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***Abstract*—Consider a binary statistical hypothesis testing problem, where n independent and identically distributed random variables Z^n are either distributed according to the null hypothesis P or the alternate hypothesis Q, and only P is known. A well-known test that is suitable for this case is the so-called Hoeffding test, which accepts P if the Kullback-Leibler (KL) divergence between the empirical distribution of Z^n and P is below some threshold. In this work, we characterize the first and second-order terms of the type-II error probability for a fixed type-I error probability for the Hoeffding test as well as for divergence tests, where the KL divergence is replaced by a general divergence. We demonstrate that, irrespective of the divergence, divergence tests achieve the first-order term of the Neyman-Pearson test, which is the optimal test when both P and Q are known. In contrast, the second-order term of divergence tests is strictly worse than that of the Neyman-Pearson test. We further demonstrate that divergence tests with an invariant divergence achieve the same second-order term as the Hoeffding test, but divergence tests with a non-invariant divergence may outperform the Hoeffding test for some alternate hypotheses Q.**

I. Introduction

Consider a binary hypothesis testing problem that decides whether a sequence of independent and identically distributed (i.i.d.) random variables Z^n is either generated from distribution P or from distribution Q. Assume that both distributions are discrete and the hypothesis test has access to P but not to Q. A suitable test for this case is the well-known *Hoeffding test* [1], which accepts P if $D_{\mathrm{KL}}(T_{Z^n}\|P) < c$, for some $c > 0$, and otherwise accepts Q. Here, T_{Z^n} is the type (the empirical distribution) of Z^n and $D_{\mathrm{KL}}(P\|Q)$ is the Kullback-Leibler (KL) divergence between P and Q [2]. In this paper, we analyze the second-order performance of the Hoeffding test as well as of Hoeffding-like tests, referred to as *divergence tests*, where the KL divergence is replaced by other divergences (see Section II for a rigorous definition).

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We focus on the asymptotic behaviour of the type-II error β_n (the probability of declaring hypothesis P under hypothesis Q) for a fixed type-I error α_n (the probability of declaring hypothesis Q under hypothesis P). When both P and Q are known, the optimal test is the likelihood ratio test, also known as the Neyman-Pearson test. For this test, the smallest type-II error β_n for which $\alpha_n \le \epsilon$ satisfies [3, Prop. 2.3]

$$-\ln \beta_n = nD_{\mathrm{KL}}(P\|Q) - \sqrt{nV(P\|Q)}\mathsf{Q}^{-1}(\epsilon) + o(\sqrt{n}) \quad (1)$$

as $n \to \infty$, where

$$V(P\|Q) \triangleq \sum_{i=1}^{k} P_i \left[\left(\ln \frac{P_i}{Q_i} - D_{\mathrm{KL}}(P\|Q)\right)^2\right] \quad (2)$$

denotes the divergence variance; $\mathsf{Q}^{-1}(\cdot)$ denotes the inverse of the tail probability of the standard Normal distribution; P_i and Q_i denote the i-th components of P and Q; and k denotes their dimension. Here and throughout this paper, we write $a_n = o(b_n)$ for two sequences $\{a_n\}$ and $\{b_n\}$ of real numbers if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. We write $a_n = O(b_n)$ if $\overline{\lim}_{n\to\infty} |\frac{a_n}{b_n}| < \infty$. By inspecting the expansion of $-\ln \beta_n$ in (1), one can define the first-order term β' and the second-order term β'' of any hypothesis test $\mathbb{T}$ as

$$\beta' \triangleq \lim_{n\to\infty} \frac{-\ln \beta_n(\mathbb{T})}{n} \quad (3)$$

and

$$\beta'' \triangleq \lim_{n\to\infty} \frac{-\ln \beta_n(\mathbb{T}) - n\beta'}{\sqrt{n}} \quad (4)$$

if the limits exist. The first-order term β' is sometimes referred to as the *error exponent*. For the Neyman-Pearson test, we have $\beta' = D_{\mathrm{KL}}(P\|Q)$ and $\beta'' = -\sqrt{V(P\|Q)}\mathsf{Q}^{-1}(\epsilon)$.

It was shown in [1] that the first-order term β' of the Hoeffding test is also $D_{\mathrm{KL}}(P\|Q)$. In other words, the Hoeffding test is first-order optimal. Recently, we have demonstrated [4] that the second-order term of the Hoeffding test is $\beta'' = -\sqrt{V(P\|Q)\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\epsilon)}$, where $\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\cdot)$ denotes the inverse of the tail probability of the chi-square distribution with $k-1$ degrees of freedom. Since $\sqrt{\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\epsilon)} > \mathsf{Q}^{-1}(\epsilon)$, it follows that the second-order performance of the Hoeffding test is worse than that of the Neyman-Pearson test.

In this paper, we analyze the second-order performance of the divergence test $\mathbb{T}^D$, which accepts P if $D(T_{Z^n}\|P) < c$, for some $c > 0$, and otherwise accepts Q. The divergence D of the divergence test $\mathbb{T}^D$ is arbitrary, so $\mathbb{T}^D$ includes the Hoeffding test as a special case when $D = D_{\text{KL}}$. We demonstrate that the divergence test $\mathbb{T}^D$ achieves the same first-order term β' as the Neyman-Pearson test, irrespective of the divergence D. Hence, $\mathbb{T}^D$ is first-order optimal for every divergence D. We further demonstrate that, for the class of *invariant divergences* [5], which includes the Rényi divergence and the f-divergence (and, hence, also the KL divergence), the divergence test $\mathbb{T}^D$ achieves the same second-order term β'' as the Hoeffding test. In contrast, we show that a divergence test $\mathbb{T}^D$ with a non-invariant divergence may achieve a second-order term β'' that is strictly better than that of the Hoeffding test for some Q and ϵ.

A. Related Work

The considered hypothesis testing problem falls under the category of *composite hypothesis testing* [6]. Indeed, in composite hypothesis testing, the test has no access to the distribution P of the null hypothesis and the distribution Q of the alternate hypothesis, but it has the knowledge that P and Q belong to the sets of distributions $\mathcal{P}$ and $\mathcal{Q}$, respectively. Our setting corresponds to the case where $\mathcal{P} = \{P\}$ and $\mathcal{Q} = \mathcal{P}^c$ (where we use the notation $\mathcal{A}^c$ to denote the complement of a set $\mathcal{A}$).

The Hoeffding test is a particular instance of the *generalized likelihood-ratio test (GLRT)* [7], which is arguably the most common test used in composite hypothesis testing. A useful benchmark for the Hoeffding test is the Neyman-Pearson test, which is the optimal test when both P and Q are known. As mentioned before, the Hoeffding test achieves the same first-order term β' as the Neyman-Pearson test, both in *Stein's regime*, where the type-I error satisfies $\alpha_n \leq \epsilon$, as well as in the *doubly-exponential regime*, where $\alpha_n \leq e^{-n\gamma}$, $\gamma > 0$; see, e.g., [1], [8]–[11]. Thus, the first-order term of the Neyman-Pearson test can be achieved without having access to the distribution Q of the alternate hypothesis. However, not having access to Q negatively affects higher-order terms. For example, for a given threshold γ, the type-I error of the Hoeffding test satisfies [11, Eq. (10)]

$$\alpha_n = n^{\frac{k-3}{2}} e^{-n\gamma}(c' + o(1)) \tag{5}$$

whereas for the corresponding Neyman-Pearson test [11, Eq. (9)]

$$\alpha_n = n^{-\frac{1}{2}} e^{-n\gamma}(c + o(1)). \tag{6}$$

Here, c and c' are constants that only depend on P, Q, and γ. Moreover, it was demonstrated in [9] that the variance of the normalized Hoeffding test statistic $nD_{\text{KL}}(T_{Z^n}\|P)$ converges to $\frac{1}{2}(k-1)$ as $n \to \infty$. Both results suggest that, for moderate n, the Hoeffding test scales unfavorably with the cardinality of P and Q, which motivated the authors of [9] to propose their *test via mismatched divergence*. The same observation can be made for Stein's regime. Indeed, as mentioned before, the second-order term of the Hoeffding test is [4]

$$\beta'' = -\sqrt{V(P\|Q)\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\epsilon)} \tag{7}$$

whereas the second-order term of the Neyman-Pearson test is [3, Prop. 2.3]

$$\beta'' = -\sqrt{V(P\|Q)}\mathsf{Q}^{-1}(\epsilon). \tag{8}$$

Since $\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\epsilon)$ is monotonically increasing in k, this again suggests an unfavorable scaling with the cardinality of P and Q.

Our setting where $\mathcal{P} = \{P\}$ and $\mathcal{Q} = \mathcal{P}^c$ was also studied by Watanabe [12], who proposed a test that is second-order optimal in some sense. The related case where only training sequences are available for both P and Q was considered in [13]. The test proposed in [13] was later shown to be second-order optimal [14].

II. Divergence and Divergence Test

A. Divergence

Let us consider a random variable Z that takes value in a discrete set $\mathcal{Z} = \{a_1, \cdots, a_k\}$ with cardinality $|\mathcal{Z}| = k \geq 2$. Let $\overline{\mathcal{P}}(\mathcal{Z})$ denote the set of probability distributions on $\mathcal{Z}$, and let $\mathcal{P}(\mathcal{Z})$ denote the set of probability distributions with strictly positive probabilities. Any probability distribution $R \in \mathcal{P}(\mathcal{Z})$ can be written as a length-k vector $R = (R_1, \cdots, R_k)^\mathsf{T}$, where $R_i \triangleq \Pr\{Z = a_i\}, i = 1, \cdots, k$. Note that this R can also be represented by its first $(k-1)$ components, denoted by the vector $\mathbf{R} = (R_1, \cdots, R_{k-1})^\mathsf{T}$, which takes value in the coordinate space

$$\Xi \triangleq \left\{ (R_1, \cdots, R_{k-1})^\mathsf{T} : R_i > 0, \sum_{i=1}^{k-1} R_i < 1 \right\}. \tag{9}$$

Given any two probability distributions $S, R \in \mathcal{P}(\mathcal{Z})$, one can define a non-negative function $D(S\|R)$, called a *divergence*, which represents a measure of discrepancy between them. A divergence is not necessarily symmetric in its arguments and also need not satisfy the triangle inequality; see [15], [16] for more details. More precisely, a divergence is defined as follows [15]:

Definition 1: Consider two distributions S and R in $\mathcal{P}(\mathcal{Z})$. A *divergence* $D \colon \mathcal{P}(\mathcal{Z}) \times \mathcal{P}(\mathcal{Z}) \to [0, \infty)$ between S and R, denoted by $D(S\|R)$, is a smooth function[1] of $\mathbf{S} \in \Xi$ and $\mathbf{R} \in \Xi$ (we may write $D(S\|R) = D(\mathbf{S}\|\mathbf{R})$) satisfying the following conditions:

1) $D(S\|R) \geq 0$ for every $S, R \in \mathcal{P}(\mathcal{Z})$.
2) $D(S\|R) = 0$ if, and only if, $S = R$.
3) When $\mathbf{S} = \mathbf{R} + \boldsymbol{\varepsilon}$ for some $\boldsymbol{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_{k-1})^\mathsf{T}$, the Taylor expansion of D satisfies

$$D(\mathbf{R} + \boldsymbol{\varepsilon}\|\mathbf{R}) = \frac{1}{2}\sum_{i,j=1}^{k-1} g_{ij}(\mathbf{R})\varepsilon_i\varepsilon_j + O(\|\boldsymbol{\varepsilon}\|_2^3) \tag{10}$$

as $\|\boldsymbol{\varepsilon}\|_2 \to 0$ for some $(k-1) \times (k-1)$-dimensional positive-definite matrix $G(\mathbf{R}) = [g_{ij}(\mathbf{R})]$ that depends on $\mathbf{R}$. In (10), $\|\boldsymbol{\varepsilon}\|_2$ is the Euclidean norm of $\boldsymbol{\varepsilon}$.

[1] We shall say that a function is *smooth* if it has partial derivatives of all orders.

4) Let $R \in \mathcal{P}(\mathcal{Z})$, and let $\{S_n\}$ be a sequence of distributions in $\mathcal{P}(\mathcal{Z})$ that converges to a distribution S on the boundary of $\mathcal{P}(\mathcal{Z})$. Then,

$$\varliminf_{n\to\infty} D(S_n \| R) > 0. \tag{11}$$

Remark 1: We follow the definition of divergence from the information geometry literature. In particular, according to [15, Def. 1.1], a divergence must satisfy the first three conditions in Definition 1. Often, the behavior of divergence on the boundary of $\mathcal{P}(\mathcal{Z})$ is not specified. In Definition 1, we add the fourth condition to treat the case of sequences of distributions $\{S_n\}$ that lie in $\mathcal{P}(\mathcal{Z})$ but converge to a distribution on the boundary of $\mathcal{P}(\mathcal{Z})$. Note that condition 4) is consistent with conditions 1) and 2).

Given a divergence D and $R \in \mathcal{P}(\mathcal{Z})$, consider the function $D(\cdot\|R)\colon \mathbb{R}^{k-1} \to \mathbb{R}$. By computing the partial derivatives of $D(S\|R)$ with respect to the first variable $\mathbf{S} = (S_1, \cdots, S_{k-1})^{\mathsf{T}}$, it follows from the third condition in Definition 1 that

$$D(S\|R) = (\mathbf{S}-\mathbf{R})^T \boldsymbol{A}_{D,\mathbf{R}}(\mathbf{S}-\mathbf{R}) + O(\|\mathbf{S}-\mathbf{R}\|_2^3) \tag{12}$$

as $\|\mathbf{S}-\mathbf{R}\|_2 \to 0$, where $\boldsymbol{A}_{D,\mathbf{R}}$ is the matrix associated with the divergence D at $\mathbf{R}$, which has components

$$a_{ij}(\mathbf{R}) \triangleq \frac{1}{2}\left.\frac{\partial^2}{\partial S_i \partial S_j} D(S\|R)\right|_{S=R}, \quad i,j = 1,\cdots,k-1. \tag{13}$$

Based on $\boldsymbol{A}_{D,\mathbf{R}}$, we can introduce the notion of an *invariant divergence*.

Definition 2: Let D be a divergence, and let $R \in \mathcal{P}(\mathcal{Z})$. Then, D is said to be an *invariant divergence* on $\mathcal{P}(\mathcal{Z})$ if the matrix associated with the divergence D at $\mathbf{R}$ is of the form $\boldsymbol{A}_{D,\mathbf{R}} = \eta \boldsymbol{\Sigma}_{\mathbf{R}}$ for a constant $\eta > 0$ (possibly depending on $\mathbf{R}$) and a matrix $\boldsymbol{\Sigma}_{\mathbf{R}}$ with components

$$\boldsymbol{\Sigma}_{ij}(\mathbf{R}) = \begin{cases} \frac{1}{R_i} + \frac{1}{R_k}, & i = j \\ \frac{1}{R_k}, & i \neq j. \end{cases} \tag{14}$$

The notion of an invariant divergence is adapted from the notion of invariance of geometric structures in information geometry; see [15], [17] for more details. The matrix $\boldsymbol{\Sigma}_{\mathbf{R}}$ represents the unique invariant Riemannian metric in $\mathcal{P}(\mathcal{Z})$ with respect to the coordinate system Ξ; see [18, Eq. (47)], [5] for more details. However, in the information geometry literature, the constant η is often required to be independent of $\mathbf{R}$. Well-known divergences, such as the KL divergence, the f-divergence, and the Rényi divergence, are invariant [19]. For an invariant divergence, (12) becomes

$$D(S\|R) = \eta(\mathbf{S}-\mathbf{R})^T \boldsymbol{\Sigma}_{\mathbf{R}}(\mathbf{S}-\mathbf{R}) + O(\|\mathbf{S}-\mathbf{R}\|_2^3) \tag{15}$$

as $\|\mathbf{S}-\mathbf{R}\|_2 \to 0$, where η is a positive constant.

There are many divergences that do not satisfy (15). An example is the *squared Mahalanobis distance*, which is of the form

$$D_{\text{SM}}(S\|R) = (\mathbf{S}-\mathbf{R})^{\mathsf{T}} \boldsymbol{W}_{\mathbf{R}}(\mathbf{S}-\mathbf{R}) \tag{16}$$

for some positive-definite matrix $\boldsymbol{W}_{\mathbf{R}}$. This divergence is non-invariant if $\boldsymbol{W}_{\mathbf{R}}$ is not a constant multiple of $\boldsymbol{\Sigma}_{\mathbf{R}}$.

For a detailed list of divergences and their properties, we refer to [19, Ch. 2].

B. General Setting and Divergence Test

We consider a binary hypothesis testing problem with null hypothesis H_0 and alternate hypothesis H_1. We assume that, under hypothesis H_0, the length-n sequence Z^n of observations is i.i.d. according to $P \in \mathcal{P}(\mathcal{Z})$; under hypothesis H_1, the sequence of observations Z^n is i.i.d. according to Q, where $Q \in \mathcal{P}(\mathcal{Z}) \setminus \{P\}$.

We next define the divergence test. To this end, we first introduce the *type distribution*, which for every sequence z^n is defined as

$$T_{z^n}(a_i) \triangleq \frac{1}{n}\sum_{\ell=1}^{n} \mathbf{1}\{z_\ell = a_i\}, \quad i = 1,\ldots,k \tag{17}$$

where $\mathbf{1}\{\cdot\}$ denotes the indicator function.

For a divergence D and a threshold $r > 0$, a *divergence test* $\mathbb{T}_n^D(r)$ for testing H_0 against the alternative H_1 is defined as follows:

Observe Z^n: if $D(T_{Z^n}\|P) < r$, then H_0 is accepted; else H_1 is accepted.

When the divergence D is the Kullback-Leibler divergence D_{KL}, the divergence test becomes the Hoeffding test, proposed by Hoeffding in [1].

For $r > 0$, define the acceptance region for H_0 as

$$\mathcal{A}_n^D(r) \triangleq \{z^n \colon D(T_{z^n}\|P) < r\}. \tag{18}$$

Then, the type-I and the type-II errors are given by

$$\alpha_n\big(\mathbb{T}_n^D(r)\big) \triangleq P^n\big(\mathcal{A}_n^D(r)^c\big) \tag{19}$$

$$\beta_n\big(\mathbb{T}_n^D(r)\big) \triangleq Q^n\big(\mathcal{A}_n^D(r)\big). \tag{20}$$

Our goal is to analyze the asymptotic behavior of the type-II error β_n when the type-I error satisfies $\alpha_n \le \epsilon$, $0 < \epsilon < 1$.

III. Main Results

The asymptotic behavior of the divergence test depends on the asymptotic behavior of the random variable $nD(T_{Z^n}\|P)$ in the limit as $n \to \infty$. For certain divergences, the limiting distribution of $nD(T_{Z^n}\|P)$ has been analyzed in the literature. For example, when D is the KL divergence, a well-known result by Wilks [20] states that $2nD_{\text{KL}}(T_{Z^n}\|P)$ converges in distribution to a chi-square random variable with $k-1$ degrees of freedom. This result generalizes to the α-divergence [21, Th. 3.1], [22, Th. 3]. In Lemma 1, we show that, for a general divergence D, $nD(T_{Z^n}\|P)$ converges in distribution to a *generalized chi-square random variable*, defined as follows:

Definition 3: The *generalized chi-square distribution* is the distribution of the random variable

$$\xi = \sum_{i=1}^{m} w_i \Upsilon_i \tag{21}$$

where w_i, $i = 1, \cdots, m$ are deterministic weight parameters and $\Upsilon_i, i = 1, \cdots, m$ are independent chi-square random variables with degree of freedom 1. We shall denote the generalized chi-square distribution with weight vector $\mathbf{w} = (w_1, \cdots, w_m)^\mathsf{T}$ and degrees of freedom m by $\chi^2_{\boldsymbol{w},m}$. If $w_i = 1$ for all i, then the generalized chi-square distribution becomes the chi-square distribution χ^2_m with degrees of freedom m.

Lemma 1: Let Z^n be a sequence of i.i.d. random variables distributed according to the distribution P of the null hypothesis, and let D be a divergence. Further let $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_{k-1})^\mathsf{T}$ be a vector that contains the eigenvalues of the matrix $\boldsymbol{\Sigma}_\mathbf{P}^{-1/2} \boldsymbol{A}_{D,\mathbf{P}} \boldsymbol{\Sigma}_\mathbf{P}^{-1/2}$, where $\boldsymbol{A}_{D,\mathbf{P}}$ is the matrix associated with the divergence D at $\mathbf{P}$ and the matrix $\boldsymbol{\Sigma}_\mathbf{P}$ is defined in (14). Then, the tail probability of the random variable $nD(T_{Z^n}\|P)$ satisfies

$$P^n(nD(T_{Z^n}\|P) \geq c) = \mathsf{Q}_{\chi^2_{\boldsymbol{\lambda},k-1}}(c) + O(\delta_n), \quad c \geq 0 \quad (22)$$

for some positive sequence $\{\delta_n\}$ that is independent of c and satisfies $\lim_{n\to\infty} \delta_n = 0$. Here, $\mathsf{Q}_{\chi^2_{\boldsymbol{\lambda},k-1}}(c) \triangleq \Pr(\xi \geq c)$ is the tail probability of the generalized chi-square random variable ξ with weight vector $\boldsymbol{\lambda}$ and degrees of freedom $k-1$.

Proof: Omitted due to space limitations. ■

We are now ready to present the main result of this paper:

Theorem 1: Let D be a divergence as defined in Definition 1, and let $0 < \epsilon < 1$. Further let $P, Q \in \mathcal{P}(\mathcal{Z})$ and $P \neq Q$. Recall that the cardinality of $\mathcal{Z}$ is $k \geq 2$. Then, for all sequences of thresholds $\{r_n\}$ satisfying

$$\alpha_n(\mathbb{T}_n^D(r_n)) \leq \epsilon \quad (23)$$

the divergence test $\mathbb{T}_n^D$ introduced in Section II-B satisfies

$$\begin{aligned} &\sup_{r_n:\, \alpha_n(\mathbb{T}_n^D(r_n)) \leq \epsilon} -\ln \beta_n\big(\mathbb{T}_n^D(r_n)\big) \\ &\quad = nD_{\mathrm{KL}}(P\|Q) - \sqrt{n}\sqrt{\mathbf{c}^\mathsf{T} \boldsymbol{A}_{D,\mathbf{P}}^{-1} \mathbf{c}} \sqrt{\mathsf{Q}^{-1}_{\chi^2_{\boldsymbol{\lambda},k-1}}(\epsilon)} \\ &\quad\quad + O(\max\{\delta_n\sqrt{n}, \ln n\}). \end{aligned} \quad (24)$$

Here, $\boldsymbol{A}_{D,\mathbf{P}}$ is the matrix associated with the divergence D at $\mathbf{P}$; the sequence $\{\delta_n\}$ was defined in (22); $\mathbf{c} = (c_1, \cdots, c_{k-1})^\mathsf{T}$ is a vector with components

$$c_i \triangleq \ln\left(\frac{P_i}{Q_i}\right) - \ln\left(\frac{P_k}{Q_k}\right), \quad i = 1, \cdots, k-1; \quad (25)$$

and $\mathsf{Q}^{-1}_{\chi^2_{\boldsymbol{\lambda},k-1}}$ is the inverse of the tail probability $\mathsf{Q}_{\chi^2_{\boldsymbol{\lambda},k-1}}$ introduced in Lemma 1.

Proof: Omitted due to space limitations. ■

Remark 2: Since the sequence $\{\delta_n\}$ tends to zero as $n \to \infty$, we have that $O(\max\{\delta_n\sqrt{n}, \ln n\}) = o(\sqrt{n})$.

Corollary 1: For the class of invariant divergences, (24) in Theorem 1 becomes

$$\begin{aligned} &\sup_{r_n:\, \alpha_n(\mathbb{T}_n^D(r_n)) \leq \epsilon} -\ln \beta_n\big(\mathbb{T}_n^D(r_n)\big) \\ &\quad = nD_{\mathrm{KL}}(P\|Q) - \sqrt{nV(P\|Q)\mathsf{Q}^{-1}_{\chi^2_{k-1}}(\epsilon)} + o(\sqrt{n}). \end{aligned} \quad (26)$$

Since the KL divergence belongs to the class of invariant divergences, it follows that (26) also characterizes the second-order performance of the Hoeffding test.

We observe from Theorem 1 that the divergence test $\mathbb{T}_n^D$ achieves the same first-order term β' as the Neyman-Pearson test, irrespective of D. In contrast, it can be shown that

$$-\sqrt{\mathbf{c}^\mathsf{T} \boldsymbol{A}_{D,\mathbf{P}}^{-1} \mathbf{c}} \sqrt{\mathsf{Q}^{-1}_{\chi^2_{\boldsymbol{\lambda},k-1}}(\epsilon)} < -\sqrt{V(P\|Q)}\mathsf{Q}_\mathcal{N}^{-1}(\epsilon). \quad (27)$$

Thus, the second-order term β'' of the divergence test $\mathbb{T}^D$ is strictly smaller than the second-order term of the Neyman-Pearson test.

In the next section, we show that there are divergences for which the divergence test outperforms the Hoeffding test for certain distributions Q of the alternate hypothesis.

IV. Second-Order Performance Comparison

In order to contrast the performances of different divergence tests, we numerically evaluate the second-order performances of $\mathbb{T}^{D_{\mathrm{KL}}}$ and $\mathbb{T}^{D_{\mathrm{SM}}}$, where D_{KL} is the KL divergence and D_{SM} is the squared Mahalanobis distance. Recall that the KL divergence is an invariant divergence. For the squared Mahalanobis distance, we shall consider (16) with $\boldsymbol{W}_\mathbf{P}$ having components

$$\boldsymbol{W}_{ij}(\mathbf{P}) = \begin{cases} \frac{1}{2P_i^2} + \frac{1}{2P_k^2}, & i = j \\ \frac{1}{2P_k^2}, & i \neq j \end{cases} \quad (28)$$

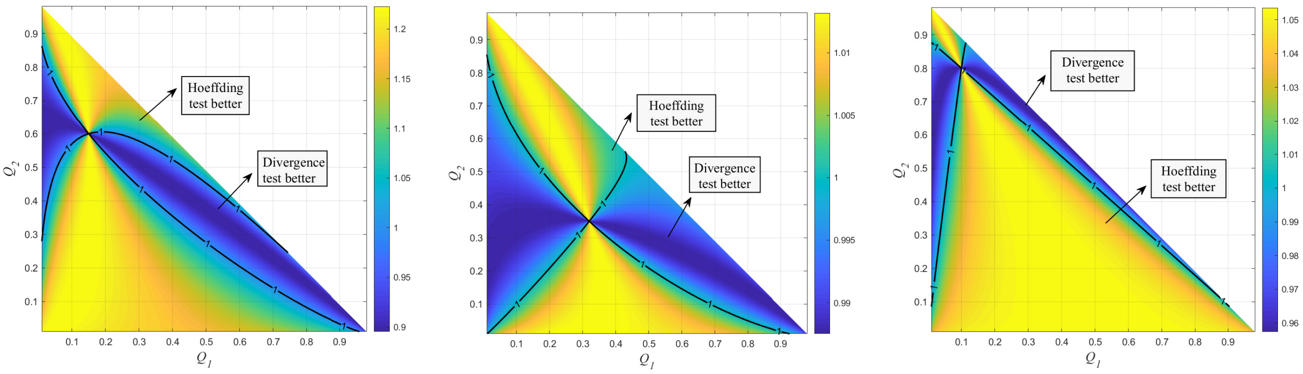
which is a non-invariant divergence. To better visualize the second-order performances, we focus on distributions with dimension $k = 3$ and represent them by the two-dimensional vectors $\mathbf{P} = (P_1, P_2)^\mathsf{T}$ and $\mathbf{Q} = (Q_1, Q_2)^\mathsf{T}$ in the coordinate space Ξ.

Since the first-order term β' of the divergence test $\mathbb{T}^D$ is not affected by the choice of D, we shall compare the second-order performances of $\mathbb{T}^{D_{\mathrm{KL}}}$ and $\mathbb{T}^{D_{\mathrm{SM}}}$ by considering the ratio of the second-order terms β'' as a function of P, Q, and ϵ:

$$\rho(P, Q, \epsilon) \triangleq \frac{\sqrt{\mathbf{c}^\mathsf{T} (\boldsymbol{W}_\mathbf{P})^{-1} \mathbf{c}} \sqrt{\mathsf{Q}^{-1}_{\chi^2_{\boldsymbol{\lambda},2}}(\epsilon)}}{\sqrt{V(P\|Q)} \sqrt{\mathsf{Q}^{-1}_{\chi^2_2}(\epsilon)}}. \quad (29)$$

If $\rho(P, Q, \epsilon) > 1$, then the second-order term of the divergence test is strictly smaller than the second-order term of the Hoeffding test, hence the Hoeffding test has a better second-order performance. In contrast, if $\rho(P, Q, \epsilon) < 1$, then the divergence test has a better second-order performance.

In Fig. 1, we plot the contour lines of the ratio $\rho(P, Q, \epsilon)$ as a function of $\mathbf{Q} \in \Xi$ for $\epsilon = 0.02$ and the three different null hypotheses $\mathbf{P} = (0.15, 0.6)$, $\mathbf{P} = (0.32, 0.35)$, and $\mathbf{P} = (0.1, 0.8)$. In the figure, the coordinate space Ξ is divided into two regions: one region is labeled as "Hoeffding test better" and includes the points $\mathbf{Q} \in \Xi$ for which $\rho(P, Q, \epsilon) > 1$; the other region is labeled as "Divergence test better" and includes the points $\mathbf{Q} \in \Xi$ for which $\rho(P, Q, \epsilon) < 1$. The solid contour line drawn in all three sub-figures shows all the points $\mathbf{Q} \in \Xi$ for which the Hoeffding test and the divergence



(a) $P = (0.15, 0.6, 0.25)$ (b) $P = (0.32, 0.35, 0.33)$ (c) $P = (0.1, 0.8, 0.1)$

Fig. 1: Second-order performance comparison between the Hoeffding test $\mathbb{T}^{D_{\mathrm{KL}}}$ and the divergence test $\mathbb{T}^{D_{\mathrm{SM}}}$ for the three different null hypotheses $P = (0.15, 0.6, 0.25)$, $P = (0.32, 0.35, 0.33)$, and $P = (0.1, 0.8, 0.1)$ and $\epsilon = 0.02$.

test have the same second-order performance. For each subfigure, the color bar on the right indicates the values of the ratio $\rho(P, Q, \epsilon)$.

Observe that there are distributions Q of the alternate hypothesis for which the Hoeffding test has a better second-order performance than the divergence test, and there are distributions Q for which the opposite is true. The set of distributions Q for which one test outperforms the other typically depends on the distribution P of the null hypothesis and on ϵ. Potentially, this behavior could be exploited in a composite hypothesis testing problem by tailoring the divergence D of the divergence test $\mathbb{T}^{D}$ to the set $\mathcal{Q}$ of possible alternate distributions.

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