \( \text{rank}(M) \).

The algebraic concept of nondegeneracy translates into uniformity in the theory of oriented matroids.

**Definition 3.15.** An oriented matroid \( M \) on \( E \) is **uniform** if every \( B \subseteq E \) with cardinality \( \text{rank}(M) \) is a basis of \( M \).

The next lemma is known as the basis exchange property.

**Lemma 3.16.** Let \( B, B' \) be bases of an oriented matroid \( M \). For every \( e \in B' \backslash B \), there exists \( f \in B \backslash B' \), such that \((B \cup e) \setminus f\) is a basis of \( M \).

**Proof.** Let \( B \) and \( B' \) be bases and \( e \in B' \backslash B \). Consider the fundamental circuit \( C := C(B, e) \). We have \( C_f \neq 0 \) for some \( f \in B \backslash B' \), otherwise \( C \subseteq B' \). Since \( C \) is unique, set \((B \cup e) \setminus f\) is independent. It is a basis as well because of its cardinality. \( \square \)

Next, we would like to encode oriented matroids in terms of the bases. The orientations of the bases of a rank \( r \) oriented matroid are given by a map \( \chi : (E^r) \to \{+,0,-\} \), where \( (E^r) \) is the set of all ordered \( r \)-subsets of \( E \). The map \( \chi \) is **alternating** if a transposition \( \pi_{ij}(B) \) for any \( B \in (E^r) \) changes the sign, i.e., we have \( \chi(\pi_{ij}(B)) = -\chi(B) \).
An ordered $B \in (E^r)$ is often handled the same way as an unordered set. So, by saying $e \in B$, we mean that $e$ is in the corresponding unordered set. For $f \in E \setminus B$, we use the terminology $B : f \to e$ for the ordered subset, which coincides with $B$, except that $e$ is replaced with $f$.

**Definition 3.17 (chirotope axioms).** A rank $r$ oriented matroid in chirotope representation is a pair $\mathcal{M} = (E, \chi)$, where the map $\chi : (E^r) \to \{+, 0, -\}$ satisfies the following axioms:

- **(B1)** $\chi$ is not identically zero,

- **(B2)** $\chi$ is alternating,

- **(B3)** If $B, B'$ in $(E^r)$ and $e \in B'$ such that
  
  \[ \chi(B : e \to f) \chi(B' : f \to e) \geq 0 \]

  for every $f \in B$, then we have

  \[ \chi(B) \chi(B') \geq 0. \]

In the literature, the terminologies basis orientation and chirotope for a map $\chi$ satisfying all these axioms are used interchangeably.

**Definition 3.18.** A chirotope of rank $r$ is a map $\chi : (E^r) \to \{+, 0, -\}$ that satisfies the axioms in Definition 3.17.

As for the vector and circuit axiomatic systems, there exist alternative axioms for chirotopes. The following strengthens the basis exchange property stated in Lemma 3.16.

**Proposition 3.19.** Let $\chi$ be a map satisfying (B1) and (B2). Then (B3) is equivalent to

- **(B3e)** if $B, B'$ in $(E^r)$ and $e \in B'$ such that
  
  $\chi(B) \chi(B') \neq 0$,

  then there is $f \in B$ such that

  \[ \chi(B : e \to f) \chi(B' : f \to e) = \chi(B) \chi(B'). \]

Theorem 3.20 below summarizes the translation from circuits to chirotopes and vice versa. For a rank $r$ oriented matroid, the theorem states the obvious fact that $B \in (E^r)$ is a basis of $\mathcal{M}$ if and only if $\chi(B) \neq 0$. This is the reason why a chirotope is sometimes called basis orientation.
3.4 Bases and chirotopes

The relationship between circuits and chirotopes is nicely illustrated by Cramer’s rule. Consider a vector configuration \( V = (v_i)_{i \in E} \in \mathbb{R}^{r \times n} \) of full row rank and submatrix \( V_B \) for some basis \( B \). Then \( V_B \) is nonsingular. According to Cramer’s rule, the unique solution \( x \) to an equation system

\[
V_B x = v_j
\]

for \( j \notin B \) is given by

\[
x_i := \frac{\det V_B : v_j \rightarrow v_i}{\det V_B},
\]

where \( V_B : v_j \rightarrow v_i \) is the matrix obtained by replacing column \( v_i \) in \( V_B \) with \( v_j \). The solution \( x \), together with \( x_j = -1 \), forms a minimal linear dependent vector in the nullspace of \( V \). For \( x_j \) of arbitrary value, the components of \( x \) must satisfy

\[
\det V_B : v_j \rightarrow v_i = -x_j x_i \det V_B.
\]

This relationship translates into the setting of oriented matroids.

**Theorem 3.20 (circuits \( \iff \) chirotopes).**

(i) Let \( \mathcal{M} = (E, \mathcal{C}) \) be a rank \( r \) oriented matroid in circuit representation. Let map \( \chi : (E^r) \rightarrow \{+, 0, -\} \) satisfy the following properties:

(p1) For every \( B \in (E^r) \), we have:

\[
\chi(B) \neq 0 \text{ if and only if } B \text{ is a basis of } \mathcal{M},
\]

(p2) \( \chi \) is alternating,

(p3) If \( B \in (E^r) \) is a basis of \( \mathcal{M} \) and \( e \in E \setminus B \), then

\[
\chi(B : e \rightarrow f) = -C_e C_f \chi(B),
\]

where \( C \) is the fundamental circuit \( C(B, e) \).

Then the map \( \chi \) is a chirotope, i.e., it satisfies the axioms in Definition 3.17.
Let $\mathcal{M} = (E, \chi)$ be a rank $r$ oriented matroid in chirotope representation. Let $C$ contain all sign vectors that are obtained by the following procedure. For every $B \in (E^r)$ with $\chi(B) \neq 0$ and $e \in E \setminus B$, extend $C$ by the opposite sign vectors $C, -C$ satisfying $C_e \neq 0$,

$$\chi(B : e \rightarrow f) = -C_e C_f \chi(B)$$

for every $f \in B$, and $C_g = 0$ for all $g \in E \setminus (B \cup e)$.

Then the set $C$ satisfies the circuit axioms in Definition 3.8.

We do not prove Theorem 3.20. The only proof we are aware of is quite involved. It uses the concept of oriented matroid duality and the fact that the vectors of two sets of sign vectors satisfying the axioms (C1), (C2), and (C3) are pairwise orthogonal if and only if both sets satisfy axiom (C4).

The following result is related to Theorem 3.20.

**Corollary 3.21.** If $\chi$ is a chirotope of an oriented matroid $\mathcal{M}$, then the opposite $-\chi$ is a chirotope of $\mathcal{M}$. Moreover, these are the only two chirotopes of $\mathcal{M}$.

### 3.5 Minors

We discuss deletions and contractions for oriented matroids. These operations yield submatroids. Let $V = (v_i)_{i \in E} \in \mathbb{R}^{r \times n}$ be a vector configuration of full row rank. The deletion by an element $e \in E$ corresponds to the removal of column $v_e$ from the vector configuration. The contraction is more involved. For any $e \in E$, let $B \in (E^r)$ be a basis containing $e$. Without loss of generality, suppose that $V_B$ equals the identity matrix $I_r$, where $v_e$ is the unit vector $e_i$. The contraction by element $e$ corresponds to the removal of column $v_e$ and row $i$ from the vector configuration $V$.

The deletion and contraction operation for oriented matroids are defined as follows. For a set $\mathcal{V}$ of sign vectors, let

$$\mathcal{V} \setminus F := \{X \setminus F : X \in \mathcal{V} \text{ and } X_F = 0\}$$
and
\[ \mathcal{V}/F := \{ X \setminus F : X \in \mathcal{V} \}. \]

**Definition 3.22.** Let \( \mathcal{M} = (E, \mathcal{V}) \) be an oriented matroid in vector representation and \( F \subseteq E \).

(i) The pair \( \mathcal{M} \setminus F := (E \setminus F, \mathcal{V} \setminus F) \) is a deletion minor of \( \mathcal{M} \).

(ii) The pair \( \mathcal{M}/F := (E \setminus F, \mathcal{V}/F) \) is a contraction minor of \( \mathcal{M} \).

The reader is asked to verify that deletion and contraction minors are actually oriented matroids by checking the vector axioms.

**Proposition 3.23.** Every deletion and contraction minor of an oriented matroid is an oriented matroid. \( \square \)

Let \( \mathcal{M} = (E, \mathcal{V}) \) be an oriented matroid in vector representation with circuits \( \mathcal{C} \). The circuits of the deletion minor \( \mathcal{M} \setminus F \) are given by
\[ \mathcal{C} \setminus F := \{ C \setminus F : C \in \mathcal{C} \text{ and } C_F = 0 \}. \]

Similarly, the circuits of the contraction minor \( \mathcal{M}/F \) are given by
\[ \mathcal{C}/F := \{ C \in \mathcal{V}/F : \text{ there is no } D \in (\mathcal{V}/F) \setminus \{ C, 0 \} \text{ such that } D \preceq C \}. \]

The operations of deletion and contraction commute. For two disjoint sets \( F, G \subseteq E \), we have
\begin{align*}
(\mathcal{M} \setminus F) \setminus G &= \mathcal{M} \setminus (F \cup G), \\
(\mathcal{M}/F)/G &= \mathcal{M}/(F \cup G) \text{ and } \\
(\mathcal{M}/F) \setminus G &= (\mathcal{M} \setminus G)/F.
\end{align*}

The rank of the minors is affected by special properties of the elements that are deleted or contracted.

**Definition 3.24.** Let \( \mathcal{M} \) be an oriented matroid on \( E \). An element \( e \in E \) is a loop if no basis contains \( e \). An element \( e \) is a coloop if it is contained in every basis.

Let \( \mathcal{M} \) be a rank \( r \) oriented matroid on \( E \). First, consider the deletion minor \( \mathcal{M} \setminus e \), where \( e \in E \) is a coloop of \( \mathcal{M} \). Note that \( B \in (E^r) \) is a basis
of $\mathcal{M}$ if and only if $B\setminus e$ is a basis of the deletion minor. Therefore, the rank of $\mathcal{M}\setminus e$ is $r - 1$. Now, suppose that $e$ is not a coloop. Then there is at least one basis $B \in (E^r)$ that does not contain $e$. Such $B$ is also a basis of the deletion minor. Therefore the rank of $\mathcal{M}\setminus e$ is $r$.

Next, consider the contraction minor $\mathcal{M}/e$, where $e \in E$ is a loop of $\mathcal{M}$. The single-element set $\{e\}$ is not independent, otherwise it could be extended by elements of another basis to a basis containing $e$. Hence, there is a single-element circuit $C$ with $C_e = +$. By the uniqueness result for fundamental circuits, the circuits $C(B,e)$ are the same for all bases $B \in (E^r)$ of $\mathcal{M}$. Namely, they are equal to $C$. Hence, the contraction minor does not have any circuit contained in bases of $\mathcal{M}$. The rank is preserved, i.e., the rank of $\mathcal{M}/e$ is $r$. Now, suppose that $e$ is not a loop. Then some basis $B \in (E^r)$ of $\mathcal{M}$ contains $e$. Consider the fundamental circuit $C := C(B,f)$ for any $f \in E\setminus B$. The sign vector $C\setminus e$ is a vector of the contraction minor. Furthermore, vector $C\setminus e$ and its opposite are the only circuits of the minor which are contained in $(B\setminus e) \cup f$. Consequently $B\setminus e$ is an independent set of the minor. It is also a basis because the minor contains $C(B,f')\setminus e$ for each other $f' \in E\setminus B$. Hence, the rank of $\mathcal{M}/e$ is $r - 1$.

In general situations of deletions and contractions by an arbitrary number of elements, the rank of the minor can be determined by proceeding element by element. The following proposition helps in determining the rank directly.

**Proposition 3.25.** Let $\mathcal{M} = (E, \chi)$ be an oriented matroid in chirotope representation and $F \subseteq E$.

(i) The rank $s$ of the deletion minor $\mathcal{M}\setminus F$ equals $\text{rank}(\mathcal{M}) - |G|$, where $G$ is any ordered, minimal subset of $F$, such that there is some basis $B$ of $\mathcal{M}$ with $B \subseteq (E\setminus F) \cup G$. The chirotope $\chi\setminus F$ of the deletion minor is given by

$$\chi\setminus F : ((E\setminus F)^*) \rightarrow \{+ , 0 , -\}, \quad B \mapsto \chi(B,G).$$

(ii) The rank $t$ of the contraction minor $\mathcal{M}/F$ equals $\text{rank}(\mathcal{M}) - |H|$, where $H$ is any ordered, maximal subset of $F$, such that there is some basis $B$ of $\mathcal{M}$ with $H \subseteq B \subseteq (E\setminus F) \cup H$. The chirotope $\chi/F$
Proof. (i). First, we prove that the rank $s$ of the deletion minor $\mathcal{M}\setminus F$ equals $\text{rank}(\mathcal{M}) - |G|$. Let $B$ be a basis of $\mathcal{M}$ with $B \subseteq (E\setminus F) \cup G$. If $B = (E\setminus F) \cup G$, then the minor is the empty matroid, and the result trivially follows. Otherwise, consider any circuit $C$ contained in $B$, i.e., they contain circuits. Since $C$ is independent in the minor, the rank $s$ of the deletion minor $M\setminus C$ such that $B \subseteq B\prime$. We extend it by elements $H$ of $G$ and consequently $B$ is an independent set of the minor. The basis $B\subseteq B\prime$ is a basis of the minor, it is an independent set of the minor. The rank $s$ of $\mathcal{M}\setminus F$ is $\text{rank}(\mathcal{M}) - |G|$.

Next, we verify that $\chi/F$ as defined in the proposition is a basis orientation of the deletion minor $\mathcal{M}\setminus F$. Consider any circuit $C\prime := C\setminus F(B', e)$ for $B' \in ((E\setminus F)^*)$ and $e \in (E\setminus F)\setminus B'$ of the minor. The elements of $C\prime$ are such that $C\prime_e = +$ and $\chi/F(B' : e \to f) = -C\prime_e C\prime_f \chi/F(B')$. We prove that $C\prime$ coincides with circuit $C := C(B' \cup G, e)$ of $\mathcal{M}$. We verify that $B' \cup G$ is actually a basis of $\mathcal{M}$. Since $B'$ is a basis of the minor, it is an independent set of $\mathcal{M}$. It can be extended to a basis of $\mathcal{M}$ by adding elements from some other basis. See proof of Lemma 3.13. Let $B$ be a basis of $\mathcal{M}$ with $B \subseteq (E\setminus F) \cup G$, which exists by definition of $G$. Observe that $G \subseteq B$. The basis $B'$ cannot be extended by any element $f \in B \cap (E\setminus F)$. We extend it by elements $g \in G$. In fact, we have to add whole $G$, otherwise we contradict to the minimality of $G$. Hence $B' \cup G$ is a basis of $\mathcal{M}$. The elements of $C$ are such that $C_e = +$, $\chi(B' : e \to f, G) = -C_e C_f \chi(B', G)$ for every $f \in B'$, and $\chi(B', G : e \to g) = -C_e C_f \chi(B', G)$ for every $g \in G$. By definition of $G$, we have $\chi(B', G : e \to g) = 0$ for all $g \in G$, and consequently $C_F = 0$. At other positions, $C$ coincides with $C\prime$. We conclude that every circuit $C\prime$ of $\mathcal{M}\setminus F$ is determined by $\chi/F$.

(ii). First, we prove that the rank $t$ of $\mathcal{M}/F$ equals $\text{rank}(\mathcal{M}) - |H|$. Let $B$ be a basis of $\mathcal{M}$ with $H \subseteq B \subseteq (E\setminus F) \cup H$. We prove that $B\setminus H$ is a basis of $\mathcal{M}/F$. Observe that $B\setminus H$ is an independent set of the minor if circuits $C := C(B, e)$ for $e \in F\setminus H$ are such that $C_{B\setminus H} = 0$. We notice
that $B \setminus H$ is a minimal subset of $E \setminus F$ such that $F \cup (B \setminus H)$ contains some basis of $\mathcal{M}$. The same argumentation as in (i) applies to the deletion minor $\mathcal{M} \setminus (E \setminus F)$. Hence, we actually have $C_{B \setminus H} = 0$, and $B \setminus H$ is an independent set of the minor. It is also a basis because circuits $C(B, e) \setminus F$ for $e \in (E \setminus F) \setminus B$ are in $\mathcal{M} / F$. The rank $t$ of $\mathcal{M} / F$ is rank$(\mathcal{M}) - |H|$. Finally, we verify that $\chi / F$ as defined in the proposition is a chirotope of the contraction minor $\mathcal{M} / F$. Consider any circuit $C' := C \setminus F(B', e)$ for $B' \in ((E \setminus F)^t)$ and $e \in (E \setminus F) \setminus B'$ of the minor. By using similar argumentation as for the proof of (i), one can prove that $B' \cup H$ is basis of $\mathcal{M}$ and that $C'$ coincides with circuit $C := C(B' \cup H, e)$ at positions in $E \setminus F$. \hfill \Box

3.6 Reorientations

In the introductory section, we have seen how an oriented matroid arising from a point configuration is represented by an oriented sphere arrangement. Given any arrangement of spheres, we get many oriented matroids by identifying spheres with elements from a ground set and giving orientations to the spheres. Actually, any labeling and orienting yields an oriented matroid. The collection of all oriented matroids that are obtained from a given arrangement builds a so-called reorientation class. A formal definition follows.

Let $\mathcal{M} = (E, V)$ be an oriented matroid in vector representation, $\pi$ a permutation of $E$, and $F \subseteq E$. For $X \in V$, sign vector $\pi \cdot X$ is the vector with $(\pi \cdot X)_{\pi(e)} = X_e$ for every $e \in E$. Sign vector $-_F X$ is the vector with $(_F X)^+ = (X^+ \setminus F) \cup (X^- \cap F)$ and $(_F X)^- = (X^- \setminus F) \cup (X^+ \cap F)$. We let $\pi \cdot V := \{ \pi \cdot X : X \in V \}$ and $-_F V := \{_-F X : X \in V \}$.

**Definition 3.26.** Let $\mathcal{M} = (E, V)$ be an oriented matroid in vector representation, $\pi$ a permutation of $E$, and $F \subseteq E$.

(i) The pair $\pi \cdot \mathcal{M} := (E, \pi \cdot V)$ is a relabeling of $\mathcal{M}$.

(ii) The pair $-_F \mathcal{M} := (E, _-F V)$ is a reorientation of $\mathcal{M}$.

We verify that relabelings and reorientations are oriented matroids by checking the vector axioms.
Proposition 3.27. Every relabeling and reorientation of an oriented matroid is an oriented matroid.

Two oriented matroids are reorientation equivalent if they can be transformed into each other by a combination of a relabeling and reorientation. For any oriented matroid, the collection of its reorientation equivalent matroids builds a reorientation class of oriented matroids.

For an ordered $B \in (E^r)$ and a permutation $\pi$ of $E$, let $\pi(B)$ be the ordered set in $(E^r)$ which is obtained from $B$ by replacing each element $e \in B$ with $\pi(e)$.

Proposition 3.28. Let $\mathcal{M} = (E, \chi)$ be a rank $r$ oriented matroid, $\pi$ a permutation of $E$, and $F \subseteq E$.

(i) The chirotope $\pi \cdot \chi$ of the relabeling $\pi \cdot \mathcal{M}$ is given by

$$\pi \cdot \chi(\pi(B)) = \chi(B)$$

for all $B \in (E^r)$.

(ii) The chirotope $-F \chi$ of the reorientation $-F \mathcal{M}$ is given by

$$-F \chi(B) = (-1)^{|F \cap B|} \chi(B)$$

for all $B \in (E^r)$.

3.7 Duality

We discuss the concept of dual or orthogonal oriented matroids. For every matroid there exists a unique dual matroid. Think of an oriented matroid that is realized by some vector configuration $V \in \mathbb{R}^{r \times n}$. The dual oriented matroid is realized by the rowspace of $V$. In other words, the vectors of the dual are given by the collection $\{\text{sign} V^T y : y \in \mathbb{R}^n\}$. Since for all $x \in \mathbb{R}^n, y \in \mathbb{R}^r$ such that $Vx = 0$, we have $(V^T y) \cdot x = y^T Vx = 0$, null- and rowspace are orthogonal to each other.

The general definition of orthogonal sign vectors is as follows.
Definition 3.29. Two sign vectors $X, Y \in \{+, 0, -\}^E$ are orthogonal if either $X \cap Y = \emptyset$ or there exist $e, f \in X \cap Y$ such that $X_e X_f = -Y_e Y_f$.

The orthogonality of two sign vectors $X$ and $Y$ is denoted by $X \perp Y$. It is well-known that the bases of orthogonal vector subspaces are the complements of each other. This motivates the following definition of dual matroids.

Definition 3.30. Let $M = (E, C)$ be an oriented matroid in circuit representation. An oriented matroid $M^* = (E, C^*)$ in circuit representation with bases

$$\{E \setminus B : B \text{ is a basis of } M\}$$

such that $C \perp D$ for all pairs $C \in C$ and $D \in C^*$ is a dual matroid of $M$.

The following uniqueness result was first proven by Bland and Las Vergnas [7].

Theorem 3.31. For every oriented matroid, there exists a dual oriented matroid. Moreover, the dual matroid is unique.

Proof. We give a construction scheme for a dual matroid in terms of chirotopes. Let $M = (E, C)$ be an oriented matroid, where $E$ has any order assigned. Consider any basis $B \in (E^r)$ and its complement $N := E \setminus B$. Both are supposed to be in order with $E$. Circuit $D := C^*(N, e)$ for any $e \in B$ of a dual $M^*$ has to be orthogonal to every circuit of $M$, in particular to the circuit $C := C(B, f)$ for any $f \in N$. Since $C_f = +$, $D_e = +$ and $C \cap D \subseteq \{e, f\}$, for the orthogonality to hold, we must have $C_e C_f = -D_e D_f$. The signs of $C_e$ and $C_f$ are determined by

$$\chi(B : f \to e) = -C_f C_e \chi(B).$$

Equivalently, the signs of $D_e$ and $D_f$ are determined by

$$\chi^*(N : e \to f) = -D_e D_f \chi^*(N).$$

We let the dual chirotope $\chi^*(N) := \text{sign}(\pi(B, N))\chi(B)$ for all $B \in (E^r)$ and ordered $N := E \setminus B$. Here, $\text{sign} \pi(B, N)$ is the permutation to bring
(B, N) in the order of E. Then, we have

\[-D_eD_f = \frac{\chi^*(N : e \to f)}{\chi^*(N)} \]

\[= \frac{\text{sign}(\pi(B : f \to e, N : e \to f))\chi(B : f \to e)}{\text{sign}(\pi(B, N))\chi(B)} \]

\[= -\frac{\chi(B : f \to e)}{\chi(B)} \]

\[= C_fC_e. \]

as desired.

Since basis B and elements e and f were arbitrarily selected, the \(\chi^*\) as defined above is the unique candidate for a chirotope of a dual matroid. It is straightforward to verify that the dual chirotope \(\chi^*\) satisfies the chirotope axioms in Definition 3.17.

It remains to prove that any pair \(C \in \mathcal{C}\) and \(D \in \mathcal{C}^*\) is orthogonal to each other. Suppose that \(C\) and \(D\) intersect, i.e., there is an \(e \in C \cap D\). Otherwise, we are done. We write \(C\) and \(D\) as fundamental circuits. We have \(C := C(B, e)\) for some basis \(B\) and \(D := C^*(N', e)\) for some basis \(N'\) of the dual candidate. Then \(B' := E \setminus N'\) is a basis of \(\mathcal{M}\) containing \(e\). We have \(\chi(B)\chi(B') \neq 0\). By chirotope axiom (B3e), there is \(f \in B\) such that \(\chi(B : e \to f)\chi(B' : f \to e) = \chi(B)\chi(B')\). The signs of \(C_e\) and \(C_f\) are determined by

\[-C_eC_f = \frac{\chi(B : e \to f)}{\chi(B)} \]

and the signs of \(D_e\) and \(D_f\) by

\[-D_eD_f = \frac{\chi^*(N' : e \to f)}{\chi^*(N')} \]

\[= -\frac{\chi(B' : f \to e)}{\chi(B')} \]

Both \(B' : f \to e\) and \(B : e \to f\) are nonzero, i.e., they are bases of \(\mathcal{M}\). It follows that \(C_f, D_f \neq 0\). Moreover, we have \(C_eC_f = -D_eD_f\). The circuit \(C\) and dual circuit \(D\) are orthogonal. \(\square\)
This proof of Theorem 3.31 shows that the chirotope of the dual is well-defined.

**Corollary 3.32.** Let $\mathcal{M} = (E, \chi)$ be a rank $r$ oriented matroid, where $E$ has any order assigned. The dual matroid $\mathcal{M}^* = (E, \chi^*)$ is given by

$$\chi^*(N) = \text{sign}(\pi(B, N))\chi(B),$$

where $B \in (E^*)$ and $N := E \setminus B$ are ordered and $\pi(B, N)$ is the permutation to bring $(B, N)$ in the order of $E$. \hfill \Box$

The circuits of the dual oriented matroid $\mathcal{M}^*$ are called cocircuits of $\mathcal{M}$. Similarly, all other matroidal objects have dual equivalents. Vectors and bases of $\mathcal{M}^*$ are covectors and cobases of $\mathcal{M}$, respectively. By definition, a loop in $\mathcal{M}^*$ is a coloop in $\mathcal{M}$ and vice versa.

The following self-duality property follows by the symmetry in the definition of a dual and the uniqueness result.

**Corollary 3.33.** Let $\mathcal{M}$ be an oriented matroid. Then $(\mathcal{M}^*)^* = \mathcal{M}$. \hfill \Box

Deletion and contraction minors are related by duality.

**Proposition 3.34.** Let $\mathcal{M}$ be an oriented matroid on $E$ and $F \subseteq E$. Then

$$(\mathcal{M}\setminus F)^* = \mathcal{M}^*/F$$

$$(\mathcal{M}/F)^* = \mathcal{M}^*\setminus F$$

**Proof.** We only prove $(\mathcal{M}\setminus F)^* = \mathcal{M}^*/F$. The second statement, if reformulated for the dual matroid, follows from the first statement and Corollary 3.33. By Proposition 3.23, both $\mathcal{M}\setminus F$ and $\mathcal{M}^*/F$ are oriented matroids. Let $\mathcal{C}$ and $\mathcal{C}^*$ be the circuits and cocircuits of $\mathcal{M}$, respectively. Pick any $C \in \mathcal{C}$ with $C_F = 0$ and $D \in \mathcal{C}^*$. First, note that $C\setminus F \perp D\setminus F$ if and only if $C \perp D$. It remains to be shown that $\mathcal{M}\setminus F$ and $\mathcal{M}^*/F$ have complementary bases. Consider $\mathcal{M}$ and let $G$ be as defined in Proposition 3.25. Observe that $B$ is a basis of $\mathcal{M}\setminus F$ if and only if $B \cup G$ is a basis of $\mathcal{M}$. Now, consider $\mathcal{M}^*$ and let $H$ be as defined in Proposition 3.25. Similarly, $N$ is a basis of $\mathcal{M}^*/F$ if and only if $N \cup H$ is a basis of $\mathcal{M}^*$. Since $\mathcal{M}$ and $\mathcal{M}^*$ have complementary bases, we can pick disjoint $G$ and
3.7 Duality

$H$ with $G \cup H = F$. It follows that $B \subseteq (E \setminus F)$ is a basis of $\mathcal{M} \setminus F$ if and only if $(E \setminus F) \setminus B$ is a basis of $\mathcal{M}^* \setminus F$. 

Recall Farkas’ Lemma from linear algebra. Consider any vector configuration $V = (v_i)_{i \in E} \in \mathbb{R}^{r \times n}$. For any $e \in E$, the system $V_{E \setminus e}x = -v_e$ and $x \geq 0$ has no solution if and only if the system $(V_{E \setminus e})^Ty \geq 0$ and $v_e^Ty > 0$ has a solution. The lemma can be translated into the setting of oriented matroids. We also give a stronger variant below.

**Lemma 3.35** (Farkas’ Lemma of OMs). Let $\mathcal{M}$ be an oriented matroid on $E$. For every $e \in E$ exactly one of the following statements holds.

(i) There is a circuit $C$ with $C \geq 0$ and $C_e = +$.

(ii) There is a cocircuit $D$ with $D \geq 0$ and $D_e = +$.

**Proof.** Both statements cannot hold because of orthogonality. We prove that at least one holds by induction on the minors. For the base case, let $F := E \setminus e$ for any $e \in E$. We prove that either $\mathcal{M} \setminus F$ or its dual $\mathcal{M}^* \setminus F$ contains the positive single-element. If element $e$ is a loop in $\mathcal{M}$, then $\mathcal{M} \setminus F$ contains it. Otherwise, element $e$ is not a coloop in $\mathcal{M}^*$. Thus, there is a cocircuit $D$ with $D_e = +$. Obviously $D \setminus F$ is contained in $\mathcal{M}^* \setminus F$. By induction hypothesis, the lemma holds for $\mathcal{M} \setminus f$, where $f \in E \setminus e$. For the sake of a contradiction, suppose that none of $\mathcal{M}$ and $\mathcal{M}^*$ contains a nonnegative circuit, which is positive at element $e$. Then each circuit $C'$ of $\mathcal{M} \setminus f$ with $C'_e = +$ has at least one negative component, and by induction hypothesis, the dual $\mathcal{M}^* \setminus f$ contains some circuit $D'$ with $D' \geq 0$ and $D'_e = +$. The corresponding $D$ in $\mathcal{M}^*$ must be such that $D_{E \setminus f} \geq 0$, $D_e = +$, and $D_f = -$. Such an argumentation also works for the other dual pair $\mathcal{M}^* \setminus f$ and $\mathcal{M} / f$. Through exactly the same reasoning, there is a $C$ in $\mathcal{M}$ with $C_{E \setminus f} \geq 0$, $C_e = +$, and $C_f = -$. If so, $C$ and $D$ contradict orthogonality. Hence, oriented matroid $\mathcal{M}$ satisfies (i) or (ii). 

**Corollary 3.36** (Generalized Farkas’ Lemma of OMs). Let $\mathcal{M}$ be an oriented matroid on $E$ and $R \cup G \cup B \cup W = E$ be any partition of $E$ with $e \in R$. Then exactly one of the following statements holds.

(i) There is a circuit $C$ with $C_e = +$, $C_R \geq 0$, $C_G \leq 0$, and $C_B = 0$.

(ii) There is a cocircuit $D$ with $D_e = +$, $D_R \geq 0$, $D_G \leq 0$, and $D_W = 0$. 

Proof. Both statements cannot hold because of orthogonality. Consider the reorientation \(-G,M\) whose dual is \(-G(M^*)\). If neither of the statements holds, then \((-G,M)\setminus B/W\) and dual matroid \((-G(M^*))\setminus W/B\) contradict Lemma 3.35.

Since every vector of an oriented matroid is a conformal composition of circuits, we conclude that vectors and covectors are orthogonal to each other. Moreover, the covectors are given by the following proposition.

**Proposition 3.37.** Let \(M = (E,C)\) be an oriented matroid in circuit representation. The covectors \(V^*\) are given by

\[\{Y \in \{+,0,-\}^E : Y \perp C \text{ for all } C \in C\}\].

**Proof.** Since every covector is a conformal composition of cocircuits, the covectors are orthogonal to the circuits. Now, consider any \(Y \in \{+, -, 0\}^E\) such that \(Y \perp C\) for all \(C \in C\). It is enough to prove that \(Y\) is a conformal composition of cocircuits, because then it is a covector. For every \(e \in Y^+\) we apply Corollary 3.36 to circuits \(C\) and cocircuits \(C^*\) with \(e \in R := Y^+\), \(G := Y^-\), \(B := \emptyset\), and \(W := Y^0\). Alternative (i) does not hold because \(Y\) would not be orthogonal to any such \(C \in C\). Hence, alternative (ii) holds. There is a cocircuit \(D \in C^*\) with \(D_e = +\) and \(D \preceq Y\). For every \(e \in Y^-\), similar conclusion can be drawn by applying the opposite variant of Corollary 3.36. Finally, we conclude that for every \(e \in Y\), there is \(D\) with \(D_e = Y_e\) and \(D \preceq Y\). The composition of all these \(D\)'s equals \(Y\). Therefore \(Y\) is conformal composition of cocircuits, and is thus a covector.

### 3.8 Extensions

An extension of an oriented matroid is the opposite of a deletion. Let \(M\) be a rank \(r\) oriented matroid on \(E\) that is realized by some vector configuration \(V = (v_i)_{i \in E} \in \mathbb{R}^{r \times n}\). A rank-preserving extension of \(M\) by an additional element \(q\) corresponds to extending \(V\) by an additional column \(v_q \in \mathbb{R}^r\).

For general oriented matroids, we get the following definition.
**Definition 3.38.** Let $\mathcal{M}$ be an oriented matroid on $E$. An oriented matroid $\widehat{\mathcal{M}}$ on $\widehat{E}$, where $\widehat{E} := E \cup q$, is an *extension* of $\mathcal{M}$ if $\widehat{\mathcal{M}} \setminus q = \mathcal{M}$.

In the remainder of this thesis, we only consider extensions, where $q$ is not a coloop in $\widehat{\mathcal{M}}$, i.e., the rank is preserved.

Consider again the realizable case. The covectors of $\mathcal{M}$ are given by $V^* = \{\text{sign} V^T y : y \in \mathbb{R}^r\}$. In other words, the normal vectors of an oriented sphere arrangement representing the dual $\mathcal{M}^*$ are the columns in $V$. The covectors of an extension $\widehat{\mathcal{M}}$ are given by $(\widehat{V})^* = \{\text{sign}(V v_q^T y) : y \in \mathbb{R}^r\}$. This basically means that an oriented sphere arrangement representing $(\widehat{\mathcal{M}})^*$ is obtained by adding an oriented sphere $S_q$ with normal vector $v_q$ to the arrangement representing $\mathcal{M}^*$. See Figure 3.5 for an example.

![Figure 3.5: A semisphere arrangement extended by an additional oriented semisphere $S_q$.](image)

Las Vergnas [48] proved that an extension of an oriented matroid is uniquely determined by a function $\sigma : C^* \rightarrow \{+, 0, -\}$, that specifies for every cocircuit of $\mathcal{M}$ whether it is supposed to lie on the positive side, on the negative side of $S_q$, or on $S_q$. Obviously, not every such function describes a valid extension of an oriented matroid.

**Definition 3.39.** A *localization* is a function $\sigma : C^* \rightarrow \{+, 0, -\}$ that describes a valid extension.
Lemma 3.40 below informally describes the collection of extensions of an oriented matroid.

For an arrangement of pseudo-semispheres representing the dual of a rank $r$ oriented matroid a vertex is the intersection of $r - 1$ semispheres, i.e., each vertex represents a cocircuit. A line is the intersection of $r - 2$ semispheres.

**Lemma 3.40.** Consider any arrangement of pseudo-semispheres representing the dual of an oriented matroid $\mathcal{M}$. A function $\sigma : \mathcal{C}^* \rightarrow \{+ , 0 , - \}$ is a localization if and only if the set $\{ D \in \mathcal{C}^* : \sigma(D) \geq 0 \}$ contains a beginning segment of the vertices on every line.

The reader is referred to [5, 48] for a more formal characterization of extensions.

Sometimes we require to construct extensions with special characteristics. Consider once more the realizable case. Let $I = (e_1 , e_2 , \ldots , e_k) \subseteq E$ be independent in $\mathcal{M}$ and $\alpha \in \{1 , -1\}^k$. Suppose that we extend the vector configuration $V$ by the vector

$$v_q := \alpha_1 v_{e_1} + e_2 v_{e_2} + e^2 \alpha_3 v_{e_3} + \ldots + e^{k-1} \alpha_k v_{e_k},$$

where $\epsilon > 0$ is sufficiently small. Extensions of this type are very structured and generalize to non-realizable oriented matroids. The related concept is known as lexicographic extension in the literature, and was first studied by Las Vergnas [48].

**Lemma 3.41.** Let $\mathcal{M}$ be an oriented matroid on $E$, set $I = (e_1 , e_2 , \ldots , e_k) \in (E^k)$ be independent in $\mathcal{M}$, and $\alpha \in \{+ , -\}^k$. Then the lexicographic extension $\widetilde{\mathcal{M}} := \mathcal{M}[I, \alpha]$ is given by $\sigma : \mathcal{C}^* \rightarrow \{+ , 0 , - \}$ with

$$\sigma(D) = \begin{cases} \alpha_i D_{e_i} & \text{if } i \text{ is minimal with } D_{e_i} \neq 0 \\ 0 & \text{if } D_{e_i} = 0 \text{ for all } i \in [k]. \end{cases}$$

**Lemma 3.42** (Todd [80]). Let $\mathcal{M}$ be an oriented matroid on $E$, set $I = (e_1 , e_2 , \ldots , e_k) \in (E^k)$ be independent in $\mathcal{M}$, and $\alpha \in \{+ , -\}^k$. Then the lexicographic extension $\widetilde{\mathcal{M}} := \mathcal{M}[I, \alpha]$ contains a circuit $C$ with $C = I \cup q$, $C_q = -$, and $C_{e_i} = \alpha_i$ for all $i \in [k]$. 


There is another issue concerning extensions and realizability. An oriented matroid may be realized by infinitely many vector configurations. Let us consider two different vector configurations $V$ and $V'$ which realize the same matroid. We construct an extension of the matroid by extending the vector configuration $V$ by some vector $v_q$. It may happen that there is no $v'_q$, such that $V'$ extended by $v'_q$ realizes the same extension. Informally speaking, some realizations may allow extensions that do not exist for other realizations. For an illustration of this scenario, we present an example given by Aichholzer, Aurenhammer, and Krasser [1]. Figure 3.6 depicts two different realizations in terms of point configurations of some oriented matroid on five elements. Realization (a) allows an extension by a point $q$, such that the convex hull of $q$ together with the two points at the bottom encloses all other points. Realization (b), on the contrary, does not allow such an extension.

![Figure 3.6: Two point configurations realizing the same oriented matroid but having different extension spaces.](image-url)
Part I

Oriented Matroid Complementarity Problem
Motivation

Many proposed solving methods for the LCP, such as pivoting algorithms, make decisions based on combinatorial properties of the underlying problem instance. Some methods have been proven to be inefficient, while for others, the algorithmic complexity remains unclear. We believe that combinatorial abstractions improve the insight into the problem structure and are a tool to simplify the analysis process. They attract attention to the combinatorial properties, while irrelevant algebraic properties are ignored.

In this part, we employ the setting of oriented matroids to study the P-matrix LCP and its subclasses. The theory of oriented matroids is well-developed and provides many valuable tools. Our overall aim is to achieve a better understanding and to improve upon the knowledge about combinatorial properties that may be exploited by some algorithm. We start by inspecting algebraic properties of the matrix classes in question and study how these translate into combinatorial properties of oriented matroids. The procedure includes combinatorially generalizing LCPs by formulating the oriented matroid complementarity problem (OMCP), which is due to Todd [80]. Since there are non-realizable oriented matroids, the obtained results properly generalize those of linear algebra. Moreover, we provide purely combinatorial proofs of the algebraic counterparts. Many results that require involved argumentation for an algebraic proof allow a surprisingly easy combinatorial proof.

Several classes of matrices and combinatorial methods have already been investigated in the setting of oriented matroids. Todd [80] generalized P- and so-called Q-matrices, and translated both Lemke’s [49] and Van...
der Heyden’s [85] algorithm into solving methods for the OMCP. Morris and Todd [63] generalized symmetric positive definite matrices. Furthermore, Fukuda and Terlaky [33] generalized sufficient matrices, the class containing the P- and positive semidefinite matrices, and proved finiteness of the criss-cross method on OMCPs with sufficient matroids.

The transition from matrices to oriented matroids and from LCPs to OMCPs is explained in Chapter 4.

In Chapter 5, we investigate several subclasses of sufficient matrices in the setting oriented matroids, in particular P- and (hidden) K-matrices. By starting with algebraic properties, we derive combinatorial counterparts and accordingly define P-matroids and (hidden) K-matroids. Among other things, we find an alternative characterization of P-matrices unpublished so far, and generalize an algebraic result for K-matrices given by Fiedler and Pták [21]. We also propose a new proper subclass of P-matroids, which we call cyclic-P-matroids. These oriented matroids arise from point configurations whose points lie on the moment curve. Hence, they are realizable. Cyclic-P-matrices, i.e., the corresponding realizations, inherit many valuable properties from the P-matrices. For instance, they are closed under principal pivot transforms and principal submatrices. The definition is such that it is possible to construct nontrivial P-matrices of arbitrary order. This allows us to conduct computational experiments with solving methods in high dimensions. Finally, we are able to give a combinatorial description of the collection of extensions by right-hand sides, which eventually turns out to be useful for complexity analysis.

Chapter 6 is devoted to solving methods for the P-matroid OMCP. We translate (simple) principal pivoting methods, which are identified by a pivot rule, into the setting of oriented matroids. We then investigate combinatorial properties satisfied by P-matroid OMCPs. By generalizing a result of [25], we prove that any K-matroid OMCP is solved by simple principal pivoting methods in a linear number of pivot steps, regardless of which pivot rule is applied. We do not discuss specific pivot rules. Discussions of the algorithmic complexity of different pivot rules is mainly done in part II by considering the combinatorial abstraction of unique-sink orientations, which is probably more appropriate for such tasks.
Chapter 4

From LCPs to OMCPs

We combinatorially generalize the LCP by formulating the oriented matroid complementarity problem (OMCP) following Todd’s approach [80].

Recall the LCP, which is for a given matrix $M \in \mathbb{R}^{n \times n}$ and vector $q \in \mathbb{R}^n$ to find vectors $w, z \in \mathbb{R}^n$ so that $w - Mz = q$, $w, z \geq 0$, and $w^Tz = 0$. As mentioned previously, any solution $(w^*, z^*)$ is such that $w_i^* = 0$ or $z_i^* = 0$ for each $i \in [n]$. Hence, finding a solution to the LCP is equivalent to finding a vector $x \in \mathbb{R}^{2n+1}$ so that

$$
[I_n \ -M \ -q \ x = 0, \\
x \geq 0, \\
x_{2n+1} = 1, \\
x_ix_{i+n} = 0 \text{ for each } i \in [n].
$$

Basically, we seek a vector in a vector subspace with certain combinatorial properties. The collection

$$
\hat{V} := \{ \text{sign } x : [I_n \ -M \ -q \ x = 0 \}
$$

is the set of vectors of an oriented matroid, and serves as the underlying oriented matroid for the OMCP.

The oriented matroids arising from LCPs are endowed with a special
structure on the ground set. We usually consider ground sets $E_{2n}$ of $2n$ elements with a fixed partition $E_{2n} = S \cup T$ into two $n$-element sets and a mapping $e \mapsto \overline{e}$ from $E_{2n}$ to $E_{2n}$ which is an involution; i.e., $\overline{\overline{e}} = e$ for every $e \in E_{2n}$ and for every $e \in S$, we have $\overline{e} \in T$. Note that this mapping constitutes a bijection between $S$ and $T$. The element $\overline{e}$ is the complement of $e$. For an $F \subseteq E_{2n}$, let $\overline{F} = \{ \overline{e} : e \in F \}$. Set $F$ is complementary if $F \cap \overline{F} = \emptyset$.

The oriented matroids we consider are of the kind $\mathcal{M} = (E_{2n}, V)$, where $S$ is a basis. In addition, we are working with extensions $\hat{\mathcal{M}} = (\hat{E}_{2n}, \hat{V})$, where $\hat{E}_{2n} := E_{2n} \cup q$ for some additional element $q$.

**Definition 4.1.** Let $\hat{\mathcal{M}} = (\hat{E}_{2n}, \hat{V})$ be an oriented matroid, where $S$ is a basis. The oriented matroid complementarity problem (OMCP) is to find a vector $X \in \{+, 0, -\}^{\hat{E}_{2n}}$ so that

\begin{align*}
X \in \hat{V}, \\
X \geq 0, \quad X_q = +, \\
X_e X_{\overline{e}} = 0 \quad \text{for every } e \in E_{2n}
\end{align*}

or to report that no such vector exists.

A vector $X \in \hat{V}$ that satisfies (4.2b) is feasible, one that satisfies (4.2c) is complementary. Note that a vector is complementary if and only if its support is a complementary set. A solution to the OMCP($\hat{\mathcal{M}}$) is any $X \in \hat{V}$ that is feasible and complementary.

It is obvious that LCPs are special cases of OMCPs. Finding a solution to an LCP($\mathcal{M}, q$) is equivalent to finding a solution to the OMCP($\hat{\mathcal{M}}$), where $\hat{\mathcal{M}}$ is the oriented matroid whose vectors $\hat{V}$ are as defined in (4.1). Clearly, if the OMCP has no solution, then the corresponding LCP has no solution. On the contrary, if there is a solution $X \in \hat{V}$, then a solution to the LCP is obtained by solving

\[
\begin{bmatrix} I_n & -M & -q \end{bmatrix} x = 0,
\]

\[
x_i = 0 \quad \text{whenever } X_i = 0,
\]

\[
x \geq 0,
\]

\[
x_{2n+1} = 1.
\]
This correspondence motivates the following definition of realizable OMCPs.

**Definition 4.2.** Let \( \widehat{M} = (\widehat{E}_{2n}, \widehat{V}) \) be an oriented matroid, where \( S \) is a basis. The oriented matroid \( \widehat{M} \) is realizable if

\[
\widehat{V} = \left\{ \text{sign } x : \begin{bmatrix} I_n & -M & -q \end{bmatrix} x = 0 \right\}
\]

for some \( M \in \mathbb{R}^{n \times n} \) and \( q \in \mathbb{R}^n \). Here, we assume that the columns of \( I_n \) are indexed by the elements of \( S \), and the columns of \(-M\) are indexed by the elements of \( T \) such that if the \( j \)th column of \( I_n \) is indexed by \( e \), then the \( j \)th column of \(-M\) is indexed by \( \bar{e} \). The pair \((M, q)\) is a realization, and \( M \) a realization matrix of \( \widehat{M} \).

This definition is atypical, but coincides with the original definition of realizability. Usually, any vector configuration \( V \in \mathbb{R}^{n \times 2n+1} \) representing \( \widehat{M} \) is a realization. In the case of OMCPs, submatrix \( V_S \) of any realization is nonsingular because \( S \) is supposed to be a basis. Consequently, the pair \((-V_S^{-1}V_T, -V_S^{-1}V_{2n+1})\) is a realization with respect to Definition 4.2.

Many combinatorial properties of an LCP\((M, q)\) are determined by the structure of the matrix \( M \) and do not depend on right-hand side \( q \). It is convenient to first study oriented matroids on \( E_{2n} \) before turning the attention to extensions on \( E_{2n} \).

**Definition 4.3.** Let \( \mathcal{M} = (E_{2n}, \mathcal{V}) \) be an oriented matroid, where \( S \) is a basis. The oriented matroid \( \mathcal{M} \) is realizable if

\[
\mathcal{V} = \left\{ \text{sign } x : \begin{bmatrix} I_n & -M \end{bmatrix} x = 0 \right\}
\]

for some \( M \in \mathbb{R}^{n \times n} \). The matrix \( M \) is a realization matrix of \( \mathcal{M} \).

Again, we remark that this definition coincides with the original definition of realizability.
Chapter 5

Classes of oriented matroids

In this chapter, we translate several classes of matrices into the setting of oriented matroids. We extract the combinatorial properties of the matrix classes in question, and accordingly define the counterparts of oriented matroids. Many known algebraic results are generalized. The last section is devoted to cyclic-P-matroids.

5.1 P-matroids

First, we consider algebraic properties of P-matrices and investigate how they translate into combinatorial properties. This procedure leads to the definition of P-matroids.

Recall that a P-matrix is a matrix whose principal minors are positive. Several conditions are equivalent to the positivity of principal minors.

Theorem 5.1. For a matrix $M \in \mathbb{R}^{n \times n}$, the following conditions are equivalent.

(a) All principal minors of $M$ are positive.
(b) Every nonzero vector \( x \in \mathbb{R}^n \) satisfies \( x_i(Mx)_i > 0 \) for some \( i \in [n] \).

(c) For every \( \sigma \in \{-1, +1\}^n \), there exists a vector \( x \in \mathbb{R}^n \) such that \( \sigma_i x_i > 0 \) and \( \sigma_i (Mx)_i > 0 \) for each \( i \in [n] \).

(d) An \( \text{LCP}(M, q) \) has exactly one solution for every \( q \in \mathbb{R}^n \).

The equivalence of (a) and (b) is due to Fiedler and Pták [21]. The equivalence of (a) and (d) was proven independently by Samelson, Thrall, and Wesler [73], Ingleton [39], and Murty [64]. Necessity of condition (c) for a P-matrix was proven by Morris [59] in the setting of oriented matroids. Proposition 5.3, together with Theorem 5.4 below, proves that (c) is actually an alternative characterization of P-matrices.

Oriented matroids that generalize P-matrices have been extensively studied by Todd [80]. The following notions and the definition of a P-matroid are motivated by condition (b) in Theorem 5.1.

A sign vector \( X \) on \( E_{2n} \) is sign-reversing (s.r.) if \( X_eX_{\overline{e}} \leq 0 \) for every \( e \in S \). If in addition \( X = E_{2n} \), the vector is totally sign-reversing (t.s.r.). Analogously, a vector \( X \) is sign-preserving (s.p.) if \( X_eX_{\overline{e}} \geq 0 \) for every \( e \in S \), and totally sign-preserving (t.s.p.) if additionally \( X = E_{2n} \).

**Definition 5.2** (Todd [80]). An oriented matroid \( M \) on \( E_{2n} \), where \( S \) is a basis, is a P-matroid if it contains no sign-reversing circuit.

In other words, every circuit \( C \) of a P-matroid is such that \( C_eC_{\overline{e}} = + \) for some \( e \in E_{2n} \). Every vector is a composition of circuits and we conclude that a P-matroid contains no sign-reversing vector. The opposite direction holds as well; an oriented matroid with no sign-reversing vector is obviously a P-matroid. Hence, we can replace the word “circuit” with “vector” in Definition 5.2.

The following result proves the correctness of the definition of a P-matroid. A proof is straightforward by using condition (b) in Theorem 5.1.

**Proposition 5.3.** Let \( M \) be an oriented matroid on \( E_{2n} \), where \( S \) is a basis.

(i) If \( M \) is realizable and there exists a realization matrix \( M \) that is a P-matrix, then \( M \) is a P-matroid.
(ii) If $M$ is a realizable P-matroid, then every realization matrix $M$ is a P-matrix.

\[ \text{Example 5.1.} \] Figure 5.1 below depicts a rank 3 P-matroid $\mathcal{M}$ in terms of an oriented semisphere arrangement. Element $t_3 \in E_{2n}$ is the “line at infinity”. Each face of the arrangement corresponds to some vector with nonnegative $t_3$ component. The semispheres have their labels on the positive side. Each vertex is labeled with the corresponding circuit. The labeling is such that the top and bottom part correspond to elements in $S$ and $T$, respectively. Pairs of complementary elements are vertically aligned.

The reader is asked to verify that there is no sign-reversing circuit or vector. The P-matrix

\[
M := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}
\]

Figure 5.1: An oriented semisphere arrangement representing a P-matroid.
is a realization matrix of $\mathcal{M}$.

Todd [80] stated several alternative characterizations for P-matroids. We restate them and add a characterization in terms of vectors and covectors, which corresponds to condition (c) in Theorem 5.1. The labeling in our result, Theorem 5.4 below, is in one-to-one correspondence with the labeling in the algebraic version. Conditions (a*), (b*), (b’*), and (c*) are conditions on the dual matroid.

For the remainder of this section, we assume that $E_{2n}$ has assigned order $s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$. Any $B \in (E_{2n}^n)$ is in natural order if it is in order with $E_{2n}$. A sign vector $X$ on $E_{2n}$ is almost-complementary if $X$ contains both $e$ and $\overline{e}$ for at most one $e \in E_{2n}$.

**Theorem 5.4.** Let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where $S$ is a basis. The following conditions are equivalent.

(a) Every complementary $B \in E_{2n}^n$ is a basis. Furthermore, there is a basis orientation $\chi$ such that $\chi(B) = (-1)^{|B \cap S|}$ for all complementary $B \in (E_{2n}^n)$ in natural order.

(b’) There is no almost-complementary sign-reversing circuit.

(b) There is no sign-reversing circuit, i.e., oriented matroid $\mathcal{M}$ is a P-matroid.

(c) Every totally sign-preserving vector on $E_{2n}$ is a vector of $\mathcal{M}$.

(d) Every OMCP($\mathcal{M}$), where $\mathcal{M}$ is an extension of $\mathcal{M}$, has exactly one solution.

(a*) Every complementary $N \in E_{2n}^n$ is a cobasis. Furthermore, there is a cobasis orientation $\chi^*$ such that $\chi^*(N) = +$ for all complementary $N \in (E_{2n}^n)$ in natural order.

(b’*) There is no almost-complementary sign-preserving cocircuit.

(b*) There is no sign-preserving cocircuit.

(c*) Every totally sign-reversing vector on $E_{2n}$ is a covector of $\mathcal{M}$.

The equivalence of conditions (a), (a*), (b’), (b’*), (b), (b*), and (d) was proven by Todd [80]. Morris [59] proved that (b) implies (c). We will
prove the whole package shortly. As a start, we illustrate these conditions on our example.

**Example 5.1 (continued).** Figure 5.2 below depicts the dual $\mathcal{M}^*$ of the previously given P-matroid $\mathcal{M}$. It satisfies the conditions (a*), (b*), (b*), and (c*). Element $s_3 \in E_{2n}$ is the “line at infinity”. Each face of the arrangement corresponds to some covector of $\mathcal{M}$ with nonnegative $s_3$ component. The semispheres have their labels on the negative side. Each vertex is labeled with the corresponding cocircuit of $\mathcal{M}$. For the moment, the dashed semisphere is to be neglected.

![Figure 5.2: An oriented semisphere arrangement representing the dual of a P-matroid.](image)

The oriented semisphere arrangements were obtained as follows. Recall that the vectors of $\mathcal{M}$ are given by

\[ \mathcal{V} := \left\{ \text{sign } x : [I_n \hspace{1em} -M] x = 0 \right\}. \]

Hence, the covectors of $\mathcal{M}$ are determined by the rowspace of the realizing
vector configuration \([I_n \ -M]\); i.e., they are given by
\[
\mathcal{V}^* := \left\{ \text{sign } y : \begin{bmatrix} M^T & I_n \end{bmatrix} y = 0 \right\}.
\]

The normal vectors of the semispheres representing \(\mathcal{M}\) are the columns in \([M^T \ I_n]\). Conversely, the normal vectors of the semispheres representing the dual \(\mathcal{M}^*\) are the columns in \([I_n \ -M]\).

Now, let us consider a realizable extension \(\widehat{\mathcal{M}} = (\widehat{E}_{2n}, \widehat{\mathcal{V}})\) of \(\mathcal{M}\), i.e., we have
\[
\widehat{\mathcal{V}} := \left\{ \text{sign } x : \begin{bmatrix} I_n & -M & -q \end{bmatrix} x = 0 \right\}
\]
for some \(q \in \mathbb{R}^n\). The covectors of \(\widehat{\mathcal{M}}\) are given by
\[
(\widehat{\mathcal{V}})^* := \left\{ \text{sign } y : \begin{bmatrix} M^T & I_n & 0 \\ q^T & 0 & 1 \end{bmatrix} y = 0 \right\}.
\]

Analogously to the previous remarks, the columns of the vector configuration realizing \((\widehat{\mathcal{M}})^*\) are the normal vectors of the oriented semispheres representing \(\widehat{\mathcal{M}}\). Unfortunately, this time the arrangement is in \(\mathbb{R}^4\), and we are not able to depict it nicely. Depicting an arrangement representing \((\widehat{\mathcal{M}})^*\), however, is still possible. It coincides with the arrangement representing \(\mathcal{M}^*\), except that an additional oriented semisphere with normal vector \(q\) is added. The dashed semisphere in Figure 5.2 belongs to the extension given by
\[
q := \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.
\]

Next, we prove Theorem 5.4. Our proof may be different from Todd’s and exploits oriented matroid duality.

**Proof of Theorem 5.4.** Let \(\mathcal{M}\) be an oriented matroid on \(E_{2n}\), where \(S\) is a basis.

(a) \(\iff\) (a*). By Corollary 3.32, any pair of dual matroids satisfies
\[
\chi^*(N) = \text{sign}(\pi(B, N))\chi(B)
\]
for basis \(B \in (E_{2n}^n)\) and ordered cobasis \(N := E_{2n} \setminus B\). If in addition \(B\) and \(N\) are complementary and in natural order, then \(\text{sign}(\pi(B, N)) = (-1)^{|B \cap S|}\). The equivalence directly follows.
5.1 P-matroids

(a) $\iff$ (b'). Note that if (a) or (b') holds, then every complementary set in $E_{2n}$ is a basis. Consider circuit $C := C(B, e)$ for any complementary basis $B \in (E_{2n})$ in natural order and $e \in E_{2n} \setminus B$. The signs $C_e$ and $C_{\overline{e}}$ are determined by $\chi(B : e \to \overline{e}) = C_e C_{\overline{e}} \chi(B)$. Observe that $C$ is not sign-reversing if and only if $C_e C_{\overline{e}} = +$; thus, if and only if $\chi(B : e \to \overline{e}) \chi(B) = -$. In other words, circuit $C$ is not sign-reversing if and only if $B$ and adjacent $B : e \to \overline{e}$ are bases and have opposite orientation. Since $B$ and $e$ were arbitrarily chosen, the equivalence follows.

(b') $\iff$ (b). Observe that (b') $\iff$ (b) obviously holds. For the other direction, assume that (b') holds. Then every complementary set in $E_{2n}$ is a basis. For the sake of a contradiction, suppose that there exists a sign-reversing circuit $C$. Then $C$ is not almost-complementary. Let $B \in E_{2n}$ be any complementary basis containing all $f \in C$, where $\overline{f} \not\in C$. Let $e$ be any element with $e, \overline{e} \in C$ and $\overline{e} \in B$. We can eliminate some element from $C$ and $D := C(B, e)$ by applying (V4c) with $G = \{f : C_f = -D_f$ and $\overline{f} \in C\} \subseteq D(C, D)$. By assumption, almost-complementary $D$ is not sign-reversing; thus, set $G$ is nonempty. Verify that the resulting vector $Z$ is sign-reversing and $Z$ contains strictly less complementary pairs than $C$. Successively apply this kind of elimination to the most recently obtained $Z$ and accordingly constructed $D$. At some point, we get a sign-reversing vector that is almost-complementary. Basically, it is a circuit and contradicts (b').

(b) $\iff$ (c*). Assume that there is no sign-reversing circuit, i.e., every circuit $C$ is such that $C_e C_{\overline{e}} = +$ for some $e \in E_{2n}$. Hence, any totally sign-reversing vector is orthogonal to every circuit, and therefore is a covector. For the opposite direction, suppose that there is a sign-reversing circuit $C$. Any totally sign-reversing vector $Y$ with $C^+ \subseteq Y^+$ and $C^- \subseteq Y^-$ is not orthogonal to $C$, and therefore not a covector.

(a*) $\iff$ (b'*) $\iff$ (b*) $\iff$ (c). For a proof of these equivalences notice that $\mathcal{M}$ satisfies (a*) if and only if $-S(\mathcal{M}^*)$ satisfies (a); analogously for (b'*) and (b'), (b*) and (b), (c*) and (c). Thus, if $\mathcal{M}$ satisfies (a*), then $-S(\mathcal{M}^*)$ satisfies (a), and also (b') by the implication (a) $\implies$ (b') proven above. Consequently, $\mathcal{M}$ satisfies (b'*). This shows (a*) $\implies$ (b'*)$. For the other implications, analogous argumentation applies.

By the implications and equivalences proven so far, the conditions (a),
(b'), (b), (c), (a*), (b*), (b'), and (c*) are equivalent. It remains to prove equivalence to condition (d).

(b) \implies (d). The proof is done by induction on minors. For the base case, we verify that the implication holds for any oriented matroid on \{s_1, t_1\}, where s_1 is a basis. By induction hypothesis, the implication holds for all minors \(\mathcal{M}\setminus e/\overline{e}\) for \(e \in E_{2n}\). Assume that there is no sign-reversing circuit. Then there is no sign-reversing circuits in any minor \(\mathcal{M}\setminus e/\overline{e}\) for \(e \in E_{2n}\). Thus, the minors satisfy (d). Let \(\widehat{\mathcal{M}}\) on \(\widehat{E}_{2n}\) be any extension of \(\mathcal{M}\). Then \(\mathcal{M}\setminus e/\overline{e}\) for any \(e \in E_{2n}\) is an extension of \(\mathcal{M}\setminus e/\overline{e}\), and contains exactly one complementary circuit \(C'\) with \(C' \geq 0\) and \(C'_q = +\). By analogous argumentation, minor \(\mathcal{M}\setminus e/\overline{e}\) contains exactly one positive complementary circuit \(D'\) with \(D'_q = +\). Let \(C\) and \(D\) be the circuits in \(\widehat{\mathcal{M}}\) with \(C_e = 0\), \(C'\{e, \overline{e}\} = C'\) and \(D_{\overline{e}} = 0\), \(D'\{e, \overline{e}\} = D'\), respectively. We are done by proving that exactly one of them is positive; i.e., a solution to the OMCP(\(\widehat{\mathcal{M}}\)). For the sake of a contradiction, suppose that \(C_eD_e \in \{+, 0\}\). Elimination of \(q\) from \(C\) and \(-D\) by applying (V4) yields a sign-reversing vector.

(d) \implies (b'). Assume that (d) holds. First, we prove that every complementary set in \(E_{2n}\) is a basis of \(\mathcal{M}\).

Note that \(\mathcal{M}\) does not contain any positive complementary circuit \(C'\). Otherwise, the extension by a loop \(q\) would contain two positive complementary circuits, namely the single-element circuit \(D\) with \(D_q = +\) and \(C \circ D\), where \(C\) is the vector of \(\widehat{\mathcal{M}}\) with \(C_q = 0\) and \(C'q = C'\).

For the sake of a contradiction, suppose that some complementary \(B \in E_{2n}^n\) is not a basis of \(\mathcal{M}\), i.e., there exists a circuit \(C'\) with \(C' \subseteq B\) and \(C'_e = +\) for some \(e \in B\). Then \(C'_e\) is independent. Consider the lexicographic extension \(\widehat{\mathcal{M}} := \mathcal{M}[C'\setminus e, -1]\). By Lemma 3.42, extension \(\widehat{\mathcal{M}}\) contains the positive complementary circuit \(D\) with \(D^+ = (C\setminus e) \cup q\). Let \(C\) be the circuit in \(\widehat{\mathcal{M}}\) with \(C_q = 0\) and \(C'q = C'\). Eliminate an element from \(D\) and \(C\) by applying (V4c) with \(G = D(D, C)\). By the previous remarks, circuit \(C\) has at least one negative entry, and thus \(G\) is not empty. The resulting vector is positive, complementary, and different from \(D\). Hence, there exist at least two solutions to the OMCP(\(\widehat{\mathcal{M}}\)), which is a contradiction.

Next, assume that there is an almost-complementary sign-reversing
circuit \( C' := C(B,e) \) in \( \mathcal{M} \). Since every complementary set in \( E_{2n}^n \) is a basis, we have \( C'_e = - \). Consider the lexicographic extension \( \mathcal{M} := \mathcal{M}[B,-1] \), which contains the positive complementary circuit \( D = \hat{C}(B,q) \) with \( D^+ = B \cup q \). Let \( C \) be the circuit in \( \mathcal{M} \) with \( C_q = 0 \) and \( C \setminus q = C' \). Eliminate \( \bar{e} \) from \( D \) and \( C \) by applying (C4s) with distinct element \( q \). If the resulting \( Z \) is positive, then both \( D \) and \( Z \) are solutions to the OMCP(\( \mathcal{M} \)) and we are done. It remains to prove that \( Z \) is actually positive. Note that \( B' := (B \setminus \bar{e}) \cup e \) is a basis of \( \mathcal{M} \). Consider the cocircuit \( Y := (\hat{C})^*(N,f) \), where \( N := \hat{E}_{2n} \setminus B' \) and \( f \in B \setminus \bar{e} \). By Lemma 3.41, we have \( Y_f Y_q = - \). Since \( Z \) and \( Y \) are orthogonal, we must have \( Z_f Z_q = + \), and thus \( Z_f = + \).

Next, we discuss relabelings and reorientations applied to P-matroids. It is well-known that P-matrices are closed under principal pivot transforms [83]. The proof is not very difficult, but uses involved properties of the Schur complement. In the setting of oriented matroids, the equivalent is much easier to prove.

A permutation \( \pi \) of \( E_{2n} \) is proper if \( \pi(e) = \pi(\bar{e}) \) for every \( e \in E_{2n} \). An \( F \subseteq E_{2n} \) is proper if \( e \in F \) implies \( \bar{e} \in F \) for every \( e \in E_{2n} \).

**Proposition 5.5.** Let \( \mathcal{M} \) be a P-matroid on \( E_{2n} \), \( \pi \) a permutation of \( E_{2n} \), and \( F \subseteq E_{2n} \), both proper. Then the following holds.

(i) The relabeling \( \pi \cdot \mathcal{M} \) is a P-matroid.

(ii) The reorientation \( -F \mathcal{M} \) is a P-matroid.

**Proof.** Any sign vector \( X \) on \( E_{2n} \) is sign-reversing if and only if the relabeled vector \( \pi \cdot X \) is sign-reversing; similarly, if and only if the reoriented vector \( -F X \) is sign-reversing. \( \square \)

In the realizable case, a proper permutation \( \pi \) such that \( \pi(e) \in \{e,\bar{e}\} \) for all \( e \in E_{2n} \) describes a principal pivot transform. Then the relabeling \( \pi \cdot \mathcal{M} \) is realized by a principal pivot transform of any realization matrix of \( \mathcal{M} \).

**Definition 5.6.** Two oriented matroids on \( E_{2n} \) are FF-equivalent if they can be transformed into each other through a combination of operations in Proposition 5.5.
Here, the expression “FF-equivalent” stands for “FacetFlip-equivalent” and will become clearer in part II of the thesis. Note that any two FF-equivalent oriented matroids belong to the same reorientation class. An oriented sphere arrangement representing the relabeling $\pi \cdot \mathcal{M}$ is obtained by relabeling the spheres in an arrangement representing $\mathcal{M}$ according to $\pi$. Similarly, an arrangement for $\mathcal{-F M}$ is obtained by exchanging the positive and negative sides of all spheres contained in $F$.

Every principal submatrix of a P-matrix, as well as of its principal pivot transforms, is again a P-matrix. We define the equivalent for oriented matroids, and formulate the generalization of this algebraic result.

**Definition 5.7.** Let $\mathcal{M}$ be a P-matroid on $E_{2n}$ and $F$ a complementary subset of $E_{2n}$. The minor $\mathcal{M}\backslash F/F$ is a principal minor of $\mathcal{M}$.

**Theorem 5.8** (Todd [80]). Every principal minor of a P-matroid is a P-matroid.

**Proof.** Let $\mathcal{M} = (E_{2n}, C)$ be a P-matroid on $E_{2n}$ and $F$ any complementary subset of $E_{2n}$. The principal minor $\mathcal{M}\backslash F/F$ contains circuits $C \backslash (F \cup \overline{F})$ for $C \in \mathcal{C}$ with $C_F = 0$. Every such $C \backslash (F \cup \overline{F})$ is sign-reversing if and only if $C$ itself is sign-reversing. \hfill $\square$

### 5.2 Z-matroids

We examine Z-matrices for the purpose of afterwards defining K-matroids. The corresponding oriented matroid generalizations are Z-matroids, whose definition was first proposed in [50].

Recall that a Z-matrix is a matrix whose off-diagonal elements are nonpositive.

**Definition 5.9.** A matroid $\mathcal{M} = (E_{2n}, C)$, where $S$ is a basis, is a Z-matroid if for every circuit $C \in \mathcal{C}$, we have:

\[
\text{if } C_T \geq 0, \text{ then } C_e = + \text{ for all } e \in S \text{ with } C_e = +.
\]  

(5.1)
The definition of Z-matroids in terms of vectors is analogous. Indeed, if any conformal composition of circuits violates (5.1), then there is always a contributing circuit violating it as well.

**Proposition 5.10.** Let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where $S$ is a basis.

(i) If $\mathcal{M}$ is realizable and there exists a realization matrix $M$ that is a Z-matrix, then $\mathcal{M}$ is a Z-matroid.

(ii) If $\mathcal{M}$ is a realizable Z-matroid, then every realization matrix $M$ is a Z-matrix.

**Proof.** We fix $E_{2n} = [2n]$ and $S = [n]$, where the complement of each $i \in S$ is the element $i + n$.

(i). Let $e_i$ denote the $i$th unit vector and $m_j$ the $j$th column of the matrix $M$. The sign pattern of the Z-matrix $M$ implies that there is no linear combination of the form

$$e_i + \sum_{j=1}^{n} x_j e_j - \sum_{j=n+1}^{2n} x_j m_j = 0,$$

where $x_j \geq 0$ for every $j > n$ and $x_{i+n} = 0$, because the $i$th entry of the left-hand side is strictly positive. Hence, there is no vector $X$ in $\mathcal{M}$ with $X_i = +$, $X_T \geq 0$, and $X_{i+n} = 0$ for any $i \in S$.

(ii). Assume that for a realizable Z-matroid $\mathcal{M}$, where $S$ is a basis, there is a realization matrix $M$ that is not a Z-matrix, i.e., there is an off-diagonal $m_{ij} > 0$. If so, there is a vector $X$ in $\mathcal{M}$ with $X_{j+n} = +$, $X_{T \setminus \{j+n\}} = 0$, and $X_i = +$. Since $X_{i+n} = 0$, such $X$ violates the Z-matroid property (5.1), which is a contradiction. Thus, no $m_{ij} > 0$ exists and $M$ has to be a Z-matrix.

In proofs, we often make use of fundamental circuits. Note that all fundamental circuits with respect to basis $S$ of a Z-matroid follow the same sign pattern.

**Lemma 5.11.** Let $\mathcal{M}$ be a Z-matroid on $E_{2n}$ and $C := C(S, e)$ for any
e ∈ T. Then

\[ C_e = +, \]
\[ C_{T \setminus e} = 0, \]
\[ C_{S \setminus \overline{e}} \leq 0. \]

**Proof.** The first and the second equality follow directly from the definition of a fundamental circuit. Thus \( C_T \geq 0 \), and the third property follows from the Z-matroid property (5.1).

Note that the conditions in the lemma are also sufficient for an oriented matroid to be a Z-matroid. Another option is to characterize a Z-matroid with respect to the dual matroid.

**Proposition 5.12.** An oriented matroid \( M \) on \( E_{2n} \), where \( S \) is a basis, is a Z-matroid if and only if for every cocircuit \( D \in C^* \), we have:

\[ \text{if } D_S \leq 0, \text{ then } D_{\overline{e}} = - \text{ for all } e \in T \text{ with } D_e = +. \] (5.2)

**Proof.** First, we prove the “only if” direction. Suppose that there is a cocircuit \( D \) that does not satisfy (5.2). Accordingly, \( D_S \leq 0 \) and there is \( e \in T \) with \( D_e = + \), but \( D_{\overline{e}} = 0 \). Then, the fundamental circuit \( C := C(S, e) \) and \( D \) are not orthogonal because \( C_{S \setminus \{e\}} \leq 0 \) by Lemma 5.11. Hence, no such \( D \) exist.

For the “if” direction, suppose that there is a circuit \( C \) with \( C_T \geq 0 \) and \( C_e = + \), but \( C_{\overline{e}} = 0 \) for some \( e \in S \). Such circuit \( C \) and the cocircuit \( D := -C^*(T, e) \) are not orthogonal because by assumption, property (5.2) is satisfied by \( D \), i.e., we have \( D_{T \setminus \{\overline{e}\}} \leq 0 \).

\[ \Box \]

### 5.3 K-matroids

The set of K-matrices is the intersection of the Z- and P-matrices. Starting with algebraic characterizations of K-matrices, which were given by Fiedler and Pták [21], we extract the combinatorics and derive properties
of the oriented matroid counterparts. The results in this section have been published [26].

**Definition 5.13.** A *K-matroid* is a P-matroid that is a Z-matroid as well.

By combining Proposition 5.3 and Proposition 5.10, we immediately get the following.

**Proposition 5.14.** Let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where $S$ is a basis.

(i) If $\mathcal{M}$ is realizable and there exists a realization matrix $M$ that is a $K$-matrix, then $\mathcal{M}$ is a K-matroid.

(ii) If $\mathcal{M}$ is a realizable K-matroid, then every realization matrix $M$ is a $K$-matrix.

The main result of this section is the combinatorial generalization of the Fiedler–Pták Theorem given next.

**Theorem 5.15** (Fiedler–Pták [21]). Let $\mathcal{M}$ be a Z-matrix. Then the following conditions are equivalent.

(a) There exists a vector $x \geq 0$ with $Mx > 0$.

(b) There exists a vector $x > 0$ with $Mx > 0$.

(c) The inverse $M^{-1}$ exists and $M^{-1} \geq 0$.

(d) Every nonzero $x \in \mathbb{R}^n$ satisfies $x_i (Mx)_i > 0$ for some $i \in [n]$, i.e., matrix $M$ is a P-matrix.

We derive oriented matroid counterparts of the conditions in Theorem 5.15. If the oriented matroid in question is realizable, then our conditions are equivalent to the conditions on the realizing matrix. In general, however, our theorem is stronger because it applies to non-realizable oriented matroids, too. As a by-product, our procedure yields a new, purely combinatorial proof of Theorem 5.15. Rather than on algebraic properties, it relies heavily on oriented matroid duality.

**Theorem 5.16.** Let $\mathcal{M}$ be a Z-matroid on $E_{2n}$. Then the following conditions are equivalent.
(a) There is a vector \( X \) with \( X_T \geq 0 \) and \( X_S > 0 \).

(b) There is a vector \( X \) with \( X > 0 \).

(c) For every circuit \( C \), we have \( C_S \geq 0 \) implies \( C_T \geq 0 \).

(d) There is no sign-reversing circuit, i.e., oriented matroid \( \mathcal{M} \) is a P-matroid.

(a*) There is a covector \( Y \) with \( Y_S \leq 0 \) and \( Y_T > 0 \).

(b*) There is a covector \( Y \) with \( Y_S < 0 \) and \( Y_T > 0 \).

(c*) For every cocircuit \( D \), we have \( D_T \geq 0 \) implies \( D_S \leq 0 \).

(d*) There is no sign-preserving cocircuit.

In order to use duality in the proof of this theorem, let us first define the reflection of an oriented matroid \( \mathcal{M} = (E_{2n}, \mathcal{V}) \) to be the pair \( \mathcal{R}(\mathcal{M}) = (E_{2n}, \mathcal{R}(\mathcal{V})) \), where \( \mathcal{R}(\mathcal{V}) := \{ \mathcal{R}(X) : X \in \mathcal{V} \} \) with

\[
(\mathcal{R}(X))_e := \begin{cases} 
X_e & \text{if } e \in S, \\
-X_{\bar{e}} & \text{if } e \in T.
\end{cases}
\]

The reflection \( \mathcal{R}(\mathcal{M}) \) is obviously an oriented matroid. Furthermore, we have \( \mathcal{R}(\mathcal{R}(\mathcal{M})) = \mathcal{M} \) because of vector axiom (V2) and \( \mathcal{R}(\mathcal{M}^*) = \mathcal{R}(\mathcal{M})^* \). Thus, we observe that

\[
\mathcal{R}(\mathcal{R}(\mathcal{M}^*)) = \mathcal{M}.
\]

**Proof of Theorem 5.16.** (a) \( \implies \) (b). Let \( X \) be as in (a). Since \( X_T \geq 0 \), the Z-matroid property (5.1) applies. As \( X_S > 0 \), we have \( X_T > 0 \).

(b) \( \implies \) (c). Let \( X \) be as in (b). Suppose that there is a circuit \( C \in \mathcal{C} \) not satisfying (c), i.e., we have \( C_S \geq 0 \), but \( C_e = - \) for at least one element \( e \in T \). We can eliminate an element from \( X \) and \( C \) by applying (V4c) with \( G = D(X, C) \). The resulting vector \( Z \) is such that \( Z_S > 0 \), but \( Z_e = 0 \) for eliminated \( e \in T \), contradicting the Z-matroid property (5.1).

(c) \( \implies \) (d). Suppose that there is a sign-reversing circuit \( C \in \mathcal{C} \), i.e., \( C_e C_{\bar{e}} \leq 0 \) for every \( e \in S \). Let \( Z^0 := C \). We apply successive circuit eliminations (C4). To get \( Z^i \), we eliminate \( e \in T \) with \( Z^i_{e-1} = - \) from \( Z^{i-1} \) and \( D := C(S, e) \). By Lemma 5.11, we have \( D_{S \setminus e} \leq 0 \). After finitely
many eliminations, we end up with a circuit \( Z^k \) with \( Z^k_T \geq 0 \). We claim that \( Z^k_S \leq 0 \). Indeed, if \( e \in S \) with \( Z^k_e = + \), then \( C_e = + \), and thus \( C_e \leq 0 \) because \( C \) is supposed to be sign-reversing. Since we never eliminate at \( e \), all circuits \( D \) used in the eliminations satisfy \( D_e \leq 0 \) as noted above. Hence \( Z^k_e \leq 0 \). On the other hand, if \( Z^k_e = 0 \), then \( Z^k_e \leq 0 \) by (5.1).

Moreover, since \( S \) is a basis, \( Z^k_e \not\in S \), and so there exists \( e \in T \) with \( Z^k_e = + \). Hence, we have \( Z^k_T \geq 0, Z^k_e = + \) for some \( e \in T \), and \( Z^k_S \leq 0 \); i.e., \(-Z^k\) violates property (c).

(d) \( \Rightarrow \) (a*). Assume that (d) holds. For every circuit \( C \) there is an \( e \in S \) with \( C_e C_e = + \). The sign vector \( Y \) with \( Y_S < 0 \) and \( Y_T > 0 \) is orthogonal to every circuit \( C \) because \( C_e C_e = -Y_e Y_e \). Hence such \( Y \) is a covector.

To finish the proof, notice that if \( M \) is a Z-matroid, then the reflection of its dual \( \mathcal{R}(M^*) \) is a Z-matroid. Furthermore, a Z-matroid \( M \) satisfies (a*) if and only if the \( \mathcal{R}(M^*) \) satisfies (a); analogously for (b*) and (b), (c*) and (c), and (d*) and (d). Thus if \( M \) satisfies (a*), then \( \mathcal{R}(M^*) \) satisfies (a), and also (b) by the implication (a) \( \Rightarrow \) (b) proven above. Consequently, \( M \) satisfies (b*). The missing implications (b*) \( \Rightarrow \) (c*), (c*) \( \Rightarrow \) (d*), and (d*) \( \Rightarrow \) (a) are proven analogously.

Principal pivot transforms of K-matrices yield P-matrices, but not necessarily Z-matrices. We can still prove the following property.

**Lemma 5.17.** Every principal minor of a K-matroid is a K-matroid.

**Proof.** Let \( M = (E_{2n}, \mathcal{C}) \) be a K-matroid. By Theorem 5.8, every principal minor of a P-matroid is a P-matroid. Thus, it is enough to prove that any principal minor \( M\setminus F/F \) for complementary \( F \subseteq E_{2n} \) is a Z-matroid, and for this, since deletions and contractions can be carried out element by element in any order, it suffices to consider the case where \( F \) is a singleton.

First, we prove that for \( e \in T \), principal minor \( M\setminus e/\overline{e} \) is a Z-matroid. The principal minor contains circuits \( C\setminus \{e, \overline{e}\} \), where \( C \in \mathcal{C} \) satisfies \( C_e = 0 \). Since every circuit in \( M \) satisfies the Z-matroid property (5.1), such circuit trivially satisfies it as well.

Secondly, let \( e \in S \) and consider minor \( M\setminus e/\overline{e} \). We apply case distinction. If \( C_{\overline{e}} = + \) for a circuit \( C \) in \( M \) with \( C_e = 0 \), then \( (C\setminus \{e, \overline{e}\})_T \geq 0 \) if and only if \( C_T \geq 0 \). As a direct consequence, \( C\setminus \{e, \overline{e}\} \) satisfies (5.1)
because $C$ does. If $C_e = -$, we can prove that there is another element $f \in T$ such that $C_f = -$ as well, i.e., we have $(C \setminus \{e, \bar{e}\})_T \not\geq 0$ and thus property (5.1) is obviously satisfied. For the sake of contradiction, suppose that there is no such $f \in T$. Consider $D := C(S, \bar{e})$, which is by the P-matroid property such that $D_{\bar{e}} = D_e = +$. Elimination of $\bar{e}$ from $D$ and $C$ by applying (C4s) with distinct element $e$ yields some circuit $Z$ with $Z_T \geq 0$, $Z_{\bar{e}} = 0$, and $Z_e = +$. Since $e \in S$, such $Z$ violates the Z-matroid property (5.1), which is a contradiction. \hfill \qed

### 5.4 Hidden Z-matroids

The class of hidden Z-matrices was introduced by Mangasarian [51] in connection with solving LCPs as linear programs.

Recall that a matrix $M \in \mathbb{R}^{n \times n}$ is a hidden Z-matrix if there exist Z-matrices $A, B \in \mathbb{R}^{n \times n}$ such that

\[
MA = B, \quad \text{and} \quad r^T A + s^T B > 0 \quad \text{for some } r, s \geq 0.
\]

The quadruple $(A, B, r, s)$ is a certificate for $M$ being a hidden Z-matrix.

Mangasarian [51] proved that if an LCP$(M, q)$, where $M$ is a hidden Z-matrix, is feasible, then the linear complementarity problem has a solution that is obtained by solving the linear program

\[
\begin{aligned}
\min \quad & p^T z \\
\text{s.t.} \quad & w - Mz = q, \\
& w, z \geq 0,
\end{aligned}
\]

where $p := r + M^T s$.

Every Z-matrix is a hidden Z-matrix. The definition of hidden Z-matrices may not look combinatorial. Nevertheless, it is possible to generalize the hidden Z-matrices in the setting of oriented matroids by looking
at the dual setting. The hidden Z-matrices themselves generalize the fol-
lowing property satisfied by all Z-matrices.

**Lemma 5.18.** Let $M \in \mathbb{R}^{n \times n}$ be a Z-matrix. If $M^T w + z > 0$ for $w, z \geq 0$, then $z_i > 0$ or $w_i > 0$ for each $i \in [n]$. Furthermore, there exists at least one such positive linear combination.

**Proof.** Let $w, z \geq 0$ such that $M^T w + z > 0$. Since each $(M^T w + z)_i > 0$ and every off-diagonal element of $M^T$ is nonpositive, we must have $z_i > 0$ or $w_i > 0$. Existence is proven by $w := 0$ and $z := 1$. \qed

This necessary condition for Z-matrices is used to define hidden Z-
matroids.

Let $\mathcal{M} = (E_{2n}, \mathcal{V})$ be an oriented matroid and $\tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{V}}^*)$, where $\tilde{E}_{2n} := E_{2n} \cup p$, an extension of the dual $\mathcal{M}^*$. A covector $Y \in \tilde{\mathcal{V}}^*$ is feasible if $Y \geq 0$ and $Y_p = +$.

**Definition 5.19.** Let $\mathcal{M}$ on $E_{2n}$ be an oriented matroid, where $S$ is a basis. The matroid $\mathcal{M}$ is a hidden Z-matroid if there exists an extension of the dual $\tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{V}}^*)$ such that

(i) there is at least one feasible $Y \in \tilde{\mathcal{V}}^*$ and

(ii) for every feasible $Y \in \tilde{\mathcal{V}}^*$, we have

$$Y_e = + \text{ or } Y_{\bar{e}} = + \text{ for every } e \in S.$$  

An oriented matroid $\tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{V}}^*)$ satisfying (i) and (ii) is a certificate for $\mathcal{M}$ being a hidden Z-matroid.

**Proposition 5.20.** Let $\mathcal{M}$ on $E_{2n}$ be an oriented matroid, where $S$ is a basis.

(i) If $\mathcal{M}$ is realizable and there exists a realization matrix $M$ that is a hidden Z-matrix, then $\mathcal{M}$ is a hidden Z-matroid.

(ii) If $\tilde{\mathcal{M}}^*$ is a realizable certificate for $\mathcal{M}$ being a hidden Z-matroid, then $\mathcal{M}$ is realizable and there exists a realization matrix $M$ that is a hidden Z-matrix.
Unfortunately, condition (ii) is not as strong as for previously discussed matroid classes. If \(M\) is a realizable hidden Z-matroid, then some realization matrices may not be hidden Z-matrices. The reason for this is that any two vector configurations that realize the same oriented matroid may have different extension spaces. Also see the remarks on page 51 in this thesis. An example of a hidden Z-matroid that is realized by both hidden Z-matrices and matrices that are not hidden Z is given in Section 5.6. As a consequence, the class of hidden Z-matroids properly generalizes hidden Z-matrices.

**Proof of Proposition 5.20.** (i). Let \(M\) be realized by some hidden Z-matrix \(M\). The matrix \(M\) satisfies 

\[
MA = B \quad \text{and} \quad r^T A + s^T B > 0 \quad \text{for some} \quad r, s \geq 0,
\]

where \(A = (a_i)_{i \in [n]}\) and \(B = (b_i)_{i \in [n]}\) are Z-matrices. Consider the vector subspace

\[
V := \left\{ \begin{pmatrix} w \\ z \\ t \end{pmatrix} : \begin{bmatrix} I_n & -M & 0 \\ 0 & r^T + s^T M & 1 \end{bmatrix} \begin{pmatrix} w \\ z \\ t \end{pmatrix} = 0 \right\}
\]

which contains the vector \((-b_i^T, -a_i^T, r^T a_i + s^T b_i)^T\) for each \(i \in [n]\). Hence, the oriented matroid \(N = (\tilde{E}_{2n}, \mathcal{V})\), where \(\mathcal{V} := \{\text{sign } x : x \in V\}\), contains for each \(e \in S\) a vector \(X\) with \(X_p = +\) and \(X_f, X_f^T \geq 0\) for all \(f \neq e\). Now, the contraction minors \(N/\{e, \bar{e}\}\) and their duals \(N^*/\{e, \bar{e}\}\) satisfy Farkas’ Lemma 3.35. Hence, for each \(e \in S\) there is no covector \(Y \in \mathcal{V}^*\) with \(Y \geq 0\), \(Y_e = Y_{\bar{e}} = 0\), and \(Y_p^T = +\). Note that the dual \(N^*\) is an extension of \(M^*\). It satisfies condition (ii) in Definition 5.19 by the previous remarks. Since \((s^T, r^T, 1)^T\) is a vector belonging to the orthogonal space \(V^\perp\), oriented matroid \(N^*\) satisfies condition (i) as well; thus, it is a certificate for \(M\) being a hidden Z-matroid.

(ii). Let \(N^* = (\tilde{E}_{2n}, \mathcal{V}^*)\) be a realizable certificate for \(M\) being a hidden Z-matroid. Suppose that the vector configuration \([M^T \quad I_n \quad -p]\) is a realization of \(N^*\). Since \(N^*\) satisfies conditions (i) and (ii) in Definition 5.19, no deletion minor \(N^*\backslash\{e, \bar{e}\}\) for \(e \in S\) contains any vector \(Y'\) with \(Y' \geq 0\) and \(Y_p' = +\). By Farkas’ Lemma 3.35, the duals \(N/\{e, \bar{e}\}\) must
contain $X'$ with $X' \geq 0$ and $X'_p = +$. Hence, matroid $\mathcal{N}$ contains for each $e \in S$ a vector $X$ with $X_p = +$ and $X_f, X_f' \geq 0$ for all $f \neq e$. Oriented matroid $\mathcal{N}$ is realized by

$$V := \left\{ \begin{pmatrix} w \\ z \\ t \end{pmatrix} : \begin{bmatrix} I_n & -M & 0 \\ 0 & p^T & 1 \end{bmatrix} \begin{pmatrix} w \\ z \\ t \end{pmatrix} = 0 \right\}.$$ 

The vector subspace $V$ contains a vector $(w^T_i, z^T_i, t_i)^T$ for each $i \in [n]$ that satisfies $(w_i)_j, (z_i)_j \geq 0$ for every $j \neq i$ and $t_i > 0$. Define Z-matrices $B := [-w_1, -w_2, \ldots, -w_n]$ and $A := [-z_1, -z_2, \ldots, -z_n]$. By the structure of $V$, we conclude that $MA = B$ and that $p^T A > 0$ because the $t_i$'s are all positive. Since $\mathcal{N}^*$ satisfies condition (i), the orthogonal space $V^\perp$ contains a feasible vector $(s^T, r^T, 1)^T$. Hence, vector $p$ can be written as $r + MTs$, and consequently $p^T A = r^T A + s^T MA = r^T A + s^T B$. Since $p^T A > 0$, we have $r^T A + s^T B > 0$. The tuple $(A, B, s, r)$ is a certificate for $M$ being a hidden Z-matrix.

We have already seen that every Z-matrix is a hidden Z-matrix. We give the generalization in the setting of oriented matroids.

**Lemma 5.21.** Every Z-matroid is a hidden Z-matroid.

**Proof.** Let $\mathcal{M}$ be a Z-matroid on $E_{2n}$. Since $S$ is a basis, elements $T$ build a cobasis. We consider the lexicographic extension $\tilde{\mathcal{M}}^* := \mathcal{M}^*[T, -1]$ of the dual by an element $p$. By Lemma 3.42, the cocircuit $D := \tilde{C}^*(T, p)$ is positive with $D^+ = T \cup p$. By construction, the extension satisfies condition (i) in Definition 5.19. Furthermore, it also satisfies condition (ii). For the sake of a contradiction, suppose that there is a feasible covector $Y \in \tilde{V}^*$ with $Y_e = Y_{\overline{e}} = 0$ for some $e \in T$. Eliminate $p$ from $D$ and $-Y$ by applying (V4s) with distinct element $e$. The resulting covector $Z$ is such that $Z_S \leq 0$, $Z_e = +$, and $Z_{\overline{e}} = 0$. Since $Z_p = 0$, sign vector $Z \setminus p$ is contained in $\mathcal{M}^* = \tilde{\mathcal{M}}^* \setminus p$. According to condition (5.2), such $Z \setminus p$ cannot belong to any dual of a Z-matroid. A contradiction is established. Hence, every feasible vector of $\tilde{\mathcal{M}}^*$ satisfies condition (ii) in Definition 5.19. Oriented matroid $\tilde{\mathcal{M}}^*$ is a certificate for $\mathcal{M}$ being a hidden Z-matroid. □
In the remainder of this section, we investigate oriented matroid operations applied to hidden Z-matroids. To start with, recall the notion of a relabeling.

**Proposition 5.22.** Let $M$ be a hidden Z-matroid on $E_{2n}$ and $\pi$ a proper permutation of $E_{2n}$. Then the relabeling $\pi \cdot M$ is a hidden Z-matroid.

**Proof.** Let $\tilde{M}^*$ be certificate for $M$ being a hidden Z-matroid. The relabeling $\pi \cdot \tilde{M}^*$ still satisfies conditions (i) and (ii) in Definition 5.19, since $\pi$ is proper. Furthermore, the relabeling is an extension of $\pi \cdot M^*$ whose dual is $\pi \cdot M$. Hence, oriented matroid $\pi \cdot \tilde{M}^*$ is a certificate for the relabeling $\pi \cdot M$ being a hidden Z-matroid. $\square$

This raises the question of whether every principal minor of a hidden Z-matroid is again a hidden Z-matroid. Unfortunately, this is not the case as the following example illustrates.

**Example 5.2.** For $n = 3$, let $M$ be the oriented matroid on $E_{2n}$ which is realized by

$$M = \begin{pmatrix} 4 & 1 & 0 \\ 2 & -4 & 2 \\ -2 & -2 & -2 \end{pmatrix}.$$

The matrix $M$ is a hidden Z-matrix with certificate $(A, B, r, s)$, where

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 2 & 0 \\ -2 & -1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 8 & -2 & -4 \\ 0 & -12 & -4 \\ 0 & 0 & 4 \end{pmatrix}, \quad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad s = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

By statement (i) in Proposition 5.20, matroid $M$ is a hidden Z-matroid.

Consider the principal minor $M \setminus t_3/s_3$, which is realized by the principal submatrix

$$D = \begin{pmatrix} 4 & 1 \\ 2 & -4 \end{pmatrix}.$$

First, we show that $D$ is not a hidden Z-matrix. We claim that given any linear combination of the form $p := D^T w + I_2 z$ for $w, z \geq 0$, vector $p$ can also be expressed as a positive linear combination, where $w_1 = z_1 = 0$. The cone $C$ spanned by the columns of $D^T$ and identity matrix $I_2$ equals
the cone spanned by \((D^T)_{:2}\) and unit vector \(e_2\) only. Therefore every \(p \in C\) allows a positive linear combination, where \(w_1 = z_1 = 0\). Hence, the principal submatrix \(D\) is not a hidden Z-matrix.

The same argumentation applies to any other realization \(D' \in \mathbb{R}^{2 \times 2}\) of \(\mathcal{M} \setminus t_3/s_3\). Since every rank 2 oriented matroid on 5 elements is realizable, we conclude that \(\mathcal{M} \setminus t_3/s_3\) is not a hidden Z-matroid.

## 5.5 Hidden K-matroids

A hidden K-matrix is a P-matrix that is also a hidden Z-matrix. Pang [67] was the first to study hidden K-matrices.

**Definition 5.23.** A hidden K-matroid is a P-matroid that is also a hidden Z-matroid.

**Corollary 5.24.** Every K-matroid is a hidden K-matroid.

**Proof.** By Lemma 5.21, every Z-matroid is a hidden Z-matroid.

Furthermore, every proper relabeling, simple principal pivot transform in particular, of a hidden K-matroid is a hidden K-matroid because both hidden Z- and P-matroids are closed under proper relabelings.

### 5.5.1 Alternative hidden K-matroid characterization

Let any matrix \(M \in \mathbb{R}^{n \times n}\), where every principal submatrix is nonsingular, be given. Pang and Chandrasekaran [69] discovered that if there is a strictly positive vector \(p\) satisfying

\[
(M_{J,J})^{-1}p_J > 0 \quad \text{for all } J \subseteq [n],
\]

then Lemke’s method with \(p\) as ray will terminate on the LCP\((M, q)\) with a solution in a linear number of pivot steps in \(n\).

They show that \(p\) satisfying (5.4) exists if \(M^T\) is a hidden K-matrix. The opposite direction was proven by Morris and Lawrence [62]. If a \(p\) satisfying (5.4) exists, then \(M^T\) is a hidden K-matrix. Condition (5.4) is basically an alternative hidden K-matrix characterization. It has a combinatorial counterpart in the setting of oriented matroids.
Let $\mathcal{M} = (E_{2n}, \mathcal{C})$ be an oriented matroid and $\widetilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{C}}^*)$, where $\tilde{E}_{2n} := E_{2n} \cup p$, an extension of the dual $\mathcal{M}^*$. A cocircuit $D \in \tilde{\mathcal{C}}^*$ is **complementary** if $D_eD_{\bar{e}} = 0$ for every $e \in E_{2n}$. Cocircuit $D \in \tilde{\mathcal{C}}^*$ is **strictly-complementary** if additionally exactly one of $D_e$ and $D_{\bar{e}}$ is nonzero for each $e \in E_{2n}$.

Note that if every complementary $B \in E_{2n}^n$ is a basis of $\mathcal{M}$, then every complementary $N \in E_{2n}^n$ is a cobasis, and thus a basis of any extension $\widetilde{\mathcal{M}}^*$ of the dual.

**Theorem 5.25.** Let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where every complementary $B \in E_{2n}^n$ is a basis. The matroid $\mathcal{M}$ is a hidden $K$-matroid if and only if there exists an extension of the dual $\widetilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{C}}^*)$, such that every complementary cocircuit $D \in \tilde{\mathcal{C}}^*$ with $D_p = +$ is feasible and strictly-complementary.

In the realizable case, the theorem has a straightforward geometric interpretation. Consider a P-matrix $M \in \mathbb{R}^{n \times n}$. Let a **dual complementary cone** be a cone which is spanned by $n$ complementary columns in the vector configuration $[MT \ I]$. Now, P-matrix $M$ is a hidden $K$-matrix if and only if the intersection of all $2^n$ dual complementary cones is nonempty and has full dimension. In this exposition, we have a different interpretation in mind. For a hidden $K$-matrix $M$ and vector $p$, we consider linear programs over $MTx \leq p$. Feasible regions of this type can be thought of as arising from an extension of the vector configuration $[MT \ I]$ by an additional column $-p$. So, if $p$ is picked according to Theorem 5.25, then the feasible region defines a combinatorial cube. Any objective function induces a unique-sink orientation of the cube, where each edge points to the incident vertex with higher objective value.

In order to simplify the proof of Theorem 5.25, we first consider Lemmas 5.26 and 5.27 below.

**Lemma 5.26.** Let $\mathcal{M}$ be a hidden $K$-matroid and $\widetilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{C}}^*)$ any certificate for $\mathcal{M}$ being a hidden $Z$-matroid. If some complementary cocircuit $C \in \tilde{\mathcal{C}}^*$ with $C_p = +$ is feasible and strictly-complementary, then every other complementary cocircuit $D \in \tilde{\mathcal{V}}^*$ with $D_p = +$ is feasible and strictly-complementary.
5.5 Hidden K-matroids

Proof. Let \( N \in \mathcal{E}_{2n}^n \) be a complementary cobasis such that \( C := \tilde{C}^*(N, p) \) is feasible and strictly-complementary. Consider an adjacent cobasis \( N' := N \setminus e \cup \bar{e} \) for any \( e \in N \) and let \( D := \tilde{C}^*(N', p) \). Proving that \( D \) is feasible and strictly-complementary is enough.

First, we prove feasibility. For the sake of a contradiction, suppose that \( D \) is not feasible. We apply case distinction. First, suppose that \( D_{\bar{e}} \geq 0 \), we eliminate \( p \) from \( -C \) and \( D \) by applying (C4). The resulting almost-complementary \( Z \) is such that \( Z_p = 0 \) and \( Z_e, Z_{\bar{e}} \leq 0 \), i.e., it is sign-preserving. Cocircuit \( Z \setminus p \) is contained in \( \mathcal{M}^* = \mathcal{M}^* \setminus p \). By definition, duals of P-matroids do not contain sign-preserving vectors. A contradiction is established and we must have \( D_{\bar{e}} = + \). Secondly, suppose that \( D_{\bar{e}} = + \), but \( D_f = - \) for other \( f \in N' \). Eliminate such an element \( f \) from \( C \) and \( D \) by applying (V4c) with \( G := D(C, D) \). The resulting vector \( Z \) is feasible such that \( Z_f = Z_{\bar{f}} = 0 \). Since \( \mathcal{M}^* \) satisfies condition (ii) in Definition 5.19, such \( Z \) does not exist. A contradiction is established. Hence, cocircuit \( D \) is feasible.

Since \( D \) is feasible, it must be strictly-complementary. Otherwise, it would violate condition (ii) in Definition 5.19.

Let \( \tilde{\mathcal{M}}^* \) be a certificate for \( \mathcal{M} \) being a hidden K-matroid. By condition (i) of Definition 5.19, there is a feasible covector \( Y \in \tilde{\mathcal{V}}^* \). Furthermore, the following holds.

Lemma 5.27. Let \( \mathcal{M} \) be a hidden K-matroid and \( \tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{\mathcal{V}}^*) \) a certificate for \( \mathcal{M} \) being a hidden Z-matroid. Then every feasible covector \( Y \in \tilde{\mathcal{V}}^* \) is a conformal composition of feasible and strictly-complementary cocircuits in \( \tilde{\mathcal{C}}^* \).

Proof. Let \( D^1 \circ D^2 \circ \cdots \circ D^k \) be any conformal decomposition of \( Y \), i.e., we have \( D^i \preceq Y \) and \( D^i \in \tilde{\mathcal{C}}^* \) for each \( i \in [k] \).

First, we prove that every \( D^i \) is feasible. Positivity is given by the fact that the decomposition is conformal. Observe that each \( D^i \) is sign-preserving. Now, if \( D^i_p = 0 \), then cocircuit \( D^i \setminus p \) is contained in \( \mathcal{M}^* \), which is the dual of a P-matroid, and thus does not contain any sign-preserving vector. Hence, every \( D^i \) is feasible.

Secondly, we prove that every \( D^i \) is strictly-complementary. Since \( \tilde{\mathcal{M}}^* \) satisfies condition (ii) of Definition 5.19, at least one of \( D^i_e \) and \( D^i_{\bar{e}} \) is
positive for each element $e \in E_{2n}$. Suppose that $D^i_e = D^j_e = +$ for some $e \in E_{2n}$. If so, the cardinality of $D^i$ is at least $n + 2$, contradicting the rank $n$ of $\tilde{M}^*$.

We conclude that every $D^i$ is feasible and strictly-complementary. □

Proof of Theorem 5.25. First, we prove the “only if” direction. Let $\mathcal{M}$ be a hidden K-matroid and $\tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{V}^*)$ a certificate for $\mathcal{M}$ being a hidden Z-matroid. Let $Y \in \tilde{V}^*$ be any feasible covector, which exists by condition (i) in Definition 5.19. By Lemma 5.27, covector $Y$ can be decomposed into feasible and strictly-complementary cocircuits. Hence, there exists at least one feasible and strictly-complementary cocircuit. Lemma 5.26 concludes the proof.

For the “if” direction, let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where every complementary $B \in E_{2n}^n$ is a basis. Assume that $E_{2n}$ has assigned order $s_1 < t_1 < s_2 < t_2 < \ldots < s_n < t_n$. Suppose that some extension of the dual $\tilde{\mathcal{M}}^* = (\tilde{E}_{2n}, \tilde{V}^*)$ is such that for each complementary $N \in E_{2n}^n$, cocircuit $\tilde{C}^*(N, p)$ is feasible and strictly-complementary. We prove that $\mathcal{M} = (\tilde{\mathcal{M}}^* \setminus p)^*$ is a hidden K-matroid.

First, we verify that $\mathcal{M}$ is a P-matroid. Let $N \in (E_{2n}^n)$ and $N' := N \setminus e \cup \bar{e}$ for some $e \in N$ be any two adjacent complementary bases of $\tilde{\mathcal{M}}^*$ in natural order. The corresponding cocircuits $C := \tilde{C}^*(N, p)$ and $D := \tilde{C}^*(N', p)$ are feasible and strictly-complementary. Eliminate $p$ from $C$ and $-D$ by applying (C4). The resulting cocircuit $Z$ is such that $Z_p = 0$ and also satisfies $Z_e = +$ and $Z_{\bar{e}} = -$, otherwise $Z \subset C$ or $Z \subset D$. Cocircuit $Z' := Z \setminus p$ is contained in $\mathcal{M}^*$, and equals cocircuit $C^*(N', e)$.

The chirotope $\chi^*$ of $\mathcal{M}^*$ is such that $\chi^*(N' : e \rightarrow \bar{e}) = -Z'_e Z_{\bar{e}} \chi^*(N')$. Since $Z'_e Z_{\bar{e}} = -$ and $N' : e \rightarrow \bar{e} = N$, we conclude that $\chi^*(N) = \chi^*(N')$. This chain of argumentation applies to all pairs of adjacent complementary bases in natural order. Hence, condition (a*) in Theorem 5.4 is satisfied, i.e., oriented matroid $\mathcal{M}^*$ is the dual of a P-matroid.

Secondly, we prove that $\mathcal{M}$ is also hidden Z-matroid. Since for each complementary $N \in E_{2n}^n$, cocircuit $\tilde{C}^*(N, p)$ is feasible, condition (i) in Definition 5.19 is trivially satisfied. Furthermore, these $\tilde{C}^*(N, p)$ are vertices of the feasible region of $\tilde{\mathcal{M}}^*$ such that each of them is connected by an edge to its $n$ adjacent complementary cocircuits. So, condition (ii) holds if no such cocircuit is connected by an edge to any almost-complementary cocircuit.
\[ D \in \tilde{V}^* \text{ with } D_p = +, \; D \geq 0, \; D_e = D_\bar{\epsilon} = + \text{ for some } e \in E_{2n}, \text{ and } \; D_f = D_{\bar{f}} = 0 \text{ for some other } f \in E_{2n}. \] For such \( D \), elimination of \( e \) from \( D \) and \(-\tilde{C}^*(N, p)\), where \( N \) is complementary and contains \( D\{\bar{e}, p\} \), by applying (V4) would yield a complementary cocircuit \( Z \) with opposite signs at some positions. Hence, the feasible region of \( \tilde{M}^* \) defines a combinatorial cube. Consequently, condition (ii) in Definition 5.19 is also satisfied. Hence, oriented matroid \( M \) is a hidden Z-matroid.

The alternative hidden K characterization allows us to prove the following interesting property.

**Theorem 5.28.** Every principal minor of a hidden K-matroid is a hidden K-matroid.

**Proof.** Let \( M \) be any hidden K-matroid and \( \tilde{M}^* \) a certificate for \( M \) being a hidden Z-matroid. Consider any principal minor \( M\{F, \bar{F}\} \) for complementary \( F \subseteq E_{2n} \). The dual of the minor is \( M^*\{\bar{F}, F\} \), and has \( \tilde{M}^*\{\bar{F}, F\} \) as an extension. Since the feasible region of \( \tilde{M}^* \) defines a combinatorial cube, the feasible region of \( \tilde{M}^*\{\bar{F}, F\} \) defines a combinatorial cube. By Theorem 5.25, oriented matroid \( M\{F, \bar{F}\} \) is a hidden K-matroid. \( \square \)

### 5.6 Cyclic-P-matroids

The classes of oriented matroids studied so far have a few disadvantages. For instance, who tells us the components of a specific circuit for any given oriented matroid? A kind of oracle probably does, which in the realizable case involves solving an equation system. From an external perspective, it is difficult to understand the structure of circuits and to tell how the circuits are related to each other, which is somewhat annoying. Another disadvantage concerns extensions. The definition of an extension is trivial, but for the analysis of algorithms, we prefer to have a combinatorial description of the collection of extensions. Such a description may be useful for formulating complexity results for a class of OMCPs.

In this section, we introduce and study the class of *cyclic-P-matroids*, for which the structure of circuits follows an easy scheme and we are able to combinatorially describe the collection of extensions. At the same time,
cyclic-P-matroids are nontrivial objects, as their combinatorial structure has potential and may be suitable for the analysis of solving methods. They inherit many interesting properties of P-matroids. They are closed under principal pivot transforms, and more generally under proper relabelings and proper reorientations. Furthermore, every principal minor of a cyclic-P-matroid is again a cyclic-P-matroid. The class contains realizable oriented matroids only, and its definition immediately leads to a construction scheme for P-matrices of arbitrary order. No other scheme to construct nontrivial P-matrices is currently known.

Cyclic-P-matroids arise from point configurations. Consider the configuration \( P := \{s_1, t_1, s_2, t_2, s_3, t_3\} \) in Figure 5.3. The points were chosen so that they realize distinct points on the moment curve in the plane. More precisely, we let \( s_1 := (x_1, x_1^2), t_1 := (x_2, x_2^2), \ldots, t_3 := (x_6, x_6^2) \) for strictly increasing \( x_1 < x_2 < \ldots < x_6 \). Any partition of the point set by a hyperplane induces a sign vector that is not sign-reversing. Hence, the collection of all sign vectors obtained by hyperplane partitioning forms a P-matroid. This generalizes to any dimension. Cyclic-P-matroids of rank \( n \) are exactly those P-matroids that arise from \( 2n \) distinct points on the moment curve in \( \mathbb{R}^{n-1} \), lifted into \( \mathbb{R}^n \) and then appropriately relabeled and reoriented.
5.6 Cyclic-P-matroids

5.6.1 Alternating matroids

We start with a discussion of the well-known alternating matroids, which are realized by points on the lifted moment curve in a straightforward way.

In the following, we consider element sets $E_n := [n]$ with assigned order $1 < 2 < \ldots < n$. Any $B \in (E_n^r)$ for $1 \leq r \leq n$ is in natural order if it is in order with $E_n$.

**Definition 5.29.** Let $r \leq n$. The unique rank $r$ oriented matroid on $E_n$ whose chirotope satisfies

$$\chi(B) = + \text{ for all } B \in (E_n^r) \text{ in natural order}$$

is an alternating matroid. We denote it by $A^{n,r}$.

The alternating matroid $A^{n,r}$ is related to the moment curve in $\mathbb{R}^{r-1}$ defined as

$$m(x) : \mathbb{R} \to \mathbb{R}^{r-1}, \ x \mapsto (x, x^2, \ldots, x^{r-1}).$$

A set of $n$ consecutive points $m(x_1), m(x_2), \ldots, m(x_n)$ on the moment curve, lifted into $\mathbb{R}^r$, provides a realization. It does not matter how the points are actually picked, it is only important that they are distinct and in increasing order. More precisely, any vector configuration

$$V \in \mathbb{R}^{r \times n} := \begin{pmatrix}
1 & 1 & \cdots & 1 \\
x_1 & x_2 & \cdots & x_n \\
x_1^2 & x_2^2 & \cdots & x_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
x_1^{r-1} & x_2^{r-1} & \cdots & x_n^{r-1}
\end{pmatrix} \quad (5.5)$$

for $x_1 < x_2 < \ldots < x_n$ is a realization of $A^{n,r}$. The matrix $V$ is the transpose of a so-called Vandermonde matrix.

**Lemma 5.30.** Every alternating matroid $A^{n,r}$ is realizable.

**Proof.** Let $V = (v_i)_{i \in E_n} \in \mathbb{R}^{r \times n}$ be as in (5.5) for strictly increasing values $x_1 < x_2 < \cdots < x_n$. For any $B \in (E_n^r)$ in natural order, consider $V_B$, the submatrix of $V$ whose $j$th column is $v_k$, where $k$ is the $j$th entry in $B$. 
Then
\[ \det V_B = \prod_{1 \leq i < j \leq n, i,j \in B} (x_j - x_i) > 0. \]

Hence, vector configuration \( V \) realizes the oriented matroid whose chirotope \( \chi \) is such that \( \chi(B) = + \) for every \( B \) in natural order. \( \square \)

Alternating matroids have many interesting properties, which were first studied by Cordovil and Duchet [12]. They are obviously uniform. The circuits are well-structured. Namely, the nonzero elements alternate in signs with respect to the natural order. Theorem 5.31 below summarizes alternative characterizations.

For a sign vector \( X \) on \( E_n \) elements \( i,j \in X \) are consequent if there is no \( k \in X \) such that \( i < k < j \). Let \( G \) contain the even elements in \( E_n \).

**Theorem 5.31.** Let \( M \) be a uniform rank \( r \) oriented matroid on \( E_n \). The following conditions are equivalent.

(a) There is a basis orientation \( \chi \) with \( \chi(B) = + \) for all \( B \in (E_n^r) \) in natural order, i.e., matroid \( M \) is the alternating matroid \( A^{n,r} \).

(b) Every circuit \( C \) is such that for all consequent \( i,j \in C \), we have \( C_i = -C_j \).

(c) Every circuit \( C \) is such that \( C_i = -C_{i+1} \) for all \( i,i+1 \in C \).

\( a^* \) There is a cobasis orientation \( \chi^* \) with \( \chi^*(N) = (-1)^{|N \cap G|} \) for all \( N \in (E_n^r) \) in natural order.

\( b^* \) Every cocircuit \( D \) is such that for all consequent \( i,j \in D \), we have \( D_i = D_j \) if \( j - i \) is odd and \( D_i = -D_j \) otherwise.

\( c^* \) Every cocircuit \( D \) is such that \( D_i = D_{i+1} \) for all \( i,i+1 \in D \).

**Proof.** (a) \( \iff \) (b). Consider any circuit \( C := C(B,j) \) for a basis \( B \in (E_n^r) \) in natural order such that \( i \in B \) and \( i,j \) are consequent in \( B \cup j \). The signs of \( C_i \) and \( C_j \) are determined by \( \chi(B : j \rightarrow i) = -C_j C_i \chi(B) \). We have \( C_j = -C_i \) if and only if \( \chi(B : j \rightarrow i) = \chi(B) \). Since \( B \) and \( i \) were arbitrarily chosen and \( B : j \rightarrow i \) is in natural order, the equivalence follows.
5.6 Cyclic-P-matroids

(b) ⇐⇒ (c). Observe that (b) ⇒ (c) obviously holds. We prove the other direction. Suppose that (c) holds and that for some circuit
\[ C := C(B, j), \]
we have \( C_i = C_j \) for a consequent pair \( i, j \geq i + 2 \in C \). Let
\[ D := C(B', j), \]
where \( B' := (B \setminus i) \cup (j - 1) \). Eliminate \( j \) from \( C \) and \(-D\) by applying (C4). Since \( D_{j-1} = -D_j \) and by uniformity of \( \mathcal{M} \), the resulting circuit \( Z \) is such that \( Z_i = Z_{j-1} \), where \( i, j - 1 \in Z \) is a consequent pair. Such argumentation successively applies. In the end, we obtain a circuit with same sign at components \( i \) and \( i + 1 \), contradicting (c).

(c) ⇒ (c*). Let \( D := C^*(N, i + 1) \) be any cocircuit, where \( i \in N \) and \( i + 1 \in E_n \setminus N \). Circuit \( C := C(B, i) \) for \( B := E_2 \setminus N \) and cocircuit \( D \) are orthogonal to each other. Since \( C_i = -C_{i+1} \), we must have \( D_i = D_{i+1} \).

(a*) ⇐⇒ (b*), (b*) ⇐⇒ (c*), (c*) ⇒ (c). For a proof of these equivalences and implications, note that \( \mathcal{M} \) satisfies (c*) if and only if \( -G(\mathcal{M}^*) \) satisfies (c); analogously for (b*) and (b), (a*) and (a). Thus, if \( \mathcal{M} \) satisfies (c*), then \( -G(\mathcal{M}^*) \) satisfies (c), and also (b) by the implication (c) ⇒ (b) proven above. Consequently, matroid \( \mathcal{M} \) satisfies (b*). This proves (c*) ⇒ (b*). For the other implications, analogous argumentation applies.

Condition (b*) in Theorem 5.31 includes a characterization of the feasible region of the dual of the alternating matroid \( A^{n,r} \). The condition is called Gale’s evenness condition in the literature and is best illustrated by an oriented hemisphere arrangement depicting the dual. See Figure 5.4.

**Corollary 5.32** (Gale’s evenness condition). A cocircuit \( D \) or its opposite of the alternating matroid \( A^{n,r} \) is positive if and only if \( j - i \) is odd for all consequent \( i, j \in D \).

**Lemma 5.33.** The dual of the alternating matroid \( A^{n,r} \) on \( E_n \) is the matroid \( -G A^{n,n-r} \).

**Proof.** Since the alternating matroid \( A^{n,r} \) satisfies condition (c*) in Theorem 5.31, the reorientation \( -G((A^{n,r})^*) \) satisfies condition (c). In other words, we have \( -G((A^{n,r})^*) = A^{n,n-r} \) and therefore \( (A^{n,r})^* = -G A^{n,n-r} \).

\[ \square \]
Lemma 5.34. Consider the alternating matroid $A^{n,r}$ on $E_n$, where $1 \leq r < n$.

(i) The deletion minor $A^{n,r} \setminus i$ is equal to the alternating matroid $A^{n-1,r}$ on $E_n \setminus i$.

(ii) The contraction minor $A^{n,r} / i$ is equal to the reorientation $-_{[i-1]}A^{n-1,r-1}$ on $E_n \setminus i$.

Proof. (i). Since $A^{n,r}$ satisfies condition (b) in Theorem 5.31, any deletion minor $A^{n,r} \setminus i$ for $i \in E_n$ satisfies condition (b). As $r < n$, element $i$ is not a coloop of $A^{n,r}$ and the rank is preserved. Hence, we have $A^{n,r} \setminus i = A^{n-1,r}$.

(ii). Since $i$ is not a loop of $A^{n,r}$, the rank of the contraction minor is $r - 1$ and its collection of circuits consists of the sign vectors $C \setminus i$, where the $C$’s are circuits of $A^{n,r}$ with $C_i \neq 0$. Since $A^{n,r}$ satisfies condition (b) in Theorem 5.31, oriented matroid $-_{[i-1]}(A^{n,r} / i)$ satisfies (b). Hence, we have $A^{n,r} / i = -_{[i-1]}A^{n-1,r-1}$. \qed
5.6 Cyclic-P-matroids

5.6.2 Definition and properties of cyclic-P-matroids

In all that follows, we consider element sets $E_{2n} := [2n]$ with assigned order $1 < 2 < \ldots < 2n$ and complementary pairs $(i, i + n)$ for $i \in [n]$.

**Definition 5.35.** A cyclic-P-matroid is a P-matroid that is reorientation equivalent to the alternating matroid $A^{2n,n}$ for any $n \geq 1$.

**Definition 5.36.** A cyclic-P-matrix is a matrix $M \in \mathbb{R}^{n \times n}$ that realizes a cyclic-P-matroid with respect to Definition 4.3.

**Corollary 5.37.** Every cyclic-P-matrix is a P-matrix

*Proof.* This follows directly from condition (ii) in Proposition 5.3.

By this definition and Proposition 5.5, the set of cyclic-P-matroids is closed under proper relabelings and proper reorientations. Note that it is a proper subset of the P-matroids. For instance, the P-matroid considered in Example 5.1 is not cyclic because it is not uniform. This can also be seen by the fact that the semisphere arrangement representing the dual does not have a region that is bounded by every semisphere, such as the positive region in Figure 5.4.

The following Theorem gives the valid relabelings and reorientations to obtain cyclic-P-matroids.

**Theorem 5.38.** An oriented matroid $\mathcal{M}$ on $E_{2n}$, where $\pi \cdot (-F\mathcal{M}) = A^{2n,n}$ for some permutation $\pi$ of $E_{2n}$ and $F \subseteq E_{2n}$, is a P-matroid if and only if $\pi$ is such that

$$\pi(e) - \pi(\overline{e}) \text{ is odd for every } e \in E_{2n},$$

and $F$ is such that for every $e \in E_{2n}$, exactly one of $e$ and $\overline{e}$ is in $F$ if

$$\left\lfloor \frac{|\pi(e) - \pi(\overline{e})|}{2} \right\rfloor$$

is even, and both or none in $F$ otherwise.
Proof. First, we prove the “only if” direction. Since alternating matroids are uniform, oriented matroid $\mathcal{M}$ is uniform. Suppose that $\pi(e) - \pi(\overline{e})$ is even for some $e \in E_{2n}$. Without loss of generality, assume that $\pi(e) > \pi(\overline{e})$ in all that follows. There exists at least one $f \in E_{2n}$ with $\pi(f) \in \pi(\overline{e}), \pi(e)$, and $\pi(f) \not\in \pi(\overline{e}), \pi(e)$. Pick any complementary $B \in E_{2n}^n$ containing $\overline{e}$ and exactly one such $f$. Let $B' := (B \setminus f) \cup \overline{f}$. Consider the circuits $C := C(B, e)$ and $D := C(B', e)$ of $\mathcal{M}$. Note that $\pi \cdot (-_F C)$ and $\pi \cdot (-_F D)$ are circuits of $\mathcal{A}^{2n,n}$, and thus satisfy condition (b) in Theorem 5.31. By the choice of $B$ and $B'$, we have $(\pi \cdot (-_F C))_{\pi(e)} \pi(\overline{e}) = -(\pi \cdot (-_F D))_{\pi(e)} \pi(\overline{e})$. Observe that $(\pi \cdot (-_F C))_{\pi(e)} = -_C e$ and accordingly for $C_{\overline{e}}, D_{\overline{e}}$, and $D_{\overline{e}}$. Hence, we conclude that $(-_F C)_e (-_F C)_{\overline{e}} = -(-_F D)_e (-_F D)_{\overline{e}}$. No matter how $F \subseteq E_{2n}$ is chosen, we have $C_e C_{\overline{e}} = -D_{\overline{e}} D_{\overline{e}}$, and thus either $C$ or $D$ is sign-reversing. Oriented matroid $\mathcal{M}$ is not a P-matroid.

Now, assume that $\mathcal{M}$ is a P-matroid. By the previous remarks, $\pi(e) - \pi(\overline{e})$ is odd for every $e \in E_{2n}$. Hence, given any $e \in E_{2n}$, if $\pi(f) \in \pi(\overline{e}), \pi(e)$ for any $f \in E_{2n}$, then $\pi(f) \in \pi(\overline{e}), \pi(e)$. Consider any circuit $C := C(B, e)$ for complementary $B \in E_{2n}^n$ and $e \in E_{2n} \setminus B$ of $\mathcal{M}$. Note that $C_e = C_{\overline{e}}$, otherwise $C$ is sign-reversing. On the other hand, circuit $\pi \cdot (-_F C)$ of $\mathcal{A}^{2n,n}$ satisfies condition (b) in Theorem 5.31. Hence, if the expression (5.6) is even, then the number of elements $f$ with $(\pi \cdot (-_F C))_{f} \not\neq 0$ and $f \in \pi(\overline{e}), \pi(e)$ is even. We have $(\pi \cdot (-_F C))_{\pi(e)} = -(\pi \cdot (-_F C))_{\pi(\overline{e})}$; i.e., $(-_F C)_e = -(-C)_{\overline{e}}$, and thus, since $C_e = C_{\overline{e}}$, exactly one of $e$ and $\overline{e}$ is in $F$. If the expression (5.6) is odd, then the number of elements $f$ with $(\pi \cdot (-_F C))_{f} \neq 0$ and $f \in \pi(\overline{e}), \pi(e)$ is odd. We have $(\pi \cdot (-_F C))_{\pi(e)} = (\pi \cdot (-_F C))_{\pi(\overline{e})}$; i.e., $(-_F C)_e = -(-_F C)_{\overline{e}}$, and thus, since $C_e = C_{\overline{e}}$, both or none of $e$ and $\overline{e}$ are in $F$.

For the “if” direction, consider an oriented matroid $\mathcal{M}$ on $E_{2n}$ with $\pi \cdot (-_F \mathcal{M}) = \mathcal{A}^{2n,n}$, where $\pi$ and $F$ satisfy the conditions in the theorem. By condition (b') in Theorem 5.4, it is enough to prove that no almost-complementary circuit of $\mathcal{M}$ is sign-reversing. Consider any circuit $C := C(B, e)$ for complementary $B \in E_{2n}^n$ and $e \in E_{2n} \setminus B$ of $\mathcal{M}$. Note that circuit $\pi \cdot (-_F C)$ of $\mathcal{A}^{2n,n}$ satisfies condition (b) in Theorem 5.31. Since $\pi(e) - \pi(\overline{e})$ is odd for every $e \in E_{2n}$, the number of elements $f$ with $(\pi \cdot (-_F C))_{f} \neq 0$ and $f \in \pi(\overline{e}), \pi(e)$ is even if and only if the value
of the expression (5.6) is even. Since $\left(\pi \cdot (-F C)\right)_{\pi(e)}(\pi \cdot (-F C))_{\pi(\bar{\pi})} = (-F C)_{\pi(e)}(-F C)_{\bar{\pi}}$, it follows that $C_e = C_{\bar{\pi}}$. \hfill \qed

A cyclic-P-matroid is basically given by an alternating path through its elements and a subset of elements to reorient. The following example illustrates the structure of the circuits of a cyclic-P-matroid and shows how to find a realizing P-matrix.

**Example 5.3.** Consider the cyclic-P-matroid $M := -F (\pi^{-1} \cdot A^{2n,n})$ for $n = 4$, where

$$
\pi := \begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 5 & 1 & 8 & 6 & 4 & 2 & 7
\end{pmatrix}
$$

and $F := \{1, 4, 5, 6, 7\}$ satisfy the conditions in Theorem 5.38. Figure 5.5 below graphically illustrates the construction of $M$.

\begin{center}
\begin{tikzpicture}
\node (a) at (0,0) {$A^{8,4}$};
\node (b) at (2,0) {$\pi^{-1}$};
\node (c) at (4,0) {$-F$};
\node (d) at (6,0) {$M$};
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tabular}{cccccccc}
1 & 2 & 3 & 4 & & & & \\
5 & 6 & 7 & 8 & & & & \\
\end{tabular}
\begin{tabular}{cccccccc}
1 & 2 & 3 & 4 & & & & \\
5 & 6 & 7 & 8 & & & & \\
\end{tabular}
\begin{tabular}{cccccccc}
1 & 2 & 3 & 4 & & & & \\
5 & 6 & 7 & 8 & & & & \\
\end{tabular}
\end{center}

\begin{center}
$C(B,e)$
\begin{tabular}{cccc}
- & 0 & + & - \\
0 & + & + & 0 \\
\end{tabular}
\end{center}

Figure 5.5: Graphical illustration of a rank 4 cyclic-P-matroid.

The elements of any circuit of $M$ alternate along the depicted path, except that the shaded elements are replaced with the opposite. We can immediately read off any fundamental circuit $C(B, e)$ of $M$. For instance, for $B := \{1, 3, 4, 6\}$ and $e := 7$, we get the depicted circuit.
Consider the vector configuration
\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 & t_8 \\
t_1^2 & t_2^2 & t_3^2 & t_4^2 & t_5^2 & t_6^2 & t_7^2 & t_8^2 \\
t_1^3 & t_2^3 & t_3^3 & t_4^3 & t_5^3 & t_6^3 & t_7^3 & t_8^3
\end{pmatrix}
\]
for \( t_1 < t_2 < \ldots < t_8 \). A \( 4 \times 4 \) P-matrix realizing \( \mathcal{M} \) is computed by first reordering the columns according to \( \pi^{-1} \), and then negating the columns in \( F \). If we let \( t := (1, 2, \ldots, 8) \), we obtain the vector configuration
\[
V := \begin{pmatrix}
-1 & 1 & 1 & -1 & -1 & -1 & -1 & 1 \\
-3 & 5 & 1 & -8 & -6 & -4 & -2 & 7 \\
-9 & 25 & 1 & -64 & -36 & -16 & -4 & 49 \\
-27 & 125 & 1 & -512 & -216 & -64 & -8 & 343
\end{pmatrix}
\]
as a realizing vector configuration of \( \mathcal{M} \). Then the matrix
\[
-(V_{[4]})^{-1}V_{[8]\setminus[4]} = \begin{pmatrix}
1/2 & -3/5 & -9/10 & -3/5 \\
5/4 & 1/2 & -1/4 & -1 \\
3/28 & -1/14 & 9/28 & -1/7 \\
-1/7 & 1/35 & -1/35 & 16/35
\end{pmatrix}
\]
is a cyclic-P-matrix.

**Lemma 5.39.** Every principal minor of a cyclic-P-matroid is a cyclic-P-matroid.

**Proof.** Let \( \mathcal{M} \) be a cyclic-P-matroid, i.e, we have \( \pi \cdot (-_F \mathcal{M}) = \mathcal{A}^{2n,n} \) for some permutation \( \pi \) of \( E_{2n} \) and \( F \subseteq E_{2n} \). By Theorem 5.8, it is enough to show that any principal minor \( \mathcal{M}\setminus e/\overline{e} \) for \( e \in E_{2n} \) is reorientation equivalent to the alternating matroid. Careful investigation yields that
\[
\mathcal{M}\setminus e/\overline{e} = (-_F (\pi^{-1} \cdot \mathcal{A}^{2n,n}))\setminus e/\overline{e}
= -_F (\pi^{-1} \cdot (\mathcal{A}^{2n,n}\setminus \pi(e)/\pi(\overline{e})))
\]
By Lemma 5.34, the minor \( \mathcal{A}^{2n,n}\setminus \pi(e)/\pi(\overline{e}) \) equals \( -_{[\pi(\overline{e})]-1} \mathcal{A}^{2n-2,n-1} \). By
replacement, we obtain
\[
M_{e/\bar{e}} = -F (\pi^{-1} \cdot (\pi(\bar{e}) - 1 \cdot A^{2n-2,n-1})) \\
= -F \oplus \pi^{-1}([\pi(\bar{e})-1]) (\pi^{-1} \cdot A^{2n-2,n-1}).
\]
Hence, the minor \(M_{e/\bar{e}}\) is reorientation equivalent to the alternating matroid \(A^{2n-2,n-1}\).

**Example 5.3 (continued).** We consider the same cyclic-P-matroid as before, and find the principal minor \(M_{e/\bar{e}}\) for \(e = 2\) and \(\bar{e} = 6\). Since \(\pi(\bar{e}) = 4\), we have \([\pi(\bar{e}) - 1] = \{1, 2, 3\}\) and consequently \(\pi^{-1}([\pi(\bar{e}) - 1]) = \{1, 3, 7\}\). Furthermore, let \(F' := F \oplus \pi^{-1}([\pi(\bar{e}) - 1]) = \{3, 4, 5, 6\}\). According to the proof of Lemma 5.39, we have
\[
M_{e/\bar{e}} = -F' (\pi^{-1} \cdot A^{2n-2,n-1}).
\]
Figure 5.6 graphically illustrates the construction of the principal minor.

![Figure 5.6: A principal minor of a rank 4 cyclic-P-matroid.](image)

**Lemma 5.40.** Let \(M\) be a cyclic-P-matroid, i.e., we have \(\pi \cdot (\_F \cdot M) = A^{2n,n}\) for some permutation \(\pi\) of \(E_{2n}\) and \(F \subseteq E_{2n}\). The dual \(M^*\) equals \(-H \cdot M\), where \(H := \{\pi^{-1}(i) : i \in E_{2n} \text{ and } i \text{ is even}\}\).

**Proof.** Observe that
\[
M^* = (\_F (\pi^{-1} \cdot A^{2n,n}))^* \\
= -F (\pi^{-1} \cdot (A^{2n,n})^*).
\]
By Corollary 5.33, we have \((A^{2n,n})^\star = _G A^{2n,n}\). Hence, by replacement

\[
M^\star = _F (\pi^{-1} \cdot (_G A^{2n,n})) \\
= -(\pi^{-1} \cdot A^{2n,n}) \\
= _{(\pi^{-1}(G))} (\pi^{-1} \cdot A^{2n,n}) \\
= _{\pi^{-1}(G)} M,
\]

where \(\pi^{-1}(G) = \{\pi^{-1}(i) : i \in E_{2n} \text{ and } i \text{ is even}\}\).

By this result, the dual of a cyclic-P-matroid is also reorientation equivalent to the alternating matroid.

**Example 5.3 (continued).** For the cyclic-P-matroid \(M\) of the running example, we have \(H = \{4, 5, 6, 7\}\). Figure 5.7 graphically illustrates the dual matroid \(M^\star\).

![Figure 5.7: The dual of a rank 4 cyclic-P-matroid.](image)

In the remainder of this section, we examine the relation of cyclic-P-matroids to other classes of P-matroids. The discussion is less formal. We make use of graphical illustrations instead of explicitly giving the relabelings and set of elements to reorient. It is always possible to deduce the formal equivalent.

The rank 2 cyclic-P-matroid depicted in Figure 5.8 is a K-matroid. This can be seen by verifying that every circuit satisfies the Z-matroid property in Definition 5.9. There is probably no cyclic-K-matroid of higher rank, but this remains to be proven.

The rank 3 cyclic-P-matroid depicted in Figure 5.9 is obviously not a K-matroid. Its dual is such that there are no elements to reorient.
Therefore, a realization of the dual is given by six points on the moment curve in the plane, lifted into $\mathbb{R}^3$. In the bottom of the figure, there are two different realizations for the dual. In realization (a), the intersection of all dual complementary cones (the shaded area in the figure) is nonempty and has full dimension, contrary to realization (b), where the intersection is empty. We conclude that the cyclic-P-matroid is a hidden K-matroid that is realized by both hidden K-matrices and P-matrices that are not hidden K.

Figure 5.8: A rank 2 cyclic-K-matroid.
Figure 5.9: A rank 3 cyclic-hidden K-matroid.
Chapter 6

P-matroid OMCPs and simple principal pivoting methods

We present the simple principal pivoting methods as solving methods for the P-matroid OMCP. These are well-established in the theory of LCPs, and can be translated into the setting of oriented matroids because they make decisions based on combinatorial properties of the underlying problem instance. They are sometimes called Bard-type methods in the literature, and were first studied by Zoutendijk [89] and Bard [4].

For an oriented matroid $\hat{\mathcal{M}}$ on $E_{2n}$, we consider the OMCP($\hat{\mathcal{M}}$). To study the algorithmic complexity of OMCPs, we must specify how the oriented matroid $\hat{\mathcal{M}}$ is made available to any algorithm. We assume that it is given by an oracle, which for a basis $B$ and an element $e \in E_{2n} \setminus B$, outputs the unique fundamental circuit $\hat{C}(B,e)$. In the realizable case, where some pair $(M,q)$ is a realization, the oracle solves the system $V_B x = -v_e$ for $x$, where $V := [I \ -M \ -q]$. It can be implemented so that it performs arithmetic operations whose number is bounded by a polynomial in $n$. Hence, an algorithm for the OMCP whose number of operations is polynomial in $n$ would obviously provide a strongly polynomial algorithm.
for the LCP. Since deciding whether a general LCP is NP-complete [11],
the existence of such an algorithm is very unlikely. In the subsequent
Section 6.1, we do, nevertheless, prove that simple principal pivoting is
efficient in cases where \( \hat{\mathcal{M}} \) is an extension of a K-matroid.

Assume that \( \hat{\mathcal{M}} \) on \( E_{2n} \) is an extension of a P-matroid. A simple principal pivoting method proceeds in pivot steps. It starts with an arbitrary complementary basis \( B \in E_{2n}^n \) and computes the circuit \( \hat{C}(B, q) \) by
using the oracle. For instance, we let \( B := S \). We require the complementarity condition (4.2c) to be an invariant. The current basis \( B \) is supposed
to be complementary at any time. If circuit \( C \) is feasible, i.e., condition
(4.2b) is satisfied as well, then the unique solution is found and the algorithm terminates. Otherwise, an adjacent basis and the corresponding
circuit is obtained as follows: pick an \( e \in B \) with \( C_e = - \) according to
some pivot rule as pivot element. The pivot element leaves the current
basis and is replaced with its complement \( \bar{e} \), i.e., we let \( B := (B \setminus e) \cup \bar{e} \).
Condition (a) in Theorem 5.4 asserts that the updated \( B \), which is complementery, is indeed a basis. The circuit \( \hat{C}(B, q) \) is computed by using the oracle, feasibility is checked, and the algorithm eventually proceeds with
the next pivot step.

**SimplePrincipalPivot**\((\hat{\mathcal{M}}, B)\)

\[
\begin{align*}
C & := \hat{C}(B, q) \\
\text{while} \ C^- \neq \emptyset \ & \text{do} \\
& \quad \text{choose} \ e \in C^- \ \text{according to a pivot rule} \\
& \quad B := B \setminus e \cup \bar{e} \\
& \quad C := \hat{C}(B, q) \\
\text{end while} \\
\text{return} \ C
\end{align*}
\]

The number of pivot steps determines the algorithmic complexity. The
number depends on the applied pivot rule, and some rules may even enter a
loop on some instances and consequently never terminate. Here, we do not
discuss specific pivot rules. The interested reader is referred to Chapter 9.

**SimplePrincipalPivot** only operates on complementary circuits. For
complexity analysis, it is important to understand the combinatorial struc-
ture of these circuits. The following theorem generalizes the unique-sink orientation property satisfied by nondegenerate P-matrix LCPs [76].

**Definition 6.1.** Let $\mathcal{M}$ be an oriented matroid on $E_{2n}$, where every complementary $B \in E_{2n}^n$ is a basis. An extension $\widehat{\mathcal{M}}$ on $\widehat{E}_{2n}$ is nondegenerate if every circuit $C = (B, q)$ for complementary basis $B$ is such that $C = B \cup q$.

**Theorem 6.2.** Let $\widehat{\mathcal{M}}$ be a nondegenerate oriented matroid on $\widehat{E}_{2n}$, where every complementary $B \in E_{2n}^n$ is a basis. Then the deletion minor $\mathcal{M} = \widehat{\mathcal{M}} \setminus q$ is a P-matroid if and only if for all pairs of complementary bases $B, B' \in E_{2n}^n$ there is an $e \in B \setminus B'$ such that $\widehat{C}(B, q)_e = -\widehat{C}(B', q)_{\bar{e}}$.

**Proof.** First, we prove the “only if” direction. Suppose that $\widehat{\mathcal{M}}$ contains complementary circuits $C := \widehat{C}(B, q)$ and $D := \widehat{C}(B', q)$ such that for all $e \in B \setminus B'$, we have $C_e = D_{\bar{e}}$. Eliminate $q$ from $C$ and $-D$ by applying (C4). The resulting $Z$ with $Z_q = 0$ is sign-reversing. Vector $Z \setminus q$ is contained in $\mathcal{M}$. Hence, oriented matroid $\mathcal{M}$ is not a P-matroid.

Next, we prove the “if” direction. Consider any pair of adjacent complementary bases $B \in (E_{2n}^n)$ and $B' := (B \setminus e) \cup \bar{e}$ for some $e \in B$ in natural order. Let $C := \widehat{C}(B, q)$ and $D := \widehat{C}(B', q)$. By assumption, the signs of $C_e$ and $D_{\bar{e}}$ are opposite. Eliminate $q$ from $C$ and $-D$ by applying (C4). The resulting circuit $Z$ is such that $Z_q = 0$. Both $Z_e$ and $Z_{\bar{e}}$ are nonzero, otherwise $Z \subset C$ or $Z \subset D$. Hence, we have $Z_e = Z_{\bar{e}}$. Vector $Z' := Z \setminus q$ is contained in $\mathcal{M}$. Actually, it is a circuit of $\mathcal{M}$. The signs of $Z'_e$ and $Z'_{\bar{e}}$ are determined by $\chi(B : \bar{e} \to e) = -Z'_{\bar{e}}Z'_e \chi(B)$, where $\chi$ is the chirotope of $\mathcal{M}$. Since $B' = B : \bar{e} \to e$, it follows that $\chi(B') = \chi(B : \bar{e} \to e) = -\chi(B)$, i.e., bases $B$ and $B'$ have opposite basis orientation. The same argumentation applies to all pairs of adjacent complementary bases in natural order. Hence, the chirotope of $\mathcal{M}$ is such that $\chi(B) = (-1)^{|B \cap S|}$ for all complementary $B \in (E_{2n}^n)$ in natural order. Oriented matroid $\mathcal{M}$ satisfies condition (a) in Theorem 5.4, and thus is a P-matroid. □

Theorem 6.2 implies that the collection of complementary circuits of a nondegenerate extension $\widehat{\mathcal{M}}$ of a P-matroid realizes a unique-sink orientation (USO) of the combinatorial $n$-cube. Through a bijective mapping, each complementary basis $B \in E_{2n}^n$ is assigned to a vertex of the cube so that it is connected by an edge to its adjacent bases $(B \setminus e) \cup \bar{e}$ for $e \in B$. 
The edges of the cube are directed according to the sign structure of the complementary circuits. From the perspective of $B$, the edge $\{B, (B\setminus e) \cup \bar{e}\}$ is outgoing if $\tilde{C}(B, q)_e = -$, and incoming otherwise. By Theorem 6.2, the direction is in correspondence with the direction computed from the viewpoint of $(B\setminus e) \cup \bar{e}$. Moreover, Theorem 6.2 implies that every face of the cube has a unique local sink. A sink is a vertex with no outgoing edges.

**Example 5.1 (continued).** Consider the extension of the P-matroid realized by P-matrix

$$M := \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix} \quad \text{and vector } q := \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$  

The reader is asked to verify that Figure 6.1 below depicts the induced USO. Observe that the orientation contains a directed cycle.

![Figure 6.1](image)

**Figure 6.1:** A USO arising from a rank 3 P-matroid extension.

In the abstraction of USOs, a simple principal pivoting method behaves as follows: it starts in a vertex of free choice and follows a directed path according to a pivot rule until the global sink is reached. The corresponding circuit is the solution to the OMCP. The length of the traveled path equals the number of pivot steps.
6.1 K-matroid OMCPs

We extend a recent result from [25] to the generalizing setting of oriented matroids. We prove that the K-matroid OMCP is solved by simple principal pivot methods in a linear number of pivot steps, regardless of which pivot rule is applied. Our proof provides a purely combinatorial proof for the original result for K-matrix LCPs, which is favourable since we avoid the machinery of matrix algebra.

The following two lemmas are required to prove the result. While the first one holds for general P-matroids, the second one is restricted to K-matroids.

Recall the SimplePrincipalPivot method formulated in the previous section. For a simplified discussion, consider the following notions and definitions.

Let $\mathbf{M}'$ be an extension of a P-matroid $\mathbf{M}$. Then every complementary $B \in E_{2n}'$ is a basis of $\mathbf{M}'$ because it is a basis of $\mathbf{M} = \mathbf{M}' \setminus q$. For any complementary basis $B \in E_{2n}'$, let $C := \hat{C}(B, q)$. An element $e \in C^-$ is a pivot element. For any pivot element $e \in C^-$, basis $(B \setminus e) \cup \bar{e}$ is the consequent basis and $\hat{C}((B \setminus e) \cup \bar{e}, q)$ the consequent circuit.

**Lemma 6.3.** Let $\mathbf{M}'$ be an extension of a P-matroid $\mathbf{M}$ and $C := \hat{C}(B, q)$ a complementary circuit. For any pivot element $e \in C^-$, the consequent circuit $D := \hat{C}((B \setminus e) \cup \bar{e}, q)$ is such that $D_\bar{e} = +$.

**Proof.** Suppose that $D_\bar{e} \leq 0$. Eliminate $q$ from $C$ and $-D$ by applying (C4). The resulting $Z$ with $Z_q = 0$ is sign-reversing and $Z \setminus q$ is contained in the P-matroid $\mathbf{M}$, which is a contradiction. Hence, we have $D_\bar{e} = +$. □

Note that Lemma 6.3 also follows from Theorem 6.2.

**Lemma 6.4.** Let $\mathbf{M}'$ be an extension of a K-matroid $\mathbf{M}$ and $C := \hat{C}(B, q)$ any complementary circuit. The consequent circuit $D := \hat{C}((B \setminus e) \cup \bar{e}, q)$ for any pivot element $e \in C^-$ is such that $D_f \geq 0$ for each $f \in B \cap T$ with $C_f \geq 0$.

**Proof.** For the sake of a contradiction, suppose that $C_f \geq 0$ and $D_f = -$ for some $f \in B \cap T$ in some consequent circuit $D$. Element $f$ is not
the pivot element $e$, since $C_e = -$. Eliminate $q$ from $C$ and $-D$ by applying (V4). The resulting vector $Z$ is such that $Z_q = 0$, $Z_e = -$, and $Z_f = +$. Furthermore, we have $Z_T = 0$ and $Z_T = -$ by Lemma 6.3. Let $F := B \setminus \{e, f\}$. Consider the principal minor $\widehat{\mathcal{M}} \setminus \overline{F} \cup q / F$ on four elements. By Lemma 5.17, the minor is a K-matroid. It contains the vector $Z' := Z \setminus (F \cup \overline{F} \cup q)$ with $Z'_e = -$, $Z'_e = -$, $Z'_f = +$, and $Z'_T = 0$. Vector $Z'$ is actually a circuit, and $-Z'$ violates the K-matroid condition (c) in Theorem 5.16. A contradiction is established.

**Theorem 6.5.** SIMPLEPRINCIPALPIVOT runs in at most $2n$ pivot steps on every extension of a K-matroid.

*Proof.* We prove that, regardless of which pivot rule is applied and which basis we start with, every element $e \in E_{2n}$ is picked at most once as the pivot element. Consider any complementary pair $(e, \overline{e})$, where $\overline{e} \in T$. It is enough to show that $\overline{e}$ enters the basis at most once. At any execution step exactly one of $e$ and $\overline{e}$ belongs to the current basis. By Lemma 6.3, when $e$ leaves and $\overline{e}$ enters the current basis, the consequent circuit $D$ is such that $D_{\overline{e}} = +$, and by Lemma 6.4, the sign of $D_{\overline{e}}$ stays nonnegative until the algorithm terminates. In other words, element $\overline{e}$ stays in the basis. □

If SIMPLEPRINCIPALPIVOT starts in basis $S$, then $n$ pivot steps at most are needed because by Lemmas 6.3 and 6.4, no element in $T$ is ever picked as pivot element.

Note that the result about the algorithmic complexity does not requires nondegeneracy.

Theorem 6.5 is nicely illustrated in the context of USOs. It basically states that any path in a USO arising from a nondegenerate K-matroid extension is of length at most $2n$. In particular, there is no directed cycle. Note that the algorithmic complexity to find the global sink does not depend on the labeling of the vertices, which means that our result can be extended to the closure under proper relabelings of K-matroids.

Let $\mathcal{M}$ be a K-matroid on $E_{2n}$ and $\pi$ be a proper permutation of $E_{2n}$. Consider the relabeling $\pi \cdot \mathcal{M}$. By Proposition 5.5, the relabeling is a P-matroid. Furthermore, there are complementary sets $S' \in E_{2n}^n$ and
$T' := \{ \overline{e} : e \in S' \}$ such that for every circuit $C$ of $\pi \cdot \mathcal{M}$, we have:

$$\text{if } C_{T'} \geq 0, \text{ then } C_{\overline{e}} = + \text{ for all } e \in S' \text{ with } C_e = +.$$ 

Proposition 5.12, the Fiedler-Pták Theorem 5.16, Lemma 5.17, and the lemmas in this section have equivalent counterparts for this closure class. The counterparts are obtained by substituting $S$ by $S'$ and $T$ by $T'$ in the original statements. Hence, we get the following.

**Corollary 6.6.** SIMPLEPRINCIPALPivot runs in at most $2n$ pivot steps on every extension of a proper relabeling of a K-matroid.

The reader may wonder why we introduced Z-matroids and K-matroids at all and did not start off with their closures. One good reason for our approach is to point out the correspondence of realizable Z-matroids and their matrix counterparts. See also Proposition 5.10. With respect to this, the main problem is that principal pivot transforms of Z-matrices are in general not Z-matrices.
Part II

Unique-Sink Orientations
Abstract models are frequently used to study mathematical problems. In part I, we employed the setting of oriented matroids to study combinatorial properties of the P-matrix LCP. In part II, we model simple principal pivoting methods in the framework of unique-sink orientations in order to study and compare different pivot rules.

A linear program induces an orientation on the polyhedron representing its feasible region in such a way that each edge points towards the incident vertex with higher objective value. These orientations are of interest in connection with pivoting methods. Every face has a unique local sink and source as well, and is obviously acyclic. Furthermore, in every d-face, there are exactly d vertex-disjoint paths from the local source to the local sink [38]. Orientations satisfying this property are called Holt-Klee. These are necessary conditions, and little is known about sufficient conditions for an orientation to be realized by a linear program. In cases where the feasible region defines a combinatorial cube, the terminology unique-sink orientation (USO) is in use, which is due to Szabó and Welzl [78]. An extensive discussion of USOs is provided by Schur’s PhD thesis [74]. USOs also serve as an abstract model for many other geometric optimization problems, such as finding the smallest enclosing ball of a given point set [78].

Stickney and Watson [76] were first in modeling the principal pivoting methods for the P-matrix LCP in the framework of USOs. In Chapter 6, we investigated simple principal pivoting methods for the P-matroid OMCP, and illustrated their behavior on a digraph, where the underly-
ing undirected graph is a combinatorial cube. Every problem instance of rank \( n \) realizes a USO of the \( n \)-cube. In this model, simple principal pivoting methods start in any vertex and move along a directed path until the sink is reached. The sink is the unique vertex with no outgoing edge and corresponds to the solution. We believe that USOs provide a valuable combinatorial abstraction to analyze and compare different pivoting strategies.

Our overall aim is to understand how algebraic properties of P-matrix LCPs translate into combinatorial properties of P-matroid OMCPs, and how the latter ones in turn translate into properties of the corresponding USOs. For each class of matroids discussed in Chapter 5, we deduce and investigate the structure of the corresponding USOs. In doing so, we hope to clarify the algorithmic implications that specific structure has.

We start with a formal introduction to the notion of USOs of combinatorial cubes, and recall how P-matroid OMCPs translate into USOs.

In Chapter 8, we consider extensions of P-, (hidden) K-, and cyclic-P-matroids and investigate the properties of the corresponding USOs. Our interest involves sufficient and necessary conditions for a USO to be an orientation arising from a P-matroid OMCP. Many of our results generalize those known for the equivalent algebraic classes. In addition, we prove that principal pivoting in hidden K-matrix LCPs and LPs, where the feasible region defines a combinatorial cube, is the same. The two problems result in the same USOs, and are thus equally difficult with respect to principal pivoting.

Chapter 9 is devoted to the study and comparison of different pivot rules. For many deterministic rules, worst-case instances have been found. Other, mainly randomized rules, remain as potentially efficient candidates. The discussion refers to many known results and contributes a few new ones. For instance, we motivate the use of some history-based pivot rule for USOs arising from LPs over combinatorial cubes, more generally for acyclic USOs. Some content is added for the purpose of completeness only, and is important to see the big picture.

Chapter 10 covers results from my master’s thesis and a joint project with our internal collaborators, Foniok, Gärtner, and Sprecher. We determine bounds on the number of P-matrix USOs, i.e., the orientations
arising from P-matrix LCPs, and their subclasses. We also compare their numbers with the sizes of USO classes having purely combinatorial definitions, such as being acyclic. The first, maybe wrong intuition, is that small USO classes are well-structured, and are thus more likely to allow a polynomial-time pivoting method. The counting is done completely in the algebraic setting, and we do not consider oriented matroids. The results have been accepted for publication [27].

Finally, in Chapter 11, we provide a database with all P-matroid USOs of the 4-cube. The collection of P-matrix USOs of the 3-cube was given by Stickney and Watson [76]. Surprisingly, for $n \leq 4$, the collection of P-matroid USOs of the $n$-cube is exactly the collection of P-matrix USOs. For higher dimensions, we do not know whether this observation still holds. While the structure of USOs arising from P-matrix LCPs is quite well understood, there is a substantial lack of construction schemes for P-matrices and more generally for P-matrix USOs. Consequently, verifying some conjecture or finding worst-case instances for some pivot rule is difficult. Providing a database of orientations may be of great use for such tasks. A paper [30] with the obtained results is in preparation.
Chapter 7

Cubes and unique-sink orientations

We formally introduce the notion of unique-sink orientations (USOs) of combinatorial cubes, and recall how P-matroid OMCPs translate into USOs.

For a binary vector $v \in \{0, 1\}^n$ and $I \subseteq [n]$, let $v \oplus I$ be the binary vector given by

$$(v \oplus I)_j := \begin{cases} 1 - v_j & \text{if } j \in I, \\ v_j & \text{if } j \notin I. \end{cases}$$

Instead of $v \oplus \{i\}$, we write $v \oplus i$.

The $n$-cube is the undirected graph $G = (V, E)$, where

$$V := \{0, 1\}^n \text{ and } E := \{\{v, v \ominus i\} : v \in V, \; i \in [n]\}.$$  

Let $\phi$ be an orientation of the $n$-cube, i.e., a digraph with underlying undirected graph $G$. If $\phi$ contains the directed edge $(v, v \oplus i)$, we write $v \xrightarrow{\phi} v \oplus i$, or simply $v \rightarrow v \oplus i$ if the orientation in question is clear from the context.

A subcube of the $n$-cube $G$ is a subgraph $G' = (V', E')$, where $V' := \{v \oplus I : I \subseteq F\}$ for some vertex $v \in \{0, 1\}^n$ and $F \subseteq [n]$ and $E' := E \cap \binom{V'}{2}$. 

113
Its dimension is $|F|$. The directed subgraph of $\phi$ induced by $V'$ is denoted by $\phi[V']$.

**Definition 7.1.** An orientation $\phi$ of the $n$-cube $G$ is a *unique-sink orientation* (USO) if every subcube $G' = (V', E')$ has a unique local *sink*, i.e., a vertex with outdegree zero in $\phi[V']$.

Note that orientations $\phi[V']$ are again USOs by definition. The orientation of the 3-cube depicted in Figure 7.1 is an example of a USO.

![Figure 7.1: A USO of the 3-cube.](image)

The following lemma can be proven by induction on the cube dimension.

**Lemma 7.2** (Stickney and Watson [76]). Let $s$ be the global sink of a USO $\phi$ of the $n$-cube. For any vertex $v$, the orientation $\phi$ contains a directed path from $v$ to $s$ whose length is the Hamming distance between $v$ and $s$.

Let $\pi$ be a permutation of $[n]$ and $L \subseteq [n]$. For any vertex $v \in \{0, 1\}^n$, the vertex $\pi \cdot v$ is the vertex with $(\pi \cdot v)_{\pi(i)} = v_i$ for all $i \in [n]$. For a given USO $\phi$ of the $n$-cube, let $(\pi, L) \cdot \phi$ be the orientation of the $n$-cube given by

$$
\pi \cdot (v \oplus L) \xrightarrow{(\pi, L) \cdot \phi} (\pi \cdot (v \oplus L)) \oplus \pi(i) \quad :\Leftrightarrow \quad v \xrightarrow{\phi} v \oplus i.
$$

Such an operation basically constitutes a relabeling of the vertices.

**Lemma 7.3.** Let $\phi$ be a USO of the $n$-cube. Then the relabeling $(\pi, L) \cdot \phi$ for arbitrary permutation $\pi$ of $[n]$ and $L \subseteq [n]$ is a USO as well.
Any two USOs that can be transformed into each other by a relabeling of the vertices are isomorphic. For a given USO, the closure of USOs obtained by relabelings builds an isomorphism class.

Let $\phi$ be any orientation of the $n$-cube and $\phi^{(F)}$ for $F \subseteq [n]$ be the orientation of the $n$-cube obtained by reversing all edges in directions contained in $F$, formally

$$v \xrightarrow{\phi^{(F)}} v \oplus i \iff \begin{cases} v \xrightarrow{\phi} v \oplus i & \text{if } i \notin F, \\ v \oplus i \xrightarrow{\phi} v & \text{if } i \in F. \end{cases}$$

The following lemma directly follows from the definition of USOs.

**Lemma 7.4** (Szabo and Welzl [78]). Let $\phi$ be a USO of the $n$-cube. Then $\phi^{(F)}$ for arbitrary $F \subseteq [n]$ is a USO as well. \hfill \qed

**Corollary 7.5.** Let $\phi$ be a USO. Every subcube $G' = (V', E')$ has a unique local source, i.e., a vertex with indegree zero in $\phi[V']$. \hfill \qed

**Definition 7.6.** Two USOs $\phi$ and $\phi'$ of the $n$-cube are FF-equivalent if there exist a permutation $\pi$ of $[n]$ and sets $L, F \subseteq [n]$ such that $\phi' = (\pi, L) \cdot (\phi^{(F)})$.

Note that $(\pi, L) \cdot (\phi^{(F)}) = ((\pi, L) \cdot \phi)^{(\pi(F))}$ for any permutation $\pi$ of $[n]$ and sets $L, F \subseteq [n]$.

Here, the expression ”FF-equivalent” stands for “FacetFlip-equivalent”, and is closely related to FF-equivalent P-matroids. In general, USOs that can be transformed into each other by facet flips belong to different isomorphism classes.

### 7.1 From P-matroid OMCPs to USOs

As mentioned, unique-sink orientations enable a graph-theoretic description of simple principal pivoting methods for the P-matroid LCP. Theorem 6.2 basically states that nondegenerate P-matroid OMCPs realize USOs. The theorem has a one-to-one translation into the following.
Theorem 7.7. Every OMCP$(\mathcal{M})$, where $\mathcal{M}$ on $E_{2n}$ is a nondegenerate extension of a P-matroid, realizes a USO $\phi$ of the $n$-cube. The orientation $\phi$ is determined by

$$v \xrightarrow{\phi} v \oplus i : \iff \begin{cases} \widehat{C}(B(v), q)_{s_i} < 0 & \text{if } v_i = 0 \\ \widehat{C}(B(v), q)_{t_i} < 0 & \text{if } v_i = 1 \end{cases} \quad (7.1)$$

for all $v \in \{0, 1\}^n$, where $B(v) := \{s_i : v_i = 0\} \cup \{t_i : v_i = 1\}$.

Example 7.1. Let $\mathcal{M}$ on $E_{2n}$ be a nondegenerate extension of the K-matroid $\mathcal{M}$ which is realized by the K-matrix

$$M := \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & -5 \\ -3 & -3 & 4 \end{pmatrix} \quad \text{and vector } q := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$ 

The USO arising from $\mathcal{M}$ is the one depicted in Figure 7.1.

In the abstraction of USOs, a simple principal pivoting method behaves as follows. It starts in a vertex of free choice and follows a directed path according to a pivot rule until the global sink is reached. The corresponding circuit is the solution to the OMCP. The length of the traveled path equals the number of pivot steps.
Chapter 8

Oriented matroid classes and USO classes

We study extensions of P-, (hidden) K-, and cyclic-P-matroids, and investigate the properties of the corresponding USOs. We are primarily interested in sufficient and necessary conditions for a USO to arise from an extension of a P-matroid. In general, such orientations are different from LP USOs. For instance, directed cycles arise.

We prove that the collection of LP USOs and USOs arising from hidden K-matrix LCPs are the same. Consequently, the hidden K-matrix LCP and the LP over combinatorial cubes are equally difficult with respect to principal pivoting.

It is possible to give a combinatorial description of the extensions of cyclic-P-matroids. However, an exact characterization of cyclic-P-matroid USOs remains an open problem. Nevertheless, such a description may turn out to be useful for computational experiments and complexity analysis. It opens the door to new research projects.

8.1 P-matroid USOs

Stickney and Watson [76] enumerated all USOs of the 3-cube and observed that some USOs do not arise from any P-matrix LCP.
Definition 8.1. A P-matrix USO is an orientation that arises from a P-matrix LCP, or equivalently from a P-matroid extension that is realizable.

Similarly, a P-matroid USO is an orientation that arises from any extension of a P-matroid. We prove the following straightforward fact.

Lemma 8.2. Every face of a P-matroid USO is a P-matroid USO.

Proof. Consider the USO $\phi$ of the $n$-cube that arises from an extension $\widehat{M}$ on $\widehat{E}_{2^n}$ of a P-matroid $M$. It is enough to prove that any $(n - 1)$-face, i.e., facet, is a P-matroid USO. The result for all other faces follows by successive application of the same argumentation. Let $V'$ contain the $2^{n-1}$ vertices $v \in \{0,1\}^n$ with $v_i = 0$ for any $i \in [n]$. The orientation $\phi[V']$ is determined by the complementary circuits of the principal minor $\widehat{M}\setminus t_i/s_i$, which is an extension of $M\setminus t_i/s_i$. By Theorem 5.8, oriented matroid $M\setminus t_i/s_i$ is a P-matroid.

In Lemma 8.2, the word “P-matroid” can be replaced with “P-matrix”, because if an extension $\widehat{M}$ on $\widehat{E}_{2^n}$ of a matroid $M$ is realizable, then $\widehat{M}\setminus e/\bar{e}$ and $M\setminus e/\bar{e}$ for $e \in E_{2^n}$ are realizable.

Next, we investigate how USOs that arise from extensions of FF-equivalent P-matroids are related to each other. Consider the following lemma and then study the subsequent example.

Lemma 8.3. For an oriented matroid $M$ on $E_{2^n}$, a permutation $\pi$ of $E_{2^n}$, and $F \subseteq E_{2^n}$, the following holds.

(i) Let $\widehat{M}$ on $\widehat{E}_{2^n}$ be an extension of $M$. Then $\pi \cdot (-F \cdot \widehat{M})$ is an extension of $\pi \cdot (-F \cdot M)$.

(ii) Let $\widehat{M}$ on $\widehat{E}_{2^n}$ be an extension of $\pi \cdot (-F \cdot M)$. Then $-F(\pi^{-1} \cdot \widehat{M})$ is an extension of $M$.  

Example 7.1 (continued). Let $\widehat{M}$ on $\widehat{E}_{2^n}$ be the K-matroid extension previously given. Consider the proper permutation

$$
\pi := \begin{pmatrix}
  s_1 & s_2 & s_3 & t_1 & t_2 & t_3 \\
  t_3 & s_2 & s_1 & s_3 & t_2 & t_1
\end{pmatrix}
$$
of \( E_{2n} \) and proper \( F := \{s_1, t_1\} \). The orientation arising from \( \pi \cdot (\mathcal{F}, \mathcal{M}) \) is depicted in Figure 8.1.

By Lemmas 7.3 and 7.4, the orientation is again a USO. By definition, it is FF-equivalent to the original USO. By Lemma 8.3, oriented matroid \( \pi \cdot (\mathcal{F}, \mathcal{M}) \) is an extension of \( \pi \cdot (\mathcal{F}, \mathcal{M}) \), which in turn is a P-matroid by Proposition 5.5. Hence, the orientation is a P-matroid USO. Since K-matroids are not closed under proper reorientations, it eventually is not a K-matroid USO. Actually it is not, because there is a directed path of length \( 7 > 2n \).

Figure 8.1: An FF-equivalent P-matroid USO of the K-matroid USO depicted in Figure 7.1.

The following result is a direct consequence from the above discussions.

**Lemma 8.4.** If \( \phi \) is a P-matroid USO of the \( n \)-cube, then every FF-equivalent USO \( (\pi, L) \cdot (\phi(F)) \) for any permutation \( \pi \) of \([n]\) and sets \( L, F \subseteq [n] \) is a P-matroid USO.

If a USO arises from a P-matroid \( \mathcal{M} \), then its FF-equivalent USOs arise from P-matroids that are FF-equivalent to \( \mathcal{M} \). Note that any oriented matroid is realizable if and only if its FF-equivalent matroids are realizable. So in Lemma 8.4, we can replace the word “P-matroid” with “P-matrix”.

It is known that every P-matrix USO satisfies the Holt-Klee condition, a condition originally defined for, and satisfied by, polytope digraphs that arise from linear programs [38].
Definition 8.5. A USO of the $n$-cube is \textit{Holt-Klee} if in every $k$-face for $1 \leq k \leq n$, there are $k$ vertex-disjoint paths from the local source to the local sink.

Theorem 8.6 (Gärtner, Morris, and Rüst [36]). \textit{Every $P$-matrix USO is Holt-Klee.} \hfill \Box

Every USO of the 2-cube satisfies the Holt-Klee condition. Figure 8.2 depicts the only two USOs of the 3-cube that are not Holt-Klee, up to isomorphism. These are the only orientations of the 3-cube that do not arise from any $P$-matrix LCP. In other words, for $n = 3$, the Holt-Klee condition is also sufficient for a USO to be a $P$-matrix USO. Alternatively, for $n \geq 4$, it is a necessary condition only. Morris [60] constructed some Holt-Klee USO of the 4-cube, which is not a $P$-matrix USO.

![Figure 8.2: The only two USOs of the 3-cube that are not Holt-Klee, up to isomorphism.](image)

By the fact that every FF-equivalent orientation of a $P$-matrix USO is again a $P$-matrix USO, the Holt-Klee condition can be strengthened.

Definition 8.7. A USO $\phi$ of the $n$-cube is \textit{strongly Holt-Klee} if every USO $\phi^{(F)}$ for $F \subseteq [n]$ is Holt-Klee.

Corollary 8.8. \textit{Every $P$-matrix USO is strongly Holt-Klee} \hfill \Box

The strengthened Holt-Klee condition still does not sufficiently describe $P$-matrix USOs. See Chapter 10 for details. It is an open question as to whether there exists an exact characterization of $P$-matrix USOs. Quite probably, there is none. Vamos [84] proved that there is no axiomatic
description of realizable oriented matroids. How then can we distinguish P-matrix USOs from non-realizable P-matroid USOs? On the other hand, we are not aware of the existence of non-realizable P-matroid USOs. The orientations in Figure 8.2 are neither P-matroid USOs, which is obvious because every rank 3 oriented matroid on 7 elements is realizable. In Chapter 11, we will observe that every P-matroid USO of the 4-cube is also a P-matrix USO, even though there exist non-realizable P-matroids of rank 4. A related open question is whether every P-matroid USO is (strongly) Holt-Klee.

8.2 K-matroid USOs

The K-matrix USOs additionally satisfy a stronger condition that is related to uniformity.

A USO of the $n$-cube is uniform if for all $i \in [n]$, we have $v \to v \oplus i$ whenever $v_i = 0$. In other words, all edges are oriented “from 0 to 1”.

**Definition 8.9.** A USO $\phi$ of the $n$-cube is locally uniform if for every subcube $V' = \{v \oplus I : I \subseteq J\}$ such that $v_J = 0$, we have:

- if $v \to v \oplus j$ for all $j \in J$, then $\phi[V']$ is uniform;

and

- if $v \oplus j \to v$ for all $j \in J$, then $\phi[V'(J)]$ is uniform.

The following theorem generalizes a result obtained by Foniok, Fukuda, Gärtner, and Lüthi [25] in the setting of oriented matroids.

**Theorem 8.10.** Every K-matroid USO is locally uniform.

**Proof.** Consider a USO of the $n$-cube arising from a nondegenerate extension $\hat{M}$ on $\tilde{E}_{2n}$ of a K-matroid $M$.

First, suppose that there is a subcube $V' = \{v \oplus I : I \subseteq J\}$ with $v_J = 0$ and $v \to v \oplus j$ for all $j \in J$. Let $B(v)$ be the complementary basis belonging to vertex $v$. Circuit $C := \hat{C}(B(v), q)$ is such that $C_{s_i} = -$ for all $i \in J$. Let $u \in V'$ and $D := \hat{C}(B(u), q)$. For the sake of a contradiction, suppose that $D_{t_k} = -$ for some $k \in J$ with $u_k = 1$. We eliminate $q$
from $-C$ and $D$ by applying (V4). The resulting vector $Z$ is such that $Z_q = 0$, $Z_s > 0$ for all $s_i \in B(v) \setminus B(u)$, and $Z_{t_k} = -$. Furthermore, for $F := B(v) \cap B(u)$, vector $Z' := Z \setminus (\overline{F} \cup q)/F$ is contained in the minor $M \setminus \overline{F}/F$. By Lemma 5.17, the minor is a K-matroid. Vector $Z'$ violates condition (c) in Theorem 5.16, which is a contradiction.

Similar argumentation works to prove the property of subcubes $V' = \{v \oplus I : I \subseteq J\}$ with $v_J = 0$ and $v \oplus j \to v$ for all $j \in J$.

In Section 6.1, we have seen that any simple principal pivoting method solves a K-matroid OMCP in at most $2n$ pivot steps, regardless of which pivot rule is applied and which basis we start with. In other words, every directed path in a K-matroid USO has length at most $2n$, and consequently the orientation is acyclic. This property is satisfied by general locally uniform USOs as well.

**Lemma 8.11** (Foniok, Fukuda, Gärtner, and Lüthi [25]). Any directed path in a locally uniform USO has length at most $2n$.

### 8.3 Hidden K-matroid OMCP is at least as hard as OMP over cubes

The collection of hidden K-matroid USOs contains the K-matroid USOs. Unfortunately, local uniformity is not preserved. Acyclicity is preserved by hidden K-matrix USOs only.

First, we discuss some issues related to the computational complexity of hidden K-matrix LCPs. Every LCP($M, q$), where $M$ is a hidden K-matrix, is obviously feasible, and vector $p$ required to solve it by the linear program (5.3) can be computed in polynomial time by using the ellipsoid method [69]. Hence, in theory there is a polynomial-time algorithm. No strongly polynomial-time method and no practically efficient method is known. Checking whether an arbitrary square matrix is hidden K can be performed in polynomial time using linear programming [68]. The complexity result for the K-matroid OMCP gives hope that some pivot rule is also efficient for the hidden K-matrix LCP. Computational experiments suggest that some randomized pivoting schemes run in expected $O(n^2)$
pivot steps. Unfortunately, we are far from proving or disproving such an observation.

In this section, yet another complexity result is derived. We prove that the hidden K-matrix USOs are exactly the LP USOs. Hence, principal pivoting schemes for hidden K-matrix LCPs and LPs over combinatorial cubes behave in the same way. The proof is partly done in the setting of oriented matroids. We reduce oriented matroid programming (OMP) over combinatorial cubes to the hidden K-matroid OMCP. The theory of OMP, originally developed by Bland [6] and Fukuda [29], is an abstraction of linear programming in the setting of oriented matroids. Our introduction is superficial; for a more thorough discussion, see [5].

Let $\mathcal{N}$ on $E$ be an oriented matroid with elements $q, p \in E$, where $p$ is not a loop and $q$ is not a coloop. An oriented matroid program, denoted by the triple $(\mathcal{N}, p, q)$, encodes the task of maximizing element $q$ over the feasible region, which is given by the feasible vectors of $\mathcal{N}^*/q$. The oriented semisphere arrangement in Figure 8.3 depicts the covectors of an OMP$(\mathcal{N}, p, q)$, where the feasible region is a combinatorial cube. The edges that bound the feasible region are directed such that they point to the incident cocircuit with “higher” objective value. The unique-sink property is preserved in such generalization of linear programming. Hence, the digraph defines a USO of a cube. A sink, i.e., a cocircuit with no outgoing edge, is a solution. All this is one-to-one transferable to ordinary

Figure 8.3: An OMP over a combinatorial 2-cube.
Theorem 8.12. Every USO of the $n$-cube that is realized by an OMP is also realized by a hidden K-matroid OMCP.

Proof. Consider an OMP$(N,p,q)$, where the feasible region is a combinatorial $n$-cube. The feasible region of any OMP is determined by the set of positive covectors $Y$ with $Y_p = +$. In our case, the region is an $n$-cube, and we can assume without loss of generality that the dual $N^*$ is a rank $n + 1$ matroid on $2n + 2$ elements. Denote its ground set by $E_{2n} \cup \{p,q\}$ such that opposite facets correspond to complementary pairs; that is, for each complementary $B \in E_{2n}^n$, the cocircuit $C^*(B \cup q,p)$ is an extreme point of the feasible region. Consider the contraction $N^*/q$. The cube property is preserved because $q$ is the element to maximize, and the matroid $N^*/q$ satisfies the conditions in Theorem 5.25. Therefore, the dual of the deletion minor $N^*/q\setminus p$ of $N^*/q$, which is $N/p\setminus q$, is a hidden K-matroid. The USO resulting from the OMP$(N,p,q)$ is found as follows. For each extreme point $C^*(B \cup q,p)$, the orientation of the incident edges is given by the circuit $C(E_{2n} \setminus B,q)$ contained in the contraction $N/p$. In the realizable case, one would speak of the reduced cost vector. This circuit is complementary and used by principal pivoting methods to solve the OMCP$(N/p)$. The matroid $N/p$ is an extension of the hidden K-matroid $N^*/q$.

We do not know whether the opposite direction holds; that is, whether every hidden K-matroid USO is realized by some OMP. There is a little lack of insight. Consider a hidden K-matroid $M$, where $\tilde{M}^*$ is a certificate for $M$ being a hidden Z-matroid, and any rank-preserving extension $\tilde{M}$ of $M$. The open question is as to whether we are able to combine the extension $\tilde{M}$ and certificate $\tilde{M}^*$ to an oriented matroid $N$ satisfying $N/p = \tilde{M}$ and $N^*/q = \tilde{M}^*$. In the realizable case, such a combination always exists. This enables us to prove that every hidden K-matrix USO is an LP USO. This result also follows by combining results from [60, 62, 69, 71], but has been explicitly stated in my master’s thesis [44] for the first time.

Theorem 8.13. A USO of the $n$-cube is realized by an LP if and only if it is realized by a hidden K-matrix LCP.
8.3 Hidden K-matroid OMCP is at least as hard as OMP over cubes

Proof. The “only if” direction follows from Theorem 8.12 and the obvious fact that every minor of a realizable oriented matroid is realizable. For a proof of the “if” direction, consider any LCP\((M, q)\), where \(M\) is a hidden K-matrix. Let \(\mathcal{M}\) on \(E_{2n}\) be the matroid that is realized by \(M\), and \(\widehat{\mathcal{M}}\) on \(\widehat{E}_{2n}\) the extension realized by the pair \((M, q)\). By Propositions 5.3 and 5.20, matroid \(\mathcal{M}\) is a hidden K-matroid, and there exists \(p \in \mathbb{R}^n\) such that \(M^T y \leq p\) defines a combinatorial cube. Let \(\widehat{\mathcal{M}}^*\) the extension of the dual of \(\mathcal{M}\), where \(M^T\) together with \(-p\) is a realization. Let \(\mathcal{N} = (E_{2n} \cup \{p, q\}, V)\) be the matroid that is realized by the vector configuration

\[
V := \begin{bmatrix}
I & -M & -q & 0 \\
0 & p^T & -q & 0
\end{bmatrix},
\]

where the last column represents element \(p\), second last element \(q\), and the other columns the complementary pairs. The dual \(\mathcal{N}^*\) is realized by

\[
V^\perp := \begin{bmatrix}
M^T & I & 0 & -p \\
q^T & 0 & 1 & 0
\end{bmatrix}.
\]

Consider the OMP\((\mathcal{N}, p, q)\), whose feasible region is given by \(\mathcal{N}^*/q = \widehat{\mathcal{M}}^*\). The feasible region is a combinatorial cube, and every complementary \(N \in E_{2n}^n\) is a basis of \(\widehat{\mathcal{M}}^*\). For feasible cocircuit \(\mathcal{C}^*(N, p)\), the orientations of the incident edges are determined by the circuit \(\widehat{C}(E_{2n} \setminus N, q)\) contained in \(\mathcal{N}/p = \widehat{\mathcal{M}}\). Hence, the USO realized by the hidden K-matroid OMCP\((\widehat{\mathcal{M}})\) is realized by the OMP\((\mathcal{N}, p, q)\), where \(\mathcal{N}\) is realizable.

Every LP USO is obviously acyclic. Therefore, by Theorem 8.13, we get the following result.

**Corollary 8.14.** Every hidden K-matrix USO is acyclic.

Fukuda [29] gave an example of an OMP over a combinatorial 3-cube that contains a directed cycle. By Theorem 8.12, we get the following.

**Corollary 8.15.** There exist hidden K-matroid USOs that contain directed cycles.

Hence, the set of hidden K-matrix USOs is a proper subset of the set of hidden K-matroid USOs.
8.4 Cyclic-P-matroid USOs

Our aim is to give a characterization of USOs arising from cyclic-P-matroids. We cannot fully accomplish this objective so far, but we are able to give a combinatorial description of the extensions of cyclic-P-matroids. By the definition of cyclic-P-matroids and Lemma 8.3, it is basically enough to describe the extensions of alternating matroids. In the end, we eventually get a scheme to enumerate all cyclic-P-matroid USOs of the \(n\)-cube for arbitrary \(n\). Computational power may still be a limiting factor.

An extension of an oriented matroid is determined by an additional oriented pseudosphere added to the pseudosphere arrangement representing the dual matroid. See also Section 3.8 and Example 5.1 on page 63. Basically, an extension is determined by a function that indicates for every cocircuit whether it is on the positive side, negative side, or lies on the added pseudosphere. Recall that a localization is a function \(\sigma : C^* \to \{+, 0, -\}\) that describes an extension. Consider Lemma 3.40, which informally describes the collection localizations of an oriented matroid.

Figure 8.4 depicts the semisphere arrangement representing the dual of the alternating matroid \(A_{n,r}\) on \(E_n = [n]\) with assigned order \(1 < 2 < \ldots < n\) for \(n = 6\) and \(r = 3\). Neglect the dashed sphere for the moment. Every vertex and line is the intersection of \(r - 1\) and \(r - 2\) semispheres, respectively. Each vertex represents a cocircuit, and is labeled with the set of \(n - (r - 1)\) semispheres, or elements, that do not contain it. Similarly, each line is labeled with the set of \(n - (r - 2)\) semispheres that do not contain it. The following observations are due to Ziegler [88].

For all \(I = \{e_1 < e_2 < \ldots < e_{k+1}\} \in E^{k+1}_n\), the \(k\)-packet is the \((k + 1)\)-tuple \(P(I) := (I\setminus e_1, I\setminus e_2, \ldots, I\setminus e_{k+1})\).

Lemma 8.16. Consider the semisphere arrangement representing the dual matroid of the alternating matroid \(A^{n,r}\). For any line \(I \in E_{2n}^{n-(r-2)}\), the vertices on \(I\) correspond to the \(n - (r - 1)\)-packet \(P(I)\).

Proof. Directly follows from condition (b*) in Theorem 5.31.

By combining Lemma 3.40 and 8.16, we finally get a description for the extensions of an alternating matroid. We are only interested in uniform
extensions. Through a perturbation argument, these are enough to build all cyclic-P-matroid USOs.

**Theorem 8.17.** Consider the alternating matroid $\mathcal{A}^{n,r}$. A function $\sigma : E_n^{n-(r-1)} \to \{+,-\}$ is a localization if and only if the subset of vertices \( \{ J \in E_n^{n-(r-1)} : \sigma(J) = + \} \) contains the beginning or ending elements in all $n-(r-1)$-packets $P(I)$ for $I \in E_n^{n-(r-2)}$.

The reader is asked to verify that the arrangement in Figure 8.4 extended by the dashed semisphere describes a localization.

Denote the set of uniform extensions of any uniform matroid $\mathcal{M}$ by $\text{ext}(\mathcal{M})$. Now, let $\mathcal{M}$ on $E_{2n}$ be a cyclic-P-matroid of rank $n$; that is, $\pi \cdot (-F,\mathcal{M}) = \mathcal{A}^{2n,n}$ for some permutation $\pi$ of $E_{2n}$ and $F \subseteq E_{2n}$ satisfying the conditions in Theorem 5.38. By Lemma 8.3, the collection of uniform extensions of $\mathcal{M}$ is given by

\[
\text{ext}(\mathcal{M}) := \{-F(\pi^{-1} \cdot \mathcal{N}) : \mathcal{N} \in \text{ext}(\mathcal{A}^{2n,n})\}.
\]
Even though there is a description of the uniform extensions of cyclic-P-matroids, the corresponding USOs may have complex combinatorial structure. Computational experiments suggest that every P-matroid USO of the 3-cube is also a cyclic-P-matroid USO. For orientations of the 4-cube, approximately 88% of the P-matroid USOs arise from cyclic-P-matroid extensions. We refer the reader to Chapter 11 for details.

A proper subclass of cyclic-P-matrix USOs is easily obtained. Namely, those that are obtained from extensions by an additional point on the moment curve.

**Proposition 8.18.** An oriented matroid $\widehat{M}$ on $\widehat{E}_{2n}$, where $\pi \cdot (-F, \widehat{M}) = A^{2n+1,n}$ for some permutation $\pi$ of $\widehat{E}_{2n}$ with $\pi(q) = q$ and $F \subseteq \widehat{E}_{2n}$, is an extension of a cyclic-P-matroid if and only if $\pi$ and $F$ restricted to $E_{2n}$ satisfy the conditions in Theorem 5.38.

**Proof.** Observe that

$$
\widehat{M}\backslash q = (-F(\pi^{-1} \cdot A^{2n+1,n}))\backslash q \\
= (-F(\pi^{-1} \cdot (A^{2n+1,n} \backslash \pi(q)))) \\
= (-F(\pi^{-1} \cdot (A^{2n+1,n} \backslash q))) \\
= (-F(\pi^{-1} \cdot A^{2n,n})),
$$

where the last equality follows from Lemma 5.34. Hence, deletion minor $\widehat{M}\backslash q$ is a P-matroid if and only if $\pi$ and $F$ restricted to $E_{2n}$ satisfy the conditions in Theorem 5.38.

We call such an extension of a cyclic-P-matroid *simple*. It is straightforward to prove that simple extensions induce acyclic USOs.

**Example 8.1.** Figure 8.5 graphically illustrates two simple extensions of a rank 3 cyclic-P-matroid. In other words, they are both obtained by an additional point $q$ on the lifted moment curve. Extension $\widehat{M}_1$ induces the so-called *Klee-Minty USO* of the 3-cube.

An option for a future project is to investigate USOs arising from lexicographic extensions of cyclic-P-matroids.
Figure 8.5: Two simple extensions of a rank 3 cyclic-P-matroid.
Chapter 9

Simple principal pivoting for USOs

We study and compare different pivot rules using the model of unique-sink orientations. For many deterministic rules, worst-case instances have been found. Other, mainly randomized rules, remain as potentially efficient candidates. The discussion references many known results and contributes a few new results. We motivate the use of some history-based pivot rule for LP USOs.

Simple principal pivoting methods applied to nondegenerate P-matroid extension follow a directed path in the corresponding USO until the sink is found. We reformulate these methods with respect to general USOs.

For any USO $\phi$ of the $n$-cube and a start vertex $v \in \{0, 1\}^n$, the simple principal pivoting method is as follows.

\[\text{SimplePrincipalPivoting}(\phi, v)\]

\[\text{while } O := \{j \in [n] : v \xrightarrow{\phi} v \oplus j\} \neq \emptyset \text{ do}\]

\[\text{choose } i \in O \text{ according to a pivot rule}\]

\[v := v \oplus i\]

\[\text{end while}\]

\[\text{return } v\]
9.1 Pivot rules and complexity issues

The \texttt{RandomEdge} rule is just a plain random walk, in each pivot step we simply pick the pivot element uniformly at random.

\begin{verbatim}
RANDOMEdge(O)
    choose i ∈ O uniformly at random
    return i
\end{verbatim}

The rule may not terminate if applied to USOs containing directed cycles. Nevertheless, it is finite in expectation [61]. The rule seems to perform well on acyclic USOs [35], but no upper bound on the algorithmic complexity is known. It behaves very badly on the highly cyclic \textit{Morris USOs} [61], which arise from LCPs\((M,q)\), where

\[
M ∈ \mathbb{R}^{n×n} := \begin{pmatrix}
1 & 2 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 2 & 0 \\
2 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and } q ∈ \mathbb{R}^n := \begin{pmatrix}
-1 \\
-1 \\
-1 \\
\vdots \\
-1 \\
-1
\end{pmatrix}.
\]

The integer \(n\) must be odd in order for \(M\) to be a P-matrix. Figure 6.1 depicts the Morris USO of the 3-cube. Observe that the orientation contains a directed cycle.

Another pivot strategy is \texttt{Murty’s} least index rule, which in each pivot step traverses the outgoing edge with least index.

\begin{verbatim}
Murty(O)
    return min(O)
\end{verbatim}

The rule always terminates, even if the USO contains cycles. This is a direct consequence of the definition of a USO and can be proven by induction on \(n\). The rule runs in a quadratic number of pivot steps on Morris USOs [25]. Unfortunately, it behaves very badly on the \textit{Klee-Minty}
USOs [65], which are realized by P-matrix LCPs $(M, q)$, where

$$M \in \mathbb{R}^{n \times n} := \begin{pmatrix}
1 & 2 & 2 & 2 & 2 & 2 \\
0 & 1 & 2 & \cdots & 2 & 2 \\
0 & 0 & 1 & 2 & 2 & 2 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0 & 1 & 2 & 2 \\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad q \in \mathbb{R}^n := \begin{pmatrix}
-1 \\
-1 \\
-1 \\
\vdots \\
-1 \\
-1 \\
-1
\end{pmatrix}.$$ 

Triangular P-matrices are hidden K-matrices. See Lemma 9.1 for details. Hence, the Klee-Minty USOs are LP USOs by Theorem 8.13 and therefore acyclic. Actually, the orientations were introduced by Klee and Minty [45] in the setting of linear programming. They proved that Dantzig’s simplex method requires an exponential number of pivot steps, and thus is not a polynomial-time method for linear programming.

Figure 9.1 depicts the Klee-Minty USO of the 3-cube. The path taken by Murty, if started in the zero vertex, has length $2^3 - 1$, and for arbitrary $n$-cubes the path has exponential length $2^n - 1$.

![Figure 9.1: The Klee-Minty USO of the 3-cube.](image)

To overcome the exponential runtime, Fukuda and Namiki [31] introduced a randomized variant of Murty.

$\text{RandomMurty}(\mathcal{O})$
if called for the first time then
    choose permutation $\pi$ of $[n]$ uniformly at random
end if
return $\pi(\min(\pi^{-1}(O)))$

Since Murty’s rule is finite, RandomMurty is finite as well. Moreover, the rule requires expected linear number of pivot steps on the Klee-Minty USOs, if started in the zero vertex [31]. For arbitrary start vertices, the rule requires an expected quadratic number of pivot steps [35]. The rule is also quadratic on the Morris USOs [25]. Furthermore, Gärtner gave a subexponential upper bound for acyclic USOs [34]. The original proof of this upper bound was formulated for the RandomFacet pivoting scheme, which is a scheme very similar to simple principal pivoting with RandomizedMurty as pivot rule.

In RandomFacet, we start in any vertex $v$. Among all facets of the USO that contain $v$, we choose a facet uniformly at random, and recursively find the local sink $s$ in that facet. If $s$ is the global sink we are done. Otherwise, we recursively find the global sink in the opposite facet of the chosen facet, starting with the vertex which is adjacent to $s$.

For a USO $\phi$ of the $n$-cube, let $S := [n]$ and $v \in \{0, 1\}^n$.

RandomFacet($\phi, S, v$)
if $S = \emptyset$ then
    return $v$
else
    choose $i \in S$ uniformly at random
    $V := \{v \oplus (S \setminus i)\}$
    $s := \text{RandomFacet}(\phi[V], S \setminus i, v)$
    if $s \oplus i \rightarrow s$ then
        return $s$
    else
        $v' := s \oplus i$
        $V' := \{v' \oplus (S \setminus i)\}$
        return RandomFacet($\phi[V'], S \setminus i, v'$)
    end if
Matoušek [52] defined a family of acyclic USOs of the $n$-cube that contains orientations, on which \textsc{RandomFacet} requires an expected subexponential number of pivot steps in the worst case [34]. Gärtner [34] proved that the expected number of pivot steps becomes quadratic if one restricts to orientations, where every 3-subcube is Holt-Klee. This bound is also tight. It is the only example of a proof that exploits the Holt-Klee condition, and shows that the condition has significant influence on the behavior of certain pivot rules.

We conclude that \textsc{RandomMurty} and \textsc{RandomFacet} remain as potentially polynomial-time candidates for solving P-matrix LCPs. Experiments suggest that they require an expected quadratic number of pivot steps on triangular P-matrices. On the other hand, a recent result by Friedmann, Hansen, and Zwick [28] indicates that \textsc{RandomFacet} may require an expected subexponential number of pivot steps in the worst case. They construct \textit{parity games (PGs)} on which the \textit{policy iteration} method with the \textsc{RandomFacet} rule needs an expected subexponential number of iterations. PGs can be reduced to Markov Decision Processes (MDPs), which in turn are solved by linear programs. Policy iteration methods for PGs are closely related to simplex algorithms for LPs. So, by this chain of reduction, the \textsc{RandomFacet} pivot rule is inefficient for linear programming. As Gärtner pointed out, the PGs constructed in the paper reduce to LPs over products of simplices, which are known to realize \textit{grid USOs} arising from GLCPs with block P-matrices [36]. Actually, LPs over products of simplices realize grid USOs arising from GLCPs with block hidden K-matrices, since Theorem 8.13 carries over to generalized LCPs. Hence, chances are high that \textsc{RandomFacet} is also inefficient for ordinary P-matrix LCPs.

Note that it is unknown whether there exist (strongly) Holt-Klee USOs on which \textsc{RandomizedMurty} or \textsc{RandomFacet} requires an expected exponential number of pivot steps.

We do not want ignore other, less studied pivot rules. Murty’s least index rule is such that, if applied to Klee-Minty USOs, it visits every vertex and traverses edges in direction $i$ exactly $2^{n-i}$ times. We would
like to motivate the use of a deterministic, history-based pivot rule for LP USOs that balances the usage of edges among all directions.

Suppose we are given a bounded LP. The feasible region is a convex polytope and the objective function induces an acyclic orientation on the underlying graph. For any start vertex \(v\), we try to reach a vertex on the “opposite” of the feasible region by following a directed path. If the feasible region is a cube, then we try to reach the antipodal vertex of \(v\) in as few pivot steps as possible. Either we reach it or we find the sink. In the former case, we then try to reach \(v\) again, that is, we would like to create a directed cycle. Since cycles do not exist in LP orientations, at some point we must end up in the sink.

The Least-UsedDirection pivot rule imitates such behavior.

**Least-UsedDirection** \((\mathcal{O})\)

```plaintext
if called for the first time then
    initialize array \(h := [0, 0, \ldots, 0]\) of length \(n\)
end if

\(i := \min\{i : i \in \mathcal{O} \text{ and } h(i) < h(j) \text{ for all } j \in \mathcal{O}\}\)

\(h[i] := h[i] + 1\)

return \(i\)
```

We prove that Least-UsedDirection is efficient for a subclass of LP USOs that includes the Klee-Minty orientations.

**Lemma 9.1** (K. [44], Tsatsomeros [81]). Every triangular P-matrix is a hidden K-matrix. \(\Box\)

**Lemma 9.2.** Consider a USO of the \(n\)-cube that arises from an LCP\((M,q)\), where \(M\) is an upper-triangular P-matrix. For all \(v \in \{0, 1\}^n\), the direction of the edge \(v \to v \oplus i\) only depends on the \(v_j\)’s for \(j \geq i\).

**Proof.** Let \(J := [i-1]\) and \(K := [n]\setminus J\). In matrix \(A_{B(v)}\) for any \(v \in \{0, 1\}^n\), the submatrix \((A_{B(v)})_{KJ}\) is all-zero. Thus, we have \((A_{B(v')}^{-1}q)i = (A_{B(v')}^{-1}q)i\) for all \(v' \in \{0, 1\}^n\) coinciding with \(v\) at positions \(j \geq i\). \(\Box\)

Consider a USO of the \(n\)-cube that arises from an LCP\((M,q)\), where \(M\) is an upper-triangular P-matrix. According to Lemma 9.2, all edges in direction \(n\) point point to the same facet. The edges in direction \(n-1\)
point to the same facet for all $v$ with $v_n = 1$, similar for all $v$ with $v_n = 0$, and so forth. It is straightforward to prove that every facet is again an upper-triangular P-matrix USO.

The lower-triangular P-matrix USOs are isomorphic to upper-triangular P-matrix USOs.

**Theorem 9.3.** The **Least-usedDirection** rule finds the sink of triangular P-matrix USOs of the $n$-cube in at most $n(n+1)/2$ pivot steps.

**Proof.** It is enough to prove the result for upper-triangular P-matrix USOs. We apply a kind of amortized analysis.

We execute the **Least-UsedDirection** pivot rule, and interrupt the algorithm as soon as

$$\min(\{i : i \in \mathcal{O} \text{ and } h(i) \leq h(j) \text{ for all } j \in \mathcal{O}\}) = 2$$

for the first time. So far, we spent $k_1 \leq n$ pivot steps. Let us deposit $n - k_1$ pivot steps in an account $a$. Since all edges in direction $n$ point to the same facet, the current vertex $v$ is in the same facet with respect to direction $n$ as the global sink. The algorithm will stay in that facet until it terminates. The facet is again an upper-triangular P-matrix USO. Hence, after at most $k_2 = a + n - 1$ another pivot steps, the current vertex is in the same facet with respect to direction $n - 1$ as the global sink. Set the account $a$ to $n + (n - 1) - k_1 - k_2 > 0$. This argumentation successively applies. We need at most $n + (n - 1) + \ldots + 1$ pivot steps in total. \hfill \square

Note that Theorem 9.3 holds for general USOs that have the same structure as triangular P-matrix USOs. Furthermore, it would be possible to formulate a similar result for USOs arising from simple extensions of cyclic-P-matroids.

A recently published result [3] states that there is no acyclic USO of the 6- and 7-cube that allows **Least-UsedDirection** to visit each vertex. Hence, its behavior on general acyclic USOs is smarter than the one of Murty’s least index rule. On the other hand, the rule does not suit USOs containing directed cycles.
Chapter 10  
Counting USOs

This chapter covers results obtained in my master thesis and a joint project with our internal collaborators Foniok, Gärtner, and Sprecher. We determine bounds on the number of USOs arising from the P-matrix LCP and its subclasses. We also compare the obtained numbers with the sizes of USO classes that have combinatorial definitions, such as being Holt-Klee. The first, maybe wrong intuition, is that small classes are well-structured, and thus are more likely to allow a polynomial-time pivoting strategy. The counting is completely done in the algebraic setting; we do not consider oriented matroids. The results have been accepted for publication [27].

First counting results were obtained by Matoušek [53], who gave asymptotic bounds on the number of all USOs and acyclic USOs. Then, Develin [19] aimed at showing that the Holt-Klee condition does not characterize the LP USOs. He proved that the number of LP USOs of the \( n \)-cube is bounded from above by \( 2^{O(n^3)} \), whereas the number of Holt-Klee USOs is bounded from below by \( 2^{\Omega(2^n/\sqrt{n})} \). The counting is usually done in the labeled sense. Nevertheless, the same asymptotic bounds stay valid for the number of USOs up to isomorphism.

Using similar means, we prove an upper bound of \( 2^{O(n^3)} \) on the number of P-matrix USOs, and observe that a slight modification of Develin’s construction scheme yields a lower bound of \( 2^{\Omega(2^n/\sqrt{n})} \) for locally uniform strongly Holt-Klee USOs. Furthermore, we provide a scheme to construct
$2^{\Omega(n^3)}$ K-matrix USOs. These results imply that the number of K-matrix, LP, and P-matrix USOs is $2^{\Theta(n^3)}$.

Previously known and new bounds on the sizes of USO classes are summarized in Table 1.1 on page 10.

As a byproduct of the counting, we also obtain asymptotic bounds for the number of USOs arising from LCPs with fixed P-matrix. For an arbitrary P-matrix, each right-hand side $q \in \{-1, +1\}^n$ induces a different USO. For some P-matrices, such as the identity matrix, $2^n$ is indeed the right number of distinct USOs. We will observe that the number of USOs arising from LCPs with fixed P-matrix is bounded from above by $2^{O(n^2)}$. This bound is also tight.

### 10.1 An upper bound for P-matrix USOs

For an upper bound on the number of distinct P-matrix USOs, the strategy is to rely on bounds on the number of regions in arrangements of polynomials.

Every USO arising from a P-matrix LCP($M, q$) is determined by the sequence

$$\sigma(M, q) := (\text{sign}(A^{-1}_{B(v)} q)_i : v \in \{0,1\}^n \text{ and } i \in [n]) .$$

Function $\sigma$ maps matrices $M \in \mathbb{R}^{n \times n}$ and right-hand sides $q \in \mathbb{R}^n$ to sequences representing the corresponding USOs.

We prove that each entry in the sequence is determined by the sign of a polynomial in the entries of $M$ and $q$. To count the number of P-matrix USOs, we count the number of distinct sequences that can arise. More precisely, we consider the arrangement formed by all the polynomials contributing to the sequence and count the number of regions. Since USOs only arise from nondegenerate P-matrix LCPs, we are not interested in sequences containing zeros; i.e., the number of faces other than regions is irrelevant. We neglect that $M$ must be a P-matrix. So, some regions may contribute to the bound even though they do not contain any point defining a P-matrix.
Lemma 10.1. Each entry of the sequence $\sigma(M, q)$ is the sign of a polynomial in the entries of $M$ and $q$ of degree at most $n$.

Proof. The entries of the matrix $A^{-1}_B(v)$ can be computed as

$$(A^{-1}_B(v))_{rs} = \frac{1}{\det A_B(v)} (-1)^{r+s} A_{sr},$$

where $A_{rs}$ is the determinant of the submatrix of $A_B(v)$ obtained by deleting the $r$th row and the $s$th column, which is a polynomial of degree at most $n - 1$. Hence

$$(A^{-1}_B(v)q)_i = \frac{1}{\det A_B(v)} \sum_{s=1}^{n} q_s \cdot (-1)^{i+s} \cdot A_{si}.$$ 

Recall that $A_B(v)$ has $|B(v)|$ columns of $-M$ and $n - |B(v)|$ columns of the identity matrix. Thus, we have $\text{sign} \det A_B(v) = (-1)^{|B(v)|}$, since $M$ is a P-matrix. Therefore

$$\text{sign}(A^{-1}_B(v)q)_i = \text{sign}((-1)^{|B(v)|} \cdot \sum_{s=1}^{n} q_s \cdot (-1)^{i+s} \cdot A_{si}),$$

which is the sign of a polynomial of degree at most $n$. \hfill $\Box$

Many upper bounds on the number of cells in an arrangement of polynomials are known. We make use of the following result.

Theorem 10.2 (Warren [86]). Let $p_1, \ldots, p_l$ be real polynomials in $k$ variables, each of degree at most $d$. If $l \geq k$, then the number of sign sequences $\sigma(x) = (\text{sign} p_1(x), \ldots, \text{sign} p_l(x))$ that consist of terms $+1, -1$ is at most $(4edl/k)^k$.

Now all is set to prove an upper bound on the number of P-matrix USOs.

Theorem 10.3. The number of distinct P-matrix USOs of the $n$-cube is at most $2^{O(n^3)}$.

Proof. By Lemma 10.1, each P-matrix USO of the $n$-cube is determined by a vector of $n2^n$ nonzero signs of polynomials of degree at most $n$. The
number of variables is $n^2 + n$, which is equal to the number of entries in matrix $M$ and vector $q$. By Theorem 10.2, there are at most

$$
\left(\frac{4e \cdot n \cdot 2^n}{n^2 + n}\right)^{n^2 + n} \leq (4e \cdot 2^n)^{n^2 + n} = 2^{O(n^3)}
$$

sequences $\sigma(M, q)$. \hfill \Box

### 10.2 A lower bound for K-matrix USOs

**Theorem 10.4.** The number of distinct K-matrix USOs of the $n$-cube is at least $2^{\Omega(n^3)}$.

**Proof.** Consider the upper triangular matrix

$$
M(\beta) = \begin{pmatrix}
1 & -1 - \beta_{1,2} & -1 - \beta_{1,3} & \ldots & -1 - \beta_{1,n} \\
0 & 1 & -1 - \beta_{2,3} & \ldots & -1 - \beta_{2,n} \\
0 & 0 & 1 & \ldots & -1 - \beta_{n-1,n} \\
0 & 0 & 0 & \ldots & 1
\end{pmatrix}
$$

and the vector $q = (-1, 1, -1, \ldots, (-1)^n)^T$. We will now examine how the choice of the $\beta_{i,j}$’s influences the USO induced by the LCP($M(\beta), q$). Our goal is to show that we can make $2^{\Omega(n^3)}$ choices, each of which induces a different USO.

From now on, we will write

$$(i, j) \prec (i', j') \text{ for } (j < j') \text{ or } (j = j' \text{ and } i > i').$$

Note that $\prec$ is a (strict) total ordering on $\{(i, j) \in [n]^2 : i < j\}$. The strategy will be to choose the values of the $\beta_{i,j}$ in the order given by $\prec$, i.e., from left to right and in each column from bottom to top. We show that for about half of the $\beta_{i,j}$’s, there is a number of choices exponential in $j - i$ such that each of these choices determines a different orientation on a certain subset of edges, which will be independent of all subsequently made choices. But first we examine the expressions that determine the orientation.
Let $B \subseteq [n]$ and, analogous to the definition of $A_B$, let $A_B(\beta)$ be the
matrix whose $j$th column is the $j$th column of $-M(\beta)$ if $j \in B$, and the
$j$th column of $I_n$ otherwise.

**Lemma 10.5.** The entries of the inverse matrix $(A_B(\beta))^{-1}$ of $A_B(\beta)$ are
given by

$$
\sigma_r \cdot ((A_B(\beta))^{-1})_{r,s} = \\
\begin{cases}
1 & \text{if } r = s, \\
0 & \text{if } r > s \text{ or if } r < s \text{ and } s \notin B, \\
2p(B,r,s) + \beta_{r,s} + t_{B,r,s}(\beta) & \text{if } r < s \text{ and } s \in B,
\end{cases}
$$

where $\sigma_r = -1$ if $r \in B$ and $\sigma_r = 1$ if $r \notin B$, $p(B,r,s) = |\{j \in B : r < j < s\}|$ and $t_{B,r,s}(\beta)$ is a polynomial in variables $\beta_{i,j}$ for $(i,j) \in \{(i,j) \in [n] \times B : i < j, (i,j) \prec (r,s)\}$ with no constant term.

**Proof.** The inverse of an upper triangular matrix is again upper triangular.
Its diagonal entries are the reciprocals of the diagonal entries of the original
matrix; in our case, they are $\pm 1$. One can easily check that in the columns
that do not belong to $B$, all the off-diagonal entries are zero (e.g., by
multiplying such a column by all the rows of $A_B(\beta)$).

So it remains to examine the above-diagonal entries in columns that
belong to $B$. The set of the indices of all such entries is $J(B) = \{(r,s) \in [n] \times B : r < s\}$. Consider the ordering $\prec$ defined above, restricted to $J(B)$.
The least element of $J(B)$ with respect to $\prec$ is $(s-1,1)$, where $s$ is the
least element of $B \setminus \{1\}$. Multiplying row $s-1$ of $A_B(\beta)$ with the $s$th
column of $(A_B(\beta))^{-1}$ reveals that $((A_B(\beta))^{-1})_{s-1,s} = \sigma_{s-1} \cdot (1 + \beta_{s-1,s})$.
Thus (10.1) holds for the $\prec$-minimum $(r,s)$ in $J(B)$.

For any other $(r,s) \in J(B)$, assume that (10.1) holds for all $(r',s') \prec (r,s)$. Multiplying the $r$th row of $A_B(\beta)$ by the $s$th column of $(A_B(\beta))^{-1}$
shows that

$$
\sigma_r \cdot ((A_B(\beta))^{-1})_{r,s} + \sum_{\substack{j \in B \\
r < j < s}} (1 + \beta_{r,j}) \cdot \sigma_j \cdot \left(2p(B,j,s) + \beta_{j,s} + t_{B,j,s}(\beta)\right) \\
+ \sigma_s \cdot (1 + \beta_{r,s}) = 0.
$$
As $\sigma_j = \sigma_s = -1$ for $j, s \in B$, we have

\[
\sigma_r \cdot \left((A_B(\beta))^{-1}\right)_{r,s} = \beta_{r,s} + 1 + \sum_{\substack{j \in B \\
 r < j < s}} (1 + \beta_{r,j}) \left(2^{p(B,j,s)} + \beta_{j,s} + t_{B,j,s}(\beta)\right)
\]

\[
= \beta_{r,s} + 1 + \sum_{\substack{j \in B \\
 r < j < s}} 2^{p(B,j,s)} + t_{B,r,s}(\beta)
\]

\[
= \beta_{r,s} + 2^{p(B,r,s)} + t_{B,r,s}(\beta),
\]

as we were supposed to show. \hfill \Box

It follows from (10.2) that $t_{B,r,s}$ satisfies this recursive formula:

\[
t_{B,r,s}(\beta) = \sum_{\substack{j \in B \\
 r < j < s}} \left(2^{p(B,j,s)} \beta_{r,j} + \beta_{j,s} + \beta_{r,j} \beta_{j,s} + t_{B,j,s}(\beta) + \beta_{r,j} t_{B,j,s}(\beta)\right) .
\]

(10.3)

Let $B \subseteq [n]$ be a basis such that, for $m = \max B$, we have $i \equiv m + 1 \pmod{2}$ for each $i \in B \setminus \{m\}$. Then $q_m \cdot q_i = -1$ for each $i \in B \setminus \{m\}$, and hence

\[
\sigma_r \cdot \left((A_B(\beta))^{-1}q\right)_r = (-1)^m \left(\beta_{r,m} + t_{B,r,m}(\beta) - \sum_{\substack{s \in B \\
 r < s < m}} (\beta_{r,s} + t_{B,r,s}(\beta))\right)
\]

\[
= (-1)^m \left(\beta_{r,m} - t'_{B,r,m}(\beta)\right)
\]

(10.4)

for all $r < m$ such that $r \equiv m + 1 \pmod{2}$; this parity condition ensures that the constant terms sum to zero. Each $t'_{B,r,m}(\beta)$ is some polynomial in variables $\beta_{i,j}$ for $(i,j) \in \{(i,j) \in [n] \times B : i < j, (i,j) \prec (r,m)\}$ with no constant term. In particular, this implies that in the corresponding USO, the orientation of the $r$th edge incident with the vertex corresponding to the basis $B$ depends only on the values of $\beta_{i,j}$ with $(i,j) \prec (r,m)$.

Now let $r, m \in [n]$, $r < m$, $r \equiv m + 1 \pmod{2}$. Let

\[
C = \{i \in [n] : r < i < m, i \equiv m + 1 \pmod{2}\}
\]
and let

\[ V' = \{(0 \oplus m) \oplus I : I \subseteq C\} \].

Note that \(|C| = (m - r - 1)/2\) and so \(|V'| = 2^{(m-r-1)/2}\).

Furthermore, suppose that the values of \(\beta_{i,j}\) are fixed for all \((i, j) \prec (r, m)\), and that these values satisfy:

\[ v, v' \in V', \ v \neq v' \implies t'_{B(v), r, m}(\beta) \neq t'_{B(v'), r, m}(\beta). \tag{10.5} \]

For each \(v \in V'\), the direction of the edge between \(v\) and \(v \oplus r\) in the USO induced by \(\text{LCP}(M(\beta), q)\) is by (10.4) determined by the sign of the difference \(\beta_{r,m} - t'_{B(v), r, m}(\beta)\). By (10.5), the currently fixed values of \(t'_{B(v), r, m}\) for \(v \in V'\) are all distinct and thus they split the reals into \(|V'| + 1\) intervals. Hence, there are \(|V'| + 1\) choices for \(\beta_{r,m}\) so that the resulting USOs will differ from one another in the orientation of at least one of these edges.

What happens, though, if we are about to choose \(\beta_{r,m}\) and (10.5) is not satisfied? Then we have to revise the choices we have made so far. Perturbing slightly each \(\beta_{i,j}\) with \((i, j) \prec (r, m)\) will not change the orientation; the next lemma implies that it will make (10.5) satisfied.

**Lemma 10.6.** Let \(r, m \in [n]\), \(r < m\), \(r \equiv m + 1 \pmod{2}\) and let \(B, B' \subseteq [n]\) be bases such that \(\max B = \max B' = m\), \(\min B > r\), \(\min B' > r\), and that \(i \equiv m + 1 \pmod{2}\) for all \(i \in (B \cup B') \setminus \{m\}\). Then the polynomial \(t'_{B,r,m}(\beta) - t'_{B',r,m}(\beta)\) is identically zero if and only if \(B = B'\).

**Proof.** First, from (10.4) we have:

\[ t'_{B,r,m}(\beta) = -t_{B,r,m}(\beta) + \sum_{s \in B \atop r < s < m} (\beta_{r,s} + t_{B,r,s}(\beta)). \tag{10.6} \]

Assume that \(B \neq B'\). Without loss of generality, there exists some \(s \in B \setminus B'\); by assumption \(s > r\). Let \(\tilde{t}_{B,r,m}(\beta_{s,m})\), \(\tilde{t}_{B',r,m}(\beta_{s,m})\) be the univariate polynomials obtained from \(t'_{B,r,m}(\beta)\), \(t'_{B',r,m}(\beta)\), respectively, by setting \(\beta_{i,j} = 0\) for all \((i, j) \neq (s, m)\). Then we can see from (10.6) and (10.3) that
\( \tilde{t}_{B',r,m}(\beta_{s,m}) \) is identically zero. On the other hand, we have

\[
\tilde{t}_{B,r,m}(\beta_{s,m}) = -t_{B,r,m}(\beta_{s,m}, 0, \ldots, 0)
\]

\[
= -\beta_{s,m} - \sum_{j \in B \atop r < j < m} t_{B,j,m}(\beta_{s,m}, 0, \ldots, 0)
\]

\[
= -2^{p(B,r,s)} \beta_{s,m}.
\]

The first inequality holds because the sum in (10.6) does not contain \( \beta_{s,m} \), and the second holds by \( \beta_{s,m} \).

Hence \( t'_{B,r,m}(\beta) - t'_B(\beta) \) is not identically zero. The converse implication is trivial. \( \square \)

The options to choose \( \beta_{r,m} \) are, of course, not independent of the values of the other \( \beta_{i,j} \)'s. However, they depend only on the \( \beta_{i,j} \)'s with \( (i,j) \prec (r,m) \). Hence it is possible to make the choices sequentially in the order given by \( \prec \); starting with \( \beta_{1,2} \) and finishing with \( \beta_{1,n} \). The values of \( \beta_{r,m} \) for \( r \equiv m \pmod{2} \) can be chosen arbitrarily, e.g., \( \beta_{r,m} = 0 \).

Therefore the number of distinct USOs induced by \( \text{LCP}(M(\beta), q) \) for various values of \( \beta_{i,j} \), as described above, is at least

\[
\prod_{m=1}^{n} \prod_{1 \leq r < m \atop r \equiv m+1 \pmod{2}} \left( 2^{(m-r-1)/2} + 1 \right) = \prod_{m=1}^{n} \prod_{i=0}^{\lfloor m/2 \rfloor - 1} (2^i + 1) = 2^{\Omega(n^3)}.
\]

Finally, it remains to show that the values of all \( \beta_{i,j} \)'s can be chosen to satisfy \( |\beta_{i,j}| < 1 \), so that \( M(\beta) \) would be a K-matrix. That follows from the next lemma.

**Lemma 10.7.** Whenever \( t'_{B,r,m} \) is defined, let

\[
\bar{\beta} = \max\{|\beta_{i,j}| : (i,j) \in [n] \times B, \; i < j, \; (i,j) \prec (r,m)\}.
\]

If \( \bar{\beta} < 1 \), then \( |t'_{B,r,m}(\beta)| < 4^{m-r+1}\bar{\beta} \).

**Proof.** By definition, \( p(B,j,s) \leq s - j - 1 \) for all eligible \( B,j,s \). Now we claim that

\[
|t_{B,r,s}(\beta)| \leq 4^{s-r}\bar{\beta}. \tag{10.7}
\]
If $s - r = 1$ or $\bar{\beta} = 0$, then by (10.3) we have $t_{B,r,s}(\beta) = 0 \leq 4^{s-r}\bar{\beta}$. Otherwise, by induction on $s - r$, again using (10.3), we have

$$(1/\bar{\beta}) \cdot |t_{B,r,s}(\beta)| \leq \sum_{j=r+1}^{s-1} 2^{p(B,j,s)} + 2(s - r - 1) + 2 \sum_{j=r+1}^{s-1} 4^{s-j} \leq 2^{s-r-1} - 1 + 2(s - r - 1) + \frac{2}{3}(4^{s-r} - 4) \leq 2^{s-r} + \frac{2}{3}4^{s-r} \leq 4^{s-r},$$

as $s - r \geq 2$. Thus (10.7) holds.

Finally, unless $\bar{\beta} = 0$, in which case $t'_{B,r,s}(\beta) = 0$, from (10.4) and (10.7) we conclude that

$$(1/\bar{\beta}) \cdot |t'_{B,r,m}(\beta)| \leq 4^{m-r} + (m - r - 1) + \frac{m-1}{3}(4^{m-r+1} - 4) \leq 4^{m-r+1}. \quad \Box$$

The first $\beta$ to be chosen is $\beta_{1,2}$, and its sign determines the direction of the edge between $(0, 1, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$. If $\beta_{1,2}$ is chosen to be $\pm(4 + \epsilon)^{-n^2}$, then all subsequent choices can be made in such a way that $|\beta_{r,s}| < 1$ for all $r, s$. \quad \Box

### 10.3 The number of USOs arising from a fixed P-matrix

In this section, we determine the number of USOs arising from LCPs with fixed P-matrix.

**Theorem 10.8.** For a P-matrix $M \in \mathbb{R}^{n \times n}$, let $u(M)$ be the number of USOs determined by LCPs $(M, q)$ for $q \in \mathbb{R}^n$. Furthermore, define $u(n) = \max_M u(M)$, where the maximum is over all P-matrices $M \in \mathbb{R}^{n \times n}$. Then

$$u(n) = 2^{\Theta(n^2)}.$$
Proof. Let us first prove the upper bound. For a fixed P-matrix $M \in \mathbb{R}^n$, we consider the sequence

$$\sigma(q) := (\text{sign}(A_{B(v)}^{-1}q)_i : v \in \{0, 1\}^n \text{ and } i \in [n]).$$

Function $\sigma$ completely describes the USOs for various $q \in \mathbb{R}^n$. Each entry in the sequence is determined by the sign of a linear function in the entries of $q$. We consider the arrangement defined by these $n2^n$ hyperplanes and count the number of regions. From Section 10.1, we know that the $LCP(M, q)$ yields a USO whenever $q$ is in some region, and for all $q$ within the same region, it yields the same USO. Thus, the number of regions in the arrangement is an upper bound for the number $u(M)$ of different USOs induced by $M$. It is well-known [20] that the number of regions in an arrangement of $l$ hyperplanes in dimension $k$ is $O(l^k)$. In our case, we have $l = n2^n$ and $k = n$, which shows that $u(M) = O((n2^n)^n) = 2^{O(n^2)}$ for all P-matrices $M$.

For the lower bound, recall that we have constructed in Section 10.2 a K-matrix $M' \in \mathbb{R}^{(n-1)\times(n-1)}$, resulting from fixing $\beta_{i,j}$ for all $j < n$, with the following property. For a suitable right-hand side $q$, $LCP(M, q)$ with

$$M = \begin{pmatrix} M' & b \\ 0 & 1 \end{pmatrix}$$

yields $2^{\Omega(n^2)}$ many different USOs in the subcube $F$ corresponding to vertices with $v_n = 1$, when $b$ is varied.

Since the subcube $F$ corresponds to the solutions of $w - Mz = q$ that satisfy $w_n = 0$, we have $z_n = q_n$ within $F$. With $w' = (w_1, \ldots, w_{n-1})^T$, $z' = (z_1, \ldots, z_{n-1})^T$ and $q' = (q_1, \ldots, q_{n-1})^T$, it follows that

$$w - Mz = q, \quad w^Tz = 0, \quad w_n = 0$$

if and only if

$$w' - M'z' = q' - bq_n, \quad w'^Tz' = 0, \quad z_n = q_n.$$

This is easily seen to imply that the induced USO in the subcube $F$ is generated by the $LCP(M', q' - bq_n)$. Thus, $u(M') = 2^{\Omega(n^2)}$, and the
We are also able to bound the number of LP USOs for a fixed feasible region defining a combinatorial $n$-cube.

The upper bound in Theorem 10.8 was obtained for P-matrices, the lower bound for K-matrices. By Theorem 8.13, the hidden K-matrix $\text{LCP}(M, q)$ realizes the USO arising from the LP $\max q^T x$ subject to $M^T x \leq p$, where $p$ is such that the feasible region defines a combinatorial cube.

**Corollary 10.9.** For an LP, where the feasible region $M^T x \leq p$ is a combinatorial cube, let $u(M, p)$ be the number of USOs obtained by various objective functions $q^T x$. Furthermore, define $u(n) = \max_{M, p} u(M, p)$, where the maximum is over all feasible regions $M^T x \leq p$ defining a combinatorial $n$-cube. Then

$$u(n) = 2^{\Theta(n^2)}.$$

**10.4 A lower bound for strongly Holt-Klee and locally uniform USOs**

A $k$-variate monotone Boolean function is a function $f : \{0, 1\}^k \to \{0, 1\}$ such that if $x \leq y$, then $f(x) \leq f(y)$. Here, $\leq$ is to be understood componentwise.

Counting monotone Boolean functions is known as Dedekind’s problem [18]. Let $M$ be the set of binary vectors of length $k$ with exactly $\lfloor k/2 \rfloor$ ones. A lower bound of $2^{{k \choose \lfloor k/2 \rfloor}}$ on the number of $k$-variate monotone Boolean functions can be obtained by taking for each subset $A \subseteq M$ the function $f_A$ given by

$$f_A(x) = 1 \text{ if and only if } \{y \in A : y \leq x\} \neq \emptyset.$$

**Theorem 10.10.** The number of acyclic, locally uniform, and strongly Holt-Klee USOs of the $n$-cube is at least $2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}} = 2^{\Omega(2^n/\sqrt{n})}$.

**Proof.** Given an $(n-1)$-variate monotone Boolean function $f$, we construct
a USO $\phi$ of the $n$-cube by setting

$$v \xrightarrow{\phi} v \oplus i \text{ if } i \neq n \text{ and } v_i = 0,$$

$$v \xrightarrow{\phi} v \oplus n \text{ if } v_n + f(v') \equiv 1 \pmod{2},$$

where $v' \in \{0, 1\}^{n-1}$ is formed by the initial $n-1$ bits of $v$.

The orientation $\phi$ is clearly acyclic because any directed walk in $\phi$ is monotone on the first $n-1$ bits.

It is easy to show local uniformity too. It suffices to consider pairs of edges $v \rightarrow v \oplus n$ and $v \rightarrow v \oplus i$ for vertices $v$ such that $v_n = v_i = 0$. Then $f(v') = 1$. Since $v \leq u$ for $u := v \oplus i$, we have $f(v') < f(u')$. Hence $f(u') = 1$ and we get $v \oplus i \rightarrow v \oplus \{i, n\}$.

For the strong Holt-Klee property, let $F \subseteq [n]$ and let $V' = \{v \oplus I : I \subseteq C\}$ be the vertex set of a subcube with $|C| =: d$. If $n \notin C$, then $\phi(F)[V']$ is isomorphic to the uniform orientation, which is well-known to satisfy the Holt-Klee condition. So suppose $n \in C$. Let $V_0 := \{v \in V' : v_n = 0\}$ and $V_1 := \{v \in V' : v_n = 1\}$. Note that $\phi(F)[V_0]$ and $\phi(F)[V_1]$ are identical if we truncate the last coordinate of their vertices, and isomorphic to the uniform USO. Now we distinguish two cases. Consider the source $t$ and the sink $s$ of $\phi(F)[V']$. First, if $t_n = s_n$, then there are $d-1$ disjoint paths from $t$ to $s$ in $\phi(F)[V_{s_n}]$ and another path obtained by concatenating the edge $t \rightarrow t \oplus n$, a path in $\phi(F)[V_{1-s_n}]$ from $t \oplus n$ to $s \oplus n$, and the edge $s \oplus n \rightarrow s$.

Secondly, let $t_n = 1 - s_n$. Without loss of generality, we may assume that $t_n = 0$ and $n \notin F$. Let $P(i_1, \ldots, i_d)$ denote the directed path $t \rightarrow t \oplus \{i_1\} \rightarrow t \oplus \{i_1, i_2\} \rightarrow \cdots \rightarrow t \oplus \{i_1, i_2, \ldots, i_d\}$. Order the elements of $C \setminus \{n\} = \{j_1, j_2, \ldots, j_{d-1}\}$ so that for $j_k \in F$ and $j_\ell \notin F$, we have $k < \ell$. Since $\phi(F)[V_0]$ and $\phi(F)[V_1]$ are both isomorphic to the uniform orientation and $t_n \neq s_n$, we have $s = t \oplus C$. Now we claim that the paths $P(j_1, j_2, \ldots, j_{d-1}, n)$, $P(j_2, j_3, \ldots, j_{d-1}, n, j_1)$, $P(j_{d-1}, n, j_1, j_2, \ldots, j_{d-2})$, $P(n, j_1, j_2, \ldots, j_{d-1})$ are vertex-disjoint directed paths from $t$ to $s$. The only non-obvious fact to show is that for any $k$, there is a directed edge $u := t \oplus \{j_k, j_{k+1}, \ldots, j_{d-1}\} \rightarrow v := u \oplus n$. Note that $u \rightarrow v$ if and only if $f(u') = 1$ and that $f(t') = f(s') = 1$. If $j_k \notin F$, then $t' \leq u'$ and so $1 = f(t') \leq f(u')$, and thus $f(u') = 1$. If, on the other
Conclusions

hand, \( j_k \in F \), then \( s' \leq u' \) and so \( 1 = f(s') \leq f(u') \), thus \( f(u') = 1 \). Hence \( u \rightarrow v \).

Therefore, the number of acyclic locally uniform strongly Holt-Klee USOs of the \( n \)-cube is lower bounded by the number of \((n - 1)\)-variate monotone Boolean functions, which concludes the proof. \( \square \)

After swapping the roles of 0 and 1 in the \( n \)th coordinate, the above construction is the same as Mike Develin’s construction [19] of USOs satisfying the Holt-Klee condition. Thus, both Develin’s and our construction yield Holt-Klee USOs, but local uniformity is obtained only in our variant.

10.5 Conclusions

From the discussions in Chapter 8, we conclude that

\[
\text{K-matrix USOs} \subseteq \text{locally uniform P-matrix USOs} \\
\subseteq \text{acyclic P-matrix USOs} \subseteq \text{P-matrix USOs} \\
\subseteq \text{strongly Holt-Klee USOs} \subseteq \text{Holt-Klee USOs}.
\]

We also have

\[
\text{K-matrix USOs} \subseteq \text{hidden K-matrix USOs} = \text{LP-USOs} \\
\subseteq \text{acyclic P-matrix USOs}.
\]

Figure 10.1 illustrates the relationship between these classes. Classes with dashed boxes have purely combinatorial definitions. It may be the case that some nonempty cells in the figure are actually empty. For instance, we do not know whether there exist locally-uniform P-matrix USOs, which are not realized by K-matrix LCPs.

The number of locally uniform USOs grows doubly exponentially with respect to the dimension. By Lemma 8.11, every path in locally-uniform USOs is short. Every pivot rule terminates in at most \( 2^n \) pivot steps. Even though the class is huge, the orientations are well-structured. The intuition that small classes allow a polynomial-time pivot rule, whereas big classes
Figure 10.1: Overview of sizes of USO classes related to the P-matrix LCP.

are less structured and therefore their orientations are hard-to-solve, may be wrong and misleading.

By studying the class sizes, one might get the impression that the class of P-matroid USOs contains many easy-to-solve orientations and relatively few that are hard-to-solve. This conjecture has been suggested by some researchers, but the computed bounds do not support it. Although the lower bound for the number of K-matrix USOs is tight to the upper bound for the number of P-matrix USOs, the latter class may contain exponentially more orientations with respect to the dimension than the former. The asymptotic bounds are tight with respect to the logarithm only.

The number of strongly Holt-Klee USOs is much bigger than the number of P-matrix USOs. We are not aware of any combinatorial property that distinguishes these classes. It is an open question as to whether there exists a combinatorial characterization of P-matrix USOs at all. Note that we are not aware of any (strongly) Holt-Klee USO on which the RandomMurty or RandomFacet pivot rule requires an expected exponential number of pivot steps. On the other hand, it is questionable
whether the strongly Holt-Klee condition suffices to prove the existence of a polynomial-time pivoting strategy for the P-matrix LCP, in case that one exists.
Chapter 11

Enumeration of P-matroid USOs of the 4-cube

Researchers are often faced with the problem of verifying some conjecture. Verification is only possible with a valuable characterization of problem instances, whose number is preferable finite. Especially in the context of LCPs, verification turns out to be a difficult task. There is a significant lack of schemes to construct P-matrices and P-matrix USOs, respectively. With respect to this, we have made some progress, e.g., realizations of cyclic-P-matroids are easily obtained.

Matrices are continuous objects, whereas many solving methods make combinatorial decisions. This was actually our preliminary motivation for the definition of P-matroids and formulation of the OMCP. Oriented matroids are discrete objects and their number is finite. Furthermore, P-matroids have a handy characterization. It seems natural to start enumerating P-matroids and their extensions.

Finschi and Fukuda [23] developed an algorithm for the enumeration of oriented matroids. Enumeration is a difficult task, as it includes the enumeration of order types of point configurations and convex polytopes.
Let OM($r, n$) be the collection of all rank $r$ oriented matroids on $n$ elements. Today’s computational power allows to enumerate the oriented matroids in OM(3, 10) and OM(4, 9). They provide a database, where the oriented matroids are grouped into reorientation classes. For an accessible and efficient encoding, each class is represented by some specific oriented matroid.

Recently, Fukuda, Miyata, and Moriyama [56] classified the oriented matroids in OM(3,9), OM(4,8), and OM(6,9) into realizable and non-realizable matroids. The realizability problem is NP-hard, even for the rank 3 case [57, 75]. Practical methods exist for oriented matroids of low rank with few elements only.

In a joint project with Miyata [30], we completed the task of enumerating the rank 4 P-matroids and the corresponding USOs. In addition, realizability is checked. We attack the problem as follows: in each reorientation class, the representing matroid is picked. For every reorientation equivalent matroid we check whether it is a P-matroid, i.e., we verify condition (a) in Theorem 5.4. Checking whether the basis orientation defines a P-matroid seems natural because every representing matroid is encoded in terms of its basis orientation. Finally, the sets of P-matroids belonging to the same reorientation class are subdivided into classes of FF-equivalent oriented matroids. For each such class, one representative is stored. Classification in this way ensures practical efficiency of the method. Note that the algorithm only considers reorientation classes representing uniform matroids. These are enough to obtain all P-matroid USOs of the 4-cube, which can be shown by a perturbation argument. The realizability problem is decided for each reorientation class.

Table 11.1 summarizes the obtained results. A short discussion follows. Explanations and results concerning extensions and USOs are postponed.

Some results are surprising and unexpected. Every reorientation class in OM(3,6) and OM(4,8) contains P-matroids. In the case of realizable oriented matroids, this has the following interpretation. Any central arrangement in $\mathbb{R}^4$ defined by eight hyperplanes in general position\(^1\) can be labeled and oriented such that it represents a P-matroid. We do not

\(^1\)A central arrangement in $\mathbb{R}^n$ is in general position if every subset of $n$ hyperplanes meets in the origin only.
know whether this observation carries over to general OM(n, 2n). Anyway, this is really bad news implying that for an oriented matroid, being a P-matroid is more a matter of correctly labeled and oriented elements than a property of the underlying combinatorial structure of the arrangement.

Every oriented matroid in OM(3, 6) is realizable, contrary to matroids in OM(4, 8). Since there is at least one P-matroid in every reorientation class, we conclude that there exist non-realizable P-matroids in OM(4, 8). An example of such a matroid was constructed by Morris [59].

The number of FF-equivalent classes of P-matroids is greater than the number of reorientation classes. Hence, some reorientation classes contain several classes of FF-equivalent P-matroids. One such class is the one containing the cyclic-P-matroids, whose representative is the alternating matroid.

Once we knew all P-matroids, we started to enumerate uniform extensions. Uniformity guarantees nondegeneracy, which in turn implies that the extensions realize USOs. By Lemma 8.3, it is enough to construct the extensions for each representative of a reorientation class. These extensions determine the extensions for each representative of FF-equivalent P-matroids. We make use of Finschi’s and Fukuda’s algorithm [23] for the generation of uniform extensions. Computing the corresponding USOs is an easy task. We continually store the USOs in a database, where storage is obviously done up to FF-equivalent USOs. Every recently obtained USO is compared to the orientations in the database and eventually added. Checking whether the USO is a P-matrix USO is done by deciding realizability for the corresponding extension in OM(4, 9).

Table 11.2 summarizes the obtained results.

<table>
<thead>
<tr>
<th></th>
<th>OM(3,6)</th>
<th>OM(4,8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>reorientation classes²</td>
<td>4</td>
<td>135</td>
</tr>
<tr>
<td>reorientation classes containing P-matroids</td>
<td>4</td>
<td>135</td>
</tr>
<tr>
<td>Classes of FF-equivalent P-matroids³</td>
<td>25</td>
<td>83,685</td>
</tr>
</tbody>
</table>

Table 11.1: The number of P-matroids.

²The number of reorientation classes that represent uniform oriented matroids.
³The number of P-matroids, up to proper relabelings and proper reorientations.
First, note that our results for the 3-cube coincide with the results obtained by Stickney and Watson [76].

Since every matroid in OM(3,7) is realizable, it is obvious that every P-matroid USO of the 3-cube is a P-matrix USO. This also holds for P-matroid USOs of the 4-cube, even though OM(4,8) contains non-realizable P-matroids. We conclude that any USO of the 4-cube arising from a non-realizable P-matroid arises from a P-matrix as well. Hence, given a general P-matroid USO, the underlying P-matroid is not uniquely determined.

Note that the class of cyclic-P-matroid USOs is huge. This suggests that a polynomial-time principal pivoting method for the cyclic-P-matrix LCP, if there is one, is difficult to find.

The completed enumeration opens the door to a broad range of new projects. As mentioned in Section 5.4 and illustrated through an example in Section 5.6, hidden K-matroids properly generalize hidden K-matrices. It is an open question as to whether hidden K- and P-matroids are actually the same. We may use our database to verify such a conjecture.

Another possibility is to analyze the relation of reorientation classes and properties of the corresponding USOs. By all means, the role of reorientation classes is unclear so far.

Having a database available allows to tackle computation experiments. We may experiment with different pivot rules and compare them to each

\begin{table}[h]
\centering
\begin{tabular}{|l|c|c|}
\hline
 & 3-cube & 4-cube \\
\hline
USOs & 19 & \text{12,640} \\
acyclic USOs [58] & 18 & 12,640 \\
P-matroid USOs & 17 & 6,910 \\
acyclic P-matroid USOs & 16 & 5,951 \\
cyclic-P-matroid USOs & 17 & 6,077 \\
Classes of FF-equivalent P-matroid USOs$^5$ & 8 & 589 \\
Classes of FF-equivalent P-matrix USOs & 8 & 589 \\
\hline
\end{tabular}
\caption{The number of P-matroid USOs.}
\end{table}

$^4$The number of USOs, up to isomorphism.

$^5$The number of P-matroid USOs, up to isomorphism and facet flips.
other. This might lead to other worst-case orientations, such as Klee-Minty USOs.

Another open project is to enumerate the LP USOs of the 4-cube. The class of hidden K-matroids may be helpful, but at the moment it is unclear how enumeration is exactly done. Computational power is one of the limiting factors.
Bibliography


Academic Curriculum Vitae

Lorenz Klaus
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Education

Supervisor: Prof. K. Fukuda.
Collaborators: Dr. J. Foniok, Dr. B. Gärtner, and Dr. H. Miyata.

Academic visit: Graduate School of Information Science and Technology, University of Tokyo.


Master’s thesis: On Classes of Unique-Sink Orientations Arising from Pivoting in Linear Complementarity.

Academic/Teaching Experience

05/2009–10/2012 Scientific assistant at IFOR.

Teaching duties, such as support and supervision of students in lectures and exercises. I was a teaching assistant of the following lectures:

- Optimization Techniques, fall 2009.
- Integer Programming, spring 2010.
- System Modeling and Optimization, fall 2010.
- Polyhedral Computation, spring 2011.
- Introduction to Optimization, fall 2011.
Talks

- On the Number of Unique-Sink Orientations Arising from Pivoting in Linear Complementarity. *Advanced Course on Optimization: Theory, Methods, and Applications*, CRM Barcelona, July 2009.


- Hidden K-matrix USOs and LP USOs are the same. *First ETH-Japan Workshop on Science and Computing*, Engelberg, March 2012.

- On Combinatorial Generalizations of the Linear Complementarity Problem. *Graduate School of Information Science and Technology*, University of Tokyo, April 2012.

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