Stable numerical schemes for magnetic induction equation with Hall effect

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Stable Numerical Schemes
for Magnetic Induction Equation
with Hall Effect

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Abstract

Magnetic reconnection (a change in magnetic field topology) is an important phenomenon in plasma physics. Standard ideal magnetohydrodynamic models are inadequate for describing reconnection and extended magnetohydrodynamic models are necessary. A popular model that accounts for fast reconnection are the Hall magnetohydrodynamic equations. In this model, the Ohm’s law for the electric field is augmented with magnetic resistivity as well as small scale effects such as electron inertia and Hall effect. In absence of explicit solution formulas, numerical simulations are essential in the study of the Hall magnetohydrodynamic equations.

Given the fact that the additional terms in the Hall magnetohydrodynamic equations vis-à-vis the ideal magnetohydrodynamic equations are only present in the evolution equation for the magnetic field, a necessary first step in the design of efficient numerical methods for the Hall magnetohydrodynamic equations is to construct stable numerical discretisation for the magnetic induction equation with Hall effect. The aim of this thesis is to develop such schemes.

In this thesis, we consider the magnetic induction equation with Hall effect in several space dimensions with given velocity fields and density distributions. We prove apriori energy estimates for the magnetic induction equation with Hall effect and show that the magnetic field is in the Sobolev space $H^1$.

Two classes of energy stable discretisation frameworks are proposed. First we design high-order semi discrete finite difference discretisation using summation by parts discrete derivative operators. Standard Runge Kutta methods are used for the time discretisation. Finite difference methods are suitable for Cartesian meshes. In order to deal with complex domain geometries, we propose energy stable discontinuous Galerkin methods to approximate the magnetic induction equation with Hall effect. These methods are based on a mixed (first order) variational formulation of the underlying system. High-order temporal extrapolation methods are proposed for an implicit-explicit IMEX time discretisation.

Due to the electron inertia term, “large” linear algebraic systems have to be solved at every time step. Efficient preconditioners need to be designed in order to invert ill-conditioned matrices, arising at each time step.

The discontinuous Galerkin methods are particularly suitable for preconditioning using auxiliary space techniques. We describe this procedure and illustrate the resulting gain in efficiency, at least in one space dimension. The finite difference and discontinuous
Galerkin methods are compared on a set of numerical experiments that demonstrate robustness.
Riassunto

La riconnessione magnetica, un cambio di topologia dei campi magnetici, rappresenta un fenomeno di interesse nella fisica dei plasmi. Modelli più raffinati si rendono necessari per descrivere fenomeni di riconnessione magnetica dal momento che il modello più comune, la magnetoidrodinamica ideale, risulta inadeguato. Tra questi quello che tiene in considerazione la riconnessione magnetica rapida è la magnetoidrodinamica con l’effetto Hall. In questa modellizzazione si generalizza la legge di Ohm considerando la resistività ed altri fenomeni che accadono su distanze brevi come quelli derivati dall’inerzia degli elettroni e l’effetto Hall. Formule di risoluzione di questi modelli generalizzati non sono conosciute, quindi ne deriva un ruolo centrale delle simulazioni numeriche nello studio delle equazioni della magnetoidrodinamica che includono l’effetto Hall.

La magnetoidrodinamica ideale e quella che include l’effetto Hall, si differenziano nel modo in cui descrivono l’evoluzione del campo magnetico. Ne deriva che un primo passo necessario nello sviluppo di metodi efficienti per la soluzione delle equazioni della magnetoidrodinamica con l’effetto Hall è quello di costruire dei metodi stabili per la soluzione dell’induzione magnetica. Lo scopo di questo lavoro è di sviluppare suddetti schemi numerici.

In questa tesi consideriamo l’induzione magnetica con l’effetto Hall assumendo che i campi di velocità e la distribuzione di densità siano dati. In questo caso possiamo dimostrare l’esistenza di stime sull’energia delle soluzioni dell’equazione che regola l’induzione magnetica con l’effetto Hall e di conseguenza anche di dimostrare che il campo magnetico è nello spazio di Sobolev $H^1$.

In questo lavoro proponiamo due classi di metodi per risolvere l’induzione magnetica con l’effetto Hall. La prima consiste in schemi semi-impliciti alle differenze finite di ordine elevato nello spazio. Per l’evoluzione nel tempo usiamo metodi standard di Runge Kutta. I metodi basati alle differenze finite sono adatti alla discretizzazione su griglie cartesiane. Nel caso di domini più complessi, abbiamo sviluppato dei metodi discontinui di Galerkin che risultano stabili possedendo delle stime sull’energia a livello discreto. Questi metodi sono basati su una formulazione variazionale (di primo ordine) del sistema di equazioni preso in considerazione. Metodi di estrapolazione di ordine elevato vengono usati per una discretizzazione temporale di tipo esplicito-implicito (IMEX).

La presenza di fenomeni di inerzia degli elettroni dà origine a sistemi lineari di grandi dimensioni che devono essere risolti ad ogni passo dell’evoluzione temporale. La necessità di invertire, ad ogni passo di tempo, le matrici mal condizionate risultanti da questi sistemi richiede lo sviluppo di efficaci precondizionatori.
I metodi discontinui di Galerkin sono idonei a esser precondizionati usando tecniche basate su spazi ausiliari. Descriveremo tali tecniche e ne illustreremo l’efficacia, almeno in una dimensione spaziale. I metodi basati sulle differenze finite e quelli basati su metodi discontinui di Galerkin sono confrontati su un serie di esperimenti numerici che ne dimostrano la robustezza.
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1. Introduction

Plasmas are becoming increasingly important in a variety of fields like astrophysics, solar physics, electrical and aerospace engineering [1]. Specific problems include the study of supernovas, accretion disks, waves in the solar atmosphere, magnetic confinement fusion, the design of plasma thrusters for spacecraft propulsion and of circuit breakers in the electrical power industry. An important phenomenon occurring in plasmas is magnetic reconnection where magnetic field lines change their topology. In this singular point where the topology changes a so-called current sheet is formed and the plasma is accelerated through a small channel.

Reconnection is a very complex event and different models compete to give a better quantitative explanation for this phenomenon. Numerical simulations are a leading tool to analyse and verify different models. A standard approach to describe plasmas is to model them as magnetised fluids with fluid motion shaping and in turn being shaped by magnetic fields. These models are termed magnetohydrodynamics (MHD) models. There are different types of MHD models that differ essentially in the treatment of the Ohm’s law. The simplest version is the ideal MHD where the electric field $E$ is given by

$$ E = -u \times B. $$ (1.1)

Here, $u$ is the fluid velocity and $B$ the magnetic field. The ideal MHD equations are extremely successful in several applications, see [3] for an overview and [12] for a recent application in solar physics. One issue with this model is that it cannot be used to describe reconnection, since one can prove the "frozen in condition" which asserts that the magnetic field lines have to follow the fluid. In this case, we do not have any dissipation, which is considered to be the main ingredient for the formation of the current sheet present during reconnection. In order to induce reconnection, a possible mechanism is magnetic resistivity resulting in the Ohm’s law:

$$ E = -u \times B + \eta J. $$ (1.2)

Here, $J$ is the current density and $\eta$ is the resistivity parameter. The resulting equations are termed the resistive MHD equations. However, the resistive MHD equations do not suffice in modeling fast magnetic reconnection. A more effective alternative is to include the Hall effect [19, 26]. Hall effect and other effects that are normally negligible become more important during reconnection. We are going now to derive the MHD equations to understand better the
Introduction

different terms present in the equation, and also to see under which assumptions this model is valid.

1.1. Derivation of the MHD Equations

To derive the MHD equations we start from a statistical description of charged particles, where the distribution function $f_s(x, v, t)$ gives us the average number of particles of kind $s$ with mass $m_s$ and electrical charge $q_s$ in the phase space at position $x$ and with velocity $v$ at time $t$. The number of particles $dN_s$ present at time $t$ in a small phase volume element $d^3x d^3v$ will be given by

$$dN_s = f_s(x, v, t) \, d^3x \, d^3v.$$

The evolution in time of the distribution function is governed by the collisional Boltzmann-Vlasov equation

$$\frac{\partial f_s}{\partial t} + v \cdot \nabla f_s + \frac{q_s}{m_s} (E + v \times B) \cdot \nabla_v f_s = \frac{\delta c f_s}{\delta t} \tag{1.3}$$

where $\delta$ denotes the delta function. The term on the right hand side will represent the collisions. A model for collisions will be given later but it is important to outline that collisions do not change the mass of one species and also they do not change the total momentum of all the species. We are going to describe the plasma not by distributions in phase space but by macroscopic quantities in physical space. To achieve this, we multiply $f_s$ by polynomial functions of $v$ and integrate over the whole velocity space. We assume that the distribution function decays fast enough at infinity, so that the different integrals are bounded. We define the following quantities

- number density, $n_s(x, t) = \int_{\mathbb{R}^3} f_s(x, v, t) \, d^3v$,
- average velocity, $u_s(x, t) = \frac{1}{n_s(x, t)} \int_{\mathbb{R}^3} v f_s(x, v, t) \, d^3v$,
- pressure tensor, $\mathbf{P}_{0,s}(x, t) = \int_{\mathbb{R}^3} m_s (v - u_s) \otimes (v - u_s) \, f_s(x, v, t) \, d^3v$,

where $\otimes$ denotes the tensor product. To derive the evolution equations of these macroscopic quantities we multiply (1.3) by 1 and $m_s v_s$ and then integrate over the velocity space. If we assume that the electric field $E$ does not depend explicitly on the velocity, and since we know collisions do not change the particle density, we obtain the conservation of mass and momentum

$$\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s u_s) = 0 \tag{1.4}$$
1.1. Derivation of the MHD Equations

\[
\frac{\partial m_s n_s \mathbf{u}_s}{\partial t} + \nabla \cdot (m_s n_s \mathbf{u}_s \otimes \mathbf{u}_s) = n_s q_s (\mathbf{E} + \mathbf{u}_s \times \mathbf{B}) - \nabla \cdot \mathbf{P}_0,s + \frac{\delta_c p_s}{\delta t}. \tag{1.5}
\]

The collision term \(\delta_c p_s/\delta t\) is the collisional drag force per volume, which gives the change in momentum due to the collisions between species. We see that through this procedure of multiplying and integrating the Boltzmann-Vlasov equation, we introduce new macroscopic quantities, since the change in time of a certain quantity depends on a higher order moment, e.g., evolution of momentum depends on pressure. We are faced with what is called the closure problem: we have to specify the unknown moments through the ones already known. In our case we can close the problem by giving an equation of state for the pressure. In the case of MHD, we will use an equation of state for ideal gases.

In MHD, we do not directly consider the mixture of two gases separately, but as an approximation, as a single species gas subject to currents, magnetic and electric fields. Let us consider a mixture of electrons labeled \(s = e\) and ions labeled \(s = i\) with respective charges \(q_e = -e\) and \(q_i = e\). We also assume, that the gas satisfies the quasi-neutrality condition, meaning that locally there are no free charges. In this case we have the same number density for ions and electrons \(n = n_i = n_e\) since the total charge \(n_i q_i + n_e q_i = e(n_i - n_e)\) is zero at every point in space.

The evolution equation of the density \(\rho := n (m_i + m_e)\) is obtained by multiplying the equation for the conservation of the number density (1.4) for each species by their respective masses, and then summing them together,

\[
\frac{\partial n (m_i + m_e)}{\partial t} + \nabla \cdot (n (m_i \mathbf{u}_i + m_e \mathbf{u}_e)) = 0,
\]

\[\Rightarrow \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \tag{1.6}\]

Here, we have introduced the fluid velocity \(\mathbf{u} := (m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_i + m_e)\).

Proceeding in the same way, we are going to derive an equation for the evolution of the total momentum \(\rho \mathbf{u}\). We sum the momentum equation (1.5) over species \(s = i, e\) and obtain

\[
\frac{\partial n (m_i \mathbf{u}_i + m_e \mathbf{u}_e)}{\partial t} + \nabla \cdot (n (m_i \mathbf{u}_i \otimes \mathbf{u}_i + m_e \mathbf{u}_e \otimes \mathbf{u}_e))
\]

\[= n e (\mathbf{u}_i - \mathbf{u}_e) \times \mathbf{B} - \nabla \cdot (\mathbf{P}_{0,i} + \mathbf{P}_{0,e}) + \frac{\delta_c p_i}{\delta t} + \frac{\delta_c p_e}{\delta t},\]

\[\Rightarrow \frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho (m_i \mathbf{u}_i \otimes \mathbf{u}_i + m_e \mathbf{u}_e \otimes \mathbf{u}_e)) = \mathbf{J} \times \mathbf{B} - \nabla \cdot (\mathbf{P}_{0,i} + \mathbf{P}_{0,e}).\]

Here, we have used the electric current density defined as \(\mathbf{J} := n e (\mathbf{u}_i - \mathbf{u}_e)\) and also the fact that the sum of the two collision terms \((\delta_c p_i)/(\delta t)\) and \((\delta_c p_e)/(\delta t)\) is zero since the total momentum is conserved for any collision.

To further simplify the equation, we introduce a new pressure tensor which is related to
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the mean speed of the fluid,

\[ \hat{P}_s(x,t) = m_s \int_R (v - u) \otimes (v - u) f_s(x,v,t) \, d^3v. \]

From the definitions of these two pressure tensors, we can express the relation between them by

\[ \hat{P}_s = \hat{P}_{0,s} + nm_s w_s \otimes w_s, \]

where we have introduced the diffusion velocity \( w_s := u_s - u \). Inserting the new pressure tensor in the equation and also substituting the velocities with the diffusive velocities \( u_s = w_s + u \) for \( s = i,e \), we get

\[ \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u + n (m_i (w_i + u) \otimes (w_i + u) + m_e (w_e + u) \otimes (w_e + u))) = J \times B - \nabla \cdot (\hat{P}_i + \hat{P}_e), \]

\[ \Rightarrow \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u + n (m_i w_i + m_e w_e) \otimes u + u \otimes n(m_i w_i + m_e w_e)), \]

\[ = J \times B - \nabla \cdot (\hat{P}_i + \hat{P}_e) \]

\[ \Rightarrow \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u) = J \times B - \nabla \cdot \hat{P}. \] (1.7)

In this case, we have defined the fluid pressure tensor as \( \hat{P} := \hat{P}_i + \hat{P}_e \) and used that the terms \( n (m_i w_i + m_e w_e) \) vanish by the definition of \( w_s \). This equation represents the conservation of momentum of the plasma. We still need some manipulation to express these conservation laws in an explicit flux form. To obtain an equivalent formulation we first use Ampère’s law \( J = \nabla \times B \) and then setting \( f = g = B \) in the vector identity

\[ \nabla (f \cdot g) = (f \cdot \nabla)g + f \times (\nabla \times g) + (g \cdot \nabla)f + g \times (\nabla \times f), \] (1.8)

we can rewrite

\[ J \times B = (B \cdot \nabla)B + \nabla \frac{|B|^2}{2}. \]

One can verify that we can write the left-hand-side of the equation as the negative of the divergence of the tensor

\[ \hat{T} := -B \otimes B + I_{3 \times 3} \frac{|B|^2}{2}. \]

By using this tensor we obtain

\[ \frac{\partial \rho u}{\partial t} + \nabla \cdot (\rho u \otimes u) = -\nabla \cdot (\hat{T} + \hat{P}). \] (1.9)
1.1. Derivation of the MHD Equations

We have seen that to obtain the conservation of momentum in plasmas, we need to sum the momentum equations (1.5) for the different species.

The Ohm’s law relates the electric field with the current in the plasma, and since the electric current density is defined to be proportional to the difference between the velocity of the species, we need to subtract a weighted version of the moment equations. Since we are subtracting the two moment equations, the collision term is not vanishing. This means that we have to model the collision terms explicitly. We assume that the drag force on the sth species is proportional to the difference between the velocity of sth and rth species

\[
\frac{\delta_c p_s}{\delta t} = -n m_s \nu_{sr} (u_s - u_r), \quad s \neq r,
\]

where \( \nu_{sr} \) is the collision frequency between the two species. Now that we have specified the collision term, we can go back to the momentum equation for the different species (1.5). We multiply the ion momentum equation by \( m_i \) and subtract the one for electron multiplied by \( m_e \), obtaining

\[
m_e m_i \left( \frac{\partial n (u_i - u_e)}{\partial t} + \nabla \cdot (n (u_i \otimes u_i - u_e \otimes u_e)) \right) = n e (m_i + m_e) E
\]

\[
+ e n (m_e u_i + m_i u_e) \times B - m_e \nabla \cdot \mathbf{P}_{0,i} + m_i \nabla \cdot \mathbf{P}_{0,e} - m_e m_i n \nu_{ie} (u_i - u_e) + m_e m_i n \nu_{ei} (u_e - u_i),
\]

\[
\Rightarrow \quad \frac{m_e m_i}{e} \left( \frac{\partial J}{\partial t} + \nabla \cdot (e n (u_i \otimes u_i - u_e \otimes u_e)) \right) = e \rho E + e n (m_e u_i + m_i u_e) \times B
\]

\[
- m_e \nabla \cdot \mathbf{P}_{0,i} - m_i \nabla \cdot \mathbf{P}_{0,e} - \frac{m_e m_i (\nu_{ei} + \nu_{ie})}{e} J.
\] (1.10)

As we already have stated before, the collisions between species do not change the total momentum of the gas, implying that the frequency of collision should be related. We know that

\[
0 = \frac{\delta_c p_i}{\delta t} + \frac{\delta_c p_e}{\delta t} = n (m_i \nu_{ie} - m_e \nu_{ei})(u_e - u_i).
\]

The last equation should be valid for all the collisions, in particular for an arbitrary difference between the velocities, hence the frequencies of the collisions satisfy \( m_i \nu_{ie} = m_i \nu_{ei} \). In this case we can rewrite the last term in (1.10) as

\[
\frac{m_e m_i (\nu_{ei} + \nu_{ie})}{e} = \frac{m_e \rho \nu_{ei}}{n e} = e \rho \eta,
\]

where we have defined the resistivity as \( \eta := (m_e \nu_{ei})/(e^2 n) \). At this point (1.10) can be rewritten as

\[
E = \eta J - \frac{n}{\rho} (m_e u_i + m_i u_e) \times B + \frac{m_e}{e \rho} \nabla \cdot \mathbf{P}_{0,i} - \frac{m_i}{e \rho} \nabla \cdot \mathbf{P}_{0,e}.
\]
**Introduction**

\[ + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left( e n (\mathbf{u}_i \otimes \mathbf{u}_i - \mathbf{u}_e \otimes \mathbf{u}_e) \right) \right). \] (1.11)

Although this equation is almost in the fluid formulation, we still have the different velocities of the species \( \mathbf{u}_s \) and the pressures relative to the species. We can now use the definition of the average fluid velocity \( \mathbf{u} = (m_i \mathbf{u}_i + m_e \mathbf{u}_e) / (m_i + m_e) \) and the definition of the electric current density \( \mathbf{J} = e n (\mathbf{u}_i - \mathbf{u}_e) \) to express the ion and electron mean velocities:

\[
\mathbf{u}_i = \mathbf{u} + \frac{m_e}{e \rho} \mathbf{J},
\]

\[
\mathbf{u}_e = \mathbf{u} - \frac{m_i}{e \rho} \mathbf{J}.
\]

We can directly use these two expressions to rewrite parts of (1.11):

\[
\frac{n}{\rho} (m_e \mathbf{u}_i + m_i \mathbf{u}_e) = \mathbf{u} + \frac{m_e - m_i}{e \rho} \mathbf{J},
\]

\[
\mathbf{u}_i \otimes \mathbf{u}_i = \mathbf{u} \otimes \mathbf{u} + \frac{m_e}{e \rho} (\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u}) + \frac{m_e^2}{e^2 \rho^2} \mathbf{J} \otimes \mathbf{J},
\]

\[
\mathbf{u}_e \otimes \mathbf{u}_e = \mathbf{u} \otimes \mathbf{u} - \frac{m_i}{e \rho} (\mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u}) + \frac{m_i^2}{e^2 \rho^2} \mathbf{J} \otimes \mathbf{J}.
\]

For the pressure terms, we first need to express them with respect to the fluid velocity \( \mathbf{u} \). To do that, we will have additional terms that depend on the diffusion velocity. The diffusion velocity can be also expressed by means of current and fluid velocity as

\[
\mathbf{w}_i = \frac{m_e}{e \rho} \mathbf{J}, \quad \mathbf{w}_e = -\frac{m_i}{e \rho} \mathbf{J}.
\]

The pressures becomes

\[
\mathbf{P}_i = \mathbf{P}_{0,i} + \frac{n m_i m_e^2}{e^2 \rho^2} \mathbf{J} \otimes \mathbf{J},
\]

\[
\mathbf{P}_e = \mathbf{P}_{0,e} + \frac{n m_e m_i^2}{e^2 \rho^2} \mathbf{J} \otimes \mathbf{J}.
\]

Now we can insert these results in (1.11) to obtain

\[
E = \eta \mathbf{J} - \mathbf{u} \times \mathbf{B} - \frac{m_e - m_i}{e \rho} \mathbf{J} \times \mathbf{B} + \frac{m_e}{e \rho} \nabla \cdot \mathbf{P}_i - \frac{m_i m_e}{e^2 \rho} \nabla \cdot \left( \frac{m_e - m_i}{e \rho} \mathbf{J} \otimes \mathbf{J} \right)
\]

\[
- \frac{m_i}{e \rho} \nabla \cdot \mathbf{P}_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \left( \mathbf{u} \otimes \mathbf{J} + \mathbf{J} \otimes \mathbf{u} + \frac{m_e - m_i}{e \rho} \mathbf{J} \otimes \mathbf{J} \right) \right)
\]

\[
= \eta \mathbf{J} - \mathbf{u} \times \mathbf{B} - \frac{m_e - m_i}{e \rho} \mathbf{J} \times \mathbf{B} + \frac{m_e}{e \rho} \nabla \cdot \mathbf{P}_i
\]
1.1. Derivation of the MHD Equations

\[- \frac{m_i}{e\rho} \nabla \cdot \vec{P}_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial J}{\partial t} + \nabla \cdot (\vec{u} \otimes \vec{J} + \vec{J} \otimes \vec{u}) \right).\]

In a plasma, the mass of the ions is much larger than that of the electrons, \(m_e \ll m_i\), so we can further approximate:

\[E = \eta J - \vec{u} \times \vec{B} + \frac{m_i}{e\rho} \nabla \cdot \vec{P}_e - \frac{m_i}{e\rho} \nabla \cdot \vec{P}_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial J}{\partial t} + \nabla \cdot (\vec{u} \otimes \vec{J} + \vec{J} \otimes \vec{u}) \right).\]

The advection term for the current can be further simplified. We rewrite it using the vector identity

\[\nabla \cdot (f \otimes g) = (\nabla \cdot f) g + (f \cdot \nabla) g\]

we get

\[\nabla \cdot (\vec{u} \otimes \vec{J} + \vec{J} \otimes \vec{u}) = (\nabla \cdot \vec{u}) \vec{J} + (\vec{u} \cdot \nabla) \vec{J} + (\vec{J} \cdot \nabla) \vec{u}.\]

We have used that the electric current density \(\vec{J}\) is divergence free. The reconnection phenomenon is driven more by electric forces and magnetic forces than by fluid effects. This means that during the reconnection we can consider the terms containing a derivative of the velocity times the current to be much smaller than the term \((\vec{u} \cdot \nabla) \vec{J}\). These term are consequentially dropped. Under these assumptions, the generalised Ohm’s law equation becomes

\[E = \eta J - \vec{u} \times \vec{B} + \frac{m_i}{e\rho} \nabla \cdot \vec{P}_e - \frac{m_i}{e\rho} \nabla \cdot \vec{P}_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial J}{\partial t} + \vec{u} \cdot \nabla \vec{J} \right).\]

The set of the fluid equations is almost complete but the system is not closed, we still have to specify the pressure tensor \(\vec{P}\) as function of the other variables. If we consider that there is enough collision in the plasma, then the pressure tensor is isotropic \(\nabla \cdot \vec{P} = \nabla p\).

We are further assuming that the scalar pressure \(p\) follows the polytropic equation of state:

\[\frac{d}{dt}(p \rho^{-\gamma}) = 0,\]

where \(\gamma\) is the polytropic index. To insert this rule in our framework of equations, we need to derive the equation for the conservation of energy in the plasma. If we take the conservation of momentum (1.7) and use the chain rule and the vector law (1.12) with \(f = \rho \vec{u}\) and \(g = \vec{u}\) we obtain

\[\rho \frac{\partial \rho}{\partial t} + \rho \frac{\partial \vec{u}}{\partial t} + (\nabla \cdot \rho \vec{u}) \vec{u} + \rho (\vec{u} \cdot \nabla) \vec{u} = \vec{J} \times \vec{B} - \nabla p \Rightarrow \rho \left( \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} \right) = \vec{J} \times \vec{B} - \nabla p,\]
where in the last step we used the mass conservation (1.6). To obtain the energy equation we multiply the above equation by \(\mathbf{u}\)

\[
\rho \mathbf{u} \cdot \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) - (\mathbf{u} \cdot \nabla) p \tag{1.15}
\]

and rewrite pointing out the different terms composing it. We start by reformulating the left hand side

\[
\rho \mathbf{u} \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \frac{1}{2} \rho \left( \frac{\partial |\mathbf{u}|^2}{\partial t} + \mathbf{u} \cdot \nabla |\mathbf{u}|^2 \right)
\]

\[
= \frac{\partial}{\partial t} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) - \frac{1}{2} |\mathbf{u}|^2 \frac{\partial \rho}{\partial t} + \rho \mathbf{u} \cdot \nabla |\mathbf{u}|^2
\]

\[
= \frac{\partial}{\partial t} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) + \frac{1}{2} |\mathbf{u}|^2 \nabla \cdot (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla |\mathbf{u}|^2.
\]

Here, we have used the mass conservation equation (1.6). Using again the vector identity (1.12) we get

\[
\rho \mathbf{u} \left( \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right) = \frac{\partial}{\partial t} \left( \frac{1}{2} \rho |\mathbf{u}|^2 \right) + \nabla \cdot \left( \frac{1}{2} \rho \mathbf{u} |\mathbf{u}|^2 \right) \tag{1.16}
\]

The left hand side of (1.15) contains the gradient of the pressure times the velocity, this term can be reformulated by means of the equation of state (1.14). Using the chain rule, we obtain

\[
\frac{dp}{dt} \rho^\gamma - \gamma p \rho^{-\gamma(p+1)} \frac{d\rho}{dt} = 0 \Rightarrow \frac{dp}{dt} = \frac{\gamma p d\rho}{\rho} \frac{d\rho}{dt}.
\]

This equation is valid on a fluid volume element at rest. Since this pressure is evaluated in a fluid moving with velocity \(\mathbf{u}\), we take into account motion writing the total derivatives as convective derivatives \(d/dt = \partial / \partial t + \mathbf{u} \cdot \nabla\) obtaining

\[
\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p = \frac{\gamma p}{\rho} \left( \frac{\partial \rho}{\partial t} + (\mathbf{u} \cdot \nabla) \rho \right).
\]

We eliminate the partial derivative in \(\rho\) using the conservation of mass (1.6)

\[
\frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p = \gamma p \nabla \cdot \mathbf{u} = \gamma \left( \nabla \cdot (\rho \mathbf{u}) - (\mathbf{u} \cdot \nabla) p \right)
\]

Solving the last equation for \((\mathbf{u} \cdot \nabla) p\) we obtain

\[
(\mathbf{u} \cdot \nabla) p = \frac{1}{\gamma-1} \frac{\partial p}{\partial t} + \frac{\gamma}{\gamma-1} \nabla \cdot \rho \mathbf{u}. \tag{1.17}
\]

The terms in (1.15) that we have already reformulated represent the contribution of
1.1. Derivation of the MHD Equations

kinetic energy and pressure to the total energy of plasma. We see that it remains to rewrite the term $\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B})$ to complete formulation of the energy equation. This last term adds the contribution of the electric forces to the energy. To extract the different parts we rearrange the terms $\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) = -(\mathbf{u} \times \mathbf{B}) \cdot \mathbf{J}$ and use the generalised Ohm’s law (1.13)

$$
(u \times B) = \eta J - E + \frac{m_i}{e \rho} J \times B - \frac{m_i}{e \rho} \nabla p_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial J}{\partial t} + (\mathbf{u} \cdot \nabla) J \right),
$$

where, exactly as we have done for the fluid pressure, we assumed that the electron pressure tensor is isotropic and diagonal. The energy term becomes

$$
\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) = \mathbf{E} \cdot (\nabla \times \mathbf{B}) - \eta |\mathbf{J}|^2 + \frac{m_i}{e \rho} (\nabla p_e) \mathbf{J} - \frac{m_e m_i}{e \rho} \frac{J}{2} \mathbf{J} \frac{\partial J}{\partial t} - \frac{m_e m_i}{2} \nabla \cdot (\mathbf{u} |\mathbf{J}|^2).
$$

Here, we have used the fact that the electric current density is divergence free, $\nabla \cdot \mathbf{J} = 0$. The term $(\nabla \cdot \mathbf{u}) |\mathbf{J}|^2$ is dropped due to similar considerations as above.

Using Ampère’s law we can rewrite the term containing the electric field as

$$
\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B})
$$

where we have used the vector identity

$$
(\nabla \cdot (f \times g)) = g \cdot (\nabla \times f) - f \cdot (\nabla \times g).
$$

The Faraday’s law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ implies

$$
\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \mathbf{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) = -\mathbf{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) - \frac{1}{2} \frac{\partial |\mathbf{B}|^2}{\partial t}.
$$

Inserting this result in (1.18) we get the contribution of electromagnetism to energy evolution as

$$
\mathbf{u} \cdot (\mathbf{J} \times \mathbf{B}) = -\mathbf{\nabla} \cdot (\mathbf{E} \times \mathbf{B}) - \frac{\partial |\mathbf{B}|^2}{\partial t} - \eta |\mathbf{J}|^2 + \frac{m_i}{e \rho} \mathbf{\nabla} \cdot (p_e \mathbf{J})
$$

$$
- \frac{m_e m_i}{2} \frac{\partial |\mathbf{J}|^2}{\partial t} - \frac{m_e m_i}{2} \mathbf{\nabla} \cdot (|\mathbf{J}|^2).
$$

Now we have the different terms that contribute to the energy of the plasma, we can define the total energy $\mathcal{E}$ collecting all the contribution together

$$
\mathcal{E} := \frac{p}{\gamma - 1} + \frac{\rho |\mathbf{u}|^2}{2} + \frac{|\mathbf{B}|^2}{2} + \frac{m_e m_i |\mathbf{J}|^2}{e^2 \rho}.
$$
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Putting (1.16), (1.17) and (1.21) together to rewrite the energy term (1.15) we obtain

$$\frac{\partial E}{\partial t} = -\nabla \cdot \left\{ \left( E + p - \frac{|B|^2}{2} \right) u + E \times B - p_e J \right\} - \eta |J|^2. \tag{1.23}$$

Finally taking (1.6), (1.9) and (1.23) we can write down the evolution equation as

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho u), \tag{1.24a}$$

$$\frac{\partial (\rho u)}{\partial t} = -\nabla \cdot \left\{ \rho \otimes u + \left( p + \frac{|B|^2}{2} \right) I_{3 \times 3} - B \otimes B \right\}, \tag{1.24b}$$

$$\frac{\partial E}{\partial t} = -\nabla \cdot \left\{ \left( E + p - \frac{|B|^2}{2} \right) u - p_e J + E \times B \right\} - \eta |J|^2, \tag{1.24c}$$

$$\frac{\partial B}{\partial t} = -\nabla \times E, \tag{1.24d}$$

and using (1.22) and (1.13) the constitutive part

$$E = \frac{p}{\gamma - 1} + \frac{\rho |u|^2}{2} + \frac{|B|^2}{2} + \frac{m_e m_i |\mathbf{J}|^2}{2}, \tag{1.25a}$$

$$E = \eta \mathbf{J} - \mathbf{u} \times \mathbf{B} + \frac{m_i}{e \rho} \mathbf{J} \times \mathbf{B} - \frac{m_i}{e \rho} \nabla p_e + \frac{m_e m_i}{e^2 \rho} \left( \frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{J} \right). \tag{1.25b}$$

This complete the description of the MHD equations with Hall effect.

1.2. Hall Induction Equations

The simulation of Hall MHD equations (1.24-1.25) is a challenging problem, since different non-linear terms come into play coupling the electromagnetic and fluid behaviours. This system can be seen as an extension of the ideal MHD problem, where additional small scale effects are taken into account. Since ideal MHD is a well studied problem, we concentrate on analysis of the new terms added to the system. Neglecting the pressure term and considering Faraday’s law (1.24d) together with the Ohm’s law (1.25b) we obtain

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \eta \nabla \times \mathbf{J} - \frac{m_i}{e \rho} \nabla \times \left( \frac{\mathbf{J} \times \mathbf{B}}{\rho} \right)$$

$$- \frac{m_e m_i}{e^2 \rho} \nabla \times \left\{ \frac{1}{\rho} \left( \frac{\partial \mathbf{J}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{J} \right) \right\}.$$
and if we use Ampère’s law, the resulting equation is

$$\frac{\partial B}{\partial t} + \frac{m_e m_i}{e^2} \nabla \times \left( \frac{1}{\rho} \nabla \times \frac{\partial B}{\partial t} \right) = \nabla \times (u \times B) - \eta \nabla \times (\nabla \times B)$$

$$- \frac{m_i m_e}{e^2} \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla) (\nabla \times B) \right) - \frac{m_i}{e} \nabla \times \left( \frac{\nabla \times B}{\rho} \times B \right).$$

(1.26)

This equation is called the magnetic induction equation with Hall effect. Instead of discussing the entire Hall MHD problem in this work, we are concentrating on the study of the sub problem of the resulting induction equation (1.26).

In chapter 2, we are going to investigate the existence and uniqueness of the solution of (1.26). We are able to prove existence introducing suitable boundary conditions and under certain regularity assumptions on the velocity $u$ and density $\rho$. Assuming a more regular solution $B$, we can show also the uniqueness of that solution. This will provide some characterisation of the solution of (1.26). In the third chapter, we will use the theoretical results obtained in chapter two to approach the problem from the point of view of finite difference schemes. The resulting scheme is effective but may be inefficient, since the formulation will not be well conditioned. We will show how by using auxiliary space preconditioning, we could not improve the performance since we could not find an efficient projection operator. We were confronted with the choice of continuing to look for a projector operator, that could have successfully preconditioned the problem, or to formulate (1.26) in another way such that the auxiliary space preconditioning would have been effective. We choose to look for a different discretisation approach, the discontinuous Galerkin (DG) formulation. We have chosen this numerical formulation not only expecting to find a suitable preconditioner for the solution of our numerical problem, but also to add versatility to the numerical method. DG method provides us a lot of freedom in the choice of the mesh used in the simulation, and also a more natural way to impose boundary conditions.

To test our method, we have first studied and simulated the one dimensional version of the problem (1.26). This one dimensional problem will be presented in chapter 4. At last, in chapter 5 we will apply the DG method to the full induction equation with Hall effect (1.26).

In the last chapter we will give an overview of challenge one has still to overcome to finally couple the discretisation of the Hall induction equations with a MHD code.
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2. The Continuous Problem

2.1. Symmetrized Equations

The main aim of this chapter is to analyse the Hall magnetic induction equation (1.26) and to derive suitable boundary conditions that will result in an energy estimate. It turns out that the advection terms $\nabla \cdot (u \times B)$ are not symmetric and impair the derivation of an energy estimate.

To overcome this problem we have to exploit a key feature of the magnetic fields. A magnetic field is normally considered to be divergence free, since, no proof was found of the existence of magnetic monopoles in nature. Hall magnetic induction equation (1.26) is derived by the application of the Faraday’s law $\frac{\partial B}{\partial t} = -\nabla \times E$ to a generalised Ohm’s law. Faraday’s law implies $\frac{\partial \nabla \cdot B}{\partial t} = 0$ for every electric field that is smooth enough, therefore this is valid also for (1.26). Implying that the Hall magnetic induction equation preserves the divergence of the magnetic field in time. If we take physically relevant initial data with $\nabla \cdot B_0 = 0$, we will have that the solution of (1.26) will satisfy

$$\nabla \cdot B = 0 \text{ for all time } t.$$  \hspace{1cm} (2.1)

Using this additional condition on the magnetic field, we will symmetrize the advection term (see [11, 17, 30]). First, we use the vector identity

$$\nabla \times (u \times B) = (B \cdot \nabla)u - B(\nabla \cdot u) + u(\nabla \cdot B) - (u \cdot \nabla)B.$$ \hspace{0.5cm} (2.2)

Then, we use the divergence constraint (2.1) and subtract $u(\nabla \cdot B)$ from (1.26) to obtain the symmetric form of the magnetic induction equations with Hall effect:

$$\frac{\partial B}{\partial t} + \alpha \nabla \times \left( \frac{1}{\rho} \nabla \times \frac{\partial B}{\partial t} \right) = (B \cdot \nabla)u - B(\nabla \cdot u) - (u \cdot \nabla)B - \eta \nabla \times (\nabla \times B)$$

$$-\alpha \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla)(\nabla \times B) \right) - \beta \nabla \times \left( \frac{\nabla \times B}{\rho} \times B \right),$$ \hspace{0.5cm} (2.3)

where $\alpha = (m_e m_i)/(e^2)$ and $\beta = m_i/e$ are positive constants that depend on plasma parameters: electron and ion masses $m_e$ and $m_i$ and elementary charge $e$.

Our goal is to solve (2.3) on an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$, but before we can do that, we have to specify the behaviour of the solution on the domain boundary respectively. Looking at the dynamics of the equation, we can identify three different components of the evolution of the magnetic field. The first two are that the magnetic
2. The Continuous Problem

field and the electric current (if $\alpha \neq 0$) are advected with velocity $\mathbf{u}$. Hence, information is transported into the domain where the velocity field on the boundary is directed into the volume of interest. This part of boundary is called the inflow boundary. More formally, assuming that the velocity field $\mathbf{u}$ is continuous for each time $t$, we can subdivide the boundary $\partial \Omega$ with outer normal $\mathbf{n}$, in two parts

$$
\Gamma_+(t) = \{ \mathbf{x} \in \partial \Omega | \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n} < 0 \}, \quad \Gamma_-(t) = \{ \mathbf{x} \in \partial \Omega | \mathbf{u}(\mathbf{x}, t) \cdot \mathbf{n} \geq 0 \}.
$$

In this case $\Gamma_+$ and $\Gamma_-$ are called the inflow and the outflow boundary. The third part of the equation that needs boundary conditions is the dissipation caused by the resistivity (if $\eta \neq 0$). In this case, we have to specify the tangential components of the magnetic field.

Putting together all the boundary conditions we have

$$
\frac{\partial \mathbf{B}}{\partial t} + \alpha \nabla \times \left( \frac{1}{\rho} \nabla \times \frac{\partial \mathbf{B}}{\partial t} \right) = (\mathbf{B} \cdot \nabla) \mathbf{u} - \mathbf{B}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla) \mathbf{B} - \eta \nabla \times (\nabla \times \mathbf{B})
$$

$$
-\alpha \nabla \times \left( \frac{1}{\rho} (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{B}) \right) - \beta \nabla \times \left( \frac{\nabla \times \mathbf{B}}{\rho} \times \mathbf{B} \right) \quad \text{in } \Omega,
$$

$$
\mathbf{B} = \mathbf{f} \quad \text{on } \Gamma_+ \quad \text{(2.6a)}
$$

$$
\sqrt{\alpha} (\nabla \times \mathbf{B}) = \mathbf{g} \quad \text{on } \Gamma_+ \quad \text{(2.6b)}
$$

$$
\eta (\mathbf{n} \times \mathbf{B}) = 0 \quad \text{on } \partial \Omega. \quad \text{(2.6d)}
$$

The vector functions $\mathbf{f}$ and $\mathbf{g}$ cannot be chosen arbitrarily but have to satisfy consistency conditions $\mathbf{n} \times \mathbf{f} = 0$ and $\mathbf{n} \cdot (\nabla \times \mathbf{f}) = \mathbf{n} \cdot \mathbf{g}$. The first condition means that the inflow value of the magnetic field also satisfies the natural boundary condition. The second condition comes from the observation that the normal part of the curl depends only on derivatives computed in the direction tangential to the surface.

In specifying the value of $\mathbf{B}$ on the inflow boundary, we have to be careful to choose this value in such a way that the solution will be physical. The condition

$$
\nabla \cdot \mathbf{B} = 0 \quad \text{on } \Gamma_+, \quad \text{(2.7)}
$$

will ensure that an initially divergence free field will remain so during the evolution.

Now that we have decoupled the induction equation from the MHD equations, we need assumptions on the density $\rho$ and velocity $\mathbf{u}$. The main assumption for us on this and the subsequent chapters is that the density is bounded away from zero, i.e., there exist $\rho_0 > 0$ such that $\rho \geq \rho_0$ for all the times. Under this assumption we can define a weighted norm as

$$
\|\mathbf{B}\|^2 = \|\mathbf{B}\|^2_{L^2(\Omega)} + \alpha \|\rho^{-1/2} \nabla \times \mathbf{B}\|^2_{L^2(\Omega)}. \quad \text{(2.8)}
$$

Further assumptions are central in proving the existence of a solution.
2.1. Symmetrized Equations

Theorem 2.1.1. Let $u$ and $\rho \in W^{1,\infty}(\Omega)$. Furthermore, assume that the time derivative of the density $\rho$ is in $L^\infty(\Omega)$ and also that there exist $\rho_0 > 0$ such that $\rho_0 < \rho$ for all times. Then the following apriori estimate for the solution of (2.6) holds:

$$
\frac{d}{dt} \|B\|^2 \leq C_0 \|B\|^2 + D_0
$$

with $C_0$ and $D_0$ being constants that depend on boundary functions $f$ and $g$, $\rho$, $\alpha$, $u$ and its derivatives only.

Proof. To simplify the notation and emphasise the symmetry of the equation, we rewrite the term $(B \cdot \nabla)u$ in the matrix form obtaining:

$$
\frac{\partial B}{\partial t} + \alpha \nabla \times \left( \frac{1}{\rho} \left( \nabla \times \frac{\partial B}{\partial t} \right) \right) = CB - (\nabla \cdot u)B - (u \cdot \nabla)B - \eta \nabla \times (\nabla \times B)
$$

$$
- \alpha \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla) (\nabla \times B) \right) - \beta \nabla \times \left( \frac{\nabla \times B}{\rho} \times B \right)
$$

where

$$
C = \begin{pmatrix}
\partial_x u^1 & \partial_y u^1 & \partial_z u^1 \\
\partial_x u^2 & \partial_y u^2 & \partial_z u^2 \\
\partial_x u^3 & \partial_y u^3 & \partial_z u^3
\end{pmatrix}
$$

To obtain the $L^2$ estimate, we multiply the equation with $B$ and integrate over $\Omega$

$$
\int_{\Omega} \frac{1}{2} \frac{\partial |B|^2}{\partial t} + \alpha B^T \nabla \times \left( \frac{1}{\rho} \left( \nabla \times \frac{\partial B}{\partial t} \right) \right) d^3x =
$$

$$
\int_{\Omega} [B^T CB - (\nabla \cdot u)|B|^2 - \frac{1}{2} (u \cdot \nabla)|B|^2 - \eta B^T \nabla \times (\nabla \times B)
$$

$$
- \alpha B^T \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla)(\nabla \times B) \right) - \beta B^T \nabla \times \left( \frac{\nabla \times B}{\rho} \times B \right) \bigg] d^3x.
$$

Integration by parts yields

$$
\frac{1}{2} \frac{d}{dt} \|B\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \int_{\Omega} \frac{1}{\rho} \frac{\partial |B|^2}{\partial t} \nabla \times B^2 d^3x = \int_{\Omega} \left[ B^T CB - \frac{1}{2} (\nabla \cdot u)|B|^2 - \frac{\eta}{\rho} \nabla \times B \bigg|_{\partial \Omega} \right] d^3x
$$

$$
- \frac{1}{2} \int_{\partial \Omega} (u \cdot \nu)|B|^2 |d s.
$$

$$
\Rightarrow \frac{1}{2} \frac{d}{dt} \|B\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \int_{\Omega} \frac{\partial}{\partial t} \left( \frac{1}{\rho} |\nabla \times B|^2 \right) d^3x = \int_{\Omega} \left[ B^T CB + \frac{\partial}{\partial t} \left( \frac{1}{\rho} \right) |\nabla \times B|^2 \right] d^3x
$$

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\[- \frac{1}{2}(\nabla \cdot \mathbf{u})|\mathbf{B}|^2 - \eta|\nabla \times \mathbf{B}|^2 - \frac{\alpha}{2\rho}(\mathbf{u} \cdot \nabla)|\nabla \times \mathbf{B}|^2 \]  \[d^3x - \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})|\mathbf{B}|^2 \, ds\]

where the boundary integral containing \(\mathbf{n} \times \mathbf{B}\) has vanished due to boundary condition (2.6d). We can then write

\[\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 \leq \int_{\Omega} \left[ \mathbf{B}^\top \mathbf{C} \mathbf{B} - \frac{1}{2}(\nabla \cdot \mathbf{u})|\mathbf{B}|^2 \right] d^3x \]

\[- \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n})|\mathbf{B}|^2 \, ds.\]

Using integration by parts we get

\[\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 \leq \int_{\Omega} \left[ \mathbf{B}^\top \mathbf{C} \mathbf{B} - \frac{1}{2}(\nabla \cdot \mathbf{u})|\mathbf{B}|^2 \right. \]

\[+ \alpha \left( \nabla \cdot \mathbf{u} - \frac{1}{2} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \frac{1}{\rho} |\nabla \times \mathbf{B}|^2 \]  \[d^3x - \frac{1}{2}\int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) \left( |\mathbf{B}|^2 + \frac{\alpha}{\rho} |\nabla \times \mathbf{B}|^2 \right) \, ds.\]

Since \(\mathbf{n} \cdot \mathbf{u}\) is positive on \(\Gamma_-\) and the values of \(\mathbf{B}\) and \(\nabla \times \mathbf{B}\) on \(\Gamma_+\) are given by boundary condition (2.6b) and (2.6c) we can state

\[\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 \leq \int_{\Omega} \left[ \mathbf{B}^\top \mathbf{C} \mathbf{B} - \frac{1}{2}(\nabla \cdot \mathbf{u})|\mathbf{B}|^2 \right. \]

\[+ \alpha \left( \nabla \cdot \mathbf{u} - \frac{1}{2} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right) \frac{1}{\rho} |\nabla \times \mathbf{B}|^2 \]  \[d^3x - \frac{1}{2}\int_{\Gamma_+} (\mathbf{u} \cdot \mathbf{n}) \left( |f|^2 + \frac{1}{\rho} |g|^2 \right) \, ds.\]

Using Cauchy-Schwartz we obtain

\[\frac{1}{2} \frac{d}{dt} \|\mathbf{B}\|^2 \leq \frac{1}{2} \left( C_1 \|\mathbf{B}\|^2_{L^2(\Omega)} + C_2 \alpha \|\rho^{-1/2}\nabla \times \mathbf{B}\|_{L^2(\Omega)} + D_0 \right) \quad (2.11)\]

where

\[C_1 = \max_{k,i} \left( \frac{\|\partial u_i / \partial x_k\|_{L^\infty(\Omega)}}{\|\partial \rho / \partial x_k\|_{L^\infty(\Omega)}} \right),\]

\[C_2 = \rho_0 \|\partial \rho / \partial t + \nabla \cdot (\rho \mathbf{u})\|_{L^\infty(\Omega)} + 2 \|\nabla \cdot \mathbf{u}\|_{L^\infty(\Omega)} \]  \[\text{and} \quad D_0 = \int_{\Gamma_+} |\mathbf{u} \cdot \mathbf{n}| \left( |f|^2 + \frac{1}{\rho} |g|^2 \right) \, ds\]

Setting \(C_0 = \max(C_1, C_2)\) we get the desired estimate. \(\square\)

**Remark 2.1.2.** It is interesting to note that if the density \(\rho\) satisfies the conservation of mass (1.24a) for a incompressible velocity field \((\nabla \cdot \mathbf{u})\) in a strong sense, then the constant \(C_2\) in (2.11) is zero. Assuming zero inflow boundary conditions we get the estimates

\[\frac{d}{dt} \|\mathbf{B}\|^2 \leq C \|\mathbf{B}\|^2_{L^2(\Omega)}.\]
2.1. Symmetrized Equations

That means that the electron inertia term has a smoothing effect. Starting with an $L^2$ divergence free magnetic field $\mathbf{B}$ on time $t = 0$, we will have a solution $\mathbf{B}$ of (2.6) in $H(\text{curl})$ for $t > 0$.

We have shown that the Magnetic induction equation with Hall effect in symmetric form possesses an energy estimate. However, the divergence of the solution of (2.3) may not be preserved exactly. Nevertheless, we have the following estimate:

**Theorem 2.1.3.** Let $\mathbf{u}$ satisfy the same conditions as in theorem 2.1.1. Furthermore, assume that the solution (2.6) satisfies the additional boundary condition (2.7), then the following apriori estimate holds:

$$\frac{d}{dt} \| \nabla \cdot \mathbf{B} \|^2_{L^2(\Omega)} \leq C \| \nabla \cdot \mathbf{B} \|_{L^2(\Omega)}$$

(2.12)

with $C$ being a constant that depend on $\mathbf{u}$ and its derivatives only.

**Proof.** Applying the divergence operator on (2.3) we obtain

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = -\nabla \cdot (\mathbf{u}(\nabla \cdot \mathbf{B})).$$

(2.13)

Using vector identity we can rewrite the right-hand side of the equation

$$\frac{\partial \nabla \cdot \mathbf{B}}{\partial t} = -\mathbf{u} \cdot \nabla(\nabla \cdot \mathbf{B}) - (\nabla \cdot \mathbf{B})(\nabla \cdot \mathbf{u}).$$

Multiplying this equation by $\nabla \cdot \mathbf{B}$ and integrating it over $\Omega$ we obtain

$$\frac{1}{2} \frac{d}{dt} \| \nabla \cdot \mathbf{B} \|^2_{L^2(\Omega)} = -\frac{1}{2} \int_\Omega \mathbf{u} \cdot \nabla(\nabla \cdot \mathbf{B})^2 d^3x - \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{B})^2 d^3x,$$

with integration by parts and boundary condition (2.7) we obtain

$$\frac{1}{2} \frac{d}{dt} \| \nabla \cdot \mathbf{B} \|_{L^2(\Omega)} = -\frac{1}{2} \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{B})^2 d^3x - \frac{1}{2} \int_{\partial \Omega} (\mathbf{u} \cdot \mathbf{n})(\nabla \cdot \mathbf{B})^2 d^3x$$

$$\leq \frac{1}{2} \| \nabla \cdot \mathbf{u} \|_{L^\infty(\Omega)} \| \nabla \cdot \mathbf{B} \|^2_{L^2(\Omega)}.$$

\(\square\)

**Corollary 2.1.4.** If the conditions for theorems 2.1.1 and 2.1.3 hold and the initial data $\mathbf{B}_0 \in [H^1(\Omega)]^3$, then the estimates imply that $\mathbf{B} \in L^\infty ((0,T), [H^1(\Omega)]^3)$.

**Remark 2.1.5.** The divergence transport equation (2.13) implies that the divergence remains zero if the initial data has zero divergence. If we assume that the initial data is divergence free, the solutions of the symmetric form (2.3) have zero divergence. These are also weak solutions of the non-symmetric form of the magnetic induction equations with Hall effect (1.26).
2. The Continuous Problem

We have proven the existence of a solution for the magnetic induction equations with Hall effect, next we consider the question of uniqueness of such solutions. We were able to show uniqueness only under additional regularity assumptions.

**Theorem 2.1.6.** Assume \( B \) is a solution of (2.6), under the same assumptions on \( u \) and \( \rho \) as in 2.1.1. Furthermore assume this solution is in \( [W^{1,\infty}(\Omega)]^3 \), then the solution is unique.

**Proof.** We are going to show, that assuming that if we have two \( H^2 \) regular solutions of (2.6), the \( L^2 \) norm of their difference will be bounded by the \( L^2 \) norm of the difference of their initial values. Thus, solutions with the same initial condition will be the same for all the times. We will use the same techniques we have used to prove the energy bound. Assume that \( B_i \in [H^2(\Omega)]^3 \) for \( i = 1, 2 \) are solutions of

\[
\frac{\partial B_i}{\partial t} + \alpha \nabla \times \frac{1}{\rho} \left( \nabla \times \frac{\partial B_i}{\partial t} \right) = CB_i - (\nabla \cdot u)B_i - (u \cdot \nabla)B_i - \eta \nabla \times (\nabla \times B_i) \\
-\alpha \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla)(\nabla \times B_i) \right) = \beta \nabla \times \left( \frac{\nabla \times B_i}{\rho} \times B_i \right),
\]

by linearity \( B := B_1 - B_2 \) satisfies

\[
\frac{\partial B}{\partial t} + \alpha \nabla \times \frac{1}{\rho} \left( \nabla \times \frac{\partial B}{\partial t} \right) = CB - (\nabla \cdot u)B - (u \cdot \nabla)B - \eta \nabla \times (\nabla \times B) \\
-\alpha \nabla \times \left( \frac{1}{\rho} (u \cdot \nabla)(\nabla \times B) \right) = \beta \nabla \times \left( \frac{\nabla \times B_1}{\rho} \times B_1 \right) + \beta \nabla \times \left( \frac{\nabla \times B_2}{\rho} \times B_2 \right).
\]

Multiplying this equation times \( B \) and integrating over \( \Omega \) and then proceeding as in the proof of Theorem 2.1.1 we get

\[
\frac{d}{dt} \|B\|^2 \leq C_0 \|B\|^2 - 2\beta \int_\Omega \left( B \cdot \nabla \times \left( \frac{\nabla \times B_1}{\rho} \times B_1 \right) - B \cdot \nabla \times \left( \frac{\nabla \times B_2}{\rho} \times B_2 \right) \right) d^3x.
\]  

(2.14)

The boundary term was neglected since \( (n \cdot u) \) is positive on \( \Gamma_+ \) and the different constants are

\[
C_0 = \max(C_1, C_2), \quad C_1 = \max_{k,i} \left( \left\| \frac{\partial u_i}{\partial x_k} \right\|_{L^\infty(\Omega)} \right),
\]

\[
C_2 = \rho_0 \left\| \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right\|_{L^\infty(\Omega)} + 2 \left\| \nabla \cdot u \right\|_{L^\infty(\Omega)}.
\]

It remains to show that the last integral is bounded. Using integration by parts we get

\[
\int_\Omega \left( B \cdot \nabla \times \left( \frac{\nabla \times B_1}{\rho} \times B_1 \right) - B \cdot \nabla \times \left( \frac{\nabla \times B_2}{\rho} \times B_2 \right) \right) d^3x = \\
\int_\Omega \frac{1}{\rho} \left( (\nabla \times B_1) \times B_1 \right) - (\nabla \times B_2) \times B_2 \right) (\nabla \times B) d^3x =
\]

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\[
\int_{\Omega} \frac{1}{\rho} \left( (\nabla \times B_1) \times (\nabla \times B_1) \right) - B_2 \cdot ((\nabla \times B_1) \times (\nabla \times B_2)) \, d^3x = \\
\int_{\Omega} \frac{1}{\rho} \left( (\nabla \times (B_1 - B_2)) \times (\nabla \times B_1) \right) - B_2 \cdot ((\nabla \times (B_1 - B_2)) \times (\nabla \times B_2)) \, d^3x = \\
\int_{\Omega} \frac{1}{\rho} \left( (\nabla \times B_1) \times (\nabla \times B_2) \right) - B_2 \cdot ((\nabla \times B_1) \times (\nabla \times B_2)) \, d^3x = \\
\int_{\Omega} \frac{1}{\rho} \mathcal{B} \cdot ((\nabla \times B_1) \times (\nabla \times B_2)) \, d^3x = \\
\int_{\Omega} \frac{1}{2\rho} \mathcal{B} \cdot [(\nabla \times B_1) \times (\nabla \times B_2) + (\nabla \times B_1) \times (\nabla \times B_2)] \, d^3x = \\
\int_{\Omega} \frac{1}{2\rho} \mathcal{B} \cdot [(\nabla \times B_1) \times (\nabla \times B_2) - (\nabla \times B_1) \times (\nabla \times B_1)] \, d^3x = \\
\int_{\Omega} \mathcal{B} \cdot \left( \frac{1}{\rho} (\nabla \times B_1) \times \left( \nabla \times \frac{B_1 + B_2}{2} \right) \right) \, d^3x
\]

where we have rearranged the terms and then used the definition of \(\mathcal{B}\). Since \(B_i \in W^{1,\infty}(\Omega)\) for \(i = 1, 2\), we set \(C_3 = \|\nabla \times \frac{B_1 + B_2}{2}\|_{L^\infty(\Omega)}\) and obtain

\[
- \int_{\Omega} \left( \mathcal{B} \cdot \nabla \times \left( \frac{\nabla \times B_1}{\rho} \times B_1 \right) - \mathcal{B} \cdot \nabla \times \left( \frac{\nabla \times B_2}{\rho} \times B_2 \right) \right) \, d^3x \leq \\
\frac{C_3}{\sqrt{\rho_0}} \int_{\Omega} |\mathcal{B} \cdot \rho^{-1/2}(\nabla \times B)| \, d^3x \leq \frac{C_3}{\rho_0^{1/2}} \|\mathcal{B}\|_{L^2(\Omega)} \|\rho^{-1/2}(\nabla \times B)\|_{L^2(\Omega)} \leq \frac{C_3}{\alpha \rho_0^{1/2}} \|\mathcal{B}\|^2
\]

by use of the lower bound on the density and of the Hölder’s inequality. Using this result in (2.14) we find

\[
\frac{d}{dt} \|\mathcal{B}\|^2 \leq C_0 \|\mathcal{B}\|^2 + \frac{C_3 \beta}{\alpha \rho_0^{1/2}} \|\mathcal{B}\|^2 \leq C_4 \|\mathcal{B}\|^2.
\]

Here \(C_4 = C_0 + \frac{C_3 \beta}{\alpha \rho_0^{1/2}}\). Using the chain rule and then dividing by \(\|\mathcal{B}\|\) we get

\[
\frac{d}{dt} \|\mathcal{B}\| \leq C_4 \|\mathcal{B}\|.
\]

That means that if \(\mathcal{B} = 0\) at \(t = 0\) it will remain so for all \(t > 0\), thus the solution is unique.

Under certain conditions we proved the existence and uniqueness of the solution of (1.26). Now our interest shifts from the continuous to the discrete point of view. In the following chapters we will try to obtain a stable discretisation of (2.6), where stability will be achieved by means of obtaining an energy estimate for the semi-discrete problem.
2. *The Continuous Problem*
3. Finite Difference Schemes

In this chapter we design finite difference schemes for the solution of (2.6) that satisfy a discrete version of the energy estimates that we have shown in the last chapter. We are constructing semi-discrete schemes, i.e., in the first step, we are discretising space keeping the time domain continuous. Then in a second step we are going to solve the resulting system of ordinary differential equations in time. To simplify the notation we are going to omit the time variable in the following.

3.1. Semi discrete Schemes

The computational domain is $\Omega = [0, L_x] \times [0, L_y] \times [0, L_z]$ and we define a uniform mesh of $N_x$ times $N_y$ times $N_z$ points with coordinates $x_i = i\Delta x$, $y_j = j\Delta y$ and $z_k = k\Delta z$. In this case $\Delta x = L_x/(N_x - 1)$, $\Delta y = L_y/(N_y - 1)$ and $\Delta z = L_z/(N_z - 1)$ are mesh widths.

The point values of the magnetic field, velocity field and density are

$\tilde{B}_{i,j,k} \sim B(x_i, y_j, z_k) \quad \tilde{u}_{i,j,k} \sim u(x_i, y_j, z_k) \quad \tilde{\rho}_{i,j,k} \sim \rho(x_i, y_j, z_k)$.

We will provide a very general discrete formulation and specify the necessary requirement that the discrete derivative should possess. As in [17, 30, 22], the key requirement for the discrete derivative is to satisfy a summation by parts (SBP) condition. The exact form of operators satisfying these requirements are presented in appendix A.

We start with one dimensional discrete derivatives using grid functions in vector form, i.e., $w = (w_0, \cdots w_{N_x-1})^T$. An approximation of the $x$ spatial derivative, $D_x$ possesses the summation by parts property (see [18, 27, 7, 20, 28]) if it can be written as $D_x = P_x^{-1}Q_x$, where $P_x$ is a diagonal $N_x \times N_x$ positive definite matrix and $Q_x$ an $N_x \times N_x$ matrix satisfying:

$$Q_x + Q_x^T = R_{N_x} - L_{N_x} \quad (3.1)$$

where $R_{N_x}$ and $L_{N_x}$ are $N_x \times N_x$ matrices: $\text{diag}(0, \cdots, 0, 1)$ and $\text{diag}(1, 0, \cdots, 0)$.

The operator $P$ defines an inner product

$$(v, w)_{P_x} = v^T P_x w \quad (3.2)$$

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3. Finite Difference Schemes

with the associated norm \( \|w\|_P = (w, w)^{1/2}_P \) that is equivalent to the norm \( \|w\| = (\Delta x \sum_k w_k^2)^{1/2} \).

Next, we define averaging operators such that we can obtain an approximate form of the product rule.

We define averaging operators as

\[
(A_x w)_i = \sum_{k=-l_i}^{r_i} \alpha_k^i w_{i+k}
\]

with \( \sum_{k=-l_i}^{r_i} \alpha_k^i = 1 \) for all \( i \). We will call the average positive symmetric, if the matrix representation of \( A_x \) is symmetric positive definite.

In [22], the following discrete product rule was shown,

**Lemma 3.1.1.** Given any function \( s \in C^2(\Omega) \), we denote his evaluation on the grid as \( \bar{s} \) and let be \( w \) a grid function. Then we can define an average operator \( \bar{A}_x \) coupled to \( D_x \) such that

\[
D_x (\bar{s} \circ w) = \bar{s} \circ D_x (w) + D_x (\bar{s}) \circ \bar{A}_x (w) + \tilde{w},
\]

where \((s \circ w)_i = s_i w_i\) and \( \|\tilde{w}\|_P \leq C \Delta x \|w\|_P \).

**Proof.** The discrete differential operator acting on a grid function \( w_i \) can also be written through sums:

\[
(D_x w)_i = \frac{1}{\Delta x} \sum_{k=-l_i}^{r_i} \beta_k^i w_{i+k}
\]

with \( \sum_{k=-l_i}^{r_i} \beta_k^i = 0 \) and \( \sum_{k=-l_i}^{r_i} k \beta_k^i = 1 \) for all \( i \). Then the operator

\[
(\bar{A}_x (w))_i := \sum_{k=-l_i}^{r_i} k \beta_k^i w_{i+k}
\]

is an average.

The residual \( \tilde{w} \) is given by

\[
\tilde{w}_i = D_x (\bar{s} \circ w)_i - (\bar{s} \circ D_x w)_i - (D_x (\bar{s}) \circ \bar{A}_x (w))_i
\]

\[
= \frac{1}{\Delta x} \sum_{k=-l_i}^{r_i} \beta_k^i \bar{s}_{i+k} w_{i+k} - \frac{\bar{s}_i}{\Delta x} \sum_{k=-l_i}^{r_i} \beta_k^i w_{i+k} - \frac{1}{\Delta x} \left( \sum_{l=-l_i}^{r_i} \beta_l^i \bar{s}_{i+l} \right) \sum_{k=-l_i}^{r_i} k \beta_k^i w_{i+k}.
\]

We expand \( \bar{s} \) with Taylor-expansion

\[
\bar{s}_{i+k} = \bar{s}_i + \Delta x \bar{s}'_i + \frac{1}{2} \Delta x^2 \bar{s}''_i k^2 + \cdots,
\]
3.1. Semi discrete Schemes

where \( s^k_i = \bar{s}''(\xi^k_i) \) with \( \xi^k_i \in (x_i, x_{i+k}) \) obtaining

\[
\tilde{w}_i = \frac{s_i}{\Delta x} \sum_{k=-l_i}^{r_i} \beta^i_k w_{i+k} + \Delta x \sum_{k=-l_i}^{r_i} k^2 \beta^i_k s^k_i w_{i+k} - \frac{s_i}{\Delta x} \sum_{k=-l_i}^{r_i} \beta^i_k w_{i+k} \\
- \left( \tilde{s}_i' + \Delta x \sum_{l=-l_i}^{r_i} \beta^l_i \tilde{s}_i' \right) \sum_{k=-l_i}^{r_i} \beta^i_k w_{i+k} = \Delta x \sum_{k=-l_i}^{r_i} \gamma^i_k w_{i+k}.
\]

Here

\[
\gamma^i_k(s) := k^3 \beta^i_k \frac{k s^k_i - \sum_{l=-l_i}^{r_i} l^2 \beta^l_i s^l_i}{2}.
\]

Since the \( \gamma^i_k(s) \) are bounded, we have that

\[
\|\tilde{w}\|_{P_x} \leq \Delta x C \|w\|_{P_x}
\]

where \( C \) depends only on the maximum of \( \gamma^i_k(s) \) and on the norm \( P_x \).

We use the above one dimensional operators to build the multidimensional operators i.e, mappings three dimensional grid functions \( w(x_i, y_j, z_k) = w_{i,j,k} \) to a column vector

\[
w = (w_{0,0,0}, w_{0,0,1}, \cdots, w_{0,0,N_x}, w_{0,1,0}, \cdots, w_{N_x,N_y,N_z})^T.
\]

We define the discrete differential operators and averages

\[
\partial_x = D_x \otimes I_y \otimes I_z, \\
\partial_y = I_x \otimes D_y \otimes I_z, \\
\partial_z = I_x \otimes I_y \otimes D_z, \\
A_x = A_x \otimes I_y \otimes I_z, \\
A_y = I_x \otimes A_y \otimes I_z, \\
A_z = I_x \otimes I_y \otimes A_z.
\]

Here \( \otimes \) is the Kronecker product. We also extend the inner product by \( P = P_x \otimes P_y \otimes P_z \) with the corresponding norm \( \|w\|_P = (w, w)_P^{1/2} \). We have generalised the one dimensional operators to three dimensions in a standard way using the identity mapping, other choice are possible but in that case inclusion of boundary condition will be a more challenging question.

At the end of this section we will briefly discuss the issue arising in generalising the operator.

We can expand lemma 3.1.1 in three dimensions:

**Corollary 3.1.2.** Given any function \( s \in C^1(\Omega) \) with \( s' \) bounded, we denote his trace on
3. Finite Difference Schemes

the grid as \( \bar{s} \) and let be \( w \) a grid function, then

\[
\| \partial_x (\bar{s} \circ w) - \bar{s} \circ \partial_x w \|_P \leq C \| w \|_P,
\]

(3.7)

where \( C \) is a constant which depends on the first order derivative of \( s \).

**Proof.** The proof of this corollary is similar to the one done before only using a substitution with a Taylor expansion of degree one. \( \square \)

As shown in [30] we have that the summation by parts property of the differential operators, coupled with the inner product \( P \) will result in a discrete version of integration by parts:

**Lemma 3.1.3.** For any grid functions \( v \) and \( w \), we have

\[
(v, \partial_x w)_P + (\partial_x v, w)_P = v^\top (R - L)w,
(v, \partial_y w)_P + (\partial_y v, w)_P = v^\top (U - D)w,
(v, \partial_z w)_P + (\partial_z v, w)_P = v^\top (T - B)w
\]

(3.8)

where \( R = R_{Nx} \otimes P_y \otimes P_z, L = L_{Nx} \otimes P_y \otimes P_z, U = P_x \otimes R_{Ny} \otimes P_z, D = P_x \otimes L_{Ny} \otimes P_z, T = P_x \otimes P_y \otimes R_{Nx} \) and \( B = P_x \otimes P_y \otimes L_{Nx} \).

**Proof.** Since \( A_k \)'s are symmetric and \( P_k \)'s diagonal, we can calculate

\[
(v, \partial_x w)_P + (\partial_x v, w)_P = v^\top \left( P_x \otimes P_y \otimes P_z \right) \left( P_x \otimes P_y \otimes P_z \right) (P_x^{-1} Q_x \otimes I_y \otimes I_z)w + \left( (P_x^{-1} Q_x \otimes I_y \otimes I_z) \right) v^\top \left( (P_x \otimes P_y \otimes P_z) \right) w = v^\top \left( Q_x \otimes P_y \otimes P_z \right) w + v^\top \left( Q_x \otimes P_y \otimes P_z \right) w = v^\top \left( (Q_x^\top + Q_x) \otimes P_y \otimes P_z \right) w = v^\top \left( R_{Nx} \otimes P_y \otimes P_z \right) w - v^\top \left( L_{Nx} \otimes P_y \otimes P_z \right) w.
\]

The proof for the other space directions follow analogously. \( \square \)

The right hand side of the above equation represents the evaluation of the grid function on the boundary of \( \Omega \).

Until now all our analysis was for scalar operators. Now we extend it to vector-valued discrete differential operators below.

**Corollary 3.1.4.** We derive from the scalar summation by parts rules the following summation by parts for vector fields

\[
(\mathbf{v}, D \cdot \mathbf{w})_P = -(D\mathbf{v}, \mathbf{w})_P + \sum_{i=1}^3 \mathbf{v}^i S^i \mathbf{w}^i,
\]
3.1. Semi discrete Schemes

\[(\tilde{v}, D \times \tilde{w})_P = (D \times \tilde{v}, \tilde{w})_P + \sum_{i,j,k=1}^{3} \epsilon_{i,j,k} \tilde{v}^i S^j \tilde{w}^k \tag{3.9} \]

where \( D = (\partial_x, \partial_y, \partial_z)^\top \), \( S^1 = \mathcal{L} - \mathcal{R} \), \( S^2 = \mathcal{U} - \mathcal{D} \), \( S^3 = \mathcal{T} - \mathcal{B} \) and \( \epsilon_{i,j,k} \) is the Levi-Civita symbol.

**Proof.** The proof of this corollary is given by the direct application of the previous theorem on discrete vector fields.

The following lemma was presented in [30] for the vector operator \( D \):

**Lemma 3.1.5.** If \( \tilde{s} \) and \( \tilde{v} \) are vector grid functions

\[(\tilde{v}, (\tilde{s} \cdot D) \circ \tilde{v})_P = \frac{1}{2} \left( \tilde{v}, (\tilde{s} \cdot D) \circ \tilde{v} - \sum_{i=1}^{3} D((\tilde{s}^i) \circ \tilde{v}) \right)_P + \frac{1}{2} \sum_{i=1}^{3} \tilde{v}^i S^i((\tilde{s}^i) \circ \tilde{v}) \tag{3.10} \]

**Proof.** See the proof of lemma 4.3 in [30] and apply it on each component of \( \tilde{s} \cdot \tilde{D} \).

In the continuous case, the key property for the preservation of divergence is the identity:

\[\nabla \cdot (\nabla \times w) = 0\]

for all \( w \in (C^2(\mathbb{R}^3))^3 \). From the definition of Kronecker product, one observes that the difference operators \( \partial_x, \partial_y \) and \( \partial_z \) commute, hence:

**Corollary 3.1.6.** Every grid function \( \tilde{w}_{i,j,k} \) coupled with \( D := (\partial_x, \partial_y, \partial_z)^\top \) satisfies

\[D \cdot (D \times \tilde{w}) = 0. \tag{3.11} \]

The proof of this corollary is straightforward.

As we have done for the continuous problem we are going to define a weighted norm

\[\|\tilde{B}\|_P^2 := \|\tilde{B}\|_P^2 + \alpha \|{\bar{\rho}}^{-1/2}D \times \tilde{B}\|_P^2. \tag{3.12} \]

Now we have all the ingredients to present two different classes of numerical schemes. The first set of schemes, termed as symmetric schemes, will discretize the symmetric version of the Hall magnetic induction equation (2.3). The second set of schemes will discretize the non-symmetric version (1.26) of the magnetic induction equations with Hall effect and will preserve a discrete version of divergence. Hence, we term it divergence preserving scheme.
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3.1.1. Symmetric Scheme

We discretize the symmetric form (2.3) of the Magnetic induction equations with Hall effect and define a semi-discrete numerical scheme as

\[
\frac{d}{dt} \tilde{B}_{i,j,k} + \alpha \left( D \times \frac{1}{\rho_{i,j,k}} \left( D \times \frac{d}{dt} \tilde{B}_{i,j,k} \right) \right) = \mathcal{A}\mathcal{V}(\tilde{B}, \tilde{u})_{i,j,k} - (\tilde{u}_{i,j,k} \cdot D) \tilde{B}_{i,j,k} - \eta (D \times (D \times \tilde{B}_{i,j,k})) - \alpha D \times \left( \left( \frac{\tilde{u}}{\rho} \right)_{i,j,k} \cdot D \right) (D \times \tilde{B}_{i,j,k}) - \beta D \times \left( \frac{1}{\rho_{i,j,k}} (D \times \tilde{B}_{i,j,k}) \times \tilde{B}_{i,j,k} \right),
\]

where

\[
\mathcal{A}\mathcal{V}(\tilde{B}, \tilde{u})_{i,j,k} = (\tilde{A}(\tilde{B}_{i,j,k}) \cdot D) \tilde{u}_{i,j,k} - \tilde{A}_x(\tilde{B}_{i,j,k}) \partial_x (\tilde{u}^1_{i,j,k}) - \tilde{A}_y(\tilde{B}_{i,j,k}) \partial_y (\tilde{u}^2_{i,j,k}) - \tilde{A}_z(\tilde{B}_{i,j,k}) \partial_z (\tilde{u}^3_{i,j,k}),
\]

\[
\tilde{A}(\tilde{B}_{i,j,k}) = (\tilde{A}_x(\tilde{B}^1_{i,j,k}), \tilde{A}_y(\tilde{B}^2_{i,j,k}), \tilde{A}_z(\tilde{B}^3_{i,j,k}))^\top.
\]

The term $\mathcal{A}\mathcal{V}$ represent the discretisation of $(B \cdot \nabla)u - B(\nabla \cdot u)$. We estimate it below,

**Lemma 3.1.7.** For $\tilde{B}_{i,j,k}$ grid function and $\tilde{u}_{i,j,k}$ a bounded grid function, we have

\[
(\tilde{B}, \mathcal{A}\mathcal{V}(\tilde{B}, \tilde{u}))_p \leq C \| \tilde{B} \|_p
\]

where $C$ depends on $\tilde{u}$ and its discrete derivative only.

**Proof.** We write (3.16) component-wise and using the triangle inequality, we obtain

\[
(\tilde{B}, \mathcal{A}\mathcal{V}(\tilde{B}, \tilde{u}))_p \leq \sum_{i=1}^{3} \left| (\tilde{B}^i, \tilde{A}_x(\tilde{B}^1))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^2))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^3))_p \right| + \left| (\tilde{B}^i, \tilde{A}_x(\tilde{B}^3))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^2))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^1))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^3))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^2))_p \right|.
\]

Since the discrete derivatives of $\tilde{u}$ are bounded, we can extract the maximum from the $p$ norms and obtain

\[
(\tilde{B}, \mathcal{A}\mathcal{V}(\tilde{B}, \tilde{u}))_p \leq C \sum_{i=1}^{3} \left| (\tilde{B}^i, \tilde{A}_x(\tilde{B}^1))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^2))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^3))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^1))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^2))_p \right| + \left| (\tilde{B}^i, \tilde{A}_x(\tilde{B}^3))_p \right| \leq C \left( \sum_{i=1}^{3} \left| (\tilde{B}^1, \tilde{A}_x(\tilde{B}^1))_p \right| + \left| (\tilde{B}^2, \tilde{A}_y(\tilde{B}^2))_p \right| + \left| (\tilde{B}^3, \tilde{A}_z(\tilde{B}^3))_p \right| \right)
\]

\[
+ \left| (\tilde{B}^2, \tilde{A}_x(\tilde{B}^1))_p \right| + \left| (\tilde{B}^3, \tilde{A}_y(\tilde{B}^3))_p \right| + \left| (\tilde{B}^1, \tilde{A}_y(\tilde{B}^2))_p \right| \leq C \left( \sum_{i=1}^{3} \left| (\tilde{B}^i, \tilde{A}_x(\tilde{B}^1))_p \right| + \left| (\tilde{B}^i, \tilde{A}_y(\tilde{B}^2))_p \right| + \left| (\tilde{B}^i, \tilde{A}_z(\tilde{B}^3))_p \right| \right).
\]
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\[ +|\langle \tilde{B}^3, \tilde{A}_y(\tilde{B}^2) \rangle_p | + |\langle \tilde{B}^1, \tilde{A}_x(\tilde{B}^3) \rangle_p | + |\langle \tilde{B}^2, \tilde{A}_z(\tilde{B}^3) \rangle_p | \]

Taking the first term and first using the equivalence of inner product, then rewriting the average operator through summation, and finally using the Cauchy inequality yields

\[ |\langle \tilde{B}^1, \tilde{A}_x(\tilde{B}^1) \rangle_p | \leq C |\langle \tilde{B}^1, \tilde{A}_x(\tilde{B}^1) \rangle | = C \sum_{i,j,k=1} r_i \sum_{l=-l_i}^{l_i} \tilde{B}_{i,j,k} \alpha_i \tilde{B}_{i,j,k}^1 + |\langle \tilde{B}^1_{i,j,k} + \alpha \rho_0 (D \times \tilde{B}) \rangle_p | \leq C \| \tilde{B} \|^2. \]

Repeating this procedure for all the terms of the sum yields

\[ \langle \tilde{B}, \mathcal{A}_V(\tilde{B}, \tilde{u}) \rangle_p \leq C \| \tilde{B} \|^2. \]

The use of the equivalence between Euclidean and P norms conclude the proof of the lemma.

This lemma is necessary to prove the energy stability of the scheme. The main theorem is:

**Theorem 3.1.8.** Let \( \tilde{u}_{i,j,k} = u(x_i, y_j, z_k) \) and \( \tilde{\rho}_{i,j,k} = \rho(x_i, y_j, z_k) \) be the point evaluation of functions in \( C^1(\Omega) \). Let \( \tilde{\rho}_{i,j,k} = \frac{\partial \rho}{\partial t}(x_i, y_j, z_k) \) be the point evaluation of a bounded function. The density should also be bounded away from zero, with \( \rho \geq \rho_0 > 0 \). Then the following estimates for the solutions (3.13) hold

\[ \frac{d}{dt} \| \tilde{B} \|^2_p \leq C_0 \| \tilde{B} \|^2_p - \text{Boundary}, \quad (3.17) \]

with \( C_0 \) being a constant that depends on the constant \( \alpha \), and on \( C^1 \) functions \( \rho \) and \( u \) and their derivatives only. The boundary term is:

\[ \text{Boundary} = \sum_{i,j,k=1}^{3} \epsilon_{i,j,k} \tilde{R}_{i,j,k} S^i S^j \tilde{S}^k + \frac{1}{2} \tilde{B}^\top S^+ \tilde{B} + \frac{\alpha}{2 \rho_0} (D \times \tilde{B})^\top S^+(D \times \tilde{B}) \]

with

\[ \tilde{R} := \alpha \left( \frac{1}{\tilde{\rho}} D \times \frac{d}{dt} \tilde{B} + \frac{\tilde{u}}{\tilde{\rho}} \cdot D \times \tilde{B} \right) + \eta (D \times \tilde{B}) + \beta \frac{D \times \tilde{B}}{\tilde{\rho}} \times \tilde{B}. \]

**Proof.** Multiplying both sides of the scheme (3.13) by \( \tilde{B} \) yields

\[ \frac{1}{2} \frac{d}{dt} \| \tilde{B} \|^2_p + \alpha \left( \tilde{B}, \left( D \times \frac{1}{\tilde{\rho}} D \times \frac{d}{dt} \tilde{B} \right) \right)_p = (\tilde{B}, \mathcal{A}_V(\tilde{B}, \tilde{u}))_p - (\tilde{B}, (\tilde{u} \cdot D) \tilde{B})_p \]
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\[ -\eta(\tilde{B}, D \times (D \times \tilde{B}))_P - \alpha \left( \tilde{B}, D \times \left( \frac{1}{\rho} (\tilde{u} \cdot D) \circ (D \times \tilde{B}) \right) \right)_P \]

\[ -\beta \left( \tilde{B}, D \times \left( \frac{D \times \tilde{B}}{\rho} \times \tilde{B} \right) \right)_P. \]

Using summation by parts of corollary 3.1.4 we get

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{B}\|_P^2 + \frac{\alpha}{2} \frac{d}{dt} \left( \frac{1}{\rho} (D \times \tilde{B}), (D \times \tilde{B}) \right)_P = (\tilde{B}, A\gamma(\tilde{B}, \tilde{u}))_P - (\tilde{B}, (\tilde{u} \cdot D) \circ \tilde{B})_P
\]

\[ + \frac{\alpha}{2} \left( \frac{d}{dt} \left( \frac{1}{\rho} \right) \circ (D \times \tilde{B}), (D \times \tilde{B}) \right)_P - \alpha \left( D \times \tilde{B}, \left( \frac{\tilde{u}}{\rho} \cdot D \right) \circ (D \times \tilde{B}) \right)_P \]

\[ -\eta\|D \times \tilde{B}\|_P^2 - \beta \left( D \times \tilde{B}, \frac{1}{\rho} \left( D \times \tilde{B} \right) \right)_P = \sum_{i,j,k=1}^3 \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{B}^k. \]

Using (3.1.7) we can estimate

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{B}\|_P^2 \leq C \|\tilde{B}\|_P^2 - (\tilde{B}, (\tilde{u} \cdot D) \circ \tilde{B})_P + \frac{\alpha \|\rho\|_{L^\infty(\Omega)}}{2\rho_0} \|\tilde{B}\|_P^2 - \alpha \left( D \times \tilde{B}, \left( \frac{\tilde{u}}{\rho} \cdot D \right) \circ (D \times \tilde{B}) \right)_P - \sum_{i,j,k=1}^3 \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{B}^k
\]

Using lemma 3.1.5 we can deal with the advection term:

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{B}\|_P^2 \leq C \|\tilde{B}\|_P^2 - \frac{1}{2} \left( \tilde{B}, (\tilde{u} \cdot D) \circ \tilde{B} - \sum_{i=1}^3 D(\tilde{u}^i \circ \tilde{B}) \right)_P
\]

\[ + \frac{\alpha \|\rho\|_{L^\infty(\Omega)}}{2\rho_0} \|\tilde{B}\|_P^2 - \alpha \left( D \times \tilde{B}, \left( \frac{\tilde{u}}{\rho} \cdot D \right) \circ (D \times \tilde{B}) - \sum_{i=1}^3 D(\tilde{u}^i \circ (D \times \tilde{B})) \right)_P \]

\[ - \sum_{i,j,k=1}^3 \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{B}^k - \frac{3}{2} \sum_{i=1}^3 \tilde{B}^i S^i (\tilde{u}^i \circ \tilde{B}) - \alpha \frac{3}{2} \sum_{i=1}^3 (D \times \tilde{B})^i S^i \left( \frac{\tilde{u}^i}{\rho} \circ (D \times \tilde{B}) \right). \]

Using Cauchy-Schwarz inequality and corollary (3.1.2) we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\tilde{B}\|_P^2 \leq C \|\tilde{B}\|_P^2 - \sum_{i,j,k=1}^3 \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{B}^k
\]

\[ - \frac{3}{2} \sum_{i=1}^3 \tilde{B}^i S^i (\tilde{u}^i \circ \tilde{B}) - \alpha \frac{3}{2} \sum_{i=1}^3 (D \times \tilde{B})^i S^i \left( \frac{\tilde{u}^i}{\rho} \circ (D \times \tilde{B}) \right). \]
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Let us analyse the first term of the contribution of the boundary

\[ \sum_{i=1}^{3} \tilde{B}^\top S^i(\tilde{u} \circ \tilde{B}) = \tilde{B}^\top S^1(\tilde{u} \circ \tilde{B}) + \tilde{B}^\top S^2(\tilde{u} \circ \tilde{B}) + \tilde{B}^\top S^3(\tilde{u} \circ \tilde{B}) \]

\[ = \tilde{B}^\top R(\tilde{u} \circ \tilde{B}) + \tilde{B}^\top U(\tilde{u} \circ \tilde{B}) + \tilde{B}^\top T(\tilde{u} \circ \tilde{B}) \]

\[ - \tilde{B}^\top L(\tilde{u} \circ \tilde{B}) - \tilde{B}^\top D(\tilde{u} \circ \tilde{B}) - \tilde{B}^\top B(\tilde{u} \circ \tilde{B}). \]

Taking the first term we write the product \((\tilde{u} \circ \tilde{B})\) in matrix form as \(U \tilde{B}\) where \(U = \text{diag}(\tilde{u})\) is a diagonal matrix. To identify the inflow and outflow part of the boundary, we split \(U = U^+ + U^-\) in a positive and negative part, where \((U^+)_{i,i} = \max(0, (U^{++})_{i,i})\) and \((U^-)_{i,i} = \min(0, (U^{--})_{i,i})\). Since \(R\) is diagonal and positive definite, the operator \(R^+ = RU^+_1\) is positive definite and the operator \(R^- = RU^-_1\) is negative definite. We have

\[ \tilde{B}^\top R(\tilde{u} \circ \tilde{B}) = \tilde{B}^\top R^+ \tilde{B} + \tilde{B}^\top R^- \tilde{B} \geq \tilde{B}^\top R^- \tilde{B}. \]

In the last step we have neglected the outflow part of the right boundary. In an analogous way we can apply this procedure to the other five boundaries. The remaining boundary term in (3.18) can be decomposed in the same way. We get the final form of (3.18)

\[ \frac{1}{2} \frac{d}{dt} \|\tilde{B}\|^2_P \leq C_1 \|\tilde{B}\|^2_P - \sum_{i,j,k=1}^{3} \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{B}^k - \frac{1}{2} \tilde{B}^\top S^+ \tilde{B} - \frac{\alpha}{2\rho} (D \times \tilde{B})^\top S^+(D \times \tilde{B}), \]

with \(S^+ := R^- + L^+ + U^- - D^+ - T^- - B^+\).

**Remark 3.1.9.** In this chapter on finite difference schemes we will not consider the problem of imposing the boundary conditions. The form of the boundary term in the energy estimate is similar to that of the continuous problem. The terms in the energy estimate arising from the boundary are localised on a one point layer at the boundary without spreading into the domain. This suggests that one could apply the Simultaneous Approximation Terms (SAT) [6] techniques to weakly impose the boundary conditions.

Since the divergence is not preserved by the symmetric scheme, we show that the divergence generated by the symmetric scheme is bounded (a similar result for the ideal magnetic induction equations was shown in [22]).

**Theorem 3.1.10.** Let \(\tilde{u}_{i,j,k} = u(x_i, y_j, z_k)\) be the point evaluation of a function \(u \in C^2\) and let the solutions of (3.13), then the following estimates hold:

\[ \frac{d}{dt} \|D \cdot \tilde{B}\|^2_P \leq C(\|D \cdot \tilde{B}\|^2_P + \|\tilde{B}\|^2) - \text{Boundary}, \]

with \(C\) a constant that depends on \(u\) and and its derivative and on the regularity of the
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grid. The boundary term is

\[\text{Boundary} := \sum_{i=1}^{3} \omega^\top S^{+i}(\tilde{u}^i \circ \omega).\]

Here \(\omega := D \cdot \tilde{B}\).

Proof. We define the discrete divergence \(\omega = D \cdot \tilde{B}\) and using the numerical scheme (3.13) with corollary 3.1.6, we obtain an equation for its evolution

\[\frac{d}{dt} \omega = D \cdot (\mathcal{A}V(\tilde{B}, \tilde{u}) - (\tilde{u} \circ D)\tilde{B}).\]  

(3.20)

Now expanding each component of \(\mathcal{A}V\) and using lemma 3.1.1, we obtain, for example for the first component:

\[\mathcal{A}V(\tilde{B}, \tilde{u})^1 = \tilde{A}_y(\tilde{B}^2) \circ \delta_y \tilde{u}^1 + \tilde{A}_z(\tilde{B}^3) \circ \delta_z \tilde{u}^1 - \tilde{A}_y(\tilde{B}^1) \circ \delta_y \tilde{u}^2 - \tilde{A}_z(\tilde{B}^1) \circ \delta_z \tilde{u}^3\]

\[= \delta_y(\tilde{B}^2 \tilde{u}^1 - \tilde{B}^1 \tilde{u}^2) - \tilde{u}^1 \circ \omega + (\tilde{u} \circ D)\tilde{B}^1 + R^{\Delta y}(\tilde{B}^1, \tilde{B}^2) + R^{\Delta z}(\tilde{B}^1, \tilde{B}^3)\]

where the residual terms are

\[R^{\Delta y}(\tilde{B}^1, \tilde{B}^2)_{i,j,k} = \Delta y \sum_{l=-l_i}^{l_i} \left( \gamma^l_j(\tilde{u}^2)\tilde{B}^1_{i,j+l,k} - \gamma^l_j(\tilde{u}^1)\tilde{B}^2_{i,j+l,k} \right),\]

\[R^{\Delta z}(\tilde{B}^1, \tilde{B}^3)_{i,j,k} = \Delta z \sum_{l=-l_i}^{l_i} \left( \gamma^l_k(\tilde{u}^3)\tilde{B}^1_{i,j+l,k} - \gamma^l_k(\tilde{u}^1)\tilde{B}^3_{i,j+l,k} \right).\]

Here the \(\gamma^l_i\) are defined in the proof of lemma 3.1.1 equation (3.5) and depend on the velocities \(u\). Applying the same technique on the two other components we obtain

\[\mathcal{A}V(\tilde{B}, \tilde{u}) - (\tilde{u} \circ D)\tilde{B} = (D \times (\tilde{u} \times \tilde{B})) - \tilde{u} \circ \omega + R\]

where

\[R = \left( \begin{array}{c} R^{\Delta y}(\tilde{B}^1, \tilde{B}^2) + R^{\Delta z}(\tilde{B}^1, \tilde{B}^3) \\ R^{\Delta x}(\tilde{B}^2, \tilde{B}^1) + R^{\Delta x}(\tilde{B}^3, \tilde{B}^1) \\ R^{\Delta x}(\tilde{B}^3, \tilde{B}^1) + R^{\Delta y}(\tilde{B}^3, \tilde{B}^2) \end{array} \right).\]

Here \(\|R\|_P \leq C \max(\Delta x, \Delta y, \Delta z)\|\tilde{B}\|_P\). Setting these results in the discrete evolution equation (3.20) and using corollary 3.1.6 we obtain

\[\frac{d}{dt} \omega = -D \cdot (\tilde{u} \circ \omega) + D R.\]
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The time evolution of the $P$ norm of the divergence is

$$\frac{d}{dt} \|\omega\|_P^2 = 2(\omega, \frac{d}{dt}\omega)_P = -2(\omega, D \cdot (\tilde{u} \circ \omega))_P + (\omega, D\mathcal{R})_P.$$

We use SBP to write

$$\frac{d}{dt} \|\omega\|_P^2 = 2(D\omega, \tilde{u} \circ \omega)_P + (\omega, D\mathcal{R})_P - \sum_{i=1}^3 \omega^\top S^i(\tilde{u}^i \circ \omega)$$

$$= 2(\tilde{u}D \circ \omega, \omega)_P + (\omega, D\mathcal{R})_P - \sum_{i=1}^3 \omega^\top S^i(\tilde{u}^i \circ \omega).$$

We know that the boundary term is bounded from below by

$$\sum_{i=1}^3 \omega^\top S^i(\tilde{u}^i \circ \omega) \leq \sum_{i=1}^3 \omega^\top S^{\pm,i}(\tilde{u}^i \circ \omega) \quad (3.21)$$

since we can split it in positive and negative, as done in the prove of the theorem above. Using Lemma 3.1.5 we can write

$$\frac{d}{dt} \|\omega\|_P^2 \leq 2(\tilde{u}D \circ \omega, \omega)_P + (\omega, D\mathcal{R})_P - \text{Boundary}$$

$$\leq C\|\omega\|_P^2 + \|\omega\|_P\|D\mathcal{R}\|_P - \text{Boundary}$$

$$\leq C\|\omega\|_P^2 + \frac{C}{\min(\Delta x, \Delta y, \Delta z)} \|\tilde{\omega}\|_P\|\mathcal{R}\|_P - \text{Boundary}$$

$$\leq C_1\|\omega\|_P^2 + C_2 \frac{\max(\Delta x, \Delta y, \Delta z)}{\min(\Delta x, \Delta y, \Delta z)} \|\omega\|_P\|\tilde{B}\|_P - \text{Boundary}$$

where the $C$’s depend on $u$ and its derivative. To obtain this result, we have again used (3.21) to get rid of the outflow boundary terms. We have also used that

$$(x, \tilde{v} \circ y)_P = (\tilde{v} \circ x, y)_P,$$

where $x$ and $y$ are scalar grid function and $\tilde{v}$ a vector grid function. We conclude the proof using Cauchy’s inequality.

Although the symmetric scheme (3.13) is stable in $H^1$, it might generate bounded divergence errors. Next, we design a scheme that preserves a discrete version of the divergence operator.
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3.1.2. Divergence Preserving Schemes

The symmetric scheme does not preserve a discrete version of the divergence constraint as it discretizes the symmetric version (2.3) of the magnetic induction equations with Hall effect. We have to discretize the non-symmetric standard version (1.26) of the magnetic induction equations with Hall effect in order to design a divergence preserving scheme. Such a scheme is

\[
\frac{d}{dt} \tilde{B}_{i,j,k} + \alpha \left( \mathbf{D} \times \frac{1}{\hat{\rho}_{i,j,k}} \mathbf{D} \times \frac{d}{dt} \tilde{B}_{i,j,k} \right) = \mathbf{D} \times (\tilde{\mathbf{u}} \times \tilde{B}_{i,j,k}) - \eta (\mathbf{D} \times (\mathbf{D} \times \tilde{B}_{i,j,k})) - \alpha \mathbf{D} \times \left( \frac{1}{\hat{\rho}_{i,j,k}} (\tilde{\mathbf{u}}_{i,j,k} \cdot \mathbf{D}) (\mathbf{D} \times \tilde{B}_{i,j,k}) \right) - \beta \mathbf{D} \times \left( \frac{1}{\hat{\rho}_{i,j,k}} (\mathbf{D} \times \tilde{B}_{i,j,k}) \times \tilde{B}_{i,j,k} \right),
\]

(3.22)

The application of corollary 3.1.6 shows that the above scheme clearly preserves the divergence:

\[
\frac{d}{dt} \mathbf{D} \cdot \tilde{\mathbf{B}} = 0.
\]

(3.23)

That means that if we have an initial data with zero discrete divergence (D·) the divergence constraint will be satisfied.

The proof for the energy stability of this scheme is more complicated since the equation is not symmetric. We introduce the following one-sided operators:

\[
D^*_x w_i = \frac{w_{i+s} - w_i}{s \Delta x},
\]

(3.24)

The operators \(D^*_y\) and \(D^*_z\) are defined analogously.

**Lemma 3.1.11.** For two grid functions \(v\) and \(w\), the following identity holds,

\[
D_x (v \circ w) = v \circ D_x w + \hat{A}_x ((D^*_x v), w),
\]

(3.25)

with \(\hat{A}_x ((D^*_x v), w)\), an average over discrete one-sided derivative of \(u\) multiplied with \(w\).

**Proof.** We write the finite difference operator as

\[
(D_x w)_i = \sum_{k=-l_i}^{r_i} \beta^i_k w_{i+k}
\]

with \(\sum_{k=-l_i}^{r_i} \beta^i_k = 0\) and \(\sum_{k=-l_i}^{r_i} k \beta^i_k = 1\). We compute the difference

\[
(D_x (v \circ w))_i - (v \circ D_x w)_i = \frac{1}{\Delta x} \sum_{k=-l_i}^{r_i} \beta^i_k v_{i+k} w_{i+k} - v_i \frac{1}{\Delta x} \sum_{k=-l_i}^{r_i} \beta^i_k w_{i+k}
\]
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\[
= \sum_{k=-r_i}^{l_i} \beta_k \frac{v_{i+k} - v_i}{k \Delta x} w_{i+k}.
\]

Defining the average \( \hat{A}_x(D_x^* v), w \) := \( \sum_{k=-q}^{q} k \beta_k (D_x^* u_i) w_{i+k} \) we note that it takes the form of the desired relation. \( \square \)

The energy bound for the divergence preserving scheme is given below:

**Theorem 3.1.12.** Let \( \tilde{u}_{i,j,k} = u(x_i, y_j, z_k) \) and \( \tilde{\rho}_{i,j,k} = \rho(x_i, y_j, z_k) \) be the point evaluation of functions in \( C^1(\Omega) \). Let \( \tilde{\rho}_{i,j,k} = \frac{\partial\tilde{\rho}}{\partial x}(x_i, y_j, z_k) \) be the point evaluation of a bounded function. The density should also be bounded away from zero, with \( \rho \geq \rho_0 > 0 \). Then the following estimates for the solutions (3.22) with zero initial discrete divergence hold

\[
\frac{d}{dt} ||\tilde{\mathbf{B}}||^2_{\mathbf{p}} \leq C_0 ||\tilde{\mathbf{B}}||^2_{\mathbf{p}} - \text{Boundary}, \tag{3.26}
\]

with \( C_0 \) being a constant that depends on the constant \( \alpha \), and on \( C^1 \) functions \( \rho \) and \( u \) and their derivatives only. The boundary term is:

\[
\text{Boundary} = \sum_{i,j,k=1}^{3} \epsilon_{i,j,k} \tilde{R}^i S^j \tilde{\mathbf{B}}^k + \frac{1}{2} \tilde{\mathbf{B}}^\top S^+ \tilde{\mathbf{B}} + \frac{\alpha}{2\rho_0} (D \times \tilde{\mathbf{B}})^\top S^+(D \times \tilde{\mathbf{B}})
\]

with

\[
\tilde{R} := \alpha \left( \frac{1}{\tilde{\rho}} D \times \frac{d\tilde{\mathbf{B}}}{dt} + \left( \frac{\tilde{\mathbf{u}} \cdot D}{\tilde{\rho}} \right) \circ D \times \tilde{\mathbf{B}} \right) + \eta (D \times \tilde{\mathbf{B}}) + \beta \frac{D \times \tilde{\mathbf{B}}}{\tilde{\rho}} \times \tilde{\mathbf{B}}.
\]

**Proof.** To prove the energy estimate, we have to symmetrize the advection terms in the scheme. This is possible since the method preserves divergence; in this case we can subtract form (3.22) \( \tilde{u}(D \cdot \tilde{\mathbf{B}}) = 0 \).

Using lemma 3.1.11 on each component, we can reformulate the discrete advection part

\[
D \times (\tilde{\mathbf{u}} \times \tilde{\mathbf{B}}) - \tilde{u}(D \cdot \tilde{\mathbf{B}}) = -(\tilde{\mathbf{u}} \cdot D) \tilde{\mathbf{B}} + \mathcal{R}(\tilde{\mathbf{B}}, \tilde{\mathbf{u}}),
\]

with

\[
\mathcal{R}(\tilde{\mathbf{B}}, \tilde{\mathbf{u}})_{i,j,k} = \\
\begin{pmatrix}
\mathcal{A}_y[(\tilde{\mathbf{u}}^2)_{i,j,k}] + \mathcal{A}_z[(\tilde{\mathbf{u}}^3)_{i,j,k}]
\end{pmatrix}.
\]

The resulting scheme is

\[
\frac{d}{dt} \tilde{B}_{i,j,k} + \alpha \left( \frac{1}{\tilde{\rho}_{i,j,k}} D \times \frac{d\tilde{\mathbf{B}}}{dt} \right)_{i,j,k} = \mathcal{R}(\tilde{\mathbf{B}}, \tilde{\mathbf{u}}) - (\tilde{\mathbf{u}} \cdot D) \tilde{\mathbf{B}} - \eta (D \times (D \times \tilde{\mathbf{B}}))
\]

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\[-\alpha \mathbf{D} \times \left( \frac{1}{\hat{\rho}_{i,j,k}} (\hat{\mathbf{u}}_{i,j,k} \cdot \mathbf{D})(\mathbf{D} \times \hat{\mathbf{B}}_{i,j,k}) \right) - \beta \mathbf{D} \times \left( \frac{1}{\hat{\rho}_{i,j,k}} (\mathbf{D} \times \hat{\mathbf{B}}_{i,j,k}) \times \hat{\mathbf{B}}_{i,j,k} \right) . \]

We see that this is very similar to the result obtained for the symmetric case. The only difference is that, instead the average term \((\hat{\mathbf{B}}, \mathcal{A}(\hat{\mathbf{B}}, \mathbf{u}))_P\) we have a residual term \((\hat{\mathbf{B}}, \mathcal{R}(\hat{\mathbf{B}}, \mathbf{u}))_P\). Then showing that the residual term is bounded:

\[(\hat{\mathbf{B}}, \mathcal{R}(\hat{\mathbf{B}}, \mathbf{u}))_P \leq C \| \hat{\mathbf{B}} \|^2_P \]

will conclude the proof. We can follow the same procedure we used in the proof of lemma 3.1.7 to bound the advection term.

The proof follows the theory presented in [23] where a generalised finite volume scheme that preserves discrete divergence is presented. The method presented there is more general in its formulation, and includes a large class of already known divergence preserving methods.

3.2. Time Stepping

For the sake of simplicity, we will consider only the case of a constant density distribution in the following, i.e., \(\rho(x, t) \equiv \rho\). Then, both the semi-discrete symmetric scheme and divergence preserving schemes can be written as

\[ \frac{d}{dt} \left( \hat{\mathbf{B}} + \alpha \mathbf{D} \times \mathbf{D} \times \hat{\mathbf{B}} \right) = \mathcal{G}(\hat{\mathbf{B}}, \hat{\mathbf{u}}), \]

Reformulating the left-hand side we obtain

\[ A \frac{d}{dt} \hat{\mathbf{B}} = \mathcal{G}(\hat{\mathbf{B}}, \hat{\mathbf{u}}), \quad (3.27) \]

where \(A = I + \alpha F\) with \(F\) the matrix representation of \(\mathbf{D} \times \mathbf{D}\). As time stepping procedure we used a second order SSP method ([14]):

\[ \hat{\mathbf{B}}^* = \hat{\mathbf{B}}^n + \Delta t A^{-1} \mathcal{G}(\hat{\mathbf{B}}^n, \hat{\mathbf{u}}), \]

\[ \hat{\mathbf{B}}^{**} = \hat{\mathbf{B}}^* + \Delta t A^{-1} \mathcal{G}(\hat{\mathbf{B}}^*, \hat{\mathbf{u}}), \]

\[ \hat{\mathbf{B}}^{n+1} = \frac{\hat{\mathbf{B}}^n + \hat{\mathbf{B}}^{**}}{2}, \quad (3.28) \]

combined with a second order spatial discretisation and a fourth order Runge Kutta method

\[ k_1 = A^{-1} \mathcal{G}(\hat{\mathbf{B}}^n, \hat{\mathbf{u}}), \]
3.3. Numerical Experiments

For simplicity, we consider the magnetic induction equations with Hall effect in two space dimensions and present numerical experiments comparing different schemes proposed above. In two space dimensions, the symmetric version (2.3) reads

\[
\begin{align*}
\frac{\partial}{\partial t} [\hat{B} + \alpha \nabla \times \nabla \times \hat{B}] &= \hat{C}_1 \hat{B} - (\hat{u} \cdot \nabla) \hat{B} - \eta \nabla \times \nabla \times \hat{B} - \alpha \nabla \times ((\hat{u} \cdot \nabla)(\nabla \times \hat{B})) - \beta \nabla \times (\hat{B} \cdot \nabla \hat{B}_3), \\
\frac{\partial}{\partial t} [\hat{B}_3 - \alpha \Delta \hat{B}_3] &= -\hat{u} \nabla \hat{B}_3 + \eta \Delta \hat{B}_3 + \alpha \nabla \cdot ((\hat{u} \cdot \nabla) \nabla \hat{B}_3) - \beta \nabla \cdot (\hat{B} \cdot (\nabla \times \hat{B})).
\end{align*}
\] (3.30)

Here, \(\hat{B} = (\hat{B}_1, \hat{B}_2)^T\), \(\hat{u} = (\hat{u}_1, \hat{u}_2)^T\) and \(\hat{C}_1 = \begin{pmatrix} \frac{\partial \hat{u}_1}{\partial x} & \frac{\partial \hat{u}_1}{\partial y} \\ \frac{\partial \hat{u}_2}{\partial x} & \frac{\partial \hat{u}_2}{\partial y} \end{pmatrix}\). We have also introduced a compact “curl” operator \(\nabla \times\) in two dimensions:

\[
\nabla \times \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}, \quad (3.31a)
\]

\[
\nabla \times \psi := \begin{pmatrix} \frac{\partial \psi}{\partial y} \\ -\frac{\partial \psi}{\partial x} \end{pmatrix}, \quad (3.31b)
\]

where \(\hat{\psi} : \mathbb{R}^3 \to \mathbb{R}^2\) and \(\psi : \mathbb{R}^3 \to \mathbb{R}\).
3. Finite Difference Schemes

3.3.1. Smooth Problems

Pure magnetic advection.

First, we test the proposed numerical schemes for the magnetic induction equations without Hall, electron inertia and resistivity terms. We take the velocity field

\[ \mathbf{u} = (-y, x) \]

in the following. Then, (3.30) with \( \eta = \alpha = \beta = 0 \) has an exact solution (see [11]) given by

\[ \hat{\mathbf{B}}(x, y, t) = R(t)\hat{\mathbf{B}}_0(R(-t)(x,y)), \tag{3.32} \]

with \( R(t) \) a rotation matrix with angle \( t \).

The initial data is

\[ \hat{\mathbf{B}}_0(x, y) = 4 \left( \frac{-y}{x} - \frac{1}{2} \right) e^{-20((x-1/2)^2+y^2)} \tag{3.33} \]

in the computational domain \( \Omega = [-2.5, 2.5] \times [-2.5, 2.5] \). We consider Neumann type non-reflecting boundary conditions. The exact solution represents the rotation of the initial hump around the domain with the hump completing one rotation in the period \( T = 2\pi \).

We will test the following four schemes: the second- and fourth-order versions of the symmetric scheme (3.13) with difference operators given in appendix A, second- and

![Convergence plots for the advection problem for the different schemes analysed.](image)

Figure 3.1.: Convergence plots for the advection problem for the different schemes analysed.
3.3. Numerical Experiments

fourth-order version of the divergence preserving scheme (3.22). The convergence plots in $L^2$ are shown in figure 3.1.

For time integration we have used a second order SSP and a standard fourth order Runge-Kutta method. The results are obtained using different mesh sizes, from 60 to 200 points. The experimental convergence orders are shown in Table 3.1 and demonstrate that the expected orders of accuracy are obtained in practice.

<table>
<thead>
<tr>
<th></th>
<th>2nd ord</th>
<th>4 ord</th>
</tr>
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<tbody>
<tr>
<td>Preserving</td>
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<td>3.98</td>
</tr>
<tr>
<td>Symmetric</td>
<td>1.89</td>
<td>3.98</td>
</tr>
</tbody>
</table>

Table 3.1.: Convergence rates for the magnetic advection problem for different schemes.

Forced Solutions

In order to test the convergence rates for various schemes for the full Magnetic induction equations with Hall effect, we add a forcing term such that the rotating hump (3.32) remains a solution of the forced equations. The Hall induction equations with the forcing term are

\[
\begin{align*}
\frac{\partial}{\partial t}[\hat{\mathbf{B}} + \alpha \hat{\mathbf{V}} \times \hat{\mathbf{V}} \times \hat{\mathbf{B}}] &= \hat{C}_1 \hat{\mathbf{B}} - (\hat{\mathbf{u}} \cdot \nabla)\hat{\mathbf{B}} - \eta \hat{\mathbf{V}} \times \hat{\mathbf{V}} \times \hat{\mathbf{B}} \\
&\quad - \alpha \hat{\mathbf{V}} \times ((\hat{\mathbf{u}} \cdot \nabla)(\hat{\mathbf{V}} \times \hat{\mathbf{B}})) - \beta \hat{\mathbf{V}} \times (\hat{\mathbf{B}} \cdot \nabla \hat{\mathbf{B}}_3) + \hat{S}(x, y, t), \\
\frac{\partial}{\partial t}[\hat{\mathbf{B}}_3 - \alpha \Delta \hat{\mathbf{B}}_3] &= -\hat{\mathbf{u}} \nabla \hat{\mathbf{B}}_3 + \eta \Delta \hat{\mathbf{B}}_3 \\
&\quad + \alpha \nabla \cdot ((\hat{\mathbf{u}} \cdot \nabla)\nabla \hat{\mathbf{B}}_3) - \beta \nabla \cdot (\hat{\mathbf{B}} \cdot (\hat{\mathbf{V}} \times \hat{\mathbf{B}}_3)) + S^3(x, y, t). 
\end{align*}
\]

(3.34a)

(3.34b)

The forcing term $\hat{S}$ is

\[
\hat{S}(x, y, t) = 160 P(x, y, t) \eta e^{-20((x \cos(t) + y \sin(t) - 1/2)^2 + (y \cos(t) - x \sin(t))^2)} \left( \frac{\sin(t) - 2y}{2x - \cos(t)} \right),
\]

$S^3(x, y, t) = 0$.

Here $P(x, y, t) = 20x \cos(t) + 20y \sin(t) - 20x^2 - 20y^2 - 3$.

The convergence results for four different schemes are presented in figure 3.2. The obtained orders of convergence are shown in Table 3.2. Again, the expected orders of convergence are obtained.
3. Finite Difference Schemes

<table>
<thead>
<tr>
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<th>2nd ord</th>
<th>4 ord</th>
</tr>
</thead>
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<td>Preserving</td>
<td>2.05</td>
<td>3.97</td>
</tr>
<tr>
<td>Symmetric</td>
<td>2.05</td>
<td>3.98</td>
</tr>
</tbody>
</table>

Table 3.2.: Convergence rates for the Forced Problem with $\eta = 0.01$, $\alpha = 0.002$ and $\beta = 0.01$

3.3.2. Hall effect solutions

Next, we test the full Magnetic induction equations with Hall effect (without any forcing) for the rotating hump problem. We set $\eta = 0.01$, $\alpha = 0.02$ and $\beta = 0.1$ and compute the solutions on a mesh $160 \times 160$ points.

We compare the results with those obtained for the pure advection of the hump.

The results are shown in figure 3.3 and demonstrate the robustness of the second-order symmetric scheme. Similar results were obtained with the divergence preserving scheme. The results show that the addition of resistivity, electron inertia and Hall effect leads to diffusion of the original hump and the creation of a non-zero $B_3$ component even if the initial $B_3$ is set to zero.

We conclude by tabulating the discrete divergence generated by the schemes for this problem in Table 3.3. As expected, the divergence preserving scheme preserves divergence to machine precision. On the other hand, the symmetric scheme does generate some spurious divergence.

The divergence errors converge quite rapidly to zero as the mesh is refined. Furthermore, there was no noticeable difference in the quality of the results for the primary solution.

Figure 3.2.: Convergence plots for the Forced Problem with $\eta = 0.01, \alpha = 0.002$ and $\beta = 0.01$
3.3. Numerical Experiments

(a) $B_1$ component for the magnetic advection problem.

(b) $B_1$ component for the Hall problem.

(c) $B_2$ component for the magnetic advection problem.

(d) $B_2$ component for the Hall problem.

(e) $B_3$ component for the magnetic advection problem.

(f) $B_3$ component for the Hall problem.

Figure 3.3.: On the right plots for solution with $\eta = 0.01$, $\alpha = 0.002$ and $\beta = 0.1$ after $T = \pi$ and on the left the advected solution after the same time.

variables between the symmetric and divergence preserving schemes.
3. Finite Difference Schemes

<table>
<thead>
<tr>
<th></th>
<th>Symmetric</th>
<th>Preserving</th>
</tr>
</thead>
<tbody>
<tr>
<td>25</td>
<td>4.6e−2</td>
<td>6.4e−17</td>
</tr>
<tr>
<td>50</td>
<td>2.4e−5</td>
<td>8.6e−17</td>
</tr>
<tr>
<td>75</td>
<td>6.2e−7</td>
<td>9.1e−17</td>
</tr>
<tr>
<td>100</td>
<td>2.5e−7</td>
<td>1.1e−16</td>
</tr>
<tr>
<td>125</td>
<td>1.4e−7</td>
<td>1.2e−16</td>
</tr>
</tbody>
</table>

Table 3.3.: Discrete divergence for the Hall problem.

3.3.3. Discontinuous Problems

The above results showed that both the symmetric as well as divergence preserving schemes worked very well for smooth problems. There was little to distinguish them. As commented before, the full MHD equations with Hall effect will contain discontinuities in the form of shock waves. We mimic this effect at the level of the magnetic induction equations with Hall effect by considering a discontinuous velocity field (note that the plasma velocities in MHD can be discontinuous):

\[
\hat{u}(x, y) = \begin{cases} 
(1) & \text{for } -\frac{6}{5} \leq x \leq \frac{6}{5} \\
(\frac{1}{2}) & \text{elsewhere}
\end{cases}
\]  

(3.35)

We take an smooth initial magnetic fields with compact support

\[
\hat{B}(x, y) = 100 \left( \begin{array}{c} y \\
-x \end{array} \right) \begin{cases} 
e^{-\frac{2}{1-x^2-y^2}} & \text{for } x^2 + y^2 \leq 0 \\
0 & \text{elsewhere}
\end{cases}
\]  

(3.36a)

\[
B_3(x, y) = 0.0
\]  

(3.36b)

over a domain \( \Omega = [-2.5, 2.5] \times [-2.5, 2.5] \) with double periodic boundary conditions. We choose \( \eta = 0.5 \), \( \alpha = 0.1 \) and \( \beta = 0.2 \) for a mesh of 200 \( \times \) 200 cells and run a simulations until \( T = 0.4 \). We will use both schemes, the divergence preserving and the symmetric scheme. In the absence of analytical solutions, we will compare the two schemes on a fixed mesh.

Comparing the two schemes in figure 3.4, we note that the symmetric scheme is more oscillatory than the divergence preserving scheme. These oscillations are small and can be seen only in the third component of the solution.

It is essential to point out that our theoretical stability results required that the underlying velocity field be smooth, at least \( C^2 \). Although the velocity field in this experiment is discontinuous, the divergence preserving scheme resolves the solution quite well, indicating its robustness on a challenging test problem. The symmetric scheme is more
3.3. Numerical Experiments

Figure 3.4.: Comparison of solutions with discontinuous velocity (3.35) for initial data (3.36). The parameters are set as $\eta = 0.5$, $\alpha = 0.1$ and $\beta = 0.2$ after $T = 0.4$. On the left we have the result of the symmetric scheme, and on the right of the divergence preserving schemes.

oscillatory on this test case, indicating that the divergence preserving scheme adds a greater amount of diffusion than the symmetric scheme.

Remark 3.3.1. We observe that in (3.28) and (3.29), we need to solve linear algebraic system at every time step and stage of the RK methods. In particular for the simplest
3. Finite Difference Schemes

time stepping for the solution of (3.27), the Euler scheme

\[ \hat{B}^{n+1} = \hat{B}^n + \Delta t A^{-1} \mathcal{G}(\hat{B}^n, \hat{u}), \]

we need to invert the matrix \( A \), which is ill conditioned. Numerical test in two dimensions showed that the condition number of \( A \) increases drastically if we increase the number of mesh points. The key issue in designing efficient time stepping procedures for the magnetic induction equations is to design a suitable preconditioner. This is complicated due to the saddle point type structure of the Maxwell’s equations. Recently, robust preconditioners have been developed for Maxwell type problems that use finite element type discretisation [16] and references therein. Our discretisation, being a finite difference discretisation, does not satisfy the requirements of this theory. Our attempt to adapt existing finite element preconditioners to this finite difference setup involved using an auxiliary space techniques [31]. The underlying idea of this procedure is to map the problem to a different function space and then solve the linear system in this auxiliary space. However, we could not find a suitable projection operator to map the complete structure of the kernel of \( A \) into this auxiliary space. Therefore we shall change perspective and present Discontinuous Galerkin discretisation of the magnetic induction equation with Hall effect in the subsequent chapters.
4. One dimensional Problem

As seen in the previous chapter, a stable finite difference discretisation of the magnetic induction equation with Hall effect resulted in having to solve (large) ill-conditioned linear systems at every time step. The design of an efficient preconditioner for this system failed as the auxiliary space technique could not be adapted to the finite element discretisation, at least in a straightforward manner.

Consequently, we will consider a discontinuous Galerkin (DG) discretisation of the magnetic induction equation with Hall effect. Apart from the potential of designing a suitable preconditioner, the DG method is able to handle complicated domain geometries discretised using unstructured grids. Furthermore, the implementation of stable boundary conditions is straightforward in the DG approach.

As a first step in designing a suitable DG method for the multi-dimensional magnetic induction equation with Hall effect, we consider the special case of one space dimension below. To do so, we set $\rho = 1$ in (2.3) and assume that the magnetic field $B$ and the velocity $u$ depend only on $z$. Further noting that the advection term $u \cdot \nabla$ reduces to $u_3 \partial / \partial z$, it is also reasonable to neglect $u_1$ and $u_2$ and set $u(z) = (0, 0, u(z))$. With these assumptions the problem (2.3) reduces to

$$\frac{\partial B_1}{\partial t} + \frac{\partial (u B_1)}{\partial z} = \eta \frac{\partial^2 B_1}{\partial z^2} + \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 B_1}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( u \frac{\partial^2 B_1}{\partial z^2} \right) \right) - \beta \frac{\partial}{\partial z} \left( B_3 \frac{\partial B_2}{\partial z} \right), \quad (4.1a)$$

$$\frac{\partial B_2}{\partial t} + \frac{\partial (u B_2)}{\partial z} = \eta \frac{\partial^2 B_2}{\partial z^2} + \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 B_2}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( u \frac{\partial^2 B_2}{\partial z^2} \right) \right) + \beta \frac{\partial}{\partial z} \left( B_3 \frac{\partial B_1}{\partial z} \right), \quad (4.1b)$$

$$\frac{\partial B_3}{\partial t} + u \frac{\partial B_3}{\partial z} = 0. \quad (4.1c)$$

The divergence free condition in one space dimension reads $(\partial B^3)/(\partial z) = 0$, hence $B^3 \equiv C$ and we can set $\tilde{\beta} = B^3 \beta$. The one dimensional restriction of the magnetic induction equation with Hall effect is

$$\frac{\partial B_1}{\partial t} + \frac{\partial (u B_1)}{\partial z} = \eta \frac{\partial^2 B_1}{\partial z^2} + \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 B_1}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( u \frac{\partial^2 B_1}{\partial z^2} \right) \right) - \tilde{\beta} \frac{\partial^2 B_2}{\partial z^2}, \quad (4.2a)$$

$$\frac{\partial B_2}{\partial t} + \frac{\partial (u B_2)}{\partial z} = \eta \frac{\partial^2 B_2}{\partial z^2} + \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 B_2}{\partial z^2} \right) + \frac{\partial}{\partial z} \left( u \frac{\partial^2 B_2}{\partial z^2} \right) \right) + \tilde{\beta} \frac{\partial^2 B_1}{\partial z^2}. \quad (4.2b)$$
4. One dimensional Problem

These equations are coupled through the Hall term. Setting $\tilde{\beta} = 0$, we see that (4.2) reduces to two copies of the following scalar one dimensional problem:

$$\frac{\partial b}{\partial t} + \frac{\partial (u b)}{\partial x} = \eta \frac{\partial^2 b}{\partial x^2} + \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 b}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( u \frac{\partial^2 b}{\partial x^2} \right) \right)$$  \hspace{1cm} (4.3)

The analysis of this equation is the main theme of this chapter.

4.1. Continuous Problem

The one dimensional equation (4.3) is augmented with the following initial condition

$$b(x, 0) = b_0(x), \hspace{1cm} (4.4)$$

and boundary conditions

$$b(0, t) = b(1, t) = 0, \hspace{1cm} (4.5a)$$

$$\sqrt{\alpha} \frac{\partial b}{\partial x}(0, t) = g^l(t), \hspace{0.5cm} \text{for} \hspace{0.5cm} t \in \Gamma_l, \hspace{1cm} (4.5b)$$

$$\sqrt{\alpha} \frac{\partial b}{\partial x}(1, t) = g^r(t), \hspace{0.5cm} \text{for} \hspace{0.5cm} t \in \Gamma_r. \hspace{1cm} (4.5c)$$

In this case we have chosen the domain to be $\Omega = [0, 1]$, and $u(x, t)$ is a given velocity. The inflow boundary is defined as:

$$\Gamma_l := \{ t \in [0, \infty[ \mid u(0, t) > 0 \}, \hspace{1cm} (4.6a)$$

$$\Gamma_r := \{ t \in [0, \infty[ \mid u(1, t) < 0 \}. \hspace{1cm} (4.6b)$$

4.1.1. Energy Estimate

As in the case of multi-dimensional magnetic induction equations with Hall effect, the one-dimensional version satisfies the following energy estimate:

**Theorem 4.1.1.** Let $u \in W^{1,\infty}([0, 1])$. Then the following a priori estimate for the solution of (4.3) satisfying boundary conditions (4.5) holds:

$$\frac{\partial}{\partial t} \left( \| b \|^2_{L^2([0, 1])} + \alpha \left\| \frac{\partial b}{\partial x} \right\|^2_{L^2([0, 1])} \right) \leq C_0 \left( \| b \|^2_{L^2([0, 1])} + \alpha \left\| \frac{\partial b}{\partial x} \right\|^2_{L^2([0, 1])} \right) + D_0 \hspace{1cm} (4.7)$$

where the constants $C_0$ and $D_0$ depend only on the velocity $u$, its derivative, and boundary functions $g^i$ with $i = l, r$. 

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4.1. Continuous Problem

Proof. Multiplying the evolution equation (4.3) with $b(x,t)$ and integrating over the domain $\Omega$ we obtain

$$\int_0^1 b \left[ \frac{\partial b}{\partial t} + \frac{\partial (u b)}{\partial x} - \eta \frac{\partial^2 b}{\partial x^2} - \alpha \left( \frac{\partial}{\partial t} \left( \frac{\partial^2 b}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( u \frac{\partial^2 b}{\partial x^2} \right) \right) \right] dx = 0,$$

$$\Rightarrow \int_0^1 \frac{1}{2} \frac{\partial b^2}{\partial t} + b \frac{\partial (u b)}{\partial x} - \eta b \frac{\partial^2 b}{\partial x^2} - \alpha b \frac{\partial}{\partial t} \left( \frac{\partial^2 b}{\partial x^2} \right) - \alpha b \frac{\partial}{\partial x} \left( u \frac{\partial^2 b}{\partial x^2} \right) dx = 0.$$

Now we apply integration by parts repeatedly and use the boundary condition (4.5a):

$$\frac{d}{dt} \left( \| b \|_{L^2([0,1])}^2 + \alpha \left\| \frac{\partial b}{\partial x} \right\|_{L^2([0,1])}^2 \right) = \int_0^1 \left( \frac{\partial u}{\partial x} \right) \left( \alpha \left( \frac{\partial b}{\partial x} \right)^2 - b^2 \right) - 2\eta \left( \frac{\partial b}{\partial x} \right)^2 \, dx$$

$$- \alpha \left( u(1,t) \left( \frac{\partial b}{\partial x}(1,t) \right)^2 - u(0,t) \left( \frac{\partial b}{\partial x}(0,t) \right)^2 \right).$$

We estimate this equation using the fact that $u$ is in $W^{1,\infty}([0,1])$. The boundary terms $u \frac{\partial b}{\partial x}$, are neglected when they are positive or are fixed by the boundary conditions (4.5b) and (4.5c) otherwise. This results in

$$\frac{d}{dt} \left( \| b \|_{L^2([0,1])}^2 + \alpha \left\| \frac{\partial b}{\partial x} \right\|_{L^2([0,1])}^2 \right) \leq \| \frac{\partial u}{\partial x} \|_{L^2([0,1])} \left( \| b \|_{L^2([0,1])}^2 + \alpha \left\| \frac{\partial b}{\partial x} \right\|_{L^2([0,1])}^2 \right)$$

$$- 2\eta \left\| \frac{\partial b}{\partial x} \right\|^2 - (\min(0,u(1,t))) g^r(t)^2 - \max(0,u(0,t))) g^l(t)^2),$$

$$\Rightarrow \frac{d}{dt} \left( \| b \|_{L^2([0,1])}^2 + \alpha \left\| \frac{\partial b}{\partial x} \right\|_{L^2([0,1])}^2 \right) \leq C_0 \left( \| b \|_{L^2([0,1])}^2 + \alpha \left\| \frac{\partial b}{\partial x} \right\|_{L^2([0,1])}^2 \right) + D_0$$

where $C_0 = \| \frac{\partial}{\partial x} u \|_{L^2([0,1])}$ and $D_0 = |(\min(0,u(1,t))) g^r(t)^2 - \max(0,u(0,t))) g^l(t)^2)|$. 

This energy estimate is crucial in developing a numerical method. In the following, we will present a numerical approximation of (4.3) such that a discrete version of the energy estimate holds.

In addition to the energy estimate, we show the following maximum principle in the case of a constant velocity field:

**Lemma 4.1.2.** Let $u$ be constant, then the solution of the Cauchy problem for (4.3), satisfies

$$\max_{x \in \mathbb{R}} \left( b(x,t) - \alpha \left[ \frac{\partial^2 b}{\partial x^2}(x,t) \right) \right) \leq \max_{x \in \mathbb{R}} \left( b_0(x) - \alpha \left[ \frac{\partial^2 b_0}{\partial x^2}(x) \right) \right).$$
4. One dimensional Problem

Proof. Since \(u\) is constant we can reformulate (4.3) as:

\[
\frac{\partial}{\partial t} \left( b - \alpha \frac{\partial^2 b}{\partial x^2} \right) + u \frac{\partial}{\partial x} \left( b - \alpha \frac{\partial^2 b}{\partial x^2} \right) = \eta \frac{\partial^2 b}{\partial x^2}.
\]

Defining

\[
\bar{b} := b - \alpha \frac{\partial^2 b}{\partial x^2}
\]

we get

\[
\frac{\partial \bar{b}}{\partial t} + u \frac{\partial \bar{b}}{\partial x} = \frac{\eta}{\alpha} (b - \bar{b}).
\]

Assume that \(x^*\) is the maximum of \(\bar{b}\) then we have

\[
\frac{d}{dt} \bar{b}(x^*) = \frac{\eta}{\alpha} (b(x^*) - \bar{b}(x^*))
\]

since \(\frac{\partial \bar{b}}{\partial x}(x^*) = 0\). Knowing that the solution of (4.8) can be written as convolution

\[
b(x, t) = \frac{1}{2\alpha} \int_{\mathbb{R}} \bar{b}(x - \tilde{x}, t) e^{-\frac{|\tilde{x}|^2}{2\alpha}} d\tilde{x},
\]

we can state that \(b \leq \bar{b}\) at maxima since the kernel \(e^{-\frac{|\tilde{x}|^2}{2\alpha}}/(2\alpha)\) is always positive. It follows that

\[
\frac{d}{dt} \bar{b}(x^*) \leq 0.
\]

\(\square\)

4.1.2. Variational Formulation

In order to construct a DG formulation of (4.3), we will rewrite (4.3) in the following mixed form:

\[
\frac{\partial b}{\partial t} + \frac{\partial (ub)}{\partial x} = \eta \frac{\partial j}{\partial x} + \alpha \frac{\partial e}{\partial x}, \quad (4.9a)
\]

\[
j = \frac{\partial b}{\partial x}, \quad (4.9b)
\]

\[
e = \frac{\partial j}{\partial t} + u \frac{\partial j}{\partial x}, \quad (4.9c)
\]
4.2. Numerical Scheme

with initial and boundary conditions

\[ b(x, 0) = b_0(x), \quad (4.10a) \]
\[ b(0, t) = b(1, t) = 0, \quad (4.10b) \]
\[ \sqrt{\alpha} j(0, t) = g_l(t) \quad \text{for} \quad t \in \Gamma_l, \quad (4.10c) \]
\[ \sqrt{\alpha} j(1, t) = g_r(t) \quad \text{for} \quad t \in \Gamma_r. \quad (4.10d) \]

To obtain a variational formulation of this problem, we multiply each equation of this system with test functions, and then integrate over \( \Omega \). The resulting formulation of the problem is:

Find \((u, j, p) \in H^1_0(\Omega) \times H^1(\Omega) \times H^1(\Omega)\) such that

\[
\int_0^1 \left( \partial_b w - (u b - \eta j - \alpha e) \frac{\partial w}{\partial x} \right) dx - (\eta q + \alpha e) w \big|_0^1 = 0 \quad \forall w \in C^1(\Omega), \quad (4.11a)
\]
\[
\int_0^1 \left( j z + b \frac{\partial z}{\partial x} \right) dx = 0 \quad \forall z \in C^1(\Omega), \quad (4.11b)
\]
\[
\int_0^1 \left( \left( e - \frac{\partial j}{\partial t} + j \left( \frac{\partial u}{\partial t} \right) \right) s + u j \frac{\partial s}{\partial x} \right) dx - u(1, t) j(1, t) s(1) \big|_{t \in [0, \infty]} = 0 \quad \forall s \in C^1(\Omega), \quad (4.11c)
\]
\[
b(., 0) = \Pi b_0. \quad (4.11d)
\]

In this formulation the boundary conditions are already included as right hand sides of equations or in the choice of the space \( H^1_0 \). The initial condition is given by (4.11d) where \( \Pi \) is the \( L^2 \)-projection.

4.2. Numerical Scheme

Our aim in this section is to present a DG discretisation of (4.11). The DG method was introduced in the 1970’s and has been very successful on efficiently discretising various PDE’s, see [2] and references therein for an historical point of view.

4.2.1. DG Formulation

A DG method approximates the variational formulation (4.11d) in the finite element space

\[
V_h = \{ f \in L^2(0, 1) \mid f|_{I_i} \in \mathbb{P}_k(I_i), i = 1, \cdots, N \}
\]
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where \( \mathbb{P}_k(I_i) \) denotes the space of polynomials in cell \( I_i \) of degree at most \( k \geq 1 \). Here, we have subdivided \( \Omega \) in \( N \) uniform intervals \( I_i = [x_{i-1}, x_i] \) with \( i = 1, \ldots, N \). The interval boundaries are given by \( x_i = i \Delta x \) where \( \Delta x = 1/N \) is the interval length. The corresponding variational formulation reads: Find \( (b_h, j_h, e_h) \in (V_h)^3 \) such that

\[
\sum_{i=1}^{N} \int_{I_i} \left( \frac{\partial b_h}{\partial t} w_h - (u b_h - \eta j_h - \beta e_h) \frac{\partial w_h}{\partial x} \right) dx + u^+(1,t)b_h(1,t)w_h(1) \\
- u^-(0,t)b_h(0,t)w_h(0) + \sum_{x_e \in \mathcal{E}^I} \hat{u}b_h|_x [w_h]|_x \\
- \sum_{x_e \in \mathcal{E}} (\eta \hat{j}h|_x + \beta \hat{e}h|_x) [w_h]|_x = 0, \quad \forall w_h \in V_h,
\]

(4.12a)

\[
\sum_{i=1}^{N} \int_{I_i} \left( j_h \frac{\partial z_h}{\partial x} \right) dx - \sum_{x_e \in \mathcal{E}^I} \hat{b}h|_x [z_h]|_x = 0, \quad \forall z_h \in V_h,
\]

(4.12b)

\[
\sum_{i=1}^{N} \int_{I_i} \left( e_h + j_h \left( \frac{\partial u}{\partial x} - \frac{\partial j_h}{\partial t} \right) s_h - j_h \frac{\partial s_h}{\partial x} \right) dx - \sum_{x_e \in \mathcal{E}^I} \hat{u}j_h|_x [s_h]|_x \\
- u(1,t)j_h(1,t)s_h(1) + u(0,t)\hat{j}h(0,t)s_h(0) \\
= u(1,t)\frac{g(t)}{\sqrt{\alpha}} s_h(1) - u^+(0,t)\frac{g(t)}{\sqrt{\alpha}} s_h(0), \quad \forall s_h \in V_h.
\]

(4.12c)

\( \mathcal{E}^I \) represents the set of all the internal edges and \( \mathcal{E} \) the set of all the edges, including the ones on the boundary of \( \Omega \). The inflow boundaries are included by means of \( v^+(x,t) := \max(v(x,t),0) \) and \( v^-(x,t) := \min(v(x,t),0) \). We have also used the jump operator

\[
\llbracket \psi_h \rrbracket |_{x_i} := \psi^+_h - \psi^-_h
\]

(4.13)

and the average operator

\[
\llbracket \psi_h \rrbracket |_{x_i} := \frac{\psi^+_h + \psi^-_h}{2}
\]

(4.14)

with \( \psi^\pm_h = \lim_{x \to x^\pm} \psi_h(x) \). On the boundary, these operators are defined as \( \llbracket \psi_h \rrbracket |_{x_0} := \psi_h(0), \| \psi_h \|_{x_N} := \psi_h(x_N), \| \psi_h \|_{x_0} := -\psi_h(0) \) and \( \| \psi_h \|_{x_N} := \psi_h(x_N) \).

Besides, we also use numerical fluxes

\[
\hat{u}b_h|_x = \hat{u}b_h(u^+, u^-, b^+_h, b^-_h), \\
\hat{u}j_h|_x = \hat{u}b_h(u^+, u^-, j^+_h, j^-_h), \\
\hat{b}h|_x = \hat{b}h(b^+_h, b^-_h, j^+_h, j^-_h, e^+_h, e^-_h), \\
\hat{j}h|_x = \hat{j}h(b^+_h, b^-_h, j^+_h, j^-_h), \\
\hat{e}h|_x = \hat{e}h(b^+_h, b^-_h, e^+_h, e^-_h).
\]
The exact form of the numerical fluxes is specified in the following.

Before we state the theorem for the energy bound, we present a well known lemma, which shows how the integration by parts acts over the sums of the intervals.

**Lemma 4.2.1.** Partial integration formulas for two functions \( \psi_h \) and \( \phi_h \) that are continuously differentiable over intervals \( I_i \) and a function \( u \) that is continuously differentiable over \([0, 1]\), can be written as:

\[
\sum_{i=1}^{N} \int_{I_i} \phi_h \frac{\partial \psi_h}{\partial x} \, dx = -\sum_{i=1}^{N} \int_{I_i} \psi_h \frac{\partial \phi_h}{\partial x} \, dx + \sum_{x_e \in \mathcal{E}} \{ \phi_h \} |_{x_e} [ \psi_h ] |_{x_e} + \sum_{x_e \in \mathcal{E}^I} \{ \psi_h \} |_{x_e} [ \phi_h ] |_{x_e},
\]

(4.15a)

\[
\sum_{i=1}^{N} \int_{I_i} \phi_h u \frac{\partial \psi_h}{\partial x} \, dx = -\sum_{i=1}^{N} \int_{I_i} \psi_h u \frac{\partial \phi_h}{\partial x} \, dx - \sum_{i=1}^{N} \int_{I_i} \psi_h \left( \frac{\partial u}{\partial x} \right) \phi_h \, dx + \sum_{x_e \in \mathcal{E}} \{ u \phi_h \} |_{x_e} [ \psi_h ] |_{x_e} + \sum_{x_e \in \mathcal{E}^I} \{ u \psi_h \} |_{x_e} [ \phi_h ] |_{x_e},
\]

(4.15b)

\[
\sum_{i=1}^{N} \int_{I_i} \psi_h u \frac{\partial \psi_h}{\partial x} \, dx = -\frac{1}{2} \sum_{i=1}^{N} \int_{I_i} (\psi_h)^2 \frac{\partial u}{\partial x} \, dx + \sum_{x_e \in \mathcal{E}^I} \{ u \psi_h \} |_{x_e} [ \psi_h ] |_{x_e} + \frac{1}{2} \sum_{x_e \in \mathcal{E}^0} \{ u \psi_h \} |_{x_e} [ \psi_h ] |_{x_e},
\]

(4.15c)

where \( \mathcal{E}, \mathcal{E}^I \) and \( \mathcal{E}^0 \) are the sets of all edges, internal edges and boundary edges respectively.

**Proof.** We start by proving (4.15a) and then show how the two other identities can be derived (4.15a).

Using integration by parts on each cell we obtain

\[
\sum_{i=1}^{N} \int_{I_i} \phi_h \frac{\partial \psi_h}{\partial x} \, dx = -\sum_{i=1}^{N} \int_{I_i} \psi_h \frac{\partial \phi_h}{\partial x} \, dx + \sum_{i=1}^{N} (\psi_h^{-}(x_i)\phi_h^{-}(x_i) - \psi_h^{+}(x_{i-1})\phi_h^{+}(x_{i-1}))
\]

\[
= -\sum_{i=1}^{N} \int_{I_i} \psi_h \frac{\partial \phi_h}{\partial x} \, dx + \sum_{x_e \in \mathcal{E}^I} (\psi_h^{-}(x_e)\phi_h^{-}(x_e) - \psi_h^{+}(x_e)\phi_h^{+}(x_e))
\]

\[
+ \psi_h(1)\phi_h(1) - \psi_h(0)\phi_h(0),
\]

where we have substituted the sum over the cells with a sum over all the internal edges.

The terms of the sum can be rewritten using jumps and averages as

\[
\psi_h^{-}(x_e)\phi_h^{-}(x_e) - \psi_h^{+}(x_e)\phi_h^{+}(x_e) = \frac{1}{2} \psi_h^{-}(x_e)[\phi_h] |_{x_e} + \frac{1}{2} \psi_h^{-}(x_e)\phi_h^{+}(x_e) + \frac{1}{2} \phi_h^{-}(x_e)[\psi_h] |_{x_e} + \frac{1}{2} \psi_h^{+}(x_e)\phi_h^{-}(x_e)
\]
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\[ + \frac{1}{2} \psi_h^+ (x_\epsilon) [\phi_h]_{x_\epsilon} - \frac{1}{2} \psi_h^+ (x_\epsilon) \phi_h^- (x_\epsilon) + \frac{1}{2} \phi_h^+ (x_\epsilon) \psi_h^- (x_\epsilon) - \frac{1}{2} \psi_h^- (x_\epsilon) \phi_h^+ (x_\epsilon) \]

\[ = \{ \{ \psi_h \} |_{x_\epsilon} [\phi_h]_{x_\epsilon} + \{ \{ \phi_h \} |_{x_\epsilon} [\psi_h]_{x_\epsilon} . \]

To get the final result, it is sufficient to rewrite the boundary term using jump and averages as \( \phi_h (0) \psi_h (0) = \{ \{ \phi_h \} |_{x_0} [\psi_h]_{x_0} \) and \( \phi_h (1) \psi_h (1) = - \{ \{ \phi_h \} |_{x_N} [\psi_h]_{x_N} \).

Inserting \( \psi_h = u \psi_h \) in (4.15a) and using the product rule we almost obtain (4.15b). To get exactly (4.15b) we have to use the fact that \( u \) is uniquely defined on the edges and therefore can be moved in or out the average and jump operators.

The last identity, (4.15c), is given by (4.15b) with \( \psi_h = \phi_h \). \( \square \)

We obtain the following estimate:

**Theorem 4.2.2.** Let \( u \in C^1 ([0, 1]) \) and \( (b_h, j_h, p_h) \in (V_h)^3 \) solutions of (4.12), then the following apriori estimate holds:

\[
\frac{d}{dt} \sum_{i=1}^N (\| b_h (., t) \|^2_{L^2(I_i)} + \alpha \| j_h (., t) \|^2_{L^2(I_i)}) \leq C_1 \sum_{i=1}^N (\| b_h (., t) \|^2_{L^2(I_i)} + \alpha \| j_h (., t) \|^2_{L^2(I_i)})
\]

\[ + C_2 (t) + \mathcal{F} (t) \]

where the constants \( C_1 \) and \( C_2 (t) \) depend only on the velocity \( u \), its derivative and boundary functions \( g \)'s. The flux term is:

\[
\frac{1}{2} \mathcal{F} (t) := \sum_{x_\epsilon \in \mathcal{E}^F} \alpha (\hat{b}_h |_{x_\epsilon} - \{ b_h \} |_{x_\epsilon}) [\hat{e}_h]_{x_\epsilon} + \sum_{x_\epsilon \in \mathcal{E}} \alpha (\hat{e}_h |_{x_\epsilon} - \{ e_h \} |_{x_\epsilon}) [b_h]_{x_\epsilon}
\]

\[ + \sum_{x_\epsilon \in \mathcal{E}^F} \eta (\hat{b}_h |_{x_\epsilon} - \{ b_h \} |_{x_\epsilon}) [j_h]_{x_\epsilon} + \sum_{x_\epsilon \in \mathcal{E}} \eta (\hat{b}_h |_{x_\epsilon} - \{ j_h \} |_{x_\epsilon}) [b_h]_{x_\epsilon}
\]

\[ - \sum_{x_\epsilon \in \mathcal{E}^F} (\hat{u} b_h |_{x_\epsilon} - \{ u b_h \} |_{x_\epsilon}) [b_h]_{x_\epsilon} - \sum_{x_\epsilon \in \mathcal{E}^F} \alpha (\hat{u} j_h |_{x_\epsilon} - \{ u j_h \} |_{x_\epsilon}) [j_h]_{x_\epsilon}. \]

**Proof.** We start by taking \( w_h = b_h \) in (4.12a) and then integrating by parts using (4.15c)

\[
0 = \sum_{i=1}^N \int_{I_i} \left( \frac{1}{2} \frac{\partial b_h^2}{\partial t} - u b_h \frac{\partial b_h}{\partial x} + (\eta j_h + \alpha e_h) \frac{\partial u}{\partial x} \right) dx + \sum_{x_\epsilon \in \mathcal{E}^F} \hat{u} b_h |_{x_\epsilon} [b_h]_{x_\epsilon}
\]

\[ - \sum_{x_\epsilon \in \mathcal{E}^F} (\eta j_h |_{x_\epsilon} + \alpha e_h |_{x_\epsilon}) [b_h]_{x_\epsilon} + u^+ (1, t) b_h (1, t)^2 - u^- (0, t) b_h (0, t)^2
\]

\[ = \sum_{i=1}^N \int_{I_i} \left( \frac{1}{2} \frac{\partial b_h^2}{\partial t} + \frac{1}{2} (\frac{\partial u}{\partial x} b_h^2 + (\eta j_h + \alpha e_h) \frac{\partial b_h}{\partial x} \right) dx
\]

\[ + \sum_{x_\epsilon \in \mathcal{E}^F} (u b_h |_{x_\epsilon} - \{ u b_h \} |_{x_\epsilon}) [b_h]_{x_\epsilon} - \sum_{x_\epsilon \in \mathcal{E}^F} (\eta j_h |_{x_\epsilon} + \alpha e_h |_{x_\epsilon}) [b_h]_{x_\epsilon}
\]

\[ + \frac{1}{2} u^+ (1, t) b_h (1, t)^2 - \frac{1}{2} u^- (1, t) b_h (1, t)^2 - \frac{1}{2} u^- (0, t) b_h (0, t)^2 \]
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\[ + \frac{1}{2} u^+(0, t) b_h(0, t)^2. \]

We can obtain a first estimate using the boundary conditions, we obtain

\[
\sum_{i=1}^{N} \int_{I_i} \left( \frac{1}{2} \frac{\partial^2 b_h^2}{\partial t^2} + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + (\eta j_h + \alpha e_h) \frac{\partial b_h}{\partial x} \right) \, dx \\
+ \sum_{x \in E^T} (u b_h|_{x_e} - \{u b_h\}|_{x_e})[b_h]|_{x_e} - \sum_{x \in E} (\eta j_h|_{x_e} + \alpha \hat{e}_h|_{x_e})[b_h]|_{x_e} \leq 0. \quad (4.17)
\]

At this point we need to substitute the terms containing \( \frac{\partial b_h}{\partial t} \) in (4.17) using (4.12b). First we set \( z_h = e_h \) and then we use partial integration (4.15b)

\[
0 = \sum_{i=1}^{N} \int_{I_i} \left( j_h e_h + b_h \frac{\partial e_h}{\partial x} \right) \, dx - \sum_{x \in E^T} \hat{b}_h|_{x_e}[e_h]|_{x_e} \\
= \sum_{i=1}^{N} \int_{I_i} \left( j_h e_h - e_h \frac{\partial b_h}{\partial x} \right) \, dx - \sum_{x \in E^T} (\hat{b}_h|_{x_e} - \{b_h\}|_{x_e})[e_h]|_{x_e} + \sum_{x \in E} \{e_h\}|_{x_e}[b_h]|_{x_e}. \quad (4.18)
\]

Now we do the same for \( z_h = j_h \)

\[
0 = \sum_{i=1}^{N} \int_{I_i} \left( j_h^2 + b_h \frac{\partial j_h}{\partial x} \right) \, dx - \sum_{x \in E^T} \hat{b}_h|_{x_e}[j_h]|_{x_e} \\
= \sum_{i=1}^{N} \int_{I_i} \left( j_h^2 - j_h \frac{\partial b_h}{\partial x} \right) \, dx - \sum_{x \in E^T} (\hat{b}_h|_{x_e} - \{b_h\}|_{x_e})[j_h]|_{x_e} + \sum_{x \in E} \{j_h\}|_{x_e}[b_h]|_{x_e}. \quad (4.19)
\]

Combining these two last results (4.18) and (4.19) in (4.17) we get

\[
\sum_{i=1}^{N} \int_{I_i} \left( \frac{1}{2} \frac{\partial b_h^2}{\partial t} + (\eta j_h + \alpha e_h j_h) \right) \, dx - \sum_{i=1}^{N} \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 b_h^2 \, dx \\
+ \sum_{x \in E^T} \beta (\hat{b}_h|_{x_e} - \{b_h\}|_{x_e})[e_h]|_{x_e} + \sum_{x \in E} \alpha (\hat{e}_h|_{x_e} - \{e_h\}|_{x_e})[b_h]|_{x_e} \\
+ \sum_{x \in E^T} \eta (\hat{b}_h|_{x_e} - \{b_h\}|_{x_e})[j_h]|_{x_e} + \sum_{x \in E} \eta (\hat{j}_h|_{x_e} - \{j_h\}|_{x_e})[b_h]|_{x_e} \\
- \sum_{x \in E^T} (u \hat{b}_h|_{x_e} - \{u b_h\}|_{x_e})[b_h]|_{x_e}. \quad (4.20)
\]

It remains to deal with the mixed term \( e_h j_h \). We use (4.12c) with \( s_h = j_h \) and partial
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Integration (4.15c)

\[ 0 = \sum_{i=1}^{N} \int_{L_i} \left( e_h j_h + j_h \left( \frac{\partial u}{\partial x} \right) j_h - \left( \frac{1}{2} \frac{\partial_j^2}{\partial t^2} - j_h \frac{\partial j_h}{\partial x} \right) \right) dx - \sum_{x_e \in \mathcal{E}^I} u_{j_h}|_{x_e}[j_h]|_{x_e} \]

\[ - u^+(1, t) j_h(1, t)^2 + u^-(0, t) j_h(0, t)^2 - u^-(1, t) g_r(t) \sqrt{\alpha} j_h(1, t) \]

\[ + u^+(0, t) \frac{g_r(t)}{\sqrt{\alpha}} j_h(0, t) = \sum_{i=1}^{N} \int_{L_i} \left( e_h j_h + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right) j_h - \frac{1}{2} \frac{\partial j_h^2}{\partial t} \right) dx \]

\[ - \sum_{x_e \in \mathcal{E}^I} \left( (u_{j_h}|_{x_e} - \{ u_{j_h} \}|_{x_e}) [j_h]|_{x_e} - \frac{1}{2} u^+(1, t) j_h(1, t)^2 + \frac{1}{2} u^-(1, t) j_h(1, t)^2 \right) \]

\[ + \frac{1}{2} u^-(0, t) j_h(0, t)^2 - \frac{1}{2} u^+(0, t) j_h(0, t)^2 - u^-(1, t) j_h(1, t) \frac{g_r(t)}{\sqrt{\alpha}} + u^+(0, t) j_h(0, t) \frac{g_r(t)}{\sqrt{\alpha}}. \]

Now we can obtain an estimate from the last equation using the definitions of the inflow boundary conditions and Cauchy-Schwarz inequality

\[ \sum_{i=1}^{N} \int_{L_i} \left( e_h j_h + \frac{1}{2} \left( \frac{\partial u}{\partial x} \right) j_h - \frac{1}{2} \frac{\partial j_h^2}{\partial t} \right) dx \geq \]

\[ + \sum_{x_e \in \mathcal{E}^I} \left( (u_{j_h}|_{x_e} - \{ u_{j_h} \}|_{x_e}) [j_h]|_{x_e} + \frac{1}{2} u^-(1, t) \frac{g_r(t)^2}{\alpha} - \frac{1}{2} u^+(0, t) \frac{g_r(t)^2}{\alpha}. \quad (4.21) \]

Subtracting (4.21) from (4.20),

\[ \frac{1}{2} \frac{d}{dt} \sum_{i=1}^{N} \left( \| b_h(., t) \|_{L^2(L_i)}^2 + \alpha \| j_h(., t) \|_{L^2(L_i)}^2 \right) \leq \sum_{i=1}^{N} \int_{L_i} \frac{1}{2} \frac{\partial u}{\partial x} \left( - (b_h)^2 + \alpha (j_h)^2 \right) dx \]

\[ + \sum_{x_e \in \mathcal{E}^I} \alpha (\hat{b}_{j_h}|_{x_e} - \{ b_{j_h} \}|_{x_e}) [e_{j_h}]|_{x_e} + \sum_{x_e \in \mathcal{E}^I} \beta (\hat{e}_{j_h}|_{x_e} - \{ e_{j_h} \}|_{x_e}) [b_{j_h}]|_{x_e} \]

\[ + \sum_{x_e \in \mathcal{E}^I} \gamma (\hat{b}_{j_h}|_{x_e} - \{ b_{j_h} \}|_{x_e}) [j_{j_h}]|_{x_e} + \sum_{x_e \in \mathcal{E}^I} \eta (\hat{j}_{j_h}|_{x_e} - \{ j_{j_h} \}|_{x_e}) [b_{j_h}]|_{x_e} \]

\[ - \sum_{x_e \in \mathcal{E}^I} \left( (u_{b_h}|_{x_e} - \{ u_{b_h} \}|_{x_e}) [b_{j_h}]|_{x_e} - \alpha \sum_{x_e \in \mathcal{E}^I} (u_{j_h}|_{x_e} - \{ u_{j_h} \}|_{x_e}) [j_{j_h}]|_{x_e} \right) \]

\[ + \frac{1}{2} v^-(1, t) g_r(t)^2 - \frac{1}{2} v^+(0, t) g_r(t)^2. \]

Using the fact that \( u \) is continuously differentiable and the definitions \( C_1 := \max_{(0,T)} \| \frac{\partial u}{\partial x} \|_{L^2((0,1))} \) and \( C_2(t) := | \alpha (\min(0, u(1, t))) g_r(t)^2 - \max(0, u(0, t)) g_r(t)^2 | \)

already given in the proof of (4.1.1) we obtain the desired estimate. \( \square \)

It is easy to see that the choice of fluxes is central in obtaining a proof of stability.
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**Corollary 4.2.3.** Assuming \( u \in C^1 \), for every choice of fluxes \( \hat{b}_h|x_e, \hat{j}_h|x_e, \hat{c}_h|x_e, \hat{u}\hat{b}_h|x_e \) and \( u\hat{j}_h|x_e \) that satisfy

\[
F(t) \leq 0 \quad \forall (b_h, j_h, e_h) \in (V_h)^3
\]

with \( F(t) \) defined in (4.16), the solutions of (4.12) satisfy

\[
\frac{d}{dt} \sum_{i=1}^{N} \left( \|b_h(\cdot,t)\|_{L^2(I_i)}^2 + \alpha \|j_h(\cdot,t)\|_{L^2(I_i)}^2 \right) \leq C_0 \sum_{i=1}^{N} \left( \|b_h(\cdot,t)\|_{L^2(I_i)}^2 + \alpha \|j_h(\cdot,t)\|_{L^2(I_i)}^2 \right) + D_0
\]

where the constants \( C_0 \) and \( D_0 \) depend only on the velocity \( u \), its derivative and the boundary functions \( g \)'s.

**Proof.** The proof follows directly by the application of the bound on \( F \) on the result of Theorem 4.2.2. \( \square \)

Next we choose the numerical fluxes in the following manner. First, the advective part of the equations necessitates the use of upwind fluxes [4]:

\[
\begin{align*}
\hat{u}b_h|x_e &= \{u \hat{b}_h\}|x_e + \chi_x \hat{b}_h|x_e, \\
\hat{u}j_h|x_e &= \{u \hat{j}_h\}|x_e + \chi_x \hat{j}_h|x_e
\end{align*}
\]

(4.22a)

(4.22b)

where \( \chi_x = \frac{|\nabla u|}{|\nabla u|} \). Then, the diffusive part of the equation is discretised using local discontinuous Galerkin (LDG) fluxes [2]:

\[
\begin{align*}
\hat{b}_h|x_e &= \{b_h\}|x_e - \alpha [b_h]|x_e, \\
\hat{j}_h|x_e &= \begin{cases} \{j_h\}|x_e - \alpha [b_h]|x_e + \beta [j_h]|x_e & \text{for } e \in E^o, \\
\{j_h\}|x_e - \alpha [b_h]|x_e & \text{for } e \in E^\partial \end{cases}, \\
\hat{c}_h|x_e &= \begin{cases} \{c_h\}|x_e - \alpha [c_h]|x_e + \beta [e_h]|x_e & \text{for } e \in E^o, \\
\{c_h\}|x_e - \alpha [b_h]|x_e & \text{for } e \in E^\partial \end{cases}
\end{align*}
\]

(4.23a)

(4.23b)

(4.23c)

Using these fluxes to compute \( F(t) \), we obtain

\[
\frac{1}{2} F(t) = \sum_{x_e \in E^x} \alpha (b_h|x_e - \{b_h\}|x_e)[b_h]|x_e + \sum_{x_e \in E} \alpha (c_h|x_e - \{c_h\}|x_e)[c_h]|x_e \\
+ \sum_{x_e \in E^x} \eta (b_h|x_e - \{b_h\}|x_e)[j_h]|x_e + \sum_{x_e \in E} \eta (j_h|x_e - \{j_h\}|x_e)[b_h]|x_e \\
- \sum_{x_e \in E^x} (\hat{u}b_h|x_e - \{u \hat{b}_h\}|x_e)[b_h]|x_e - \sum_{x_e \in E^x} \beta (\hat{u}j_h|x_e - \{u \hat{j}_h\}|x_e)[j_h]|x_e \\
= - \sum_{x_e \in E} \chi_x \{b_h\}|x_e^2 + \{j_h\}|x_e^2 \sum_{x_e \in E} (\alpha \chi_x + \eta \chi_x^2)[b_h]|x_e^2.
\]
4. One dimensional Problem

If we choose positive penalty parameters $\alpha^j$ and $\alpha^e$, we get $\mathcal{F}(t) \leq 0$ and the resulting DG method is energy stable.

Remark 4.2.4. It is important to notice that the choice of fluxes satisfying the condition $\mathcal{F}(t) \leq 0$ required by the Corollary 4.2.3 is not unique. Our decision seemed to us the most natural, but other fluxes, like interior penalty [2], are an acceptable choice.

4.2.2. Matrix Formulation

In this section we represent the space $V_h$ by means of basis functions and reformulate the problem (4.12) in matrix form. We define a basis for $V_h$

$$\psi_s^i(x) := \begin{cases} \phi^s\left(\frac{2x}{\Delta x} - 2i + 1\right) & \text{for } x \in I_i \\ 0 & \text{for } x \notin I_i \end{cases} \quad (4.24)$$

for $i = 1, \cdots, N$ and $s = 0, \cdots, k$. In this case $(\phi^s(x))_{s=0,\cdots,k}$ is polynomial of order $k$ on the interval $[-1, 1]$. We express the discrete solutions $(b_h, p_h, j_h)$ by means of this basis function

$$b_h(x, t) = \sum_{i=1}^{N} \sum_{s=0}^{k} b^i_s(t) \psi^s_i(x) = \sum_{I=1}^{(k+1)N} b^I(t) \psi^I(x) \quad (4.25a)$$

$$e_h(x, t) = \sum_{i=1}^{N} \sum_{s=0}^{k} e^i_s(t) \psi^s_i(x) = \sum_{I=1}^{(k+1)N} e^I(t) \psi^I(x) \quad (4.25b)$$

$$j_h(x, t) = \sum_{i=1}^{N} \sum_{s=0}^{k} j^i_s(t) \psi^s_i(x) = \sum_{I=1}^{(k+1)N} j^I(t) \psi^I(x) \quad (4.25c)$$

where $I = s + 1 + k(i - 1)$ is a global index over all the degrees of freedom. Setting the test functions $w_h, z_h$ and $s_h$ equal to $\psi^I$ for $I = 1, \cdots, N(k+1)$ in (4.12) we get a system of equations. We can rewrite it in matrix form putting the coefficients in vector form $b = (b^1_h, \cdots, b^{(k+1)N}_h)^\top$, $e = (e^1_h, \cdots, e^{(k+1)N}_h)^\top$, $j = (j^1_h, \cdots, j^{(k+1)N}_h)^\top$:

$$M \dot{b} - (S(u) - E(u) - Q(c_e) - Bc)b + (\eta \alpha^j + \alpha \alpha^p)Qb + (S - E - bQ)(\eta j + \alpha e) = 0, \quad (4.26a)$$

$$M \dot{j} + (S - E + bQ)b = 0, \quad (4.26b)$$

$$Me - M \dot{j} + (S(u) + M \frac{\partial u}{\partial x}) - E(u) - Q(c_e) - Bc)j = g. \quad (4.26c)$$

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In this case we have used the fluxes defined in (4.22). The different $(k+1)N \times (k+1)N$ matrices and right hand-sides are defined as follows:

\[
(M(y))_{I,J} := \sum_{i=1}^{N} \int_{I_i} y(x) \psi_J(x) \psi_I(x) \, dx,
\]

\[
(S(y))_{I,J} := \sum_{i=1}^{N} \int_{I_i} y(x) \psi_J(x) \frac{\partial \psi_I(x)}{\partial x} \, dx,
\]

\[
(E(y))_{I,J} := \sum_{x_e \in \mathcal{E}} \left\{ \left[ y(x) \psi_J(x) \right]_{x_e} \left[ \psi_I(x) \right]_{x_e} \right\},
\]

\[
(\bar{E}(y))_{I,J} := \sum_{x_e \in \mathcal{E}} \left\{ \left[ y(x) \psi_J(x) \right]_{x_e} \left[ \psi_I(x) \right]_{x_e} \right\},
\]

\[
(Q(y))_{I,J} := \sum_{x_e \in \mathcal{E}} \left\{ \left[ y(x) \psi_J(x) \right]_{x_e} \left[ \psi_I(x) \right]_{x_e} \right\},
\]

\[
(\bar{Q}(y))_{I,J} := \sum_{x_e \in \mathcal{E}} \left\{ \left[ y(x) \psi_J(x) \right]_{x_e} \left[ \psi_I(x) \right]_{x_e} \right\},
\]

\[
(Bc)_{I,J} := u^{-}(0,t)\psi_J(0)\psi_I(0) - u^{+}(1,t)\psi_J(1)\psi_I(1),
\]

\[
(g)_{I} := u^{-}(1,t)g_r(t)\psi_I(1) - u^{+}(0,t)g_l(t)\psi_I(0).
\]

To simplify the notation, we have omitted to write the function $y(x)$ when $y(x) = 1$, for example $M = M(1)$. Since the mass matrix $M$ for the DG method is block diagonal and the dimension of the block is $k+1$, the inverse can be computed explicitly. So defining some auxiliary matrices

\[
Z := M^{-1}(S(u) - E(u) - Q(\epsilon) - Bc),
\]

\[
\tilde{Z} := M^{-1}(S(u) + M\frac{\partial u}{\partial x}) - E(u) - Q(\epsilon) - Bc,
\]

\[
W := M^{-1}(S - \bar{E} - bQ) \text{ and }
\]

\[
\tilde{W} := M^{-1}(S - \bar{E} + bQ),
\]

we can reformulate (4.26) as

\[
\dot{b} - Zb + (\eta a^j + \alpha a^p)\bar{Q}b + W(\eta j + \beta e) = 0,
\]

\[
\dot{j} = -\tilde{W}b,
\]

\[
e - \dot{j} + \tilde{Z}j = M^{-1}g.
\]

Eliminating the auxiliary variables from the system results in

\[
(\mathbb{I} - \alpha W\tilde{W})\dot{b} = -(\eta a^e + \alpha a^j)\bar{Q}b + \eta W\tilde{W}b + Zb - \alpha W\bar{Z}\tilde{W}b + \alpha WM^{-1}g \quad (4.29)
\]

where $\mathbb{I}$ is the identity matrix.
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4.2.3. Time Stepping

To motivate the challenges that arise in deriving a suitable time stepping procedure for (4.29), we start with the special case \( u = 0, \alpha = 0 \) and \( g^i = 0 \) for \( i = l, r \) in (4.26) obtaining

\[
\dot{b} = -\eta a\bar{Q}b + \eta W\tilde{W}b.
\]

Using implicit Euler discretisation, we obtain

\[
(I - \eta \Delta t(W\tilde{W} - a\bar{Q}))b^{n+1} = b^n, \quad \Rightarrow \quad A_{DG}(\eta \Delta t)b^{n+1} = b^n \quad (4.30)
\]

with \( A_{DG}(\gamma) := I - \gamma(W\tilde{W} - a\bar{Q}) \). We know that the matrix \( A_{DG} \) is ill-conditioned and needs a suitable preconditioner to invert it efficiently. We will design such a preconditioner in this chapter.

Next, in order to utilize the preconditioner developed for \( A_{DG} \), we introduce the following implicit temporal discretisation (4.26)

\[
(I - (\alpha + \Delta t\eta)W\tilde{W} + \Delta t(\eta a^j + \alpha a^c)\bar{Q})b^{n+1} = \Delta t\beta W M^{-1} g^n + (I + \Delta t(Z - \alpha WZ\tilde{W}))b^n.
\]

We choose penalty parameters \( a^j = a \) and \( a^c = a/\Delta t \) and obtain

\[
A_{DG}(\alpha + \Delta t\eta)b^{n+1} = \Delta t\alpha W M^{-1} g^n + (I + \Delta t(Z - \alpha WZ\tilde{W}))b^n \quad (4.31)
\]

as a suitable system of equations to solve at every time step.

4.2.4. Auxiliary Space Preconditioner

In this section, we concentrate on a sub-problem of (4.31), i.e., solving in an efficient way

\[
A_{DG}(\gamma)\omega = \mathbf{l} \quad (4.32)
\]

In this section, the parameter \( \gamma \in \mathbb{R} \), \( \mathbf{l} \) is a known load vector and \( \omega \) the unknown vector. The linear problem (4.32) can realised as a DG discretisation of the continuous problem

\[
\omega(x) - \gamma \frac{\partial^2 \omega}{\partial x^2}(x) = l(x), \quad x \in (0, 1), \quad (4.33a)
\]

\[
\omega(0) = \omega(1) = 0 \quad (4.33b)
\]

where the vectors \( \omega \) and \( \mathbf{l} \) are the coefficient vectors of the sought function \( \omega(x) \) and given function \( l(x) \).
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Conforming Galerkin Discretisation (CG)

The conforming discretisation of (4.33) is obtained by writing it in a variational formulation. We find \( \omega \in H^1_0([0,1]) \) such that

\[
\int_0^1 (\omega(x)z(x) - \gamma \frac{\partial \omega}{\partial x}(x) \frac{\partial z}{\partial x}(x)) \, dx = \int_0^1 l(x)z(x) \, dx \quad \forall z \in H^1_0([0,1]),
\]

and then approximating \( H^1_0([0,1]) \) with a discrete space \( \bar{V}_h \subseteq H^1_0([0,1]) \). The conforming approximate solution \( \bar{\omega}_h \) of (4.33) is given by this variational form: find \( \bar{\omega}_h \in \bar{V}_h \)

\[
\int_0^1 \bar{\omega}_h(x)\bar{z}_h(x) \, dx - \gamma \int_0^1 \frac{\partial \bar{\omega}_h}{\partial x}(x) \frac{\partial \bar{z}_h}{\partial x}(x) \, dx = \int_0^1 l(x) \bar{z}_h(x) \, dx \quad \forall \bar{z}_h \in \bar{V}_h.
\] (4.34)

The space \( \bar{V}_h \) is the continuous finite element space

\[
\bar{V}_h = \{ f \in C_0(0,1) \mid f|_{I_i} \in P_k(I_i), i = 1, \ldots, N \}.
\]

\( P_k(I_i) \) denotes the space of polynomials in cell \( I_i \) of degree at most \( k \geq 2 \). Choosing basis functions \( (\bar{\psi}_I)_{I=1,\ldots,Nk-1} \) and setting \( \bar{\omega}_h(x) = \sum_{I=1}^{Nk-1} \bar{\omega}_I \bar{\psi}_I(x) \) we can rewrite the problem (4.34) in matrix form

\[
M_{CG}\bar{\omega} + S_{CG}\bar{\omega} = \bar{l}.
\]

Here we have written the coefficients in vector form \( \bar{\omega} = (\bar{\omega}_1, \ldots, \bar{\omega}_N)^\top \) and \( \bar{l} = (\bar{l}_1, \ldots, \bar{l}_{Nk-1})^\top \). The definition of the load coefficients is

\[
\bar{l}^S := \int_0^1 l(x)\bar{\psi}_S(x) \, dx,
\]

and of the mass and stiffness matrices are

\[
(M_{CG})_{I,J} := \int_0^1 \bar{\psi}_I(x)\bar{\psi}_J(x) \, dx,
\]

\[
(S_{CG})_{I,J} := \int_0^1 \frac{\partial}{\partial x} \bar{\psi}_I(x) \frac{\partial}{\partial x} \bar{\psi}_J(x) \, dx.
\]

One has to solve the linear system

\[
A_{CG}(\gamma)\bar{\omega} = \bar{l} \tag{4.35}
\]

where \( A_{CG}(\gamma) := M_{CG} + \gamma S_{CG} \). This problem is normally easier to solve than the DG formulation, since it has less degrees of freedom and furthermore, one can use efficient preconditioning techniques like Multigrid methods for (4.35). We will use the conforming linear system (4.35) to solve the DG linear system (4.32).
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Prolongation Operator

We need a prolongation operator \( \Pi : \bar{V}_h \rightarrow V_h \) which connects the auxiliary space \( \bar{V}_h \) to our DG space \( V_h \). Since the auxiliary space is included in the DG space \( (\bar{V}_h \subset V_h) \), we choose this operator to be the natural inclusion operator. Using the basis of both spaces, we see that we can uniquely represent the basis of the CG space through the basis of the DG space as

\[
\tilde{\psi}_I(x) = \sum_{J=1}^{(k+1)N} p_{I,J} \psi_J(x).
\]

We define the matrix

\[
(P)_{I,J} := (\Pi(\tilde{\psi}_I))_J = p_{I,J}
\]

to map one space to the other. In this case the “restriction” operator will be given by the adjoint of the prolongation \( R = P^\top \).

Smoothing Operator

To build the preconditioner, we still need some kind of smoothing operator that takes cares of the most oscillatory parts of the residuum vector. Our choice is to use forward and backward Gauss-Seidel methods, where the solution of \( Ax = b \) is computed by splitting \( A \) as \( A = L + D + U \) where \( L \) is lower triangular part, \( U \) the upper triangular part and \( D \) the diagonal part. The Algorithm 1. is forward Gauss-Seidel in the case

\[
\textbf{Algorithm 1} \quad x=\text{Smother}(x_0, b, B, C, K)
\]

\[
x \leftarrow x_0
\]

\[
\text{for } i = 1 \rightarrow K \text{ do}
\]

\[
\tilde{x} \leftarrow B\tilde{x} = Cx + b
\]

\[
x \leftarrow \tilde{x}
\]

\[
\text{end for}
\]

\[
\text{return } x
\]

that the matrices are \( B = L + D \) and \( C = -U \), and backward Gauss-Seidel when they are \( B = U + D \) and \( C = -L \).

Preconditioner

We couple together the different ingredients we have described until now, to obtain a solution of (4.32). Starting from an initial guess \( \omega_0 \) we use an iterative process: First, we use a few forward Gauss-Seidel steps, then we compute the residuum and restrict it
to the conforming space. We solve the related linear problem in the conforming space
and in the last step we prolong the solution in the DG space and apply a few steps of
backwards Gauss-Seidel. This procedure is summarised in Algorithm 4.2.4.

What remains to be specified in this case is how to solve the problem in the space \( \bar{V}_h \).

Algorithm 2 \( \omega = \text{Solve}(\omega_0, b, A_{DG}, P, A_{CG}, K, Q) \)

\[
x \leftarrow \omega_0 \\
L, U, D \leftarrow \text{decompose}(A_{DG}) \\
\text{for } i = 1 \rightarrow Q \text{ do} \\
\quad x \leftarrow \text{Smoother}(x, b, L + D, -U, K) \\
\quad r \leftarrow b - A_{DG}x \\
\quad \bar{\rho} \leftarrow P^T r \\
\quad \bar{\kappa} \leftarrow A_{CG}\bar{\kappa} = \bar{\rho} \\
\quad x \leftarrow x + P\bar{\kappa} \\
\quad x \leftarrow \text{Smoother}(x, b, U + D, -L, K) \\
\text{end for} \\
\omega \leftarrow x \\
\text{return } \omega
\]

For the one dimensional problem (4.35) we can use multigrid methods [21] or also direct
solvers that are competitive for the one dimensional problem.

4.2.5. Higher Order Time Integration

In the previous part, we have described how we can build a good preconditioner for the
first order explicit-implicit discretisation of our semi-discrete problem (4.29), but the
question of higher order time discretisation remains open. We will use extrapolation
methods to obtain higher order temporal accuracy. This technique is based on the
existence of an asymptotic expansion of the error. We write \( \bar{b}_{h}^{n+1} \) for the exact solution
of (4.29) after time \( \Delta t = \bar{N}\tau \) with initial condition \( b_h(0) = b^n \) and we write \( \bar{b}_{h}^{n+1} \) for the
solution to the same initial data computed with the numerical scheme (4.31) and time
step \( \tau \). Henrici showed in 1962 [15] that an expansion of the form

\[
b_{h}^{n+1} = b_h^n + e_0 \tau + e_1 \tau^2 + \cdots + e_{k-2} \tau^{k-1} + E(\Delta t, \tau)\tau^k
\]

(4.36)

with \( \|E(\Delta t, \tau)\| = O(\tau) \) exists. We call \( \bar{\chi}(\tau) \) the pure polynomial part of the expansion
(4.36):

\[
\bar{\chi}(\tau) := \bar{b}_{h}^{n+1} + e_0 \tau + e_1 \tau^2 + \cdots + e_{k-2} \tau^{k-1}
\]

The main idea is to use \( k \) different time steps \( \tau_1 > \tau_2 > \cdots > \tau_k > 0 \) to compute
different values \( b_{h_i}^{n+1} \) and then use them to build up the unique polynomial reconstruction
4. One dimensional Problem

\( \chi \) of degree \( k - 1 \) such that

\[
\chi(\tau_i) = b_{\tilde{N}_i} \quad \text{for } i = 1, \cdots, k.
\]

The difference between the pure polynomial part and the interpolated polynomial at the nodes is

\[
\bar{\chi}(\tau_i) - \chi(\tau_i) = E(\Delta t, \tau_i)\tau_i^k.
\]

One can show (see for example [9]) that the difference between the two polynomials \( \chi \) and \( \bar{\chi} \) at \( \tau = 0 \) is bounded by

\[
\|\chi(0) - \bar{\chi}(0)\| \leq C \max_{1 \leq i \leq k} \|E(\Delta t, \tau_i)\| \tau_i^k,
\]

where the constant \( C \) is the Lebesgue constant and depends only on the choice of \( \tau_i \). Since \( \bar{\chi}(0) \) is the exact solution at time \( \Delta t \) and \( E(\Delta t, \tau) = O(\tau) \) we obtain

\[
\|\chi(0) - b_{n+1}^h\| = O((\Delta t)^{k+1})
\]

when we chose \( \tau_1 = \Delta t \). We have shown that it is possible to obtain a higher order approximation from the interpolation polynomial. We still have to chose the \( \tau_i \)'s. One possibility is to choose them by trying to minimise the constant \( C \) in front of the error (4.37). In this case, we do not take in account the computational time of the full algorithm. The smaller the \( \tau_i \)'s, the larger are the \( \tilde{N}_i \), the number of sub-steps needed to get to \( \Delta t \). Since it is more important to minimise the number of linear systems to solve at each time step, we take the \( \tau_i \) as large as possible, that means \( \tau_i = \frac{\Delta t}{i} \) for \( i = 1, \cdots, k \). An easy computation gives us the total number of linear systems to solve for an increment of \( \Delta t \) with order \( k \): to compute every value \( b_{\tilde{N}_i}^h \) we need \( \tilde{N}_i \) steps, then summing over \( k \) we get a total of

\[
\tilde{N}_{Tot} := \sum_{i=1}^{k} \tilde{N}_i = \sum_{i=1}^{k} i = \frac{k(k + 1)}{2}
\]

implicit steps to solve. The interpolation at point 0 is obtained with the Aitken-Neville scheme

\[
\begin{align*}
T_{1,1} \quad &\downarrow \quad \vdots \quad \downarrow \\
T_{2,1} \quad \to & \quad T_{2,2} \\
\vdots \quad & \quad \ddots \\
T_{k-1,1} \quad \to & \quad \cdots \quad \to \quad T_{k-1,2} \\
T_{k,1} \quad \to & \quad \cdots \quad \to \quad T_{k,k-1} \quad \to \quad T_{k,k}
\end{align*}
\]

(4.38)
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where \( T_{i,1} = b_i^N \) for \( i = 1, \ldots, k \) and

\[
T_{i,l+1} = \frac{i}{l} T_{i,l} + \frac{l - i}{l} T_{i-1,l} \quad \text{for} \quad l = 1, \ldots, k - 1.
\]

The final result \( T_{k,k} \) is the \( \chi(0) \) value we were looking for.

**Stability**

We will now study the stability of this extrapolation technique. We do so by reformulating our problem in the framework of [10]. We can rewrite (4.31) as the implicit-explicit discrete solution of

\[
\dot{y} = W y + Z y + \tilde{t}.
\]

(4.39)

We have introduced a shorter notation in this stability sub section, where \( y := b_h, \)

\( T := I - \beta (W \tilde{W} - a Q), \)

\( W := \eta (W \tilde{W} - a Q), \)

\( Z := Z - \beta W Z \tilde{W} \) and \( \tilde{t} := \beta W M^{-1} g. \)

Taking the notation of [10], we write

\[
\dot{y} = f(t, y) = A y + g(t, y),
\]

(4.40)

setting \( A = T^{-1} W. \) The numerical methods analysed in the article [10] are written as

\[
(I - \gamma_i \Delta t A) u_i = \Delta t f(t^n + c_i \Delta t, u_i) + \Delta t A \sum_{j=1}^{i-1} \gamma_{i,j} k_j \quad i = 1, \ldots, m,
\]

\[
y^{n+1} = y^n + \sum_{i=1}^{m} b_i k_i.
\]

(4.41)

So applying our implicit-explicit method, we obtain

\[
\frac{y^{n+1} - y^n}{\Delta t} = A y^{n+1} + g(t^n, y^n),
\]

\[
\Rightarrow (I - \Delta t A) y^{n+1} = y^n + \Delta t g(t^n, y^n).
\]

Writing \( g(t, y) = f(t, y) - A y \) we get

\[
(I - \Delta t A) y^{n+1} = y^n + \Delta t f(t^n, y^n) - A y^n = (I - \Delta t A) y^n + \Delta t f(t^n, y^n)
\]

\[
\Rightarrow y^{n+1} = y^n + (I - \Delta t A)^{-1} \Delta t f(t^n, y^n).
\]

We note that this equation is exactly (4.31) reformulated in the form (4.41) with \( \gamma_1 = b_1 = 1, \gamma_{1,1} = a_{1,1} = 0 \) for the first order scheme. For higher order methods we interpolate between values \( v^q, \) which are solutions at time \( \Delta t \) computed with time steps \( \Delta t/q. \) We
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set \( v_0^q = y^n \) and then we compute the \( v^q = v_q^q \) with \( q \) sub-steps

\[
(1 - \frac{\Delta t}{q} A) \hat{k}_i = \Delta t f(t^n + (i - 1) \frac{\Delta t}{q}, v_{i-1}^q),
\]

\[
v_i^q = y^n + \frac{1}{q} \sum_{j=1}^{i} \hat{k}_i, \tag{4.42a}
\]

with \( i = 1, \cdots, q \). The \( \gamma \)'s and \( a \)'s in (4.41) can be obtained in a general form. Although, it is not possible to provide a general form of the \( b \)'s, we can compute them explicitly after fixing the order of the extrapolation method. As example, we can obtain the coefficients for the second order method. In this case we have \( m=3, \gamma_{i,j}=0 \),

\[
(c_i) = \begin{pmatrix} 0 \\ 0 \\ 1/2 \end{pmatrix}, \quad (\gamma_i) = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \quad (a_{i,j}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}
\]

and from the extrapolation scheme we get

\[
(b_i) = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}.
\]

From [10], we see that the stability of this time stepping method depends on the stability function \( R(z) \). This function is obtained by setting \( f(t,y) = Ay \) in (4.41) and representing the final step as

\[
y^{n+1} = R(\Delta t A)y^n.
\]

To compute this stability, the internal stability functions \( R^q \)

\[
v^q = R^q(\Delta t A)y^n
\]

are essential.

**Lemma 4.2.5.** The step stability functions \( v_i^q = R_i^q(\Delta t A)y^n \) of the extrapolation method (4.42) are given by

\[
R_i^q(z) = \frac{1}{1 - (\frac{1}{q})i}.
\]

**Proof.** We first prove the case for \( i = 1 \), then we show the general form by induction. For \( i = 1 \) substituting \((1 - z/q)\hat{k}_1 = zy^n \) in \( v_1^q = y^n + \hat{k}_1/q \) we obtain

\[
v_1^q = y^n + \frac{1}{q} \frac{qz}{q - z} y^n = \frac{q}{q - z} y^n \Rightarrow R_1^q(z) = \frac{1}{1 - z/q}.
\]
4.2. Numerical Scheme

To prove the equation for $i$ we first substitute $(1 - z/n)\tilde{k}_{i-1} = zv_{i-1}^q$ in $v_i^q = u_{i-1} + \tilde{k}_{i-1}/q$ obtaining

$$v_i^q = (1 + \frac{1}{q} \cdot \frac{qz}{q - z}) u_{i-1}.$$ 

Using the induction step $u_{i-1} = \frac{1}{1 - (z/q)} y^n$ we obtain the result for $i$. □

Using the lemma, it is easy to see that the internal stability functions are

$$R^q(z) = \frac{1}{(1 - \left(\frac{z}{q}\right))^q}.$$ 

(4.43)

We can use this result to compute the stability function $R(z)$ for any arbitrary order $k$ by using the extrapolation tableau (4.38) with $T_{i,l} = R^l(z)$ for $l = 1, \ldots, k$. Using this technique for our example, the second order method, we obtain

$$R(z) = \frac{z^2 + 4z - 4}{(z - 2)^2(z - 1)}.$$ 

(4.44)

Assuming that the matrices $A$ and $M^{-1}Z$ have the properties

$$\langle x, Ax \rangle \leq \mu \|x\|^2 \quad \forall x \in \mathbb{R}^{kN},$$ 

(4.45a)

$$\|M^{-1}Z\| \leq L.$$ 

(4.45b)

we can give a stability condition for the method. Summarising two result of [10], we have

**Lemma 4.2.6.** Let $R$ be an $A$-stable stability function of the extrapolation method, and let $|R(iy)| < 1$ for $y \neq 0$ and $|R(\infty)| < 1$, then there exist constants $C < 0$ and $1 \leq \omega < \infty$ such that numerical solutions $y$ and $z$ of differential equation (4.39) which satisfies (4.45) with $\mu + \omega L \leq 0$, satisfy

$$\|y^{n+1} - z^{n+1}\| \leq \|y^n - z^n\|$$

for all time steps $0 < \Delta t < C/L$.

**Proof.** This proof is the result combination of Corollary 8. and Theorem 13. of [10] □

Now we take $R(z)$ to be the function (4.44) computed for the second order extrapolation method. From figure 4.1 it is clear that region $|R(z)| > 1$ lies to the right of the $x$-axis. It remains to check if it crosses or touches the axis. We compute

$$|R(iy)|^2 = \frac{16 + 24y^2 + y^4}{(1 + y^2)(4 + y^2)^2}$$
4. One dimensional Problem

![Figure 4.1: Stability region for $R(z)$ given in (4.44). The line represents the boundary $|R(z)| = 1$](image)

and notice that the polynomial $y^4 + 24y + 16$ possesses only pure imaginary roots. Implying that $R$ is A-stable and $|R(iy)| < 1$. Since the degree of the polynomial in the denominator of $R(z)$ is larger than the one in the nominator, we have $|R(\infty)| = 0 < 1$. That means that all the conditions of the lemma are satisfied. The second order extrapolation method is stable if $\Delta t$ is enough small and $-\mu$ is not too close to $L$.

The last condition, $|R(\infty)| < 1$ is proven to be of crucial importance to obtain contractivity. It has been shown that if it is not satisfied, then the method will not be contractive for any $\omega$. This condition is satisfied by all the extrapolation functions, since the stability function $R(z)$ is a linear combination of the internal stability functions $R^q(z)$ and all of them satisfy $|R^q(\infty)| = 0$ for $q = 1, \ldots$.

This theory give us some insight on the stability of the numerical method, but does not provide us quantitative results for $\omega$ and $C$ (explicit calculation is only possible for $k = 1$). In our numerical test we will use an heuristic approach to set the CFL condition.

**Remark 4.2.7.** In the following section we are showing numerical results for the one dimensional magnetic induction equation with Hall effect (4.2). It is important to show that this system is energy stable.

**Corollary 4.2.8.** Let $u \in W^{1,\infty}([0, 1])$ and $B_1, B_2$ be solutions of (4.2) and satisfy boundary conditions

$$B_i(0, t) = B_i(1, t) = 0,$$

$$\sqrt{\alpha} \frac{\partial B_i}{\partial x}(0, t) = g_l^i(t), \quad \text{for} \quad t \in \Gamma_l,$$

$$\sqrt{\alpha} \frac{\partial B_i}{\partial x}(1, t) = g_r^i(t), \quad \text{for} \quad t \in \Gamma_r,$$
4.3. Numerical Experiments

for \( i = 1, 2 \), then the following apriori estimate holds

\[
\frac{\partial}{\partial t} \sum_{i=1}^{2} \left( \| B_i \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_i \|_{L^2([0,1])}^2 \right) \leq C_0 \sum_{i=1}^{2} \left( \| B_i \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_i \|_{L^2([0,1])}^2 \right) + D_0
\]

where the constants \( C_0 \) and \( D_0 \) depend only on the velocity \( u \), its derivative, and boundary functions \( g_k^i \) with \( k = l, r \) and \( i = 1, 2 \).

Proof. Multiplying (4.2a) by \( B_1 \) and (4.2b) by \( B_2 \) integrating over \([0,1]\) and proceeding as in the proof of theorem 4.2.2 we get

\[
\frac{d}{dt} \left( \| B_1 \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_1 \|_{L^2([0,1])}^2 \right) \leq C_0 \left( \| B_1 \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_1 \|_{L^2([0,1])}^2 \right) + D_1 - \beta \int_0^1 B_1 \frac{\partial^2 B_2}{\partial x^2} \, dx
\]

and

\[
\frac{d}{dt} \left( \| B_2 \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_2 \|_{L^2([0,1])}^2 \right) \leq C_0 \left( \| B_2 \|_{L^2([0,1])}^2 + \alpha \| \frac{\partial}{\partial x} B_2 \|_{L^2([0,1])}^2 \right) + D_1 + \beta \int_0^1 B_2 \frac{\partial^2 B_1}{\partial x^2} \, dx
\]

Using integration by parts and summing the two equations concludes the proof.

The DG method (4.12), time stepping (4.31) and preconditioner outlined in section 4.2.4 can be easily extended to (4.2) componentwise and by an explicit time discretisation of the Hall term.

4.3. Numerical Experiments

We will present a series of tests to first validate the method we have developed, and then to show some examples to illustrate the dynamics of the one dimensional magnetic induction equations with Hall effect (4.12).

We have chosen to implement a method of order two (k=2). As basis functions for the DG space \( V_h \), we have used normed Legendre polynomials up to order 1. We made this choice to exploit the orthogonality property of this polynomial, resulting in the mass matrix being proportional to the identity matrix. For the CG space \( \bar{V}_h \), we take standard hat functions, which are piece-wise linear functions on the intervals \( I_i \).

We consider the following special cases.
4. One dimensional Problem

1. Advection

Although, setting $\alpha = \eta = 0$ for the problem (4.3) results in the conservative advection equation:

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} = 0 \quad \Rightarrow \quad \frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} = \frac{\partial u}{\partial x} b.$$  

For the sake of simplicity and to highlight the role of advection, we discard the left-hand and present results with the pure advection:

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} = 0.$$  

The solution of this equation is given by $b(x, t) = b(x(0)) = b_0(x_0)$, with characteristic $x(t)$. We consider two cases for which the solution: $x_0 = x - ct$ if $u(x) = c$ and $x_0 = x e^{-ct}$ for $u(x) = cx$. As initial data we take the function

$$b_0(x) = \begin{cases} 
(1 - (4x^2 - 1))^4 & 0 \leq x \leq \frac{1}{2} \\
0 & \frac{1}{2} < x \leq \frac{1}{2} 
\end{cases}$$

For these pure advective problems, the natural boundary conditions is to specify the value of the function $b$ at the inflow boundary. For these examples, we chose to set it equal to zero.

2. Heat Equation

If set $u = 0$ and $\alpha = 0$, we get a pure diffusive problem

$$\frac{\partial b}{\partial t} = \eta \frac{\partial^2 b}{\partial x^2},$$

$$b(x, 0) = b(1, 0) = 0,$$

This is the one dimensional heat equation with thermal diffusivity $\eta$. Using eigenfunction of the Laplace operator we can take an initial function

$$b_0(x) = \sin(\pi x) + \frac{1}{2} \sin(2\pi x)$$

for which the exact solution takes the form:

$$b(x, t) = e^{-\pi^2 \eta t} \sin(\pi x) + \frac{1}{2} e^{-4\pi^2 \eta t} \sin(2\pi x).$$

3. Advection Diffusion

For the special case of $\alpha = 0$ and of a constant velocity $u(x) = c$, we can compute
4.3. Numerical Experiments

a solution for

\[
\frac{\partial b}{\partial t} + c \frac{\partial b}{\partial x} = \eta \frac{\partial^2 b}{\partial x^2},
\]

\[b(0, t) = b(1, t) = 0\]

with initial data:

\[b_0(x) = e^{\frac{c}{2\eta}}(\sin(\pi x) + \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(3\pi x)).\]

In that case the solution is

\[b(x, t) = e^{\frac{c}{2\eta}}(e^{-\lambda_1 t}\sin(\pi x) + \frac{e^{-\lambda_2 t}}{2}\sin(2\pi x) + \frac{e^{-\lambda_3 t}}{4}\sin(3\pi x)) \quad (4.46)\]

with \(\lambda_k = \frac{c^2 + 4\pi^2 k^2 \eta^2}{4\eta}\).

![Figure 4.2.: Convergence plot for problems 1. to 4. at time \(T = 1\). The reference triangle has a slope of 2, i.e. convergence order of 2.](image)

Before we analysing the results of the numerical tests, we set the CFL condition. As we have shown before, we known that for small enough time steps, the method extrapolation is stable. However, we do not have an explicit quantitative value of the time step. Our assumptions are based on the analysis of the sub problems contained in the full equation. We note that an important ingredient in these equations is advection which is solved in an explicit manner. So we can compute the CFL number for constant velocity \(u\) with the first order scheme and use it also for the other schemes. Taking (4.31) with \(\eta = \alpha = 0\) we obtain

\[b^{n+1} = Zb^n.\]
4. One dimensional Problem

A stable choice for the time stepping will be \( \Delta t = 0.95 \Delta x / \| Z \| \). The norm of \( Z \) is computed with power iteration. The different results of the problem presented before are given in Fig. 4.2. We note that for all the four different examples considered here the expected order of convergence of two is obtained.

In the absence of explicit solutions of (4.3) we modify slightly the equation (4.3) adding a source term \( f(x, t) \)

\[
\frac{\partial b}{\partial t} + \frac{\partial u b}{\partial x} = \eta \frac{\partial b}{\partial \mathbf{x}} + \alpha \left( \frac{\partial b}{\partial \mathbf{x} t} + \frac{\partial}{\partial x} \left( u \frac{\partial b}{\partial \mathbf{x}} \right) \right) + f \tag{4.47}
\]

We can chose this additional term in such a way that a chosen function \( b(x, t) \) solves (4.47). Since \( \alpha \) should be a small correction term in the dynamics of the equation, we will use the solutions of the of the advection diffusion equation. Setting \( b(x, t) = e^{\frac{c^2}{\eta} \pi^2 - \lambda_k t} \sin(k \pi x) \) in (4.47) we get

\[
f_k(x, t, \alpha) = \frac{\alpha}{16 \eta^3} e^{\frac{c^2}{\eta} \pi^2 - \lambda_k t} \left( l_k \sin(k \pi x) - d_k \left( \frac{d_k}{4} \sin(k \pi x) + l_k \cos(k \pi x) \right) \right)
\]

where \( l_k := 2 kc \) and \( d_k := c^2 - (l_k/c)^2 \). Due to the linearity of the equation we see that (4.47) with \( f(x, t, \beta) = f_1(x, t, \beta) + f_2(x, t, \beta)/2 + f_3(x, t, \beta)/4 \) will have (4.46) as solution. The convergence results are presented in figures 4.2 and 4.3 and show the expected convergence rate.

To illustrate the solutions, we take a bell shaped smooth function as initial data

\[
b_0(x) = e^{-100(x-0.3)^2} \tag{4.48}
\]
4.3. Numerical Experiments

and solve (4.3) on a uniform mesh of 1025 cells with a fixed velocity $u = 3/4$. We show in figure 4.4 the solution at time $T = 0.5$ for different choice of parameters. The first one is pure advection, as $\eta = \alpha = 0$. In this case, the wave is transported to the left without changes in shape. The second choice is to switch on the diffusion term setting $\eta = 0.01$ and keeping $\alpha = 0$. In this case the information is transported to the left but it is also spread out on the domain due to the effect of the diffusive term. The last case involves using both diffusion and electron inertia $\eta = 0.01$ and $\alpha = 0.005$. In this case, the wave is translated to the left but the electron inertia is acting against the diffusion and we obtain a function that more localised than in the case of advection diffusion.

To observe the effect of $\alpha$, we are going to solve the same problem as before with the inertial parameter which is five times more than the diffusion, i.e., $\eta = 0.01$ and $\alpha = 0.05$. In figure 4.4, we see that the case the effect of diffusion is drastically reduced and we observe almost only advection.

To show the performance of the preconditioner presented before in section 4.2.4, we solve (4.3) at time $T = 0.5$ for parameters $\eta = 0.01$, $\alpha = 0.05$ and with inital data (4.48). In Figure 4.6, we show the computational time for different number of mesh points, with
Figure 4.5.: Solutions of (4.3) with initial data (4.48) velocity $u = 3/4$ at time $T = 0.5$, as pure advection $\eta = \alpha = 0$, advection diffusion $\eta = 0.01$ and $\alpha = 0.005$ and also advection diffusion with electron inertia $\eta = 0.01$ and $\alpha = 0.1$. 
4.3. Numerical Experiments

Figure 4.6.: Computational time comparison for solutions of (4.3) with initial data 4.48 velocity $u = 3/4$ at time $T = 0.5$ with parameters $\eta = 0.01, \alpha = 0.005$ and $\beta = 0$ with and without preconditioning.

and without the preconditioner. We observe a gain of more than one order of magnitude in efficiency.

As a last test we solve the system of one dimensional equations (4.2). In this case we choose the same initial function $B_0(x) = e^{-100(x-0.3)^2}$ for $B_1$ and $B_2$. We set the velocity to be $u = 3/4$ and compare the pure advection ($\eta = \alpha = \beta = 0$), with uncoupled inertial advection diffusion system ($\eta = \alpha = 0.01$ and $\beta = 0$) and also with coupled inertial advection diffusion system ($\eta = \alpha = \beta = 0.01$). The results are shown in figure 4.7, where we observe that effect of coupling is to increase the diffusion in one component, in this case $B_2$, and reduce it in the other, $B_1$. The result is that the function $B_1$ is advected almost without being diffused. These results in one dimension show us that electron inertia and Hall effect allow the advection of one component almost without any diffusive losses. That means that in a diffusive regime, where the frozen in condition for the magnetic field is no more valid and a current sheet can be formed, we obtain a almost pure advection at least for one component. This absence of diffusive losses could be one of the leading factors in obtaining a fast reconnection rate.
4. One dimensional Problem

Figure 4.7.: Solutions of (4.47) with initial data 4.48 velocity $u = 3/4$ at time $T = 0.5$, as pure advection $\eta = \alpha = \beta = 0$, uncoupled inertial advection diffusion $\eta = \alpha = 0.01$ and $\beta = 0$ and coupled inertial advection also coupled inertial advection advection diffusion with $\eta = \alpha = \beta = 0.01$. 

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5. Discontinuous Galerkin Methods

In the previous chapter we have shown an energy stable DG method to solve the one dimensional magnetic induction equation with Hall effect. There we also showed that we were able to obtain an efficient preconditioner to solve the resulting linear algebraic system using auxiliary space techniques.

In this chapter, we are going to describe the DG procedure to approximate the multi-dimensional magnetic induction equation with Hall effect (2.6). To do so, we introduce auxiliary variables $J$, $E_1$ and $E_2$ and rewrite (2.6) as a first-order system

$$\frac{\partial B}{\partial t} - (B \cdot \nabla) u + B(\nabla \cdot u) + (u \cdot \nabla) B = - \eta \nabla \times J - \alpha \nabla \times E_1 - \beta \nabla \times E_2,$$

(5.1a)

$$J = \nabla \times B,$$

(5.1b)

$$\rho E_1 = \frac{\partial J}{\partial t} + (u \cdot \nabla) J,$$

(5.1c)

$$\rho E_2 = J \times B,$$

(5.1d)

$$B = f \text{ on } \Gamma_+,$$

(5.1e)

$$\sqrt{\alpha} J = g \text{ on } \Gamma_+,$$

(5.1f)

$$\eta (n \times B) = 0 \text{ on } \partial \Omega.$$  

(5.1g)

The DG formulation is based on a variational formulation corresponding to (5.1) namely, find $(B, J, E_1, E_2) \in H(\text{curl}, \Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ such that

$$\int_{\Omega} \left[ \left( \frac{\partial B}{\partial t} - C B \right) \nabla W - B(\nabla \cdot W) W + (\eta J + \alpha E_1 + \beta E_2)(\nabla \times W) \right] d^3x =$$

$$- \int_{\Gamma_+} (n \cdot u)f \nabla W ds \quad \forall W \in C^\infty(\Omega),$$

(5.2a)

$$\int_{\Omega} [J S - B(\nabla \times S)] d^3x = 0 \quad \forall S \in C^\infty(\Omega),$$

(5.2b)

$$\int_{\Omega} \left[ (\rho E_1 + (\nabla u) J - \frac{\partial J}{\partial t}) Z + J(\nabla \cdot W) Z \right] d^3x = \int_{\Gamma_+} (n \cdot u) \frac{g}{\sqrt{\alpha}} Z ds \quad \forall Z \in C^\infty(\Omega),$$

(5.2c)

$$\int_{\Omega} (\rho E_2 - J \times B) K d^3x = 0 \quad \forall K \in C^\infty(\Omega).$$

(5.2d)
5. Discontinuous Galerkin Methods

Here the boundary conditions are already included in the variational formulation and in the choice of the space $V$. In this case,

$$C = \begin{pmatrix}
\partial_x u^1 & \partial_y u^1 & \partial_z u^1 \\
\partial_x u^2 & \partial_y u^2 & \partial_z u^2 \\
\partial_x u^3 & \partial_y u^3 & \partial_z u^3
\end{pmatrix}.$$

5.1. Numerical Schemes

We are going to approximate (5.2) on the finite element space $V_h = (V_h)^3$ where

$$V_h = \{ f \in L^2(\Omega) | f|_K \in S^k(K), \forall K \in T_h \}.$$

In this case $T_h$ is a shape regular triangulation of $\Omega$ into tetrahedra and/or parallelopipeds, with possible hanging nodes. The space $S^k(K)$ is the space $P^k(K)$ of polynomial of degree at most $k$ in $K$ if $K$ is a tetrahedron and the space $Q^k(K)$ of polynomial of degree at most $k$ in each variable in $K$ if $K$ is a parallelopiped.

The DG scheme for the magnetic induction equation with Hall effect is:

Find $(B_h, J_h, E_{h,1}, E_{h,2}) \in (V_h)^4$ such that

$$(\frac{\partial B_h}{\partial t} - CB_h, W_h)_{T_h} - (B_h, (u \cdot \nabla)W_h)_{T_h} + ((\eta J_h + \alpha E_{1,h} + \beta E_{2,h}), \nabla \times W_h)_{T_h}$$

$$+ \sum_{i=1}^3 \langle \hat{u} B_i, [W_i] N \rangle_{E \cap \Gamma^+} - \langle \eta \hat{J}_h + \alpha \hat{E}_{1,h} + \beta \hat{E}_{2,h}, [W_h] T \rangle_{\mathcal{E}}$$

$$= -\langle (n \cdot u)f, W_h \rangle_{\mathcal{E}^0 \cap \Gamma^+}$$

$$(J_h, S_h)_{T_h} - (B_h, \nabla \times S_h)_{T_h} + (\hat{B}_h, [S_h] T)_{\mathcal{E}} = 0$$

$$\forall W_h \in (V_h)^3, \forall S_h \in (V_h)^3, \quad (5.3a)$$

$$(\rho E_{1,h} + (\nabla u) J_h - \frac{\partial J_h}{\partial t}), Z_h)_{T_h} + (J_h, (u \cdot \nabla) Z_h)_{T_h} - \sum_{i=1}^3 \langle \hat{u} J_i, [Z_i] N \rangle_{E \cap \Gamma^+}$$

$$= \langle (n u) g, Z_h \rangle_{\mathcal{E}^0 \cap \Gamma^+}$$

$$\forall Z_h \in (V_h)^3, \forall K_h \in (V_h)^3. \quad (5.3c)$$

Here to simplify the notation we have introduced some inner products

$$(f, g)_{T_h} := \sum_{K \in T_h} \int_K f \cdot g \, d^3x, \quad (5.4a)$$

$$\langle f, g \rangle_{\mathcal{E}} := \sum_{e \in \mathcal{E}} \int_e f \cdot g \, ds. \quad (5.4b)$$
where $S$ is a set of the faces. These inner products are analogously defined for scalar functions. The set of the faces of the triangulation is subdivided in two main groups. We denote $K^+$ and $K^-$ as two elements of triangulation $\mathcal{T}_h$, then the internal faces $\mathcal{E}^I$ are the set of the non empty interior of $\partial K^+ \cap \partial K^-$. The boundary faces $\mathcal{E}^\partial$ are the set of the non empty interior of $\partial K \cap \partial \Omega$ where $K$ is an element of $\mathcal{T}_h$. The set of all the faces will be denoted as $\mathcal{E} = \mathcal{E}^I \cup \mathcal{E}^\partial$.

We have also used jump, and average operators. Evaluating $v \in (V_h)^3$ on an internal face $e \in \mathcal{E}$, we denote $K^+$ and $K^-$ the elements sharing $e$, $n^\pm$ the normal unit vector pointing outwards to $K^\pm$, and set $v^\pm = v|_{K^\pm}$. We have defined averages and tangential jumps of $v$ at $x \in e$ as

$$[\langle f \rangle]_T := \begin{cases} \frac{1}{2}(f^+ + f^-) & \text{if } e \in \mathcal{E}^I, \\ f & \text{if } e \in \mathcal{E}^\partial, \end{cases} \quad (5.5a)$$

$$[f]_T := \begin{cases} n^+ \times f^+ + n^- \times f^- & \text{if } e \in \mathcal{E}^I, \\ n \times f & \text{if } e \in \mathcal{E}^\partial. \end{cases} \quad (5.5b)$$

For a scalar function $s \in V_h$ we have defined the average and the normal jump at $x \in e$ as

$$\left\langle \left\{ \frac{1}{2}(s^+ + s^-) \right\} \right\rangle := \begin{cases} \frac{1}{2}(s^+ + s^-) & \text{if } e \in \mathcal{E}^I, \\ s & \text{if } e \in \mathcal{E}^\partial, \end{cases} \quad (5.6a)$$

$$\left\{ \left\{ n^+ s^+ + n^- s^- \right\} \right\} := \begin{cases} n^+ s^+ + n^- s^- & \text{if } e \in \mathcal{E}^I, \\ n s & \text{if } e \in \mathcal{E}^\partial. \end{cases} \quad (5.6b)$$

Besides, we also use numerical fluxes

$$\hat{u} \mathbf{B}_e|_e = \hat{u} \mathbf{B}_h(u^+, u^-, (B^+_h)^+, (B^-_h)^-),$$
$$\hat{u} \mathbf{J}_e|_e = \hat{u} \mathbf{J}_h(u^+, u^-, (J^+_h)^+, (J^-_h)^-),$$
$$\mathbf{B}_h|_e = \mathbf{B}_h(B^+_h, B^-_h, J^+_h, J^-_h),$$
$$\mathbf{J}_h|_e = \mathbf{J}_h(B^+_h, B^-_h, J^+_h, J^-_h),$$
$$\hat{E}_{1,h}|_e = \hat{E}_{1,h}(B^+_h, B^-_h, E^+_h, E^-_h),$$
$$\hat{E}_{2,h}|_e = \hat{E}_{2,h}(B^+_h, B^-_h, E^+_h, E^-_h),$$

for $i = 1, 2, 3$ and $e$ a face element. The exact form of the numerical fluxes is specified in the following.

Before stating the energy estimate we introduce the well known rules to apply integration by parts on a discrete piecewise continuous space.

**Lemma 5.1.1.** Let $f, g \in (V_h)^3$, $s, q \in V_h$ and $u \in (C^0(\Omega))^3$, then the following rules hold

$$(f, \nabla \times g)|_{T_h} = (g, \nabla \times f)|_{T_h} + \langle \langle f \rangle \rangle, [g]_T \epsilon - \langle \langle g \rangle \rangle, [f]_T \epsilon^T, \quad (5.7a)$$
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\[(s, \mathbf{u} \cdot \nabla q)_{\Gamma_h} = -(s, \mathbf{u} \cdot \nabla q)_{\Gamma_h} - (s(\nabla \cdot \mathbf{u}), q)_{\Gamma_h} + \langle \{\{\mathbf{u}s\}\}, [q]_N \rangle_{\mathcal{E}_{\gamma}} + \langle \{\{\mathbf{u}q\}\}, [s]_N \rangle_{\mathcal{E}_{\gamma}} \quad (5.7b)\]

**Proof.** If we apply integration by parts on \((f, \nabla \times g)_{\Gamma_h}\) we get

\[(f, \nabla \times g)_{\Gamma_h} = (g, \nabla \times f)_{\Gamma_h} + \sum_{K \in \mathcal{T}_h} \int_{\partial K} f(n \times g) \, ds \]

\[= (g, \nabla \times f)_{\Gamma_h} + \sum_{e \in \mathcal{E}_{\Gamma}} \int_e \left[ f^+(n^+ \times g^+) + f^-(n^- \times g^-) \right] \, ds \]

\[+ \sum_{e \in \mathcal{E}_{\Gamma}} \int_e f(n \times g) \, ds \]

where we have substituted the sum over the boundary of the cell \(K\) of the triangulation with a sum over the faces. We now use (5.6) to reformulate the sum over inner faces as

\[f^+(n^+ \times g^+) + f^-(n^- \times g^-) = f^+[g]_T - g^-[f]_T = \{\{f\}\}[g]_T - \{\{g\}\}[f]_T,\]

obtaining the desired result. For the second equation we are following the same principle, using integration by parts on \((s, \mathbf{u} \cdot \nabla q)_{\Gamma_h}\) we obtain

\[(s, \mathbf{u} \cdot \nabla q)_{\Gamma_h} = -(s, \mathbf{u} \cdot \nabla q)_{\Gamma_h} - (s(\nabla \cdot \mathbf{u}), q)_{\Gamma_h} + \sum_{K \in \mathcal{T}_h} \int_{\partial K} (n \times g) q \, ds.\]

We rewrite again the sum over the boundaries of \(K\) as sums over faces

\[\sum_{K \in \mathcal{T}_h} \int_{\partial K} (n \times g) q \, ds = \sum_{e \in \mathcal{E}_{\Gamma}} \int_e \left[ (n \times g^+) \cdot (n^+ q^+) + (n \times g^-) \cdot (n^- q^-) \right] \, ds \]

\[+ \sum_{e \in \mathcal{E}_{\Gamma}} \int_e (n \times g) q \, ds = \sum_{e \in \mathcal{E}_{\Gamma}} \int_e \left[ (n \times g^+) \cdot \{g\}_N + (n \times g^-) \cdot \{s\}_N \right] \, ds + \langle \{\{\mathbf{u}g\}\}, [g]_N \rangle_{\mathcal{E}_\gamma} \]

\[= \langle \{\{\mathbf{u}s\}\}, [q]_N \rangle_{\mathcal{E}_\gamma} + \langle \{\{\mathbf{u}q\}\}, [s]_N \rangle_{\mathcal{E}_\gamma}.\]

Here we have used the definitions of jumps and averages (5.6). \(\square\)

**Corollary 5.1.2.** Let \(s \in V_h\) and \(\mathbf{u} \in (C^0(\Omega))^3\), then the following rule hold

\[(s, \mathbf{u} \cdot \nabla s)_{\Gamma_h} = -\frac{1}{2} (s(\nabla \cdot \mathbf{u}), s)_{\Gamma_h} + \langle \{\{\mathbf{u}s\}\}, [s]_N \rangle_{\mathcal{E}_{\gamma}} + \frac{1}{2} \langle \{\{\mathbf{u}s\}\}, [s]_N \rangle_{\mathcal{E}_{\gamma}} \quad (5.8)\]

**Proof.** We get this result by applying (5.7b) directly with \(q = s\). \(\square\)

The approximate solution generated by the DG discretisation (5.3) satisfies the energy estimate.
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Theorem 5.1.3. Let \( u \) and \( \rho \) be in \( W^{1,\infty}(\Omega) \). Furthermore, assume that the time derivative of the density is in \( L^\infty(\Omega) \) and also that there exists \( \rho_0 > 0 \) such that \( \rho_0 < \rho \) for all times. Then the following apriori estimate for the solution of (5.3) holds:

\[
\frac{\partial}{\partial t} \left( \| B_h \|_{L^2(\tau_h)} + \| \rho^{-1/2} J_h \|_{L^2(\tau_h)} \right) \leq C_0 \left( \| B_h \|_{L^2(\tau_h)} + \| \rho^{-1/2} J_h \|_{L^2(\tau_h)} \right) + D_0 + \mathcal{F} \tag{5.9}
\]

with \( C_0 \) and \( D_0 \) being constants that depend on boundary functions \( f \) and \( g \), \( \rho \), \( \alpha \), \( u \) and their derivatives. The flux term is given by

\[
\frac{1}{2} \mathcal{F} = \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon \tau - \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon e \cap \Gamma - \langle \eta (\{ B_h \} - \hat{B}_h), [J_h]_N \rangle \varepsilon \tau - \eta \langle \{ B_h \} - \hat{B}_h, [J_h]_N \rangle \varepsilon e \]

\[
\quad + \alpha \langle \{\{ u J_h \} - \hat{u} J_h \}, [J_h]_N \rangle \varepsilon \tau + \alpha \langle \{\{ u J_h \} - \hat{u} J_h \}, [J_h]_N \rangle \varepsilon e \cap \Gamma - \langle \eta (\{ B_h \} - \hat{B}_h), [J_{1,h}]_T \rangle \varepsilon e \]

\[
\quad + \alpha \langle \{\{ B_h \} - \hat{B}_h, [E_{1,h}]_T \rangle \varepsilon e = \langle \alpha (\{ E_{1,h} \} - \hat{E}_{1,h}), [B_h]_T \rangle \varepsilon e \tag{5.10}
\]

Proof. We set \( W_h = B_h \) in (5.3a) and use (5.8) for each component of \( B_h \) obtaining

\[
\frac{1}{2} \frac{\partial}{\partial t} \| B_h \|_{L^2(\tau_h)} - \left( C - \frac{1}{2} (\nabla u) B_h, B_h \right) _{\tau_h} - \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon \tau - \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon e \cap \Gamma - \]

\[
\quad - \langle \{\{ n \cdot u \} f, B_h \rangle \varepsilon e + \langle \{\{ n \cdot u \} f, B_h \rangle \varepsilon e + \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon e \cap \Gamma + \]

Using Cauchy-Schwartz inequality and the definition of the inflow boundary, we obtain

\[
\quad - \langle \{\{ n \cdot u \} f, B_h \rangle \varepsilon e + \langle \{\{ n \cdot u \} f, B_h \rangle \varepsilon e + \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon e \cap \Gamma +
\]

Here we have rewritten the second term using the definition of jumps and averages on the boundary. Using the last inequality and the fact that the derivatives of the velocity are bounded we can rewrite (5.11) as

\[
\frac{1}{2} \frac{\partial}{\partial t} \| B_h \|_{L^2(\tau_h)} + \langle \{\{ n \cdot u \} f, B_h \rangle \varepsilon e + \sum_{i=1}^{3} \langle \{\{ u B_h^i \} - \hat{u} B_h^i \}, [B_h^i]_N \rangle \varepsilon e \cap \Gamma +
\]

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\[
- \sum_{i=1}^{3} \left( \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \sum_{i=1}^{3} \left( \frac{1}{2} \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \langle \eta \tilde{J}_h + \alpha \tilde{E}_{1,h} + \beta \tilde{E}_{2,h}, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \right) \right) \\
\leq \frac{C_1}{2} \|B_h\|_{L^2(T_h)} + \frac{1}{2} \langle \mathbf{n} \cdot \mathbf{u} | \mathbf{f} \rangle_{\mathcal{E}^\Gamma} + (5.12)
\]

where \( C_1 = \max_{k,i} \left( \| \frac{\partial u}{\partial x_k} \|_{L^\infty(T_h)} \right) \). Now we have to estimate the second term. Let start with the diffusion term. Setting \( S_h = J_h \) in (5.3b) and using (5.7a) we obtain

\[
\frac{1}{2} \|J_h\|_{L^2(T_h)} = (B_h, \nabla \times J_h)_{T_h} - \langle \tilde{B}_h, [J_h]_N \rangle_{\mathcal{E}^\Gamma} \\
= (J_h, \nabla \times B_h)_{T_h} + \left( \langle \{B_h\} - \tilde{B}_h, [J_h]_T \rangle_{\mathcal{E}^\Gamma} - \langle \{J_h\}, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \right)
\]

Solving for \((J_h, \nabla \times B_h)_{T_h}\) and inserting this result in (5.12) we obtain

\[
\frac{1}{2} \frac{\partial}{\partial t} \|B_h\|_{L^2(T_h)} + ((\alpha \mathbf{E}_{1,h} + \beta \mathbf{E}_{2,h}), \nabla \times B_h)_{T_h} \\
- \sum_{i=1}^{3} \left( \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \sum_{i=1}^{3} \left( \frac{1}{2} \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \eta \langle \{J_h\} - \tilde{J}_h, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \right) \right) \\
+ \eta \langle \{J_h\} - \tilde{J}_h, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \leq \frac{C_1}{2} \|B_h\|_{L^2(T_h)} + \frac{1}{2} \langle \mathbf{n} \cdot \mathbf{u} | \mathbf{f} \rangle_{\mathcal{E}^\Gamma} + (5.13)
\]

To handle the \( \mathbf{E}_{i,h} \) term, we are again using (5.3b) and (5.7a) to obtain:

\[
(J, \mathbf{E}_{i,h})_{T_h} = (B_h, \nabla \times \mathbf{E}_{i,h})_{T_h} - \langle \tilde{B}_h, [E_{i,h}]_T \rangle_{\mathcal{E}^\Gamma} \\
= (E_{i,h}, \nabla \times B_h)_{T_h} + \left( \langle \{B_h\} - \tilde{B}_h, [E_{i,h}]_T \rangle_{\mathcal{E}^\Gamma} - \langle \{E_{i,h}\}, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \right)
\]

setting \( \mathbf{S}_h = \mathbf{E}_{i,h} \) with \( i = 1, 2 \). Inserting these results in (5.13) we obtain

\[
\frac{1}{2} \frac{\partial}{\partial t} \|B_h\|_{L^2(T_h)} + ((\alpha \mathbf{E}_{1,h} + \beta \mathbf{E}_{2,h}), \mathbf{J}_h)_{T_h} \\
- \sum_{i=1}^{3} \left( \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \sum_{i=1}^{3} \left( \frac{1}{2} \langle \{uB_h^i\} - \tilde{uB}_h^i, [B_h^i]_N \rangle_{\mathcal{E}^\Gamma} - \eta \langle \{J_h\} - \tilde{J}_h, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \right) \right) \\
- \eta \langle \{J_h\} - \tilde{J}_h, [B_h]_T \rangle_{\mathcal{E}^\Gamma} \leq \frac{C_1}{2} \|B_h\|_{L^2(T_h)} + \frac{1}{2} \langle \mathbf{n} \cdot \mathbf{u} | \mathbf{f} \rangle_{\mathcal{E}^\Gamma} + (5.14)
\]

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Setting $K_h = \frac{1}{\rho} J_h$ in (5.3d) we obtain
\[
(E_{2,h}, J_h)_{\Gamma_h} = (\rho^{-1/2} J_h \times B_h, J_h)_{\Gamma_h} = 0
\]
since the vector product is orthogonal to $J_h$. We can insert this result in (5.14) getting:
\[
\frac{1}{2} \frac{\partial}{\partial t} \|B_h\|_{L^2(\Gamma_h)} + \alpha (E_{1,h}, J_h)_{\Gamma_h}
- \sum_{i=1}^{3} \langle \langle \|J_b^i\| - u_b J_b^i, [B_h^i, N] \rangle_{\Gamma_h} \rangle
- \sum_{i=1}^{3} \langle \langle \|J_b^i\| - u_b J_b^i, [B_h^i, N] \rangle_{\Gamma_h} \rangle
- \eta\langle \langle \|J_b^i\| - B_b, [J_h^i, N] \rangle_{\Gamma_h} \rangle
- \alpha\langle \langle \|J_b^i\| - B_b, [J_h^i, N] \rangle_{\Gamma_h} \rangle
- \beta\langle \langle \|J_b^i\| - B_b, [E_{2,h}, N] \rangle_{\Gamma_h} \rangle
\leq \frac{C_1}{2} \|B_h\|_{L^2(\Gamma_h)} + \frac{1}{2} \langle \langle \text{n} \cdot u, f \rangle_{\Gamma_h} \rangle.
\]  
(5.15)

At this point it remains to estimate the $E_{1,h}$ term. We set $Z_h = \frac{1}{\rho} J_h$ in (5.3c) obtaining
\[
(E_{1,h}, J_h)_{\Gamma_h} + \left( \frac{\nabla u}{\rho} J_h, J_h \right)_{\Gamma_h} + (J_h, (u \cdot \nabla)(\rho^{-1} J_h))_{\Gamma_h} - \left( \frac{\partial J_h}{\partial t}, J_h \right)_{\Gamma_h}
- \sum_{i=1}^{3} \langle \langle \rho^{-1} u J_b^i, [J_b^i, N] \rangle_{\Gamma_h} \rangle
- \frac{1}{\rho} \langle \langle \rho \nabla \cdot u - u \cdot \nabla \rho - \frac{1}{2} \frac{\partial \rho}{\partial t}, J_h \rangle_{\Gamma_h} \rangle
- \sum_{i=1}^{3} \langle \langle \rho^{-1} u J_b^i, [J_b^i, N] \rangle_{\Gamma_h} \rangle
= \langle \langle \rho^{-1} \frac{\text{n} \cdot u}{\sqrt{\alpha}}, J_h \rangle_{\Gamma_h} \rangle.
\]

With (5.8) we get
\[
(E_{1,h}, J_h)_{\Gamma_h} + \left( \frac{\nabla u}{\rho} - \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) \right) J_h, J_h \right)_{\Gamma_h} - \frac{1}{2} \frac{\partial}{\partial t} \|\rho^{-1/2} J_h\|_{L^2(\Gamma_h)}
+ \sum_{i=1}^{3} \langle \langle \rho^{-1} (\|u J_b^i\| - u J_b^i), [J_b^i, N] \rangle_{\Gamma_h} \rangle
+ \sum_{i=1}^{3} \langle \langle \rho^{-1} (\frac{1}{2} \|u J_b^i\| - u J_b^i), [J_b^i, N] \rangle_{\Gamma_h} \rangle
+ \sum_{i=1}^{3} \langle \langle \rho^{-1} \frac{1}{2} \|u J_b^i\| - u J_b^i), [J_b^i, N] \rangle_{\Gamma_h} \rangle
= \langle \langle \rho^{-1} (\text{n} \cdot u) \frac{\text{n}}{\sqrt{\alpha}}, J_h \rangle_{\Gamma_h} \rangle.
\]

Again we use the definition of inflow boundaries and the Cauchy-Schwartz inequality.
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Since the density and velocity are bounded we can write:

\[
(\mathbf{E}_{1,h}, \mathbf{J}_h)_{\mathcal{T}_h} + \frac{1}{2} \frac{\partial}{\partial t} \| \rho^{-1/2} \mathbf{J}_h \|_{L^2(\mathcal{T}_h)} + \sum_{i=1}^{3} \langle \rho^{-1}(\{ u \mathbf{J}_h \} - \mathbf{u} \mathbf{J}_h^i, [\mathbf{J}_h^i]_N) \rangle_{\mathcal{T}_h} + \sum_{i=1}^{3} \langle \rho^{-1}(\{ u \mathbf{J}_h \}^i - \mathbf{u} \mathbf{J}_h^i, [\mathbf{J}_h^i]_N) \rangle_{\mathcal{T}_h} \geq C_2 \| u \mathbf{J}_h \|_{L^2(\mathcal{T}_h)}^2
\]

where \( C_2 = \| \nabla \cdot u \|_{L^\infty(\mathcal{T}_h)} + \rho_0^{-1} \| \nabla \cdot (\rho u) \|_{L^\infty(\mathcal{T}_h)} \). Inserting this result in (5.15) we get the desired result

\[
\frac{1}{2} \frac{\partial}{\partial t} \| \mathbf{B}_h \|_{L^2(\mathcal{T}_h)} + \frac{\alpha}{2} \| \rho^{-1/2} \mathbf{J}_h \|_{L^2(\mathcal{T}_h)}^2
\]

We need to choose numerical fluxes to obtain energy stable DG methods as follows

**Corollary 5.1.4.** For every choice of fluxes \( \mathbf{B}_h, \mathbf{J}_h, \mathbf{E}_{1,h}, \mathbf{E}_{2,h}, \mathbf{u} \mathbf{B}_h, \mathbf{u} \mathbf{J}_h \), with

\[
F \leq 0
\]

where \( F \) given in (5.10). Assume that \( u \in W^{1,\infty}(\Omega) \) and furthermore assume that the density \( \rho \) is in \( W^{1,\infty}(\Omega) \) and also that there exist \( \rho_0 > 0 \) such that \( \rho_0 < \rho \) for all times, the following estimate for the solutions of (5.3) holds

\[
\frac{\partial}{\partial t} \left( \| \mathbf{B}_h \|_{L^2(\mathcal{T}_h)} + \| \mathbf{J}_h \|_{L^2(\mathcal{T}_h)} \right) \leq C_0 \left( \| \mathbf{B}_h \|_{L^2(\mathcal{T}_h)} + \| \mathbf{J}_h \|_{L^2(\mathcal{T}_h)} \right) + D_0
\]

with \( C_0 \) and \( D_0 \) being constants that depend on boundary functions \( \mathbf{f} \) and \( \mathbf{g} \), \( \rho \), \( \alpha \), \( u \) and its derivatives only.

**Proof.** This follows by direct application of theorem 5.1.3. 

\[ \square \]
Different stable choices for the fluxes are possible. As we have done in the one-dimensional case, we can split the problem in two parts. For the advective part we have taken the upwind flux [4], for each component $i = 1, 2, 3$ setting

\[
\widehat{uB}_h^i = \{uB_h^i\} + c_e[B_h^i]_N, \quad (5.18a)
\]
\[
\widehat{uJ}_h^i = \{uJ_h^i\} + c_e[J_h^i]_N. \quad (5.18b)
\]

In this case, the value $c_e$ is set as $c_e = |n \cdot u|/2$. For the diffusive part we have chosen the local discontinuous Galerkin (LDG) fluxes proposed in [25] to solve the curl curl problem. In our case we have

\[
\tilde{B}_h = \{B_h\} + b[B_h]_T \quad (5.19a)
\]
\[
\tilde{J}_h = \begin{cases}
\{J_h\} + b[J_h]_T - a^j[B_h]_T & e \in E^I \\
\{J_h\} - a^j[B_h]_T & e \in E^\partial
\end{cases} \quad (5.19b)
\]
\[
\tilde{E}_{1,h} = \begin{cases}
\{E_{1,h}\} + b[E_{1,h}]_T - a^j[B_h]_T & e \in E^I \\
\{E_{1,h}\} - a^j[B_h]_T & e \in E^\partial
\end{cases} \quad (5.19c)
\]
\[
\tilde{E}_{2,h} = \begin{cases}
\{E_{2,h}\} + b[E_{2,h}]_T - a^2[B_h]_T & e \in E^I \\
\{E_{2,h}\} - a^2[B_h]_T & e \in E^\partial
\end{cases} \quad (5.19d)
\]

Here $a^0, a^j, a^1, a^2$ and $b$ are positive penalty parameters. In this case we can easily show, inserting these fluxes in (5.10) that we have

\[
\frac{1}{2} \mathcal{F} = - \sum_{i=1}^3 \left( c_e \langle [B_h^i]_N, [B_h^j]_N \rangle_{E^I} + \frac{a}{\rho_0} \langle c_e \langle [J_h^i]_N, [J_h^j]_N \rangle_{E^I} \right)
\]
\[
- \langle \eta a^j + \alpha a^1 + b a^2 \rangle \langle [B_h]_T, [B_h]_T \rangle_{E^\partial} \leq 0.
\]

That means that this choice of fluxes give us an energy stable semi-discretisate DG scheme for the problem (2.6).

### 5.1.1. Matrix Formulation

For simplicity, we consider the case of constant density and set $\rho = 1$ for the remainder of this chapter. Furthermore, we consider the magnetic induction equation with Hall effect in two space dimensions (3.30).

As specified before we need a uniform triangulation $\mathcal{T}_h$. We subdivide $\Omega = [0, 1]^2$ in $N^2$ squares $K_{i,j} = [(i-1)h, ih] \times [(j-1)h, jh]$ with $i, j = 1, \cdots, N$ and mesh size $h = 1/N$.

To express $V_h = (V_h)^3$, we fix a basis of $V_h$. Choosing a basis $\hat{\phi}_i(x)$ with $i = 0, \cdots, k$ for the polynomial of degree $k$ on $[0, 1]$, we build a basis of $V_h$ setting

\[
\hat{\psi}_{i,j}^{L} (x, y) = \begin{cases}
\hat{\phi}_i \left( \frac{x-x_j}{h} \right) \hat{\phi}_j \left( \frac{y-y_j}{h} \right), & \text{for } (x, y) \in K_{i,j} \\
0 & \text{for } (x, y) \notin K_{i,j}
\end{cases}
\]
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where \( x_I = (I - 1)h \) and \( y_J = (J - 1)h \). We set a global index \( K \) for this basis defining a bijective mapping from local indices \((I, J, i, j)\) to \( i(I, J, i, j) = i + (J - 1)(k + 1) + (I - 1)(k + 1)^2 + (J - 1)(k + 1)^2 N\). We will then write \( \psi^K(x, y) := \tilde{\psi}^{i,j}_{I,J}(x, y) \) for \((I, J, i, j) = i^{-1}(K)\).

The solution of 5.3 can be expressed through this basis by setting:

\[
\begin{align*}
B^i_h &:= \sum_{K=1}^{N_{tot}} B^K_i(t) \psi^K(x, y), \\
J^i_h &:= \sum_{K=1}^{N_{tot}} J^K_i(t) \psi^K(x, y), \\
E^{i,1}_1,h &:= \sum_{K=1}^{N_{tot}} E^{i,1}_K(t) \psi^K(x, y), \\
E^{i,2}_1,h &:= \sum_{K=1}^{N_{tot}} \hat{E}^{i,2}_K(t) \psi^K(x, y),
\end{align*}
\]

for \( i = 1, 2, 3 \) and where \( N_{tot} = N^2(k+1)^2 \) is the dimension of \( \mathcal{V}_h \). In the following we are denoting with an underbar the vector form of the coefficients, e.g., \( \mathbf{B}^i = (B^i_1, \cdots, B^i_{N_{tot}})^\top \) and \( \mathbf{B} = (\mathbf{B}^1, \mathbf{B}^2, \mathbf{B}^3)^\top \). Some computation shown in detail in Appendix B, reduces the system (5.3) with fluxes (5.19) and (5.18) in to form (B.14):

\[
\begin{align*}
\mathbf{B}_t - (Y_1 - \xi Y_2)\mathbf{B} + Y_3(\eta \mathbb{I} + \alpha \mathbb{E} + \beta \hat{\mathbb{E}}) &= \mathbf{F}, \quad (5.20a) \\
\mathbb{I} &= Y_4 \mathbf{B}, \quad (5.20b) \\
\mathbb{E} &= \mathbb{J} - Y_5 \mathbb{I} - \alpha^{-1/2} \tilde{\mathbb{G}}, \quad (5.20c) \\
\hat{\mathbb{E}} &= Y_6(\mathbf{B}) \mathbb{J} \quad (5.20d)
\end{align*}
\]

where the different matrices are explicitly given in the appendix. In the following, we emphasise that the matrices \( Y_1 \) and \( Y_3 \) result from the advective part in the equation: \( Y_1 \) is the discretisation of the directional derivative acting on the magnetic field and \( Y_3 \) is the directional derivative acting on the current. The nonlinear effect is given by the matrix \( Y_6 \), which is the discrete form of the Hall term and depends on the magnetic field.

The matrices \( Y_2, Y_3 \) and \( Y_4 \) are the discrete form of the curl curl operator, where in \( Y_3 \) and \( Y_4 \) are included the jumps terms depending on parameter \( b \) which penalises discontinuities in the auxiliary variables. The matrix \( Y_2 \) penalises directly the tangential jumps in the magnetic field. We can control these jumps with the constant \( \xi \), which is given by different \( a' \)'s, i.e., \( \xi = \eta a^1 + \alpha a^1 + \beta a^2 \).

Since we have used LDG discretisation we eliminate the auxiliary variables. This would not have been the case if we would have used other fluxes like interior penalty. After the elimination we obtain

\[
(\mathbb{I} + \alpha Y_5 Y_4)\mathbb{B}_t - (Y_1 - \xi Y_2 - Y_3(\eta \mathbb{I} - \alpha Y_5 + \beta Y_6(\mathbf{B})))\mathbf{Y}_4)\mathbf{B} = \mathbf{F} + \alpha^{1/2} \tilde{\mathbb{G}}.
\]
5.1. Numerical Schemes

Now it remains to discretize time. We will use the same implicit explicit approach used for the one dimensional model. We will rewrite the time derivative as difference between the value at time $t^{n+1}$ and the one at time $t^n$. Then the advective and nonlinear parts are evaluated at time $t^n$ and the diffusive parts at $t^{n+1}$. The resulting full discrete problem is:

$$(I + \alpha Y_3 Y_4) \frac{B^{n+1} - B^n}{\Delta t} + (\xi Y_2 + \eta Y_3 Y_4)B^{n+1} - (Y_1 + Y_3(\alpha Y_5 - \beta Y_6(B))Y_4)B^n = \tilde{F}^n + \alpha^{1/2}\tilde{G}^n.$$ 

Bringing all the $t^n$ terms on the right-hand side we get

$$(I + \alpha Y_3 Y_4 + \Delta t(\xi Y_2 + \eta Y_3 Y_4))B^{n+1} = \Pi(B^n)$$

where $\Pi(B^n) = (I + \alpha Y_3 Y_4)B^n + \Delta t(\tilde{F}^n + \alpha^{1/2}\tilde{G}^n) + (Y_1 + Y_3(\alpha Y_5 - \beta Y_6(B))Y_4)B^n$. At this point we have to set the different penalty parameters $a$’s. In [25], it is shown that in the case of the curl curl problem, one has to set them proportional to the inverse of the parameter in front of the curl curl operator. We are also going to set the parameters inversely proportional to the time step. Setting the penalty parameters as

$$a' = \frac{a}{3\Delta t\eta}, \quad a_1 = \frac{a}{3\alpha\Delta t} \quad \text{and} \quad a_2 = \frac{a}{3\Delta \beta}.$$ 

The final form is:

$$A_{DG}(\alpha + \Delta t\eta)B^{n+1} = \Pi(B^n)$$

where $A_{DG}(\gamma) = I + aY_2 + \gamma Y_3 Y_4$. This matrix represents the DG discretisation of the problem $\omega + \gamma \nabla \times \nabla \times \omega$ which can be used to develop a suitable preconditioner.

**Remark 5.1.5.** The topic of an efficient preconditioner for $A_{DG}$ is still an open research problem. At the moment, we have tried to apply the auxiliary space preconditioner techniques described for the one dimensional case, for this multidimensional problem. As auxiliary space $\tilde{V}_h$, we have used Nedelec element of the first kind [24], which are a conforming discretisation of $H(\text{curl})$.

Since this discrete space is included in the DG space ($\tilde{V}_h \subset V_h$), the prolongation operator $\Pi$ is set to be the inclusion operator. The results we obtained were not satisfactory although numerical experiments showed that the dimensions of the kernel of linear system arising from the discretisation of the curl operator in the DG method is the same as the one obtained for the linear system arising from the curl operator discretized using the conforming space. The curl curl operator, is adding at the level of the mass matrices, a penalty terms for the tangential jump. We have noticed that this contribution is not local as expected. This impairs the effectiveness of the preconditioner, since this kind of contribution is not mapped by our prolongation to the conforming space and also cannot be resolved with a small number of iterations of the local smoother. A more detailed analysis need to be performed to solve this problem.
5. Discontinuous Galerkin Methods

5.2. Numerical Experiments

As for the finite difference schemes, we are going to illustrate different simulations to first validate the numerical schemes and then to show the behaviour of the magnetic induction equation with Hall effect.

5.2.1. Smooth Problems

Pure Magnetic Advection

We have already seen in section 3.3.1 that choosing a rotating velocity field

\[ \hat{u} = (-y, x)^\top \]

the solution of (3.30) at time \( t \) with \( \eta = \alpha = \beta = 0 \) is given by (3.32). This is the solution of the Cauchy problem, but choosing an initial data with compact support and taking \( \Omega \) enough large and zero inflow boundary conditions we obtain the same result. This is valid if \( \Omega \) contains \( S(t) = \{x \in \mathbb{R}^2 | R(-t)x \in \text{supp}(B_0(x)) \} \) for all times. In this case \( R(t) \) is the rotation with angle \( t \).

We are choosing a bell shaped function in \( C^4 \) with compact support

\[ \text{bell}(r) = \begin{cases} 4^6 \left( \frac{1}{4} - |r| \right)^5 & |r| \leq \frac{1}{4} \\ 0 & |r| > \frac{1}{4} \end{cases} \]

and build the initial magnetic field as

\[ B_0(x, y) = \left( \begin{array}{c} -y \\ x - \frac{1}{2} \end{array} \right) \text{bell}((x - 1/2)^2 + y^2). \] (5.22)

The domain \( \Omega = [-2.5, 2.5]^2 \) satisfy the condition \( S(t) \in \Omega \) for all \( t \). We choose, as we have done for the finite difference test in chapter 3, to simulate one rotation \( T = 2\pi \) for different mesh size with \( N \) from 20 to 200.

Forced Solution

We are going to modify the equation with a source term, to be able to obtain some convergence results for the full magnetic induction equation. The modified form for the two dimensional magnetic induction equation with Hall effect is given in (3.34). Setting

\[ \dot{S}(x, y, t) = P(2x - \cos(t), 2y - \sin(t)) \left( \begin{array}{c} \sin(t) - 2y \\ 2x - \cos(t) \end{array} \right) \]

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5.2. Numerical Experiments

\[ \hat{S}^3 = 0 \]

with

\[
P(x, y) = \begin{cases} 
960(x^2 + y^2 - 1)^3(x^2 + y^2 + 1/3) & x^2 + y^2 \leq 1 \\
0 & x^2 + y^2 > 1 
\end{cases}
\]

will force the solution of (3.34) to be (3.32) with \( B_0 \) given in (5.22). Since we still do not have a suitable preconditioner we are running our simulations for smaller resolutions from 20 to 100 cells up to the final time \( T = \pi/4 \). The results for the forced problem and the pure advection problem are shown in figure 5.1. We observe that in the case of the advection a super-convergence of order \( k + 3/2 \) is obtained. For the forced problem the expected order of convergence of \( k \) is obtained.

5.2.2. Hall effect solutions

Next, we test the full Magnetic induction equations with Hall effect (without any forcing) for the rotating hump problem. We set \( \eta = 0.01, \alpha = 0.02 \) and \( \beta = 0.01 \) and compute the solutions on a mesh \( 120 \times 120 \) points with piecewise linear polynomial.

We compare the results with the obtained obtained with the finite difference code of chapter 3. The finite difference solution is computed on the same mesh with the SBP operators of order 2. The results are shown in figure 5.2 and demonstrate the robustness of the DG scheme. We do not observe any difference in the first and second component of the magnetic field the main difference between the two results is at the level of the third component, which is induced by the Hall effect. We observe that at this resolution the DG method, although converging only with order one is less diffusive than the finite

![Figure 5.1.](image)

Figure 5.1.: Convergence plots the advection problem on the left and on the right for the forced problem with \( \eta = 0.01, \alpha = 0.002 \) and \( \beta = 0.01 \)
5. Discontinuous Galerkin Methods

difference scheme of order two. This improvement at the level of the small scale effects could be central in the formation of reconnection at the right time scale. The third component of $\mathbf{B}$ differs also in the shapes obtained by the two methods. This could be caused by the difference in treatment of the boundary conditions. The finite difference code is imposing non-reflecting Neumann boundary condition and the DG is weakly setting the natural boundary condition on $\mathbf{B}$. 
5.2. Numerical Experiments

(a) $B_1$ solved with Finite Difference Methods.

(b) $B_1$ solved with DG Methods.

(c) $B_2$ solved with Finite Difference Methods.

(d) $B_2$ solved with DG Methods.

(e) $B_3$ solved with Finite Difference Methods.

(f) $B_3$ solved with DG Methods.

Figure 5.2.: We are giving numerical results for the solution of (2.6) with $\eta = 0.01$, $\alpha = 0.002$ and $\beta = 0.01$ after $T = \frac{\pi}{4}$. On the left the solution is obtained with the finite difference method and on the right with the discontinuous Galerkin method.
5. Discontinuous Galerkin Methods
6. Conclusion

Ideal MHD equations have been very successfully applied in modelling many interesting phenomena in astrophysics, solar physics, electrical and aerospace engineering. However, a major drawback of this model is the "frozen-in" condition, i.e., the magnetic field lines have to follow the fluid motion. Hence, ideal MHD cannot model magnetic reconnection, which involves large changes in the topology of the magnetic field.

Since magnetic reconnection [1] is considered to be a very important physical effect in many applications in plasma physics, one has to go beyond the ideal MHD model.

Adding magnetic resistivity to the ideal MHD equations is the simplest approach. However, the magnetic reconnection rates obtained with the resulting resistive MHD equations are too slow to account for the fast reconnection observed in practice. A more elaborate approach entails correcting the Ohm’s law by adding small scale effects such as electron inertia and the Hall effect. The resulting version of the MHD equations are termed Hall MHD equations (1.24-1.25) and have been shown to provide more realistic fast reconnection rates [8].

As the Hall MHD equations are highly nonlinear and complicated, it is not possible to obtain explicit analytical solution formulas. Numerical simulations are the main tool for the study of these equations. The ideal MHD equations (1.1) are a system of hyperbolic conservation laws and many efficient numerical methods exist for approximating them in a robust manner [29]. Compared to the ideal MHD equations, the Hall MHD equations include magnetic resistivity, electron inertia and the Hall effect. These corrections to the ideal MHD equations need to be discretized in a stable and efficient manner. Then, these corrections can be coupled with ideal MHD codes to obtain suitable numerical schemes for the Hall MHD equations.

A necessary first step in constructing efficient numerical methods for the Hall MHD equations involves designing suitable discretisation of the magnetic induction equation with Hall effect (1.26). The main aim of this thesis has been to analyse the magnetic induction equation with Hall effect and to design stable numerical methods to approximate them.

To this end, we have derived a priori estimates on the magnetic energy for the magnetic induction equation with Hall effect in chapter 2. The energy estimates provide a bound on the magnetic field in the Sobolev space $H(curl)$. To do so, we symmetrize the advective parts of the equations using the divergence constraint. The divergence constraint coupled with the bound in $H(curl)$ ensure that the magnetic field is in $L^\infty((0,T); H^1)$. 
6. Conclusion

Furthermore, we also show uniqueness of solutions to the magnetic induction equation with Hall effect, provided that the magnetic field is sufficiently regular.

In chapter 3, we construct finite difference schemes for the magnetic induction equation with Hall effect that satisfy a discrete version of the energy estimate (2.9). The schemes use summation by parts operators as discrete spatial derivatives and are of two types, one discretising the symmetric version of the equations (2.3) and another discretising the standard form of the equations (1.26). Both schemes are shown to satisfy a discrete energy estimate (3.26). In addition, the symmetric finite difference scheme satisfies a discrete divergence bound. On the other hand, the scheme (3.22) discretising the standard form of the equation (1.26) is designed to be discrete divergence free. Numerical experiments illustrating the robustness of the finite difference schemes are presented in chapter 3.

The presented finite difference schemes are semi-discrete, i.e., discrete in space and continuous in time. We use standard Runge Kutta methods to design a fully discrete extension. However, because of the electron inertia term involving a mixed spatio-temporal derivative, large linear algebraic systems need to be solved within every stage for every time step. As it happens, the resulting matrices are ill-conditioned. Consequently, suitable preconditioners have to be developed in order to increase the efficiency of the whole algorithm.

The matrix, that needs to be preconditioned, contains a discrete form of the curl(curl) operator. Such matrices have been preconditioned efficiently using multigrid methods [16], provided that they are generated from conforming finite element discretisation of the curl(curl) operator. We tried to use auxiliary space techniques by mapping the approximate solutions from our finite difference space to a conforming finite element space using suitable prolongation operators. Unfortunately, this approach failed on account of the mismatch between the kernels of finite difference and finite element discretisation of the curl(curl) operator.

The above considerations suggest using finite element discretisations of the magnetic induction equation with Hall effect to be able to precondition the resulting matrices at every time step. Given the fact that the magnetic induction equation with Hall effect involves advection of both the magnetic field as well as the current, a discontinuous Galerkin finite element method is preferable to a conforming finite element method. Furthermore, using conforming finite elements methods when dealing with nonlinear problems such as the full Hall MHD equations is problematic as the solutions may contain discontinuities such as shock waves.

Motivated by the possible availability of efficient preconditioners as well as the ability to handle complex domain geometries (discretized using unstructured grids), we design discontinuous Galerkin discretisation of the magnetic induction equation with Hall effect. The design is split into two stages.

In stage I (presented in chapter 4), we consider the simplified version of the magnetic induction equation with Hall effect in one space dimension. A mixed (first order) varia-
tional formulation is introduced and discretized using a discontinuous Galerkin method, based on piecewise smooth (discontinuous across elements) polynomials. The method is shown to be energy stable and the approximate solution are shown to be bounded in $H^1$.

A fully discrete scheme is developed using a high order extrapolation method based on implicit-explicit (IMEX) Euler time steps. Suitable stability criteria are imposed in order to determine the time step.

An added advantage of the discontinuous Galerkin approach was success in finding a suitable preconditioner for the resulting matrices using auxiliary space techniques. Numerical tests demonstrating the discontinuous Galerkin discretisation in one space dimension are also presented.

In stage II (presented in chapter 5), we extend the discontinuous Galerkin discretisation to approximate the magnetic induction equation with Hall effect in several space dimensions using a mixed (first order) variational formulation. Again, an energy estimate is proven and the approximate solution is shown to be in $H(\text{curl})$. The extrapolation method is used to obtain a fully discrete implicit-explicit (IMEX) time stepping scheme (explicit for the magnetic field, current advection and Hall effect and implicit in resistivity and electron inertia). We also attempted to use an auxiliary space techniques (analogous to the one dimensional version) in this case. Although the kernels of the conforming finite element space, based on the Nedelec edge element of the first type [24] and of the discontinuous Galerkin discretisation of the curl(curl) operators are the same, the technique did not result in an efficient algorithm due to global coupling of the discontinuous Galerkin jump operators. Numerical experiments comparing the finite difference and the discontinuous Galerkin discretisations are presented in chapter 5.

Summarising the main results of this thesis are:

1. Analysis of the magnetic induction equation with Hall effect resulting in an energy ($H^1$) bound.
2. Design of high-order finite difference discretisations satisfying a discrete version of the energy bound.
3. Design of high-order discontinuous Galerkin discretisations.
4. IMEX time stepping using an extrapolation method.
5. Development of efficient preconditioners for the discontinuous Galerkin discretisation, using auxiliary space techniques, at least in one space dimension.

Clearly, more needs to be done in order to provide an efficient discretisation of the Hall MHD equations. In particular we need to

1. Increase the efficiency of the preconditioner for the discontinuous Galerkin discretisation in several space dimensions by dealing with the possible spurious modes generated by the globally coupled discontinuous Galerkin jump operators.
6. Conclusion

2. Testing the finite difference and (particularly) the discontinuous Galerkin code on realistic space time dependent velocity fields and density distributions, such as those generated from an ideal MHD code. The key issue here will be the ability of the discontinuous Galerkin code to deal with possibly discontinuous velocity fields and density distributions.

3. Coupling the discontinuous Galerkin discretisation of the magnetic induction equation with Hall effect with a high-resolution finite volume or discontinuous Galerkin discretisation of the ideal MHD equations. The coupling can be performed using a splitting method, such as the one proposed in [13].

4. Efficient parallelisation of the discontinuous Galerkin code, such that it can scale up to large numbers of processors. The availability of an efficient preconditioner as an input to an iterative linear algebraic solver is absolutely crucial for scalability of the parallelism.

5. Testing the Hall MHD code on benchmark test cases such the GEM challenge [5] in order to obtain realistic fast reconnection rates.
A. Finite Difference Operators

The different operators used in our numerical experiment, are based on one dimensional operators coupled together with Kronecker product. The one dimensional operators are given for $q = x, y, z$ in matrix form:

- **Second order central difference**
  
  $$D_{q}^{(2)} = P_{q}^{-1}Q = \frac{1}{2\Delta q} \begin{pmatrix} -2 & 2 \\ -1 & 0 & 1 \\ \vdots & \ddots & \ddots \\ -1 & 0 & 1 \\ -2 & 2 \end{pmatrix}, \quad P_{q} = \Delta q \begin{pmatrix} \frac{1}{2} & \cdots & \cdots \\ 1 & \cdots & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}.$$ 

- **Fourth order central difference**
  
  $$D_{q}^{(4)} = P_{q}^{-1}Q = \frac{1}{\Delta q} \begin{pmatrix} -1 & 1 & \frac{1}{12} & -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} \\ \frac{1}{12} & 0 & \frac{1}{2} & \frac{2}{3} & -\frac{1}{12} & \cdots & \cdots \\ -\frac{2}{3} & 0 & \frac{2}{3} & -\frac{1}{12} & \cdots & \cdots \\ \frac{1}{12} & 0 & \frac{2}{3} & -\frac{1}{12} & \cdots & \cdots \\ -1 & 1 \end{pmatrix}, \quad P_{q} = \Delta q \begin{pmatrix} \frac{1}{2} & \cdots & \cdots \\ 1 & \cdots & \cdots \end{pmatrix}.$$ 

Combining this operators we obtain the two spatial discretisation used in the numerical experiments.
We give the discrete derivative for the $x$ direction, the ones for the other spatial directions are defined analogously.
Standard second and fourth order operator are

$$\partial_{x} = D_{x}^{(k)} \otimes I_{y} \otimes I_{z} \quad k = 2, 4$$
A. Finite Difference Operators

where $I_q$ are the identity matrices.
B. Two Dimensional Matrix Formulation

We are going to use the discrete space $V_h$ given in 5.1.1 to express the system of ordinary differential equations arising from (5.3) in matrix form. We see in (3.30) that the dynamics of the first two components of the magnetic field are almost decoupled from the dynamics of the third one. These two equations are connected by the nonlinear term and the electron inertia. This motivate us, to set $S = (S_1, S_2)^T$ to describe the two first components and the small letter $s = S_3$ for the third component of a three dimensional coefficient vectors $(S_1, S_2, S_3)^T$. We start to use this notation to rewrite (5.3a) with fluxes (5.18) and (5.19), getting

\[ M^{22} \hat{B}_t - M^{22}(C_1) \hat{B} - S^{22} u + S^{21}(\eta \hat{B} + \alpha \hat{E} + \beta \hat{E}) + E^{22} \hat{B} + Q^{22} \hat{B} = \hat{F} \quad (B.1a) \]

\[ M^{11} h_t - M^{12}(C_2) \hat{B} - S_0 h + S^{12}(\eta \hat{B} + \alpha \hat{E} + \beta \hat{E}) + E_0 h + Q_0 h = \hat{f} \quad (B.1b) \]

with

\[ C_1 = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{\partial u_1}{\partial y} \\ \frac{\partial u_2}{\partial x} & \frac{\partial u_2}{\partial y} \end{pmatrix}, \quad C_2 = \begin{pmatrix} \frac{\partial u_3}{\partial x} & \frac{\partial u_3}{\partial y} \end{pmatrix} \]

In this case, the matrices are defined blockwise as

\[ M^{n,m}(A(x,y)) = \begin{pmatrix} M(A_{1,1}(x,y)) & \cdots & M(A_{1,m}(x,y)) \\ \vdots & \ddots & \vdots \\ M(A_{n,1}(x,y)) & \cdots & M(A_{n,m}(x,y)) \end{pmatrix}, \]

\[ S^{22}_u = \begin{pmatrix} S_u & 0 \\ 0 & S_u \end{pmatrix}, \quad S^{21} = \begin{pmatrix} -Dy \\ Dx \end{pmatrix}, \quad S^{12} = \begin{pmatrix} Dy & -Dx \end{pmatrix}, \]

\[ E^{22}_u = \begin{pmatrix} E_u & 0 \\ 0 & E_u \end{pmatrix}, \quad E^{21} = \begin{pmatrix} -\bar{E}_x \\ \bar{E}_y \end{pmatrix}, \quad E^{12} = \begin{pmatrix} \bar{E}_x & -\bar{E}_y \end{pmatrix}, \]

\[ Q^{22}_u = \begin{pmatrix} Q_u & 0 \\ 0 & Q_u \end{pmatrix}, \quad Q^{22}_I = \begin{pmatrix} Q_x & 0 \\ 0 & Q_y \end{pmatrix}, \quad Q^{22} = \begin{pmatrix} \bar{Q}_x & 0 \\ 0 & \bar{Q}_y \end{pmatrix}, \]

\[ Q_I = \bar{Q}_x + \bar{Q}_y, \quad Q = Q_x + Q_y. \]
B. Two Dimensional Matrix Formulation

where \( A(x,y) \) is an \( n \) times \( m \) matrix. The different matrices are

\[
(M(f))_{i,j} := (f \psi^i, \psi^j)_{T_h},
\]
\[
(S_u)_{i,j} := \left( (u_1 \frac{\partial \psi^i}{\partial x} + u_2 \frac{\partial \psi^i}{\partial y}), \psi^j \right)_{T_h},
\]
\[
(E_u)_{i,j} := (\langle \{ \mathbf{u} \psi^i \}, [\psi^j]_N \rangle_{\partial y},\partial^+),
\]
\[
(E_q)_{i,j} := (\langle \{ \mathbf{q} \psi^i \}, [\psi^j]_N \rangle_{\partial y},\partial^+, (E_q)_{i,j} := (\langle \{ \mathbf{q} \psi^i \}, [\psi^j]_N \rangle_{\partial y},\partial^+),
\]
\[
(Q_u)_{i,j} := ([\mathbf{u}]_j, \mathbf{q} \psi^i \rangle_{\partial y},\partial^+, (Q_u)_{i,j} := ([\mathbf{u}]_j, \mathbf{q} \psi^i \rangle_{\partial y},\partial^+),
\]
\[
(D_x)_{i,j} := \left( \frac{\partial \psi^i}{\partial x}, \psi^j \right)_{T_h}, (D_y)_{i,j} := \left( \frac{\partial \psi^i}{\partial y}, \psi^j \right)_{T_h},
\]

with \( q = x, y \). The subsets \( \mathcal{E}^x \) and \( \mathcal{E}^y \) are the subset of edges aligned with the \( x^- \) or \( y^- \) axis, respectively. If the matrix \( a \) is the identity matrix \( I \) we write \( M^{n,n} \) instead of \( M^{n,n}(I) \). The vector \( \mathbf{F} \) is defined as

\[
\mathbf{F}^i = -\langle (\mathbf{nu}) \mathbf{f}^i, \psi^j \rangle_{\partial y},\partial^+.
\]

The mass matrices \( M \) are block diagonal with dimension \((k + 1)^2\) with \( k \) being the polynomial degree of the basis. Thus, we can invert the matrices explicitly for moderate \( k \). Multiplying the two equations in (B.1) by \((M^{22})^{-1}\) and \((M^{11})^{-1}\) and collecting terms together resulting

\[
\begin{align*}
\tilde{B}_t + (\xi \tilde{Q}^{22} - \tilde{Z}^{22}(C^1)) \tilde{B} + \eta(\tilde{W}^{21} - b \tilde{Q}^{22}) + \alpha(\tilde{W}^{21} - b \tilde{Q}^{22}) &= \tilde{\eta}, \\
\beta(\tilde{W}^{21} - b \tilde{Q}^{22}) &= \tilde{\beta}, \\
\end{align*}
\]

We have introduced the tilde notation where \( \tilde{T} = (M^{n,n})^{-1}T \), with \( T \) being a general matrix with \( n \) rows, and also

\[
\xi := \eta a^1 + \alpha a^2 + \beta a^3,
\]
\[
Z^{22}(C) := (M^{22}(C) + S^{22}_u - E^{22}_u - Q^{22}_u),
\]
\[
Z(C) := (M(C) + S_u - E_u - Q_u),
\]
\[
W^{kj} := S^{kj} - E^{kj},
\]
\[
W^{kj} := S^{kj} - E^{kj}.
\]
We use the same strategy to rewrite the terms (5.3b) and (5.3c) with fluxes (5.18) and (5.19), similar computation yields

\[ \mathbf{j} = \tilde{W}^{21}_l \mathbf{b} - b \tilde{Q}^{22}_l \mathbf{b}, \]  
(B.10a)

\[ \mathbf{j} = \tilde{W}^{12}_l \mathbf{b} - b \tilde{Q}^{12}_l \mathbf{b}, \]  
(B.10b)

and

\[ \mathbf{E} = \tilde{\mathbf{j}} - \tilde{Z}^{22}_l (C_3) \tilde{\mathbf{j}} - \alpha^{-1/2} \mathbf{G}, \]  
(B.11a)

\[ \mathbf{e} = \tilde{\mathbf{j}} - \tilde{Z}^{22}_l (C_4) \tilde{\mathbf{j}} - \alpha^{-1/2} \mathbf{G}. \]  
(B.11b)

Here

\[ C_3 = \begin{pmatrix} \nabla \cdot \mathbf{u} & 0 \\ 0 & \nabla \cdot \mathbf{u} \end{pmatrix}, \quad C_4 = \nabla \cdot \mathbf{u} \]

and

\[ (G^i)_I = -\langle (\mathbf{n} \cdot \mathbf{u}) g^i, \psi^I \rangle_{\mathcal{E} \cap \Gamma}. \]

We have now to reformulate the last term (5.3d), where the non linearity appears. We obtain

\[ M^{22}_l \hat{\mathbf{E}} - \tilde{P}^{22}_l (\mathbf{b}) - \tilde{P}^{21}_l (\mathbf{B}) \mathbf{j} = 0, \]  
(B.12a)

\[ M^{11}_l \hat{\mathbf{e}} - \tilde{P}^{12}_l (\mathbf{B}) \tilde{\mathbf{j}} = 0, \]  
(B.12b)

where

\[ P^{22}_l (\mathbf{b}) = \begin{pmatrix} 0 & P^{(b)} \\ P^{(b)} & 0 \end{pmatrix}, \quad P^{21}_l (\mathbf{B}) = \begin{pmatrix} -P(B_2) \\ P(B_1) \end{pmatrix}, \quad P^{12}_l (\mathbf{B}) = \begin{pmatrix} P(B_2) & -P(B_1) \end{pmatrix}. \]

The matrix \( P(S) \) is build by vector \( S \) through

\[ (P(S))_I,J = \sum_{K=1}^{N_{tot}} T_{I,J,K} S_K, \]

where the tensor \( T \) is given by

\[ T_{I,J,K} = \langle \psi^I \psi^K, \psi^J \rangle_{\mathcal{T}_h}. \]

Multiplying with the inverse of the mass matrices we get

\[ \hat{\mathbf{E}} - \tilde{P}^{22}_l (\mathbf{b}) - \tilde{P}^{21}_l (\mathbf{B}) \mathbf{j} = 0, \]  
(B.13a)

\[ \hat{\mathbf{e}} - \tilde{P}^{12}_l (\mathbf{B}) \tilde{\mathbf{j}} = 0. \]  
(B.13b)
B. Two Dimensional Matrix Formulation

We recombine (B.9), (B.10), (B.11) and (B.13) using vectors $S = (S_1, S_2, S_3)^T$ getting

\[
\begin{align*}
\mathbb{B}_t - (Y_1 - \xi Y_2) \mathbb{B} + Y_3 (\eta \mathbb{I} + \alpha \mathbb{E} + \beta \tilde{\mathbb{E}}) &= \tilde{\mathbb{F}}, \quad (B.14a) \\
\mathbb{I} &= Y_4 \mathbb{B}, \quad (B.14b) \\
\mathbb{E} &= \mathbb{J}_t - Y_5 \mathbb{I} - \alpha^{-1/2} \tilde{\mathbb{G}}, \quad (B.14c) \\
\tilde{\mathbb{E}} &= Y_6 \mathbb{J} \quad (B.14d)
\end{align*}
\]

where

\[
\begin{align*}
Y_1 &= \begin{pmatrix} \tilde{Z}^{22}(C_1) & 0 \\ M^{12} & \tilde{Z} \end{pmatrix}, \quad Y_2 = \begin{pmatrix} \tilde{Q}^{22} & 0 \\ 0 & \tilde{Q} \end{pmatrix}, \quad Y_3 = \begin{pmatrix} -b \tilde{Q}_t^{22} & \tilde{W}_t^{21} \\ \tilde{W}_t^{12} & -b \tilde{Q}_t \end{pmatrix}, \\
Y_4 &= \begin{pmatrix} -b \tilde{Q}_t^{22} & \tilde{W}_t^{21} \\ \tilde{W}_t^{12} & -b \tilde{Q}_t \end{pmatrix}, \quad Y_5 = \begin{pmatrix} \tilde{Z}^{22}(C_3) & 0 \\ 0 & \tilde{Z}(C_4) \end{pmatrix}, \\
Y_6 &= \begin{pmatrix} \tilde{P}^{22}(b) & \tilde{P}^{21}(\tilde{B}) \\ \tilde{P}^{12}(\tilde{B}) & 0 \end{pmatrix}, \quad \mathbb{F} = \begin{pmatrix} \tilde{F} \\ \tilde{f} \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} \tilde{G} \\ \tilde{g} \end{pmatrix}.
\end{align*}
\]
References


References


