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Journal Article

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Publication date:
2007-08

Permanent link:
https://doi.org/10.3929/ethz-b-000065989

Originally published in:
Geometriae Dedicata 128(1), https://doi.org/10.1007/s10711-007-9188-y

Geom Dedicata (2007) 128:167–176
DOI 10.1007/s10711-007-9188-y

ORIGINAL PAPER

Twisted Weyl groups of Lie groups and nonabelian cohomology

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Received: 28 September 2006 / Accepted: 31 August 2007 / Published online: 22 September 2007

Abstract For a cyclic group A and a connected Lie group G with an A-module structure (with the additional assumptions that G is compact and the A-module structure on G is 1-semisimple if $A \cong \mathbb{Z}$), we define the twisted Weyl group $W = W(G, A, T)$, which acts on T and $H^1(A, T)$, where T is a maximal compact torus of G_0^A, the identity component of the group of invariants G^A. We then prove that the natural map $W\backslash H^1(A, T) \to H^1(A, G)$ is a bijection, reducing the calculation of $H^1(A, G)$ to the calculation of the action of W on T. We also prove some properties of the twisted Weyl group W, one of which is that W is a finite group. A new proof of a known result concerning the ranks of groups of invariants with respect to automorphisms of a compact Lie group is also given.

Keywords Lie group · Twisted Weyl group · Nonabelian cohomology · Twisted conjugate action

Mathematics Subject Classification (2000) 20J06 · 22E15 · 57S15 · 57S20

1 Introduction

Let A be a group, and let G be a connected Lie group with an A-module structure. It is proved in [1] that, if A is finite, then there exists a maximal compact subgroup K of G which is an A-submodule of G, and for every such K, $H^1(A, K) \to H^1(A, G)$ is a bijection. This reduces the calculation of $H^1(A, G)$ to the calculation of $H^1(A, K)$ for K compact.

Based on some results in [1], in this paper we go further along this direction. We reduce the calculation of $H^1(\mathbb{Z}, G)$ for G compact, and $H^1(\mathbb{Z}/n\mathbb{Z}, G)$ for G general, to the calculation of the action of the twisted Weyl group on a compact torus of G.

Let A be a cyclic group, that is, A is isomorphic to $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$, and let G be a connected Lie group with an A-module structure. In the case of $A \cong \mathbb{Z}$, we always assume that G is compact and the $\mathbb{Z}$-module structure on G is 1-semisimple, that is,

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$$\ker(1 - d\sigma) = \ker((1 - d\sigma)^2)$$

for a generator σ of A. Let T be a maximal compact torus of G_0^A, the identity component of $G^A = \{g \in G | a(g) = g, \forall a \in A\}$. Then it is proved in [1] that the natural map $H^1(A, T) \to H^1(A, G)$ is surjective (see Theorems 4.1 and 5.1 in [1]).

Let σ be a generator of A. Denote

$$Z_\sigma(T) = \{g \in G | gt\sigma(g)^{-1} = t, \forall t \in T\},$$

$$N_\sigma(T) = \{g \in G | gT\sigma(g)^{-1} = T\},$$

then $Z_\sigma(T)$ and $N_\sigma(T)$ are closed subgroups of G, $Z_\sigma(T)$ is normal in $N_\sigma(T)$. Define the *twisted Weyl group* $W = W(G, A, T)$ by

$$W = N_\sigma(T)/Z_\sigma(T).$$

Then it can be proved that W is independent of the choice of σ, and, as an abstract group, W is independent of the choice of T. It can also be proved that W is a finite group.

Since T is abelian and A acts trivially on T, $H^1(A, T)$ may be identified with the set of cocycles $Z^1(A, T)$, which consists of homomorphisms $A \to T$. The twisted Weyl group W acts on $Z^1(A, T) \cong H^1(A, T)$ by

$$(w.\alpha)(a) = g\alpha(a)a(g)^{-1},$$

where $w \in W, \alpha \in Z^1(A, T), a \in A, g \in w$. Denote by $W\backslash H^1(A, T)$ the space of W-orbits in $H^1(A, T)$. Then the natural map $H^1(A, T) \to H^1(A, G)$ reduces to a map $W\backslash H^1(A, T) \to H^1(A, G)$.

The main result of this paper is the following assertion.

Theorem 1.1 *Suppose A is a cyclic group, G is a connected Lie group with an A-module structure. If $A \cong \mathbb{Z}$, we assume that G is compact and the $\mathbb{Z}$-module structure on G is* 1*-semisimple. Let T be a maximal compact torus of G_0^A, $W = W(G, A, T)$ the associated twisted Weyl group. Then the map $W\backslash H^1(A, T) \to H^1(A, G)$ is a bijection.*

Upon choosing a generator of A, we may identify $H^1(A, T)$ with T (if $A \cong \mathbb{Z}$) or the finite subgroup $E_n(T) = \{t \in T | t^n = e\}$ of T (if $A \cong \mathbb{Z}/n\mathbb{Z}$). So Theorem 1.1 reduces the calculation of $H^1(A, G)$ to the calculation of the action of W on T or $E_n(T)$. (This action will be defined directly in Sect. 3.)

Some special cases of Theorem 1.1 are known. In the case of $A \cong \mathbb{Z}/2\mathbb{Z}$, G is a complex reductive group, and $\mathbb{Z}/2\mathbb{Z}$ acts on G by complex conjugation with respect to a real form of G, the bijectivity of the map $W\backslash H^1(\mathbb{Z}/2\mathbb{Z}, T) \to H^1(\mathbb{Z}/2\mathbb{Z}, G)$ was proved by Borovoi [2]. Mohrdieck [5] proved a result on conjugacy classes in non-connected semisimple algebraic groups over algebraically closed field, which implies the special case of Theorem 1.1 that $A \cong \mathbb{Z}$ and a generator of A, as an automorphism of the compact group G, is induced from an automorphism of the Dynkin diagram of G.

Our proof of Theorem 1.1 relies on the notion of twisted conjugate actions of Lie groups associated with automorphisms. Recall that for a Lie group G with an automorphism σ, the twisted conjugate action of G on itself associated with σ is defined by

$$\tau_g(h) = gh\sigma(g)^{-1}.$$

Two elements $g_1, g_2 \in G$ are σ*-conjugate* if they lie in the same orbit of the twisted conjugate action associated with σ. The utility of this notion is based on the following fact: If A is cyclic with a generator σ, then two cocycles $\alpha_1, \alpha_2 \in Z^1(A, G)$ are cohomologous if and only if $\alpha_1(\sigma)$ and $\alpha_2(\sigma)$ are σ-conjugate.

Similar to the definition of $W(G, A, T)$, we can define the twisted Weyl group $W(G, \sigma, T)$ associated with an individual automorphism σ of G, which acts naturally on the maximal compact torus T of G_0^σ. The proof of Theorem 1.1 is based on the following result.

Theorem 1.2 *Let G be a connected compact Lie group with a 1-semisimple automorphism σ, T a maximal torus of G_0^σ. Then two elements $t_1, t_2 \in T$ are σ-conjugate if and only if they lie in the same $W(G, \sigma, T)$-orbit.*

To prove Theorem 1.2, we need the following known result.

Theorem 1.3 *Let σ_1, σ_2 be automorphisms of a connected compact Lie group G. If $\sigma_2 \circ \sigma_1^{-1}$ is an inner automorphism of G, then* $\operatorname{rank} G_0^{\sigma_1} = \operatorname{rank} G_0^{\sigma_2}$.

Although the proof of Theorem 1.3 is implicitly included in Gantmacher [3], the author can not find an explicit reference for it. Using properties of twisted conjugate actions, we give a new proof of Theorem 1.3 in Sect. 2. In Sect. 3 we define the twisted Weyl group of a Lie group associated with an automorphism, prove some of its properties, and give the proof of Theorem 1.2. The proof of Theorem 1.1 is given is Sect. 4.

2 A preliminary result

We discuss in this section a known result concerning groups of invariants with respect to automorphisms of a compact Lie group, which will be used in the proofs of some results in later sections.

For a connected Lie group G with an automorphism σ, we denote by $G^\sigma = \{g \in G | \sigma(g) = g\}$ the group of invariants, and denote by G_0^σ the identity component of G^σ.

Theorem 2.1 *Let σ_1, σ_2 be automorphisms of a connected compact Lie group G. If $\sigma_2 \circ \sigma_1^{-1}$ is an inner automorphism, then* $\operatorname{rank} G_0^{\sigma_1} = \operatorname{rank} G_0^{\sigma_2}$.

Theorem 2.1 was implicitly proved in [3] (see also [6], Chapt. 4, Sect. 4). But the author can not find an explicit reference. Here we give a new proof of this result, with the aid of some properties of twisted conjugate actions. This may also be viewed as an application of twisted conjugate actions.

We first prove the following basic result.

Lemma 2.2 *Suppose a Lie group G acts smoothly on a smooth manifold M. Let $m = \dim M$, d be the maximal dimension of G-orbits in M. If N is a closed submanifold of M such that the intersection of every G-orbit with N is a nonempty and discrete subset of M, then* $\dim N = m - d$.

Proof Denote the action of G on M by $\varphi : G \times M \to M$. For $x \in M$, denote the G-orbit containing x by O_x. Let $M' = \{x \in M | \dim O_x = d\}$. Then for a point $x \in M$, $x \in M'$ if and only if the differential of the map $\varphi(\cdot, x) : G \to M$ has maximal rank. So $M \setminus M'$ can be locally described as the zero locus of a smooth function on M. Then M' is open in M.

Since φ is transversal to M, $\varphi^{-1}(N)$ is a closed submanifold of $G \times M$. Denote by $\pi_2 : G \times M \to M$ the projection to the second factor, and consider the restriction $\pi_2|_{\varphi^{-1}(N)} : \varphi^{-1}(N) \to M$ of π_2 to $\varphi^{-1}(N)$. By Sard's Theorem, we can choose a regular value $p \in M'$

of $\pi 2|_{\varphi^{-1}(N)}$. Since $O_p \cap N$ is nonempty, there exists $g \in G$ such that $q = \varphi(g, p) \in N$. Since (g, p) is a regular point of $\pi 2|_{\varphi^{-1}(N)}$, $T_{(g,p)}\varphi^{-1}(N) + T_{(g,p)}G \times \{p\} = T_{(g,p)}G \times M$. This implies $T_qN + T_qO_p = T_qM$. So N intersects O_p transversally at q. But $N \cap O_p$ is discrete, so $\dim N + \dim O_p = m$. Note that $p \in M'$ implies $\dim O_p = d$, we have $\dim N = m - d$. □

Proof of Theorem 2.1 Let τ_1, τ_2 be the twisted conjugate actions associated with σ_1 and σ_2, respectively. We first assume that G is semisimple. Let T_1, T_2 be maximal tori of $G_0^{\sigma_1}$ and $G_0^{\sigma_2}$, respectively. Then by Theorem 6.1 in [1], the intersection of every τ_1-orbit (resp. τ_2-orbit) with T_1 (resp. T_2) is nonempty and finite. By Lemma 2.2, we have

$$\dim T_1 = \dim G - \text{the maximal dimension of } \tau_1\text{-orbits},$$

$$\dim T_2 = \dim G - \text{the maximal dimension of } \tau_2\text{-orbits}.$$

But by Proposition 6.1 in [1], τ_1 and τ_2 are equivalent. So the maximal dimension of τ_1-orbits and that of τ_2-orbits coincide. Thus we have $\dim T_1 = \dim T_2$, that is, $\operatorname{rank} G_0^{\sigma_1} = \operatorname{rank} G_0^{\sigma_2}$.

For the general case, let G_s be the semisimple part of G, and let G_t be the identity component of the center of G, which is a compact torus. Then G_s and G_t are invariant under σ_1 and σ_2. Since $\sigma_2 \circ \sigma_1^{-1}$ is inner, $(G_t)_0^{\sigma_1} = (G_t)_0^{\sigma_2}$. But we have proved that $\operatorname{rank}(G_s)_0^{\sigma_1} = \operatorname{rank}(G_s)_0^{\sigma_2}$. So

$$\operatorname{rank} G_0^{\sigma_1} = \operatorname{rank}(G_s)_0^{\sigma_1} + \dim(G_t)_0^{\sigma_1} = \operatorname{rank}(G_s)_0^{\sigma_2} + \dim(G_t)_0^{\sigma_2} = \operatorname{rank} G_0^{\sigma_2}.$$

□

3 Twisted Weyl groups associated with automorphisms

In order to define the twisted Weyl group associated with an A-module structure on a Lie group, in this section we first consider the twisted Weyl group associated with a single automorphism of a Lie group.

Let G be a connected Lie group with an automorphism σ. For technical reasons, we always make the following assumptions.

(1) If G is compact, we assume that σ is 1-semisimple, that is,

$$\ker(1 - d\sigma) = \ker((1 - d\sigma)^2);$$

(2) If G is noncompact, we assume that σ is of finite order.

Note that an automorphism of finite order is 1-semisimple. Let τ be the twisted conjugate action of G on itself associated with σ, which is defined by $\tau_g(h) = gh\sigma(g)^{-1}$. For a closed subgroup H of G, denote

$$Z_\sigma(H) = Z_{\sigma,G}(H) = \{g \in G | \tau_g(t) = t, \forall t \in H\},$$
$$N_\sigma(H) = N_{\sigma,G}(H) = \{g \in G | \tau_g(H) = H\}.$$

Then $Z_\sigma(H)$ and $N_\sigma(H)$ are closed subgroups of G, and $Z_\sigma(H)$ is normal in $Z_\sigma(H)$. Now let T be a maximal compact torus of G_0^σ. Define the *twisted Weyl group associated with* σ by

$$W(G, \sigma, T) = N_\sigma(T)/Z_\sigma(T).$$

We will denote $W(G, \sigma, T)$ by $W(\sigma, T)$, or $W(T)$, or simply W, if the omitted data are explicit from the context.

Proposition 3.1 *As an abstract group, $W(T)$ is independent of the choices of T.*

Proof Let T' be another maximal compact torus of G_0^σ. By Proposition 5.1 in [1], there exists $g \in G_0^\sigma$ such that $T' = gTg^{-1}$. It is easily verified that $Z_\sigma(T') = gZ_\sigma(T)g^{-1}$, $N_\sigma(T') = gN_\sigma(T)g^{-1}$. So $W(T) \cong W(T')$. □

Proposition 3.2 *(i) $Z_\sigma(T) \subset G^\sigma$;*
(ii) The Lie algebras of $Z_\sigma(T)$ and $N_\sigma(T)$ coincide;
(iii) If G is compact, then the Lie algebras of $Z_\sigma(T)$ and $N_\sigma(T)$ coincide with the Lie algebra of T.

Proof (i) Let $g \in Z_\sigma(T)$, then for every $t \in T$, $\tau_g(t) = t$. In particular, $\tau_g(e) = g\sigma(g)^{-1} = e$, that is, $\sigma(g) = g$. Hence $g \in G^\sigma$.
(ii) If X belongs to the Lie algebra of $N_\sigma(T)$, then for every $s \in \mathbb{R}$ and every $t \in T$, $\tau_{e^{sX}}(t) = e^{sX}te^{-sd\sigma(X)} \in T$. This implies $(1 - \mathrm{Ad}(t)d\sigma)X \in \mathfrak{t}$, the Lie algebra of T. Since $\mathfrak{t} \subset \ker(1 - \mathrm{Ad}(t)d\sigma)$, $(1 - \mathrm{Ad}(t)d\sigma)^2X = 0$. Since the mutually commutative endomorphisms $d\sigma$ and $\mathrm{Ad}(t)$ are 1-semisimple, $\mathrm{Ad}(t)d\sigma$ is 1-semisimple. So we have $(1 - \mathrm{Ad}(t)d\sigma)X = 0$ for every $t \in T$, that is, X belongs to the Lie algebra of $Z_\sigma(T)$.
(iii) Suppose G is compact. By (i), the identity component $Z_\sigma(T)_0$ of $Z_\sigma(T)$ is contained in G_0^σ. So $Z_\sigma(T)_0 \subset Z_\sigma(T) \cap G_0^\sigma = Z_{G_0^\sigma}(T) = T$, where $Z_{G_0^\sigma}(T)$ is the centralizer of T in G_0^σ. But $T \subset Z_\sigma(T)_0$, so $Z_\sigma(T)_0 = T$. Hence the Lie algebra of $Z_\sigma(T)$ coincides with the Lie algebra of T, and then (iii) follows from (ii).

□

Remark 3.1 Suppose G is compact. If σ is the identity, we have $Z_\sigma(T) = Z_G(T) = T$. But this does not hold for general σ, although $Z_\sigma(T)$ and T have the same Lie algebra. For example, let $G = U(3)$ with automorphism σ defined by $\sigma(g) = \overline{g}$. Then $G^\sigma = O(3)$, $G_0^\sigma = SO(3)$. $T = \mathrm{diag}(SO(2), 1)$ is a maximal torus of G_0^σ, $g = \mathrm{diag}(1, 1, -1) \in Z_\sigma(T)$, but $g \notin T$.

Remark 3.2 It is not necessary that $N_\sigma(T) \subset G^\sigma$. Consider also the example in the above remark. Then $h = \mathrm{diag}(i, i, 1) \in N_\sigma(T)$, but $h \notin G^\sigma$.

By (ii) of Proposition 3.2, the twisted Weyl group W is finite if G is compact. We claim that it is also finite in the noncompact case. To prove this, we need some preliminaries, some of which are also used in the next section.

Suppose G is a connected Lie group with an automorphism σ of finite order. By Theorem 3.1 in [1], there always exists a maximal compact subgroup K of G which is σ-invariant. We first prove two lemmas.

Lemma 3.3 *Let G be a connected Lie group with an automorphism σ of finite order, and let K be a σ-invariant maximal compact subgroup of G. Then every $g \in G$ admits a decomposition $g = kp$ such that $k \in K$, and such that for every $k_1, k_2 \in K$, if $\tau_g(k_1) = k_2$, then $\tau_k(k_1) = k_2$, $\tau_p(k_1) = k_1$.*

Proof Let $\mathfrak{g}$ and $\mathfrak{k}$ be the Lie algebras of G and K, respectively. By [4], Chapter XV, Theorem 3.1, there exist linear subspaces $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}_1 \oplus \ldots \oplus \mathfrak{m}_r$ such that $\mathrm{Ad}(k)(\mathfrak{m}_i) = \mathfrak{m}_i, \forall k \in K, i \in \{1, \ldots, r\}$, and such that the map $\varphi : K \times \mathfrak{m}_1 \times \cdots \times \mathfrak{m}_r \to G$

defined by $\varphi(k, X_1, \ldots, X_r) = ke^{X_1} \cdots e^{X_r}$ is a diffeomorphism. For $g \in G$, write g as $g = ke^{X_1} \cdots e^{X_r}$, where $k \in K$, $X_i \in \mathfrak{m}_i$. Let $k_1, k_2 \in K$ such that $\tau_g(k_1) = gk_1\sigma(g)^{-1} = k_2$. Rewrite this equality as $k_1^{-1}gk_1 = k_1^{-1}k_2\sigma(g)$, then we have

$$(k_1^{-1}kk_1)e^{\mathrm{Ad}(k_1^{-1})X_1} \cdots e^{\mathrm{Ad}(k_1^{-1})X_r} = (k_1^{-1}k_2\sigma(k))e^{d\sigma(X_1)} \cdots e^{d\sigma(X_r)},$$

that is,

$$\varphi(k_1^{-1}kk_1, \mathrm{Ad}(k_1^{-1})X_1, \ldots, \mathrm{Ad}(k_1^{-1})X_r) = \varphi(k_1^{-1}k_2\sigma(k), d\sigma(X_1), \cdots, d\sigma(X_r)).$$

Since φ is a diffeomorphism, we have $k_1^{-1}kk_1 = k_1^{-1}k_2\sigma(k)$, that is, $\tau_k(k_1) = k_2$. Let $p = e^{X_1} \cdots e^{X_r} = k^{-1}g$, then $\tau_p(k_1) = \tau_{k^{-1}} \circ \tau_g(k_1) = \tau_{k^{-1}}(k_2) = k_1$. This proves the lemma. □

Lemma 3.4 *Let G be a connected Lie group with an automorphism σ of finite order, and let K be a σ-invariant maximal compact subgroup of G. Then for every closed subgroup H of K, $N_{\sigma,G}(H) = N_{\sigma,K}(H) \cdot Z_{\sigma,G}(H)$.*

Proof It is obvious that $N_{\sigma,K}(H) \cdot Z_{\sigma,G}(H) \subset N_{\sigma,G}(H)$. To prove the converse, let $g \in N_{\sigma,G}(H)$. Write $g = kp$ as in Lemma 3.3. Then for every $h_1 \in H$, $h_2 = \tau_g(h_1) \in H \subset K$. So $\tau_k(h_1) = h_2 \in H$, $\tau_p(h_1) = h_1$, that is, $k \in N_{\sigma,K}(H)$, $p \in Z_{\sigma,G}(H)$. Hence $g \in N_{\sigma,K}(H) \cdot Z_{\sigma,G}(H)$. This proves $N_{\sigma,G}(H) \subset N_{\sigma,K}(H) \cdot Z_{\sigma,G}(H)$. □

Corollary 3.5 *Let G be a connected Lie group with a maximal compact group K, H a closed subgroup of K. Then $N_G(H) = N_K(H) \cdot Z_G(H)$.* □

Proposition 3.6 *Let G be a connected Lie group with an automorphism σ of finite order. Then*

(i) For every maximal compact torus T of G_0^σ, there exists a σ-invariant maximal compact subgroup K of G such that T is a maximal torus of K_0^σ;
(ii) For every σ-invariant maximal compact subgroup K of G and every maximal torus T of K_0^σ, T is a maximal compact torus of G_0^σ.

Proof (i) Suppose σ is of order n. Then $\mathbb{Z}/n\mathbb{Z}$ acts on G by $(m, g) \mapsto \sigma^m(g)$. Consider the semidirect product $G \rtimes \mathbb{Z}/n\mathbb{Z}$. Then $T \rtimes \mathbb{Z}/n\mathbb{Z}$ is a compact subgroup of $G \rtimes \mathbb{Z}/n\mathbb{Z}$. Let $\widetilde{K}$ be a maximal compact subgroup of $G \rtimes \mathbb{Z}/n\mathbb{Z}$ containing $T \rtimes \mathbb{Z}/n\mathbb{Z}$, and let K be the identity component of $\widetilde{K}$. Then K is a maximal compact subgroup of G containing T. Since $T \subset G_0^\sigma$, $T \subset K_0^\sigma$. Moreover, since T is a maximal compact torus of G_0^σ, T is a maximal torus of K_0^σ.

(ii) Suppose K is a σ-invariant maximal compact subgroup of G, T is a maximal torus of K_0^σ. It is obvious that T is a compact torus of G_0^σ. Let T' be a maximal compact torus of G_0^σ containing T. We want to prove that $T' = T$.
By (i), there is a σ-invariant maximal compact subgroup K' of G such that T' is a maximal torus of $(K')_0^\sigma$. By [4], Chapter XV, Theorem 3.1, there exists $g \in G$ such that $gK'g^{-1} = K$. Then $gT'g^{-1}$ is a maximal torus of $g(K')_0^\sigma g^{-1}$. But for $k \in K$, $k \in g(K')^\sigma g^{-1} \Leftrightarrow \sigma(g^{-1}kg) = g^{-1}kg \Leftrightarrow k \in G^{\mathrm{Inn}(g\sigma(g)^{-1})\circ\sigma}$, where $\mathrm{Inn}(g\sigma(g)^{-1})$ is the inner automorphism of G induced by $g\sigma(g)^{-1}$. Note that $\mathrm{Inn}(g\sigma(g)^{-1})(K) = g\sigma(g)^{-1}K\sigma(g)g^{-1} = g\sigma(g^{-1}Kg)g^{-1} = gK'g^{-1} = K$, that is, K is $\mathrm{Inn}(g\sigma(g)^{-1})$-invariant. So $g(K')^\sigma g^{-1} = K^{\mathrm{Inn}(g\sigma(g)^{-1})\circ\sigma}$, and then $g(K')_0^\sigma g^{-1} = K_0^{\mathrm{Inn}(g\sigma(g)^{-1})\circ\sigma}$. By Corollary 3.5, $g\sigma(g)^{-1} \in N_G(K) = K \cdot Z_G(K)$. So $\mathrm{Inn}(g\sigma(g)^{-1})|_K$ is an inner automorphism of K. By Theorem 2.1, $\mathrm{rank}\, K_0^\sigma = \mathrm{rank}\, K_0^{\mathrm{Inn}(g\sigma(g)^{-1})\circ\sigma}$, that is, $\dim T = \dim gT'g^{-1} = \dim T'$. Hence $T' = T$. □

Let G be a connected Lie group with an automorphism σ of finite order, and let T be a maximal compact torus of G_0^σ. By Proposition 3.6, we can choose a σ-invariant maximal compact subgroup K of G containing T. The natural inclusion $N_{\sigma,K}(T) \hookrightarrow N_{\sigma,G}(T)$ induces a natural map $W(K, \sigma|_K, T) \to W(G, \sigma, T)$.

Proposition 3.7 *Under the above conditions, the natural map $W(K, \sigma|_K, T) \to W(G, \sigma, T)$ is an isomorphism.*

Proof By Lemma 3.4, we have

$$\begin{aligned} W(G, \sigma, T) &= N_{\sigma,G}(T)/Z_{\sigma,G}(T) \\ &= N_{\sigma,K}(T) \cdot Z_{\sigma,G}(T)/Z_{\sigma,G}(T) \\ &\cong N_{\sigma,K}(T)/N_{\sigma,K}(T) \cap Z_{\sigma,G}(T) \\ &= N_{\sigma,K}(T)/Z_{\sigma,K}(T) \\ &= W(K, \sigma|_K, T). \end{aligned}$$

It is obvious that the inverse of the above isomorphism coincides with the natural map $W(K, \sigma|_K, T) \to W(G, \sigma, T)$. This proves the proposition. □

Proposition 3.8 *$W(G, \sigma, T)$ is a finite group.*

Proof First assume G is compact. Since $Z_\sigma(T)$ and $N_\sigma(T)$ are closed subgroups of G, they have finitely many connected components. But by (ii) of Proposition 3.2, their identity components coincide. So $W(G, \sigma, T) = N_\sigma(T)/Z_\sigma(T)$ is finite.

Now suppose G is noncompact. By Proposition 3.6, there exists a σ-invariant maximal compact subgroup K of G such that T is a maximal torus of K_0^σ. By Proposition 3.7, $W(G, \sigma, T) \cong W(K, \sigma|_K, T)$. But we have proved that $W(K, \sigma|_K, T)$ is finite. So $W(G, \sigma, T)$ is finite. □

The twisted conjugate action τ associated with σ induces naturally an action of W on T, defined by

$$w.t = \tau_g(t),$$

where $w \in W(G, \sigma, T)$, $t \in T$, $g \in w$. If σ is of finite order and n is a positive integer which is divisible by the order of σ, then the finite subgroup $E_n(T) = \{t \in T | t^n = e\}$ of T is invariant under the natural action of W on T. In fact, if $t \in E_n(t)$, $g \in N_\sigma(T)$, then $\tau_g(t)^n = (gt\sigma(g)^{-1})^n = (gt\sigma(g^{-1}))\sigma(gt\sigma(g^{-1})) \cdots \sigma^{n-1}(gt\sigma(g^{-1})) = gt^n\sigma^n(g^{-1}) = e$, that is, $\tau_g(t) \in E_n(T)$. So W acts naturally on $E_n(T)$.

The following result is important for us to prove the main result in this paper.

Theorem 3.9 *Let G be a connected compact Lie group with a 1-semisimple automorphism σ, T a maximal torus of G_0^σ, $W = W(G, \sigma, T)$. Then two elements $t_1, t_2 \in T$ are σ-conjugate if and only if they lie in the same W-orbit.*

Proof The "if" part is obvious. To prove the converse, assume that $t_1, t_2 \in T$ are σ-conjugate, that is, there exists $g \in G$ such that $t_2 = \tau_g(t_1)$. Let $G_{t_1} = \{h \in G | \tau_h(t_1) = t_1\}$, and let $(G_{t_1})_0$ be the identity component of G_{t_1}. It is obvious that $T \subset (G_{t_1})_0$. Note that $h \in G_{t_1} \Leftrightarrow ht_1\sigma(h)^{-1} = t_1 \Leftrightarrow h = t_1\sigma(h)t_1^{-1} \Leftrightarrow h \in G^{\mathrm{Inn}(t_1)\circ\sigma}$, that is, $G_{t_1} = G^{\mathrm{Inn}(t_1)\circ\sigma}$. By Theorem 2.1, $\mathrm{rank} G_0^\sigma = \mathrm{rank}(G_{t_1})_0$. So T is a maximal torus of $(G_{t_1})_0$.

We claim that $g^{-1}Tg \subset (G_{t_1})_0$. In fact, for every $t \in T$, we have $\tau_{g^{-1}tg}(t_1) = \tau_{g^{-1}} \circ \tau_t \circ \tau_g(t_1) = \tau_{g^{-1}} \circ \tau_t(t_2) = \tau_{g^{-1}}(t_2) = t_1$. So $g^{-1}Tg \subset G_{t_1}$. But $g^{-1}Tg$ is connected, so $g^{-1}Tg \subset (G_{t_1})_0$.

Since T and $g^{-1}Tg$ are maximal tori of $(G_{t_1})_0$, there exists $x \in (G_{t_1})_0$ such that $xTx^{-1} = g^{-1}Tg$. Let $y = gx$, then $y^{-1}Ty = T$, and then $\tau_{y^{-1}}(T) = y^{-1}T\sigma(y) = y^{-1}Tyy^{-1}t_2\sigma(y) = Tx^{-1}g^{-1}t_2\sigma(g)\sigma(x) = T\tau_{x^{-1}}(t_1) = Tt_1 = T$. So $y \in N_\sigma(T)$. But $\tau_y(t_1) = \tau_g \circ \tau_x(t_1) = \tau_g(t_1) = t_2$. So t_1 and t_2 lie in the same W-orbit. This completes the proof of the theorem. □

4 Twisted Weyl groups and nonabelian cohomology

In this section we define the twisted Weyl group associated with an A-module structure on a Lie group, and give the proof Theorem 1.1.

Let A be a cyclic group, that is, $A \cong \mathbb{Z}$ or $A \cong \mathbb{Z}/n\mathbb{Z}$, and let G be a connected Lie group with an A-module structure. *If $A \cong \mathbb{Z}$, we always assume that G is compact and the $\mathbb{Z}$-module structure on G is 1-semisimple*, that is, the action of a generator of $\mathbb{Z}$ on G is 1-semisimple. Let T be a maximal compact torus of G^A. Then for every generator σ of A, we can construct the twisted Weyl group $W(G, \sigma, T)$ associated with σ.

Proposition 4.1 *Under the above conditions, $W(G, \sigma, T)$ is independent of the choice of the generator σ of A.*

Proof It is sufficient to prove that $Z_\sigma(T)$ and $N_\sigma(T)$ are independent of the choice of σ. By (i) of Proposition 3.2, $Z_\sigma(T) \subset G^A$. So $Z_\sigma(T) = Z_{G^A}(T)$, which is obviously independent of the choice of σ.

To prove that $N_\sigma(T)$ is independent of the choice of σ, let σ' be a generator of A different from σ. If $A \cong \mathbb{Z}$, then $\sigma' = \sigma^{-1}$. Let $g \in N_\sigma(T)$, then

$$\begin{aligned} gT\sigma'(g)^{-1} &= gT\sigma^{-1}(g^{-1}) = \sigma^{-1}(\sigma(g)Tg^{-1}) \\ &= \sigma^{-1}((gT\sigma(g)^{-1})^{-1})) = \sigma^{-1}(T^{-1}) = T. \end{aligned}$$

So $g \in N_{\sigma'}(T)$. Hence $N_\sigma(T) \subset N_{\sigma'}(T)$. By symmetry, we have $N_\sigma(T) = N_{\sigma'}(T)$. If $A \cong \mathbb{Z}/n\mathbb{Z}$, then there exist positive integers r and s such that $\sigma' = \sigma^r, \sigma = (\sigma')^s$. Let $g \in N_\sigma(T)$, then

$$\begin{aligned} gT\sigma'(g)^{-1} = gT\sigma^r(g^{-1}) &= (gT\sigma(g^{-1}))(\sigma(g)T\sigma^2(g^{-1}))\cdots(\sigma^{r-1}(g)T\sigma^r(g^{-1})) \\ &= (gT\sigma(g^{-1}))\sigma(gT\sigma(g^{-1}))\cdots\sigma^{r-1}(gT\sigma(g^{-1})) = T^r = T. \end{aligned}$$

Hence $N_\sigma(T) \subset N_{\sigma'}(T)$. By symmetry, we also have $N_\sigma(T) = N_{\sigma'}(T)$. □

In virtue of Proposition 4.1, we can define the *twisted Weyl group associated with the A-module G* by

$$W(G, A, T) = W(G, \sigma, T).$$

By Proposition 3.1, $W(G, A, T)$, as an abstract group, is independent of the choice of T. We will simply denote $W(G, A, T)$ by W if the omitted data are explicit from the context.

Now consider the cohomology $H^1(A, T)$. Since T is abelian and A acts trivially on T, $H^1(A, T)$ coincides with the set of cocycles $Z^1(A, T)$, and an element $\alpha \in Z^1(A, T)$ is a homomorphism $\alpha : A \to T$. Then $W(G, A, T)$ acts naturally on $Z^1(A, T) \cong H^1(A, T)$ by

$$(w.\alpha)(a) = g\alpha(a)a(g)^{-1},$$

where $w \in W(G, A, T), \alpha \in Z^1(A, T), a \in A, g \in w$.

Since A is cyclic, a homomorphism $\alpha : A \to T$ is determined by its value $\alpha(\sigma)$ on a generator σ of A. So if a generator σ of A is chosen, we have natural identifications $T \cong Z^1(\mathbb{Z}, T) \cong H^1(\mathbb{Z}, T)$ and $E_n(T) \cong Z^1(\mathbb{Z}/n\mathbb{Z}, T) \cong H^1(\mathbb{Z}/n\mathbb{Z}, T)$. Under these identifications, it is obvious that the natural action of $W(G, \mathbb{Z}, T)$ on $H^1(\mathbb{Z}, T)$ coincides with the natural action of $W(G, \sigma, T)$ on T, and the natural action of $W(G, \mathbb{Z}/n\mathbb{Z}, T)$ on $H^1(\mathbb{Z}/n\mathbb{Z}, T)$ coincides with the natural action of $W(G, \sigma, T)$ on $E_n(T)$.

The natural inclusion $T \hookrightarrow G$ induces a natural map $H^1(\mathbb{Z}, T) \to H^1(\mathbb{Z}, G)$, which obviously reduces to a map $W\backslash H^1(\mathbb{Z}, T) \to H^1(\mathbb{Z}, G)$. Now we are prepared to prove the main result of this paper.

Theorem 4.2 *Let A be a cyclic group, G a connected Lie group with an A-module structure. If $A \cong \mathbb{Z}$, we assume that G is compact and the $\mathbb{Z}$-module structure on G is 1-semisimple. Let T be a maximal compact torus of G_0^A, $W = W(G, A, T)$ the associated twisted Weyl group. Then the map $W\backslash H^1(A, T) \to H^1(A, G)$ is a bijection.*

Proof By Theorems 4.1 and 5.1 in [1], the natural map $H^1(A, T) \to H^1(A, G)$ is surjective, so $W\backslash H^1(A, T) \to H^1(A, G)$ is surjective.

To prove the injectivity, we first assume $A \cong \mathbb{Z}$. Choose a generator σ of $\mathbb{Z}$. Then there is an identification $H^1(\mathbb{Z}, T) \cong T$, under which the natural action of $W(G, \mathbb{Z}, T)$ on $H^1(\mathbb{Z}, T)$ coincides with the natural action of $W(G, \sigma, T)$ on T. Suppose $t_1, t_2 \in T \cong H^1(\mathbb{Z}, T)$ have the same image in $H^1(\mathbb{Z}, G)$ under the natural map $H^1(\mathbb{Z}, T) \to H^1(\mathbb{Z}, G)$. Then t_1 and t_2 are σ-conjugate. By Theorem 3.9, t_1 and t_2 lie in the same $W(G, \sigma, T)$-orbit. So $W\backslash H^1(\mathbb{Z}, T) \to H^1(\mathbb{Z}, G)$ is injective.

Now assume that $A \cong \mathbb{Z}/n\mathbb{Z}$. By Proposition 3.6, there exists a maximal compact subgroup K of G which is a $\mathbb{Z}/n\mathbb{Z}$-submodule such that T is a maximal torus of $K_0^{\mathbb{Z}/n\mathbb{Z}}$. By Theorem 3.1 in [1], the natural map $H^1(\mathbb{Z}/n\mathbb{Z}, K) \to H^1(\mathbb{Z}/n\mathbb{Z}, G)$ is bijective. So it is sufficient to prove that $W\backslash H^1(\mathbb{Z}/n\mathbb{Z}, T) \to H^1(\mathbb{Z}/n\mathbb{Z}, K)$ is injective. Let σ' be a generator of $\mathbb{Z}/n\mathbb{Z}$. Then there is an identification $H^1(\mathbb{Z}/n\mathbb{Z}, T) \cong E_n(T)$. Suppose $t_1, t_2 \in E_n(T) \cong H^1(\mathbb{Z}/n\mathbb{Z}, T)$ have the same image in $H^1(\mathbb{Z}/n\mathbb{Z}, K)$ under the natural map $H^1(\mathbb{Z}/n\mathbb{Z}, T) \to H^1(\mathbb{Z}/n\mathbb{Z}, K)$. Then t_1 and t_2 are $\sigma'|_K$-conjugate. By Theorem 3.9, t_1 and t_2 lie in the same $W(K, \sigma'|_K, T)$-orbit. But by Proposition 3.7, $W(G, \sigma', T) \cong W(K, \sigma'|_K, T)$, and it is obvious that the natural actions of $W(G, \sigma', T)$ and $(K, \sigma'|_K, T)$ on T coincide. So $W\backslash H^1(\mathbb{Z}, T) \to H^1(\mathbb{Z}, G)$ is injective. □

Remark 4.1 By Proposition 3.6, for a connected Lie group G with a $\mathbb{Z}/n\mathbb{Z}$-module structure, the family $\{T | T$ is a maximal compact torus of $G_0^{\mathbb{Z}/n\mathbb{Z}}\}$ coincides with the family $\bigcup_K \{T | T$ is a maximal torus of $K_0^{\mathbb{Z}/n\mathbb{Z}}\}$, where K runs through all maximal compact subgroups of G which are $\mathbb{Z}/n\mathbb{Z}$-submodules.

Acknowledgement The author would like to thank Jiu-Kang Yu for kind guidance.

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